15.455x Mathematical Methods of Quantitative Finance

Week 5: Continuous-Time Finance (continued)

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Finance at MIT
Where ingenuity drives results

From SDE to PDE:
The Black-Scholes equation

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Black-Scholes equation

But wait! The derivation assumed that the number of shares was **constant**.

- However, having found the equation for *V*, the condition

$$\Delta = \frac{\partial V}{\partial S}$$

means that the number of shares is **dynamic** and changing **continuously**...which **completely contradicts** the assumption we used to derive the result.

Specifically, we took $\mathrm{d}(\Delta S) = \Delta\,\mathrm{d}S$

instead of

$$\mathrm{d}(\Delta S) = \Delta\,\mathrm{d}S + S\,\mathrm{d}\Delta + \mathrm{d}\Delta\,\mathrm{d}S$$

Self-financing dynamic replication

Nevertheless, the result is correct. Let's consider a different approach...

Three asset portfolio:

- Underlying security (e.g., stock)
- Derivative (e.g., vanilla call or put)
- "Cash" (e.g., short-term risk-free bond)

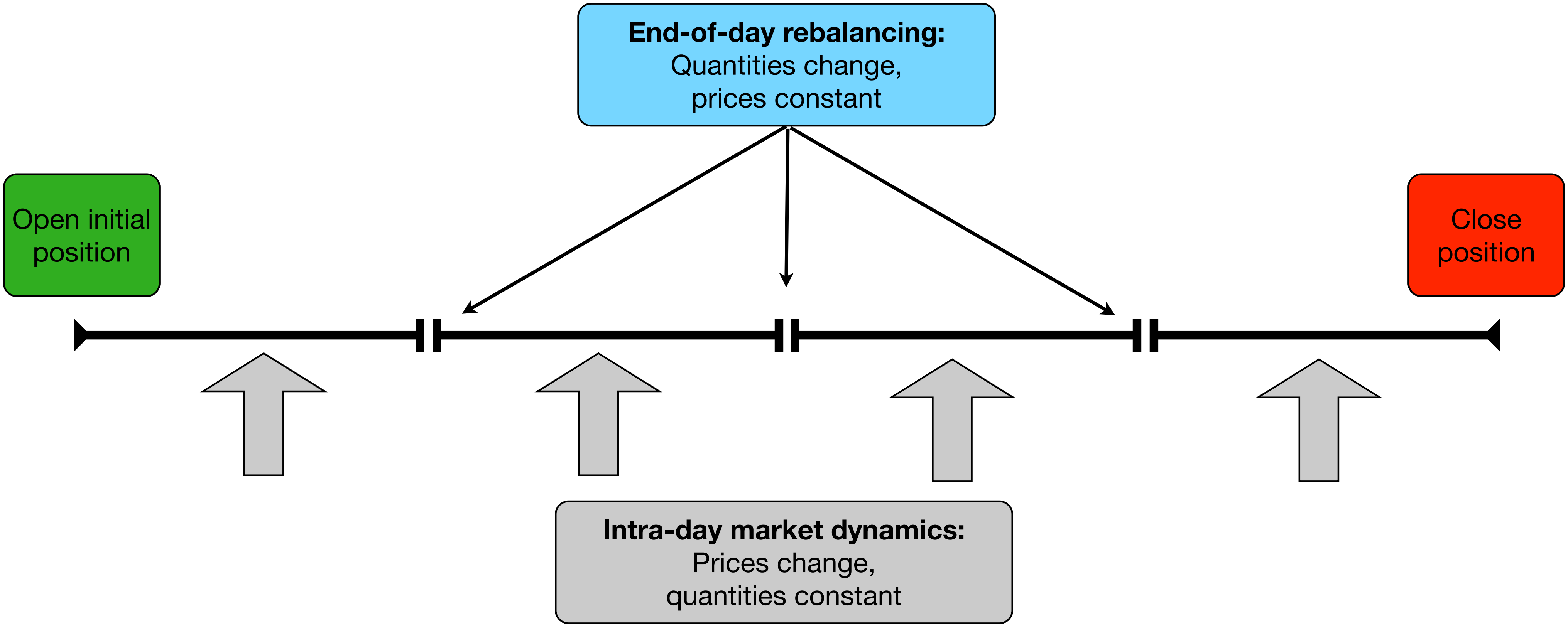
Dynamic trading strategy

- "Buy and hold" option position (e.g., long one call contract)
- Rebalance stock position, based on price and time to expiry
- Stock purchases and sales flow in and out of cash account
- Cash account earns or borrows at risk-free rate

Assumptions

- Unlimited credit start out with no money, but a lot of credit
- Lend or borrow at single rate *r*
- Stock can be traded in fractional quantities
- Full use of short sale proceeds something we have to keep collateral
- No dividends
- No transaction costs
- No trading delays or market impacts

Discrete hedging and portfolio rebalancing



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Rebalancing condition for self-financing portfolio

- Consider a portfolio X with initial value **zero** that holds stocks and bonds (a.k.a. "cash") in quantities q and C. At the initial time, the portfolio balance is

$$X_0 = q_0 S_0 + C_0 M_0 = 0$$

- Rebalancing **does not change** the portfolio value. It simply exchanges bonds for stock of equal value at the prevailing market price. In order to rebalance the portfolio at time t (to a new quantity q_t),

$$\begin{aligned} X_t^{\text{post}} - X_t^{\text{pre}} &= S_t(q_t - q_{t-1}) + M_t(C_t - C_{t-1}) = 0 \\ &= S_{t-1}(q_t - q_{t-1}) + M_{t-1}(C_t - C_{t-1}) \\ &\quad + (S_t - S_{t-1})(q_t - q_{t-1}) + (M_t - M_{t-1})(C_t - C_{t-1}) \end{aligned}$$

- In continuous time we therefore have this **self-financing condition**:

$$S\mathrm{d}q + M\mathrm{d}C + \mathrm{d}S\mathrm{d}q + \mathrm{d}M\mathrm{d}C = 0$$

Dynamic hedging and portfolio balance

- Now consider a self-financing portfolio consisting of a single derivative contract of value *V* plus **dynamically rebalanced** stock+bond positions.

$$\pi = V + qS + CM$$

- The portfolio's change in value **between** two time periods in which no trading or rebalancing takes place is

$$\pi_t - \pi_{t-1} = (V_t - V_{t-1}) + q_{t-1}(S_t - S_{t-1}) + C_{t-1}(M_t - M_{t-1})$$

- In continuous time,

self–financing condition: 0

$$\begin{aligned} \mathrm{d}\pi &= \mathrm{d}V + \mathrm{d}(qS + CM) \\ &= \mathrm{d}V + (q\mathrm{d}S + C\mathrm{d}M) + \boxed{(S\mathrm{d}q + M\mathrm{d}C + \mathrm{d}q\mathrm{d}S + \mathrm{d}C\mathrm{d}M)} \\ &= \mathrm{d}V + (q\mathrm{d}S + C\mathrm{d}M) \\ &= \mathrm{d}V + q\mathrm{d}S + rCM\mathrm{d}t \\ &= \mathrm{d}V + q\mathrm{d}S + r(\pi - V - qS)\mathrm{d}t \end{aligned}$$

Dynamic hedging and portfolio balance

- We now can look for a **dynamic** trading strategy for $q = q(t)$ that eliminates the risk in the portfolio. The reason to expect this is possible is that there is **only one stochastic driver** for both the stock and derivative. Indeed,

the stock

V: value of derivative

Let $V = V(t, S)$ and $\mathrm{d}S = (\mu S)\mathrm{d}t + (\sigma S)\mathrm{d}B$.

S: underlying asset of derivative

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S}\right)\mathrm{d}S$$

$$\begin{aligned}\mathrm{d}\pi &= \mathrm{d}V + q\mathrm{d}S + r(\pi - V - qS)\mathrm{d}t \\ &= \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} + q\right)\mathrm{d}S + r(\pi - V - qS)\mathrm{d}t\end{aligned}$$

- So let's choose $q(t)$ so that the coefficient of the stochastic dS term vanishes.

$$q = -\frac{\partial V}{\partial S} = -\Delta$$

- We do **not** assume it is constant. This formula tells us how to **replicate** V's payoff using only stocks and bonds.

Black-Scholes equation

- With this dynamic choice of q, the portfolio is now risk-free. Since its initial value was zero, the value must remain zero at all times to avoid arbitrage. Therefore

we started with a portfolio with initial value 0. It's self–financing. There's no money coming in or out. So it began with value 0. It stays 0. The rate of change is 0

$$\pi = 0, \quad \mathrm{d}\pi = 0 = \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} + q\right)\mathrm{d}S + r(\pi - V - qS)\mathrm{d}t$$

$$= \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} - rV + rS\frac{\partial V}{\partial S}\right)\mathrm{d}t$$

which leads to the Black-Scholes PDE:

$$\boxed{\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}$$

Summary of some key formulas

- Itô process: $\mathrm{d}X = a\,\mathrm{d}t + b\,\mathrm{d}B$
- Itô formula:
$$\begin{aligned}\mathrm{d}F &= \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\mathrm{d}t \\ &= \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right)\mathrm{d}t + b\frac{\partial F}{\partial X}\mathrm{d}B\end{aligned}$$
- Stock price: $\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}B$
$$\mathrm{d}(\log S) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}B$$
- Black-Scholes: $\Delta = \partial V/\partial S, \quad \mathrm{d}\pi = r\pi\mathrm{d}t,$
$$\boxed{\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}$$

Black-Scholes PDE

- Linear equation obeys superposition: sum of two solutions is also a solution

- Boundary conditions: typically specify **terminal** value of payoff at fixed future time and solve for **initial** value. we know what they're worth when they expire. We want to know what they're worth before expiration.

- Assumptions: lognormal distribution, geometric Brownian motion
  - Conditions on more general Itô process?
  - Time-varying volatility? constant perhaps become dynamic over time
  - Linearity in price variable?
  - Allowed range of price variable?

Black-Scholes and beyond

- Continuous **dividend yield**: no longer purely self-financing, adjust cash flows:

$$\mathrm{d}\pi = (rV - \Delta S(r - D))\mathrm{d}t$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0$$

- Note: not simply a substitution for risk-free rate. The holder of the stock receives dividends; the holder of the unexercised option does not.

Black-Scholes and beyond

- Currency options: receive foreign interest rate, equivalent to cash inflow, so substitute

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - r^*)S\frac{\partial V}{\partial S} - rV = 0$$

- Commodity options: cost of carry *q* is cash outflow.

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r + q - D)S\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes and beyond

- Options on futures:
  - Change variables from spot to forward $\mathcal{F} = e^{r(T-t)}S$
  - Substitute and use (ordinary) chain rule to get an even simpler equation

$$\boxed{\frac{\partial V}{\partial t} + \frac{\sigma^2 \mathcal{F}^2}{2}\frac{\partial^2 V}{\partial \mathcal{F}^2} - rV = 0}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  - Note that the first-derivative term, related to carry costs, drops out in terms of simpler, more natural variables which already include the effect.
  - Compare with diffusion equation $\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial x^2} = 0$

Fixed income modeling

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Fixed income modeling

- Interest rates are dynamic...and stochastic
- Interest rates are not tradable assets
- Interest rates exhibit a term structure
- Interest rate term structures are dynamic...and stochastic
- Bond prices across instruments are determined in relation to the same set of rates
- Interest rate derivatives depend on all of the above

United States | 97) Settings | Sovereign Debt Monitor

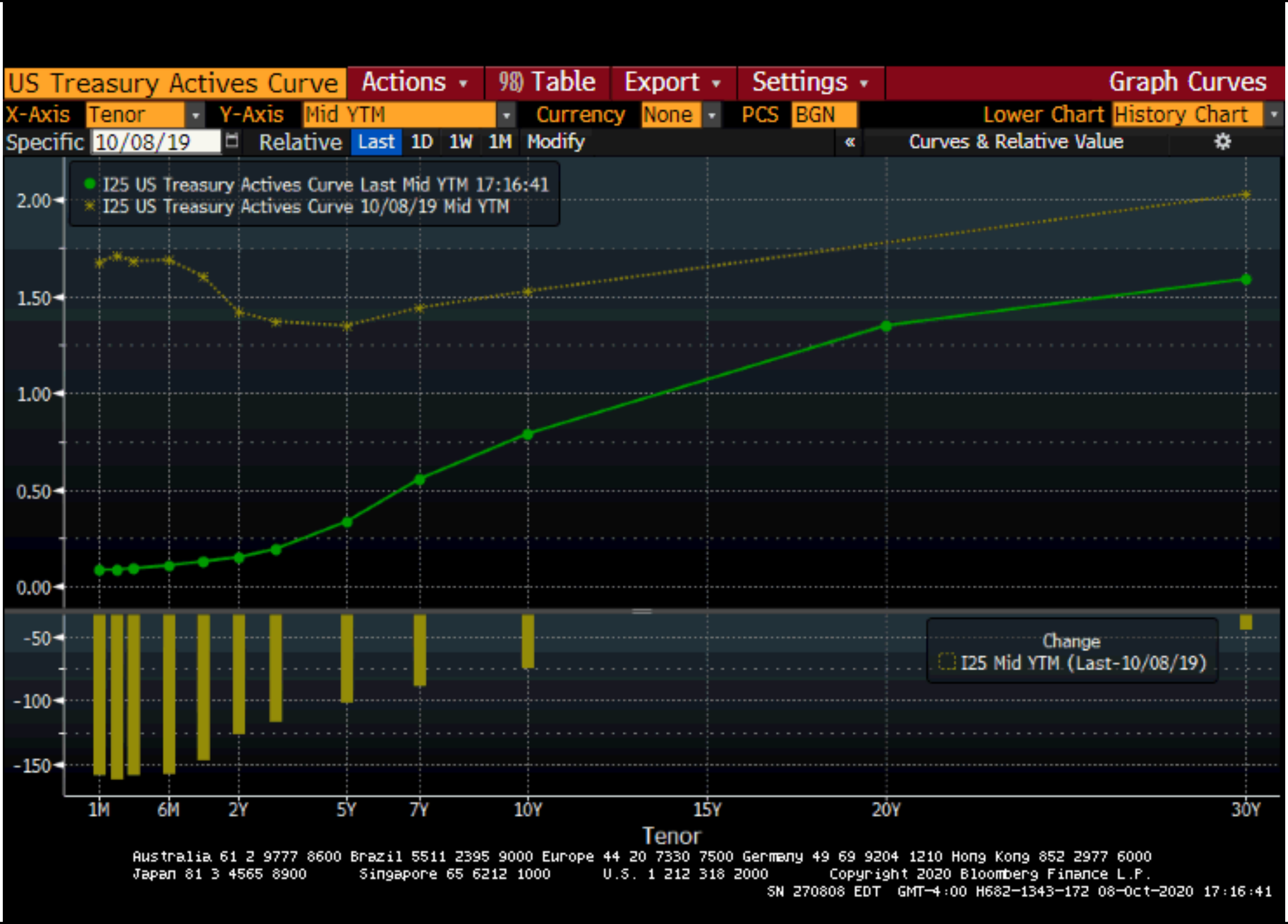
Historical Data Range: 3 Months

Security		Price		Yield	Chg	#SDΔ/day	Low	Range (Avg / Now)	High	Avg	+/-	#SD	RSI
1) Benchmarks													
10) T 0 1/8 09/22	2Y	99-30 3/8	c	0.151	+0.0	0.0	0.105		0.163	0.139	+1.2	+0.9	62.4
11) T 0 1/8 10/23	3Y	99-25 1/4	c	0.196	-0.3	-0.1	0.112		0.201	0.163	+3.3	+1.7	64.6
12) T 0 1/4 09/25	5Y	99-18 1/4	c	0.337	-0.5	-0.2	0.190		0.342	0.274	+6.4	+2.2	64.2
13) T 0 3/8 09/27	7Y	98-25+	c	0.551	-0.7	-0.2	0.357		0.558	0.463	+8.8	+2.3	64.7
14) T 0 5/8 08/30	10Y	98-15+	c	0.785	-0.2	0.0	0.507		0.787	0.653	+13.3	+2.2	64.5
15) T 1 1/8 08/40	20Y	96-03	c	1.350	-0.3	0.0	0.954		1.353	1.151	+19.9	+2.0	66.2
16) T 1 3/8 08/50	30Y	94-31+	c	1.587	+0.1	0.0	1.186		1.589	1.377	+21.0	+2.1	66.4
2) Curves													
18) 2yr-10yr			c	63.6	-0.2	0.0	40.3		63.9	51.6	+12.0	+2.2	64.2
19) 2yr-20yr			c	120.3	-0.3	0.0	85.0		121.1	101.6	+18.7	+2.0	65.6
20) 2yr-30yr			c	143.9	-0.1	0.0	108.0		144.8	124.0	+19.9	+2.1	65.6
21) 5yr-10yr			c	45.0	+0.3	0.1	31.0		45.7	38.1	+6.8	+1.7	62.3
3) Butterflies													
23) 2Y-5Y-10Y			c	-26.5	-0.4	-0.1	-31.7		-18.9	-25.0	-1.5	-0.5	53.9
24) 5Y-10Y-20Y			c	-12.2	--	--	--		--	--	--	--	--
4) Inflation													
26) US B/E 10YR			c	1.734	+2.2	0.2	1.382		1.809	1.613	+12.1	+1.1	60.1
27) TII0 1/8 07/30	10Y	111-00+	c	-0.949	-2.4	-0.3	-1.106		-0.762	-0.965	+1.6	+0.2	57.8
5) CDS spread													
29) CDS EUR SR 5Y			c	19.6	+0.1	0.1	19.5		24.0	22.6	-3.0	-2.2	47.7

News | Economic Events

02:00 04:00 06:00 08:00 10:00 12:00 14:00 16:00 18:00 20:00 22:00

	Curve Id	1M	3M	2Y	5Y	10Y	20Y	30Y
11)	I25	0.089	0.094	0.152	0.338	0.786	1.352	1.588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98) Table
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Y-Axis Mid YTM
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PCS BGN
Lower Chart History Chart
Specific 10/08/19
Relative Last 1D 1W 1M Modify
Curves & Relative Value
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I25 US Treasury Actives Curve 10/08/19 Mid YTM
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1.50
1.00
0.50
0.00
-50
-100
-150
Change
I25 Mid YTM (Last-10/08/19)
1M
6M
2Y
5Y
7Y
10Y
15Y
20Y
30Y
Tenor
Australia 61 2 9777 8600 Brazil 5511 2395 9000 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 4565 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2020 Bloomberg Finance L.P.
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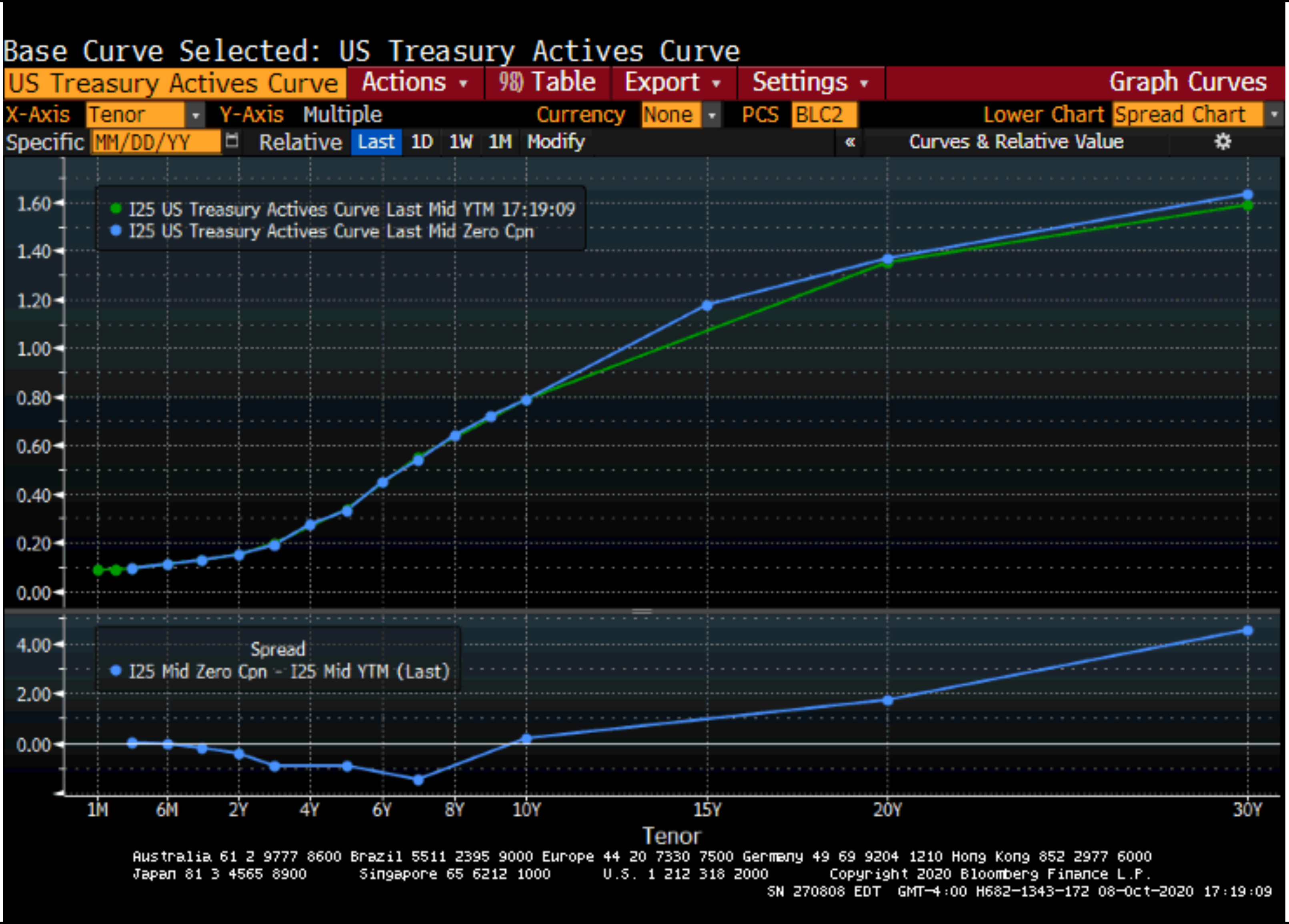

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US Treasury Actives Curve
Actions
98) Table
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Tenor
Y-Axis
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Currency
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PCS
BLC2
Lower Chart
Spread Chart
Specific
MM/DD/YY
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Last
1D
1W
1M
Modify
Curves & Relative Value
I25 US Treasury Actives Curve Last Mid YTM 17:19:09
I25 US Treasury Actives Curve Last Mid Zero Cpn
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
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Spread
I25 Mid Zero Cpn - I25 Mid YTM (Last)
4.00
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2Y
4Y
6Y
8Y
10Y
15Y
20Y
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Tenor
Australia 61 2 9777 8600 Brazil 5511 2395 9000 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 4565 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2020 Bloomberg Finance L.P.
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Bond pricing

The pricing of bonds and other interest-rate instruments presents a few new twists

- Which interest rates to use?
  - There is no longer a single "risk-free rate" to use for discounting or risk-neutral pricing
  - All risk-free bonds of various maturities are on an equal footing
- Interest rates are neither constant in time, nor deterministic
- Interest rates are not traded quantities
- Bonds and payoffs are interrelated across maturities, hence expected to be constrained by no-arbitrage principles

Bond pricing

- Idea: can we model or price all bonds, of all maturities, in a way that
  - Avoids arbitrage
  - Relies only on a fixed number of stochastic factors

- For example, is it possible to derive an arbitrage-free model of interest rates and bond prices such that the prices depend only on the **short** infinitesimal duration. **rate**, where that rate is given as an Itô process? We want

maturity of the bond

$$V = V(t, T, y_t)$$
$$\mathrm{d}y_t = a\,\mathrm{d}t + b\,\mathrm{d}B$$

- If so, the short-rate process determines prices, and the prices determine the yield curve.

Bond pricing: one-factor model

- Can we find a PDE for bond prices, say for zero-coupon bonds?
  - What are the hedging instruments?
  - How do we generalize our previous no-arbitrage derivations?
  - Use as many instruments as needed to cancel independent stochastic factors...
  - ...but because rates are not traded instruments, the solutions will not be completely determined

- One-factor model
  - All bond prices are assumed to depend on a single stochastic variable, the short rate *y*. ZCB: zero–coupon bonds
  - Use two ZCB, of different maturities, to create a dynamically re-balanced portfolio that is risk-free
  - For **each bond**, Itô's formula gives $$\mathrm{d}V_i = \left[\frac{\partial V_i}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2}\right]\mathrm{d}t + \frac{\partial V_i}{\partial y}\,\mathrm{d}y$$

Bond pricing: one-factor model

by eliminating dy

- A portfolio that (instantaneously) eliminates risk is given by

$$\pi = q_1 V_1 + q_2 V_2, \quad \frac{q_1}{q_2} = -\frac{\partial V_2/\partial y}{\partial V_1/\partial y}$$

- Then over a short interval, the portfolio earns the risk-free short rate

$$\mathrm{d}\pi = q_1\,\mathrm{d}V_1 + q_2\,\mathrm{d}V_2 = y\pi\,\mathrm{d}t,$$

earning over an infinitesimal period of time and y is the short–term interest rate

$$\sum_{i=1,2} q_i \left[\frac{\partial V_i}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2}\right]\mathrm{d}t = y\left(q_1 V_1 + q_2 V_2\right)\,\mathrm{d}t$$

- A simple choice of q enables us to separate V_1 and V_2

$$q_1 = \frac{1}{\partial V_1/\partial y}, \quad q_2 = -\frac{1}{\partial V_2/\partial y}$$

Bond pricing: one-factor model

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- Equating coefficients of dt and grouping together terms in V_1 and V_2, we get

one equation for two unknowns, v1 and v2

Because the expression on the left depends on T1, the second expression depends on T2. But these two expressions are equal to each other.

$$\boxed{\frac{\frac{\partial V_1}{\partial t}+\frac{b^2}{2}\frac{\partial^2 V_1}{\partial y^2}-yV_1}{\frac{\partial V_1}{\partial y}}=\frac{\frac{\partial V_2}{\partial t}+\frac{b^2}{2}\frac{\partial^2 V_2}{\partial y^2}-yV_2}{\frac{\partial V_2}{\partial y}}}=f(t,y)$$

f(t, y) does not depend on T1 or T2: independent of the particular bond specific (such as maturity)

time and short rate

- Since the first fraction, on the left-hand side, has all the dependence on T_1 while the second fraction has the dependence on T_2, both expressions must be **independent** of the bond's maturity.
- Therefore each side must be equal to a function *f(t,y)* of *t* and *y* alone, and **all maturities** satisfy the same PDE,

all bonds

$$\frac{\partial V_i}{\partial t}+\frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2}-yV_i-f(t,y)\frac{\partial V_i}{\partial y}=0, \qquad V_i(T,y)=1.$$

at maturity pay off $1

if y is a constant,

$$\partial V/\partial t - yV = 0,$$

$$V(t,y)=e^{(t-T)y}$$

Bond pricing: one-factor model

- We can interpret the unspecified function *f(t,y)* in terms of the **risk premium** for the bond *V*. The bond's Itô process can be written, using the PDE, as

$$\begin{aligned} \mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial y^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial y}\right)\mathrm{d}y \\ &= \left[\left(yV + f(t,y)\frac{\partial V}{\partial y}\right) + a\frac{\partial V}{\partial y}\right]\mathrm{d}t + \left[b\frac{\partial V}{\partial y}\right]\mathrm{d}B \end{aligned}$$

- Its excess return per unit of risk is

$$\frac{\mathrm{d}V - \overbrace{yV\,\mathrm{d}t}^{\text{risk–free}}}{\underbrace{b\frac{\partial V}{\partial y}}_{\text{risk}}} = \left[\boxed{\frac{a+f}{b}}\right]\mathrm{d}t + \mathrm{d}B$$

- So the d*t* coefficient represents the **market price of risk**.

Bond pricing: one-factor model

- The market price of risk is related to the function *f(t,y)* by

$$\eta \equiv \frac{a+f}{b}, \quad f = b\eta - a$$

  - It is identical for all bonds, and it represents the extra **deterministic** rate of return per unit of randomness.
  - from market prices
    Unfortunately, it is unobservable. This generally happens whenever the stochastic factor is not directly related to the price of a traded security. (Compare with derivation of PDE for stock derivatives.).
    stock price was directly observable. And we didn't have any ambiguity.
  - Common approaches are to either fit parameters from market observed prices or to make assumptions about functional form that lead to tractable solutions.

Interest rate models

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Models of the spot rate

- Ho & Lee

$$dy = \psi(t)\,\mathrm{d}t + \sigma\ \mathrm{d}B$$

  - Time-dependent, deterministic drift term
  - No-arbitrage prices
  - Yield curve fitting, e.g.,

observed market prices

$$\psi(t) = -\frac{\partial^2}{\partial t^2}\log \boxed{V_{\mathrm{mkt}}(t_0, t)} + \sigma^2 (t - t_0)$$

by picking different functional forms of psi,
we can try to fit observed market prices

- Vasicek

Ornstein–Uhlenbeck

$$dy = \alpha(\bar{y} - y)\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

  - Single-factor mean-reversion

- Hull & White

$$dy = [\alpha(\bar{y} - y) + \psi(t)]\ \mathrm{d}t + \sigma\,\mathrm{d}B$$

  - Generalizes the models above
  - Model yield curve and volatility

- Cox-Ingersoll-Ross

$$dy = \alpha(\bar{y} - y)\,\mathrm{d}t + \sigma\sqrt{y}\ \mathrm{d}B$$

Bond pricing: one-factor model

- Example: spot rate follows an Ornstein-Uhlenbeck process, and the market price of risk is constant or linear in the spot rate:
  - $\mathrm{d}y = \alpha(\bar{y} - y)\,\mathrm{d}t + \sigma\,\mathrm{d}B$
  - $\eta = c_0 + c_1 y$
  - $$-f(t,y) = a - b\eta = \alpha(\bar{y} - y) - \sigma(c_0 + c_1 y) = \alpha'(\bar{y}' - y) \qquad \begin{cases} \alpha' &= \alpha + \sigma c_1 \\ \bar{y}' &= \frac{1}{\alpha'}(\alpha\bar{y} - \sigma c_0) \end{cases}$$

- Estimation of model parameters $\{\sigma, \alpha, \bar{y}, c_0, c_1\}$ *vs.* $\{\sigma, \alpha', \bar{y}'\}$
  - Infer from short-rate dynamics
  - Infer from observed bond prices

- Apply model to new bonds, new periods, and new interest-rate derivatives

Models related to the diffusion process

interest rates, unlike stock prices, don't diffuse. They don't go all over the place. In fact, they don't vary that much at all.

- Pure diffusion not a good description of interest rates
  - Mean-reversion dynamics is one way to prevent rate from diffusing too far away

$$\mathrm{d}y = \alpha(\bar{y} - y)\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

  - Can change variables to **transform** into a related random walk model

use Ito's lemma to change from variable y to variable z

$$y = e^{-\alpha t} z,$$

$$\mathrm{d}y = -\alpha y\,\mathrm{d}t + e^{-\alpha t}\,\mathrm{d}z = \alpha(\bar{y} - y)\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

$$\implies \mathrm{d}z = e^{\alpha t}\left[(\alpha\bar{y})\,\mathrm{d}t + \sigma\,\mathrm{d}B\right]$$

if the model were purely deterministic, we would have a long–term average interest rate, and we might expect to see exponential behavior

Mean reversion models

- Integrate the SDE to find $\mathrm{d}z = e^{\alpha t}\left[(\alpha\bar{y})\,\mathrm{d}t + \sigma\,\mathrm{d}B\right]$

$$z(t) - z_0 = \bar{y}(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s}\,\mathrm{d}B_s,$$

$y = e^{-\alpha t} z,$

$$y(t) = y_0 e^{-\alpha t} + \bar{y}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s}\,\mathrm{d}B_s$$

if sigma = 0 (absence of volatility) , when t --> 0, y(t) --> y0; when t goes to infinity, y(t) --> bar(y) (mean reversion)

- The moments have smooth limits as *t* goes to infinity. The mean value relaxes exponentially to its long-term average

because mean of dB is zero and expectation is linear

$$\begin{aligned}\mathbb{E}[y(t)] &= y_0 e^{-\alpha t} + \bar{y}(1 - e^{-\alpha t}) \\ &= y_0 + (\bar{y} - y_0)\left(1 - e^{-\alpha t}\right)\end{aligned}$$

when t --> 0, y(t) --> y0; when t goes to infinity, y(t) --> bar(y) (mean reversion)

Mean reversion models

- For the variance,

$$
\begin{aligned}
\operatorname{Var}(y(t)) &= \sigma^2 e^{-2\alpha t} \int e^{\alpha s} \int e^{\alpha s'} \mathbb{E}\left[\mathrm{d}B_s \, \mathrm{d}B_{s'}\right] \\
&= \sigma^2 e^{-2\alpha t} \int e^{\alpha s} \int e^{\alpha s'} \delta(s - s') \, \mathrm{d}s \, \mathrm{d}s' \\
&= \sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha s} \, \mathrm{d}s \\
&= \frac{\sigma^2}{2\alpha} \left(1 - e^{-2\alpha t}\right) \to \frac{\sigma^2}{2\alpha}
\end{aligned}
$$

dB at two time are uncorrelated with each other, unless their times coincide

Bond pricing

- The bond pricing PDE for this 1-factor model takes the form (dropping primes)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - yV + \alpha(\bar{y} - y)\frac{\partial V}{\partial y} = 0$$

  - Try a solution for a zero-coupon bond of the form

$$V(t, y) = e^{f(t) - yg(t)}$$

    and apply boundary conditions.

  - What behaviors and term structures are possible?
  - Upward sloping? Inverted? Oscillating? ....?
  - Do they fit the market?

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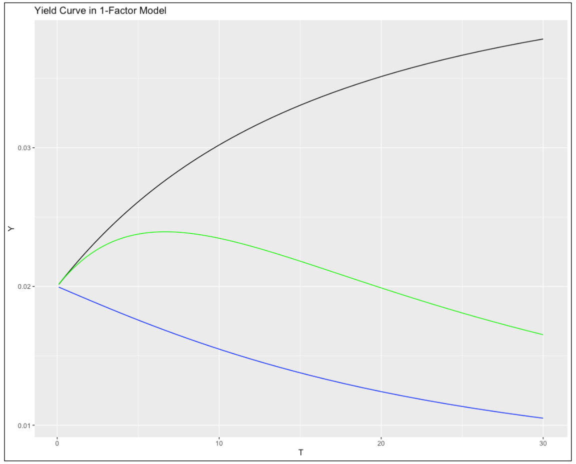


$\alpha = 0.1, \sigma = 0.01, \bar{y} = 0.05, y_0 = 0.02$
$\sigma = 0.03(\text{green}), \bar{y} = 0.01(\text{blue})$

moment of general normal distribution

$$\mathrm{E}\left[X^k\right] = \mathrm{E}\left[(\mu+\sigma Z)^k\right] = \sum_{m=0}^{k} \binom{k}{m} \mu^m \sigma^{k-m} \mathrm{E}\left[Z^{k-m}\right]$$

$$\mathrm{E}\left[Z^{2m}\right] = \frac{(2m)!}{2^m m!}$$

$$\mathrm{E}\left[X^4\right] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

Solving PDEs

moment of standard normal distribution

$$M_X(t) = \mathrm{E}\left[e^{tX}\right] = \int_{x=-\infty}^{\infty} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{x=-\infty}^{\infty} \frac{e^{-\left(x^2-2tx+t^2\right)/2} e^{t^2/2}}{\sqrt{2\pi}} dx = e^{t^2/2} \int_{x=-\infty}^{\infty} \frac{e^{-(x-t)^2/2}}{\sqrt{2\pi}} dx.$$

$$M_X(t) = e^{t^2/2}.$$

$$\mathrm{E}\left[X^k\right] = \left[\frac{d^k M_X(t)}{dt^k}\right]_{t=0}$$

$$\mathrm{E}[X] = 0$$
$$\mathrm{E}\left[X^2\right] = 1$$
$$\mathrm{E}\left[X^3\right] = 0$$
$$\mathrm{E}\left[X^4\right] = 3$$

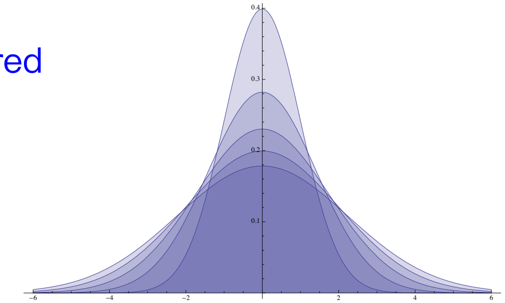
Finance at MIT
Where ingenuity drives results

Preview of some key results

- The Black-Scholes PDE is closely related to the diffusion equation, whose solution is Gaussian

t plays the role of sigma squared

$$\boxed{\frac{\partial p_0}{\partial t} = \frac{1}{2}\frac{\partial^2 p_0}{\partial z^2}} \qquad \boxed{p_0(z,t) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}}$$



- The Gaussian solution can be used to price derivatives with **any** terminal payoff:

$$V(S,t) = \int p(S_T, T; S, t) \boxed{V(S_T, T)} \mathrm{d}S_T$$

integration variable is with respect to St over all possible future terminal values of the stock price

- The same equation can be used to compute default probabilities, price exotic options, and describe random walks in the presence of barriers.

Changing variables: 3 easy steps

- Write derivative in terms of its forward or future value

express things in terms of its forward or future value

$$V(S,t) = e^{-r(T-t)} U(S,t) \implies \frac{\partial U}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} = 0$$

- Substitute "backward" variable for time evolution and replace price with its logarithm:

$$\tau \equiv T - t, \quad \xi \equiv \log S, \quad S = e^{\xi}, \quad \xi \in [-\infty, \infty] \implies \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial \xi^2} - \left(r - \frac{\sigma^2}{2}\right) \frac{\partial U}{\partial \xi} = 0$$

- Finally, clean up last term by defining

$$x \equiv \xi + (r - \sigma^2/2)\tau = \log S + (r - \sigma^2/2)\tau \implies \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial x^2} = 0$$

Diffusion equation

- A solution to the diffusion equation will give a solution to Black-Scholes via (reversing) substitution of variables
  - **Terminal** values of Black-Scholes (i.e., the payoff functions) correspond to **initial** values of the diffusion equation
  - A special solution:

$$U(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-x^2/(2\sigma^2\tau)} \qquad \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} = 0$$

  - Verify that it satisfies equation:

$$\eta = \tau\sigma^2 \qquad \frac{\partial U}{\partial \eta} - \frac{1}{2}\frac{\partial^2 U}{\partial x^2} = 0$$

$$\frac{\partial U}{\partial x} = \left(-\frac{x}{\sigma^2\tau}\right)U, \quad \frac{\partial^2 U}{\partial x^2} = \left(-\frac{1}{\sigma^2\tau} + \frac{x^2}{\sigma^4\tau^2}\right)U, \quad \frac{\partial U}{\partial \tau} = \left(-\frac{1}{2\tau} + \frac{x^2}{2\sigma^2\tau^2}\right)U = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2}$$

Diffusion equation, random walks, and probability

This **special** solution can be used to obtain the **general** solution:

$$\text{If } p(z,t=0)=f(z), \text{ then general solution given by}$$

expectation of f(z') on Normal(z,t)

$$p(z,t)=\int \boxed{p_0(z-z',t)} f(z')\mathrm{d}z' = \frac{1}{\sqrt{2\pi t}}\int e^{-(z-z')^2/2t} f(z')\mathrm{d}z'$$

kernel function

Examples:

$$f(z)=z^2$$ p(z,t)=z^2+t

$$f(z)=e^{az}$$

$$f(z)=\cos(\lambda z)$$

$$f(z)=\theta(z-\kappa)=\begin{cases}1, & z>\kappa\\ 0, & z<\kappa\end{cases}$$

$$\frac{\partial p}{\partial t}-\frac{1}{2}\frac{\partial^2 p}{\partial z^2}=0$$
$$p(z,T)=z^3$$
$$\tau=t-T$$
$$\frac{\partial p}{\partial \tau}-\frac{1}{2}\frac{\partial^2 p}{\partial z^2}=0$$
$$p(z,0)=z^3$$
$$p(z,\tau)=z^3+3z\tau$$
$$p(z,t)=z^3+3z(t-T)$$

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2}=0$$
$$\tau=\sigma^2(T-t)$$
$$\frac{\partial V}{\partial \tau}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}=0$$
$$V(S,\tau=0)=\begin{cases}1 & S<K\\ 0 & S>K\end{cases}$$
$$V(S,\tau)=\int p_0(S-w,\tau)V(w,\tau=0)\mathrm{d}w$$
$$V(S,\tau)=\frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{K} e^{-(S-w)^2/2\tau}\mathrm{d}w$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(K-S)/\sqrt{\tau}} e^{-u^2/2}\,\mathrm{d}u$$
$$=\Phi\left(\frac{K-S}{\sqrt{\tau}}\right)$$
$$V(S,t)=\Phi\left(\frac{K-S}{\sigma\sqrt{T-t}}\right)$$