A Note on the Sum of Non-Identically Distributed Doubly Truncated Normal Distributions

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Abstract

It is proved that the sum of n independent but non-identically distributed doubly truncated Normal distributions converges in distribution to a Normal distribution. It is also shown how the result can be

applied in estimating a constrained mixed effects model.

*Keywords: Truncated Normal Distribution, Lindeberg-Feller Theorem, Lindeberg Condition, Constrained Mixed Effects Model,

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \tag{1}$$

Constrained Mixed Effects Model,

Hotivation

It is our observation that modern statistical models are heavily dependent on Normal distributions. For example, consider a simple linear regression model: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad (1)$ where the error term ϵ_i is assumed to follow $N(0, \sigma^2)$, so it follows that $y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$. Hence, maximum likelihood estimation (MLE) can be applied to estimate the unknown parameters $\beta_0, \beta_1, \sigma^2$. Another example is the linear mixed effects (LME) model (McCulloch and Neuhaus (2014)), say we have a mixed intercept and slope model given below $y_{i,j} = (\beta_0 + \beta_{0,j}) + (\beta_1 + \beta_{1,j})x_{i,j} + \epsilon_{i,j}, \qquad (2)$ where i and j indicate the row number and group number, respectively. It is assumed that $\beta_{0,j} \sim N(0, \eta_0^2), \beta_{1,j} \sim N(0, \eta_1^2), \epsilon_{i,j} \sim N(0, \sigma^2). \qquad (3)$ In other words, only with the Normality assumptions of both the error term and the random effects from we have the result that $y_{i,j}$ is also Normally distributed, upon which nearly all of the modern

$$y_{i,j} = (\beta_0 + \beta_{0,j}) + (\beta_1 + \beta_{1,j})x_{i,j} + \epsilon_{i,j},$$
(2)

$$\beta_{0,j} \sim N(0, \eta_0^2), \beta_{1,j} \sim N(0, \eta_1^2), \epsilon_{i,j} \sim N(0, \sigma^2).$$
 (3)

 ϵ an we have the result that $y_{i,j}$ is also Normally distributed, upon which nearly all of the modern statistical inference methods are built. The underlying reason is simple: suppose $x_i \sim N(\mu_i, \sigma_i^2)$, it is straightforward to show the weighted sum of independent $\alpha_i x_i$ is still Normally distributed with α_i as known constant. Mathematically,

$$\sum_{i=1}^{n} \alpha_i x_i \sim N\left(\sum_{i=1}^{n} \alpha_i \mu_i, \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2\right). \tag{4}$$

The above property is rather neat and elegant. Without such a well-behaved property, the analytical expression of the exact distribution of $y_{i,j}$ will be unavailable under the LME model. In addition, many

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other continuous distributions do not have such a nice behavior. That being said, a Normal distribution is unconstrained. Technically speaking, under the LME model with Normality assumptions, there is no control on the sign of the overall slope $\beta_1 + \beta_{1,j}$ even if researchers have some prior knowledge about its range.

We propose to use the truncated Normal distribution (Horrace (2005)) on the random effects so that the overall slope $\beta_1 + \beta_{1,j}$ will be bounded. However, significant difficulty has been observed by switching from the Normality assumption to the truncated Normality assumption: the sum of truncated Normal distribution is analytically intractable, and there is a lack of study on its large sample property. Hence, in this note we attempt to show that the sum of n independent but non-identically distributed doubly truncated Normal (DTN) distributions converges in distribution to a Normal distribution. Therefore, inference based on the Normality of $y_{i,j}$ can still be applied when n is sufficiently large.

Horrace (2005) studied one-sided truncated Normal (TN) distribution, and the authors presented some analytical results about it. Robert (1995) talked about how to simulate truncated Normal variables. More recently, Cha (2015) discussed more properties about TN in his PhD thesis. The rest of this article is organized as follows. We will give more analytical results about TN and DTN in Section 2. The main results are presented in Section 3, and how the results can be applied to estimate an constrained LME model is discussed in Section 4.

2. Preliminaries

A truncated Normal (TN) distribution is parameterized by 4 parameters: location, μ ; scale, η ; lower bound a; upper bound b. The Normal distribution is a special case of it when $a = -\infty$ and $b = \infty$. The probability density function (PDF) of a $\mathcal{TN}(\mu, \eta^2, [a, b])$, with $\eta > 0$, is given by

$$f_{\mathcal{TN}}(x; \mu, \eta^2, a, b) = \begin{cases} \frac{1}{\eta} \frac{\phi(\xi)}{\Phi(b') - \Phi(a')}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and cumulative distribution function (CDF) of the standard Normal distribution, i.e.,

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\xi^2\right), \text{ and } \Phi(\xi) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right],$$

respectively, and

$$\xi \triangleq \frac{x-\mu}{\eta}, \ a' \triangleq \frac{a-\mu}{\eta}, \ \text{and} \ b' \triangleq \frac{b-\mu}{\eta}.$$

The mean and variance of $x \sim \mathcal{TN}(\mu, \eta^2, [a, b])$ are known and given by Olive (2008):

$$\mathbf{E}[x] = \mu + \frac{\phi(a') - \phi(b')}{\Phi(b') - \Phi(a')} \eta,$$

$$\mathbf{Var}[x] = \eta^{2} \left[1 + \frac{a'\phi(a') - b'\phi(b')}{\Phi(b') - \Phi(a')} - \left(\frac{\phi(a') - \phi(b')}{\Phi(b') - \Phi(a')} \right)^{2} \right].$$

In this paper, we are particularly interested in a special case, namely the symmetric doubly truncated normal (DTN) distributions $\mathcal{TN}(\mu, \eta^2, [\mu - \rho\eta, \mu + \rho\eta])$ with $\rho > 0$, denoted by $\mathcal{DTN}(\mu, \eta^2, \rho)$. It is a special case of a TN with $a = \mu - \rho\eta$, $b = \mu + \rho\eta$, i.e., the lower bound and upper bound are symmetric around mean μ . The properties of a DTN distribution is given by Lemma 1.

Lemma 1. Suppose $x \sim \mathcal{DTN}(\mu, \eta^2, \rho)$, the following results hold

(i). The density function is

$$f_{\mathcal{DTN}}(x; \mu, \eta^2, \rho) = \left\{ \begin{array}{l} \frac{1}{\eta} \frac{\phi(\xi)}{2\Phi(\rho) - 1}, & x \in [\mu - \rho\eta, \mu + \rho\eta] \\ 0, & otherwise \end{array} \right\}$$

(ii). The expectation is

$$\mathbf{E}[x] = \mu,$$

(iii). The variance is

$$\mathbf{Var}[x] = \eta^2 \left[1 - \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} \right],$$

The proof is omitted as it is straightforward to verify the above results. Note that we define DTN distributions with $\rho > 0$. In fact, when $\rho = 0$, it becomes a deterministic value, and hence the variance is 0. This is indeed consistent with the fact that

$$\lim_{\rho \to 0} \left[1 - \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} \right] = 1 - \lim_{\rho \to 0} \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} = 1 - \lim_{\rho \to 0} \frac{2\phi(\rho) + 2\rho\phi'(\rho)}{2\phi(\rho)} = 0$$

where the second equal sign is due to L'H \hat{o} pital's rule. We report some properties regarding the DTN distribution in Lemma 2.

Lemma 2. Let $x \sim \mathcal{DTN}(\mu, \eta^2, \rho)$ with $\rho > 0$, the following results hold.

- (i). $x \mu \sim \mathcal{DTN}(0, \eta^2, \rho)$.
- (ii). For any $x, y \in [\mu \rho \eta, \mu + \rho \eta]$, if $x + y = 2\mu$ then $f_{\mathcal{DTN}}(x; \mu, \eta^2, \rho) = f_{\mathcal{DTN}}(y; \mu, \eta^2, \rho)$.
- (iii). $\operatorname{Var}[x] \leq \eta^2$.
- (iv). Suppose $x' \sim \mathcal{DTN}(\mu, \eta^2, \rho')$, then $\mathbf{Var}[x] \leq \mathbf{Var}[x']$ if $\rho \leq \rho'$.
- (v). If $x \sim \mathcal{DTN}(0, \eta^2, \rho)$. Define $x' = k_0 + k_1 x$, then $x' \sim \mathcal{DTN}(k_0, k_1^2 \eta^2, \rho)$.

Proof. (i) and (ii) are obvious from properties of standard Normal distribution PDF $\phi(\xi)$. To prove (iii), we notice that $2\Phi(\rho) > 1$ for all $\rho > 0$. Since $\phi(\rho) > 0$, we have

$$\frac{2\rho\phi(\rho)}{2\Phi(\rho)-1} > 0, \ \forall \rho > 0.$$

Therefore,

$$\mathbf{Var}[x] = \eta^2 \left[1 - \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} \right] \le \eta^2.$$

To prove (iv), we examine the following function

$$g(\rho) \triangleq \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1}.$$

It is clear this function is differentiable on $(0, +\infty)$. Noticing that the derivate of $\phi(\rho)$, $\phi'(\rho) = -\rho\phi(\rho)$, the derivative of $g(\rho)$ is written as

$$g'(\rho) = \frac{2\phi(\rho) + 2\rho\phi'(\rho)}{2\Phi(\rho) - 1} - \frac{4\rho[\phi(\rho)]^2}{[2\Phi(\rho) - 1]^2}$$

$$= \frac{[2\phi(\rho) + 2\rho\phi'(\rho)][2\Phi(\rho) - 1] - 4\rho[\phi(\rho)]^2}{[2\Phi(\rho) - 1]^2}$$

$$= \frac{2\phi(\rho)\left[(1 - \rho^2)(2\Phi(\rho) - 1) - 2\rho\phi(\rho)\right]}{[2\Phi(\rho) - 1]^2}$$

We further let

$$t(\rho) \triangleq (1 - \rho^2)(2\Phi(\rho) - 1) - 2\rho\phi(\rho).$$

It is clear that $t(\rho)$ is continuous in ρ and t(0) = 0. We also have

$$t'(\rho) = -2\rho(2\Phi(\rho) - 1) < 0,$$

for all $\rho > 0$. Therefore, $t(\rho) < 0$ for all $\rho > 0$. It implies that $g'(\rho) < 0$ for all $\rho > 0$. Therefore, $g(\rho)$ is a monotonically decreasing function on $(0, \infty)$. Hence (iv) holds readily. For (v), it is straightforward to verify

$$f(x') = (1/k_1)f(\frac{x-k_0}{k_1}) \tag{5}$$

$$= (1/k_1) \frac{1}{\eta} \frac{\phi(\frac{x-k_0-0}{k_1\eta})}{2\Phi(\rho)-1} \tag{6}$$

$$= \frac{1}{k_1 \eta} \frac{\phi(\frac{x-k_0}{k_1 \eta})}{2\Phi(\rho) - 1} \tag{7}$$

The last equation is the PDF of $\mathcal{DTN}(k_0, k_1^2 \eta^2, \rho)$

3. Main Results

It is worth pointing out that, while the sum of independent non-identically distributed Normal random variables is Normally distributed, it is not the case for DTNs. The exact distribution of the sum of independent non-identically DTNs is analytically intractable. However, the following Normality results hold.

Theorem 3. Suppose $x_i \sim \mathcal{DTN}(\mu_i, \eta_i^2, \rho_i)$ are independent with $\eta_i < +\infty$, for $i = 1, 2, \dots, n$. Assume also the following conditions hold.

- (i) The ρ_i 's are bounded from above, i.e., there exists a ρ such that $\rho_i \leq \overline{\rho}$ for all $i = 1, 2, \dots, n$.
- (ii) The μ_i 's are bounded from above, i.e., there exists a μ such that $\mu_i \leq \overline{\mu}$ for all $i = 1, 2, \dots, n$.
- (iii) The η_i 's are bounded from above and below, i.e., there exist $\overline{\eta}$ and $\underline{\eta}$ such that $\underline{\eta} \leq \overline{\eta}$ for all $i = 1, 2, \dots, n$.

Then

$$\frac{1}{s_n} \sum_{i=1}^n (x_i - \mu_i) \stackrel{d}{\to} \mathcal{N}(0, 1),$$

as $n \to \infty$, where

$$s_n = \sum_{i=1}^n \mathbf{Var}[x_i].$$

Proof. For the proof, we will use the well known Lindeberg-Feller theorem (Zolotarev, 1967): Suppose that x_1, x_2, \cdots are independent random variables such that $\mathbf{E}[x_n] = \mu_n$ and $\mathbf{Var}[x_n] = \sigma_n^2 < \infty$ for all $n = 1, 2, \cdots$. Define:

$$y_n = x_n - \mu_n,$$

$$t_n = \sum_{i=1}^n y_i,$$

$$s_n^2 = \mathbf{Var}[t_n] = \sum_{i=1}^n \sigma_i^2.$$

If the Lindeberg condition

for every
$$\epsilon > 0$$
, $\frac{1}{s_n^2} \sum_{i=1}^n \mathbf{E}[y_i^2 \cdot \mathbf{1}_{|y_i| \ge \epsilon s_n}] \to 0 \text{ as } n \to \infty$ (8)

is satisfied, then

$$\frac{T_n}{s_n} \stackrel{d}{\to} \mathcal{N}(0,1).$$

For Theorem 3 to hold, it suffices to verify the Lindeberg condition (8). First, by Lemma 2, item (iii), we have

$$\mathbf{Var}[x_i] \leq \eta_i^2 < \infty.$$

We let $y_i = x_i - \mu_i$ for all $i = 1, 2, \dots, n$. Next, since $\rho_i \ge \rho$ for each $i = 1, 2, \dots$, by Lemma 2 item (iv), we have

$$\mathbf{Var}[x_i] \geq \eta_i^2 \left[1 - \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} \right] \geq \underline{\eta}^2 \left[1 - \frac{2\rho\phi(\rho)}{2\Phi(\rho) - 1} \right] \triangleq v.$$

It follows that $s_n \ge nv$ for all $n = 1, 2, \cdots$. By Lemma 2, $y_i \sim \mathcal{DTN}(0, \eta_i, \rho_i)$. Therefore, for any given $\epsilon > 0$ and for each $i = 1, 2, \cdots$, we have

$$\mathbf{E}[y_i^2 \cdot \mathbf{1}_{|y_i| > \epsilon s_n}] = \int_{-\rho_i \eta_i}^{\rho_i \eta_i} y^2 f_{\mathcal{D}\mathcal{T}\mathcal{N}}(y; 0, \eta_i, \rho_i) \cdot \mathbf{1}_{|y| > \epsilon s_n} dy$$

$$= \begin{cases} 0 & \text{if } \epsilon s_n \ge \rho_i \eta_i \\ 2 \int_{\epsilon s_n}^{\rho_i \eta_i} y^2 f_{\mathcal{D}\mathcal{T}\mathcal{N}}(y; 0, \eta_i, \rho_i) dy & \text{otherwise} \end{cases}$$

Notice that, for all $n \geq \frac{\rho \overline{\eta}}{\epsilon v}$ we have

$$\epsilon s_n \geq \epsilon nv \geq \rho \overline{\eta} \geq \rho_i \eta_i, \forall i = 1, 2, \cdots$$

and hence

$$\sum_{i=1}^{n} \mathbf{E}[y_i^2 \cdot \mathbf{1}_{|y_i| > \epsilon s_n}] = 0. \quad \Box$$

Therefore Lindeberg condition (8) holds, and thus Theorem 3 follows readily.

Remark 4. The assumptions that η_i 's and ρ_i 's are bounded from above can be relaxed to the following growth condition:

$$\lim_{i \to \infty} \frac{\rho_i \eta_i}{i} = 0,$$

which allows the sequences of ρ_i 's and η_i 's to be unbounded with controlled growth rate as i goes to infinity.

In addition, it is also straightforward to verify the following corollary to Theorem 3.

Corollary 5. Let $x_i \sim \mathcal{DTN}(\mu_i, \eta_i^2, \rho_i)$, $i = 1, 2, \cdots$ be independent with μ_i 's, η_i 's, and ρ_i 's satisfying conditions in Theorem 3. Let $\beta_i, i = 1, 2, \cdots$, be real numbers bounded from below and above, i.e., there exist $\overline{\beta}$ and $\underline{\beta}$ satisfying $\underline{\beta} \leq \beta_i \leq \overline{\beta}$ for all $i = 1, 2, \cdots$. Then,

$$\frac{1}{s_n} \sum_{i=1}^n \beta_i(x_i - \mu_i) \xrightarrow{d} \mathcal{N}(0, 1),$$

as $n \to \infty$, where

$$s_n = \sum_{i=1}^n \beta_i^2 \mathbf{Var}[x_i].$$

Corollary 5 indicates that the (weighted) sum of finitely many independent but non-identically distributed DTNs converges in distribution to a Normal distribution. The conditions can further be relaxed to the following remark.

Remark 6. We can further relax the condition for Corollary 5. It holds as long as

$$\lim_{i \to \infty} \frac{\rho_i \eta_i \beta_i}{i} = 0.$$

4. Application to Constrained Mixed Effects Model

Suppose there are g groups, indexed by $\ell = 1, \dots, g$, the mixed effects model (McCulloch and Neuhaus (2014)) is given by

$$\mathbf{y}^{\ell} = X^{\ell} \boldsymbol{\beta} + Z^{\ell} \boldsymbol{\gamma}^{\ell} + \boldsymbol{\varepsilon}^{\ell}, \tag{9}$$

where

$$\boldsymbol{\varepsilon}^{\ell} \sim \mathcal{N}(\mathbf{0}_{m_{\ell}}, \sigma^{2} \mathbf{I}_{m_{\ell}}) \tag{10}$$

and m_{ℓ} is the sample size for group ℓ , the total size is $m = \sum_{\ell=1}^{g} m_{\ell}$, $\mathbf{0}_{m_{\ell}}$ is a size m_{ℓ} column vector with 0 as all of its elements. $\mathbf{I}_{m_{\ell}}$ is a identity matrix with size m_{ℓ} . For the random effects $\gamma_{\ell,i}$, we assume they are independent and follow the distribution

$$\gamma_{\ell,i} \sim \mathcal{DTN}(0, \varsigma_i^2, [-\beta_i, \beta_i]), i = 1, \dots, p,$$
(11)

where $\beta_i > 0$, for each $i = 1, \dots, p$. p is the number of columns for which the random effects are considered. Each $\gamma_{\ell,i}$ is mathematically constrained within its corresponding $[-\beta_i, \beta_i]$. Hence, the overall coefficient of group ℓ and column i calculated as $\beta_i + \gamma_{\ell,i} \geq 0$. This way, we can guarantee that the overall coefficient will be non-negative. One can follow a similar procedure if a non-positive sign is needed. Following the results in Section 3, we have

$$\mathbf{E}[y^{\ell}|X^{\ell},\boldsymbol{\beta}] = X^{\ell}\boldsymbol{\beta},$$

$$\mathbf{Var}[y^{\ell}|X^{\ell},Z^{\ell},\boldsymbol{\beta}] = Z^{\ell}\Lambda(Z^{\ell})^{T} + \sigma^{2}\mathbf{I}_{m_{\ell}},$$

where

$$\Lambda = \operatorname{diag}\left[\left(\varsigma_i^2 \left[1 - \frac{2\rho_i \phi(\rho_i)}{2\Phi(\rho_i) - 1}\right]\right)_{i=1}^p\right].$$

Therefore, we have

$$y^{\ell} \xrightarrow{d} \mathcal{N}(X^{\ell}\boldsymbol{\beta}, Z^{\ell}\Lambda(Z^{\ell})^{T} + \sigma^{2}\mathbf{I}_{m_{\ell}}),$$

and MLE can be used for parameter estimation.

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