The **MSE EQUIVALENCY THEOREM**, next, provides the basis for the application of rigorous error control over **HPS interval coalescence** (of the continuous speculation of the quasi-stationary decision bit) in order to detect the presence of quasi-stationary conditions along some arbitrary length interval.

During stationary conditions, the residuals of the smoothed super-heterodyned signals (with respect to the generated forecast) behave like uncorrelated noise processes. During quasi-stationary conditions we attempt to continuously fit and verify for the presence or departure from such model. If the quasi-stationary model is a fit to a speculated interval, the behavior of the error correlation during such would somewhat resemble the behavior expected of (true)-stationary conditions. As a result, error correlation becomes **relatively** small. On the other hand, if the behavior of these signals in the current interval is not stationary (e.g., a trend exist or a large errors induced by a piece-wise constant model), the accumulation of this quadratic error correlation becomes non-negligible.

The theorem is used

1) to construct a consistent confidence error bound associated with the accumulative behavior of quasi-stationary conditions (continuously speculated in terms of μ_i, σ_i for the corresponding outlooks) in terms of the behavior of an amortized quadratic form of the resultant error correlation of the two tracking signals used to speculate the presence of quasi-stationary conditions;
2) and to detect via an quadratic error form any departures from this model (i.e., the presence of quasi-stationary conditions along an arbitrary length monitored via the predicted behavior of accumulation of residual errors during the interval).

Let $\langle y(i) \rangle$ be an arbitrary signal;
Let $\langle y(i-\tau) \rangle$ be its $\langle \tau \rangle$-delayed version;
Let $\langle f(i-\tau) \rangle$ be a CLT-smoothed version of the $\langle \tau \rangle$-delayed version of $\langle y(i) \rangle$;
Let α be a confidence level.
Let $\langle \vec{v}' \rangle$ be an arbitrary finite interval of size m'. Then, at an α confidence level, the maximum error permissible $\langle MSE_{max}(i)/m' \rangle$ along an interval $\langle \vec{v}' \rangle$ of signal $\langle f(i) \rangle$ if **approximate τ-invariance** exists across interval $\langle \vec{v}' \rangle$ of signal $\langle f(i) \rangle$ is bounded by

$$\left\langle MSE\left(i|(m,m')\right)\right\rangle \le \sqrt{m'\cdot t_{\max}^{2}+2\cdot\left\langle \hat{\mu}\left[\left\langle \zeta\left(i|(m,m')\right)\right\rangle|m'\right]\right\rangle^{2}} \cdot \frac{\left\langle \hat{\mu}\left[\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle|m'\right]\right\rangle}{m'} \qquad \textbf{(5.8)}$$

where $\langle \zeta(i)/(m,m') \rangle$ represents an error correlation given by

$$\left\langle \zeta\left(i|(m,m')\right)\right\rangle=\frac{\left\langle \hat{\varepsilon}_{fast}(i)\right\rangle\cdot\left\langle \hat{\varepsilon}_{slow}(i)\right\rangle}{\left\langle \hat{\sigma}_D(i)|(m,m')\right\rangle^{2}}, \qquad \textbf{(5.9)}$$

$\langle \mu[\langle \sigma_D(i)/(m,m') \rangle] \rangle$ represents the average of the pooled standard deviations across the interval $\langle \vec{v}' \rangle$, and where $t_{max} \equiv t(m+m'-2, \alpha/2)$.

SKETCHED PROOF TO THE MSE EQUIVALENCY THEOREM

1. The t-test $\langle t^*(i)/(m,m')\rangle \le t_{max}$ is defined as

$$\frac{\langle \Delta\hat{\mu}(i)|(m,m')\rangle}{\langle \hat{\sigma}_D(i)|(m,m')\rangle} \le t_{\max} .$$

2. Which for our super-heterodyned signals becomes, $\dfrac{\langle \hat{\mu}[\langle y(i\text{-}\tau)\rangle|m]\rangle\text{-}\langle \hat{\mu}[\langle y(i)\rangle|m']\rangle}{\langle \hat{\sigma}_D(i)|(m,m')\rangle} \le t_{\max}$,

3. ***Restating in terms of the residuals of the signals with respect to the forecast ‹mon*(i)› signal,*** **(2)** is also equivalent to

$$\frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle\text{-}\langle \hat{\varepsilon}_{slow}(i)\rangle\right)}{\langle \hat{\sigma}_D(i)|(m,m')\rangle} \le t_{\max}$$, where $\langle \dot{\varepsilon}_{slow}(i)\rangle = \langle mon^*(i)\rangle\text{-}\langle \mu[\langle y(i\text{-}\tau)\rangle/m]\rangle$,

$\langle \dot{\varepsilon}_{fast}(i)\rangle = \langle mon^*(i)\rangle\text{-}\langle \mu[\langle y(i)\rangle/m']\rangle$.

4. Squaring both sides of **(3)** results in

$$\frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle\text{-}\langle \hat{\varepsilon}_{slow}(i)\rangle\right)^2}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2} \le t_{\max}^2 .$$

5. In turn, **(4)** is equivalent to

$$\frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2+\langle \hat{\varepsilon}_{slow}(i)\rangle^2\text{-}2\cdot\langle \hat{\varepsilon}_{fast}(i)\rangle\langle \hat{\varepsilon}_{slow}(i)\rangle\right)}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2} \le t_{\max}^2 .$$

6. By collecting terms in **(5)**,

$$\left(\frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2+\langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2}\text{-}2\frac{\langle \hat{\varepsilon}_{fast}(i)\rangle\cdot\langle \hat{\varepsilon}_{slow}(i)\rangle}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2}\right) \le t_{\max}^2 .$$

7. By letting $\langle \zeta(i|(m,m'))\rangle = \dfrac{\langle \hat{\varepsilon}_{fast}(i)\rangle\cdot\langle \hat{\varepsilon}_{slow}(i)\rangle}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2}$; **(6)** can be rewritten as

$$\frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2+\langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2} \le t_{\max}^2+2\cdot\langle \zeta(i|(m,m'))\rangle .$$

8. Next we sum (6) over some finite interval of arbitrary. An amortization of **(6)** is done over an interval, generating two cases, negligible error correlation and significant error correlation, for the former, if quasi-stationary conditions remain across the currently speculated interval this inequality must be satisfied along the entire quasi-stationary interval:

$$\sum_i \frac{\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2} \leq \sum_i t_{\max}^2 + 2 \cdot \sum_i \langle \zeta(i|(m,m'))\rangle \; .$$

9. Moreover, if the error correlation is negligible along the entire duration of a quasi-stationary interval, the behavior of the error residuals as noise of the currently speculated quasi-stationary interval should also behave as approximately noise along any subinterval of size ***m'***, where ***m'*** controls the amortization effort over the accumulated quadratic error. Therefore, constraining our view to the most recent **windowed outlook** of size ***m'*** over **(8)**, we get that:

$$\sum_{i=i'-m'}^{i'} \frac{\left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\langle \hat{\sigma}_D(i)|(m,m')\rangle^2} £ \sum_{i=i'-m'}^{i'} t_{\max}^2 + 2\sum_{i=i'-m'}^{i'} \langle \zeta(i|(m,m'))\rangle \; .$$

10. The summations on the left hand size can be simplified by recognizing that (A) the pooled variance described by the denominator need be approximately constant along the interval for quasi-stationary conditions to be upheld, and (B) that the pooled variance along the windowed outlook can be estimated by the average pooled variance during the windowed outlook. That is,

$$E\left[\hat{\mu}\langle\langle \sigma_D(i|(m,m'))\rangle\rangle - \langle\langle \sigma_D(i|(m,m'))\rangle\rangle\right] \to 0$$

11. Using the mean pooled variance of (**10**) as a constant denominator for the left hand side in (**9**), allows the numerator in the left hand side of (**9**) to be simplified as the summation of the quadratic residuals over the windowed outlook of size ***m'*** over the currently speculated quasi-stationary segment. That is

$$\sum_{i=i'-m'}^{i'} \frac{\left(\langle \varepsilon_{fast}(i)\rangle^2 + \langle \varepsilon_{slow}(i)\rangle^2\right)}{\left\langle \langle \sigma_D(i|(m,m'))\rangle^2 \right\rangle} \to \frac{1}{\left\langle \left[\hat{\mu}\langle \sigma_D(i|(m,m'))\rangle\right]^2 \right\rangle} \cdot \sum_{i=i'-m'}^{i'} \left(\langle \varepsilon_{fast}(i)\rangle^2 + \langle \varepsilon_{slow}(i)\rangle^2\right)$$

12. It follows that for the currently speculated interval to be a quasi-stationary segment, then **(9)** must reduce into **(11)** paving the way into the following relationship between three quadratic terms (sums of quadratic residuals, average pooled deviation, and maximal confidence), and the aforementioned error correlation:

$$\frac{\sum_{i=i'-m'}^{i'} \left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\langle \hat{\mu}[\langle \hat{\sigma}_D(i)|(m,m')\rangle|m']\rangle^2} \leq \sum_{i=i'-m'}^{i'} t_{\max}^2 + 2\sum_{i=i'-m'}^{i'} \langle \zeta(i|(m,m'))\rangle \; .$$

13. Taking roots on both sides of **(12)** we get that:

$$\sqrt{\frac{\sum_{i=i'-m'}^{i'} \left(\langle \hat{\varepsilon}_{fast}(i)\rangle^2 + \langle \hat{\varepsilon}_{slow}(i)\rangle^2\right)}{\langle \hat{\mu}[\langle \hat{\sigma}_D(i)|(m,m')\rangle|m']\rangle^2}} \leq \sqrt{\sum_{i=i'-m'}^{i'} t_{\max}^2 + 2\sum_{i=i'-m'}^{i'} \langle \zeta(i|(m,m'))\rangle}$$

14. Recognizing the quadratic error form in (13) as a mean squared error form (but rather defined now in terms of two signals as opposed to one (that is, the smoothed signal and its ‹*τ*›-delayed smoothed counter-part), lets to the following definition:

$$\left\langle MSE\left(i\middle|\left(m,m'\right)\right)\right\rangle = \sum_{i=i'-m'}^{i'}\left(\left\langle\hat{\varepsilon}_{fast}(i)\right\rangle^{2}+\left\langle\hat{\varepsilon}_{slow}(i)\right\rangle^{2}\right)\Big/m' .$$

15. Substituting **(14)** into **(13)**, it follows that:

$$\frac{m'\cdot\left\langle MSE\left(i\middle|\left(m,m'\right)\right)\right\rangle}{\left\langle\hat{\mu}\left[\left\langle\hat{\sigma}_{D}(i)\middle|(m,m')\right\rangle\middle|m'\right]\right\rangle} \le \sqrt{m'\cdot t_{\max}^{2}+2\cdot\left\langle\hat{\mu}\left[\left\langle\zeta\left(i\middle|\left(m,m'\right)\right)\right\rangle\middle|m'\right]\right\rangle^{2}}$$

16. Therefore,

$$\left\langle MSE\left(i\middle|\left(m,m'\right)\right)\right\rangle \le \sqrt{m'\cdot t_{\max}^{2}+2\left\langle\hat{\mu}\left[\left\langle\zeta\left(i\middle|\left(m,m'\right)\right)\right\rangle\middle|m'\right]\right\rangle^{2}}\cdot\frac{\left\langle\hat{\mu}\left[\left\langle\hat{\sigma}_{D}(i)\middle|(m,m')\right\rangle\middle|m'\right]\right\rangle}{m'} \blacksquare$$