Computational Linear Algebra

Notes and Questions for MATH 3325

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January 23, 2025

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Chapter 1

Solving Linear Systems

1.1 Warmup Problems

You should try to do the following activity right away.

Activity 1.1.1

(a) Solve:

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(b) Draw a graph of the solution set of the equation: $3x_1 - 2x_2 = 6$.

Hint. If a solution has $x_1 = a$, what is x_2 or viceversa?

(c) Draw a graph of the solution set of the equation: $-x_1 + x_2 = 1$.

(d) Graph the solution sets from the two previous steps together. How does your answer to part 1.1.1.a compare to your graph?

(e) Solve:

$$\begin{aligned} 2x_1 - 2x_2 &= 6 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(f) Solve:

$$\begin{aligned} 2x_1 - 2x_2 &= -2 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(g) Wait, what just happened? Explain the results of the previous two parts. What do the graphs of the corresponding solution sets look like in relation to the graphs of the equations?

(h) What are the possible intersections of two lines? Clearly state your conjecture which adresses ALL possibilities.

Throughout this course we will be doing many of the same things you did in the previous questions, but we will do them in a more general setting that will allow us to solve *many* new and old problems.

1.1.1 Application Warmup Problem

Here is an application problem where you can see the relevence of your work you just did.

Activity 1.1.2 Introduction to Traffic Flow. Consider the following diagram of a network with the flows indicated:

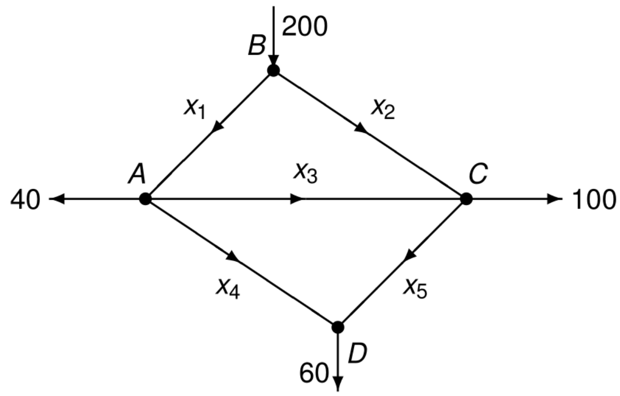


Figure 1.1.1 A network with directions of flow

If all the flows, x_1, x_2, x_3, x_4, and x_5, are all nonnegative, what is the ***largest possible value*** for x_3?

Hint. To answer this consider the following:

1. For each node (A, B, C, D) set up a conservation equation. Remember the flow in must equal the flow out.
2. How many linear equations do you have? How many unknowns?
3. Use your conservation equations for A and D to solve for x_3 in terms of x_1 and x_5. How might we solve for x_2 and x_4 in terms of x_3 and x_5?
4. Is our solution unique?

1.2 Solving Linear Systems

1.2.1 Elementary Operations

Objectives

- To understand the language and tools of efficently solving linear systems of equations
- To understand how to use matrices to store information about and solve linear systems
- To understand how echelon forms will give a form for equivalent systems of equations that will allow us to characterize the types of solutions to the system

Our first discussion of linear algebra will cover the ideas of efficiently solving a system of linear equations and matrix operations.

A system of m linear equations in n variables can be written:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m \end{array}$$

The term a_{ij} is the **coefficient** of the j-th variable (denoted x_j) in the i-th equation. In these notes, we will only consider real values for the coefficients of our linear systems, i.e. $a_{ij} \in \mathbb{R}$. A **solution** is a choice of variable values that satisfies *all* equations in the system. A solution is *not* a particular variable value but must include a choice for *all* variables in the system. The **solution set** for a system of equations is the set of all possible solutions. We will have many ways to describe solutions to a system this semester but they all specify the values of x_1, x_2, ..., and x_n, typically as an ordered n-tuple (x_1,x_2, ..., x_n).

Activity 1.2.1 Is $(1, 2, 3)$ a solution to the following system?

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & 7x_3 & = & 22 \end{array}$$

The previous problem shows how easy it is to check if a set of variable values is a solution. However, *finding* a solution or the set of all solutions is harder but very important to many problems. Generally speaking, the process of finding the solution set for a system of equations is to trade the system of equations you have for an **equivalent** system (a system with the same solution set).

Activity 1.2.2 For each pair of equations given, state whether E_1 is equivalent to E_2.

(a) $E_1 : x^2 - 1 = 0$ and $E_2 : x - 1 = 0$

(b) $E_1 : x^2 - 2x + 1 = 0$ and $E_2 : x - 1 = 0$

(c) $E_1 : e^x = 1$ and $E_2 : x^3 + x^2 + x = 0$

Hopefully it will be easier to explicitly write the solution set of the new equivalent system.

Definition 1.2.1 An **elementary operation** on a system of equations is an operation of the form:

1. multiplying an equation by a non-zero scalar
2. switching two equations
3. adding a multiple of one equation to another equation

$\diamondsuit$

Activity 1.2.3 For this question, we will consider the following system of linear equations:

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4$$
$$b_1x_1 + b_2x_2 + b_3x_3 = b_4$$

(a) Multiply the second equation in our system by negative three and state the *new* system of equations.

(b) Write a few sentences about why the new system of equations given in the previous part is equivalent to the original system.

(c) Write out the equation obtained by multiplying the second equation in the original system by a non-zero scalar (which we will call k) and adding to the first equation.

(d) Replace the second equation in the original system with your answer to the previous part, which we will call System 2. Prove that System 2 is equivalent to the original system. In other words, you need to show that (c_1, c_2, c_3) is a solution of the equations S_1:

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= a_4 \\ b_1x_1 + b_2x_2 + b_3x_3 &= b_4 \end{aligned}$$

if and only if (c_1, c_2, c_3) is a solution to System 2.

Activity 1.2.4 Solve the following systems just using elementary operations. Remember to show your work.

(a)

$$\begin{aligned} 2y + z &= 4 \\ x - 3y + 2z &= 5 \\ 2x + y &= -2 \end{aligned}$$

(b)

$$\begin{aligned} 3x - 2y - z &= 0 \\ 2x + y + z &= 10 \\ x + 4y + 3z &= 20 \end{aligned}$$

(c)

$$\begin{aligned} 3x - 2y - z &= 0 \\ 2x + y + z &= 10 \\ x + 4y + 3z &= 10 \end{aligned}$$

A system of equations is **consistent** if there exists at least one solution to the system. In other words, a consistent system of equations has a nonempty solution set. A system that is not consistent is said to be **inconsistent**.

In Activity 1.2.4, note that you didn't change anything but the *coefficients* in the system of equations as you traded one system for another. Some of the coefficients probably became zero, but you didn't really eliminate any variables or consider a totally different problem. We will use matrices to efficiently store, and manipulate the coefficients in a system of linear equations, since they are all that matter for now. Matrices will have *many* uses in this and other courses, and we will use capital letters like A and B to denote matrices. Matrices will be rectangular arrays with the same number of entries in each row and the same number of entries in each column. The size of a matrix is given (in order) as the number of rows by the number of columns, so a 3 by 2 matrix has 3 rows and 2 columns.

In order to specify what **entry** we are referring to in a matrix, we need an ordered pair of indices telling us the number of the row and number of the column to look in respectively. For instance, if

$$B = \begin{bmatrix} 1 & 5 & 0 \\ \heartsuit & \bigstar & \blacklozenge \\ \pounds & \circledR & \maltese \end{bmatrix},$$

then the $(3,2)$ entry of B is in the third row and 2nd column. You could also write this as $B_{3,2} = \circledR$. The ***i*-th row** of a matrix A will be denoted $row_i(A)$ and the ***j*-th column** will be denoted $column_j(A)$.

In order to distinguish **vectors** (as being more than just n by 1 matrices), we will use the arrow notation and lower case symbols like $\vec{u}$ and $\vec{v}$ to denote vectors. Unless otherwise stated, we will use column vectors. For instance, if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$, then the second **component** of $\vec{v}$ is the scalar v_2. The size of a vector in $\mathbb{R}^n$ is the number of components the vector has. In later work, we will deal with a *much* more general notion of vectors that will *not* have components like vectors in $\mathbb{R}^n$.

The **coefficient matrix** of a linear system of m equations in n variables is a m by n matrix whose (i,j) entry is the coefficient of the j-th variable, x_j, in the i-th equation of the system. The **augmented matrix** of a linear system of m equations in n variables is a m by $(n+1)$ matrix whose first n columns are the coefficient matrix of the system and the last column is the constant terms from the right side of each equation.

The system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m \end{array}$$

has a coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

and an augmented matrix of

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{array}\right]$$

For some properties of the system of equations, we need only look at the coefficient matrix but others will need the augmented matrix. It is important to know the difference and be able to state which corresponding matrix you are using in your work.

Question 1.2.2 For each system given, write the coefficent matrix.

(a)

$$3x_1 - 2x_2 = 6$$

$$\begin{aligned} 2x_1 - 2x_2 &= -2 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(b)

$$3x_1 - 2x_2 - 4x_3 = 6$$

(c)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= 6 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(d)

$$\begin{aligned} x &= 5 \\ 4x &= -1 \\ -2x &= 10 \end{aligned}$$

(e)

$$\begin{aligned} 7x &= 3 \\ 2y &= 1 \\ 3z &= -2 \end{aligned}$$

(f)

$$\begin{aligned} 3r - 5s + t &= 2 \\ -6r + 10s - 2t &= 3 \end{aligned}$$

□

Question 1.2.3 For each system given, write the corresponding augmented matrix.

(a)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= -2 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(b)

$$3x_1 - 2x_2 - 4x_3 = 6$$

(c)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= 6 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(d)

$$x = 5$$

$$\begin{aligned} 4x &= -1 \\ -2x &= 10 \end{aligned}$$

(e)

$$\begin{aligned} 7x &= 3 \\ 2y &= 1 \\ 3z &= -2 \end{aligned}$$

(f)

$$\begin{aligned} 3r - 5s + t &= 2 \\ -6r + 10s - 2t &= 3 \end{aligned}$$

□

The elementary operations on equations outlined above will correspond to elementary row operations on matrices as well. Specifically, an **elementary row operation** on a matrix is an operation of the form:

- multiplying a row by a non-zero scalar
- switching two rows
- adding a multiple of one row to another row

Question 1.2.4 Using the matrix A given below, perform each of the following row operations:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 0 & -1 \\ 0 & 5 & -1 & 2 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_3$
2. $-3R_2 \to R_2$
3. $2R_2 + R_1 \to R_1$

□

We now have operations to trade our system of equations for an equivalent system, but we have not stated a way to make sure that the solution set will be easy to explicitly state from our new equivalent system. The following matrix forms will be useful for determining solution sets and various other properties of the corresponding system of equations.

Definition 1.2.5 A rectangular matrix is in **row echelon form** if it has the following three properties:

- All nonzero rows are above any rows of all zeros.
- Each **leading entry** (being the first non-zero entry) of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

If a matrix in row echelon form satisfies the following additional properties, then we say the matrix is in **reduced row echelon form**:

- The leading entry in each nonzero row is 1.

- Each leading 1 is the only nonzero entry in its column.

◊

The leading entry in a nonzero row of the row echelon form is called a **pivot**. The column in which a pivot occurs is called a **pivot column** and the corresponding variable is a **basic variable** or **pivot variable**. A variable corresponding to a column in which the coefficient matrix does *not* have a pivot are called **free variables**. While the echelon form is needed to find where pivots will occur, we will sometimes refer to pivot positions of a matrix even when the matrix is not in echelon form.

Question 1.2.6 For each of the following matrices, determine if the matrix is in row echelon form, reduced row echelon form, or neither. If the matrix is in row echelon or reduced row echelon form, treat the matrix like an augmented matrix and describe each variable as either a free variable or a pivot variable.

1. $$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

2. $$A_2 = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

3. $$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

4. $$A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

5. $$A_5 = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

□

Theorem 1.2.7 *The reduced row echelon form of a rectangular matrix is unique.*

It is important to note that the row echelon form of a matrix is *not* unique.

Activity 1.2.5 Give an example of a matrix M that has the following properties. If such a matrix cannot exist, explain why.

(a) M satisfies the first two properties of row echelon form but does not satisfy the third.

(b) M satisfies the first and third properties of row echelon form but does not satisfy the second.

(c) M satisfies the second and third properties of row echelon form but does not satisfy the first.

(d) M satisfies the three properties of row echelon form but does not satisfy the first property of reduced row echelon form.

(e) M satisfies the properties of row echelon form and the first property of reduced row echelon form but does not satisfy the second property of reduced row echelon form.

Example 1.2.8

(a) In this example, we will list out ALL of the possible row echelon forms of a two by two matrix. We will use the symbols ■ for a non-zero entry and $*$ for an entry that can be any real number. We will also use 0 and 1 for entries that must be 0 and 1.

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

You should think carefully to see if there are any possibilities missing here.

(b) In this example, we will list out ALL of the possible reduced row echelon forms of a two by two matrix. We will use the symbols ■ for a non-zero entry and $*$ for an entry that can be any real number. We will also use 0 and 1 for entries that must be 0 and 1.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

You should think carefully to see if there are any possibilities missing here.

□

Activity 1.2.6 List out all possible row echelon forms of 3 by 4 matrices using the symbols ■ for a pivot, $*$ for a non-pivot entry (possibly 0), and 0 (when an entry *must* be 0). For each of these, list out which variables are pivot variables and which are free variables.

Hint. There are 15 possible.

Activity 1.2.7 List out all possible reduced row echelon forms of 3 by 4 matrices using the symbols ■ for a pivot, $*$ for a non-pivot entry (possibly 0), and 0 (when an entry *must* be 0). What value must the ■ entries be? For each of these, list out which variables are pivot variables and which are free variables.

Activity 1.2.8 Solve each of the following systems by converting to an augmented matrix and using elementary row operations to reduce the augmented matrix to reduced row echelon form. With each reduced row echelon form, put a box around all pivot entries. Use the system of equations corresponding to the reduced row echelon form to write out the solution set for each system.

(a)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= -2 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= 6 \\ -x_1 + x_2 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} 4x - y + 3z &= 5 \\ 3x - y + 2z &= 7 \end{aligned}$$

(d)

$$\begin{aligned} 7x - 11y - 2z &= 3 \\ 8x - 2y + 3z &= 1 \end{aligned}$$

(e)

$$\begin{aligned} 3r - 5s + t &= 2 \\ -6r + 10s - 2t &= 3 \end{aligned}$$

Investigation 1.2.9 Once you have the augmented matrix for a system of linear equations in reduced row-echelon form, how do you use it to determine the solution set for the system? Write a step-by-step procedure that is general enough to be used on any system of linear equations. Be aware of any implicit assumptions you're making (and try to avoid them).

Two of the most important questions we will consider this semester are:

1. Is the system consistent?
2. If a solution exists, is the solution unique?

Investigation 1.2.10 Look back at your results so far and try to figure out what properties of the system (or corresponding matrices) will help us answer question 1 and which properties of the system will help us answer question 2. Write a conjecture about each question.

1.3 Consistency and Uniqueness Theorems

Objectives

- To understand when a system of equations will be consistent in terms of the number of pivot variables
- To understand when a system of equations will have a unique solution in terms of the number of pivot variables
- To understand how to use the reduced row echelon form of the augmented matrix for a linear system to easily write out the solution set

In class, we came up with statements of the following two theorems:

Theorem 1.3.1 Consistency Theorem. *A system of equations is consistent if and only if the row echelon form of its augmented matrix has no pivot entries in the rightmost column. Equivalently, a system of equations is inconsistent if and only if the row echelon form of its* augmented *matrix has a pivot entry in the rightmost column.*

Theorem 1.3.2 Uniqueness Theorem. *A system of m equations with n variables has a unique solution if and only if its* augmented *matrix has n pivot entries and no pivot entry in the rightmost column.*

1.3.1 Writing Solution Sets

Activity 1.3.1 For the matrix below, verify that the matrix is in rref (reduced row echelon form) and treat the matrix as an augmented matrix for a system of linear equations. Write out the corresponding system of equations. Use this system of equations to write each variable in terms of just free variables and constants.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -4 & 0 & -1 & 5 \\ 0 & 1 & 4 & 0 & 3 & 0 & 2 & -6 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

Question 1.3.3 Under what conditions would your process for the previous activity not work? In other words, when would it not be possible to write each variable in terms of just free variables and constants. □

1.3.2 Determining Consistency/Uniqueness of Solutions

Activity 1.3.2 Give an example matrix that fits each of the following conditions:

1. A 3 by 4 augmented matrix corresponding to a linear system with a unique solution
2. A 3 by 4 augmented matrix corresponding to a consistent linear system of equations that does NOT have a unique solution
3. A 3 by 4 augmented matrix corresponding to an inconsistent system of linear equations

Activity 1.3.3 Using the statement of the Consistency Theorem and Uniqueness Theorem, treat each of your answers to Activity 1.2.6 as an *augmented* matrix of a linear system of equations and state:

1. whether each corresponding system of equations will be consistent, inconsistent, or you can't tell.
2. whether each corresponding system of equations will have a unique solution, multiple solutions, no solutions, or you can't tell.

Activity 1.3.4 Using the statement of the Consistency Theorem and Uniqueness Theorem, treat each of your answers to Activity 1.2.6 as a *coefficient* matrix of a linear system of equations and state:

1. whether each corresponding system of equations will be consistent, inconsistent, or you can't tell.
2. whether each corresponding system of equations will have a unique solution, multiple solutions, no solutions, or you can't tell.

Hint. You will probably need to restate the theorems or think about how coefficient matrices are different to augmented matrices!

1.4 Geometric Interpretations and Applications

In this section we will look at some examples of geometric interpretation of solutions to a system of linear equations, then we will look at a few examples of common application problems related to our study of linear systems.

1.4.1 Geometric Interpretation of a Solution Set

Recall from earlier, that the solution set of a linear equation in two variables was a line in $\mathbb{R}^2$ (the plane) and that the solution set of a system of two equations in two variables was possibly a point, a line, or empty. Similarly, the solution set for a linear equation in three variables will be a plane in 3-space ($\mathbb{R}^3$).

Activity 1.4.1

1. List out all the possible ways two planes can intersect in a three dimensional space.

2. List out all the possible ways three planes can intersect in a three dimensional space.

3. List out all the possible ways four planes can intersect in a three dimensional space.

4. List out all the possible ways five planes can intersect in a three dimensional space.

We don't usually draw what a solution set of a linear equation in four variables looks like because drawing in four dimensions is difficult. The graph is a single linear equation in four variables would be called a hyperplane in 4-space. Although we don't draw m hyperplanes in n-space, the intersections of hyperplanes will work very similarly to the pictures we can draw in 3-space (also known as $\mathbb{R}^3$).

We can use the open source computer algebra system SageMath to plot things, and we can even do it right here in the course notes. Click the button to plot a plane below.

```
var('x,y');
plot3d(3*x-2*y,(x,-10,10),(y,-10,10),color='red')
```

Plotting the equations, $3x-2y-z=0$, $2x+y+z=10$, and $x+4y+3z=20$ in red, yellow, and green respectively gives:

```
var('x,y');
p1=plot3d(3*x-2*y,(x,-10,10),(y,-10,10),color='red')
p2=plot3d(10-2*x-y,(x,-10,10),(y,-10,10),color='yellow')
p3=plot3d((1/3)*(20-x-4*y),(x,-10,10),(y,-10,10),color='green')
show(p1+p2+p3,aspect_ratio=(1,1,.2))
```

Investigation 1.4.2 Does your answer to Task 1.2.4.b make sense with this plot? Explain.

Question 1.4.1 For each of the systems in Activity 1.2.8, use SageMath to draw a plot of each of the equations in the system and write a sentence for each system about why the plot and your answer to Activity 1.2.8 make sense.

Hint. You can edit the code block above and click the button again, and it will update the graph. □

If you remember parametric equations of lines and planes in space from multivariable calculus, then we will return to those ideas soon

1.4.2 Applications and Linear Algebra

Many network or physical problems are diagramed by a figure that displays how different parts are connected and how much of something can flow between different nodes. A particularly common diagram is that of electric circuits. We will look at a couple of laws now that help us set up a system of equations for common circuit types.

An electrical network is a specialized network where we specify the locations of resistors, batteries, devices powered by sources, and others.The goal is often to determine the current through various locations of the network. In balancing a network we use two specific laws: ***Current*** and ***Voltage***.

- Current Law: sum of the currents flowing into any node is equal to the sum of the currents flowing out of that node. This is often called the conservation of flow.
- Voltage Law: The sum of the voltage drops around any circuit is equal to the total voltage around the the circuit, most likely provided by batteries or other power source.

The above laws are attributed to Gustav Kirchhoff and are called *Kirchhoff's Laws*. We should also mention Ohm's Law, which describes the force, in volts, associated with the current (amps) passing through a resistor (ohms). Namely,

$$V = IR$$

Example 1.4.2 Consider the electrical network shown here

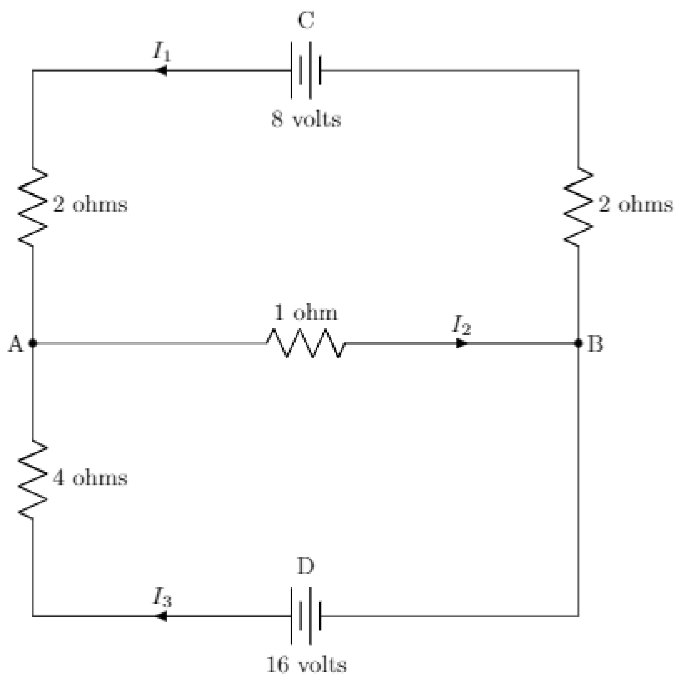


Figure 1.4.3 An electrical circuit

We can set up the equations for the three currents using our voltage and current law in conjunction with Ohm's Law. This yields

$$\begin{aligned} I_1 \quad -I_2 \quad +I_3 &= 0 \\ 4I_1 \quad +I_2 \qquad\quad &= 8 \\ I_2 \quad +4I_3 &= 16 \end{aligned}$$

The first equation is using our conservation of flow. The second equation is using Ohm's law with our voltage law around the circuit CABC. The third equation is similar, but around the circuit DABD.

The rref of the corresponding augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & |1 \\ 0 & 1 & 0 & |4 \\ 0 & 0 & 1 & |3 \end{bmatrix}$$

Thus we have a unique solution of $I_1 = 1, I_2 = 4, I_3 = 3$. □

Activity 1.4.3 For this activity, we will be considering the following circuit

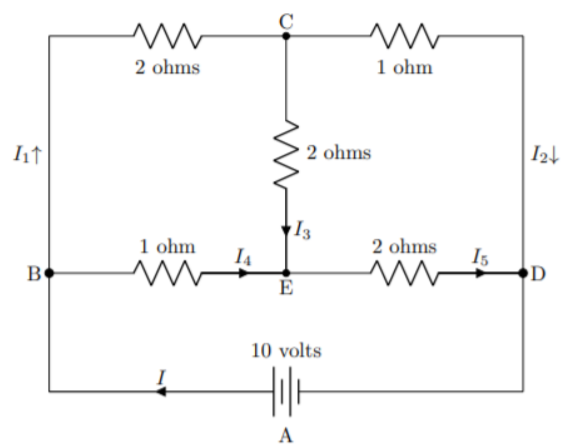


Figure 1.4.4 An electrical circuit

(a) Write out the current equations for each of the four junctions

(b) Write out the voltage drop equations for three of the basic circuits.

(c) Use Python to input the corresponding augmented matrix and solve the system based on the rref. Explain the meaning of your solution.

Another common application of linear systems is balancing a physical system according to conservation of some property. For instance, in chemical reactions, the number of different atoms of an element does not change, rather the way they are arranged in molecules is what changes. Thus the number of each kind of molecule must be the same on the right and left side of a reation equation.

Example 1.4.5 Let's look at the simple chemical equation for creating water from hydrogen and oxygen.

$$\alpha H_2 + \beta O_2 \to \gamma H_2O$$

We want to know how many molecules of each type are needed to go into the reaction and how many will come out. Note that the coefficents in this setting must be positive integers since we cannot have a fraction of a molecule.

If we consider the number of hydrogen atoms in the reaction, we get

$$2\alpha = 2\gamma$$

If we consider the number of oxygen atoms in the reaction, we get

$$2\beta = \gamma$$

Thus we get the following system

$$\begin{array}{ccccc} 2\alpha & & -2\gamma & = 0 \\ & 2\beta & -\gamma & = 0 \end{array}$$

which has augmented form

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

which can be reduced to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{bmatrix}$$

Notice that there is ***NOT*** a unique solution to system of equations, but rather we can have solutions of the form

$$\begin{array}{ll} \alpha & = \gamma \\ \beta & = \frac{1}{2}\gamma \\ \gamma & = \gamma(free) \end{array}$$

Therefore, the *smallest integer solution* is when $\gamma = 2$, which gives the following chemical reaction

$$2H_2 + O_2 \to 2H_2O$$

□

Activity 1.4.4 Consider the chemical reaction

$$wNH_3 + xO_2 \to yN_2 + zH_2O$$

(a) Write out equations for nitrogen, hydrogen, and oxygen atoms

(b) Input the corresponding augmented matrix into Python and use the rref to write out the solution set

(c) Write out the simplest form of the chemical reaction equation.

Activity 1.4.5 Find the coefficents for quadratic polynomial of the form $y = c_0 + c_1 x + c_2 x^2$ that goes through the points (-1,3), (2,2), and (3,5).

1.5 Vectors and Vector Calculations

In order to distinguish **vectors** (as being more than just n by 1 matrices), we will use the arrow notation and lower case symbols like $\vec{u}$ and $\vec{v}$ to denote vectors. Unless otherwise stated, we will use column vectors. For instance, if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$, then the second **component** of $\vec{v}$ is the scalar v_2. The size of a vector in $\mathbb{R}^n$ is the number of components the vector has. In later work, we will deal with a *much* more general notion of vectors that will *not* have components like vectors in $\mathbb{R}^n$. Recall that two vectors in $\mathbb{R}^n$ are equal if and only if all of their components are equal.

Geometrically, we will view vectors in $\mathbb{R}^n$ as an arrow which change in the i-th coordinate being given by the i-th component of the vector. For instance, the vector $\langle 1, 2, 3\rangle$ will have a plot in 3-dimensions that looks like

```
@interact
def _():
  var('t')
  vec1=arrow((0,0,0),(1,2,3),color='blue', width=3,)
  vec1=parametric_plot3d((t,0,0),(t,-3,3),color="black")+vec1
  vec1=parametric_plot3d((0,t,0),(t,-3,3),color="black")+vec1
  vec1=parametric_plot3d((0,0,t),(t,-3,3),color="black")+vec1
  show(vec1)
```

Vectors do not have a particular beginning or ending point so all of the blue vectors in the following plot are representations of $\langle 1, 2, 3\rangle$.

```
L=[(1,-2,1),(-1,2,1),(2,0,0),(1,2,-2),(-1,-3,-1)]
@interact
def _():
  var('t')
  vec1=arrow((0,0,0),(1,2,3),color='blue', width=3)
  for a in L:
    vec1+=arrow(a,(1+a[0],2+a[1],3+a[2]),color='blue',
        width=3)
  vec1=parametric_plot3d((t,0,0),(t,-3,3),color="black")+vec1
  vec1=parametric_plot3d((0,t,0),(t,-3,3),color="black")+vec1
  vec1=parametric_plot3d((0,0,t),(t,-3,3),color="black")+vec1
  show(vec1)
```

Vectors can be added together to measure the net change (done by completing one vector, then the other). Algebraically, vector addition is done componentwise. If $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$, then $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle$.

```
@interact
def _2(v=('v', input_grid(1, 3, default=[[1,2,3]],
    to_value=lambda x: vector(flatten(x)))),w=('w',
    input_grid(1, 3, default=[[-2,1,-1]], to_value=lambda x:
    vector(flatten(x))))):
  var('t')
  vec1=parametric_plot3d((t,0,0),(t,-3,3),color="black")
  vec1=parametric_plot3d((0,t,0),(t,-3,3),color="black")+vec1
  vec1=parametric_plot3d((0,0,t),(t,-3,3),color="black")+vec1
  vec1+=arrow((0,0,0),v,color='red')+text3d('v',(v[0]/2+0.5,v[1]/2+0.5,v[2]/2),fontsize=
  vec1+=arrow(v,v+w,color='blue')+text3d('w',(v[0]+w[0]/2+0.5,v[1]+w[1]/2+0.5,v[2]+w[2]/
  vec1+=arrow((0,0,0),v+w,color='green')+text3d('v+w',(v[0]/2+w[0]/2-0.5,v[1]/2+w[1]/2-0
  show(vec1)
```

Many other vector operations can be done componentwise, such as scalar multiplication and subtraction. If $k \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$, then $k\vec{v} = \langle kv_1, kv_2, \ldots, kv_n\rangle$. If $\vec{v}, \vec{w} \in \mathbb{R}^n$, then $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n\rangle$. Geometrically, scalar multiplication will stretch (and flip if $k < 0$) the arrow for a vector.

1.5.1 The Dot Product

Definition 1.5.1 Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, then the **dot product** of $\vec{v}$ with $\vec{w}$ is the scalar value given by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

$\diamondsuit$

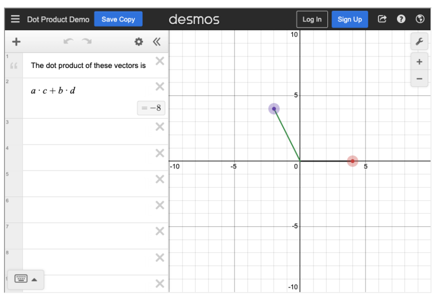
Sometimes we are interested in the total amount of change and not what direction a vector is in. The **magnitude** (or length) of a vector $\vec{v} \in \mathbb{R}^n$ is given by the following

$$\|\vec{v}\| = \sqrt{\sum_{j=1}^{n} (v_i)^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

A **unit vector** is a vector of length 1.

Definition 1.5.2 Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, then the vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$. $\diamondsuit$

Activity 1.5.1 Use the Desmos interact embeded below to change the end points of our two vectors. Based on what you see about the value of the dot product for different configurations of vectors, answer the questions below



(a) How does changing the length of the vectors change the value of the dot product?

(b) How does changing the angle of the vectors change the value of the dot product?

Definition 1.5.3 The **projection of a vector** $\vec{v}$ onto a vector $\vec{w}$ gives the vector part of $\vec{v}$ that is parallel to $\vec{w}$ and is computed as

$$proj_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}\left(\frac{\vec{w}}{\|\vec{w}\|}\right)$$

This is sometimes called the vector projection. The right most part of the definition of the projection vector shows that the projection of $\vec{v}$ onto $\vec{w}$ will be a scalar $(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|})$times the unit vector in the direction of $\vec{w}$. The scalar $\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}$ is referred to as the scalar projection of $\vec{v}$ onto $\vec{w}$.

The vector projection of $\vec{v}$ onto $\vec{w}$ measures the vector part of $\vec{v}$ that is parallel to $\vec{w}$, where as the scalar projection of $\vec{v}$ onto $\vec{w}$ measures the length of $\vec{v}$ that is parallel to $\vec{w}$. If you subtract the projection of $\vec{v}$ onto $\vec{w}$ from $\vec{v}$ $(\vec{v} - proj_{\vec{w}}\vec{v})$, the result will the part of $\vec{v}$ that is orthogonal to $\vec{w}$ because $\vec{v} - proj_{\vec{w}}\vec{v}$ has subtracted out ALL of $\vec{v}$ that is parallel to $\vec{w}$. $\diamondsuit$

Theorem 1.5.4 *If θ is the angle between two vectors $\vec{v}$ and $\vec{w}$, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$$

Question 1.5.5 Let $\vec{u} = \langle 3, 1, 2 \rangle$, $\vec{v} = \langle -1, 2, 0 \rangle$, and $\vec{w} = \langle 4, -3, -1 \rangle$.

1. $||\vec{w}|| =$
2. Find a unit vector that is in the opposite direction of $\vec{w}$
3. $3\vec{u} + \vec{v} - 2\vec{w} =$
4. Can you choose a and b such that $a\vec{u} + b\vec{v} = \vec{w}$?
5. Does $\vec{w} \cdot \vec{v} \cdot \vec{u}$ make sense? Why or why not?
6. What angle does $\vec{v}$ make with $\vec{u}$?
7. What angle does $\vec{v}$ make with the z-axis?
8. How much of $\vec{v}$ is parallel to $\vec{u}$?
9. How much of $\vec{u}$ is parallel to $\vec{v}$?
10. How much of $\vec{w}$ is parallel to $\vec{u}$?
11. How much of $\vec{u}$ is orthogonal to $\vec{v}$?
12. Find a unit vector that is orthogonal to $\vec{v} = \langle -1, 2, 5 \rangle$. How many such vectors are there?

□

1.6 Vector Equations

Definition 1.6.1 A **linear combination** of a set S is a vector of the form

$$\sum_{i=1}^{n} c_i \vec{v_i} = c_1 \vec{v_1} + c_2 \vec{v_2} + ... + c_k \vec{v_k}$$

where $\vec{v_i} \in S$ and $c_i \in \mathbb{R}$. Note that

$$\sum_{i=1}^{n} c_i \vec{v_i}$$

will not usually be in S even though $\vec{v_i} \in S$. ♢

Investigation 1.6.1 Prove that the system of equations given by

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m
\end{array}$$

has the same set of solutions as the vector equation

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In other words, prove that $(c_1, c_2, ..., c_n)$ is a solution to the vector equation iff $(c_1, c_2, ..., c_n)$ is a solution to the system of linear equations. Make sure you clearly connect the ideas in your proof and do not make an argument that these equations look similar.

Activity 1.6.2

(a) Solve the following vector equation:

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

(b) Give an example of a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has no solution or explain why no such vector exists.

(c) Give an example of a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

(d) Give an example of a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

(e) Give an example of a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has no solutions or explain why no such vector exists.

(f) Give an example of a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

Activity 1.6.3

(a) Can you write $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$? Justify your answer.

(b) Can you write $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$? Justify your answer.

(c) Can you write $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$? Justify your answer.

(d) Can you write $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$? Justify your answer.

You can use the idea from Activity 1.3.1 to write the solution set as a vector of the variables $x_1, \ldots, x_n$ where each variable is written in terms of the free variables and constants. This vector form in terms of the free variables is called the parametric form of the solution set.

Question 1.6.2 Give the parametric form of the solution set for the system given by the augmented matrix below.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -4 & 0 & -1 & 5 \\ 0 & 1 & 4 & 0 & 3 & 0 & 2 & -6 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

□

Activity 1.6.4

(a) Write each of the locations given by a red dot as a linear combination of $\vec{u}$ and $\vec{v}$.

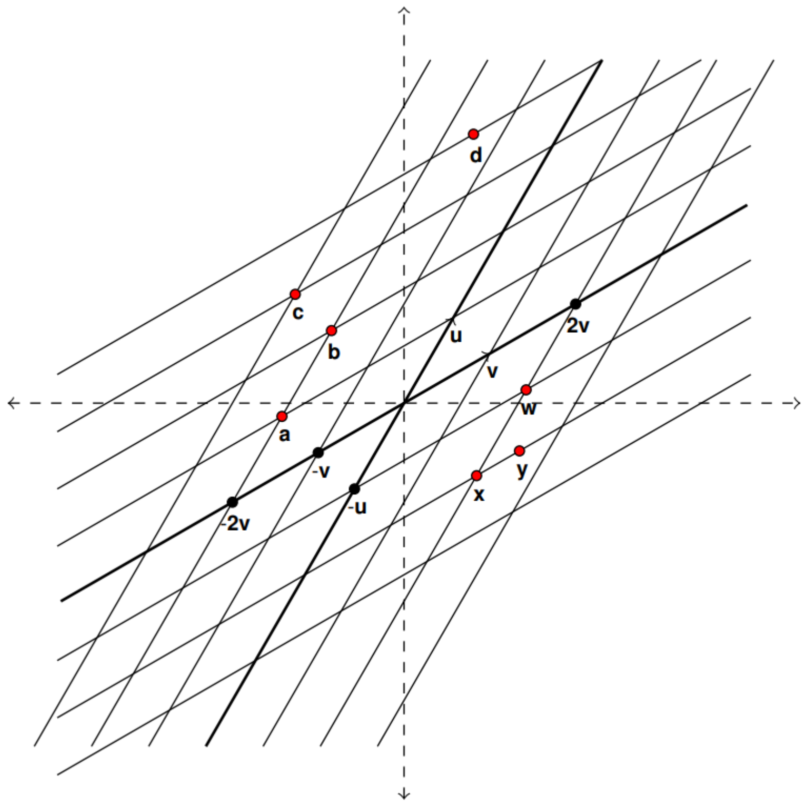


Figure 1.6.3

(b) Can you write every location in the plane of Figure 1.6.3 as a linear combination of $\vec{u}$ and $\vec{v}$? Either explain why you can write every point as a linear combination of $\vec{u}$ and $\vec{v}$ or give a point that cannot be written as a linear combination of $\vec{u}$ and $\vec{v}$.

1.7 Span (both a noun and a verb)

1.7.1 Span as a Noun

Definition 1.7.1 Let S be a set of vectors, $S = \{\vec{v}_1, \vec{v}_2, \ldots\}$. We define the **span** of S, denoted $Span(S)$, as the set of all linear combinations of vectors from S. That is,

$$Span(S) = Span(\{\vec{v}_1, \vec{v}_2, \ldots\}) = \{\vec{w} | \vec{w} = \sum_i c_i \vec{v}_i\}$$

◊

Activity 1.7.1 Look back at Activity 1.6.2 and Activity 1.6.3 and restate each of the questions in terms of span. For instance, part 1 of Activity 1.6.2 could be stated as "Show that $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$."

Note that the set S might not be finite but the number of vectors involved in the summation for a linear combination is finite. Also, remember to treat $Span(S)$ as a set and not a vector. Remember that the use of span in $Span(S)$ is a noun.

Activity 1.7.2

1. How many vectors are in $Span(\{\langle 1,1\rangle, \langle 1,-1\rangle\})$?
2. Is there any vector in $\mathbb{R}^2$ that is not in $Span(\{\langle 1,1\rangle, \langle 1,-1\rangle\})$?
3. How many vectors are in $Span(\{\langle 1,1,1\rangle, \langle 1,-1,1\rangle\})$?
4. Is there any vector in $\mathbb{R}^3$ that is not in $Span(\{\langle 1,1,1\rangle, \langle 1,-1,1\rangle\})$?
5. Try to write out the set of vectors in $Span(\{\langle 1,1,1\rangle, \langle 1,-1,1\rangle\})$? Hint: write the corresponding system of equations, then use the solution set of this system to write out the exact vector form of $Span(\{\langle 1,1,1\rangle, \langle 1,-1,1\rangle\})$.
6. Is there any vector in $\mathbb{R}^2$ that is not in $Span(\{\langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\})$?

The following are equivalent questions:

- Is a vector $\vec{b}$ in $Span(S)$?
- Does $\vec{b} = \sum_i (c_i\vec{v}_i)$ have a solution?

A few other related questions are:

- When will there be a solution to $\vec{b} = \sum_i (c_i\vec{v}_i)$?
- When will there be a UNIQUE solution to $\vec{b} = \sum_i (c_i\vec{v}_i)$?
- How can we describe $Span(S)$ as a collection of vectors?

1.7.2 Span as a Verb

Definition 1.7.2 A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ spans a vector space V if $Span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}) = V$. In other words, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ spans a vector space V if every vector in V can be written as a linear combination from the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$. ♢

Activity 1.7.3

(a) Does $\{\langle 1,2\rangle, \langle 3,6\rangle\}$ span $\mathbb{R}^2$?

(b) Does $\{\langle 1,2\rangle, \langle 3,4\rangle\}$ span $\mathbb{R}^2$?

(c) Does $\{\langle 1,2,3\rangle, \langle 4,5,6\rangle\}$ span $\mathbb{R}^3$?

(d) Does $\{\langle 1,2,3\rangle, \langle 4,5,6\rangle, \langle 0,1,0\rangle\}$ span $\mathbb{R}^3$?

1.8 Linear Independence

We have seen how vector equations relate to a system of equations and how to frame different questions in terms of whether it is possible to find a linear combination from a set S that equals a target vector. This was the same as asking if our target vector was in the span of S. When we looked at vector equations, we also looked at whether there was a unique linear combination or whether there are many ways to write a target vector as a linear combination.

In this section we will introduce the idea of linear independence and how that relates to the uniqueness of these linear combinations. Let's consider the homogeneous vector equation:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$

Notice that this ***always*** has a solution. What is it?

Is it possible to have another solution to this equation? Let's generate a couple of examples.

Activity 1.8.1

(a) Give a set of two vectors from $\mathbb{R}^3$, $\vec{v}_1$ and $\vec{v}_2$, such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ has more than one solution. Justify your answer.

(b) Give a set of two vectors from $\mathbb{R}^3$, $\vec{v}_1$ and $\vec{v}_2$, such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ has only one solution. How do you know there is only one solution?

Definition 1.8.1 A set of vectors S is **linearly independent** if the only linear combination of elements of S that equals the zero vector is the trivial linear combination. In other words, S being a linear independent set implies that if $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k} = \vec{0}$ where $\vec{v_i} \in S$, then all $c_i = 0$.

A set of vectors S is **linearly dependent** if the set is not linearly independent. More specifically, there exists a solution to $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k} = \vec{0}$ where $\vec{v_i} \in S$ and at least one of the $c_j \neq 0$. $\diamondsuit$

Activity 1.8.2

(a) Is the set $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$ linearly independent?

(b) Is the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ linearly independent?

(c) Choose a vector $\vec{v}$ so that the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} \right\}$ is linearly independent, where $\vec{v} \in \mathbb{R}^3$.

(d) Is your choice of $\vec{v}$ in $Span\left(\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \right)$? Show why or why not.

Chapter 2

Matrices

2.1 Matrix Products and Equations

Objectives

- To compute matrix-vector products and understand how this relates to linear combinations
- To understand how to convert between systems of linear equations, vector equations, and matrix equations
- To understand how the matrix-vector product gives rise to linear transformations

2.1.1 Matrix-Vector Products

Definition 2.1.1 We define a **matrix-vector product** as follows: If A is a m by n matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then the **matrix-vector product** is given by

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\mathbb{R}^m$. ◊

Investigation 2.1.1 If A is a m by n matrix, then $A\vec{x} \in \mathbb{R}^{\diamondsuit}$ for what value of $\diamondsuit$?

It should not surprise you that you can multiply a scalar multiple of a vector by a matrix by factoring out the scalar. In mathematical notation, $A(k\vec{v}) = k(A\vec{v})$. Additionally, you can apply the scalar multiplication to the matrix. In other words, $A(k\vec{v}) = k(A\vec{v}) = (kA)\vec{v}$. These kind of manipulations will be discussed more when we work with matrix operations later, but you may find these facts useful in your work right now. You should take time to write out the details of any of these arithmetic ideas that you think would be useful in your work.

Activity 2.1.2 Let

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 5 & 1 \\ -2 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(a) Write out the matrix vector product of A and $\vec{x}$ as a linear combination of the columns of A.

(b) Find $A\vec{x}$

(c) Compute all of the other matrix vector products that will be allowed with the matrices A, B, C and $\vec{x}, \vec{y}, \vec{z}$.

The matrix A can be seen from a column vector form as $A = [\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]$ which means that we can think of the product of A and $\vec{x}$ as $A\vec{x} = [\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$ $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$.

Investigation 2.1.3

(a) Write out the k-th component of the resulting vector of the product

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Solution. $a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n$

(b) How can you express the result of the matrix-vector product in terms of $\vec{x}$ and the rows of A?

Solution. The k-th component of the matrix-vector product is the dot product of row k of A with $\vec{x}$.

(c) How can you express the result of the matrix-vector product in terms of $\vec{x}$ and the columns of A?

Solution. One way to view this is as a linear combination of the columns of A with the coefficient on the k-th column of A being x_k.

2.1.2 The Matrix Equation

Based on the above definition of the matrix vector product, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, then by Investigation 1.6.1, $A\vec{x} = \vec{b}$ has the same solution set as the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Investigation 2.1.4 Write each of the following as a matrix equation, a vector equation, and system of equations. You need to write out the exact corresponding vector equation, matrix equation, and system of equations, *not* some equivalent form.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(b) $a_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

(c)

$$\begin{aligned} 2x_1 + 3x_2 \quad\quad &= 7 \\ -x_1 + x_2 + 4x_3 &= 0 \\ 5x_1 - 6x_2 - x_3 &= 2 \end{aligned}$$

Investigation 2.1.5 Rephrase Activity 1.6.2 as matrix equations.

2.1.3 Linear Transformations

Our definition of a matrix vector product suggests that the matrix-vector product 2.1.1 of a m by n matrix will transform vectors from $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. In this way, we can define a function as follows.

Definition 2.1.2 Let A be a m by n matrix. Then we define $T_A : \mathbb{R}^n \to \mathbb{R}^m$, where $T_A(\vec{x}) = A\vec{x}$. $\Diamond$

Using what we know of solving linear systems, vector equations, and thier relationship to matrix vector products, we note the following:

- If there is no soltuion to the matrix equation $A\vec{x} = \vec{b}$, then we say that $\vec{b}$ is NOT in the range of the linear transformation T_A.
- If there is always a solution to the matrix equation $A\vec{x} = \vec{b}$, then we say that the map T_A completely covers $\mathbb{R}^m$. In other words, the range of T_A is all of $\mathbb{R}^m$.
- If whenever a solution exists, the solution is unique, then we say that the map T_A is one-to-one.

Linear Transformations are very important ways to understand how a vector space is changed under the operation given. Linear transformations are important because they preserve linear combinations. In other words,

$$T_A(a_1\vec{v}_1 + a_2\vec{v}_2) = a_1T_A(\vec{v}_1) + a_2T_A(\vec{v}_2)$$

Activity 2.1.6 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

(a) The linear transformation T_A takes $\mathbb{R}^m$ to $\mathbb{R}^n$ for what values of m and n?

(b) Compute $T_A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

(c) Compute $T_A\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$.

(d) Compute $T_A\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

(e) Compute T_A for any other vectors you might want to and write a few sentences about how vectors are transformed by T_A.

Activity 2.1.7 Let $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$.

(a) The linear transformation T_B takes $\mathbb{R}^m$ to $\mathbb{R}^n$ for what values of m and n?

(b) Compute $T_B\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

(c) Compute $T_B\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(d) Compute $T_B\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

(e) Compute T_B for any other vectors you might want to and write a few sentences about how vectors are transformed by T_B.

2.2 Matrix Operations

Objectives

- To compute matrix opertions like addition, subtraction, scalar multiplication, transpose, and matrix multiplication
- To understand how to compute matrix operations for an entry, row, or column
- To determine if matrices fir specific forms like upper triangular, lower triangular, symmetric, skew symmetric, or identity

2.2.1 Addition and Transposition

Investigation 2.2.1 Finish the following sentences.

(a) Vectors are equal if...

(b) Matrices are equal if...

(c) A scalar is...

Just as you can add two vectors in $\mathbb{R}^n$ componentwise, you can add two matrices entry-wise. For this reason, it only makes sense to add two matrices if they are the same size. You can also define scalar multiplication of a matrix entry-wise.

Investigation 2.2.2 Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -7 \\ 4 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \\ 4 & 2 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Is B a scalar multiple of A? Why or why not?

(b) $2A - 3C =$

(c) $-(A + C) + 2B =$

(d) $(84A + 16B - 12C)_{2,1} =$

Investigation 2.2.3 Symbolically, $(A + B)_{i,j} =$ ________ and $(kA)_{i,j} =$ ________

Definition 2.2.1 Let A be a m by n matrix. The transpose of A, denoted A^T, is a n by m matrix such that $(A^T)_{ij} = (A)_{ji}$. $\Diamond$

There are a couple of ways to think about the transpose. First, you can think about flipping the matrix across the main diagonal (the elements of the form $A_{i,i}$). You can also view the transpose of a matrix as switching the rows and columns (but preserving the order). In other words, the i-th row of A^T is the i-th column of A.

Investigation 2.2.4 Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \end{bmatrix}$

(a) $A^T =$

(b) $B^T =$

(c) $A^T + B =$

(d) $B^T + A =$

Investigation 2.2.5 Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -7 \\ 4 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \\ 4 & 2 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) $A + B^T =$

(b) $((C - B)^T + A)^T =$

Theorem 2.2.2 *If A and B are m by n matrices, then $(A + B)^T = A^T + B^T$.*

Investigation 2.2.6 What dimensions should A have in order to be able to add A to A^T?

Theorem 2.2.3 *For all matrices A, $(A^T)^T = A$.*

A matrix A is **symmetric** if $A^T = A$.

Theorem 2.2.4 *The sum of two symmetric m by n matrices is symmetric.*

Theorem 2.2.5 *If A is a symmetric matrix, then kA is symmetric.*

2.2.2 Matrix Multiplication

Earlier we saw how to multiply a m by n matrix by a vector from $\mathbb{R}^n$. We will discuss how to define matrix multiplication with multiple interpretations.

Let A be an m by n matrix and let $\vec{x_1}$ and $\vec{x_2}$ be vectors from $\mathbb{R}^n$. Earlier we defined what $A\vec{x_1}$ and $A\vec{x_2}$ meant. If we build a n by 2 matrix B using $\vec{x_1}$ and $\vec{x_2}$ as the columns, then we can define AB, read as "A times B", to be

$$AB = A[\vec{x_1} \quad \vec{x_2}] = [A\vec{x_1} \quad A\vec{x_2}]$$

The above definition is just distributing our matrix-vector product across the columns of B. In a similar fashion, given any n by k matrix

$$B = [\vec{b_1} \quad \vec{b_2} \quad \cdots \quad \vec{b_k}]$$

where $\vec{b_i}$ is the i-th column of B, we can define

$$AB = [A\vec{b_1} \quad A\vec{b_2} \quad \cdots \quad A\vec{b_k}]$$

In particular, this means that if AB makes sense, then we can calculate just the i-th column of AB without calculating all of AB. Namely, the i-th column of AB is $A\ column_i(B)$, which is written symbolically as $column_i(AB) = A\ column_i(B)$.

Activity 2.2.7 Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -7 \\ 4 & 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \\ 4 & 2 & 6 \end{bmatrix}$.

(a) Write out the columns of B as vectors where $B = [\vec{x}_1 \vec{x}_2] \vec{x}_3]$.

(b) Compute the following: $A\vec{x}_1$ $A\vec{x}_2$ $A\vec{x}_3$

(c) Use the results of your previous work to compute AB

(d) Take a moment to look back on exactly what calculation you had to do to get the value of $(AB)_{2,3}$. Write out exactly which parts of A and B are used in your calculation.

Formally, we can define the product of a m by n matrix A with a n by k matrix B to be the m by k matrix AB such that

$$(AB)_{i,j} = \sum_{l=1}^{n} (A)_{i,l}(B)_{l,j}$$

This formula looks difficult, but what it really tells us is that the (i, j) entry of AB is really the dot product of the i-th row of A with the j-th column of B. Remember the **dot product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ is just the sum of the products of the components. Namely,

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

This idea lets us calculate the matrix product AB one entry at a time. Continuing this idea will lead us to see that the i-th row of the product AB can be calculated as $row_i(AB) = row_i(A)B$.

Note that in general $AB \neq BA$, even when both products make sense.

Investigation 2.2.8

(a) What sizes of matrices can you add to a m by n matrix?

(b) What sizes of matrices can you multiply on the right of a m by n matrix?

(c) What sizes of matrices can you multiply on the left of a m by n matrix?

Investigation 2.2.9 If $A \in M_{m \times n}$, when does it make sense to multiply by A^T?

Investigation 2.2.10 Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$.

(a) What is the size of AB?

(b) Compute just the first column of AB.

(c) Write the first column of AB as a linear combination of the columns of A. Be sure to check your work.

(d) Solve the matrix equation $A\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(e) Compute just the second row of AB

Investigation 2.2.11 Let $A = \begin{bmatrix} 3 & 2 & 1 & 5 & 6 \\ 4 & 1 & 3 & 2 & -1 \\ 0 & 2 & 5 & 6 & 7 \\ 8 & 2 & 3 & 1 & 4 \end{bmatrix}$ and $B =$

$$\begin{bmatrix} 5 & -2 & 2 & 4 \\ 6 & 2 & 3 & 6 \\ 4 & -1 & 7 & 14 \\ 2 & 0 & -2 & -4 \\ 1 & 1 & 2 & 4 \end{bmatrix}$$

(a) $A_{2,3} =$

(b) $B_{1,4} =$

(c) $(AB)_{2,3} =$

(d) $row_2(AB) =$

(e) $column_3(AB) =$

Investigation 2.2.12 Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$. Compute AB and BA.

Investigation 2.2.13 Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$. Compute AB and BA.

You can approach proving the following theorem by showing matrix equality entry-wise or column-wise or row-wise.

Theorem 2.2.6 *For all matrices A, B, and C such that the addition and multiplication of matrices below makes sense,*

$$(A + B)C = AC + BC.$$

Investigation 2.2.14 Give 2 different examples of 3 by 3 matrices A and B such that $AB = BA$.

Investigation 2.2.15 Give 2 different examples of 3 by 3 matrices A and B such that $AB \neq BA$.

Theorem 2.2.7 *For all matrices A, B such that AB is defined, $(AB)^T = B^T A^T$.*

2.2.3 Special Types of Matrices

A **square** matrix is a matrix that has the same number of rows and columns. A m by n matrix A is said to be **upper triangular** if $A_{i,j} = 0$ whenever $i > j$. Similarly, a matrix A is **lower triangular** if $A_{i,j} = 0$ whenever $i < j$. We usually consider square matrices when we talk about upper or lower triangular, but it may be helpful to consider non-square cases.

Investigation 2.2.16 Give an example of a matrix that is upper triangular but is not in echelon form. If one does not exist, explain why.

Investigation 2.2.17 Give an example of a matrix that is in echelon form but is not upper triangular. If one does not exist, explain why.

Investigation 2.2.18 Can a matrix be upper *and* lower triangular? Either give an example or explain why there cannot exist one.

Diagonal matrices are matrices whose only nonzero entries are on the diagonal. Specifically, a matrix A is diagonal if $A_{i,j} = 0$ whenever $i \neq j$.

Investigation 2.2.19 Give an example of a matrix that is diagonal but not in echelon form.

The n by n **identity matrix**, denoted Id_n, is the unique matrix such that $Id_n\vec{x} = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$. In fact the entries of Id_n are easily computed in terms of the Dirac delta function. Specifically $(Id_n)_{i,j} = \delta_{i,j}$, where

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Investigation 2.2.20 Write out Id_5 and use it to prove that for any $\vec{v} \in \mathbb{R}^5$ the product of Id_5 and $\vec{v}$ will always be $\vec{v}$.

Investigation 2.2.21 Superstar Bonus Question. Prove that Id_5 is the only matrix that has the property from the problem above.

2.3 Inverse Matrices

In this section, we will only consider square matrices.

Definition 2.3.1 A matrix $A \in M_{n\times n}$ is **invertible** if there exists a matrix B such that $AB = Id_n$ and $BA = Id_n$. The inverse matrix of A is denoted A^{-1}. ♢

Be careful that you do not use the notation A^{-1} until you have shown that A is invertible. By inverse, we mean the multiplicative inverse for a matrix. A matrix that is not invertible is called a **singular matrix**. A **non-singular matrix** is an invertible matrix.

In the next couple of sections we will examine the following two questions:

- How can you efficently calculate the inverse matrix for a given A?
- How can you determine when a matrix is invertible without finding its inverse?

2.3.1 Computing Inverses

Investigation 2.3.1 We will look at a way to find the inverse matrix of A in terms of the matrix-vector product and how that can be used as a representation of matrix multiplication.

(a) We want to find a matrix C such that $AC = Id$. So let's expand C as columns.

$$AC = A[\vec{c}_1\ \vec{c}_2\ \dots\ \vec{c}_n] = [A\vec{c}_1\ A\vec{c}_2\ \dots\ A\vec{c}_n]$$

Using this perspective on the equation $AC = Id$, we get

$$A\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ A\vec{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \cdots,\ A\vec{c}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

How would we find out if there were any solutions to these matrix equations?

(b) How could you find solutions to all of these matrix equations all at once?

In general computing the inverse of a matrix takes more time and operations than solving a system of equations. For this reason, it is generally easier to find and solve a related system of equations problem than to compute the inverse matrix. We will outline a few ways to find inverse matrices and compute a few small examples.

Investigation 2.3.2 Any sequence of elementary row operations that reduces A to Id_n also transforms Id_n into A^{-1}.

The previous result shows that computing inverses is equivalent to a row reduction problem. In particular, if A is invertible, then reducing $[A \mid Id_n]$ to reduced row echelon form will produce the matrix $[Id_n \mid A^{-1}]$.

Activity 2.3.3

(a) Use the result of the previous investigation to find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Check your work by computing $AB = Id_n$ and $BA = Id_n$ for the matrix you think is the inverse of A.

(b) Use the result of the previous investigation to find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Check your work by computing $AB = Id_n$ and $BA = Id_n$ for the matrix you think is the inverse of A.

(c) Use the idea above to compute the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Be sure to note any assumptions you will need to make in order to reduce $[A \mid Id_n]$ to $[Id_n \mid A^{-1}]$.

(d) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & -1 \end{bmatrix}$, find A^{-1} and check that $AA^{-1} = Id_3$.

Checkpoint 2.3.2 If $A = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}$, find A^{-1} and use your answer to solve $A\vec{x} = \vec{b}$ if:

(a) $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(b) $\vec{b} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

(c) $\vec{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

(d) $\vec{b} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Investigation 2.3.4 Your friend Nick, who thinks he knows everything, claims that $(AB)^{-1} = A^{-1}B^{-1}$ because that's how exponents work. Is he right? Justify your answer like you are going to have to convince Nick.Mulitply Nick's Result on both sides by the matrix (AB) and simplify.

Investigation 2.3.5 Nick makes another concerning statement about how algebra works with matrices. Specifically, he claims that if the product of two matrices is zero, then one of the two matrices must be the zero matrix. He writes $AB = 0 \Rightarrow A = 0$ or $B = 0$ on the board to justify his idea. Is he wrong again or do matrices work like this?Come up with an example of matrices A and B that will demonstrate is claim his false.

2.4 Invertible Matrix Theorem

Investigation 2.4.1 In many texts there is a long list of equivalent conditions for when a square matrix is invertible. Below is a list of some of these conditions that we have talked about or proven. Go back through your notes and questions and cite when we connected two of the ideas in the list. For instance, parts *a*) and *b*) are linked by Investigation 2.3.2

Before stating this major theorem, we should explain what the phrase "the following are equivalent" (sometimes written "TFAE" in scratchwork or on the board) means. A theorem of this type is essentially a giant if and only if theorem. Specifically, each statement in the theorem is true or each statement in the theorem is false. It is not possible for some to be true and some to be false. In a theorem with, say, three statements, we often prove that statement 1 implies statement 2, statement 2 implies statement 3, and statement three implies statement 1. Then you can start at any statement and reach any other statement, showing that if one is true, all the others must be true. However, with longer lists, we sometimes have to prove things a bit more piecemeal.

Theorem 2.4.1 The Invertible Matrix Theorem. *Let A be a n by n matrix. The following are equivalent:*

a) A is an invertible matrix.

b) A is row equivalent to Id_n.

c) A has n pivots.

d) $A\vec{x} = \vec{0}$ has only the trivial solution.

e) The linear transformation $\vec{x} \to A\vec{x}$ is one-to-one.

f) The linear transformation $\vec{x} \to A\vec{x}$ is onto.

g) $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.

h) The columns of A form a linearly independent set.

i) The columns of A span $\mathbb{R}^n$.

j) The rows of A form a linearly independent set.

k) The rows of A span $\mathbb{R}^n$.

l) A^T is invertible.

Investigation 2.4.2 Two important ideas in this course that have been tied to many different methods or ideas are 1) consistent systems of linear equations and 2) invertible matrices. These two ideas are a bit different though. Give an example of a consistent system of linear equations (in matrix equation form $A\vec{x} = \vec{b}$) where the coefficient matrix A is a non-invertible square matrix.

2.5 Determinants

Determinants will be an incredibly useful tool in quickly determining several important properties of square matrices. We will first look at how to compute determinants and later outline the important properties that determinants have. While some of you may have been taught some rules for how to compute determinants of 2 by 2 and 3 by 3 matrices, I encourage you to understand how to compute determinants in general.

2.5.1 Computing Determinants

Definition 2.5.1 The **determinant** is a function from n by n matrices to the real numbers ($\det : M_{n\times n} \to \mathbb{R}$).

If A is a 1 by 1 matrix, $A = [A_{1,1}]$, then $\det(A) = A_{1,1}$.

For $n \geq 2$, the determinant of a n by n matrix is given by the following formula in terms of determinants of $(n-1)$ by $(n-1)$ matrices:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} (A_{1,j}) \det(A^*_{1,j})$$

where $A^*_{i,j}$ is the $(n-1)$ by $(n-1)$ matrix obtained by deleting the i-th row and j-th column of A. The term $(-1)^{i+j}\det(A^*_{i,j})$ is called the (i,j) **cofactor of** A. $\Diamond$

The above definition uses cofactor expansion along the first row.

Investigation 2.5.1 In this question, we will unpack the determinant formula above for a 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Rather than using the summation notation of the formula above, write out the two terms in $\det(A)$.

(b) $A^*_{1,1} =$

(c) $A^*_{1,2} =$

(d) $A_{1,1} =$

(e) $A_{1,2} =$

(f) $(-1)^{1+1} =$

(g) $(-1)^{1+2} =$

(h) $\det(A) =$

Your answer to the previous problem will be useful in calculating determinants of 3 by 3 matrices. We will use the theorem below without proving it.

Theorem 2.5.2 *The determinant can be computed by cofactor expansion along any row or column. Specifically the cofactor expansion along the k-th row is given by*

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} (A_{k,j}) \det(A^*_{k,j})$$

and the cofactor expansion along the k-th column is given by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+k} (A_{i,k}) \quad \det(A^*_{i,k}).$$

Checkpoint 2.5.3 Use cofactor expansion along the first column of $A =$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ to compute $\det(A)$.

Checkpoint 2.5.4 Use cofactor expansion along the second row of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ to compute $\det(A)$. Did you get the same answer as the previous question?

Checkpoint 2.5.5 Compute the determinant of $B = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$. How does your answer compare with the previous problem?

Checkpoint 2.5.6 Compute the determinant of $C = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$.

Checkpoint 2.5.7 Compute the determinant of $D = \begin{bmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{bmatrix}$.

Checkpoint 2.5.8 Compute the determinant of the following matrices:

(a) $\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 4 \\ -3 & 5 & 0 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix}$

(b) $2\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 4 \\ -3 & 5 & 0 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix}$

Activity 2.5.2

(a) Find $det(A)$ when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(b) Find $det(B)$ when

$$B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$$

(c) Find $det(C)$ when

$$C = \begin{bmatrix} 6 & 6 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

(d) Find $det(D)$ when

$$D = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

2.5.2 Properties of Determinants

Investigation 2.5.3 Prove that if A has a row of zeros, then $\det(A) = 0$.

Investigation 2.5.4 Prove that $\det(Id_n) = 1$.

We will now state several useful properties of determinants. We will defer the proofs until later in the course. You may use these theorems *unless a problem specifically asks you to prove one of them*, in which case, the problem will note that you may not use the theorem to prove it.

Theorem 2.5.9 *The determinants of elementary matrices have the following values:*

(a) If E_1 multiplies a row by a scalar α, then $\det(E_1) = \alpha$.

(b) If E_2 adds α times a row to another row, then $\det(E_2) = 1$.

(c) If E_3 swaps two rows, then $\det(E_3) = -1$.

Theorem 2.5.10

(a) If A and B are n by n, then $\det(AB) = \det(A)\det(B)$.

(b) If A is n by n and k is a scalar, then $\det(kA) = k^n \det(A)$.

(c) The determinant of an upper or lower triangular matrix is the product of its diagonal entries.

$$\det(L) = \prod_{i=1}^{n} (L)_{i,i}$$
$$\det(U) = \prod_{i=1}^{n} (U)_{i,i}$$

(d) The determinant of a diagonal matrix is the product of its diagonal entries. If D is diagonal, then

$$\det(D) = \prod_{i=1}^{n} (D)_{i,i}.$$

(e) $\det(A) = \det(A^T)$

(f) If the matrix A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$

(g) A matrix A is invertible iff $\det(A) \neq 0$.

The final property of the theorem above should be included in The Invertible Matrix Theorem!

Theorem 2.5.11 *Let A be an $n \times n$ matrix. We have that $\det(A) = 0$ iff $A\vec{x} = \vec{0}$ has solutions such that $\vec{x} \neq \vec{0}$.*

2.6 A Motivating Example

Activity 2.6.1 We want to look at the effect that a matrix has when we look at the matrix's effect on an entire vector space. Let's start small and look at

$$A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$$

We want to look at what happens to different directions (measured with unit vectors) when we use the function $T_A(\vec{x}) : \vec{x} \to A\vec{x}$.

(a) Use Python to construct a bunch of unit vectors centered at the origin. Plot these vectors.

(b) Use Python to calculate $T_A(\vec{x})$ for each of your unit vectors. Plot these vectors. What do you notice about these results?

(c) Where does T_A send $\begin{bmatrix}1\\0\end{bmatrix}$? Why should this make sense on your plot?

(d) Where does T_A send $\begin{bmatrix}0\\1\end{bmatrix}$? Why should this make sense on your plot?

(e) How does the area traced out by all unit vectors starting at the origin change under T_A? Calculate the determinant of A and compare to your change in area.

(f) Where does T_A send $\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$? Why should this make sense on your plot?

(g) Are there other directions that work like this?

(h) So $\vec{v}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$ are important directions. Will there be any other directions that are just scalar multiplication under the action of T_A?

(i) The set $\{\vec{v}_1, \vec{v}_2\}$ spans all of $\mathbb{R}^2$. How can we justify this statement?

(j) We can write any vector in $\mathbb{R}^2$ as a linear combination of the set $\{\vec{v}_1, \vec{v}_2\}$. Use this idea to describe $T_A(\vec{w})$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

(k) Can we do all of these steps for other matrices? Great question Dr. Long. You deserve a raise and come cookies. You may proceed to the next section to see the answer.

2.7 Eigenvalues and Eigenvectors

Definition 2.7.1 An **eigenvector** of a matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. The scalar λ is called an **eigenvalue** of A if there exists a nonzero solution to $A\vec{x} = \lambda\vec{x}$. $\Diamond$

Investigation 2.7.1 Which of the following vectors are an eigenvector of $A = \begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$? For any vectors that are eigenvectors of A, give the eigenvalue.

(a) $\vec{v_1} = \begin{bmatrix}1\\2\end{bmatrix}$

(b) $\vec{v_2} = \begin{bmatrix}-1\\1\end{bmatrix}$

(c) $\vec{v_3} = \begin{bmatrix}3\\-1\end{bmatrix}$

(d) $\vec{v_4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(e) $\vec{v_5} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As a hint for the following two problems, it will suffice to try to find an eigenvector of the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$. You might first convince yourself that for these matrices, no eigenvector can have first component 0.

Investigation 2.7.2 Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Try to find an eigenvector with eigenvalue 3. In other words, find a vector $\vec{v}$ such that $A\vec{v} = 3\vec{v}$.

Investigation 2.7.3 Let $A = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$. Try to find an eigenvector with eigenvalue -3. In other words, find a vector $\vec{v}$ such that $A\vec{v} = -3\vec{v}$.

As a hint to proving this, look back to Theorem 2.5.11

Theorem 2.7.2 *Let A be a square matrix. We have that* $\det(A - \alpha Id) = 0$ *iff α is an eigenvalue of A.*

If A is a n by n matrix, then $\det(A - tId)$ will be a n-th degree polynomial in t, which we call the **characteristic polynomial of** A. The previous theorem shows that finding roots of the characteristic polynomial is the same as finding eigenvalues.

Investigation 2.7.4 For each of the following matrices:

1. write out the characteristic polynomial
2. give all eigenvalues
3. for each eigenvalue, find an eigenvector

You should do the first two by hand to get a feel for finding the characteristic polynomial. After that, I have provided a SageMath cell you can modify to get the characteristic polynomial quickly, but you will need to work from there to find eigenvalues and eigenvectors.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

```
A = matrix(2,2,[1,2,3,4])
A.charpoly('t')
```

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

Hint. Work smarter, not harder, on this part!

A root α of a polynomial (in t) has **(algebraic) multiplicity** k if k is the largest integer such that $(t-\alpha)^k$ is a factor. Which, if any, of the eigenvalues you found above have algebraic multiplicity greater than 1?

Investigation 2.7.5 Prove that a nonzero vector, $\vec{v}$, is an eigenvector of A with eigenvalue λ if and only if $\vec{v}$ is in the null space of $A - \lambda Id$.

Solution. ($\Rightarrow$) If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}$. By algebra, this means that $A\vec{v} - \lambda\vec{v} = \vec{0}$, and hence $(A - \lambda Id)\vec{v} = \vec{0}$. Thus, $\vec{v}$ is in the null space of $A - \lambda Id$.

($\Leftarrow$) If $\vec{v} \in Null(A - \lambda Id)$, then $(A - \lambda Id)\vec{v} = 0$. Hence, $A\vec{v} - \lambda\vec{v} = \vec{0}$, or $A\vec{v} = \lambda\vec{v}$. Thus, $\vec{v}$ is an eigenvector of A with eigenvalue λ.

Investigation 2.7.6 Prove that if $\vec{v}$ is an eigenvector of A, then $\alpha\vec{v}$ is also an eigenvector of A (when $\alpha \neq 0$).

Solution. Since $\vec{v}$ is an eigenvector of A, there is a scalar λ such that $A\vec{v} = \lambda\vec{v}$. By properties of matrix multiplication, we thus have

$$A(\alpha\vec{v}) = \alpha A\vec{v} = \alpha\lambda\vec{v} = \lambda(\alpha\vec{v}).$$

As $\alpha \neq 0$, this shows that $\alpha\vec{v}$ is an eigenvector with the same eigenvalue.

Investigation 2.7.7 Prove that if $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of A with the same eigenvalue, then $\vec{v_1}+\vec{v_2}$ is also an eigenvector of A. What is the eigenvalue of $\vec{v_1} + \vec{v_2}$?

Solution. Let λ be the associated eigenvalue. We have that $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$. Thus, we have

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2).$$

Therefore, $\vec{v}_1 + \vec{v}_2$ is an eigenvector with the same eigenvalue.

Definition 2.7.3 If λ is an eigenvalue of A, then the **eigenspace of** λ, E_λ, is the set of vectors $\vec{x}$ such that $(A - \lambda Id_n)\vec{x} = \vec{0}$. ♢

Activity 2.7.8 Determine all eigenspaces for each matrix.

(a) $\begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$

Investigation 2.7.9

(a) Let $A = \begin{bmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 2 \end{bmatrix}$. Show that A only has an eigenvalue of 2. What is the algebraic multiplicity of the eigenvalue 2?

(b) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 3? If so, give a choice of a, b, and c that does so.

(c) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 2? If so, give a choice of a, b, and c that does so.

(d) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 1? If so, give a choice of a, b, and c that does so.

Chapter 3

Applications of Linear Algebra

3.1 Discrete Dynamical Systems

A **dynamical system** is a pair (X, R) where X is the set of states a system can be in and R is a rule for how the system evolves or changes. This can feel like a really abstract and general statement, so let's look at some real life examples and some simple math examples that we can easily work with.

Example 3.1.1 Let our state space be the set of all possible collections of position, velocities, and times of the sun and nine planets (Pluto is still a planet to me...). You might store each planet's position, velocity, and time as a vector then we would say that the ten vectors are the current state of the system and the rule for evolution would be the the force of gravity. In this way, each state leads to the next by following the dynamical rule.

The field of dynamical systems would study questions like 1)if this system always has a solution, 2)what properties solutions typically have, or 3) what is the long term behavior of solution? Another important part of the system above would be an initial condition, or the state of the system at beginning.

You often label this as time zero and state the values for the different planets' position and velocity at time zero. This leads to even more questions like "If the initial configuration of planets was a bit different, would the long term behavior still be the same?" This example is a continuous dynamical system since we look how the states of the system evolve in terms of a continuous variable (time in this case). This particular system is quite complicated and has been a focus of science, philosophy, and religion for quite some time. □

Example 3.1.2 Let us look at a much simplier example and do some calculations. Let our dynamical system be $(\mathbb{R}, f(x) = x^2)$. This means that our state space is the set of real numbers and our current state evolves according to the rule $x \to x^2$.

Notice here that we can only apply our rule (apply the function$f$) in discrete amounts. So if we start with the initial value $x_0 = a$, then our next state will be $x_1 = f(x_0 = a) = a^2$. Our study really becomes about the sequence $x_0, x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)), x_3 = f(x_2) = f(f(f(x_0))), \ldots$. This is called a **discrete dynamical system** because we can measure the state of the system (and it's evolution) only at discrete values.

Take a few minutes to find the solution sequence for $x_0 = 2$. Try to write out what the long term behavior of this solution sequence is. Will all initial values have this same long term behavior? How many different long term behaviors can you find? □

Example 3.1.3 You may think a game like chess is also a dynamical system but that is not the case. You could consider all the different ways that pieces could be configured on the board as a state space for the game, there is no rule for how the configuration must change. This is what makes chess an interesting game. □

3.1.1 Predator-Prey Systems

In this section we will look at a basic type of two dimensional discrete dynamical system that models the populations of a predator and a prey, which we will call foxes (F) and rabbits, R. We will construct a discrete dynamcial system that describe the amount of foxes and rabbits in the next year based on the amount of foxes and rabbits in the current year. In other words,

$$F_{n+1} = g(F_n, R_n)$$

$$R_{n+1} = h(F_n, R_n)$$

Let's figure out reasonable functions for g and h under the following ideas:

- If there are no rabbits, then some of the foxes will die in the next year (starvation)
- If there are rabbits, then the fox populations grows in the next year based on the interaction of the species (predation)
- If there are no foxes, the rabbit population will grow by some proportion in the next year
- If there are foxes, then the rabbit population will decrease based on the interaction of the species (being eaten)

Functions that fit these simple rules might be of the form

$$F_{n+1} = g(F_n, R_n) = -aF_n + bR_n$$

$$R_{n+1} = h(F_n, R_n) = cR_n - dF_n$$

So choosing coefficients would give the relative weight of each of these rules on the change in each population. For instance,

$$F_{n+1} = -0.2F_n + 0.13R_n$$

$$R_{n+1} = 1.05R_n - 2.1F_n$$

Note here that we can use linear algebra to analyse this type of problem because our sequence now looks like

$$\vec{x}_0 = \begin{bmatrix} F_0 \\ R_0 \end{bmatrix}, \vec{x}_1 = A\vec{x}_0, \vec{x}_2 = A\vec{x}_1 = A(A\vec{x}_0), \ldots$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In other words, the solution to this two dimensional, discrete dynamical system is of the form $\vec{x}_n = A^n\vec{x}_0$.

3.1.2 Difference Equations and Linear Algebra

A dynamical system with an n-dimensional vector for the state and dynamical rule given by $\vec{x}_{n+1} = A\vec{x}_n$ is called a linear **difference equation**.

Let examine how eigenvalues and eigenvectors could help us easily understand the long term behavior of a linear difference equation. In particular, we will assume that we have a difference equation given by a n by n matrix A and that we can find a set of n linearly independent eigenvectors, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. This is actually a really big assumption and is NOT true in general, so the discussion below will not be enough to analyze the general case. Because we have a set of n linearly independent vectors from $\mathbb{R}^n$, we can put them as columns of a matrix and apply the Invertible Matrix Theorem to demonstrate that this set will also span all of $\mathbb{R}^n$. This means that no matter what initial values we take for our system, we can write that initial value as a linear combination of the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. In other words, there is a solution to the vector equation

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

for every $\vec{x}_0 \in \mathbb{R}^n$. Because these vectors in our spanning set are not just some vectors, but rather are eigenvectors of A, we will be able to write out the rest of the sequence and understand the long term behavior regardless of initial values of our system.

Looking at $\vec{x}_1$ we get the following:

$$\begin{aligned}\vec{x}_1 = A\vec{x}_0 &= A(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) \\ &= c_1(A\vec{v}_1) + c_2(A\vec{v}_2) + \ldots + c_n(A\vec{v}_n)\end{aligned}$$

Since each of the $\vec{x}_i$ are eigenvectors, $A\vec{v}_i = \lambda_i\vec{v}_i$. Thus,

$$\vec{x}_1 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \ldots + c_n\lambda_n\vec{v}_n$$

In this same way, we can look at the k-th iteration of our system and get

$$\begin{aligned}\vec{x}_k = A^k\vec{x}_0 &= A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) \\ &= c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \ldots + c_n\lambda_n^k\vec{v}_n\end{aligned}$$

Notice that the only thing changing under iteration is the power of the eigenvalue. Once we figure out how much of our inital value vector is in the direction of each of our eigenvectors, then that amount does NOT change during the evolution of our system! The only thing changing is that each eigenvector direction is getting stretched or shrunk by the eigenvalue at each step. So when will these different parts grow or shrink?

Activity 3.1.1 In this activity, we want to go through all of the parts of the argument above and its geometric meaning for the difference equation described by $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

(a) What are the eigenvalues and eigenvectors of A?

(b) How can we write the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ as a linear combination of the eigenvectors of A?

(c) How can we write the vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ as a linear combination of the eigenvectors of A?

(d) Show that if $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$, then $A\vec{w} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$.

(e) What is the long term behavior of the solution with initial value $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Activity 3.1.2 In this activity, we want to go through all of the parts of the argument above and its geometric meaning for the difference equation described by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(a) What are the eigenvalues and eigenvectors of A?

(b) How can we write the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ as a linear combination of the eigenvectors of A?

(c) Show that if $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$, then $A\vec{w} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$.

(d) What is the long term behavior of the solution with initial value $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

3.1.3 Types of Solutions to Two Dimensional Linear Difference Equations

Question 3.1.4 What are the fixed points of the difference equation $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. □

Question 3.1.5 What are the fixed points of the difference equation $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. □

Since the long term behavior of these type of systems depends on the eigenvalues, we wil try to talk about all of the possible cases of eigenvalues and the corresponding behaviors. Remember that we need to pay attention to how λ^k changes as k increases.

- $|\lambda_1| < 1$ and $|\lambda_2| < 1$
- $|\lambda_1| > 1$ and $|\lambda_2| > 1$
- $|\lambda_1| < 1$ and $|\lambda_2| > 1$
- $|\lambda_1| = 1$ and $|\lambda_2| < 1$
- $|\lambda_1| = 1$ and $|\lambda_2| > 1$
- $|\lambda_1| = 0$ and $|\lambda_2| < 1$
- $|\lambda_1| = 0$ and $|\lambda_2| > 1$
- What other possibilities are there?

Activity 3.1.3 Use your new found appreaciation of eigenvalues and eigenvectors to describe the general solution and behavior of solutions to the difference equation $\vec{x}_{k+1} = A\vec{x}_k$ with each of the following A.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.999 \end{bmatrix}$

(c) $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

(g) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(h) $A = \begin{bmatrix} -0.2 & 0.13 \\ -2.1 & 1.05 \end{bmatrix}$

Definition 3.1.6 An **attractor** or attracting fixed point is a fixed point of a dynamical system where all nearby points converge to the fixed point. These are also called **sinks**.

A **repeller** or repelling fixed point is a fixed point of a dynamical system where all nearby points move away from the fixed point. These are also called **sources**.

A **saddle** fixed point is a fixed point of a dynamical system where some nearby points converge to the fixed point and other nearby points move away from the fixed point. ♢

Question 3.1.7 If we consider the dynamical system given by $(\mathbb{R}, f(x) = -x^2)$, what are the fixed points of this system and what behaviors do they exhibit? □

Activity 3.1.4 Use your work from the earlier activity to describe the fixed points and thier behavior for the difference equation $\vec{x}_{k+1} = A\vec{x}_k$ with each of the following A.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.999 \end{bmatrix}$

(c) $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

(g) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(h) $A = \begin{bmatrix} -0.2 & 0.13 \\ -2.1 & 1.05 \end{bmatrix}$

3.2 Change of Coordinates

In the previous section, we saw how advantageous our use of eigenvalues and eigenvectors was to describing the long term behavior of the linear discrete dynamical systems. We will take a few minutes here to set up the algebra of what was going on more fully.

3.2.1 Same Eigenvalues, Different Eigenvectors

Activity 3.2.1

(a) Find the eigenvalues for the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$. For each of the eigenvalues of these matrices, you need to find an eigenvector.

(b) Hopefully you noticed that these matrices have the same eigenvalues but different eigenvectors. Let $\vec{v}_1$ be the eigenvector of A corresponding to $\lambda_1 = \frac{1}{2}$ and $\vec{v}_2$ be the eigenvector of A corresponding to $\lambda_2 = 2$. Let $\vec{w}_1$ be the eigenvector of B corresponding to $\lambda_1 = \frac{1}{2}$ and $\vec{w}_2$ be the eigenvector of B corresponding to $\lambda_2 = 2$.

Find a matrix C such that $C\vec{v}_1 = \vec{w}_1$ and $C\vec{v}_2 = \vec{w}_2$.

(c) Find a matrix D such that $D\vec{w}_1 = \vec{v}_1$ and $D\vec{w}_2 = \vec{v}_2$.

(d) Compute the matrix CAD.

(e) Explain what just happened with matrix product CAD...

Activity 3.2.2

(a) Let's try to reverse engineer what just happened in the previous activity. Can we come up with the matrix that has eigenvalues $\lambda_1 = 1.5$ and $\lambda_2 = -1/3$ with eigenvectors of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$?

Think about what parts of the corresponding matrix parts are given by the information above. Once you have guessed at how to write the corresponding CAD matrix. Test your answer by checking what the eigenvalues and eigenvectors are.

3.3 Continuous Dynamical Systems

A **dynamical system** is a pair (X, R) where X is the set of states a system can be in and R is a rule for how the system evolves or changes. We will look at some dynamical systems where the rule of evolution will describe how the state of the system changes in terms of a continuous parameter. Let's look at some examples.

Example 3.3.1 When examining an electrical circuit with a resistor, capacitor, and an inductor, it is useful to look at how the current (a measure of the flow of electricity) changes as a function of time. The dynamical system in this case would consist of (X, R) where X is the set of possible functions with input t and R is the rule given by the differential equation $L\dfrac{d^2I}{dt^2} + R\dfrac{dI}{dt} + \frac{1}{C}I = 0$.

In this equation the current is $I(t)$ and the constants R, L, and C are the resistance, inductance, and capacitance of the individual components of the circuit. $\square$

Example 3.3.2 If you are looking at the position of an object moving under the force of gravity and under air-resistence, your dynamical system would consist of (X, R) where

- X is the set of vectors of the form $\vec{w}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ where $x(t), y(t), z(t)$ are continuous functions of t
- R is the rule of evolution given by $m\dfrac{d^2w}{dt^2} = -\omega\dfrac{dw}{dt} + mg\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\square$

Example 3.3.3 Wave Equation. Many different physical phenomena satisfy a very famous differential equation:

$$\frac{\partial^2 g}{\partial t^2} = c^2(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2})$$

The state of the system is given by some function $g(x, y, z, t)$ that may vary in both space and time coordinates. This kind of system is called a partial differential equation since there is not A derivative for a multivariable function and the change in our system depends on the various partial derivatives of our function. $\square$

Example 3.3.4 Heat Equation. Many different physical phenomena satisfy another very famous differential equation:

$$\frac{\partial g}{\partial t} = \alpha(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2})$$

The state of the system is given by some function $g(x, y, z, t)$ that may vary in both space and time coordinates. This is another partial differential equation. $\square$

Activity 3.3.1 For this activity, we want to look at the following 2D continuous dynamical system.

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = \frac{1}{3}y$$

(a) What would a solution look like to this system?

(b) Give a solution to this system.

(c) Give all possible solutions to this system.

(d) What is the solution with $x(0) = 1$ and $y(0) = -1$?

None of the stuff in the previous problem seems like linear algebra, so why are we doing this stuff? The answer is that we can expand our notion of what a "vector" is and use the idea that we would like to express solutions to these systems as linear combinations of our "nice" solutions.

3.3.1 Linear Systems of Ordinary Differential Equations

In this subsection, we will look at systems of Linear ODEs of the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 && + \ldots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 && + \ldots + a_{2n}x_n \\
&\vdots \ddots && \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 && + \ldots + a_{nn}x_n
\end{aligned}$$

with initial values given by $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$.

Notice that this system has the following properties:

- no forcing term (the right hand side of the system does not explicitly depend on t)
- constant coefficients
- linear solutions (solutions are linear combinations of each other)

If you look around at other books and online resources, you will see that the solution to the system given by $\frac{d\vec{x}}{dt} = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, will be of the form: $\vec{x}(t) = exp(At)\vec{x_0}$. The term $exp(At)$ is called the matrix exponential of A.

The solution to a linear continuous dynamical system involves evaluating a matrix exponential. This is not a straightforward task and the evaluation algorithm is highly suspect in many situations. In fact, one of the most cited papers in all of applied mathematics is written by Van Loan and Moler (founder of Matlab) titled19 Dubious Ways to Compute the Exponential of a Matrix[1] written in 1978. This paper and idea was so important in computational science and applied mathematics that it was revised by the same authors and updated 25 years later titled appropriately 19 Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later[2]. In short, the papers primary contribution is to show that there is no accepted way to evaluate a matrix exponential for all matrices and the algorithm choice is matrix dependent. Dr. Beauregard's research takes a particular interest in symplectic approximations as they preserve fundamental physical quantities.

Let's start with the same assumption we did for discrete dynamical systems: We will assume that we have continuous dynamical system given by a n by n matrix A (with rule given by $\frac{d\vec{x}}{dt} = A\vec{x}$) and that we can find a set of n linearly independent eigenvectors of A, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, with eigenvalues

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$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Further, let's define two matrices

$$D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and $V = [\vec{v}_1, \ldots, \vec{v}_n]$. From our work on change of coordinates, you should recognize that $A = VDV^{-1}$. So,

$$\begin{aligned} A^k =&(VDV^{-1})(VDV^{-1})\ldots(VDV^{-1}) \\ =&VD^kV^{-1} \end{aligned}$$

where $D^k = diag(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$. We can use our knowledge of power series to write exponentials using

$$e^\alpha = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$$

Notice that all this requires to apply to a matrix is that powers of the matrices have to make sense and the scalar multiplication by $\frac{1}{k!}$ also needs to make sense. So we can define the matrix exponential as

$$e^{At} = V \sum_{k=0}^{\infty} \frac{D^k}{k!} V^{-1} = Ve^{Dt}V^{-1} = V diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})V^{-1}$$

Note here that $e^{At}\vec{x}_0$ will be a vector (by matrix vector product) and thus our solution to $\dfrac{d\vec{x}}{dt} = A\vec{x}$ is given by $\vec{x}(t) = e^{At}\vec{x}_0 = Ve^{Dt}V^{-1}\vec{x}_0$.

This looks a bit like our solutions to the discrete dynamical systems but still different. The vector $V^{-1}\vec{x}_0$ is a solution to what matrix equation? If $\vec{c} = V^{-1}\vec{x}_0$, then $\vec{c}$ is the solution to $V\vec{c} = \vec{x}_0$!!! You should recognize that $\vec{c}$ is the vector of coefficients for the vector equation $c_1\vec{v_1} + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{x}_0$! The vector $\vec{c} = V^{-1}\vec{x}_0$ comes from writing the initial condition of our system as a linear combination of the eigenvectors of A!

Our solutions to $\dfrac{d\vec{x}}{dt} = A\vec{x}$ are of the form

$$\vec{x}(t) = e^{At}\vec{x}_0 = Ve^{Dt}\vec{c} = \sum_{j=1}^{n} c_j e^{\lambda_j t}\vec{v}_j$$

Look at how much of this is determined by the algebra of problems you already know how to do. Which parts of this will determine the long term behavior of solutions? When will you have fixed point(s)? When will the fixed point(s) be attractors? repellors? saddles?

Example 3.3.5 Let $A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$. As we have seen in our previous work, A has eigenvalues of $\lambda_1 = 1/2$ and $\lambda_2 = 2$ with a choice of eigenvectors given by $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The system of differential equations that corresponds to this matrix A is given by:

$$\begin{aligned} \frac{dx_1}{dt} =&1.25x_1 - 0.75x_2 \\ \frac{dx_2}{dt} =& -0.75x_1 + 1.25x_2 \end{aligned}$$

Using our tools from earlier, we can see that the solutions of this system can be written in the vector form

$$\begin{aligned}\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \sum_{j=1}^{n} c_j e^{\lambda_j t} \vec{v}_j \\ &= c_1 e^{\frac{1}{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

If we wanted to find the particular solution with $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then we need to solve

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which has a solution of $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. So the particular solution with $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is

$$\vec{x} = \frac{3}{2} e^{\frac{1}{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

□

Activity 3.3.2 To arouse a deeper interest into the study of linear systems, let us borrow from a classic example of relationships, first presented by Steven Strogatz in 1988 and then further illustrated by J.C. Sprott in 2004.

Now we know the story of Romeo and Juliet. In our situation, Romeo is desperately in love with Juliet, but Juliet is more fickle than what Shakespeare had in mind. In fact, the more Romeo loves Juliet, the more Juliet wants to run away and hide. This discourages Romeo and he starts to love Juliet less, strangely this is exactly the moment that Juliet finds Romeo more attractive and she begins to love him. On the other hand, Romeo notices again that Juliet is warming up to Romeo and he begins to warm up to her as well. Will the eager beaver (Romeo) ever find true love with the ever fickle and cautious lover (Juliet)?

Let $R(t)$ and $J(t)$ be the love (or hate, when negative) that each person has for each other, respectively, at time t. Let $a, b, c, d > 0$. The love/hate relationship is governed by the dynamical system:

$$\begin{aligned}\frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= -cR + dJ\end{aligned}$$

Consider the case where $a = 0$, $b = c = d = 1$ and answer the following:

- Determine the eigenvalues and eigenvectors for the coefficient matrix.
- Write down the general solution using the eigenvalues and eigenvectors.
- Use Euler's formula to simplify the result completely to determine the real-valued solution to the position function.
- Plot $R(t)$ and $J(t)$ over time. Plot $R(t)$ versus $J(t)$. Contrast this to a phase portrait.
- How does the situation change if $a = 3$? What of $a > 3$?

Activity 3.3.3

(a) For each of the matrices below, state the general solution for the system of differential equations given by $\frac{d\vec{x}}{dt} = A\vec{x}$ and find the particular solution with $\vec{x}(1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

(b) $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $A_3 = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5 \end{bmatrix}$

(d) $A_4 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

3.3.2 Converting Higher Order DEs to Systems

Since we have such a nice description and clean algebra to solve systems of differential equations of the form $\frac{d\vec{x}}{dt} = A\vec{x}$, it is common to write other types of questions in terms of a system of first order differential equations. For example, if we consider the damped harmonic oscillator (an object moving on a spring with friction), then the system follows the differential equation

$$m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} - kx$$

where $x(t)$ is the function of time that measures the position of the object (as measured from the rest length of the spring), m is the mass of the object, k is the spring constant, and α is the coefficent of friction for the object. This is called a second order differential equation because it involves a second derivative of the objective function, but we can rewrite this as first order differential equation of the vector $\vec{y}(t) = \begin{bmatrix} \frac{dx}{dt}(t) \\ x(t) \end{bmatrix}$. In particular,

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} -\frac{\alpha}{m}\frac{dx}{dt} - \frac{k}{m}x \\ \frac{dx}{dt} \end{bmatrix}$$

Which takes the form $\frac{d\vec{y}}{dt} = A\vec{y}$ for $A = \begin{bmatrix} -\frac{\alpha}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}$. Thus by analyzing the second component of our solution, $\vec{y}(t)$, we can give the solution to $m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} - kx$

Example 3.3.6 If we consider the second-order, ordinary differential equation given by

$$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} - 2x$$

then we are trying to find a scalar function $x(t)$ that satisfies the second order equation. The corresponding first-order, vector differential equation will be

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} -5\frac{dx}{dt} - 2x \\ \frac{dx}{dt} \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ x \end{bmatrix}$$

The corresponding matrix ($\begin{bmatrix} -5 & -2 \\ 1 & 0 \end{bmatrix}$) has eigenvalues of $\frac{-5\pm\sqrt{17}}{2}$ which will have eigenvectors of $\begin{bmatrix} -5 \pm \sqrt{17} \\ 2 \end{bmatrix}$. So the general solution to our vector equation will be

$$\vec{y(t)} = c_1 e^{(\frac{-5+\sqrt{17}}{2})t} \begin{bmatrix} -5 + \sqrt{17} \\ 2 \end{bmatrix} + c_2 e^{(\frac{-5-\sqrt{17}}{2})t} \begin{bmatrix} -5 - \sqrt{17} \\ 2 \end{bmatrix}$$

The solution to

$$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} - 2x$$

will be the second component of our vector solution, namely $x(t) = c_1 e^{(\frac{-5+\sqrt{17}}{2})t}(2) + c_2 e^{(\frac{-5-\sqrt{17}}{2})t}(2)$. You can write this a little more simply because you can absorb the constants into the c_1 and c_2 to get $x(t) = c_1 e^{(\frac{-5+\sqrt{17}}{2})t} + c_2 e^{(\frac{-5-\sqrt{17}}{2})t}$.

What is the long term behavior of these solutions? Does the behavior depend on c_1 and c_2? □

3.4 Graphics and Linear Algebra

Graphics and computer-based image generation are a large class of problems that utilize a lot of the tools we have discussed in this course. We will be (over)simplifying many of these ideas but you can understand the basics now and learn the linear algebra tools about the more sophisticated approaches needed to work on these ideas later.

Example 1: Transforming Basic 2D Images It is vital to know a little about how images are created/stored. The long and the short of it is that the different image file types correspond to a different method of storing data about how to create the image. This is by no means a comprehensive list of relevant ideas but enough to get you to see how broadly applicable the elements we have talked about are.

- SVG/Scalable Vector Graphics[1]: Instead of pixel based descriptions of an image, SVG files store the image with curves, lines, and other mathematical graphs relative to a grid.
- PS/Postscript: is simplified instructions on how to create a figure
- Fonts: instructions on how to create the shape and width of each part of a character. This is like parameterizations of the curves to make the character along with an idea of width as you move along the curve of the character.
- Fourier Transform of an image: https://plus.maths.org/content/fourier-transforms-images Images might not be stored as an array of colors but might be more effi-

[1] en.wikipedia.org/wiki/SVG