Técnicas de Simulación en Computadoras en LaTeX

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Ejercicio 1

En la ecuación de transferencia de calor utilizada en la clase

$$\nabla\left(k\nabla T\right)=-Q$$

Modifique toda la formulación de la aplicación del MEF, considerando que Q ya no es constante, sino que corresponde a la siguiente función:

$$Q=Q(x)=2x^2-3x^2$$

Comente en qué cambia con respecto al caso de 1 dimensión.

Solución

Interpolación

$$N_1=1-\varepsilon-\eta$$
$$N_2=\varepsilon$$
$$N_3=\eta$$
$$T=N_1T_1+N_2T_2+N_3T_3=\boldsymbol{NT}$$
$$x_0=\approx(1-\varepsilon-\eta)x_1+\varepsilon x_2+\eta x_3\equiv(x_2+x_1)\epsilon+(x_3-x_1)\eta+x_1$$
$$y_0=\approx(1-\varepsilon-\eta)y_1+\varepsilon y_2+\eta y_3\equiv(y_2+y_1)\epsilon+(y_3-y_1)\eta+y_1$$
$$\boldsymbol{X}=[x_0\ \ y_0]$$
$$\boldsymbol{\varepsilon}=[\varepsilon\ \ \eta]$$
$$\nabla_x=\begin{bmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{bmatrix}$$
$$\nabla_\varepsilon=\begin{bmatrix}\frac{d}{d\varepsilon}\\ \frac{d}{d\eta}\end{bmatrix}$$

Discretización

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=-Q_x$$
$$\frac{d}{dy}\left(k\frac{dT}{dy}\right)=-Q_y$$
$$\nabla_x(k\nabla_x(\boldsymbol{NT}))+2x^2-3x^2\approx0$$

Calculo residual

$$\nabla_x(k\nabla_x(\boldsymbol{NT}))+2x^2-3x^2=\Re$$

Método de los residuos ponderados

$$\int_{\Omega} \xi_i w_i d_{\Omega} = 0$$

Método de Galerkin

$$W_i = N_i$$

$$\int_{\Omega} \boldsymbol{N}^T \left[\nabla_x (k \nabla_x (\boldsymbol{NT})) + 2x^2 - 3x^2\right] d\Omega = 0$$

Forma fuerte

$$\underbrace{k \int_{\Omega} \boldsymbol{N}^T \left[\nabla_x (\nabla_x (\boldsymbol{NT}))\right] d\Omega \boldsymbol{T}}_{\text{Primera parte}} = \underbrace{- \int_{\Omega} (2x^2 - 3x^2 d) \boldsymbol{N}^T d\Omega}_{\text{Segunda parte}}$$

Integración por partes

$$\underbrace{k \int_{\Omega} \boldsymbol{N}^T \left[\nabla_x (\nabla_x (\boldsymbol{NT}))\right] d\Omega \boldsymbol{T}}_{\text{Primera parte}} \Rightarrow \underbrace{\boldsymbol{N}^T \boldsymbol{k} \nabla_x \boldsymbol{NT}|_{\Gamma}}_{\text{Termino natural}} - k \int_{\Omega} \nabla_x \boldsymbol{N}^T \nabla_x \boldsymbol{N} d\Omega \boldsymbol{T}$$

$$\nabla_x \boldsymbol{N} = \frac{dN}{dx} = \left(\frac{dx}{d\varepsilon}\right)^{-1} \frac{dN}{d\varepsilon} = \left[\nabla_\varepsilon \boldsymbol{X}\right]^{-1} \nabla_\varepsilon \boldsymbol{N}$$

$$\left(\frac{dx}{d\varepsilon}\right)^{-1} = \left[\nabla_\varepsilon \boldsymbol{X}\right]^{-1} = \left(\begin{bmatrix} \frac{d}{d\varepsilon} \\ \frac{d}{d\eta} \end{bmatrix} \begin{bmatrix} x_0 & y_0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}^{-1}$$

$$\frac{dN}{d\varepsilon} = \nabla_\varepsilon \boldsymbol{N} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\nabla_x \boldsymbol{N} = \frac{1}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}} \begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

Si por la propiedad $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$ entonces

$$\nabla_x \boldsymbol{N}^T = \frac{1}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$k \int_{\Omega} \nabla_x \boldsymbol{N}^T \nabla_x \boldsymbol{N} d\Omega T =$$

$$\underbrace{\frac{k}{\left(\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}\right)^2}}_{\text{Det}} \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}^T} \underbrace{\begin{bmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}}_{\boldsymbol{A}^T} \underbrace{\begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \int_{\Omega} d\Omega \boldsymbol{T}$$

$$k \int_\Omega \nabla_x \boldsymbol{N}^T \nabla_x \boldsymbol{N} d\Omega T = \frac{k}{\mathsf{Det}^2} \boldsymbol{B}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{B} \int_\Omega d\Omega \boldsymbol{T}$$

Integración

$$\underbrace{-\int_\Omega \left(2x^2 - 3x^2\right) \boldsymbol{N}^T d\Omega}_{\text{Segunda Parte}} = -\iint \left(2x^2 - 3x^2\right) \begin{bmatrix} 1-\varepsilon-\eta \\ \varepsilon \\ \eta \end{bmatrix} dxdy$$

Transformación geométrica

$$dxdy = |\boldsymbol{J}| d\varepsilon d\eta$$

$$\boldsymbol{J} = \left(\nabla_\varepsilon \boldsymbol{X}\right)^T = \boldsymbol{X}^T \nabla_\varepsilon^T = \underbrace{\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}}_{\text{Jacobiano}}$$

$$\underbrace{-\int_\Omega \left(2x^2 - 3x^2\right) \boldsymbol{N}^T d\Omega}_{\text{Segunda Parte}} = -\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} \iint \left(2x_0^2 - 3x_0^2\right) \begin{bmatrix} 1-\varepsilon-\eta \\ \varepsilon \\ \eta \end{bmatrix} d\varepsilon d\eta$$

$$= -\boldsymbol{J} \int_0^1 \int_0^1 \left(2x_0^2 - 3x_0^2\right) \begin{bmatrix} 1-\varepsilon-\eta \\ \varepsilon \\ \eta \end{bmatrix} d\varepsilon d\eta$$

$$= -\boldsymbol{J} \int_0^1 \int_0^1 \left(2\left[(x_2 - x_1)\varepsilon + (x_3 - x_1)\eta + x_1\right]^2 - 3\left[(x_2 - x_1)\varepsilon + (x_3 - x_1)\eta + x_1\right]^2\right) \begin{bmatrix} 1-\varepsilon-\eta \\ \varepsilon \\ \eta \end{bmatrix} d\varepsilon d\eta$$