15.455x Mathematical Methods of Quantitative Finance

Week 9: Optimal decision making and optimal strategies

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Finance at MIT

Where ingenuity drives results

Calculus of variations

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Calculus of variations

Bellman's principle of optimality originates in a classic problem in continuous optimization, the **principle of least action**.

- Let the Lagrangian *L* be a function of *q(t)* and its time derivative.
- Minimize its integral **over all possible paths**, subject to the **constraint** that all paths have the same initial and terminal values.

q: the amount you have in inventory

function of different functions q

$$L = L(q, \dot{q}), \quad S = \int_0^T L(q, \dot{q})\ \mathrm{d}t$$

we want to solve for a function of time q(t), not a single point

Vary q as $q(t) = q(t, \alpha) = q(t, 0) + \alpha x(t)$,

with $x(0) = X(T) = 0$. Then

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}\alpha} &= \int \left[\frac{\partial L}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha}\right] \mathrm{d}t \\ &= \int \left[\frac{\partial L}{\partial q} x(t) + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t} x(t)\right] \mathrm{d}t \\ &= \int \boxed{\left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]} x(t)\ \mathrm{d}t \\ &\quad + \left[\frac{\partial L}{\partial \dot{q}} x(t)\right]_{t=0}^{t=T} = 0\end{aligned}$$

the only way the integral is 0 for any x(t) is that the quantity of red square vanishes

Calculus of variations

- Euler-Lagrange equations for *q(t)*:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- Example: Newton's second law *F* = *ma*

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q), \quad \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \quad \frac{\partial L}{\partial q} = -V'(q),$$

$$m\ddot{q} = -\underbrace{V'(q)}_{\text{force}}.$$

$$\text{Harmonic oscillator: } V(q) = \frac{1}{2}kq^2, \quad F = -V'(q) = -kq,$$

special case of V(q)

$$m\ddot{q} = -kq, \quad q(t) = A\sin(\omega t + B), \quad \omega = \sqrt{k/m}$$

Calculus of variations

- Example: Almgren & Chriss. sell fast is bad sell slow is bad
  - Key idea: penalize both **trading speed** (market impact) and **inventory** (market risk)
  - Approximate the discrete sum by an integral

$$U = \frac{\tilde{\eta}}{\Delta t} \sum (\Delta q_k)^2 + \lambda \sigma^2 (\Delta t) \sum q_k^2 + \text{const.} \implies S = \frac{1}{2} \int \left[\tilde{\eta} \dot{q}^2 + \lambda \sigma^2 q^2 \right] \, \mathrm{d}t$$

  - Similar to harmonic oscillator with wrong-sign potential.

$$\eta \ddot{q} = \lambda \sigma^2 q \implies q(t) = e^{\pm \kappa t}, \quad \kappa = \sqrt{\lambda \sigma^2 / \tilde{\eta}}$$

$$q(0) = Q,\ q(T) = 0 \implies q(t) = Q \frac{\sinh \kappa (T - t)}{\sinh \kappa T}$$

Dynamic programming

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Dynamic programming

- Multiperiod optimization: find **sequence of decisions** over future horizons which minimizes expected **cost** function.
  - Finite and infinite (or indefinite)
  - Optimal solutions may require incurring short-term losses to obtain best long-term return.

- Dynamic programming is a technique applicable to problems with **overlapping subproblems** of identical form, amenable to recursive formulation.
- Example: portfolio rebalancing
  - Is optimal long-term policy to iterative optimal short-term policy?
  - As prices evolve, weights depart from initial optimality – even if return forecasts are unchanged
  - Rebalancing requires paying transaction costs; non-trading incurs suboptimality costs

Recursive structure: a simple example

Find the number of **minimum-length paths** in this graph.
with same length

- Let N denote number of paths from origin.
- 1-step relationship:

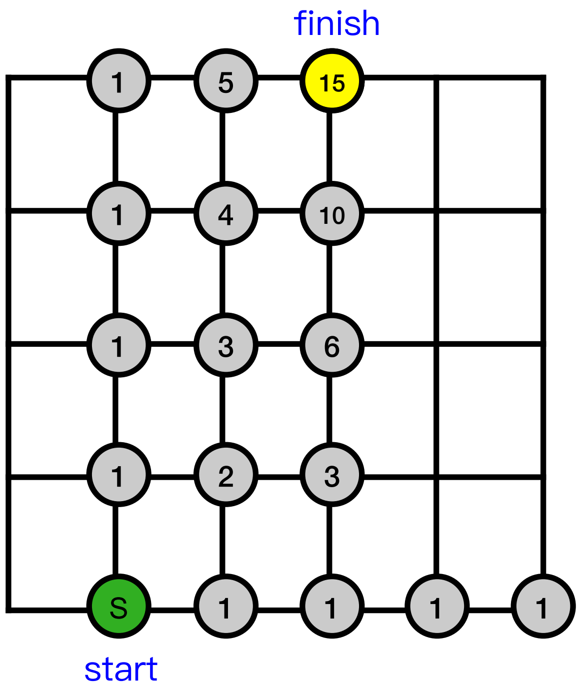
$$N(i,j) = N(i,j-1) + N(i-1,j)$$

- Boundary condition:

$$N(i,0) = N(0,j) = 1$$

Procedure:

- Label boundary with N=1 at each node
- Move to interior, with each node assigned sum of values from incoming nodes



Don't be greedy

- **Goal:** Find the path from top to bottom with greatest sum.

9

3 3

2 8 10

4 11 5 1

2 6 11 12 6

Don't be greedy

- **Goal:** Find the path from top to bottom with greatest sum.
- **"Greedy"** algorithm:
  - At decision point, take **locally** optimal next step.
  - Sum = 39

9

2 3

2 8 10

4 11 5 1

2 6 11 12 6

Don't be greedy

short–term optimal that lead us to no longer having access to states that might be much more profitable in the future.

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- **Goal:** Find the path from top to bottom with greatest sum.
- **Solution:**
  - At boundary, add value of node above
  - At interior point, choose greater of two nearest ancestors and add.
  - Find best terminal value and climb back up tree.
- Sum = 42

9
2 3
2 8 10
4 11 5 1
2 6 11 12 6

9
11 12
13 20 22
17 31 27 23
19 37 $\boxed{42}$ 39 29

Aspects of dynamical programming problems

- State variables
- Actions, control
- Stochastic evolution
  - Markov process
  - Effects of actions and environment are **fully known**
  - Boundary conditions
- Rewards, costs, discounting (tradeoff: short vs. long-term cost)
- **Optimal policy**: action/decision rules for long-term objective

Setup for Markov decision processes

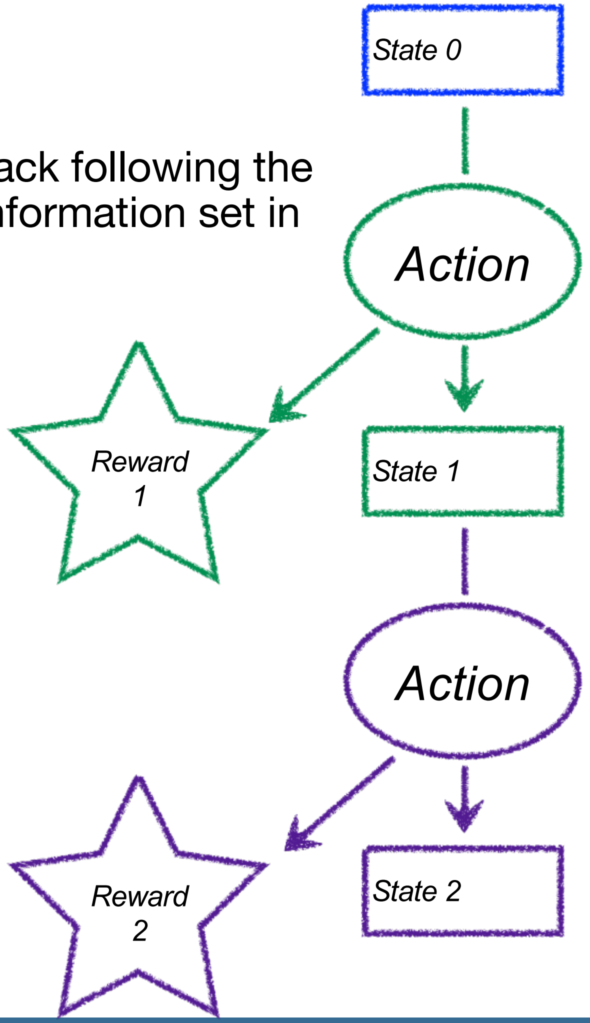
- States
- Actions
- Rewards
- Goal
- Values
- Performance

Definitions and notation:

- States $s \in S_t$: states available at time t.
- Actions $a \in \mathcal{A}(S_t)$: actions available from states at time t.
- Transition probabilities $p(s'|s,a)$: probability to end in state s' after taking action a from initial state s.
- Rewards $r(s,a,s')$: reward (distribution) resulting from taking action a from initial state s leading to successor state s'.
- Policy $\pi(a|s)$: probability (distribution) for action a to be selected, given initial state s.
- Value function $V_\pi(s)$: value of being in state s under policy π.
- State-action value function $q_\pi(s,a)$: value of taking action a under policy π from a given state s.

Rewards

- Rewards and punishments, received as feedback following the implementation of each action, form the key information set in determining policy
- Evolution may be **stochastic**
  - Randomness in **environment**
  - Randomness in **policy**
- Examples:
  - 1-period return on security holding
  - Penalty for breaching risk limit



Value functions

- Value functions store current estimates of the benefits of taking an action while in a specific state while following a given policy.

- **State values**: *v(s)* value of being in state *s.*

$$v_\pi(s) \equiv \mathbb{E}_\pi[G_t | S_t = s], \quad G_t = \sum_{i=0}^{\infty} \gamma^i R_{t+1+i}$$

- **State-action values**: *q(s,a)* value of taking action *a* when starting from state *s.*

$$q_\pi(s,a) \equiv \mathbb{E}_\pi[G_t | S_t = s, A_t = a],$$

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$$

- Value functions satisfy MDP recursion relations

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) \left[r(s,a,s') + \gamma v_\pi(s') \right]$$

Recursion and optimality

- Value functions satisfy MDP recursion relations, relating values to successor and predecessor actions

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \left[r(s, a, s') + \gamma v_\pi(s')\right]$$

- In reinforcement learning, neither the probabilities nor the future values are known.
- Policy evaluation: given a policy, solve for the value function
- Policy optimization: since value functions give an ordering over policies, select the policy which has the highest value

$$v_*(s) = \max_\pi v_\pi(s), \quad \forall s \in \mathcal{S}$$

$$q_*(s, a) = \max_\pi q_\pi(s, a), \quad \forall s \in \mathcal{S}, \forall a \in \mathcal{A}$$

Example: Execution costs (Bertsimas & Lo)

- Find the **optimal trading policy** to buy a quantity *Q* of securities over a time horizon *T*; that is, find the **best** sequence of *q*'s that satisfies boundary conditions.

$$\min_{\{q_s\}} \mathrm{E}\left[\sum_{t=1}^{T} q_t \underset{\text{price}}{p_t}\right], \qquad \sum_{t=1}^{T} q_t = Q$$

- Objective function: minimize **expected execution cost**

Example: Execution costs (Bertsimas & Lo)

- Each trade produces a **permanent price impact** proportional to the size of the trade.

market impact term

$$\boxed{p_t = p_{t-1} + \epsilon_t + \theta q_t}$$

- Theta: cost coefficient (fixed)
- Epsilon: zero mean noise process
  - Arithmetic Brownian motion in absence of trading
- *p, W*: **state variables** – reflect information already known
- *q*: **control variable** – freely chosen; goal to find optimal sequence

$$W_t = Q - \sum_{s=1}^{t-1} q_s$$

Dynamic programming principle

- Goal: find a sequence of *q*'s that minimizes expected cost.
- Dynamic programming principle: frame as a **recursive optimization** so that the optimal sequence, **at each step**, is **also optimal** for all the remaining shares.
- Optimal policy satisfies (at each time step)

number of shares we have available that we need to move

$$V_t(p_{t-1}, W_t) = \min_{q_t} \mathbb{E}_t\left[q_t p_t + V_{t+1}(p_t, W_{t+1})\right]$$

price available

Recursive solution

- Start from **terminal condition** and work backwards
- In the last period, there is no minimization to do since **there is no choice** but to trade any remaining shares: $q_T = W_T$
- Therefore the terminal condition is

we don't have any choice. It's fixed

$$
\begin{aligned}
V_T(p_{T-1}, W_T) &= \min_{q_T} \mathbb{E}_T[q_T p_T] = \mathbb{E}_T\left[q_T(p_{T-1} + \epsilon_T + \theta q_T)\right] \\
&= q_T(p_{T-1} + \theta q_T) \\
&= W_T(p_{T-1} + \theta W_T)
\end{aligned}
$$

- Explicit **formula for** V in terms of state variables **known** at T.

Recursive solution

- For the next-to-last period, substitute this result into recursion

$$
\begin{aligned}
V_{T-1}(p_{T-2}, W_{T-1}) &= \min_{q_{T-1}} \mathbb{E}_{T-1}[q_{T-1}p_{T-1} + V_T(p_{T-1}, W_T)] \\
&= \min_{q_{T-1}} \mathbb{E}_{T-1}[q_{T-1}p_{T-1} + W_T(p_{T-1} + \theta W_T)] \\
&= \min_{q_{T-1}} \mathbb{E}_{T-1}[(q_{T-1} + W_T)p_{T-1} + \theta W_T^2]
\end{aligned}
$$

- Three steps forward with from this expression:
  - **Compute** the expectation
  - **Minimize** result over all possible q_{T-1}
  - Express result in terms of state variables **known at time *T-1***

Recursive solution

- Eliminate late-time state variables using their recursion relations

$$\begin{aligned} p_{T-1} &= p_{T-2} + \theta q_{T-1} + \epsilon_{T-1}, \\ W_T &= W_{T-1} - q_{T-1} \end{aligned}$$

- Substituting, we obtain an expression **quadratic** in q_{T-1}

$$\begin{aligned} V_{T-1}(p_{T-2}, W_{T-1}) &= \min_{q_{T-1}} \mathbb{E}_{T-1}[W_{T-1}(p_{T-2} + \theta q_{T-1} + \epsilon_{T-1}) + \theta(W_{T-1} - q_{T-1})^2] \\ &= \min_{q_{T-1}} \left(W_{T-1}(p_{T-2} + \theta q_{T-1}) + \theta(W_{T-1} - q_{T-1})^2\right) \\ &= \min_{q_{T-1}} \left(\theta(q_{T-1}^2 - W_{T-1} q_{T-1}) + \text{constant}\right) \end{aligned}$$

Recursive solution

- Minimizing the quadratic function gives $\widehat{q}_{T-1} = W_{T-1}/2$
- Evaluating the function **at the minimum** gives

$$
\begin{aligned}
V_{T-1}(p_{T-2}, W_{T-1}) &= W_{T-1}\left(p_{T-2} + \theta(W_{T-1}/2)\right) + \frac{1}{4}\theta W_{T-1}^2 \\
&= W_{T-1}\left(p_{T-2} + \frac{3}{4}\theta W_{T-1}\right)
\end{aligned}
$$

- This is the desired closed-form expression for *T-1*.

Recursive solution

- Now proceed one more step and substitute $W_{T-1} = W_{T-2} - q_{T-2}$ to get

$$V_{T-2}(p_{T-3}, W_{T-2})$$
$$= \min_{q_{T-2}} \mathbb{E}_{T-2}\left[W_{T-2}(p_{T-3} + \theta q_{T-2} + \epsilon_{T-2}) + \frac{3}{4}\theta(W_{T-2} - q_{T-2})^2\right]$$
$$= \min_{q_{T-2}} \left(W_{T-2}(p_{T-3} + \theta q_{T-2}) + \frac{3}{4}\theta(W_{T-2} - q_{T-2})^2\right)$$

- This is also a quadratic function, with minimum value at

$$\widehat{q}_{T-2} = W_{T-2}/3$$

Recursive solution

- Substituting,

$$
\begin{aligned}
V_{T-2}(p_{T-3}, W_{T-2}) &= W_{T-2}\left(p_{T-2} + \theta(W_{T-2}/3)\right) + \frac{1}{3}\theta W_{T-2}^2 \\
&= W_{T-2}\left(p_{T-3} + \frac{2}{3}\theta W_{T-2}\right)
\end{aligned}
$$

Recursive solution

- Continuing this way, step-by-step, we find at *T-k* that

$$\widehat{q}_{T-k} = W_{T-k}/(k+1)$$

while

$$V_{T-k}(p_{T-k-1}, W_{T-k}) = W_{T-k}\left(p_{T-k-1} + \frac{k+2}{2k+2}\theta W_{T-k}\right)$$

Recursive solution

- This continues all the way to the beginning, for which *k = T-1*,

$$\widehat{q}_1 = W_1/T = Q/T,$$
$$V_1(p_0, W_1) = W_1\left(p_0 + \frac{T+1}{2T}\theta W_1\right)$$

- Therefore the **optimal first trade size** is *1/T* of the total *Q.*
- The optimal **second** trade size is *1/(T-1)* of the **remainder**, so...
- The optimal solution in general is to divide *Q* into **equal slices**:

$$\boxed{q_t = Q/T}$$

Recursive solution

- The expected cost at the initial time is therefore

$$
\begin{aligned}
V_1(p_0, W_1) &= \mathbb{E}\left[\sum q_t p_t\right] = \mathbb{E}\left[\frac{Q}{T}\sum p_t\right] = \frac{Q}{T}\sum \mathbb{E}[p_t] \\
&= \frac{Q}{T}\left(Tp_0 + \theta\left(\frac{Q}{T}\right)(1+2+\cdots+T)\right) \\
&= Qp_0 + \theta\left(\frac{Q}{T}\right)^2 \frac{T(T+1)}{2} \\
&= Qp_0 + \frac{\theta}{2}Q^2(1+1/T)
\end{aligned}
$$

acquisition cost = mark to market cost + deviation

Frictionless expected cost

Expected transaction costs

Solution for optimal execution

- Expected transaction cost is **finite**, even if trade is spread over a large number of intervals

$$\mathcal{C} = \frac{\theta}{2} Q^2 (1 + 1/T), \qquad \lim_{T \to \infty} \mathcal{C} = \frac{\theta}{2} Q^2$$

- Compare with alternative of trading all shares in initial period:

$$V_1(p_0, W_1) = \mathbb{E}\left[q_1 p_1\right] = Q(p_0 + \theta Q), \qquad q_t = \begin{cases} Q & t = 1, \\ 0 & t > 1 \end{cases}$$

$$= Qp_0 + \theta Q^2,$$

$$\mathcal{C} = \theta Q^2$$

Dynamic programming and optimal execution

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Summary

- Cost of optimal execution is half the cost of immediate execution.
- Cost is **quadratic** in total trade quantity
  - Small for small *Q*
- Impact is permanent: action of each trade affects all subsequent trades
- Analysis did not account for: temporary impact, opportunity cost of delay, volatility of final value, risk preferences

we're buying shares because we have a signal. We know we have good forecast for the future price. We think that the value is going to go up. And the longer we wait, yeah, we might get a lower cost. But we might lose out of the expected gain in the returns.

Optimal trading strategies

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Optimal trading strategies

- Strategic trader models designed to elucidate
  - Market structure
  - Spreads
  - Liquidity
  - Price discovery

- A useful strategy for a trader or portfolio manager should
  - Minimize costs
  - Minimize risks
  - Maximize utility

- Approach
  - Minimize cost for given risk, dependent on risk preferences

Strategy variables

- Goal of trading program is to sell (or buy) a block of ***Q* shares in time *T*.**
- Subdivide total **trade horizon** into N subperiods of length $\Delta t = T/N$.
- Trade at times $t_k = k\Delta t,\ k=0,\ldots,N$.
- Let q_k be holdings at time k, with boundary conditions $q_0=Q,\ q_N=0$.
- Let $\Delta q_k = q_{k-1} - q_k$ be quantity sold in k^{th} period
- A **strategy** is an algorithm for assigning valid values to Δq_k based on **information available** up to t_{k-1}.

Price process during trading

- **Drift**: the normal rate increase or decrease, absent trading
  - Assume zero over trade horizon for first approximation
  - Inclusion important for capturing **opportunity costs** of informed trading

- **Diffusion**: volatility creates **uncertainty** in final execution price

- **Market impact**: change in price due to trading
  - Assume function of average **trading speed**, $\Delta q_k / \Delta t$
  - Permanent: offset to price observed at all future times (at least up to T)
  - Temporary: affects **price received** during a given period but is not part of subsequent price observations.

Price process during trading

Market impact: change in price due to trading

- Permanent: offset to price observed at all future times (at least up to *T*)

$$p_k = p_{k-1} + \sigma z_k (\Delta t)^{1/2} - g \overbrace{\left(\frac{\Delta q_k}{\Delta t} \right)}^{\text{speed}} \overbrace{\Delta t}^{\text{duration}}$$

Arithmetic Brownian motion

Permanent market impact

- Temporary: affects **price received** during a given period but is not part of subsequent price observations.

$$\tilde{p}_k = p_{k-1} - h \left(\frac{\Delta q_k}{\Delta t} \right)$$

Temporary impact function

Trading costs

The total trading cost, or **implementation shortfall**, is the difference between initial value and the actual final proceeds of trading

Diffusion

Permanent impact

$$
\begin{aligned}
\mathcal{C} = Q(p_0 - \bar{p}_{\text{eff}}) &= Qp_0 - \sum_{k=1}^{N} (\Delta q_k)\tilde{p}_k \\
&= \sum q_k \left[\sigma z_k (\Delta t)^{1/2} - g\left(\frac{\Delta q_k}{\Delta t}\right) \Delta t \right] \\
&\quad - \sum (\Delta q_k) h\left(\frac{\Delta q_k}{\Delta t}\right)
\end{aligned}
$$

Period impact

Implementation shortfall

while you sell, price drop, you can not earn at initial price

- Lower effective price received, measured ex-post.
  - Effective price can actually be greater, due to diffusion random variable realized
- All trades are assumed to be executed and trade list filled, regardless of price movement
  - Don't consider unfilled limit orders. market orders vs limit orders
- Before trading, realized effective price is a random variable. So optimize cost, variance – or both.

$$
\begin{aligned}
\mathrm{E}\left[\mathcal{C}\right] &= Q(p_0 - \bar{p}_{\mathrm{eff}}) \\
&= \Delta t \sum q_k g\left(\frac{\Delta q_k}{\Delta t}\right) + \sum (\Delta q_k) h\left(\frac{\Delta q_k}{\Delta t}\right) \\
\mathrm{Var}[\mathcal{C}] &= (\sigma^2 \Delta t) \sum q_k^2
\end{aligned}
$$

Linear impact functions

- Let $g(v) = \gamma v$
  - Final, permanent price changes depend on total *Q*
  - Early trades push down price of later trades

$$p_k = p_0 + \sigma(\Delta t)^{1/2} \sum z_k - \gamma(Q - q_k)$$

The more we trade early, the more it hurts us later on. If we trade more slowly at the beginning, there's less impact on the later trades, but there will be greater variance.

Permanent impact

Contribution to expected shortfall from permanent impact function is

$$\Delta t \sum q_k g\left(\frac{\Delta q_k}{\Delta t}\right) = \gamma \sum q_k \Delta q_k$$

$$= \gamma \sum q_k (q_{k-1} - q_k)$$

$$= \frac{1}{2}\gamma Q^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} (\Delta q_k)^2$$

$$\sum_{k=1}^{N} (\Delta q_k)^2 = \sum_{k=1}^{N} (q_{k-1} - q_k)^2$$

$$= \sum q_k^2 + \sum q_{k-1}^2 - 2\sum q_{k-1} q_k$$

$$= q_0^2 + 2\sum q_k^2 - 2\sum q_{k-1} q_k$$

$$= Q^2 - 2\sum q_k (q_{k-1} - q_k)$$

Temporary impact

- Let the temporary impact function be linear as well

$$h\left(\frac{\Delta q_k}{\Delta t}\right) = c_1 \operatorname{sign}(\Delta q_k) + \frac{\eta}{\Delta t}\Delta q_k$$

- Constant coefficient reflects fixed costs plus bid/ask spread

- The contribution to expected cost during a period is **quadratic**:

never any positive benefit to trading. If you're a market maker and you are providing liquidity to the markets , then you can make money on your trades. We're going to assume that that's not the case here.

$$(\Delta q_k) h\left(\frac{\Delta q_k}{\Delta t}\right) = c_1|\Delta q_k| + \frac{\eta}{\Delta t}(\Delta q_k)^2$$

lose money regardless of whether we're buying or selling

$$= c_1|\Delta q_k| + \frac{1}{2}c_2(\Delta q_k)^2, \quad \eta = \frac{1}{2}c_2\Delta t$$

Expected shortfall under linear impact

object: decrease market impact (cost)

- Combining contributions,

$$\mathrm{E}\left[\mathcal{C}\right] = \frac{1}{2}\gamma Q^2 + c_1 \sum |\Delta q_k| + \frac{1}{2}(c_2 - \gamma) \sum (\Delta q_k)^2$$

- If all trades have the same sign (i.e., no buying back during a sell program), then

$$\mathrm{E}\left[\mathcal{C}\right] = \underbrace{\left(\frac{1}{2}\gamma Q^2 + c_1 Q\right)}_{\textit{Constant}} + \underbrace{\frac{1}{2}(c_2 - \gamma) \sum (\Delta q_k)^2}_{\textit{Sign depends on parameter values}}$$

Case: minimum impact strategy

- The minimum cost is obtained by dividing trades into equal sizes:

$$q_k = \frac{Q}{N}(N-k), \quad \Delta q_k = \frac{Q}{N}$$

- This linear trajectory minimizes the sum of squares, so

$$\begin{aligned} \mathrm{E}\left[\mathcal{C}\right] &= \left(\frac{1}{2}\gamma Q^2 + c_1 Q\right) + \frac{1}{2}(c_2 - \gamma)Q^2/N \\ &= c_1 Q + \frac{1}{2}Q^2\left(\frac{c_2}{N} + \gamma\left(1 - \frac{1}{N}\right)\right) \end{aligned}$$

Case: minimum impact strategy

- The variance can be large

$$
\begin{aligned}
\mathrm{Var}[\mathcal{C}] &= \sigma^2(\Delta t)\sum_{k=1}^{N} q_k^2 \\
&= \sigma^2 \frac{T}{N}\frac{Q^2}{N^2}\left(\sum_{\ell=1}^{N-1}\ell^2 = \frac{(N-1)N(2N-1)}{6}\right) \\
&= \frac{1}{3}\sigma^2 Q^2 T\left(1-\frac{1}{N}\right)\left(1-\frac{1}{2N}\right)
\end{aligned}
$$

Case: minimum impact strategy

- Large-*N* limit

$$\mathrm{E}\left[\mathcal{C}\right] \to c_1 Q + \frac{1}{2}\gamma Q^2$$

$$\mathrm{Var}[\mathcal{C}] \to \frac{1}{3}\sigma^2 Q^2 T$$

Case: minimum variance strategy

- For linear impact, the variance will be zero (the minimum) if the stock is sold all at once

$$\mathrm{E}\left[\mathcal{C}\right] = c_1 Q + \frac{1}{2} c_2 Q^2$$
$$\mathrm{Var}[\mathcal{C}] = 0$$

- Expect cost impact will be large

Optimal strategies

- In general, the optimal strategy depends on the impact functions and the trader **risk aversion**.

- Minimize the expected cost with a penalty for risk

$$U = \mathrm{E}\left[\mathcal{C}\right] + \lambda \mathrm{Var}[\mathcal{C}]$$

- The risk parameter measures the marginal value of exchanging higher costs for lower risk, and vice versa
  - Minimum impact corresponds to $\lambda \to 0$.
  - Minimum variance corresponds to $\lambda \to \infty$.

lambda

- The one-parameter family of optimal strategies forms the **efficient trading frontier**.

Optimal strategies

- We can solve for closed form solution in the case of linear impact functions *g* and *h.*

$$U = \frac{\tilde{\eta}}{\Delta t} \sum (\Delta q_k)^2 + \lambda \sigma^2 (\Delta t) \sum q_k^2 + \text{const.}$$

- Minimizing *U,*

$$\frac{\partial U}{\partial q_i} = 2\Delta t \left(\lambda \sigma^2 q_i - \frac{\tilde{\eta}}{(\Delta t)^2} (q_{i-1} - 2q_i + q_{i+1}) \right) = 0$$

Optimal strategies

- This is a discrete difference equation
  - In continuous time, a second-order differential equation

$$\frac{q_{i-1} - 2q_i + q_{i+1}}{(\Delta t)^2} = \left(\frac{\lambda\sigma^2}{\tilde{\eta}}\right) q_i$$

  - Solution with boundary conditions $q_0 = Q, \quad q_N = 0$

$$q_i = \frac{\sinh(\kappa(N - i)\Delta t)}{\sinh(\kappa T)} Q$$

The parameter κ is defined by

$$\cosh(\kappa\Delta t) = 1 + \frac{(\Delta t)^2}{2}\left(\frac{\lambda\sigma^2}{\tilde{\eta}}\right)$$

Results

- **Trajectories** defined for each period and λ.
- Time scale:
  - Note that 1/κ defines the lifetime of the trading trajectory.
  - If it is much less than *T*, then the trade schedule will complete early, on **its own time scale**.
- **Extensions**:
  - Drift & information
  - General impact functions
  - Multiple securities
  - Correlation

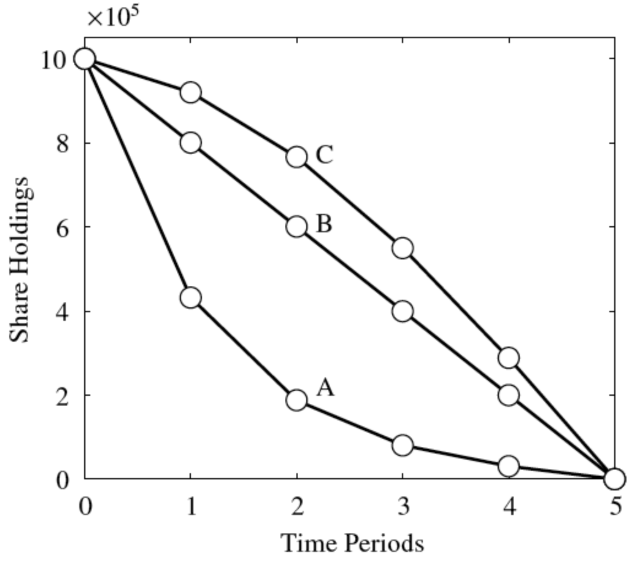

FIGURE 2. Optimal trajectories. The trajectories corresponding to the points shown in Figure 1. A: $\lambda = 2 \times 10^{-6}$, B: $\lambda = 0$, C: $\lambda = -2 \times 10^{-7}$.

A: risk aversion B: risk–neutral C: risk preference

Source: Almgren & Chriss (2000)

balance: fast (more market impact/less uncertainty) VS slow (less market impact/more uncertainty)

Efficient trading frontier

- The **efficient frontier** defines the least-cost strategies for given level of risk.

  equal slicing strategy
- The "naive" strategy at point *B* has lowest cost.
- Since curve is flat near minimum, tradeoffs are valuable moving to the left.

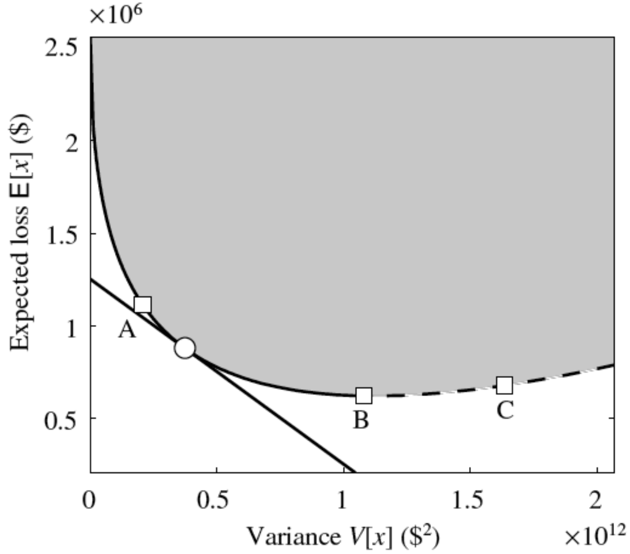

FIGURE 1. The efficient frontier. The parameters are as in Table 1. The shaded region is the set of variances and expectations attainable by some time-dependent strategy. The solid curve is the efficient frontier; the dashed curve is strategies that have higher variance for the same expected costs. Point B is the ''naïve'' strategy, minimizing expected cost without regard to variance. The straight line illustrates selection of a specific optimal strategy for $\lambda = 10^{-6}$. Points A, B, C are strategies illustrated in Figure 2.

Source: Almgren & Chriss (2000)

References and further reading

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