BOUNDED ESTIMATION OF ARBITRARY TIME-SCALE QUASI-STATIONARY TIME SEGMENTS UNDER CONSISTENT CONFIDENCE

A quasi-stationary time segment in mean and variance is detected through the continuous speculation of a quasi-stationary decision bit. The **MSE EQUIVALENCY THEOREM**, next, provides the basis for the application of rigorous error control over the sequential speculation under consistent confidence levels. Its application enforces bounded error and confidence over the presence of quasi-stationary conditions along any segment that repeatedly complies with the computed bound.

To detect quasi-stationary conditions, the original signal is split into two versions, a ***tao***-delayed version and a non-delayed version. This principle is known as **super-heterodyning** and was introduced by Armstrong in 1917 for significantly enhancing discrimination power over a noisy signal. Then, each of these signals is smoothed using a moving average window whose size complies with Central Limit Theorem (CLT) convergence requirements. The resulted signals are used to track **recent past** and **present** outlooks of the original input signal.

During both true and quasi stationary conditions, the first two moments of both these tracking signals exhibit constancy. Moreover, because both tracking signals are obtained from maximal likelihood estimators, the residuals of these signals when computed with respect to any other maximal likelihood estimator version of the original input signal (in particular, a smoothed version of the original input signal behave like uncorrelated noise processes (for true stationary conditions) and (approximately uncorrelated noise processes) noise residuals.

Given this insight, during quasi-stationary conditions we attempt to continuously fit and verify for the presence or departure from such model (that is, an approximately uncorrelated noise processes in the residuals computed with respect to the recent-past and present). The theorem provides the basis to simultaneously compute a bound to verify compliance to this model by **both** tracking signals (i.e., smoothed recent past and present outlooks over the input signal) as well as departure of this model (by either one of these smoothed outlooks) under specifiable confidence levels.

Effectively, if the quasi-stationary model is a fit to a speculated interval, the behavior of the error correlation during such would somewhat resemble the behavior expected of (true)-stationary conditions. As a result, error correlation becomes **relatively** small. On the other hand, if the behavior of these signals in the current interval is not stationary (e.g., a trend exist or a large errors induced by a piece-wise constant model), the accumulation of this quadratic error correlation becomes non-negligible.

The theorem is used

1) to construct a consistent confidence error bound associated with the accumulative behavior of the repeated speculation of quasi-stationary conditions (computed in terms of μ_i, σ_i for both tracking outlooks) in terms of the behavior of an amortized quadratic form of the resultant error correlation of the residuals derived from **both** tracking signal with respect to the quasi-stationary forecast; and
2) to detect via an **quadratic error form** any departures from this model (i.e., the stable presence of quasi-stationary conditions along a random-duration interval monitored in terms of the behavior of its amortized residuals across some finite outlook into such).

Let α be a confidence level.
Let ‹$y(i)$› be an arbitrary signal.
Let ‹$y(i-\tau)$› be its ‹τ›-delayed version;
Let ‹$f(i-\tau)$› be a CLT-smoothed version of the ‹τ›-delayed version of ‹$y(i)$›. In particular, let

‹$\mu(‹y(i)›)|(m')$› be the m-degree smoothed version of ‹$y(i)$› and
‹$\mu(‹y(i-\tau)›)|(m)$› be the m-degree smoothed version of ‹$y(i)$›.

Let ‹v'› be an arbitrary finite interval of size m'.

Then, at α confidence level,

- the maximum error permissible ‹$MSE_{max}(i)|m'$›
- along any finite interval ‹v'› of the original signal ‹$f(i)$›
- if **approximate τ-invariance** exists across interval ‹v'› of signal ‹$f(i)$›
- is bounded by

$$\left\langle MSE\left(i\middle|\left(m,m'\right)\right)\right\rangle \leq \sqrt{m'\times t_{\max}^2+2\times\left\langle\hat{\mu}\left[\left\langle\zeta\left(i\middle|\left(m,m'\right)\right)\right\rangle\middle|m\right]\right\rangle^2}\times\frac{\left\langle\hat{\mu}\left[\left\langle\hat{\sigma}_D\left(i\right)\middle|\left(m,m'\right)\right\rangle\middle|m'\right]\right\rangle}{m'} \tag{5.8}$$

Where:

- ‹$\zeta(i)|(m,m')$› represents an error correlation given by

$$\left\langle\zeta\left(i\middle|\left(m,m'\right)\right)\right\rangle=\frac{\left\langle\hat{\varepsilon}_{fast}(i)\right\rangle\times\left\langle\hat{\varepsilon}_{slow}(i)\right\rangle}{\left\langle\hat{\sigma}_D\left(i\right)\middle|\left(m,m'\right)\right\rangle^2}, \tag{5.9}$$

- ‹$\mu[‹\sigma D(i)|(m,m')›]$› represents the average pooled standard deviation across the interval ‹v'›, and
- $t_{max} \equiv t(m+m'-2, \alpha/2)$ (that is, a paired t-test confidence for unknown means, variance for unequal populations)

Sketched Proof to the MSE Equivalency Theorem

1. Taking the smoothed **recent past** and **present** outlooks into the input signal as sampled populations, a simple but relevant paired t-test for comparing two unequal size populations with unknown mean, unknown and possibly unequal variance is given as ‹$t^*(i)|(m,m')$›$\leq t_{max}$ being defined as a difference between sampled means divided by their estimated pooled standard deviation juxtaposed against the area under the t-distribution with degrees $(m+m'-2, \alpha/2)$. That is,

$$\frac{\left\langle \Delta\hat{\mu}(i)|(m,m')\right\rangle}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle} \leq t_{\max}$$

.

2. For our smoothed super-heterodyned signal set, the test is becomes,

$$\frac{\left\langle \hat{\mu}\left[\left\langle y(i-\tau)\right\rangle|m\right]\right\rangle-\left\langle \hat{\mu}\left[\left\langle y(i)\right\rangle|m'\right]\right\rangle}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle} \leq t_{\max}$$

,

3. Let's refer to the generated quasi-stationary forecast as ***‹mon*(i)› (*** generated by *any* selection function ***g(‹μ(‹y(i-τ)›)|(m)›, ‹μ(‹y(i)›)|(m')›))***. By definition, the residuals of the forecast when computed with respect to the smoothed tracking signals are given by:

- ***‹έslow(i)› =‹mon*(i)›-‹μ[‹y(i-τ)›|m]›***,
- ***‹έfast(i)›=‹mon*(i)›-‹μ[‹y(i)›|m']›,*** respectively.

Now, restating **(2)** in terms of these residuals is equivalent to:

$$\frac{\left(\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle-\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle\right)}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle} \leq t_{\max}$$

.

4. Squaring both sides of **(3)** results in

$$\frac{\left(\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle-\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle\right)^2}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle^2} \leq t_{\max}^2$$

.

5. In turn, **(4)** is equivalent to

$$\frac{\left(\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle^2+\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle^2-2\times\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle\right)}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle^2} \leq t_{\max}^2$$

.

6. By collecting terms in **(5)**,

$$\left(\frac{\left(\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle^2+\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle^2\right)}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle^2}-2\frac{\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle\times\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle^2}\right) \leq t_{\max}^2$$

.

$$\left\langle \zeta\left(i \middle| \left(m,m' \right) \right) \right\rangle = \frac{\left\langle \hat{\varepsilon}_{fast}(i) \right\rangle \times \left\langle \hat{\varepsilon}_{slow}(i) \right\rangle}{\left\langle \hat{\sigma}_D\left(i \right) \middle| \left(m,m' \right) \right\rangle^2}$$

7. By letting ; **(6)** can be rewritten as

$$\frac{\left(\left\langle \hat{\varepsilon}_{fast}(i) \right\rangle^2 + \left\langle \hat{\varepsilon}_{slow}(i) \right\rangle^2 \right)}{\left\langle \hat{\sigma}_D\left(i \right) \middle| \left(m,m' \right) \right\rangle^2} \leq t_{\max}^2 + 2 \times \left\langle \zeta\left(i \middle| \left(m,m' \right) \right) \right\rangle$$

.

8. Next we sum (6) over some finite interval of arbitrary. An amortization of **(6)** is done over an interval, generating two cases, negligible error correlation and significant error correlation, for the former, if quasi-stationary conditions remain across the currently speculated interval this inequality must be satisfied along the entire quasi-stationary interval:

$$\sum_i \frac{\left\langle \hat{\varepsilon}_{fast}(i) \right\rangle^2 + \left\langle \hat{\varepsilon}_{slow}(i) \right\rangle^2}{\left\langle \hat{\sigma}_D\left(i \right) \middle| \left(m,m' \right) \right\rangle^2} \leq \sum_i t_{\max}^2 + 2 \times \sum_i \left\langle \zeta\left(i \middle| \left(m,m' \right) \right) \right\rangle$$

.

9. Moreover, if the error correlation is negligible along the entire duration of a quasi-stationary interval, the behavior of the error residuals as noise of the currently speculated quasi-stationary interval should also behave as approximately noise along any subinterval of size ***m'***, where ***m'*** controls the amortization effort over the accumulated quadratic error. Therefore, constraining our view to the most recent **windowed outlook** of size ***m'*** over **(8)**, we get that:

$$\sum_{i=i'-m'}^{i'} \frac{\left(\left\langle \hat{\varepsilon}_{fast}(i) \right\rangle^2 + \left\langle \hat{\varepsilon}_{slow}(i) \right\rangle^2 \right)}{\left\langle \hat{\sigma}_D\left(i \right) \middle| \left(m,m' \right) \right\rangle^2} £ \sum_{i=i'-m'}^{i'} t_{\max}^2 + 2 \sum_{i=i'-m'}^{i'} \left\langle \zeta\left(i \middle| \left(m,m' \right) \right) \right\rangle$$

.

10. The summations on the left hand size can be simplified by recognizing that (A) the pooled variance described by the denominator need be approximately constant along the interval for quasi-stationary conditions to be upheld, and (B) that the pooled variance along the windowed outlook can be estimated by the average pooled variance during the windowed outlook. That is, there exist a finite bound ε such that:

$$E\left[\hat{\mu}\left\langle \left\langle \sigma_D\left(i \middle| \left(m, m' \right) \right) \right\rangle \right\rangle - \left\langle \left\langle \sigma_D\left(i \middle| \left(m, m' \right) \right) \right\rangle \right\rangle \right] \leq \varepsilon$$

11. Using the mean pooled variance of (**10**) as a constant denominator for the left hand side in (**9**), allows the numerator in the left hand side of (**9**) to be simplified as the summation of the quadratic residuals over the windowed outlook of size ***m'*** over the currently speculated quasi-stationary segment. That is

$$\sum_{i=i'-m'}^{i'} \frac{\left(\langle \varepsilon_{fast}(i)\rangle^2 + \langle \varepsilon_{slow}(i)\rangle^2\right)}{\left\langle \langle \sigma_D(i|(m,m'))\rangle^2 \right\rangle} \to \frac{1}{\left\langle \left[\hat{\mu}\langle \sigma_D(i|(m,m'))\rangle\right]^2 \right\rangle} \cdot \sum_{i=i'-m'}^{i'} \left(\langle \varepsilon_{fast}(i)\rangle^2 + \langle \varepsilon_{slow}(i)\rangle^2\right)$$

12. It follows that for the currently speculated interval to be a quasi-stationary segment, then **(9)** must reduce into **(11)** paving the way into the following relationship between three quadratic terms (sums of quadratic residuals, average pooled deviation, and maximal confidence), and the aforementioned error correlation:

$$\frac{\sum_{i=i'-m'}^{i'} \left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\left\langle \hat{\mu}\left[\langle \hat{\sigma}_D(i)|(m,m')\rangle|m'\right]\right\rangle^2} \le \sum_{i=i'-m'}^{i'} t_{\max}^2 + 2\sum_{i=i'-m'}^{i'} \langle \zeta(i|(m,m'))\rangle$$

.

13. Taking roots on both sides of **(12)** we get that:

$$\sqrt{\frac{\sum_{i=i'-m'}^{i'} \left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\left\langle \hat{\mu}\left[\langle \hat{\sigma}_D(i)|(m,m')\rangle|m'\right]\right\rangle^2}} \le \sqrt{\sum_{i=i'-m'}^{i'} t_{\max}^2 + 2\sum_{i=i'-m'}^{i'} \langle \zeta(i|(m,m'))\rangle}$$

14. Recognizing the quadratic error form in (13) as a mean squared error form (but rather defined now in terms of two signals as opposed to one (that is, the smoothed signal and its **‹τ›**-delayed smoothed counter-part), lets to the following definition:

$$\langle MSE(i|(m,m'))\rangle = \sum_{i=i'-m'}^{i'} \left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right) \Big/ m'$$

.

15. Substituting **(14)** into **(13)**, it follows that:

$$\frac{m' \times \langle MSE(i|(m,m'))\rangle}{\left\langle \hat{\mu}\left[\langle \hat{\sigma}_D(i)|(m,m')\rangle|m'\right]\right\rangle} \le \sqrt{m' \times t_{\max}^2 + 2 \times \left\langle \hat{\mu}\left[\langle \zeta(i|(m,m'))\rangle|m'\right]\right\rangle^2}$$

16. Therefore,

$$\left\langle MSE\left(i \middle| \left(m,m'\right)\right)\right\rangle \leq \sqrt{m' \times t_{\max}^{2} + 2\left\langle \hat{\mu}\left[\left\langle \zeta\left(i \middle| \left(m,m'\right)\right)\right\rangle \middle| m'\right]\right\rangle^{2}} \times \frac{\left\langle \hat{\mu}\left[\left\langle \hat{\sigma}_{D}(i) | (m,m')\right\rangle \middle| m'\right]\right\rangle}{m'}$$

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