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15.455x Mathematical Methods of Quantitative Finance

Week 6: Continuous-Time Finance (continued)

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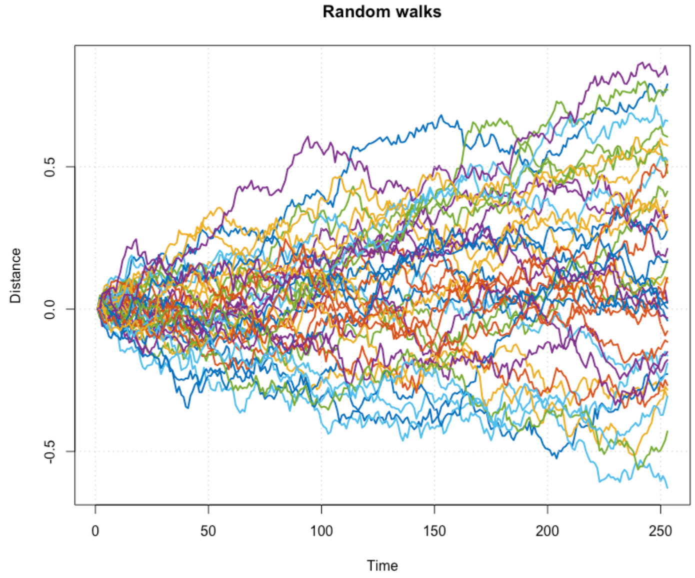
Probability density for random walks



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Probabilities for random walks

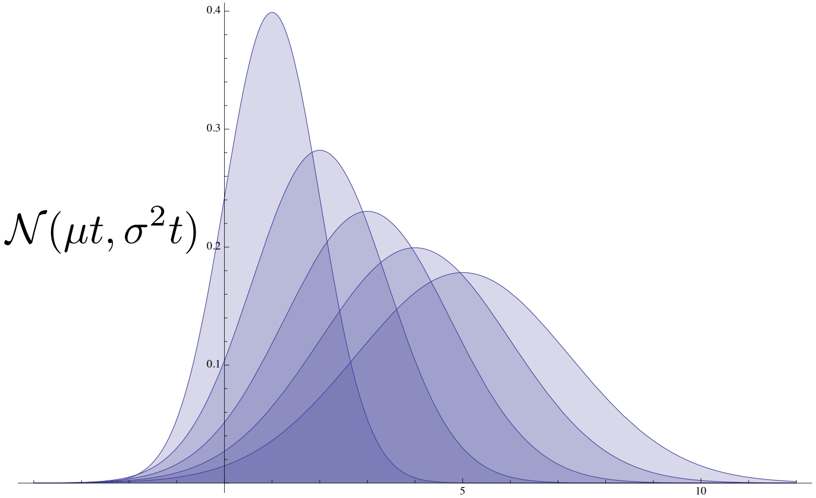
And as time goes on, the peak moves to the right. As the distribution moves to the right, variance broadens out over time.

- Since a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)},$$

a time-dependent stochastic process where $X_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ has probability density

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(x-\mu t)^2/(2\sigma^2 t)}$$



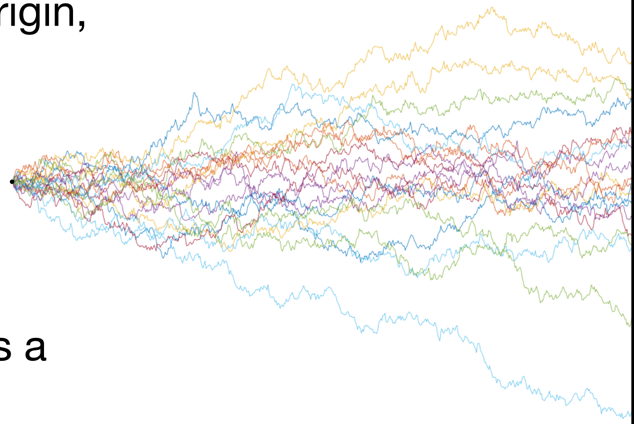
- This function satisfies the partial differential equation

$$\frac{\partial p}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial p}{\partial x} = 0$$

Probabilities for random walks

- More generally, for a random walk that begins elsewhere than the origin,

$$p(x_T, T; x_0, t_0) = \frac{1}{\sqrt{2\pi\sigma^2(T - t_0)}} \exp\left[-\frac{\left[(x_T - x_0) - \mu(T - t_0)\right]^2}{2\sigma^2(T - t_0)}\right].$$



- Even though the starting point isn't random, this can be analyzed as a function of its initial coordinates.
  - Notice that it depends only on coordinate differences.
  - It satisfies the "backward" equation

T–t0, xT–x0

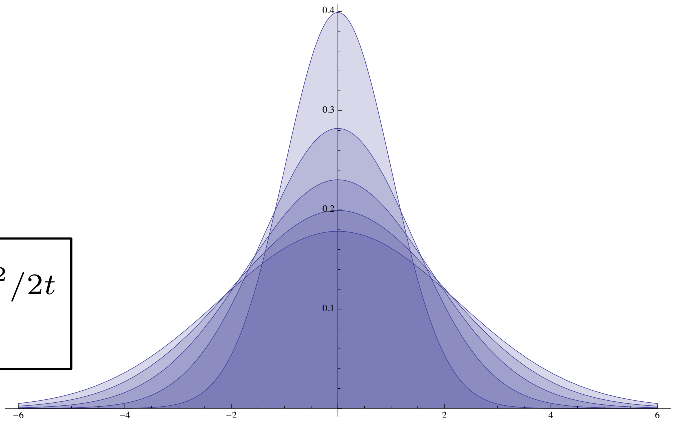
$$\frac{\partial p}{\partial t_0} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x_0^2} + \mu\frac{\partial p}{\partial x_0} = 0 \qquad \frac{\partial p}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial p}{\partial x} = 0$$

Diffusion equation, random walks, and probability

- In the special case of pure Brownian motion, $\mu = 0, \sigma = 1$ the probability density obeys the diffusion equation

$$\frac{\partial p_0}{\partial t} = \frac{1}{2}\frac{\partial^2 p_0}{\partial z^2}$$

- The PDE has many solutions
- The Gaussian solution $\boxed{p_0(z,t) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}}$



describes probability **concentrated at the origin** initially that **diffuses** over time.

- Increasing likelihood that the endpoint for the walk will be found far from its starting point.
- Only defined for $t > 0$ due to the square root.

Diffusion equation, random walks, and probability

- This **special** solution can be used to obtain the **general** solution:

For initial conditions $p(z, t=0) = f(z)$ the general solution is given by

$$p(z,t) = \int p_0(z-w,t) f(w)\,\mathrm{d}w = \frac{1}{\sqrt{2\pi t}} \int e^{-(z-w)^2/2t} f(w)\,\mathrm{d}w$$

- Examples:

$$f(z) = z^2$$
$$f(z) = e^{az}$$
$$f(z) = \cos(\lambda z)$$
$$f(z) = \theta(z-\kappa) = \begin{cases} 1, & z > \kappa \\ 0, & z < \kappa \end{cases}$$

Diffusion equation, random walks, and probability

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- Verify solution and initial conditions:
  - $$\left[\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right] p(z,t) = \int \left(\left[\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right] p_0(z-w,t)\right) f(w)\,\mathrm{d}w = 0;$$
  - $$\begin{aligned} \lim_{t\to 0} p(z,t) &= \lim_{t\to 0} \frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} f(z+u\sqrt{t})\,\mathrm{d}u, \quad \text{using } u = (w-z)/\sqrt{t} \\ &= f(z) \end{aligned}$$

Special functions

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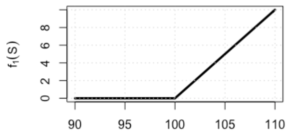
A few special functions

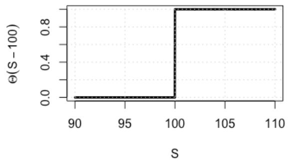
Let's pause to define a few convenient functions, starting by re-writing the familiar payoff function for a call option using absolute value.

$$f_1(S) = \max(S-K,0) = \frac{1}{2}\left(|S-K| + S - K\right)$$



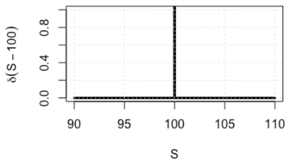
- The slope of the payoff function is the **step function**, which takes values either zero or one.

$$\frac{\mathrm{d}}{\mathrm{d}S} f_1(S) \equiv \theta(S-K) = \begin{cases} 1 & \text{if } S > K, \\ 0 & \text{otherwise} \end{cases}$$



- The derivative of the step function is the **Dirac delta function**, which is zero almost everywhere – and also has unit area "under the curve"!

$$\frac{\mathrm{d}^2}{\mathrm{d}S^2} f_1(S) \equiv \delta(S-K) = \begin{cases} 0 & \text{if } S \neq K, \\ \infty & \text{otherwise} \end{cases}$$



Dirac delta function

- Limit of Gaussian as width goes to zero
  - Singular at zero
  - Integral for area under the curve is one

$$\delta(x) = \lim_{t\to 0} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & x = 0 \end{cases}$$

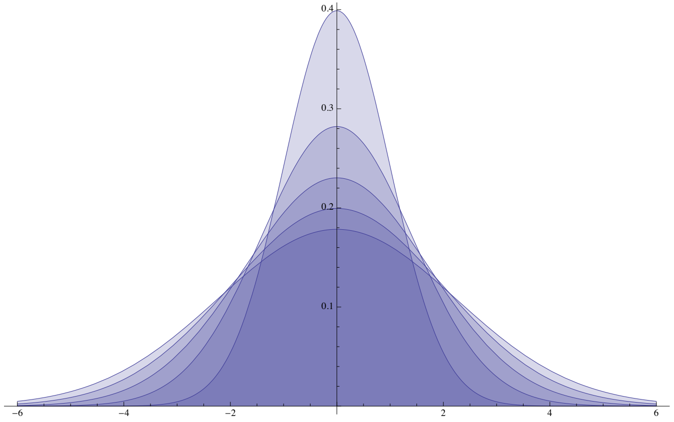
$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

- Assigns to any function it is integrated against its value at zero

- Properly speaking, a "generalized function" or functional

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-y) f(x)\,\mathrm{d}x = f(y)$$



Green's functions

Modifying the special solution slightly gives the **Green's function** that can be used to construct solutions to an **inhomogeneous equation**. Define

$$G(z,t) = p_0(z,t)\theta(t) = \frac{\theta(t)}{\sqrt{2\pi t}}e^{-z^2/2t},$$
$$\mathcal{D}G(z,t) = p_0(z,t)\delta(t) = \delta(z)\delta(t).$$

Then if there is a fixed function *h*(*z*,*t*) on the right hand side, *G* gives a solution:

$$p(z,t) = \int G(z-z',t-t')h(z',t')\mathrm{d}z'\mathrm{d}t' = \int_0^\infty \int_{-\infty}^\infty \frac{e^{-(z-z')^2/(2(t-t'))}}{\sqrt{2\pi(t-t')}}h(z',t')\mathrm{d}z'\mathrm{d}t'$$
$$\mathcal{D}p(z,t) = \int \delta(z-z')\delta(t-t')h(z',t')\mathrm{d}z'\mathrm{d}t' = h(z,t),$$
$$\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial z^2} = h(z,t)$$

Reflections, barriers, and survival probabilities

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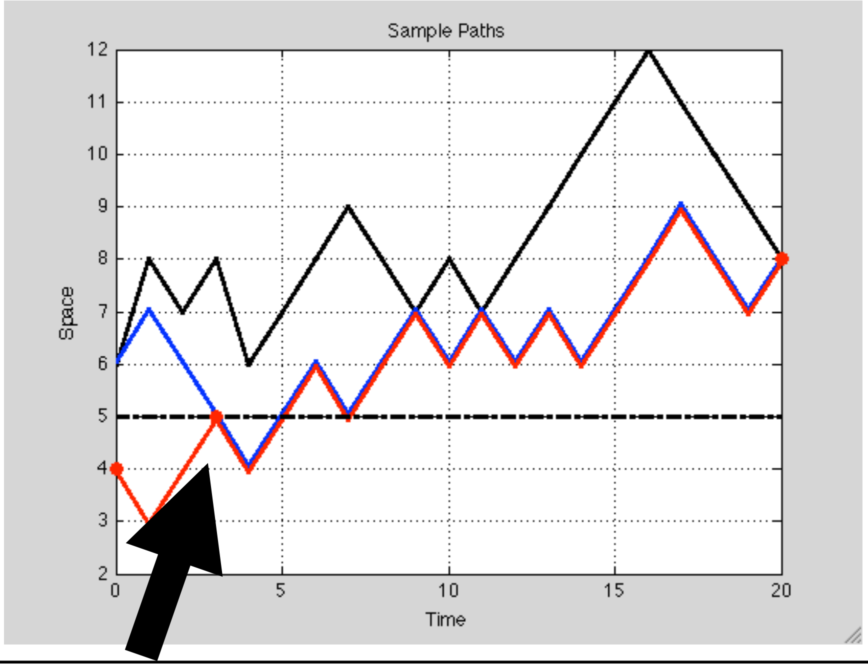
Survival probabilities

What is probability to get from point A to point B... without ever hitting point C?

- "Absorbing barrier" to represent events such as default
  - Mean time to hit barrier?
  - Probability to not have hit through time *t*?

- Method of images
  - Compute unrestricted probability to go from A to B
  - **Subtract** unrestricted probability to go from A* to B, where A* is the image point, i.e., reflection below the barrier of the point A.



*Reflect portion of blue path at **first passage** through barrier to get red path*

Survival probabilities

Probability to arrive without crossing barrier at z*, without drift:

$$p_s(z,t) = p_0(z - z_0, t) - p_0\left(z - [2z^* - z_0], t\right)$$
$$= \frac{1}{\sqrt{2\pi t}}\left(e^{-(z-z_0)^2/2t} - e^{-(z-[2z^*-z_0])^2/2t}\right)$$

A random walk takes steps of size +–1 with equal likelihood. What is the probability that a random walker starting at location 6 arrives at location 8 in exactly 20 steps, without ever reaching the location 5?

- The survival probability density obeys boundary condition

$$p_s(z^*, t) = 0$$

To get from 6 to 8 requires 11 positive steps and 9 negative ones. To get from the " image point" 4 where the reflected paths begin to 8 requires 12 positive steps and 8 negative ones.
Do subtract of two probabilities

- Therefore the complete solution for $t > 0$ is

$$p_s(z,t) = \begin{cases} \frac{1}{\sqrt{2\pi t}}\left(e^{-(z-z_0)^2/2t} - e^{-(z+z_0-2z^*)^2/2t}\right) & z > z^*, \\ 0 & z \leq z^*. \end{cases}$$

Survival probabilities

- Probability to arrive, **including drift** term, breaks symmetry. mu is not equal to 0
- Use boundary condition $p_s(z^*, t) = 0$ to determine constant prefactor in "image" term

C

$$p_s(z,t) = p(z - z_0, t) - Cp(z - [2z^* - z_0], t)$$ linear combination of solution is also a solution

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}}\left(e^{-(z-\mu t - z_0)^2/2\sigma^2 t} - Ce^{-(z-\mu t + z_0 - 2z^*)^2/2\sigma^2 t}\right), \quad C = e^{-2\mu(z_0 - z^*)/\sigma^2}$$

- Integrate over all non-defaulting results, above the barrier, at time *t*

$$p_s(t) = \int_{z^*}^{\infty} p_s(z,t)\mathrm{d}z$$

$$= \Phi\left(\frac{\mu t + z_0 - z^*}{\sqrt{\sigma^2 t}}\right) - e^{-2\mu(z_0 - z^*)/\sigma^2}\Phi\left(\frac{\mu t - z_0 + z^*}{\sqrt{\sigma^2 t}}\right)$$

$$\boxed{\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\mathrm{d}z}$$

Survival probabilities

Application: corporation **non-default probability** for corporate bond pricing

$$
\begin{aligned}
z &= \text{firm value} = D + E \\
z^* &= \text{firm debt} = D \\
z_0 &= \text{firm current value}, \quad z_0 > z^*
\end{aligned}
$$

mu

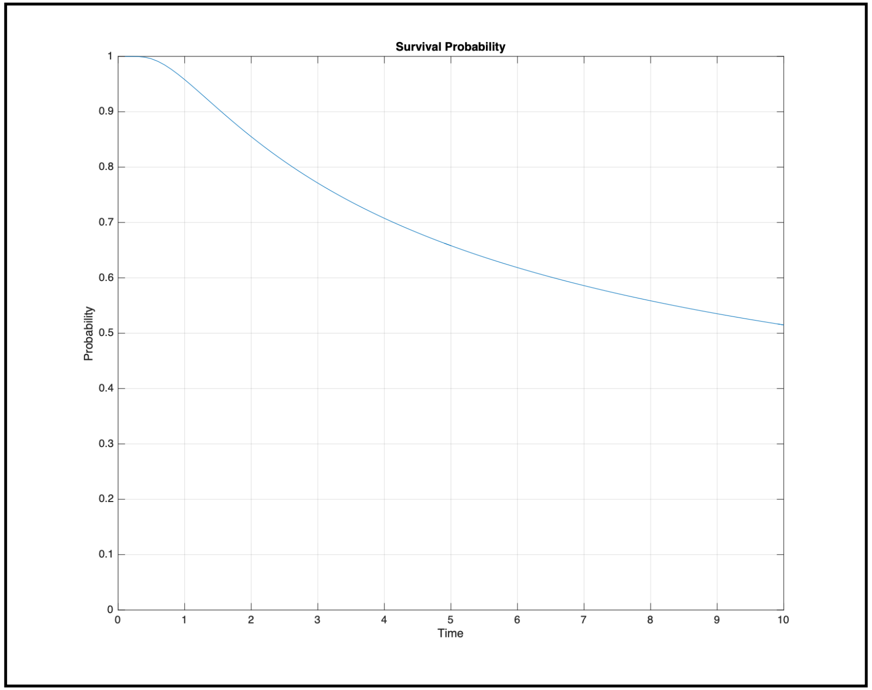
- How important is it to have high growth rate vs. high initial buffer to protect against default?
- What is required buffer, given growth rate, so that 10-year default probability is less than 25%?
- What is optimal capital structure to fund growth and minimize default probability?

Survival probabilities

- Sample parameter values (cf. Wise & Bhansali)

$$\mu = 0.01$$
$$\sigma = 0.25$$
$$z_0 - z^* = 0.5$$

- Default entirely due to chance of value diffusion below barrier, absent other sources of business shocks.



the longer we wait, the more it goes down because you have more and more chances for something bad to happen

Probability densities and expectations

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Stock price diffusion

We can also ask about more general future payoffs and expectations.

- The future **expected value** of a function on random paths satisfies the **same differential equation** as the probability density, considered as a function of its initial values.

- Consider the probability density function of the standard stock price path defined by

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}B$$

The probability $p(S_T, T; S, t)$ satisfies

the probability of arriving at ST a time T given that we started at price S at this time t satisfies a differential equation

$$\frac{\partial p}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S} = 0$$

What to expect when you're expecting

So consider the future expectation of a function of S_T:

payoff on a call option where T represents the time when the option expires

$$\mathrm{E}_t\left[f(S_T)\right] = \int p(S_T, T; S, t) f(S_T)\mathrm{d}S_T = F(S,t)$$

T: terminal time

t: earlier time

- The expectation is itself a function of the initial (or current) values of S,t and satisfies the same differential equation, along with the limiting value

$$\lim_{t\to T} F(S,t) = \int \delta(S_T - S) f(S_T)\mathrm{d}S_T = f(S)$$

no longer random because we're evaluating at time big T what the value will be a time big T.

- For the expectation of a terminal payoff, consider the equation satisfied by its present value

at terminal time T

$$V(S,t) = e^{-r(T-t)}F(S,t) = e^{-r(T-t)}\mathrm{E}_t\left[f(S_T)\right] = e^{-r(T-t)}\mathrm{E}_t\left[V(S_T, T)\right]$$

Drift away

V satisfies a PDE **similar** to Black-Scholes, **except** with a μ-dependent drift

$$\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} - rV = 0$$

the discounting factor adds an extra term minus rV

- *V* would **exactly** satisfy the Black-Scholes PDE if it were instead based on an Itô process where **the drift is replaced by the risk-free rate** risk–neutral pricing

$$\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}B$$

- With respect to this evolution equation, the present value of a Black-Scholes contract is given by the expectation of its discounted payoff:

$$e^{-rt}V(S,t) = \mathrm{E}_t\left[e^{-rT}V(S_T,T)\right]$$

t: earlier time T: later time

Black-Scholes solutions

- One method for computing option prices is to evaluate the expectation numerically using Monte Carlo techniques to average over a large number of appropriate paths.

- Another method is to apply the probability density formulas directly. Returning to the original variables for stock price, time, etc.,

$$V(s,t) = \cancel{\int p(S_T,T;S,t)V(S_T,T)\,\mathrm{d}S_T} \quad e^{-r(T-t)} \int p\,(S_T, T; S, t)\, V\,(S_T, T)\, dS_T$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int e^{-(x-x')^2/2\sigma^2(T-t)} f(x')dx',$$

where $f(x') = g(S') = \max(S' - K, 0)$

for a vanilla call option of strike price K expiring at time T.

Black-Scholes solution

So $V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{x'=\log K}^{\infty} e^{-(x-x')^2/2\sigma^2(T-t)}(e^{x'} - K)dx'$

$$= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$

how far away we are from the at the money forward value

where $d_\pm \equiv \frac{\log(S/Ke^{-r(T-t)})}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}$ and $\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, \mathrm{d}z$

The "risk-neutral" probability density describes the diffusion of a hypothetical asset with the same volatility as *S* but with drift rate *r*:

$$p_{RN}(S_T, T; S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}S_T} \exp\left[-\frac{\left(\log(S_T/S) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right]$$

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Greeks and exotics

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The Greeks

It is customary to define various partial derivatives of the solution, including

$$\text{Delta } \Delta \equiv \partial V/\partial S = \begin{cases} \Phi(d_+), & \text{call} \\ \Phi(-d_+) = \Delta_{\text{call}} - 1, & \text{put} \end{cases}$$

$$P + S = C + Ke^{-rT}$$

differentiate this with respect to S

$$\text{Gamma } \Gamma \equiv \partial^2 V/\partial S^2 = \frac{\Phi'(d_+)}{\sigma S \sqrt{T-t}},$$

$$\text{Vega } v \equiv \partial V/\partial \sigma = \Phi'(d_+) S \sqrt{T-t}$$

- The delta and gamma can be given their own probability/diffusion representation. The vega, which is the derivative with respect to a *parameter*, cannot.

Black-Scholes solutions: exotic options

Likewise, different payoff functions lead directly to a value formula by plugging into the integral. Example:

For a **binary call option**, with payoff $f(x') = g(S') = \theta(S' - K) = \begin{cases} 1, & S' \geq K, \\ 0, & S' < K \end{cases}$

$$V(S,t) = e^{-r(T-t)}\Phi(d_-)$$

which is directly related to the probability of the stock finishing in the money at time *T*...under the risk-neutral measure. This is **not** the real-world probability, which depends on μ

$$p_\mu(S_T, T; S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}S_T} \exp\left[-\frac{\left(\log\left(S_T/Se^{(\mu-\sigma^2/2)(T-t)}\right)\right)^2}{2\sigma^2(T-t)}\right]$$

Black-Scholes solutions: exotic options

Example: consider a power option whose payoff is a fixed power of *S:* $X_T = S_T^2$

$$X = S^2, \quad \log X = 2\log S, \quad \mathrm{d}(\log X) = 2\ \mathrm{d}(\log S)$$

$$X_t = S_0^2 e^{2\left[(\mu-\sigma^2/2)t+\sigma\sqrt{t}Z\right]},$$

mu replaced by r

$$\mathbb{E}^Q[X_T] = S_0^2 e^{2(r-\sigma^2/2)T} e^{2\sigma^2 T},$$

take present value: e^(–rT)

$$V = S_0^2 e^{rT+\sigma^2 T}.$$

where we used the risk-neutral measure and made use of the moment-generating function for Gaussian random variables

$$Y \sim \mathcal{N}(\mu, \sigma^2) \implies f(\lambda) = \mathbb{E}[e^{\lambda Y}] = e^{\lambda\mu+\lambda^2\sigma^2/2}$$

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American options

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American exercise

- For American options, there are additional considerations. The owner of the option has the right to exercise at any time, not just at T.

- Should the option be exercised early? If so, when? Since the owner might no longer hold the option at T, we cannot simply apply the earlier formulas.

American perpetual put

Example: consider a put option that **never** expires.

- Its payoff upon exercise, at all times, is max(*K* - *S, 0*), where *K* is the strike price. The value is time-independent, so it satisfies

$$\frac{(\sigma S)^2}{2}\frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + rS\frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0$$

stock price itself always satisfies the Black–Scholes equation. But we also know that that can't be part of the solution because it grows as S grows to infinity. And the value of a put should be a decreasing function in the stock price

- Let's try a solution of power-law form

$$V(S) = S^\alpha \quad \Longrightarrow \quad (\alpha^2 - \alpha)\frac{\sigma^2}{2} + \alpha r - r = 0 \quad \Longrightarrow \quad \alpha = 1 \text{ or } -2r/\sigma^2$$

- Since the solution must vanish for increasing *S* (and assuming *r* > *0*),

$$V(S) = cS^{-2r/\sigma^2}$$

American perpetual put

- For $S > K$, don't exercise.
  - However if S is far below K, it could be advantageous to exercise.
  - (Special case: if the stock price S decreases to zero, the option's value can never go higher so there is no point waiting any longer)

- Boundary condition: the option's value will equal its exercise value when

$$V(\hat{S}) = K - \hat{S} \implies V(S) = (K - \hat{S}) \left(\frac{S}{\hat{S}} \right)^{-2r/\sigma^2}$$

- The option writer must assume that the buyer will choose to maximize *V:*

$$\left. \frac{\partial V}{\partial \hat{S}} \right|_{S=\hat{S}} = 0 \implies \hat{S} = \frac{K}{1 + \sigma^2/2r},$$

$$V(S) = \frac{K\sigma^2/2r}{1 + \sigma^2/2r} \left(\frac{S}{K}(1 + \sigma^2/2r) \right)^{-2r/\sigma^2}$$

Measures, martingales, and Monte Carlo

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Measures and martingales

- An Itô process is a martingale if and only if it has **zero drift**. Measure for Brownian motion.

to get the risk–neutral measure, we take the drift rate mu and we replace it by r

$$\mathbb{E}_t[X_{t'}] = X_t, \quad t < t' \implies \mathbb{E}_t[\mathrm{d}X_t] = 0$$
$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t \implies a = 0$$

- Now consider a discounted price process

$$F = e^{-rt} S \text{ where } \mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B$$

Then

$$\frac{\partial F}{\partial S} = e^{-rt}, \quad \frac{\partial^2 F}{\partial S^2} = 0, \quad \frac{\partial F}{\partial t} = -re^{-rt}S,$$
$$\frac{\mathrm{d}F}{F} = (\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}B \text{ is a martingale iff } \mu = r.$$

Risk-neutral pricing

What is the measure for **risk-neutral pricing**?

- Under measure Q, expected return of risky assets equals risk-free rate, i.e.,

$$\mathbb{E}_t^Q\left[\frac{\mathrm{d}S_t}{S_t}\right] = r\mathrm{d}t$$

- How do we find the measure Q? Let's write

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_t} &= r\mathrm{d}t + (\mu - r)\mathrm{d}t + \sigma\mathrm{d}B \\ &= r\mathrm{d}t + \sigma\mathrm{d}B^Q, \text{ where } \mathrm{d}B^Q \equiv \left(\frac{\mu - r}{\sigma}\right)\mathrm{d}t + \mathrm{d}B\end{aligned}$$

- Then the new differential is a martingale:

$$\begin{aligned}\mathbb{E}_t^Q\left[\mathrm{d}B^Q\right] &= 0, \\ \mathrm{Var}(\mathrm{d}B^Q) &= \mathrm{d}t\end{aligned}$$

Risk-neutral pricing

- Heuristic: replace drift with risk-free rate to get risk-neutral process: $\mu \to r$

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^Q$$

$$\mathrm{d}(\log S_t) = \left(r - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t^Q,$$

$$\log S_T/S_0 \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

- Analogous to discrete-time binomial model results: use risk-neutral, not objective, probabilities to determine pricing.

Risk-neutral pricing

All (no-arbitrage) traded assets have discounted price process that are martingales

$$e^{-rt} X_t = \mathbb{E}_t^Q \left[e^{-rT} X_T \right]$$

- For a call option, when interest rate is constant,

Q–measure: letting the paths evolve according to Ito processes, but where we replace the drift rate mu by the risk–free rate r

$$C_t = e^{-r(T-t)} \mathbb{E}_t^Q \left[\max(S_T - K, 0) \right]$$

payoff of call option

- Monte Carlo implementation: generate ensemble of equiprobable price paths using **risk-neutral** drift and volatility parameters, compute terminal payoffs, and take average of their discounted present value.

Monte Carlo pricing

- More generally, price any contract from its terminal values, allowing risk-free rate to vary with time

$$V_t = \mathbb{E}_t^Q \left[\frac{V_T}{\beta_T / \beta_t} \right] \qquad \begin{cases} \mathbb{E}[\cdot] & \text{Sum over paths, equal weights} \\ Q: & \text{Use } r \text{ in evolution} \\ V_T & \text{Terminal value of paths} \\ \beta_T/\beta_t & \text{Discounting } e^{\int_t^T r(s)\,\mathrm{d}s} \end{cases}$$

general

special case: e^(rT)

Monte Carlo pricing

- Generate an ensemble of **risk-neutral paths**
  - Use risk-free rate for drift
  - Use random number generation so that all paths are equally probable **under risk-neutral measure**
- Determine terminal payoffs
- Compute discounted present value of average over paths

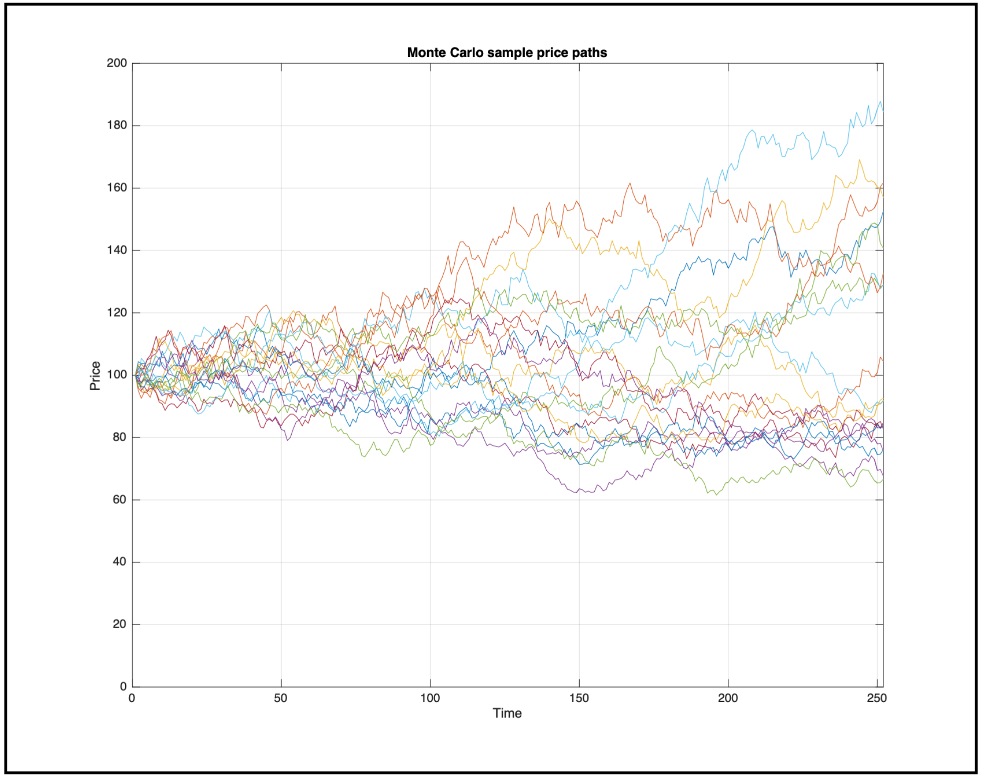
```
MCprice <- function(Price, Strike, Rate, Time, Volatility, Steps, Paths) {
#
# Monte Carlo pricer for vanilla options [8/12/2021 pfm]
# Input arguments use consistent units, e.g., annualized
# Price:    current price of underlying
# Strike:   strike price of option contract
# Rate:     risk-free rate
# Time:     time to expiration
# Volatility
# Steps:    number of time steps in discretization
# Paths:    number of Monte Carlo simulation paths for sampling measure

S0                                        <- Price
K                                         <- Strike
rf                                        <- Rate
T                                         <- Time
sigma                                     <- Volatility
Nt                                        <- Steps
Np                                        <- Paths
dt                                        <- T/Nt

# Select independent, standardized shocks.  For example,
z                    <- matrix(sign(rnorm(Nt*Np)),ncol=Np)

# Define IID returns for each step and path under risk-neutral measure
# INSERT CODE HERE

# Construct stochastic paths and price process
# INSERT CODE HERE

# Define payoff values for derivatives
# INSERT CODE HERE

# Compute call and put values as discounted expected payoffs under RN measure
# INSERT CODE HERE

# Return values
return(data.frame(call=Call,put=Put))
}
```

Monte Carlo pricing

Accuracy, limits, and convergence

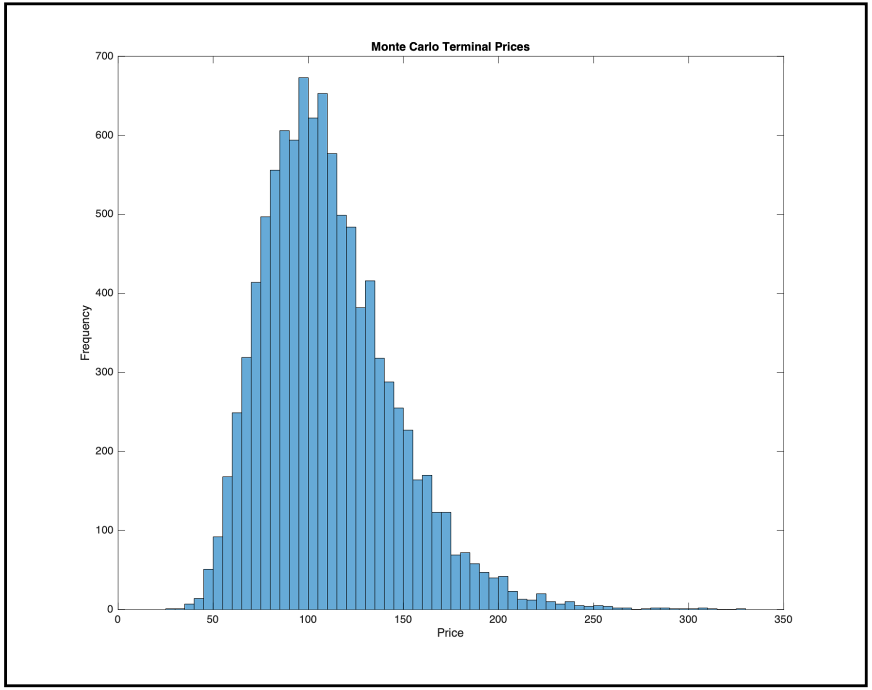
- Discrete time steps
- Finite number of sample paths

```
S0 <- 100; K <- 100; T <- 1; rf <- 0.1;
sigma <- 0.3; Nt <- 252; Np <- 1e4;

MCprice(S0,K,rf,T,sigma,Nt,Np)

      call      put
1 16.93101 7.051155

library(RQuantLib)

EuropeanOption("call",S0,K,0,rf,T,sigma)$value

[1] 16.73413

EuropeanOption("put",S0,K,0,rf,T,sigma)$value

[1] 7.217875
```



Monte Carlo pricing

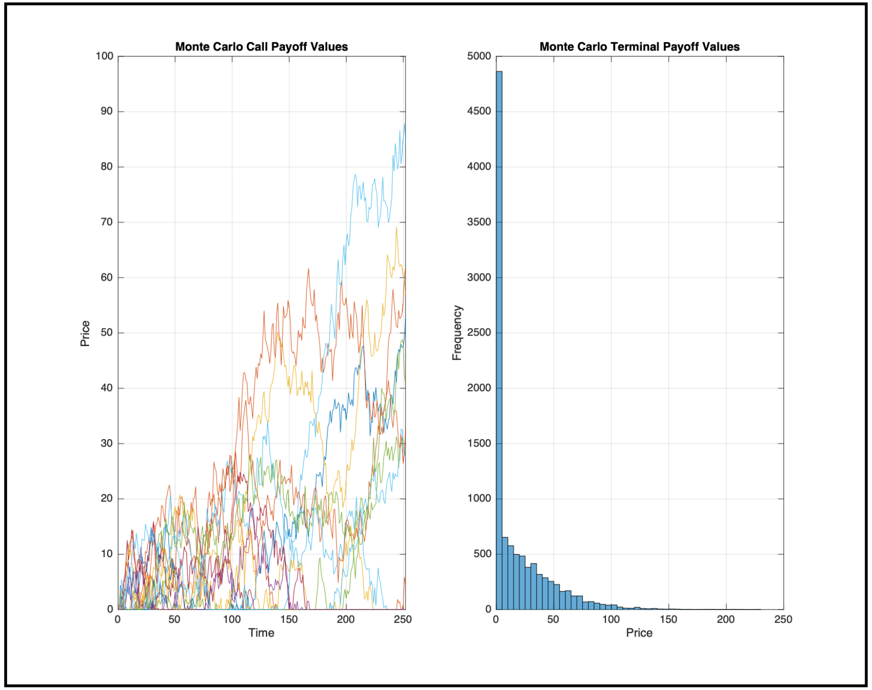
- Price paths lognormally distributed
- Mean value based on risk-neutral, not objective, drift rate risk–free rate
- Volatility identical



Monte Carlo pricing

Implementation of measure:

- Since all paths **equally probable under *Q* measure**, compute option value using simple arithmetic average of discounted payoffs.



Itô processes in higher dimensions

Finance at MIT

Where ingenuity drives results

Itô's lemma: multiple stochastic variables

- For multiple Itô processes, formula generalizes.

$$\mathrm{d}X_i = a_i(t, X_1, X_2, \ldots)\,\mathrm{d}t + b_i(t, X_1, X_2, \ldots)\,\mathrm{d}B_i$$

$$\mathrm{d}F = \frac{\partial F}{\partial t}\,\mathrm{d}t + \sum \frac{\partial F}{\partial X_i}\,\mathrm{d}X_i + \frac{1}{2}\sum \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial X_i \partial X_j}\,\mathrm{d}t$$

- Applications
  - Multiple assets, such as a stock index or portfolio
  - Multiple factors, reducing independent sources of correlation
  - Risk models, to determine sources of risk priced in the market
  - Term-structure models for interest rates and derivatives
  - ...

Itô's lemma: multiple stochastic variables

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- Heuristic "rule of thumb" for correlated Brownian motions

$$\begin{aligned}
(\mathrm{d}B_i)^2 &\to \mathrm{d}t, \\
(\mathrm{d}B_i)(\mathrm{d}B_j) &\to \rho_{ij}\,\mathrm{d}t, \\
(\mathrm{d}X_i)^2 &\to b_i^2\ \mathrm{d}t \\
(\mathrm{d}X_i)(\mathrm{d}X_j) &\to \rho_{ij} b_i b_j\ \mathrm{d}t
\end{aligned}$$

Itô's lemma

- Example: consider two independent stochastic variables, and

$$F = X_1 X_2 \implies \mathrm{d}F = X_1\,\mathrm{d}X_2 + X_2\,\mathrm{d}X_1 + (\,\mathrm{d}X_1)(\,\mathrm{d}X_2),$$

$$\frac{\mathrm{d}F}{F} = \frac{\mathrm{d}X_1}{X_1} + \frac{\mathrm{d}X_2}{X_2} + \left(\frac{\mathrm{d}X_1}{X_1}\right)\left(\frac{\mathrm{d}X_2}{X_2}\right)$$

- Geometric Brownian motions: $dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}(dY)^2 + \frac{\partial^2 F}{\partial X \partial Y}(dX)(dY)$

$$\frac{\mathrm{d}X_i}{X_i} = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_i$$

$$\frac{\mathrm{d}F}{F} = (\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2)\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1 + \sigma_2\,\mathrm{d}B_2$$

Itô's lemma

- Example: consider two independent stochastic variables. How does **ratio** evolve?

need all 2nd order terms in the taylor expansion

$$F = \frac{X_2}{X_1} \implies \mathrm{d}F = \frac{\mathrm{d}X_2}{X_1} - \frac{X_2\,\mathrm{d}X_1}{X_1^2} + \frac{1}{2}\left(\frac{2X_2}{X_1^3}\right)(\mathrm{d}X_1)^2 - \left(\frac{1}{X_1^2}\right)(\mathrm{d}X_1)(\mathrm{d}X_2)$$

$$\frac{\mathrm{d}F}{F} = \frac{\mathrm{d}X_2}{X_2} - \frac{\mathrm{d}X_1}{X_1} + \left[\frac{b_1^2}{X_1^2} - \frac{\rho_{12} b_1 b_2}{X_1 X_2}\right]\mathrm{d}t,$$

$$\text{If } \rho_{12} = 0 \implies \quad = \left(\mu_2 - \mu_1 + \sigma_1^2\right)\mathrm{d}t + \sigma_2\,\mathrm{d}B_2 - \sigma_1\,\mathrm{d}B_1$$

- If drift coefficients are equal, then growth rate of 2 vs. 1 is positive. However the same is true of the inverse. Contradiction?
- Application: changes of base currency

References

- Books
  - Capinski and Zastawniak (2003) – "*Mathematics for Finance,*" Springer
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