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Linear Algebra:
Notes and Problems

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Contents

To the Student

In this course, you will learn about linear algebra by solving a carefully designed sequence of problems. It is important that you understand **every** problem and proof. As hard as it is to imagine, you will occasionally want to have more questions to work on in order to fully understand the ideas in this class. Feel free to ask me about additional problems to consider when you are stuck. The notes and problems will refer back to previous parts, so I suggest you keep a binder with the notes and your work together and bring all of these materials to class and any office hours. I also suggest having several colored pens or pencils to help you draw and label your work so it will be easily understood. Your written work needs to be legible and large enough so that someone else can easily read and understand what you are doing.

Unlike mathematics courses you have had in the past, solving a problem in this course will **always** have two components:

- Find a solution
- Explain how you know that your solution is correct.

This will help prepare you to use mathematics in the future, when there will <u>not</u> be someone to tell you if your solution is correct (or not). That is also why it will be primarily up to you (the students) to assess the correctness of work done and presented in class. This means that using the proper language (both written and spoken) and specificity are keys to effective communication. This class will teach you about the specificity and precision of mathematical language, so it is important that you practice and work on this. In order for you to understand the ideas in this class you will need to evaluate other people's ideas as well as your own for correctness. The work in this class is *not about getting an answer* but rather **making sense** of the process and ideas involved in the problems. For this reason, justification in your work and ideas is very important. Why you did or tried something is just as important as what you did or what result you got. In fact, clearly articulating your thought process will make you a more efficient thinker.

You are not alone in this course. The role of the instructor is to guide the discussion and make sure you have the resources to be successful. While this new learning environment can be a bit unsettling for students at first, you will get comfortable as you do more problems and get feedback from other students and the instructor. I am also here to help you outside of class time and expect you to find a way to get the help you need, whether face to face or over email. You will find that once you have thought about a problem for a little while, it will only take a little push to get unstuck.

Some Notation:

- $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of natural numbers
- $\mathbb{Z} = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$ is the set of integers
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{R}^n$ is the set of vectors or ordered n-tuples with n components from $\mathbb{R}$. $\mathbb{R}^n$ can also be viewed as the set of points in n-space. For instance, $\mathbb{R}^2$ would be the cartesian plane and $\mathbb{R}^3$ would correspond to three dimensional space
- $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ is the set of complex numbers
- A, B, C, ... denote matrices
- $\vec{u}$, $\vec{v}$, $\vec{w}$, ... denote vectors (which starting in Chapter 2 will not necessarily be in $\mathbb{R}^n$)
- $\vec{0}_V$ is the zero vector in vector space V
- $\emptyset$ is the null set or set that contains no elements
- $M_{m\times n}$ is the set of matrices with m rows and n columns
- DNMS is an acronym for *Does Not Make Sense*
- DNE is an acronym for *Does Not Exist*

Definitions will be bolded for some terms and others will have their own heading and number. Many definitions and comments will be numbered so that everyone will be able to refer to them in work.

You will need to make some plots in this class and it will be advantageous to use a computer algebra system in some problems. For this reason, I suggest that, if you have not already, you should make an account on CoCalc at https://cocalc.com/ (formerly the Sage MathCloud). Additionally, CoCalc will allow you to easily write your work in LaTeX, a wonderful typesetting program. A homework bonus will be given to students who write their work in LaTeX.

Chapter 1

Efficiently Solving Systems of Linear Equations and Matrix Operations

1.1 Warmup Problems

Question 1. *Solve:*

$$3x_1 - 2x_2 = 6$$
$$-x_1 + x_2 = 1$$

Question 2. *Draw a graph of the solution set of the equation:* $3x_1 - 2x_2 = 6$ *(Hint: If a solution has* $x_1 = a$*, what is* x_2 *or viceversa?)*

Question 3. *Draw a graph of the solution set of the equation:* $-x_1 + x_2 = 1$

Question 4. *Graph the solution sets from Questions 2 and 3 together. How does your answer to Question 1 compare to your graph?*

Question 5. *Solve:*

$$2x_1 - 2x_2 = 6$$
$$-x_1 + x_2 = 1$$

Question 6. *Solve:*

$$2x_1 - 2x_2 = -2$$
$$-x_1 + x_2 = 1$$

Question 7. *Wait... what just happened? Explain the results of the previous two problems. What do the graphs of the corresponding solution sets look like in relation to the graphs of the equations?*

Question 8. *What are the possible intersections of two lines? Clearly state your conjecture.*

Throughout this course we will be doing many of the same things you did in the previous questions, but we will do them in a more general setting that will allow us to solve **many** new and old problems.

1.2 Solving Linear Systems

Our first chapter will cover the ideas of efficiently solving a system of linear equations and matrix operations.

A system of m linear equations in n variables can be written:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

The term a_{ij} is the **coefficient** of the j-th variable (denoted x_j) in the i-th equation. In these notes, we will only consider real values for the coefficients of our linear systems, i.e. $a_{ij} \in \mathbb{R}$. A **solution** is a choice of variable values that satisfies all equations in the system. A solution is *not* a particular variable value but must include a choice for all variables in the system. The **solution set** for a system of equations is the set of all possible solutions. We will have many ways to describe solutions to a system this semester but they all specify the values of x_1, x_2, ..., and x_n, typically as an ordered n-tuple (x_1, x_2, ..., x_n).

Question 9. *Is* $(1,2,3)$ *a solution to the following system?*

$$\begin{array}{ccccccc} 1x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & 7x_3 & = & 0 \end{array}$$

The previous problem shows how easy it is to check if a set of variable values is a solution. However, *finding* a solution or the set of all solutions is harder but very important to many problems. Generally speaking, the process of finding the solution set for a system of equations is to trade the system of equations you have for an **equivalent** system (a system with the same solution set).

Question 10. *For each pair of equations given, state whether E_1 is equivalent to E_2.*

a) $E_1 : x^2 - 1 = 0$ $\qquad E_2 : x - 1 = 0$

b) $E_1 : x^2 - 2x + 1 = 0$ $\qquad E_2 : x - 1 = 0$

c) $E_1 : e^x = 1$ $\qquad E_2 : x^3 + x^2 + x = 0$

Hopefully it will be easier to explicitly write the solution set of the new equivalent system. An **elementary operation** on a system of equations is an operation of the form:

a) multiplying an equation by a non-zero scalar

b) switching two equations

c) adding a multiple of one equation to another equation

Question 11. *For this question, we will consider the following system of linear equations:*

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4$$
$$b_1x_1 + b_2x_2 + b_3x_3 = b_4$$

*a) Multiply the second equation in our system by negative three and state the **new** system of equations.*

b) Write a few sentences about why the new system of equations given in the previous part is equivalent to the original system.

c) Write a few sentences about why switching the order in which equations are presented in a system does not change the set of solutions.

d) Write out the equation obtained by multiplying the second equation in the original system by a non-zero scalar (which we will call k) and adding to the first equation.

e) Replace the second equation in the original system with your answer to the previous part, which we will call System 2. Prove that System 2 is equivalent to the original system. In other words, you need to show that (c_1, c_2, c_3) is a solution of the equations S_1:

$$a_1x_1 + a_2x_2 + a_3x_3 = a_4$$
$$b_1x_1 + b_2x_2 + b_3x_3 = b_4$$

if and only if (c_1, c_2, c_3) *is a solution to System 2.*

Remark 12. *For those of you new to the term "**if and only if**" (sometimes abbreviated **iff**), if and only if denotes a biconditional statement. A statement of the form "P iff Q" implies "if P, then Q" and "if Q, then P". If you are asked to prove a biconditional statement you need to prove **both** conditional statements. The good news is that if you have a biconditional statement as a theorem, then you can use either (or both) of the conditional statements in your work.*

Question 13. *Solve the following systems just using elementary operations. Remember to show your work.*

a)

$$\begin{aligned} 2y+z&=4\\ x-3y+2z&=5\\ 2x+y&=-2 \end{aligned}$$

b)

$$\begin{aligned} 3x-2y-z&=0\\ 2x+y+z&=10\\ x+4y+3z&=20 \end{aligned}$$

c)

$$\begin{aligned} 3x-2y-z&=0\\ 2x+y+z&=10\\ x+4y+3z&=10 \end{aligned}$$

A system of equations is **consistent** if there exists at least one solution to the system. In other words, a consistent system of equations has a nonempty solution set. A system that is not consistent is said to be **inconsistent**.

In Question 13, note that you didn't change anything but the *coefficients* in the system of equations as you traded one system for another. Some of the coefficients probably became zero, but you didn't really eliminate any variables or consider a totally different problem. We will use matrices to efficiently store, and manipulate the coefficients in a system of linear equations, since they are all that matter for now. Matrices will have *many* uses in this and other courses, and we will use capital letters like A and B to denote matrices. Matrices will be rectangular arrays with the same number of entries in each row and the same number of entries in each column. The size of a matrix is given (in order) as the number of rows by the number of columns, so a 3 by 2 matrix has 3 rows and 2 columns.

In order to specify what **entry** we are referring to in a matrix, we need an ordered pair of indices telling us the number of the row and number of the column to look in respectively. For instance, if $B = \begin{bmatrix} 1 & 5 & 0 \\ \heartsuit & \bigstar & \blacklozenge \\ \pounds & \circledR & \maltese \end{bmatrix}$, then the $(3,2)$ entry of B is in the third row and 2nd column. You could also write this as $B_{3,2} = \circledR$. The ***i*-th row** of a matrix A will be denoted $row_i(A)$ and the ***j*-th column** will be denoted $column_j(A)$.

In order to distinguish **vectors** (as being more than just n by 1 matrices), we will use the arrow notation and lower case symbols like $\vec{u}$ and $\vec{v}$ to denote vectors. Unless otherwise stated, we will use column vectors. For instance, if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$, then the second **component** of $\vec{v}$ is the scalar v_2. The size of a vector in $\mathbb{R}^n$ is the number of components the vector has. In Chapter 2, we will deal with a *much* more general notion of vectors that will *not* have components like vectors in $\mathbb{R}^n$.

The **coefficient matrix** of a linear system of m equations in n variables is a m by n matrix whose (i, j) entry is the coefficient of the j-th variable, x_j, in the i-th equation of the system. The **augmented matrix** of a linear system of m equations in n variables is a m by $(n+1)$ matrix whose first n columns are the coefficient matrix of the system and the last column is the constant terms from the right side of each equation.

The system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

has a coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

and an augmented matrix of

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

For some properties of the system of equations, we need only look at the coefficient matrix but others will need the augmented matrix. It is important to know the difference and be able to state which corresponding matrix you are using in your work.

Question 14. *What is the coefficient matrix for the previous systems? Give the coefficient matrix for Question i as A_i. (Hint: You should have 10 answers to this question from Questions 1, 2, 3, 5, 6, 9, 11, and three from Question 13.)*

Question 15. *What is the augmented matrix for the previous systems? Give the augmented matrix for Question i as A_i. (Hint: You should have 10 answers to this question.)*

The elementary operations on equations outlined above will correspond to elementary row operations on matrices as well. Specifically, an **elementary row operation** on a matrix is an operation of the form:

- multiplying a row by a non-zero scalar
- switching two rows
- adding a multiple of one row to another row

We now have operations to trade our system of equations for an equivalent system, but we have not stated a way to make sure that the solution set will be easy to explicitly state from our new equivalent system. The following matrix forms will be useful for determining solution sets and various other properties of the corresponding system of equations.

Definition 16. *A rectangular matrix is in **row echelon form** if it has the following three properties:*

a) All nonzero rows are above any rows of all zeros.

*b) Each **leading entry** (being the first non-zero entry) of a row is in a column to the right of the leading entry of the row above it.*

c) All entries in a column below a leading entry are zeros.

*If a matrix in row echelon form satisfies the following additional properties, then we say the matrix is in **reduced row echelon form**:*

d) The leading entry in each nonzero row is 1.

e) Each leading 1 is the only nonzero entry in its column.

The leading entry in a nonzero row of the row echelon form is called a **pivot**. The column in which a pivot occurs is called a **pivot column** and the corresponding variable is a **basic variable** or **pivot variable**. A variable corresponding to a column in which the coefficient matrix does <u>not</u> have a pivot are called **free variables**. While the echelon form is needed to find where pivots will occur, we will sometimes refer to pivot positions of a matrix even when the matrix is not in echelon form.

Theorem 17. *The reduced row echelon form of a rectangular matrix is unique.*

It is important to note that the row echelon form of a matrix is <u>not</u> unique.

Question 18. *Give an example of a matrix M that has the following properties. If such a matrix cannot exist, explain why.*

a) M satisfies properties a and b of row echelon form but does not satisfy property c.

b) M satisfies properties a and c of row echelon form but does not satisfy property b.

c) M satisfies properties b and c of row echelon form but does not satisfy property a.

d) M satisfies properties a, b, and c of row echelon form but does not satisfy property d of reduced row echelon form.

e) M satisfies properties a, b, c, and d of reduced row echelon form but does not satisfy property e of reduced row echelon form.

Question 19. *List out all possible row echelon forms of 3 by 4 matrices using the symbols $\blacksquare$ for a pivot, $*$ for a non-pivot entry (possibly 0), and 0 (when an entry <u>must</u> be 0). For each of these, list out which variables are pivot variables and which are free variables. Hint: There are 15 possible.*

Question 20. *List out all possible reduced row echelon forms of 3 by 4 matrices using the symbols $\blacksquare$ for a pivot, $*$ for a non-pivot entry (possibly 0), and 0 (when an entry <u>must</u> be 0). What value must the $\blacksquare$ entries be? For each of these, list out which variables are pivot variables and which are free variables.*

Question 21. *Solve each of the following systems by converting to an augmented matrix and using elementary row operations to reduce the augmented matrix to reduced row echelon form. With each reduced row echelon form, put a box around all pivot entries. Use the system of equations corresponding to the reduced row echelon form to write out the solution set for each system.*

a)

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ 2x_1 - 2x_2 &= -2 \\ -x_1 + x_2 &= 1 \end{aligned}$$

b)

$$3x_1 - 2x_2 = 6$$

$$2x_1 - 2x_2 = 6$$
$$-x_1 + x_2 = 1$$

c)

$$4x - y + 3z = 5$$
$$3x - y + 2z = 7$$

d)

$$7x - 11y - 2z = 3$$
$$8x - 2y + 3z = 1$$

e)

$$3r - 5s + t = 2$$
$$-6r + 10s - 2t = 3$$

Question 22. *Once you have the augmented matrix for a system of linear equations in reduced row-echelon form, how do you use it to determine the solution set for the system? Write a step-by-step procedure that is general enough to be used on any system of linear equations. Be aware of any implicit assumptions you're making (and try to avoid them).*

Two of the most important questions we will consider this semester are:

1) Is the system consistent?

2) If a solution exists, is the solution unique?

Question 23. *Look back at your results so far and try to figure out what properties of the system (or corresponding matrices) will help us answer question 1 and which properties of the system will help us answer question 2. Write a conjecture about each question.*

The space below is for you to write a statement of the theorem that the class decides is the best for determining consistency and uniqueness.

Theorem 24. *Classwide statement of consistency theorem:*

Theorem 25. *Classwide statement of uniqueness theorem:*

The following two proofs are difficult and one of the reasons we will be resubmitting proofs.

Question 26. *Prove the classwide statement of the consistency theorem.*

Question 27. *Prove the classwide statement of the uniqueness theorem.*

Question 28. *Using the statement of the classwide uniqueness and consistency theorems, treat each of your answers to Question 19 as a coefficient matrix of a linear system of equations and state:*

a) whether each corresponding system of equations will be consistent, inconsistent, or you can't tell.

b) whether each corresponding system of equations will have a unique solution, multiple solutions, no solutions, or you can't tell.

Question 29. *Using the statement of the classwide uniqueness and consistency theorems, treat each of your answers to Question 19 as an augmented matrix of a linear system of equations and state:*

a) whether each corresponding system of equations will be consistent, inconsistent, or you can't tell.

b) whether each corresponding system of equations will have a unique solution, multiple solutions, no solutions, or you can't tell.

1.2.1 Geometric Interpretation of a Solution Set

Recall from Questions 2 through 8, that the solution set of a linear equation in two variables was a line in $\mathbb{R}^2$ (the plane) and that the solution set of a system of two equations in two variables was possibly a point, a line, or empty. Similarly, the solution set for a linear equation in three variables will be a plane in 3-space ($\mathbb{R}^3$).

Question 30. *a) List out all the possible ways two planes can intersect in a three dimensional space.*

b) List out all the possible ways three planes can intersect in a three dimensional space.

c) List out all the possible ways four planes can intersect in a three dimensional space.

d) List out all the possible ways five planes can intersect in a three dimensional space.

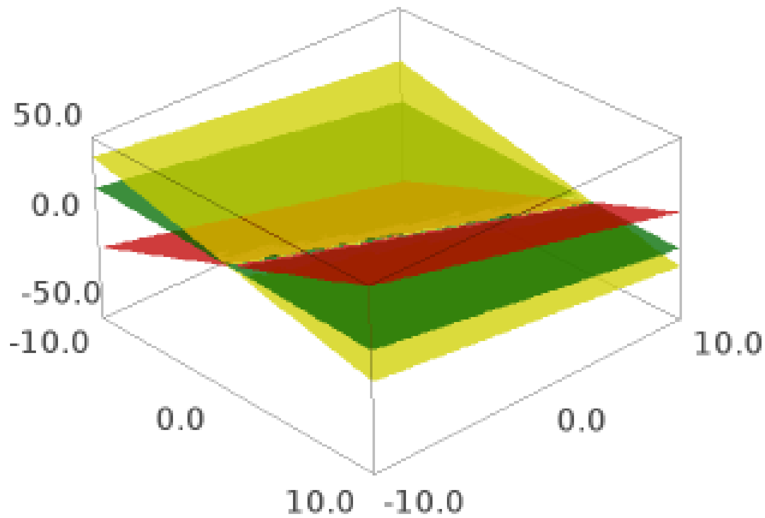
We don't usually draw what a solution set of a linear equation in four variables looks like because drawing in four dimensions is difficult. The graph os a single linear equation in four variables would be called a hyperplane in 4-space. Although we don't draw m hyperplanes in n-space, the intersections of hyperplanes will work very similarly to the pictures we can draw in 3-space (also known as $\mathbb{R}^3$).

Question 31. *Why does the graph of a linear equation in n variables need to be a flat $n-1$ dimensional hyperplane?*

Recall that the following commands will plot the plane given by $3x-2y-z=0$ in Sage:

```
var('x,y');
plot3d(3*x-2*y,(x,-10,10),(y,-10,10),color='red')
```

Plotting the equations, $3x-2y-z=0$, $2x+y+z=10$, and $x+4y+3z=20$ in red, yellow, and green respectively gives:



Question 32. *Does your answer to Question 13 part b make sense with this plot? Explain.*

Question 33. *For each of the systems in Question 21, use Sage to draw a plot of each of the equations in the system and write a sentence for each system about why the plot and your answer to Question 21 make sense.*

If you remember parametric equations of lines and planes in space from multivariable calculus, then we will return to those ideas in Section 1.4.

1.3 Vector and Matrix Equations

Recall that two vectors in $\mathbb{R}^n$ are equal if and only if all of their components are equal. [1]

Question 34. *Prove that the system of equations given by*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

has the same set of solutions as the vector equation

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

In other words, prove that $(c_1, c_2, ..., c_n)$ *is a solution to the vector equation iff* $(c_1, c_2, ..., c_n)$ *is a solution to the system of linear equations. Make sure you clearly connect the ideas in your proof and do not make an argument that these equations look similar.*

Question 35. *Solve the following vector equation directly:*

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Question 36. *Give an example of a vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *such that the equation*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has no solution or explain why no such vector exists.

Question 37. *Give an example of a vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *such that the equation*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

Question 38. *Give an example of a vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *such that the equation*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

Question 39. *Give an example of a vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *such that the equation*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has no solutions or explain why no such vector exists.

Question 40. *Give an example of a vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *such that the equation*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$$

has exactly 1 solution or explain why no such vector exists.

Definition 41. *A* ***linear combination*** *of a set S is a vector of the form*

$$\sum_{i=1}^{n} c_i\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

where $\vec{v}_i \in S$ *and* $c_i \in \mathbb{R}$*. Note that*

$$\sum_{i=1}^{n} c_i\vec{v}_i$$

will not usually be in S even though $\vec{v}_i \in S$*.*

Question 42. *Can you write* $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ *as a linear combination of* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$*? Justify your answer.*

Question 43. *Repeat the previous problem for* $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Question 44. *Can you write* $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ *as a linear combination of* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$*? Justify your answer.*

Definition 45. *We define a **matrix-vector product** as follows:*

If A is a m by n matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$*, then the **matrix-vector product** is given by*

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Question 46. *If A is a m by n matrix, then $A\vec{x} \in \mathbb{R}^{\diamondsuit}$ for what value of $\diamondsuit$?*

It should not surprise you that you can multiply a scalar multiple of a vector by a matrix by factoring out the scalar. In mathematical notation, $A(k\vec{v}) = k(A\vec{v})$. Additionally, you can apply the scalar multiplication to the matrix. In other words, $A(k\vec{v}) = k(A\vec{v}) = (kA)\vec{v}$. These kind of manipulations will be discussed more when we work with matrix operations later, but you may find these facts useful in your work right now. You should take time to write out the details of any of this arithmetic ideas that you think would be useful in your work.

Question 47. *a) Write out the k-th component of the resulting vector of the product*

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

b) How can you express the result of the matrix-vector product in terms of $\vec{x}$ and the rows of A?

c) How can you express the result of the matrix-vector product in terms of $\vec{x}$ and the column of A?

Based on the above definition of the matrix vector product, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, then by Question 34, $A\vec{x} = \vec{b}$ has the same solution set as the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Question 48. *Write each of the following as a matrix equation, a vector equation, and system of equations. You need to write out the exact corresponding vector equation, matrix equation, and system of equations, <u>not</u> some equivalent form.*

a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

b) $a_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

c)

$$2x_1 + 3x_2 = 7$$
$$-x_1 + x_2 + 4x_3 = 0$$
$$5x_1 - 6x_2 - x_3 = 2$$

1.4 Solution Sets of Linear Systems

In this section, we will talk about efficient and clear ways to express the set of solutions to a linear system of equations.

Question 49. *Prove that if a system of linear equations has two distinct solutions, then the system has infinitely many solutions.*

Question 50. *For each of the systems in Question 21, use the reduced row echelon form to solve for each pivot (basic) variable in terms of the free variables and constant terms. By substituting in your new expressions for the pivot variables into the vector* $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$*, you will get a vector solely in terms of the free variables. You can now write the solution set as a linear combination of vectors with each free variable as a coefficient. For instance, if a system had free variables* x_2 *and* x_5*, then the parametric form would look like* $\vec{u} + x_2\vec{v} + x_5\vec{w}$*. This is called the* ***parametric form*** *of the solution set for the system, and is really a linear combination of the vectors* $\vec{u}$*,* $\vec{v}$*, and* $\vec{w}$ *in the example.*

Question 51. *Solve each of the following systems and present your solution set in parametric form.*

a)

$$\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

b)

$$\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

c)

$$\begin{aligned} 4x - y + 3z &= 0 \\ 3x - y + 2z &= 0 \end{aligned}$$

d)

$$\begin{aligned} 7x - 11y - 2z &= 0 \\ 8x - 2y + 3z &= 0 \end{aligned}$$

e)

$$\begin{aligned} 3r - 5s + t &= 0 \\ -6r + 10s - 2t &= 0 \end{aligned}$$

Definition 52. *A system of linear equations is* ***homogeneous*** *if the matrix form of the system, $A\vec{x}=\vec{b}$ has $\vec{b}=\vec{0}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$. If $\vec{b}\neq\vec{0}$, then the corresponding system is* ***nonhomogeneous***.

Question 53. *Using your answers to Questions 50 and 51 as a guide, state and prove a theorem that discusses how the solution set to a homogeneous system is related to the solution set of the non-homogenous system.*

Question 54. State and prove *a theorem that describes under what conditions of the matrix A the system $A\vec{x}=\vec{b}$ will have a solution for every $\vec{b}$. Essentially, you need to fill in the blank of the following statement: If ______________________________, then $A\vec{x}=\vec{b}$ will have a solution for every $\vec{b}\in\mathbb{R}^m$.*

Definition 55. *The* ***column space of a matrix*** *A, denoted $Col(A)$ is the set of vectors that can be written as a linear combination of the columns of A. If A is m by n, then $Col(A)=\{\vec{b}\in\mathbb{R}^m|A\vec{x}=\vec{b}$ for some $\vec{x}\in\mathbb{R}^n\}$.*

Theorem 56. *The pivot columns of a matrix A generate $Col(A)$. This means that if $\vec{v}\in Col(A)$, then $\vec{v}$ can be written as a linear combination using only the pivot columns of A.*

Note that this theorem uses the pivot columns of A and **not** the pivot columns of the *echelon form* of A. Even though you need the echelon form to figure out which columns have pivots, you should use the appropriate columns of A in your linear combination.

Definition 57. *The* ***null space of a matrix****, denoted $Null(A)$, is the set of vectors that when multiplied by the matrix give the zero vector. In other words, $Null(A)$ is the solution set of the homogeneous equation $A\vec{x}=\vec{0}$.*

Question 58. *Let $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$. Describe the sets $Col(A)$ and $Null(A)$ using a parametric form.*

Question 59. *Let $A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$. Describe the sets $Col(A)$ and $Null(A)$ using a parametric form.*

Question 60. *Let*

$$A=\begin{bmatrix}-3&6&-1&1&-7&0\\1&-2&2&3&-1&0\\2&-4&5&8&-8&0\end{bmatrix}\sim\begin{bmatrix}1&-2&0&-1&0&0\\0&0&1&2&0&0\\0&0&0&0&1&0\end{bmatrix}$$

What is the reduced row echelon form of $\begin{bmatrix} -3 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$*? You should use the information given above and* ***not*** *a lot of calculations.*

Question 61. *Let*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Describe $Col(A)$ *and* $Null(A)$ *using a parametric form using as* <u>*few*</u> *vectors as possible.*

Question 62. *Under what conditions on a m by n matrix, A, will* $Col(A)$ *be all of* $\mathbb{R}^m$*?*

Remember that $Col(A)$ and $Null(A)$ are usually very different sets, in fact, they aren't always in the same parent set. If A is a m by n matrix, then $Col(A)$ is in $\mathbb{R}^{\spadesuit}$ and $Null(A)$ is in $\mathbb{R}^{\clubsuit}$, for what values of $\clubsuit$ and $\spadesuit$?

Question 63. *Find an example of a* 2 *by* 2 *matrix where* $Col(A)$ *is the same set as* $Null(A)$.

Question 64. *Given a matrix A with echelon form* $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$*:*

a) *What is the minimum number of vectors that will be needed to give the parametric form of* $Col(A)$*?*

b) *What is the minimum number of vectors that will be needed to give the parametric form of* $Null(A)$*?*

c) $Col(A)$ *is a subset of* $\mathbb{R}^{\spadesuit}$ *for what value of* $\spadesuit$*?*

d) $Null(A)$ *is a subset of* $\mathbb{R}^{\spadesuit}$ *for what value of* $\spadesuit$*?*

Question 65. *Given a matrix A with echelon form* $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$*:*

a) *What is the minimum number of vectors that will be needed to give the parametric form of* $Col(A)$*?*

b) *What is the minimum number of vectors that will be needed to give the parametric form of* $Null(A)$*?*

c) $Col(A)$ is a subset of $\mathbb{R}^{\spadesuit}$ for what value of $\spadesuit$?

d) $Null(A)$ is a subset of $\mathbb{R}^{\spadesuit}$ for what value of $\spadesuit$?

Question 66. *Write a sentence to explain your answer to each part of the following question. Given a matrix A, how many vectors will be needed to give the parametric form of*

- *$Col(A)$*
- *$Null(A)$*

Question 67. *Can every vector in $\mathbb{R}^3$ be written as a linear combination of the columns of $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & 8 & -1 & 7 \end{bmatrix}$? Justify your answer.*

1.5 Applications

Solving linear systems of equations can be used to balance equations of chemical reactions. For instance, if propane (C_3H_8) and oxygen (O_2) react to give off carbon dioxide (CO_2) and water (H_2O), you can write something like $C_3H_8 + O_2 \to CO_2 + H_2O$ but this is not balanced because there are three carbon atoms going into this reaction and only one coming out. Instead we look at $x_1(C_3H_8) + x_2(O_2) \to x_3(CO_2) + x_4(H_2O)$, where x_i stands for the number of molecules of each reactant needed to balance the chemical equation. We can then solve a system of equations where each equation comes from balancing a different element involved in the reaction.

Question 68. *Give the system of equations described by the burning of propane above. Solve the system and check that this balances the chemical equation.*

Question 69. *Balance the following equations:*

a) $B_2S_3 + H_2O \to H_3BO_3 + H_2S$

b) $KMnO_4 + MnSO_4 + H_2O \to MnO_2 + K_2SO_4 + H_2SO_4$

Question 70. *Should the systems used to balance a chemical equation have a unique solution? Why or why not? Consider how many equations you will have and how many variables.*

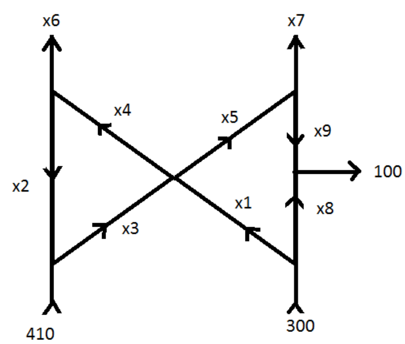
Question 71. *In studying the urban-rural migration patterns of Springfield, you notice that in any given year 70% of the population in urban areas stays* *in urban areas, 20% moves to suburban areas, and 10% moves to rural areas. For the population in a suburban area, the respective percentages moving to urban, suburban, and rural areas are 5%, 80%, and 15%. For the population in a rural area, the respective percentages are 11%, 14%, and 75%.*

a) *If u_t, s_t, and r_t are the percentages of the population in an urban, suburban, and rural areas in year t, set up a system of linear equations for u_{t+1}, s_{t+1}, and r_{t+1} in terms of u_t, s_t, and r_t.*

b) *Write your answer to part a) as a matrix equation where $\vec{x}_t = \begin{bmatrix} u_t \\ s_t \\ r_t \end{bmatrix}$ and $\vec{x}_{t+1} = \begin{bmatrix} u_{t+1} \\ s_{t+1} \\ r_{t+1} \end{bmatrix}$.*

c) *Use Sage or another computational tool to compute $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{10}$, when $\vec{x}_0 = \begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$.* [2]

Question 72. *Networks (traffic, electrical, etc.) can be studied by setting up a system of linear equations corresponding to the different junctions. Specifically, at each junction, the amount of stuff coming in (flow in) must be the same as the amount of stuff going out (flow out). If $x1, \ldots, x9$ are the amount of stuff flowing between junctions, set up a system of equations corresponding to the network given below.*



Question 73. *Is there only one way that the flow of stuff can satisfy the information given by the network graph above? Why or why not?*

Question 74. *Bonus: Solve the system of equations that is your answer to Question 72. Describe how the parts of your solution set correspond to the network.*

1.6 Matrix operations

Question 75. *Finish the following sentences.*

a) Vectors are equal if ...

b) Matrices are equal if ...

c) A scalar is ...

Just as you can add two vectors in $\mathbb{R}^n$ componentwise, you can add two matrices entry-wise. For this reason, it only makes sense to add two matrices if they are the same size. You can also define scalar multiplication of a matrix entry-wise.

Question 76. *Let* $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -7 \\ 4 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \\ 4 & 2 & 6 \end{bmatrix}$, *and* $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Is B a scalar multiple of A? Why or why not?

b) $2A - 3C =$

c) $-(A+C)+2B =$

d) $(84A+16B-12C)_{2,1} =$

Question 77. *Symbolically,* $(A+B)_{i,j} =$ ____________ *and* $(kA)_{i,j} =$ ____________

Definition 78. *Let A be a m by n matrix. The transpose of A, denoted* A^T*, is a n by m matrix such that* $(A^T)_{ij} = (A)_{ji}$.

There are a couple of ways to think about the transpose. First, you can think about flipping the matrix across the main diagonal (the elements of the form $A_{i,i}$). You can also view the transpose of a matrix as switching the rows and columns (but preserving the order). In other words, the i-th row of A^T is the i-th column of A.

Question 79. *Let* $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 4 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \end{bmatrix}$

a) $A^T =$

b) $B^T =$

c) $A^T + B =$

d) $B^T + A =$

Question 80. *Let* $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -7 \\ 4 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 0 \\ 3 & 0 & -21 \\ 4 & 2 & 6 \end{bmatrix}$, *and* $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

a) $A + B^T =$

b) $((C - B)^T + A)^T =$

Question 81. *Prove that* $(A+B)^T = A^T + B^T$ *where A and B are m by n matrices.*

Question 82. *What dimensions should A have in order to be able to add A to* A^T*?*

Question 83. *Prove that* $(A^T)^T = A$.

A matrix A is **symmetric** if $A^T = A$.

Question 84. *Prove that the sum of two symmetric m by n matrices is symmetric.*

Question 85. *Prove that kA is symmetric if A is symmetric.*

1.6.1 Special Types of Matrices

A **square** matrix is a matrix that has the same number of rows and columns. A m by n matrix A is said to be **upper triangular** if $A_{i,j} = 0$ whenever $i > j$. Similarly, a matrix A is **lower triangular** if $A_{i,j} = 0$ whenever $i < j$. We usually consider square matrices when we talk about upper or lower triangular, but it may be helpful to consider non-square cases.

Question 86. *Give an example of a matrix that is upper triangular but is not in echelon form. If one does not exist, explain why.*

Question 87. *Give an example of a matrix that is in echelon form but is not upper triangular. If one does not exist, explain why.*

Question 88. *Can a matrix be upper* <u>*and*</u> *lower triangular? Either give an example or explain why there cannot exist one.*

Diagonal matrices are matrices whose only nonzero entries are on the diagonal. Specifically, a matrix A is diagonal if $A_{i,j} = 0$ whenever $i \neq j$.

Question 89. *Give an example of a matrix that is diagonal but not in echelon form.*

The n by n identity matrix, denoted Id_n, is the unique matrix such that $Id_n\vec{x} = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$. In fact the entries of Id_n are easily computed in terms of the Dirac delta function. Specifically $(Id_n)_{i,j} = \delta_{i,j}$, where

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Question 90. *Write out Id_5 and use it to prove that for any $\vec{v} \in \mathbb{R}^5$ the product of Id_5 and $\vec{v}$ will always be $\vec{v}$.*

Question 91. *<u>Superstar Bonus Question:</u> Prove that Id_5 is the only matrix that has the property from the problem above.*

1.6.2 Matrix Multiplication

Earlier we saw how to multiply a m by n matrix by a vector from $\mathbb{R}^n$. We will discuss how to define matrix multiplication with multiple interpretations.

Let A be an m by n matrix and let $\vec{x_1}$ and $\vec{x_2}$ be vectors from $\mathbb{R}^n$. Earlier we defined what $A\vec{x_1}$ and $A\vec{x_2}$ meant. If we build a n by 2 matrix B using $\vec{x_1}$ and $\vec{x_2}$ as the columns, then we can define AB, read as "A times B", to be

$$AB = A[\vec{x_1} \quad \vec{x_2}] = [A\vec{x_1} \quad A\vec{x_2}]$$

The above definition is just distributing our matrix-vector product across the columns of B. In a similar fashion, given any n by k matrix

$$B = [\vec{b_1} \quad \vec{b_2} \quad \cdots \quad \vec{b_k}]$$

where $\vec{b_i}$ is the i-th column of B, we can define

$$AB = [A\vec{b_1} \quad A\vec{b_2} \quad \cdots \quad A\vec{b_k}]$$

In particular, this means that if AB makes sense, then we can calculate just the i-th column of AB without calculating all of AB. Namely, the i-th column of AB is $Acolumn_i(B)$, which is written symbolically as $column_i(AB) = Acolumn_i(B)$.

Formally, we can define the product of a m by n matrix A with a n by k matrix B to be the m by k matrix AB such that

$$(AB)_{i,j} = \sum_{l=1}^{n} (A)_{i,l}(B)_{l,j}$$

This formula looks difficult, but what it really tells us is that the (i, j) entry of AB is really the dot product of the i-th row of A with the j-th column of B. Remember the **dot product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ is just the sum of the products of the components. Namely,

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

This idea lets us calculate the matrix product AB one entry at a time. Continuing this idea will lead us to see that the i-th row of the product AB can be calculated as $row_i(AB) = row_i(A)B$.

Note that in general $AB \neq BA$, even when both products make sense.

Question 92. *a) What sizes of matrices can you add to a m by n matrix?*

b) What sizes of matrices can you multiply on the right of a m by n matrix?

c) What sizes of matrices can you multiply on the left of a m by n matrix?

Question 93. *If $A \in M_{m \times n}$, when does it make sense to multiply by A^T?*

Question 94. *Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$.*

a) What is the size of AB?

b) Compute just the first column of AB.

c) Write the first column of AB as a linear combination of the columns of A. Be sure to check your work.

d) Solve the matrix equation $A\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

e) Compute just the second row of AB

Question 95. *Let $A = \begin{bmatrix} 3 & 2 & 1 & 5 & 6 \\ 4 & 1 & 3 & 2 & -1 \\ 0 & 2 & 5 & 6 & 7 \\ 8 & 2 & 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 2 & 4 \\ 6 & 2 & 3 & 6 \\ 4 & -1 & 7 & 14 \\ 2 & 0 & -2 & -4 \\ 1 & 1 & 2 & 4 \end{bmatrix}$*

a) $A_{2,3} =$

b) $B_{1,4} =$

c) $(AB)_{2,3} =$

d) $row_2(AB) =$

e) $column_3(AB) =$

Question 96. *Let* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$. *Compute AB and BA.*

Question 97. *Let* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -2 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$. *Compute AB and BA.*

Question 98. *Prove that* $(A+B)C = AC+BC$ *for matrices A, B, and C such that the addition and multiplication of matrices makes sense. You can approach this problem by showing matrix equality entry-wise or column-wise or row-wise.*

Question 99. *Give 2 different examples of 3 by 3 matrices A and B such that* $AB = BA$.

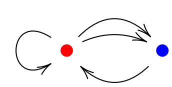
Question 100. *Give 2 different examples of 3 by 3 matrices A and B such that* $AB \neq BA$.

Question 101. *Prove* $(AB)^T = B^T A^T$.

Question 102. *Matrices are good for efficiently storing information. For instance, the adjacency matrix of a directed graph works as follows:*

- *If A is n by n, the corresponding directed graph has n vertices.*
- *There are* $(A_{i,j})$ *edges from vertex i to vertex j.*

For instance, if $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, *then* $\mathscr{G}_B$ =



a) *Draw* $\mathscr{G}_A$, *the directed graph with adjacency matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

b) *Compute* $A^2 = AA$ *and describe what* $(A^2)_{2,1}$ *means in terms of* $\mathscr{G}_A$, *the directed graph of A. What would* $(A^2)_{1,2}$ *or* $(A^2)_{3,2}$ *mean?*

c) *What does* $(A^n)_{i,j}$ *mean for* $n \geq 1$?

Chapter 2

Vector Spaces

Vector spaces are the primary objects that we study in linear algebra. Generally speaking, a vector space is a collection of objects (called vectors) that preserves linear relationships; that is, the objects work well under vector addition and scalar multiplication. As you will see shortly, vectors are not always going to be the column vectors of Chapter 1 or viewed geometrically as arrows from one point to another.

Definition 103. *A* ***vector space****, V, is a nonempty set of objects called* vectors *with two operations called addition and scalar multiplication such that the following hold for all $\vec{u},\vec{v},\vec{w} \in V$ and $c,d \in \mathbb{R}$:*

a) *If $\vec{u},\vec{v} \in V$, then $\vec{u}+\vec{v} \in V$.*

b) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

c) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

d) *There exists a vector $\vec{0}_V$ such that $\vec{v}+\vec{0}_V = \vec{v}$.*

e) *For each $\vec{u} \in V$, there is a vector $-\vec{u} \in V$ such that $\vec{u}+(-\vec{u}) = \vec{0}$.*

f) *If $\vec{u} \in V$ and $c \in \mathbb{R}$, then $c\vec{u} \in V$.*

g) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

h) $(c+d)\vec{v} = c\vec{v}+d\vec{v}$

i) $c(d\vec{v}) = (cd)\vec{v}$

j) $1\vec{v} = \vec{v}$

You can refer to these properties as

a) *closure of vector addition*

b) commutativity of vector addition

c) associativity of vector addition

d) existence of the zero vector

e) existence of the additive inverse

f) closure of scalar multiplication

g) distributive property of scalar multiplication across vector addition

h) distributive property of scalar addition across scalar multiplication (of a vector)

i) associativity of scalar multiplication

j) existence of scalar multiplicative identity

This is the definition for a *real* vector space since the scalars come from $\mathbb{R}$, the real numbers. Sometimes it will be useful for us to consider complex vector spaces (scalars come from $\mathbb{C}$), but unless otherwise stated, you should assume that you are working with a real vector space.

Question 104. *In order to gain an appreciation of definitions, use only the above definition to prove the following results:*

a) The zero vector is unique. You can begin this by supposing that there exists some $\vec{w}$ such that $\vec{x}+\vec{w}=\vec{x}$ for every $\vec{x}\in V$, then you need to show that $\vec{w}$ must equal $\vec{0}_V$.

b) The additive inverse of a vector is unique.

Example 105. *The real numbers, $\mathbb{R}$, are a vector space since all of the above properties hold.*

Example 106. *Real valued vectors, $\mathbb{R}^n$, are a vector space since all of the above properties hold when vector addition and scalar multiplication are done componentwise. We can think of the vectors in $\mathbb{R}^n$ as a directed line segment (an arrow) or as a point in n-dimensional space.*

Question 107. *Show why $\mathbb{Z}^n$, the set of vectors with n integer components is not a vector space.*

Question 108. *Is $\mathbb{C}^n$ a real vector space? Why or why not?*

Question 109. *Is $\mathbb{R}^n$ a complex vector space? Why or why not?*

Example 110. *The set of m by n matrices over the real numbers,* $M_{m\times n}(\mathbb{R})$ *or simply* $M_{m\times n}$*, is a vector space since all of the above properties hold when "vector" addition and scalar multiplication are done entry wise. The quotes are around vector in the previous sentence because you may not always think of matrices as being vectors but using the properties from Section 1.6, you can treat matrices as vectors in the general sense.*

Question 111. *The set of polynomials (in variable t) of degree at most n is denoted by* $\mathbb{P}_n$*.*

a) *Is* $t^2-4t\in\mathbb{P}_2$*?*

b) *Is* $3t^2+t\in\mathbb{P}_3$*?*

c) *Is* $t^2-t+1\in\mathbb{P}_1$*?*

d) *Write* $\mathbb{P}_n$ *as a set using set builder notation. Be sure you have a statement that you can use to verify if an object is in your set or not.*

e) *Prove that* $\mathbb{P}_n$ *is a real vector space.*

Example 112. *The following sets are also vector spaces:*

a) *The set of all polynomials (in variable t) denoted* $\mathbb{P}$*.*

b) $F(S,\mathbb{R})$*, the set of functions from a set S to the real numbers. We will take a closer look at this vector space in the next problem.*

c) $\{\vec{0}\}$*, the* ***trivial vector space****.*

Question 113. *We are going to take a look at the vector space* $V=F(\{a,b,c\},\mathbb{R})$ *to get used to our more general way of thinking about vectors and vector spaces. You should think of the vector space V as the set of functions with domain* $\{a,b,c\}$ *and codomain* $\mathbb{R}$*. In other words, we are looking at the set of functions that only use a, b, and c as inputs and have outputs of real numbers.*

- *Let* g_1 *be the function that takes a, b, and c to* 3*,* -2*, and* 0 *respectively.*
- *Let* g_2 *be the function that takes a, b, and c to* -2*,* 7*, and* 1 *respectively.*
- *Let* g_3 *be the function that takes a, b, and c to* 1*,* 1*, and* 1 *respectively.*
- *Let* g_4 *be the function that takes a, b, and c to* 0*,* 0*, and* 0 *respectively.*
- *Fill in the blank:* $g_2(b)=$ ______
- *Fill in the blank:* $g_3(a)=$ ______

- *Fill in the blank:* $g_1(c) =$ ______

a) *Does it make sense to add the inputs of these functions? Explain.*

b) *Does it make sense to add the outputs of these functions? Explain.*

c) *Let* g_5 *be the function that takes* 5, 1, *and* 0 *to a, b, and c respectively. Is* $g_5 \in V$*?*

d) *Describe the function* $g_1 + g_2$*. In other words, give the outputs for all possible inputs and write a sentence about how you built* $g_1 + g_2$ *in terms of* g_1 *and* g_2*.*

e) *Describe the function* $3g_3$*.*

f) *What function when added to* g_2 *will give* g_4*?*

g) *Can you write* g_1 *as a linear combination of the set* $\{g_2, g_3, g_4\}$*? Explain why or why not.*

h) *Can you write* g_4 *as a linear combination of the set* $\{g_2, g_3, g_1\}$*? Explain why or why not.*

Question 114. a) *Write a sentence or two about what property makes a vector* $\vec{v} \in V$ *the zero vector for* V*, called* $\vec{0}_V$*.*

b) *What is the zero vector for the vector space* $M_{m\times n}$*? Remember to state your answer as an element of* $M_{m\times n}$*.*

c) *What is the zero vector for the vector space* $\mathbb{P}_n$*? Remember to state your answer as an element of* $\mathbb{P}_n$*.*

d) *What is the zero vector for the vector space* $\mathbb{P}$*? Remember to state your answer as an element of* $\mathbb{P}$*.*

e) *What is the zero vector for the vector space* $F(\mathbb{R},\mathbb{R})$*? Remember to state your answer as an element of* $F(\mathbb{R},\mathbb{R})$*.*

2.1 Subspaces

As Question 111 shows, it can be very tedious to prove that a set is indeed a vector space. A **subspace** of a vector space is a subset that is itself a vector space. Since most of the properties of the vector spaces we look at get inherited from some larger vector space, it is often easier to show that a set is a vector space by showing it is a subspace of the appropriate parent vector space.

Theorem 115. *A subset H of a vector space V is a subspace if and only if the following are true:*

a) The zero vector of V is in H; $\vec{0}_V \in H$.

b) H is closed under vector addition; if $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$.

c) H is closed under scalar multiplication; if $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$.

This theorem is so useful because we can prove a set is a vector space by checking only **three** properties instead of the ten that are involved in the definition. The reason that we do not need to check these other properties is that by using this subspace, we already have proven the proper rules of arithmetic from the parent space. Additionally, since we are using the same rules for scalar multiplication and vector addition as the parent space, we must also have the same scalars as the parent space.

Question 116. *Use the preceding theorem to prove that $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.*

Question 117. *Is $\mathbb{R}$ a subspace of $\mathbb{C}$? Why or why not?*

Question 118. *Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Why or why not?*

Question 119. *Is the set of points on the plane given by $z = 0$ a subspace of $\mathbb{R}^3$? Why or why not?*

Question 120. *Is the set of points on the plane given by $z = 1$ a subspace of $\mathbb{R}^3$? Why or why not?*

Question 121. *Draw a plot of the points in $\mathbb{R}^2$ given by $\{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 | x_1 x_2 \geq 0\}$. Is $\{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 | x_1 x_2 \geq 0\}$ a subspace of $\mathbb{R}^2$? Why or why not?*

Question 122. *Is $Sym_{n\times n}$, the set of symmetric n by n matrices a subspace of $M_{n\times n}$? Why or why not?*

Question 123. *Prove or disprove: The set of odd functions on $\mathbb{R}$ ($f(-t) = -f(t)$ for every $t \in \mathbb{R}$) a subspace of $F(\mathbb{R}, \mathbb{R})$.*

Question 124. *If A is a m by n matrix, prove that the solution set to the homogeneous equation $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.*

Question 125. *Prove that if H and K are subspaces of some vector space V, then the set $H \bigcap K$ is a subspace of V as well.*

Question 126. *Prove or Disprove: the set of* 2 *by* 2 *matrices with at least one zero entry is a subspace of* $M_{2\times 2}$.

Question 127. *Prove or Disprove: the set of matrices of the form* $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ *is a subspace of* $M_{2\times 2}$.

Question 128. *Prove or disprove: The set of quadratic polynomials of the form* at^2+b *is a subspace of the vector space of polynomials.*

2.2 Span

Recall that a **linear combination** of the set $\{\vec{v}_1, ..\vec{v}_k\}$ is a vector of the form

$$\sum_{i=1}^{k} c_i\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k$$

Note that some of the c_i may be zero. In other words, not every vector in a set needs to be part of a linear combination from that set.

Question 129. *Can you write* $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ *as a linear combination of* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$*? Remember that you need to justify your work on every problem.*

Question 130. *Repeat the previous problem for* $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, *and* $\vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Question 131. *Can you write* $2+4t$ *as a linear combination of* $1+t$ *and* $-1+t$*?*

Question 132. *Can you write* $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ *as a linear combination of* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$*?*

The **span** of a set of vectors S, denoted $span(S)$ is the set of <u>all</u> possible linear combinations of S. A set S is said to **span or generate a vector space** V if $span(S) = V$.

Question 133. *If* $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$, *is* $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \in span(S)$*?*

Question 134. *If* $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$*, is* $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in span(S)$*?*

Question 135. *If* $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$*, is* $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in span(S)$*?*

Question 136. *If* $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$*, is* $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in span(S)$*?*

Question 137. *Is* $1 - t^2$ *in the span of* $\{3, 4 + t + t^2, 5 - t\}$*?*

Question 138. *For what value(s) of* α *and* β *is* $\vec{p} = \begin{bmatrix} \beta \\ -2 \\ \alpha \\ -4 \end{bmatrix}$ *a solution to* $A\vec{x} = \vec{b}$ *if* $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$ *and* $\vec{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$*?*

Question 139. *Is* $\vec{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$ *in the span of the set of columns of* $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$*? If so, what are the coefficients?*

Question 140. *Prove that if* S *is a set of* k *vectors from a vector space* V*, then* $span(S)$ *is a subspace of* V*.*

Question 141. *Find a finite set of vectors that generates each of the following vector spaces (be sure to show why your set works):*

a) $\mathbb{R}^3$

b) $\mathbb{P}_2$

c) $Sym_{n\times n}$

Question 142. *Show that the set* $\{1+t, t+t^2, 1+t^3, t+t^2+t^3\}$ *spans all of* $\mathbb{P}_3$*. Hint: Come up with a system of equations that you will need to solve and use your theorems from Chapter 1.*

Question 143. *Geometrically describe the span of* $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$.

Question 144. *Geometrically describe the span of* $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Question 145. *Does the span of* $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ *have to go through the origin?*

Question 146. *Does the span of* $\{\vec{v}_1, \ldots, \vec{v}_k\}$ *where* $\vec{v}_i \in \mathbb{R}^n$ *have to go through the*
origin?

2.3 Linear Independence

Definition 147. *A set of vectors S is* ***linearly independent*** *if the only linear combination of the zero vector is the trivial linear combination. In other words, S being a linear independent set implies that if* $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$ *where* $\vec{v}_i \in S$*, then all* $c_i = 0$.

A set of vectors S is ***linearly dependent*** *if the set is not linearly independent. More specifically, there exists a solution to* $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$ *where* $\vec{v}_i \in S$ *and at least one of the* $c_j \neq 0$.

Question 148. *Is the set* $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$ *linearly independent?*

Question 149. *Is the set* $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ *linearly independent?*

Question 150.

a) *Choose a vector* $\vec{v}$ *so that the set* $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} \right\}$ *is linearly independent, where* $\vec{v} \in \mathbb{R}^3$.

b) *Is your choice of* $\vec{v}$ *in Span* $\left(\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \right)$? *Show why or why not.*

Question 151. *Is* $\{2+t^2, 1+t^2\}$ *a linearly dependent set in* $\mathbb{P}_2$*?*

Question 152. *Is* $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ *a linearly independent set in* $M_{2\times 2}$*?*

Question 153. *Prove that* $\{1+t, t+t^2, 1+t^2\}$ *is linearly independent.*

Question 154. *Prove that if a set* S *of a vector space* V *contains* $\vec{0}_V$*, then* S *is linearly dependent.*

Question 155. *If* A *is a* m *by* n *matrix, then the columns of* A *form a linearly independent set if and only if* A *has* ______ *pivot columns. Completely justify your response.*

Question 156. *Prove the following statement: If* $M = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ *is linearly independent, then any subset of* M *is linearly independent.*

Question 157. *Prove or disprove: If* $M = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ *is linearly dependent, then any subset of* M *is linearly dependent.*

Question 158. *Prove that if* $\vec{u}$ *is in the span of* S*, then* $S \bigcup \{\vec{u}\}$ *is linearly dependent.*

The following two questions are a wonderful summary of the difference between and the importance of linear dependence and linear independence.

Question 159. *Prove that if* S *is a linearly dependent set, then any* $\vec{w} \in span(S)$ *can be written as a linear combination from* S *in more than one way.*

Question 160. *Prove that if* S *is a linearly independent set, then any* $\vec{w} \in span(S)$ *can be written as a linear combination from* S *in only one way.*

2.4 Linear Transformations

Linear transformations are the "nice" functions from a vector space to a vector space. In particular, linear transformations preserve the operations of scalar multiplication and vector addition.

Definition 161. *A function* T *from a vector space* V *to a vector space* W *is a* ***linear transformation*** *if for every* $\vec{v}_1, \vec{v}_2 \in V$ *and* $c \in \mathbb{R}$

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- $T(c\vec{v}_1) = cT(\vec{v}_1)$

Question 162. *Prove that the map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$ is linear, where A is an m by n real valued matrix.*

Eventually we will be able to state a lot of linear transformations as a **matrix transformation** like in the problem above, but we will not be able to do this in general.

Question 163. *Prove that the map $T : \mathbb{P} \to \mathbb{P}$ given by $T(f) = \dfrac{df}{dt}$ is linear. You may use your calculus knowledge.*

Question 164. *For each of the following functions, determine if the function is a linear transformation. Remember to justify your reasoning and answers.*

a) $f_1 : \mathbb{P} \to \mathbb{R}$ *where* $f_1(\vec{p}) =$ *the degree of the polynomial* $\vec{p}$

b) $f_2 : \mathbb{P} \to \mathbb{R}$ *where* $f_2(\vec{p}) = \vec{p}(t = 1)$

c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ *where* $f_3(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a+b \\ a-b \\ b+1 \end{bmatrix}$

d) $f_4 : \mathbb{R}^3 \to \mathbb{R}^2$ *where* $f_4(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = \begin{bmatrix} a+b \\ a-c \end{bmatrix}$

e) $f_5 : \mathbb{R}^3 \to \mathbb{R}^2$ *where* $f_5(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = \begin{bmatrix} a+b \\ c^2 \end{bmatrix}$

Question 165. *Prove that if T is a linear transformation and a set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent, then the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.*

Question 166. *Give a counterexample to the following statement: If T is a linear transformation and a set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, then the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly independent.*

Question 167. *Prove that if T is a linear transformation from V to W, then $T(\vec{0}_V) = \vec{0}_W$.*

Question 168. *If a linear transformation, T, sends the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ and sends the vector $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, find the following:*

- $T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$
- $T\left(\begin{bmatrix}0\\5\end{bmatrix}\right)$
- $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$

Question 169. *Find a matrix A such that for the transformation in the previous problem $T(\vec{x}) = A\vec{x}$.*

Definition 170. *If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the **standard matrix presentation** of T is a m by n matrix*

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad \dots \quad T(\vec{e}_n)]$$

*where $\vec{e}_i$ is the **i-th elementary vector** of $\mathbb{R}^n$. Note that $(\vec{e}_i)_j = \delta_{i,j}$, where δ is the Dirac delta function defined by*

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The vector $\vec{e}_i$ can also be thought of as the i-th column of Id_n, the n by n identity matrix. Because of how we defined the standard matrix presentation, only transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ will have standard matrix presentations. In particular, the standard matrix presentation keeps track of where the standard basis vectors ($\vec{e}_i$) go under the transformation T.

Question 171. *Write out $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ from $\mathbb{R}^3$. What is the result of multiplying $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ by $\vec{e}_1$? What about $\vec{e}_2$? $\vec{e}_3$?*

What would this mean for the following matrix product:

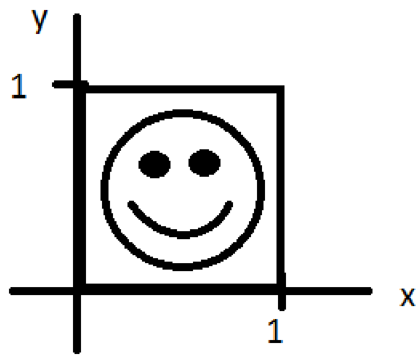
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$$

Question 172. *Determine the standard matrix presentation A for the following T:*

- *$T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects points over the vertical axis*
- *$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points clockwise by $\pi/2$*

- *$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points by π and then flips points over the vertical axis*

Question 173. *Draw what the image of the picture below will look like after applying given the linear transformations. It may help to look at where $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix}$ get mapped by T.*



a) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects points across the vertical axis*

b) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points clockwise by $\pi/2$ (around the origin)*

c) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ and $A = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$*

d) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ and $A = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$*

e) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ and $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$*

f) *$T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\vec{x}) = A\vec{x}$ and $A = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$*

Question 174. *If a linear transformation, T, sends the vector $\begin{bmatrix}1\\1\end{bmatrix}$ to $\begin{bmatrix}-2\\2\end{bmatrix}$ and sends the vector $\begin{bmatrix}-1\\1\end{bmatrix}$ to $\begin{bmatrix}0\\2\end{bmatrix}$, find the following:*

- $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ *Hint: How can you write $\begin{bmatrix}1\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$?*

- $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$
- *Find the standard matrix presentation for T*

Question 175. *Let $T_{\vec{0}}$ be the function from V to W such that $T(\vec{x}) = \vec{0}_W$ for every $\vec{x} \in V$. Let Id_V be the identity map on V, $Id_V(\vec{x}) = \vec{x}$ for every $\vec{x} \in V$.*

a) *Prove that $T_{\vec{0}}$ is linear.*

b) *Prove that Id_V is linear.*

The **range** of a linear transformation $T : V \to W$ is the set of things in the codomain W that are the output of T for some input. That is $range(T) = \{\vec{y} \in W | \vec{y} = T(\vec{x})$ for some $\vec{x} \in V\}$. The **null space**, or **kernel**, of a linear transformation $T : V \to W$ is the set of inputs that get mapped to the zero vector of W. That is $Null(T) = \{\vec{x} \in V | T(\vec{x}) = \vec{0_W}\}$.

Question 176. *Is $\vec{b} = \begin{bmatrix}0\\2\\1\end{bmatrix}$ in the range of the linear transformation $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}1 & 2\\3 & 4\\0 & 0\end{bmatrix}$? Justify your answer without doing any matrix operations. Hint: write the corresponding matrix equation as a vector equation.*

Question 177. *Let $A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$. Find the range and null space of T where $T(\vec{x}) = A\vec{x}$. Remember to state the range and null space so that the reader can most easily verify whether a vector is in the set or not.*

Question 178. *Let T from $\mathbb{R}^2$ to $\mathbb{P}_2$ be given by $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = a + (a+b)t + (a-b)t^2$.*

a) *Prove T is linear.*

b) *Compute the range of T.*

c) *Compute the null space of T.*

Question 179. *Let V be the vector space of polynomials in x and y.*

a) *Show the transformation T that maps f to $\frac{\partial f}{\partial x}$ is a linear transformation.*

b) *Compute the null space of T.*

c) *Compute the range of T.*

Question 180. *Let T be a linear transformation from V to W. Prove that $null(T)$ is a subspace of V.*

Question 181. *Let T be a linear transformation from V to W. Prove that $range(T)$ is a subspace of W.*

A function $f : C \to D$ is **one to one** if whenever $f(x) = f(y)$, then $x = y$. This means that each input gets sent to a different output by the function f. Alternately, you can say a one to one function does not map two different inputs to the same output.

A function $f : C \to D$ is **onto** if every element of D has some input that is mapped to it. In other words, a map f is onto if the range of f is all of D.

Question 182. *For each of the functions from $\mathbb{R}$ to $\mathbb{R}$ below state whether the function is either 1-1 but not onto, onto but not 1-1, 1-1 and onto, or not 1-1 and not onto.*

a) $f(x) = e^x$

b) $f(x) = x$

c) $f(x) = x^2$

d) $f(x) = 1 - x$

e) $f(x) = x^2(1 - x)$

f) $f(x) = sin(x)$

g) $f(x) = x^3$

Question 183. *Let T from $\mathbb{R}^2$ to $\mathbb{P}_2$ be given by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)t + (a-b)t^2$.*

a) *Is T one-to-one?*

b) *Is T onto?*

Question 184. *Give an example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ that is one to one.*

Question 185. *Give an example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is onto.*

Question 186. *Give an example of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ that is onto.*

Question 187. *If the set of columns of a m by n matrix A are linearly independent, does the set of columns of A span all of $\mathbb{R}^m$?*

Question 188. *If the set of columns of a m by n matrix A are linearly independent, is the range of $T(\vec{x}) = A\vec{x}$ all of $\mathbb{R}^m$?*

Theorem 189. *A linear transformation $T : V \to W$ is onto iff $range(T) = W$.*

Question 190. *Prove that for T a linear transformation from V to W, $null(T) = \{\vec{0}\}$ iff T is 1-1.*

2.5 Applications

Definition 191. *Let $\mathscr{C}^n(\mathbb{R},\mathbb{R})$, or simply $\mathscr{C}^n$ be the set of functions from $\mathbb{R}$ to $\mathbb{R}$ that are n times continuously differentiable.*

Question 192. *Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. A solution to the system of differential equations:*

$$\frac{dx}{dt} = a_1 \ x(t) + a_2 \ y(t)$$
$$\frac{dy}{dt} = a_3 \ x(t) + a_4 \ y(t)$$

is a choice of x and y as functions of t such that both differential equations are satisfied.

a) If the pair of functions $(g(t), h(t))$ is a solution to the system above, what does this imply about the derivatives of g and h? Be very specific.

The solution set to the given set of differential equations will be a subset of the ordered pairs of differentiable functions; specifically the solutions will be in the set $(\mathscr{C}^2)^2 = \{(x(t), y(t)) | x, y \in \mathscr{C}^2\}$.

b) Prove that the set of solutions to the system above is a subspace of the vector space $(\mathscr{C}^2)^2$.

c) Consider the system of differential equations given by

$$\frac{dx}{dt} = a_1 \ x(t) + a_2 \ y(t)$$

$$\frac{dy}{dt} = a_3 \ \ x(t) + 1$$

Is the set of solutions to this system a subspace of $(\mathscr{C}^2)^2$? Be sure to justify why or why not.

d) *Consider the system of differential equations given by*

$$\frac{dx}{dt} = (x(t))^2 + a_2 \ \ y(t)$$

$$\frac{dy}{dt} = a_3 \ \ x(t) + a_4 \ \ y(t)$$

Is the set of solutions to this system a subspace of $(\mathscr{C}^2)^2$? Be sure to justify why or why not.

The previous result is especially important in a differential equations class because finding the solution set of the system of differential equations can reduce to finding a few solutions that spans a large enough space.

Question 193. a) *Let S be the set of solutions to the differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx(t) = 0$. Prove that $S = \{f \in F(\mathbb{R}, \mathbb{R}) | a\frac{d^2f}{dt^2} + b\frac{df}{dt} + cf(t) = 0\}$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.*

b) *If $f_1(t)$ and $f_2(t)$ are solutions to the differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c(x(t)) = g(t)$, then prove that $f_1 - f_2$ is a solution to the homogeneous differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c(x(t)) = 0$.*

c) *Conclude that the solution set of the non-homogeneous differential equation is of the form $y(t) + s(t)$, where y is a solution to the nonhomogeneous differential equation and $s(t) \in S$, where S is the solution set to the homogeneous differential equation.*

The previous problem is analogous to your work on Question 53.

Question 194. a) *Show that the transformation T from $\mathscr{C}^2$ to $\mathscr{C}^2$ given by $T(f) = a\frac{d^2f}{dt^2} + b\frac{df}{dt} + cf(t)$ is linear.*

b) *What is $Null(T)$?*

Chapter 3

Connecting Ideas

3.1 Basis and Dimension

In the previous chapter, we saw that if a set S spans a vector space V, then S is big enough to write everything in V (as a linear combination of S). We also saw that a linearly independent set S had a unique way to represent elements in $span(S)$ (Question 160). A convenient and minimal way to describe a vector space is to give a set of vectors that spans all of V but does not include anything extra.

Definition 195. *A basis for a vector space V is a set of vectors that is linearly independent and spans V.*

In this way, a basis is big enough (spans V) and contains nothing extra (linearly independent).

Question 196. *Can a set of 4 vectors be a basis for $\mathbb{R}^3$? Why or why not? Be sure to justify using ideas from Chapter 1 and not any theorems past this point.*

Question 197. *Can a set of 2 vectors be a basis for $\mathbb{R}^3$? Why or why not?*

Theorem 198. *If there exists a basis for a vector space V with n vectors, then every basis of V must have exactly n vectors.*

The previous theorem does not imply that there is only one basis for a vector space, just that any two bases for the same vector space will have the exact same number of vectors. The idea that every basis for a vector space V has the same number of vectors gives rise to the idea of dimension.

Definition 199. *If a vector space V has a basis with a finite number of elements, n, then we say that V has* ***dimension*** *n or that V is n**-dimensional**, also written as $dim(V) = n$.*

Question 200. *Show that* $\{\vec{e}_1, ..., \vec{e}_n\}$ *is a basis for* $\mathbb{R}^n$ *and thus that* $\mathbb{R}^n$ *is an n-dimensional vector space.*

Question 201. *Give a set of 3 different vectors in* $\mathbb{R}^3$ *that are not a basis for* $\mathbb{R}^3$*. Be sure to show why the set does not satisfy the definition of a basis.*

Question 202. *Give a basis for* $\mathbb{P}_3$ *and compute the dimension of* $\mathbb{P}_3$*.*

Question 203. *What is* $dim(\mathbb{P}_n)$*? Be sure to justify.*

Question 204. *Recall that the set* $\{\vec{0}\}$ *is the trivial vector space. What is a basis for* $\{\vec{0}\}$*? What is* $dim(\{\vec{0}\})$*?*

Theorem 205. *If* V *is an n-dimensional vector space and* S *is a set with exactly n vectors, then* S *is linearly independent if and only if* S *spans* V*.*

This is an *enormously* helpful theorem since we know that a linearly independent set of n vectors from a n-dimensional vector space is a basis (no need to show spanning). This goes the other way as well, namely if a set of n vectors, S, spans a n-dimensional vector space, V, then S is a basis for V (no need to show linear independence).

Question 206. *Prove that* $\{1+t, t+t^2, 1+t^2\}$ *is a basis for* $\mathbb{P}_2$*.*

Question 207. *Give two different bases for* $M_{2\times 2}$*.*

Question 208. *What is the dimension of* $Sym_{3\times 3}$*, the vector space of symmetric 3 by 3 matrices?*

Question 209. *What is the dimension of* $Sym_{n\times n}$*?*

Question 210. *What is the dimension of* $\mathbb{P}$*?*

Question 211. *a) Prove that* $H = \{\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} | t \in \mathbb{R}\}$ *is a subspace of* $\mathbb{R}^3$*.*

b) Is $Span(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}) = H$*?*

c) What dimension is H*?*

3.1.1 rank and nullity

Recall from Question 180, if $T : V \to W$ is linear, then $Null(T)$ and $range(T)$ are subspaces of V and W respectively.

Definition 212. *The **rank** of a transformation T is $dim(range(T))$ and the **nullity** of T is $dim(Null(T))$.*

Question 213. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

a) *Find $rank(T)$ and $nullity(T)$ where $T(\vec{x}) = A\vec{x}$.*

b) *Find a basis for $Null(T)$.*

c) *Find a basis for $range(T)$.*

Question 214. *Let T from $\mathbb{R}^2$ to $\mathbb{P}_2$ be given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)t + (a-b)t^2$.

a) $rank(T) =$

b) $nullity(T) =$

c) *Find a basis for $Null(T)$.*

d) *Find a basis for $range(T)$.*

Question 215. *Let $T : \mathbb{P}_3 \to \mathbb{R}^2$ be given by* $T(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$. *Compute $rank(T)$ and $nullity(T)$.*

Theorem 216 (Dimension Theorem)**.** *Let T be a linear transformation from V to W with V a n-dimensional vector space. $rank(T) + nullity(T) = n$.*

If T is a matrix transformation ($T(\vec{x}) = A\vec{x}$), then the $rank(T) = rank(A) = dim(Col(A))$ and $nullity(T) = nullity(A) = dim(Null(A))$.

Question 217. *Using Question 66 and other previous work, prove the Dimension Theorem for $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$, where A is a m by n matrix.*

Question 218. *List out all possible echelon forms of 3 by 3 matrices using the symbols ■ for pivot, $*$ for non-pivot entry (possibly 0), and 0 if an entry must be 0. For each form, give the rank of the matrix and the dimension of the null space.*

3.1.2 Coordinate Vectors relative to a basis

Given an ordered basis $\beta = \{\vec{v}_1, ..., \vec{v}_k\}$ of a vector space V, the **coordinate vector of $\vec{x}$ relative to β**, denoted $[\vec{x}]_\beta$, is a vector of the coefficients of the unique way to write $\vec{x}$ as a linear combination of β. Namely, if $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k$, then $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$.

Question 219. *For each of the following vectors, compute* $[\vec{v}]_\beta$ *where* $\beta = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$

a) $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

b) $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

c) $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

d) $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

e) $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Question 220. *In the previous problem, you wrote out the coordinate vectors relative to* $\beta = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$. *Note that the first three vectors you used form a basis as well, which we will call* $\gamma = \{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \}$.

a) Compute $[\vec{v}]_\gamma$ *for* $v = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

b) *The coordinate vectors of γ relative to β can be used to make the* ***change of basis matrix*** *from β to γ. Specifically, the change of basis matrix from β to γ is given by $[[\vec{\gamma}_1]_\beta[\vec{\gamma}_2]_\beta[\vec{\gamma}_3]_\beta]$. Use your work from the previous question, to construct the change of basis matrix from β to γ.*

c) *Multiplying by this change of basis matrix will convert a coordinate vector relative to β to a coordinate vector relative to γ. Verify that if you multiply your change of basis matrix from β to γ by $[\vec{v}]_\beta$ you get $[\vec{v}]_\gamma$ where $v = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.*

The above process of constructing a change of basis matrix works for any two bases of the same vector space (even if the vector space is not $\mathbb{R}^n$.

Question 221. *For each of the following vectors, compute $[\vec{v}]_\beta$ where $\beta = \{1+t, t+t^2, 1+t^2\}$*

a) $\vec{v} = 2+2t$

b) $\vec{v} = 4-t^2+t$

c) $\vec{v} = 3$

d) $\vec{v} = t$

e) $\vec{v} = 6t^2$

The coordinate vector allows us to state problems in a vector space like $\mathbb{P}_n$ in the same way we state problems in $\mathbb{R}^n$.

Question 222. *For each of the following vectors, compute $[\vec{v}]_\beta$ where $\beta = \left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$*

a) $\vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $\vec{v} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

3.2 Invertible Matrices

In this section, we will only consider square matrices. A matrix $A \in M_{n\times n}$ is **invertible** if there exists a matrix B such that $AB = Id_n$ and $BA = Id_n$.

The inverse matrix of A is denoted A^{-1}. Be careful that you do not use the notation A^{-1} until you have shown that A is invertible.

3.2.1 Elementary Matrices

Recall that an elementary row operation on a matrix is an operation of the form:

- multiplying a row by a non-zero scalar
- switching two rows
- adding a multiple of one row to another row

Elementary matrices are obtained by performing an elementary operation on the identity matrix.

Question 223. *Give the elementary matrix obtained by performing the given operation on Id_3 (These are 4 separate questions):*

a) Scaling the first row by α

b) Switching the second and third rows

c) Adding 3 times the 2nd row to the 1st row

d) Adding 3 times the 1st row to the 2nd row

Question 224. *Check that your answer to the previous question does the desired operation by multiplying each of the four previous elementary matrices by $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Which side do you multiply the elementary matrix on to correspond to row operations?*

Question 225. *Compute (and verify) the inverse of each of the elementary matrices from the previous problems. (Hint: Think about how you would go backwards for each of the elementary operations.*

Your work on the previous questions should convince you that elementary matrices are invertible and that multiplying by an elementary matrix produces the same result as having performed the corresponding elementary row operation. Elementary matrices offer a way of keeping track of elementary operations.

Theorem 226. *Elementary matrices are invertible and the inverse matrix is an elementary matrix corresponding to the inverse elementary operation.*

Theorem 227. *If A and B are invertible n by n matrices, then AB is an invertible n by n matrix. Further, $(AB)^{-1} = B^{-1}A^{-1}$.*

Question 228. *Prove that if A can be reduced to Id_n by elementary row operations, then A is invertible.*

Question 229. *Give all values of k where $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & k & 4 \\ 3 & 5 & 1 \end{bmatrix}$ will be invertible.*

Question 230. *Give all values of k where $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & k & 4 \\ 3 & -1 & 1 \end{bmatrix}$ will be invertible.*

Question 231. *How many pivots must a matrix A have in order to be row reducible to Id_n? Justify using previous results.*

Question 232. *Prove that if A is invertible, then $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.*

Question 233. *Prove or disprove: If A and B are invertible n by n matrices, then $A + B$ is invertible.*

Question 234. *Prove that if A is invertible, then A^T is invertible.*

3.2.2 Computing Inverses

In general computing the inverse of a matrix takes more time and operations than solving a system of equations. For this reason, it is generally easier to find and solve a related system of equations problem than to compute the inverse matrix. We will outline a few ways to find inverse matrices and compute a few small examples.

Question 235. *If a matrix A is row reduced to Id_n by elementary row operations corresponding (in order of use) to elementary matrices E_1, E_2, ... , E_k, give an expression for A^{-1}.*

Question 236. *Use your answer to the previous question to prove the following:*

Any sequence of elementary row operations that reduces A to Id_n also transforms Id_n into A^{-1}.

The previous result shows that computing inverses is equivalent to a row reduction problem. In particular, if A is invertible, then reducing $[A \mid Id_n]$ to reduced row echelon form will produce the matrix $[Id_n \mid A^{-1}]$.

Question 237. *Use the idea above to compute the inverse of* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$*. Be sure to note any assumptions you will need to make in order to reduce* $[A \mid Id_n]$ *to* $[Id_n \mid A^{-1}]$.

Question 238. *If* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & -1 \end{bmatrix}$*, find* A^{-1} *and check that* $AA^{-1} = Id_3$.

Question 239. *If* $A = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}$*, find* A^{-1} *and use your answer to solve* $A\vec{x} = \vec{b}$ *if:*

a) $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) $\vec{b} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

c) $\vec{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

d) $\vec{b} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

3.2.3 Invertible Matrix Theorem

Question 240. *In many texts there is a long list of equivalent conditions for when a square matrix is invertible. Below is a list of some of these conditions that we have talked about or proven. Go back through your notes and questions and cite when we connected two of the ideas in the list. For instance, parts a) and b) are linked by Question 228*

Theorem 241 (The Invertible Matrix Theorem)**.** *Let A be a n by n matrix. The following are equivalent statements (either all True or all False):*

a) A is an invertible matrix.

b) A is row equivalent to Id_n.

c) A has n pivots.

d) $rank(A) = n$

e) $nullity(A) = 0$

f) $A\vec{x} = \vec{0}$ *has only the trivial solution.*

g) *The linear transformation* $\vec{x} \to A\vec{x}$ *is one-to-one.*

h) *The linear transformation* $\vec{x} \to A\vec{x}$ *is onto.*

i) $A\vec{x} = \vec{b}$ *has a solution for every* $\vec{b} \in \mathbb{R}^n$.

j) *The columns of* A *form a linearly independent set.*

k) *The columns of* A *span* $\mathbb{R}^n$.

l) *The columns of* A *are a basis for* $\mathbb{R}^n$.

m) A^T *is invertible.*

Question 242. *Two important ideas in this course that have been tied to many different methods or ideas are 1) consistent systems of linear equations and 2) invertible matrices. These two ideas are a bit different though. Give an example of a consistent system of linear equations (in matrix equation form* $A\vec{x} = \vec{b}$*) where the coefficient matrix* A *is a non-invertible square matrix.*

3.3 Invertible Linear Transformations

Definition 243. *A linear transformation* T *from* V *to* W *is invertible if there exists a linear transformation* U *from* W *to* V *such that* $T \circ U = Id_W$ *and* $U \circ T = Id_V$.

Alternative definition: A linear transformation T from V to W is invertible if T is one-to-one and onto.

Question 244. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. *Is* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *given by* $T(\vec{x}) = A\vec{x}$ *an invertible linear transformation?*

Question 245. *Let* T *from* $\mathbb{R}^2$ *to* $\mathbb{P}_2$ *be given by* $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)t + (a-b)t^2$. *Is* T *an invertible linear transformation?*

Question 246. *Let* T *from* $\mathbb{R}^3$ *to* $\mathbb{P}_2$ *be given by* $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a+c) + (a+b)t + (a-b)t^2$. *Is* T *an invertible linear transformation?*

3.4 LU factorization of matrices

Question 247. *a) Reduce* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *to* <u>*echelon*</u> *form (not reduced row echelon form). How many row operations did you use?*

b) Compute the elementary matrix (or matrices) corresponding to the row operation(s) from the previous part.

c) Compute the inverse of the elementary matrix from the previous problem and multiply the result by the echelon form you found in part b). What is your result and why does this make sense?

d) Let L be the inverse elementary matrix from part c) and let U be the echelon form from part a). Solve $L\vec{y} = \vec{b}$ *for* $\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

e) Now solve $U\vec{x} = \vec{y}$.

f) Now solve $A\vec{x} = \vec{b}$.

Question 248. *a) Reduce* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *to* <u>*echelon*</u> *form. How many row operations did you use?*

b) Compute the elementary matrix (or matrices) corresponding to the row operation(s) from the previous problem.

c) Compute the inverse of the elementary matrix (or product of matrices) from part b) and multiply your answer by the echelon form you found in part a). What is your result and why does this make sense?

d) Let L be the inverse elementary matrix product from part c) and let U be the echelon form from part a). Solve $L\vec{y} = \vec{b}$ *for* $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

e) Now solve $U\vec{x} = \vec{y}$.

f) Now solve $A\vec{x} = \vec{b}$.

The preceding two problems can be generalized to show how row operations will conveniently reduce any matrix into a product of an upper- and a lower-diagonal matrix. This *LU* decomposition has certain advantages when solving linear systems using a computer, especially for large systems.

3.5 Determinants

Determinants will be an incredibly useful tool in quickly determining several important properties of square matrices. We will first look at how to compute determinants and later outline the important properties that determinants have. While some of you may have been taught some rules for how to compute determinants of 2 by 2 and 3 by 3 matrices, I encourage you to understand how to compute determinants in general.

3.5.1 Computing Determinants

Definition 249. *The **determinant** is a function from n by n matrices to the real numbers (det : $M_{n\times n} \to \mathbb{R}$). If A is a 1 by 1 matrix, $A = [A_{1,1}]$, then $det(A) = A_{1,1}$. For $n \geq 2$, the determinant of a n by n matrix is given by the following formula in terms of determinants of $(n-1)$ by $(n-1)$ matrices:*

$$det(A) = \sum_{j=1}^{n} (-1)^{1+j}(A_{1,j})\ \ det(A^*_{1,j})$$

where $A^*_{i,j}$ is the $(n-1)$ by $(n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

*The term $A_{i,j}\ \ det(A^*_{i,j})$ is called the (i,j) **cofactor of** A.*

The above definition uses cofactor expansion along the first row.

Question 250. *In this question, we will unpack the determinant formula above for a 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.*

a) Rather than using the summation notation of the formula above, write out the two terms in $det(A)$.

b) $A^*_{1,1} =$

c) $A^*_{1,2} =$

d) $A_{1,1} =$

e) $A_{1,2} =$

f) $(-1)^{1+1} =$

g) $(-1)^{1+2} =$

h) $det(A) =$

Your answer to the previous problem will be useful in calculating determinants of 3 by 3 matrices.

Theorem 251. *The determinant can be computed by cofactor expansion along any row or column. Specifically the cofactor expansion along the k-th row is given by*

$$det(A) = \sum_{j=1}^{n} (-1)^{k+j} (A_{k,j}) \;\; det(A^*_{k,j})$$

and the cofactor expansion along the k-th column is given by

$$det(A) = \sum_{i=1}^{n} (-1)^{i+k} (A_{i,k}) \;\; det(A^*_{i,k})$$

Question 252. *Use cofactor expansion along the first column of* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *to compute* $det(A)$.

Question 253. *Use cofactor expansion along the second row of* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *to compute* $det(A)$. *Did you get the same answer as the previous question?*

Question 254. *Compute the determinant of* $B = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$. *How does your answer compare with the previous problem?*

Question 255. *Compute the determinant of* $C = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$.

Question 256. *Compute the determinant of* $D = \begin{bmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{bmatrix}$.

Question 257. *Compute the determinant of the following matrices:*

a) $\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 4 \\ -3 & 5 & 0 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix}$

b) $2\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 4 \\ -3 & 5 & 0 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix}$

3.5.2 Properties of Determinants

Question 258. *Prove that if A has a row of zeros, then* $det(A) = 0$*.*

Question 259. *Prove that* $det(Id_n) = 1$*.*

Question 260. *Use your results from the previous subsection to state what the determinants of the three different kinds of elementary matrices are. (You do not need to prove the general case, just give a clear, correct statement.)*

Theorem 261. *a) If A and B are n by n, then* $det(AB) = det(A)det(B)$*.*

b) The determinant of an upper or lower triangular matrix is the product of its diagonal entries.

$$det(L) = \prod_{i=1}^{n} L_{i,i}$$

$$det(U) = \prod_{i=1}^{n} U_{i,i}$$

c) The determinant of a diagonal matrix is the product of its diagonal entries. If D is diagonal, then

$$det(D) = \prod_{i=1}^{n} D_{i,i}$$

d) $det(A) = det(A^T)$

e) A matrix A is invertible iff $det(A) \neq 0$*. This property should be included in the Invertible Matrix Theorem. In fact, you should go write it in as part (n) of Theorem 241.*

Question 262. *Let* $A = E_1E_2E_3$ *be a four by four matrix, where*

- E_1 *is the elementary matrix that adds two times the 2nd column to the 3rd column*
- E_2 *is the elementary matrix that switches the second and fourth rows*
- E_3 *is the elementary matrix that scales the first row by* $\frac{1}{2}$

Compute $det(A)$.

Question 263. *Let $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$. Compute $det(L)$, $det(U)$, and $det(A)$. What relationship should these determinants have?*

Question 264. *Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.*

a) Compute $det(A)$.

b) Let A_1 be the matrix obtained by replacing the first column of A with $\vec{b}$. Compute $det(A_1)$.

c) By similar replacement of the second and third columns, find A_2 and A_3. Compute $det(A_2)$ and $det(A_3)$.

d) Solve $A\vec{x} = \vec{b}$.

e) Write $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in terms of $det(A_1)$, $det(A_2)$, $det(A_3)$, and $det(A)$.

*The previous problem is an example of **Cramer's Rule**, which allows you to write the unique solution of $A\vec{x} = \vec{b}$ (for a square invertible matrix A) in terms of determinants.*

Question 265. *Prove: $det(A) = 0$ iff $A\vec{x} = \vec{0}$ has solutions such that $\vec{x} \neq \vec{0}$.*

3.6 Eigenvalues and Eigenvectors

Definition 266. *An **eigenvector** of a matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. The scalar λ is called an **eigenvalue** of A if there exists a nonzero solution to $A\vec{x} = \lambda\vec{x}$.*

Question 267. *Which of the following vectors are an eigenvector of $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$? For any vectors that are eigenvectors of A, give the eigenvalue.*

a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b) $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c) $\vec{v}_3 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

d) $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

e) $\vec{v}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Question 268. *Let* $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$*. Try to find an eigenvector with eigenvalue* 3*. In other words, find a vector* $\vec{v}$ *such that* $A\vec{v} = 3\vec{v}$*.*

Question 269. *Let* $A = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$*. Try to find an eigenvector with eigenvalue* -3*. In other words, find a vector* $\vec{v}$ *such that* $A\vec{v} = -3\vec{v}$*.*

Question 270. *Prove:* $det(A - \alpha Id) = 0$ *iff* α *is an eigenvalue. Hint: Look at Question 265.*

If A *is a* n *by* n *matrix, then* $det(A - tId)$ *will be a* n*-th degree polynomial in* t*, which we call the* ***characteristic polynomial of*** A*. The previous question shows that finding roots of the characteristic polynomial is the same as finding eigenvalues.*

Question 271. *Find each of the following matrices: write out the characteristic polynomial, give all eigenvalues, and for each eigenvalue, find an eigenvector.*

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

e) $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

A root α of a polynomial (in t) has (algebraic) multiplicity k if k is the largest integer such that $(t-\alpha)^k$ is a factor.

Question 272. *Prove that a nonzero vector, $\vec{v}$, is an eigenvector of A with eigenvalue λ if and only if $\vec{v}$ is in the null space of $A - \lambda Id$.*

Question 273. *Prove that if $\vec{v}$ is an eigenvector of A, then $\alpha\vec{v}$ is also an eigenvector of A (when $\alpha \neq 0$).*

Question 274. *Prove that if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with the same eigenvalue, then $\vec{v}_1 + \vec{v}_2$ is also an eigenvector of A. What is the eigenvalue of $\vec{v}_1 + \vec{v}_2$?*

Definition 275. *If λ is an eigenvalue of A, then the* ***eigenspace of*** λ*, E_λ, is the set of vectors $\vec{x}$ such that $(A - \lambda Id_n)\vec{x} = \vec{0}$. The previous two questions along with the inclusion of $\vec{0}$ give the following theorem.*

Theorem 276. *If λ is an eigenvalue of $A \in M_{n\times n}$, then E_λ is a subspace of $\mathbb{R}^n$.*

Question 277. *Prove that $dim(E_\lambda) \geq 1$ for every eigenvalue λ.*

Question 278. *a) Let $A = \begin{bmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 2 \end{bmatrix}$. Show that A only has an eigenvalue of 2. What is the algebraic multiplicity of the eigenvalue 2?*

b) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 3? If so, give a choice of a, b, and c that does so.

c) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 2? If so, give a choice of a, b, and c that does so.

d) Can you pick a, b, and c, so that the eigenspace of 2 has dimension 1? If so, give a choice of a, b, and c that does so.

3.6.1 Diagonalizability

Definition 279. *A matrix A is **diagonalizable** if there exists an invertible matrix Q such that $A = QDQ^{-1}$ where D is a diagonal matrix.*

Theorem 280. *A matrix $A \in M_{n\times n}$ is diagonalizable iff A has n linearly independent eigenvectors. In fact, the matrix Q that will diagonalize A will have the n linearly independent eigenvectors as its columns.*

*The question becomes when can we find n linearly independent eigenvectors for a matrix A. It turns out that **if you can** find n linearly independent eigenvectors for A, then the matrix Q has columns given by these eigenvectors and the diagonal matrix will have the eigenvalues on the diagonal. In particular, if the i-th column of Q has eigenvalue λ_i, then $D_{i,i} = \lambda_i$.*

Question 281. *Can you diagonalize $A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$? If so, give a basis of eigenvectors, give corresponding choices for Q, Q^{-1}, and D, then use these to demonstrate how $A = QDQ^{-1}$.*

Question 282. *Can you diagonalize $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$? If so, give a basis of eigenvectors, give corresponding choices for Q, Q^{-1}, and D, then use these to demonstrate how $A = QDQ^{-1}$.*

Question 283. *Prove that if $\vec{v}_1$ is an eigenvector with eigenvalue λ_1 and $\vec{v}_2$ is an eigenvector with eigenvalue $\lambda_2 \neq \lambda_1$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.*

The following theorem relies on the preceding question and the fact that the dimension of every eigenspace is at least 1.

Theorem 284. *If a n by n matrix A has n distinct eigenvalues, then A is diagonalizable.*

Question 285. *The converse of this theorem is not true in that there diagonalizable matrices that do not have distinct eigenvalues. Give an example of a matrix that is diagonalizable but does not have distinct eigenvalues. Remember that diagonal matrices are diagonalizable.*

Theorem 286. *A n by n matrix A is diagonalizable iff the sums of the dimensions of its eigenspaces is n.*

Question 287. *Give an example of a matrix that is not diagonalizable. Justify your claim.*

Question 288. *Let A be a 4 by 4 matrix.*

a) *How many eigenvalues can A have?*

b) *For each of the possible number of eigenvalues in the previous part, write out all of the possible dimensions of each of the eigenspaces. For instance: if A has 4 distinct eigenvalues, then the only possibility is that each eigenspace has dimension 1 (why is that?).*

c) *Which of the cases from the previous problem correspond to A being diagonalizable?*

3.6.2 Eigenvalues and Eigenvectors of Linear Transformations

The structure of eigenvalues, eigenvectors, and even diagonalizability can be generalized to linear transformations if we consider a square matrix A as a transformation $\vec{x} \to A\vec{x}$.

Definition 289. *An **eigenvector** of a linear transformation $T : V \to V$ is a nonzero vector $\vec{x} \in V$ such that $T(\vec{x}) = \lambda\vec{x}$ for some scalar λ. The scalar λ is called an **eigenvalue** of T if there exists a nonzero solution to $T(\vec{x}) = \lambda\vec{x}$.*

Question 290. *Examine each of the following transformations of $\mathbb{R}^2$ geometrically and find all eigenvalues and eigenvectors of the transformation. You should not try to construct and use a transformation matrix but rather think about what kinds of vectors will be mapped to a scalar multiple of themselves. Only non-zero vectors that are mapped to a scalar multiple of themselves are eigenvectors.*

a) *T_1 flips points over the horizontal axis.*

b) *T_2 flips points over the line $y = mx$.*

c) *T_3 rotates points by π counterclockwise.*

d) *T_4 rotates points by $\frac{\pi}{3}$ counterclockwise.*

e) *T_5 shears points horizontally by 2. In other words, $T_5(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T_5(\vec{e_2}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.*

f) *T_6 projects points onto the vertical axis.*

Question 291. *What are the eigenvalues and eigenvectors of the transformation $T : \mathbb{P} \to \mathbb{P}$ given by $T(f) = \dfrac{df}{dt}$?*

Question 292. *Let T be the transformation of $\mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Describe geometrically what the linear transformation T does.*

The next question demonstrates why we need to consider complex eigenvalues and eigenvectors even when the martix entries are real numbers.

Question 293. *What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$? You need to consider complex numbers for both the eigenvalues and eigenvectors. Be sure to check your eigenvalues and eigenvectors.*

For the previous problem, we would technically need to work in a complex numbers to do the algebra, but we don't want to dwell on the algebra of complex vector spaces (which is actually not very different.) Instead, we would like to investigate what is happening geometrically when we have complex eigenvalues for matrices with real number entries.

Question 294. *What do you think the scalar multiplication by $2i$ is doing in the previous problem? Think about a geometric answer and consider Question .*

Question 295. *Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates around the origin by $\frac{\pi}{2}$ clockwise and then scales vectors by a factor of 2. Find A, the standard matrix for T and determine if A is diagonalizable.*

Question 296. *Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates around the origin by θ counterclockwise. Find A, the standard matrix for T (in terms of θ).*

Determine for which values of θ the matrix A will be diagonalizable.

Chapter 4

Inner Product Spaces

4.1 Inner Products

Recall the dot product of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ *and* $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ *is the sum of the products of the components. Namely,*

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i = \vec{v}^T \vec{w}$$

The dot product of a vector $\vec{x} \in \mathbb{R}^n$ *with itself gives the length of the vector squared,* $\vec{x} \cdot \vec{x} = ||\vec{x}||^2$. *The dot product is the familiar example of an inner product on a real vector space.*

If $z = a + bi \in \mathbb{C}$, *the conjugate of* z *is denoted* $\bar{z}$ *and computed as* $\bar{z} = a - bi$.

Definition 297. *An **inner product** on a vector space* V *is a function from* $V \times V$ *to* $\mathbb{R}$ *for real vector spaces (*$\mathbb{C}$ *for complex vector spaces), denoted by* $\langle *, * \rangle$, *such that for all* $\vec{x}, \vec{y}, \vec{z} \in V$ *and* $c \in \mathbb{R}$ *(or* $\mathbb{C}$*):*

a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ *(or* $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$ *for* $\mathbb{C}$*)*

b) $c\langle \vec{x}, \vec{y} \rangle = \langle c\vec{x}, \vec{y} \rangle$ *and* $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

c) $\langle \vec{x}, \vec{x} \rangle \geq 0$

*A vector space with a defined inner product is called an **inner product space**.*

Example 298. *a)* $\mathbb{R}^n$ *with the dot product defined above is an inner product space.*

b) $C([0,1])$, *the set of continuous functions on the interval* $[0,1]$, *is an inner product space when*

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\ dt$$

c) *Frobeinus Inner Product on Matrices: If* $A,B \in M_{m\times n}(\mathbb{R})$, *then*

$$\langle A,B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}B_{i,j}$$

is an inner product on $M_{m\times n}(\mathbb{R})$.

Definition 299. *Two non-zero vectors* $\vec{x}$ *and* $\vec{y}$ *in an inner product space are* ***orthogonal*** *if* $\langle \vec{x},\vec{y}\rangle = 0$.

Question 300. *Find 3 different vectors in* $\mathbb{R}^2$ *that are orthogonal to* $\begin{bmatrix}1\\2\end{bmatrix}$.

Question 301. *Find 3 different vectors in* $\mathbb{R}^3$ *that are orthogonal to* $\begin{bmatrix}1\\2\\-1\end{bmatrix}$.

Question 302. *Find a vector in* $C([0,1])$ *that is orthogonal to* $f(t) = t$.

Question 303. *Find a vector in* $C([0,1])$ *that is orthogonal to* $f(t) = 1$.

Question 304. *Find a vector in* $M_{2\times 3}(\mathbb{R})$ *that is orthogonal to* $A = \begin{bmatrix}0 & 2 & 1\\1 & -3 & 0\end{bmatrix}$.

Definition 305. *For vectors in* $\mathbb{R}^n$, *the* ***projection*** *of* $\vec{x}$ *onto* $\vec{y}$ *computed with the following:*

$$proj_{\vec{y}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\right)\vec{y}$$

Question 306. a) *Compute* $proj_{\vec{u}}\vec{v}$ *with* $\vec{u} = \langle 2,2\rangle$ *and* $\vec{v} = \langle 1,3\rangle$.

b) *Plot* $\vec{u}$, $\vec{v}$, *and* $proj_{\vec{u}}\vec{v}$ *starting at the origin.*

c) *Write a few sentences about what the projection measures geometrically.*

Inner product spaces are useful because the same argument we made in the previous problem about how much of one vector is in the direction of another can be generalized to vector spaces that do not have the geometric interpretation of arrows in space.

4.2 Orthogonal Complements

*A set of vectors is **orthogonal** if every pair of distinct vectors in the set is orthogonal.*

Question 307. *Give an orthogonal set of 3 non-zero vectors in $\mathbb{R}^5$.*

Let W be a subspace of an inner product space V. The orthogonal complement of W, denoted $W^\perp$, is the set of vectors in V that are orthogonal to every vector in W.

Question 308. *Let $W = span(\{\begin{bmatrix}1\\1\end{bmatrix}\})$. What is $W^\perp$?*

Question 309. *Let $W = span(\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\})$. What is $W^\perp$?*

Question 310. *Let $W = span(\{f(t) = t\})$ be a subspace of $C([0,1])$. What is $W^\perp$?*

Question 311. *Prove that if W is a subspace of an inner product space V, then $W^\perp$ is a subspace of V.*

4.3 Orthonormal Bases.

The elementary vectors of $\mathbb{R}^n$, $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$, form a basis for $\mathbb{R}^n$. Even better than that, the basis has only unit vectors and is orthogonal as a set (each pair of vectors is orthogonal to each other). These properties are very fundamental to how you worked with vectors before you started this class and why $\mathbb{R}^n$ has such nice geometric intuition built in. The fundamental idea of this section is understanding a procedure for how to **build** *a basis that is an orthogonal set and has vectors of "length" one.*

Question 312. *In this question, you will build an orthonormal basis of $\mathbb{R}^3$ from the ordered set $\beta = \{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\end{bmatrix}\}$. Orthonormal means that the set is orthogonal and contains only unit vectors.*

a) We will construct the orthonormal basis $\gamma = \{\vec{\gamma}_1, \vec{\gamma}_2, \vec{\gamma}_3\}$ by going through the elements in β in order. In other words, we will consider

$\vec{\beta}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ *first. Find* γ_1*, a unit vector in the direction of* $\vec{\beta}_1$*. This will be our first unit basis vector in* γ*.*

b) *We now want to consider* $\vec{\beta}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$*. Is* $\vec{\beta}_2$ *orthogonal to* $\vec{\gamma}_1$*?*

c) *We didn't get lucky, so we will have to take out the part of* $\vec{\beta}_2$ *that is NOT orthogonal to* $\vec{\gamma}_1$*. In other words, we need to find the projection of* $\vec{\beta}_2$ *onto* $\vec{\gamma}_1$*. Compute* $proj_{\vec{\gamma}_1}\vec{\beta}_2$*.*

d) *In order to take out the part of* $\vec{\beta}_2$ *that is NOT orthogonal to* $\vec{\gamma}_1$*, we should subtract* $proj_{\vec{\gamma}_1}\vec{\beta}_2$ *from* $\vec{\beta}_2$*. Find* $\vec{\beta}_2 - proj_{\vec{\gamma}_1}\vec{\beta}_2$ *and verify that this difference IS orthogonal to* $\vec{\gamma}_1$*.*

e) *Since* $\vec{\beta}_2 - proj_{\vec{\gamma}_1}\vec{\beta}_2$ *is orthogonal to* $\vec{\gamma}_1$*, we define* $\vec{\gamma}_2$ *be the unit vector in the direction of* $\vec{\beta}_2 - proj_{\vec{\gamma}_1}\vec{\beta}_2$*. Write out the set* $\{\vec{\gamma}_1, \vec{\gamma}_2\}$*.*

f) *All that's left to do is take* $\vec{\beta}_3$ *and make* $\vec{\gamma}_3$*, a unit vector that is orthogonal to both* $\vec{\gamma}_1$ *and* $\vec{\gamma}_2$*. Find the appropriate projections of* $\vec{\beta}_3$ *in order to subtract out the parts of* $\vec{\beta}_3$ *that is not orthogonal to* $\vec{\gamma}_1$ *and* $\vec{\gamma}_2$*. Then find the unit vector in the direction of the difference to get* $\vec{\gamma}_3$*.*

g) *Verify that* $\gamma = \{\vec{\gamma}_1, \vec{\gamma}_2, \vec{\gamma}_3\}$ *is an orthonormal basis for* $\mathbb{R}^3$*.*

Question 313. *Go through the same process above to create an orthonormal basis of* $\mathbb{P}_2$ *from the basis* $\beta = \{1, t, t^2\}$ *using the inner product and projection formula given by*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\ dt$$

and

$$proj_{\vec{y}}\vec{x} = \left(\frac{\langle \vec{x}, \vec{y}\rangle}{\langle \vec{y}, \vec{y}\rangle}\right)\vec{y}$$