Técnicas de Simulación en Computadoras en LaTeX

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Ejercicio 1

En la ecuación de transferencia de calor utilizada en la clase

$$\frac{d}{dx}\left(\frac{d}{dx}T\right) = -Q$$

Modifique toda la formulación de la aplicación del MEF, considerando que Q ya no es constante, sino que corresponde a la siguiente función:

$$Q = Q(x) = x^2 - 3x$$

Solución

Mallado

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -(x^2 - 3x), \qquad k \rightarrow \text{permeabilidad térmica } \textit{constante}$$

$$\text{donde } T = [N_i \;\; N_{i+1}] \begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} \Rightarrow \hat{T} \approx \mathbf{NT}, \qquad \mathbf{N}_{(x)}$$

Discretización

$$\frac{d}{dx}\left(k\frac{d(\mathbf{NT})}{dx}\right) \approx -x^2 + 3x$$

Residuos pasamos de aproximación a igualación

$$\frac{d}{dx}\left(k\frac{d(\mathbf{N})}{dx}\mathbf{T}\right) + x^2 - 3x = \xi, \qquad \xi \rightarrow \text{resto o residuo}$$

Método de los residuos ponderados $\int_\Omega \xi_i w_i d\Omega = 0$

$$\int_\Omega \mathbf{W} \cdot \left[\frac{d}{dx}\left(k\frac{d(\mathbf{N})}{dx}\mathbf{T}\right) + x^2 - 3x\right] d\Omega = 0$$

$$\text{donde } \frac{d(\mathbf{N})}{dx} \Rightarrow \frac{d}{dx}[N_i \;\; N_{i+1}] \Rightarrow \left[\frac{dN_i}{dx} \;\; \frac{dN_{i+1}}{dx}\right] \Rightarrow \left[\left(-\frac{1}{x_{i+1} - x_i}\right)\left(\frac{1}{x_{i+1} - x_i}\right)\right]$$

$$\int_\Omega \left[\left(\begin{bmatrix} W_i \\ W_{i+1} \end{bmatrix} \cdot \frac{d}{dx}\left[-\frac{k}{x_{i+1} - x_i} \;\; \frac{k}{x_{i+1} - x_i}\right]\right)\begin{bmatrix} t_i \\ t_{i+1} \end{bmatrix} + x^2 - 3x\right] d\Omega = 0$$

Método de Galerkin $\Rightarrow W_i = N_i$

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^t \cdot \left[\frac{d}{dx}\left(k\frac{d(\mathbf{N})}{dx}\mathbf{T}\right) + x^2 - 3x\right] dx = 0 \qquad \textbf{Strong form}$$

Utilizando integración por partes $\int u\, dv = uv - \int v\, du$

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^t \cdot \left[\frac{d}{dx}\left(k\frac{d(\mathbf{N})}{dx}\mathbf{T}\right)\right] dx + \int_{x_i}^{x_{i+1}} \mathbf{N}^t(x^2 - 3x)dx = 0$$

$$U = \mathbf{N}^t \qquad dU = \frac{d}{dx}\mathbf{N}^t$$

$$dV = \frac{d}{dx}\left(k\frac{d(\mathbf{N})}{dx}\mathbf{T}\right) \qquad V = k\frac{d(\mathbf{N})}{dx}\mathbf{T}$$

$$\mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T} - \int \frac{d}{dx}\mathbf{N}^t k\frac{d}{dx}\mathbf{N}\mathbf{T}dx + (x^2-3x)\int \mathbf{N}^t dx = 0$$

$$\mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma} - \int_{x_i}^{x_{i+1}} \frac{d}{dx}\mathbf{N}^t k\frac{d}{dx}\mathbf{N}\mathbf{T}dx + (x^2-3x)\int_{x_i}^{x_{i+1}} \mathbf{N}^t dx = 0 \qquad \textbf{Weak form}$$

Retomando la forma débil

$$k\int_{x_i}^{x_{i+1}} \frac{d}{dx}\mathbf{N}^t \frac{d}{dx}\mathbf{N}\mathbf{T}dx = \int_{x_i}^{x_{i+1}} (x^2-3x)\mathbf{N}^t dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

donde $\frac{d}{dx}\mathbf{N}^t \Rightarrow \frac{d}{dx}[N_1 \;\; N_2]^t \Rightarrow \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{x_{i+1}-x_i} \\ \frac{1}{x_{i+1}-x_i} \end{bmatrix} \Rightarrow \frac{1}{x_{i+1}-x_i}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

donde $\frac{d}{dx}\mathbf{N} \Rightarrow \frac{1}{x_{i+1}-x_i}[-1 \;\; 1]$

donde $\frac{d}{dx}\mathbf{N}^t \frac{d}{dx}\mathbf{N} \Rightarrow \frac{1}{(x_{i+1}-x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$k\int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1}-x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{T}dx = \int_{x_i}^{x_{i+1}} (x^2-3x)\mathbf{N}^t dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$k\frac{1}{(x_{i+1}-x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(x_{i+1}-x_i)\mathbf{T}dx = \int_{x_i}^{x_{i+1}} (x^2-3x)\begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix}dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$k\frac{1}{(x_{i+1}-x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}dx = \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_i(x^2-3x) \\ N_{i+1}(x^2-3x) \end{bmatrix}dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$k\frac{1}{(x_{i+1}-x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}dx = \begin{bmatrix} \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{x_{i+1}-x_i}(x^2-3x) \\ \int_{x_i}^{x_{i+1}} \frac{x-x_i}{x_{i+1}-x_i}(x^2-3x) \end{bmatrix}dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$k\frac{1}{(x_{i+1}-x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}dx = \begin{bmatrix} -\frac{(x_i-x_{i+1})(3(x_i^2)+2(x_i)(x_{i+1}-6)+(x_{i+1}-6)(x_{i+1}))}{12} \\ \frac{(x_i-x_{i+1})((x_i^2)+2(x_i)(x_{i+1}-3)+3(x_{i+1}-4)(x_{i+1}))}{12} \end{bmatrix}dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$k\frac{1}{(x_{i+1}-x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}dx =$$

$$\frac{(x_i-x_{i_1})}{12}\begin{bmatrix} -(3(x_i^2)+2(x_i)(x_{i+1}-6)+(x_{i+1}-6)(x_{i+1}) \\ ((x_i^2)+2(x_i)(x_{i+1}-3)+3(x_{i+1}-4)(x_{i+1})) \end{bmatrix}dx + \mathbf{N}^t k\frac{d(\mathbf{N})}{dx}\mathbf{T}\bigg|_{\Gamma}$$

$$\mathbf{kT} = \mathbf{b}$$

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Para cada elemento i

$$\mathbf{k}^e = \begin{bmatrix} k^e_{i1} & k^e_{i2} \\ k^e_{i3} & k^e_{i4} \end{bmatrix}$$

$$\mathbf{b}^e = \begin{bmatrix} b^e_{i1} \\ b^e_{i2} \end{bmatrix}$$

Ensamblaje

Elemento	**i**	**i+1**
1	1	2
2	2	3
3	3	4

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14}+k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24}+k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12}+b_{21} \\ b_{22}+b_{31} \\ b_{32} \end{bmatrix}$$

$$\mathbf{KT} = \mathbf{B}$$

Condiciones de contorno

Condiciones de Dirichlet: $T = T_0$ en Γ_D, $\Gamma_D \subseteq \Gamma$

$$\text{donde } \mathbf{N}^t k \frac{d(\mathbf{N})}{dx} \mathbf{T}\Big|_\Gamma = \begin{bmatrix} 10 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

Condiciones de Neumann: $\frac{dT}{dx} = T_0$ en Γ_N, $\Gamma_N \subseteq \Gamma$

$$\text{donde } \mathbf{N}^t k \frac{d(\mathbf{N})}{dx} \mathbf{T}\Big|_\Gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14}+k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24}+k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} 10 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12}+b_{21} \\ b_{22}+b_{31} \\ b_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$