Midterm 2 Suggested Answers

1. A model for traffic treats density of cars u as a conserved variable and models the velocity as $v = 1 - u$. The flux of cars is thus $uv = u(1-u)$ and the hyperbolic conservation law is

$$u_t + \Big[u(1-u)\Big]_x = 0.$$

Now cars can hold one or more people (or perhaps none in the near future). Suppose we would also like to know how people move along the highway so we introduce a new conserved variable ρ If ρ is the density of people per unit length of highway, the flux of people is ρv so we get a conservation equation

$$\rho_t + \Big[\rho(1-u)\Big]_x = 0.$$

(a) As a function of u and ρ, what wave speeds are present in this coupled system of equations?

(b) Given a Riemann problem $U_L = [u_L, \rho_L]$ and $U_R = [u_R, \rho_R]$, what is the entropy condition for a shock to exist between U_L and U_* or between U_* and U_R?

(c) In case the entropy condition above is satisfied, what is the Rankine-Hugoniot condition across the shock?

Answer: (a) The flux Jacobian is

$$f'(u,\rho) = \begin{bmatrix} 1-2u & 0 \\ -\rho & 1-u \end{bmatrix}.$$

This is diagonal so the eigenvalues $\lambda_0 = 1-2u$ and $\lambda_1 = 1-u$ can be read off the diagonal.

(b) A shock in the 0-wave satisfies the entropy inequality when

$$1 - 2u_L = \lambda_0(U_L) \geq s_0 \geq \lambda_0(U_*) = 1 - 2u_*.$$

This simplifies to $u_L \leq u_*$. A shock in the 1-wave is admissible when $1 - u_* = \lambda_1(U_*) \geq s_1 \geq \lambda_1(U_R) = 1 - u_R$ which reduces to $u_* \leq u_R$.

(c) The Rankine-Hugoniot condition across the 0-wave is

$$\begin{aligned} s_0 \begin{bmatrix} u_* - u_L \\ \rho_* - \rho_L \end{bmatrix} &= \begin{bmatrix} u_*(1-u_*) - u_L(1-u_L) \\ \rho_*(1-u_*) - \rho_L(1-u_L) \end{bmatrix} \\ &= \begin{bmatrix} (u_* - u_L)(1 - u_* - u_L) \\ \rho_*(1-u_*) - \rho_L(1-u_L) \end{bmatrix} \end{aligned}$$

so $s_0 = 1 - u_* - u_L$ (from the first equation) when $u_* \neq u_L$. This wave speed satisfies the eigenvalue bounds involving λ_0. Across the 1-wave we have $s_1 = 1 - u_R - u_*$ if $u_* \neq u_R$, but this does not satisfy the eigenvalue bounds involving λ_1. Alternatively, we may observe that the first equation does not contain ρ at all, therefore u only has a discontinuity across the 0-wave and thus $u_* = u_R$. Either way, $u_* = u_R$ so the second equation yields $s_1 = 1 - u_R$. We can find ρ_* from the 0-wave,

$$\begin{aligned} (1 - u_R - u_L)(\rho_* - \rho_L) &= \rho_*(1-u_R) - \rho_L(1-u_L) \\ \rho_* &= \rho_L \frac{u_R}{u_L}. \end{aligned}$$

2. The equilibrium diffusion equation

$$-\nabla \cdot \big(\kappa \nabla u\big) = f(x,y,z) \text{ on } \Omega \subset \mathbb{R}^3 \qquad\qquad u|_{\partial\Omega} = 0$$

is solved in three dimensions using a conservative finite difference method (approximating κ at staggered points).

(a) If κ is independent of u, this equation is linear and can be discretized to yield the matrix equation $Au = b$. How many nonzeros per row are present in the matrix A?

(b) If κ depends on ∇u, as in the p-Laplacian

$$\kappa(\nabla u) = \left(\frac{\epsilon^2}{2} + \frac{\nabla u \cdot \nabla u}{2}\right)^{(p-2)/2},$$

our discrete system will have the form $F(u) = 0$. To compute $F(u)$, we need to compute the full gradient ∇u at staggered points such as $(x - h/2, y, z)$. The aligned component

$$u_x(x - h/2, y, z) \approx \frac{u(x,y,z) - u(x-h,y,z)}{h}$$

is simple, but the transverse components are trickier. In 2D, we might approximate transverse derivatives using a scheme such as

$$u_y(x - h/2, y) \approx \frac{u(x - h/2, y + h) - u(x - h/2, y - h)}{2h}$$

where $u(x - h/2, y + h) \approx \frac{1}{2}\big[u(x - h, y + h) + u(x - h, y)\big]$. For the 3D problem, how many nonzeros per row are present in the Jacobian matrix

$$J = \frac{\partial F}{\partial u}?$$

(c) If κ is independent of u, but discontinuous, what order of convergence can we expect from the discretization above under grid refinement $h \to 0$?

Answer: (a) In 3D, we need fluxes at the six faces of the dual cube $[-h/2, h/2]^3$. Each of these fluxes is computed by directional derivatives of the form

$$\kappa(-h/2, 0, 0)\frac{u(0,0,0) - u(-h,0,0)}{h},$$

each of which contains the center point and one neighbor. Summing over the 6 faces, our stencil depends on a total of 7 grid values of u. Consequently, the matrix has 7 nonzeros per row, except possibly for boundary conditions.

(b) With the transverse derivatives included, our gradients ∇u at staggered points depend on all points

$$\Big\{(x,y,z) : x,y,z \in \{-h, 0, h\} \text{ and } xyz = 0\Big\}$$

where the $xyz = 0$ condition excludes the 8 "corners" from the $3^3 = 27$ possible points, leaving $27 - 8 = 19$ nonzeros in the Jacobian.

(c) If κ is discontinuous, the true gradient ∇u will have a jump at the discontinuity so that $\kappa \nabla u \cdot \hat{n}$ is continuous across the interface with normal $\hat{n}$. The error in our pointwise formulas for gradient can thus be $O(1)$ and after integrating over the surface of an element of size h, $O(h)$.

3. Conservative reconstruction is similar to polynomial interpolation/regression except that instead of pointwise values, we ask for the average value in each cell to match the average that is input. When building a linear function, the average value on a cell is equal to the pointwise value at the centroid.

   (a) Given input data u_{-1}, u_0, u_1 at cells of width 1 centered at $-1, 0, 1$, find a polynomial $a_0 + b_0 x$ such that $\int_{-1/2}^{1/2} a_0 + b_0 x = u_0$ and that minimizes

$$\left(\int_{-3/2}^{-1/2} a_0 + b_0 x - u_{-1} \right)^2 + \left(\int_{1/2}^{3/2} a_0 + b_0 x - u_1 \right)^2 .$$

   (b) Using the reconstruction above, write expressions for the value of this reconstruction at $x = \pm 1/2$ in terms of u_{-1}, u_0, u_1.

   (c) Use a similar reconstruction in the cells centered at -1 and 1 and evaluate at $x = \pm 1/2$ to provide inputs $u^L_{\pm 1/2}, u^R_{\pm 1/2}$ to the Riemann problem at these interfaces. Solve the Riemann problem for the linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

   to provide a numerical flux $\tilde{f}(u^L_{\pm 1/2}, u^R_{\pm 1/2})$ at each interface.

   (d) Using the divergence theorem and numerical fluxes above,

$$\int_{-1/2}^{1/2} \frac{du}{dx} = \tilde{f}(u^L_{1/2}, u^R_{1/2}) - \tilde{f}(u^L_{-1/2}, u^R_{-1/2})$$

   write an expression for the time evolution of the average value $u_0(t)$ as a function of the average values in the neighboring cells,

$$\frac{du_0(t)}{dt} = h(u_{-2}, u_{-1}, u_0, u_1, u_2).$$

   Is this a familiar method?

Answer: (a) The equality constraint for u_0 determines $a_0 = u_0$. Using the property that the integral of a linear function over an interval is the value at the centroid, we seek b_0 that minimizes

$$\left(u_0 + b_0(-1) - u_{-1}\right)^2 + \left(u_0 + b_0(1) - u_1\right)^2 = \Big(b_0 - (u_0 - u_{-1})\Big)^2 + \Big(b_0 - (u_1 - u_0)\Big)^2 .$$

The minimizer is

$$b_0 = \frac{(u_0 - u_{-1}) + (u_1 - u_0)}{2} = \frac{u_1 - u_{-1}}{2} .$$

(b) Evaluating the reconstruction, we have

$$\begin{aligned} u^R_{-1/2} &= \tilde{u}_0(-1/2) = u_0 - \frac{u_1 - u_{-1}}{4} \\ u^L_{1/2} &= \tilde{u}_0(+1/2) = u_0 + \frac{u_1 - u_{-1}}{4} . \end{aligned}$$

(c) The other inputs to the Riemann problems are

$$\begin{aligned} u^L_{-1/2} &= \tilde{u}_{-1}(-1/2) = u_{-1} + \frac{u_0 - u_{-2}}{4} \\ u^R_{1/2} &= \tilde{u}_1(+1/2) = u_1 - \frac{u_2 - u_0}{4} . \end{aligned}$$

The solution to each Riemann problem is

$$\tilde{f}(u_L, u_R) = u_L$$

because the advection is to the right at velocity 1. Consequently,

$$\begin{aligned} \tilde{f}_{-1/2} &= u_{-1} + \frac{u_0 - u_{-2}}{4} \\ \tilde{f}_{1/2} &= u_0 + \frac{u_1 - u_{-1}}{4} . \end{aligned}$$

(d) Differencing the above, we have

$$\begin{aligned}\dot{u}_0 = \tilde{f}_{-1/2} - \tilde{f}_{1/2} &= u_{-1} + \frac{u_0 - u_{-2}}{4} - u_0 - \frac{u_1 - u_{-1}}{4} \\ &= -\frac{1}{4}u_{-2} + \frac{5}{4}u_{-1} - \frac{3}{4}u_0 - \frac{1}{4}u_1.\end{aligned}$$

You probably haven't seen this method before, though it has similar structure to the stencil

$$\texttt{fdstencil(0, [-2, -1, 0, 1])} = \begin{bmatrix} \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

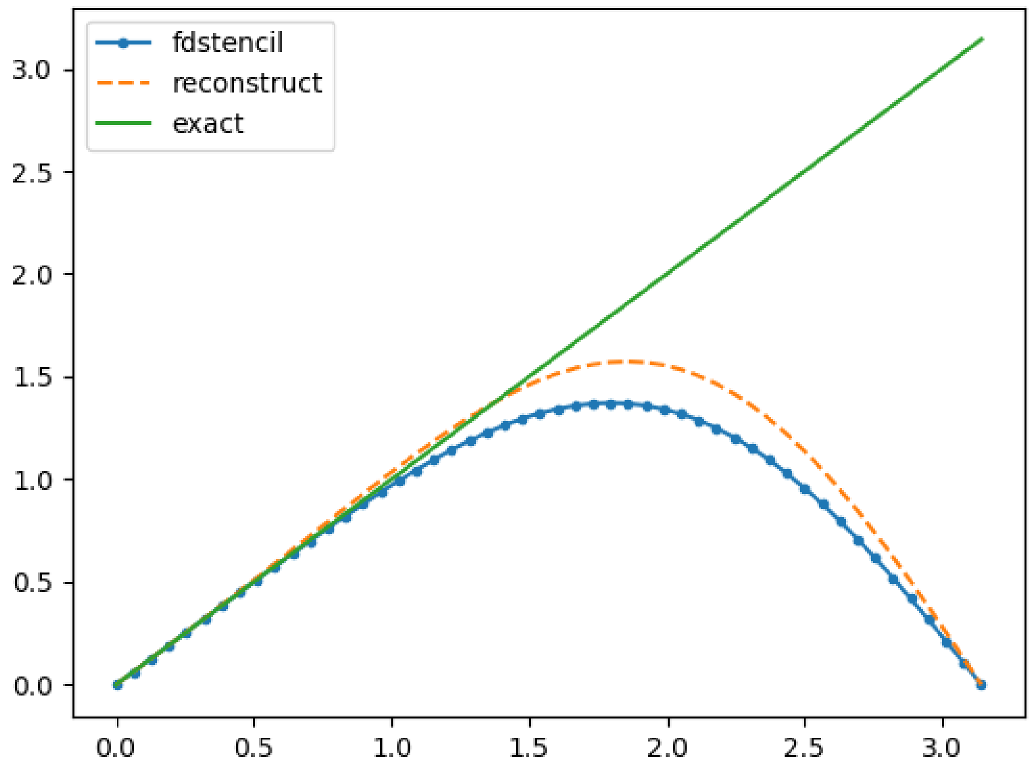

Figure 1: Dispersion diagram for the 4-point baised finite difference stencil compared to the reconstruction stencil we have found here.