Exam: (Numbering) - Ans2 Cal6- Unit – Delete EXCEL Permanently

Math, Probability, Statistics, Review of Linear Algebra:

$\int u dv = uv - \int v du ; \int a^{kx}\, dx = \frac{a^{kx}}{k * \ln(a)} + C; \int x e^{cx} dx = e^{cx}(\frac{cx-1}{c});$

$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} + \text{erf}(x) + C; -\frac{dI(a)}{da} = \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}};$

$Var\ (X + a) = Var\ (X); Var\ (aX) = a^2 Var(X); Var[\sum X_i] = \sum Var[X_i] + \sum_{i=1}^{N}\sum_{i\neq j}^{N} Cov[X_i, X_j]; Var[X_1 + X_2] = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2);$ $Cov(X, a) = 0; Cov(aX + bY, cW + dV) = acCov(X, W) + adCov(X, V) + \cdots;$

$Cov(X, X) = Var(X) = \sigma^2 = E[X^2] - E^2[X]; = E[(X-\mu)^2]$

$Cov(X, Y) = E[(X - \bar{X})(Y - \bar{Y})] = E(X * Y) - E(X) * E(Y); E[aX] = aE[X]$

$Corr: \rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \in [-1, +1]; E[(X + Y)^2] = E[X^2] + E[Y^2] + 2E[X * Y];$

$\mu_p = \Sigma w_i \mu_i; \sigma_p^2 = \Sigma w_i^2 Var(R_i) + 2\Sigma w_i w_j Cov(R_i, R_j) = \Sigma w_i^2 \sigma_i^2 + 2\sum_{i<j}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$

$Arithmetic: S_n = \frac{n}{2}[2a + (n-1)d]; eg: 1 + 2 + \cdots + n = n(n+1)/2$

$Geometric: S = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, for\ |r| < 1, S_n = \begin{cases} a\frac{1-r^n}{1-r}, r \neq 1 \\ an,\ \ r = 1 \end{cases}$

$(x+y)^3 = x^3 + y^3 + 3x^2y + 3xy^2;\ (x+y)^4 = x^4 + y^4 + 4x^3y + 4xy^3 + 6x^2y^2$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots;\ f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots$

Second-order ODE solving: $ay'' + by' + cy = 0 \to y(t) = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} \to plug\ in: y = e^{at} \to \alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Partial: $U = f(x(t), y(t)) \to \frac{\partial U}{\partial t} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t}; cos' = -sin, ; sin' = cos;$

L'Hospital's Rule: $if\ \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} or \frac{\pm\infty}{\pm\infty} then: \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Taylor: $dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial x \partial y}dxdy + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(dy)^2 ..$

$d(XY) = XdY + YdX + dXdY; U = f(x(t), y(t)) \to \frac{\partial U}{\partial t} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t}$

Itô lemma: multiple stochastic variables

$dX_i = a_i(t, X_1, X_2, \ldots)dt + b_i(t, X_1, X_2 \ldots)dB_i$

$dF = \frac{\partial F}{\partial t}dt + \Sigma\frac{\partial F}{\partial X_i}dX_i + \frac{1}{2}\Sigma\rho_{ij}b_i b_j \frac{\partial^2 F}{\partial X_i \partial X_j}dt$

$Itô: V = f(S, t);\ F = g(V) \to dF = \frac{\partial F}{\partial V}dV + \frac{1}{2}\frac{\partial^2 F}{\partial V^2}(dV)^2; x = V, y = 0$

$Rules: (dB_i)^2 \to= dt; dB_i dB_j \to \rho_{ij}dt; (dX_i)^2 \to b_i^2 dt; dX_i dX_j \to \rho_{ij} b_i b_j dt$

Matrix: $(cA)^T = cA^T; (A + B)^T = A^T + B^T; (A^T)^T = A; (AB)C = A(BC); AB \# BA; I_m A = A = AI_n; A_{mn}O_{np} = O_{mp}; A^{-1}A = AA^{-1} = I; (A^{-1})^T = (A^T)^{-1}; (AB)^T = B^T A^T; (AB)^{-1} = B^{-1}A^{-1}; r(A) = r(A^T) \leq \min(n, s); r(AB) \leq \min(r(A), r(B)); r(AA^T) = r(A)$; **r(A)= #indp column A**

Matrix m x n (m=rows, n=columns) (Amxp *Bpxn = ABmxn; Identity matrix = I [1…0; 0 1 0…] (A*I = A; A*A-1=I), $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, kA = [ka_{ij}]; \det(A) = ad - bc; Det\ (MM') = (DetM)(DetM'); Det(cM) = c^n Det(M);\ Ker\ T = \{v \in V | Tv = 0\} \subset V; \dim V = \dim(Im\ T) + \dim(Ker\ T)$; *Cannot mix up row & column when transform*

Week-1:

Probability Distribution satisfies: $Prob(a < X < b) = \int_a^b p(x)dx;$

$P(x) \geq 0, \sum_1^n p(x_k) = 1, \int_{-\infty}^{+\infty} p(x)dx = 1;$

Expected value (expectations) = **weighted** value by **probability**

$E[f(X)] = \sum_{k=1}^{n} f(x_k)p(x_k);\ E[f(x)] = \int_{-\infty}^{+\infty} f(x)p(x)dx;$

Mean = E of random var itself: $\mu \equiv E[X] = \int_{-\infty}^{+\infty} xp(x)dx; \mu = \sum_1^n x_k p(x_k)$

Moment: $\mu_l \equiv E[X^l] = \int_{-\infty}^{+\infty} x^l p(x)dx$; (standardize = $/\sigma^l$ ≡ dimensionless)

l^{th} **central** moment: $m_l = E[(X-\mu)^l] = \int_{-\infty}^{+\infty} (x-\mu)^l p(x)dx$

$Variance: \sigma^2 = Var(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2 = \int (x-\mu)^2 p(x)dx$

$Skewness: S(x) = E[\left(\frac{X-\mu}{\sigma}\right)^3]; Kurtosis\ (excess): K(x) \equiv \frac{E[(X-\mu)^4]}{\sigma^4} - 3; \kappa \geq -1, \kappa_{Gaussian} = 0 \to E[z^4] = 3\ where, z \sim \mathcal{N}(0,1)$

$Uniform: p(x) = \begin{cases} 1,\ x \in [0,1] \\ 0, otherwise \end{cases},\ \ \mu = \frac{1}{2}; \mu_l = \frac{1}{l+1}; \sigma^2 = \frac{1}{12};$

$Binomial\ (Bernoulli, 0\ or\ 1, sucess\ @\ prob\ p, k\ sucess\ in\ n\ trial)$

$f(k; n, p) = \binom{n}{k}p^k q^{n-k}; q = 1 - p, \binom{n}{k} = \frac{n!}{k!\,(n-k)!};$

$\mu_{Bino} = np; \sigma_{Bino}^2 = npq = E[X^2] - E^2[X]; E[X^2] = (np)^2 + np(1-p);$

(bino) scaling: $z_k = \frac{x_k - np}{\sqrt{npq}} \sim \mathcal{N}(0,1)$ for all n, $f(k; n, p) \approx f(z_k) = \frac{1}{\sqrt{2\pi}}e^{-z_k^2/2}$

$Gaussian\ (Normal): p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \sim \mathcal{N}(\mu, \sigma^2); \int p_G dx = E[X^0 = 1] = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = 1; Scaling: z = \frac{x-\mu}{\sigma} \sim \mathcal{N}(0,1), p(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}, ; \mu_Z = 0, \sigma_Z = 1, E[z^4] = 3$

$Normality \equiv Linear: Y_{Norm} = aX_{Norm} + b \to E[Y] = a\mu + b, var(Y) = a^2\sigma^2$

$P(Z \leq z) = F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-z'^2/2}dz' = \Phi(z), with\ standardize: z = \frac{x-\mu}{\sigma}$

CDF of std normal: $\Phi(y) = P(Y \leq y) = P(Y < y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-t^2/2}dt$

Poisson: *avg arrival time,* ***k*** *events in the next continuous time interval* ***t***

$p(k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}; p(k; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}; \mu = \lambda, E[X^2] = \lambda + \lambda^2, Var[X] = \lambda$

$Cauchy: p(x) = \frac{A}{(\pi A)^2 + x^2} \sim \frac{A}{x^2}; \mu\ \&\ \sigma^2: undefined.$

Geometric: $p(x = k) = q^{k-1}p \to \mu = \frac{1}{p}, Var[X] = \frac{q}{p^2}; q = 1 - p$

Exponential: $p(x) = \lambda e^{-\lambda x} \to \mu = \frac{1}{\lambda}, Var[X] = 1/\lambda^2$

Log normal: $r_t = \log(P_t/P_{t-1}) \sim N(\mu, \sigma^2); P_t = P_{t-1}e^{r_t};\ P_T = P_0 e^{r_1 + \cdots + r_T}; r(T) = \Sigma r_i \sim N(T\mu, T\sigma^2); \mu_{log} = \exp\left(\mu + \frac{\sigma^2}{2}\right) - 1; \sigma_{log}^2 = (e^{\sigma^2} - 1) * e^{2\mu+\sigma^2}; \log(S_T/S_0) = \sum_{t=1}^{T} r_t \to E[\log(S_T/S_0)] = \sum_1^T E[r_t] = T\mu$

Law of Large Number (LLN): n ↗ the probability of μ deviate from np → 0

Central Limit Theorem (CLT): n ↗ for *fixed p*, the distribution → Gaussian (N)

PS1.1 (Bino): E [T|1st success or waiting time] = 1/p; $E[T^r] = \left(\frac{p}{q}\right)\left[q\frac{d}{dq}\right]^r \frac{1}{1-q}$

PS1.2 (RW): $S_T = \Sigma X_i; X_t = az_t + b; z \sim N(0,1) \to E[S_T] = bT, \sigma_{S_T} = a\sqrt{T}$

Week-2:

Time series models: $(z_t \sim IID(0,1);\ Cov(z_t, z_{t'}) = \delta_{tt'} = 0\ [t = t']\ or\ 1)$

RWM (Random Walk Model): $r_t = \mu + \sigma z_t; X_T = \sum_1^T r_t \sim \mathcal{N}(\mu T, \sigma^2 T)$;

MA (Moving avg, ∈ **past z) 1st order**: $r_t \equiv \mu + \sigma z_t + \phi z_{t-1}; \phi = const$

GARCH (Volatility): $r_t \equiv \mu + \sigma_t z_t = \mu + \epsilon_t; \sigma_t \updownarrow, \epsilon_t \sim N(0, \sigma_t^2)$

AR (Auto regressive ∈ **past return**): $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t$;

AR (1) mean reversion: $R_t - \mu = -\lambda(R_{t-1} - \mu) + \sigma z_t; \mu = \frac{c_0}{1-c_1}, \lambda = -c_1, |\lambda| < 1; E[R_t] = \mu; Var[R_t] = \gamma_0 = \frac{\sigma^2}{1-\lambda^2}; E[z_t(R_{t-1} - \mu)] = 0$

AR (1) = MA of infinite: $Y_t = \frac{R_t - \mu}{\sigma} \to Y_t = \sum_{k=0}^{\infty}(-\lambda)^k z_{t-k}; E[z_s Y_t]_{t<s} = 0$

***AR1 Lag-k autocovariance** coefficient:* $\gamma_k \equiv Cov(R_t, R_{t-k}) = E[(R_t - \mu)(R_{t-k} - \mu)] = -\lambda \gamma_{k-1} = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k \sigma^2}{1-\lambda^2}, for\ any\ k > 0$;

ARMA(p,q): $r_t = (c_0 + c_1 r_{t-1} + .. + c_p r_{t-p}) + \sigma z_t + (\phi_1 z_{t-1} + .. + \phi_q z_{t-q})$

AR (depend on past) = **PACF** sharp cut-off & **ACF** slow decay. **MA (depend on shock)** = **ACF** sharp cut-off & a **PACF** slow decay

Stationary: joint distribution of all its value = invariant (∉ t): t→ t+s

Weak stationary (RW, MA, AR):

(1) $\mu, \sigma^2 = const$ (2) $\boldsymbol{ACF}$: $E[X_t X_s]\ or\ E[X_t X_{t+k}] \notin t\ (or\ can\ shift$ 't-s')

PS2.2 **(Couple processes)**: $x' = x_t + y_t; y' = x_t - y_t \to x = 0.5(x' + y')$..

Week-3: Time Series Models

Covariance-stationary process: $\gamma_k = Cov(r_t, r_{t-k}); \gamma_0 = Var(r_t); \rho_k = Corr(r_t, r_{t-k}) = \gamma_k/\gamma_0$; e.g: $Var(r_t + r_{t-1}) = 2Var(r_t)(1 + \rho_1)$

Forecasting: conditioned observe **E[Yt] @I_t**: $Y_t \equiv observed = const$

Optimal forecast (Granger) is the *conditional mean*: $f_{t,h} = E[x_{t+h}|I_t]$ (cost function is symmetric & convex) → **Forecast error**: $e_{t+h} = x_{t+h} - f_{t,h}$

Mean-squared forecast error: $\boldsymbol{MSFE}\ (f_{t,h}) = E[e_{t,h}^2] = E[(x_{t+h} - f_{t,h})^2]$

Calibrate the Bino (tree): $S_t = S_{t-1}e^{r_t}: r_t = (logu, p)\ or\ (logd, q = 1-p)$- or with Bernoulli variable: $r_t = a + bx_t, x_t = (1, p)\ or\ (0, 1-p)$. Then, with real data (μ, σ): $\mu = E[r_t] = a + pb, \sigma^2 = Var(r_t) = b^2 pq \to$

$$a = \mu - \sigma\sqrt{\frac{p}{q}}, b = \frac{\sigma}{\sqrt{pq}}; \log u = \mu + \sigma\sqrt{\frac{q}{p}}, \log d = \mu - \sigma\sqrt{\frac{p}{q}}$$

Gambler's ruin: recursive, discrete (bet **$b** ea), boundary (total wealth **$a**)

Qx = **prob of ruin** (loss all) from capital **$x**: $Q_x = pQ_{x+b} + qQ_{x-b}$ with $Q_0 = 1, Q_a = 0$: **For p=q=1/2**: $Q_x = 1 - x/a;\ prob\ of\ sucess: \boldsymbol{P_x = 1 - Q_x}$;

For p#q : $Q_x = \frac{(q/p)^{a/b} - (q/p)^{x/b}}{(q/p)^{a/b} - 1}$, if a =∞ then $Q_x = \begin{cases} 1, if\ p \le q \\ (q/p)^x, if\ p > q \end{cases}$

Duration of game: $D_x = pD_{x+1} + qD_{x-1} + 1, 0 < x < a \to D_x = x(a - x)\ if\ p = q; if\ p\#q: D_x = \frac{x}{q-p} - \frac{a}{q-p} * \frac{1-(q/p)^x}{1-(q/p)^a}$

PS3.1: $\boldsymbol{MSFE}\ (f_{t,h}) = E[e_{t,h}^2] = E\left[(x_{t+h} - f_{t,h})^2\right] = E[(\sigma z_t)^2] = \sigma^2$

PS3.2c: $\eta, \epsilon \in IID \to Cov(\eta, \epsilon) = Cov(\eta_t, \eta_{t+k}) = 0$ **BUT** $E[\eta_t, \eta_{t+k}] \ne 0$

PS3.3b: $P_{2\ sucess}, no\ fail = P_x * (p * 1 + q * P_{(a-x)}): 0.2 * (\frac{1}{2} * 1 + \frac{1}{2} * \frac{124}{125})$

Week-4: Continuous time Stochastics *[z, IID, ~N(0,1)]* - $dB_t = z\sqrt{\Delta t}$

Brownian: $\Delta t = \frac{T}{n}; \lambda = \sqrt{\Delta t} = \sqrt{T/n}; \epsilon_t \equiv \lambda z_t; \lim_{\Delta t \to 0} B_{\Delta t, T} \sim \boldsymbol{\mathcal{N}(0, T)}$;

$E[(dB_t)^{odd}] = \mu = 0; Var[dB_t] = dt;\ E[dB_t^4] = 3(dt)^2;\ dB_t \sim \mathcal{N}(0, dt)$;

Itô lemma: $dX_t = a(X,t)\ dt + b(X,t)\ dB_t\ ; (dB_t)^2 \to dt; (dX_t)^2 \to b^2 dt$

$$dF(X,t) = \left(\frac{\partial F}{\partial t}dt + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}dt\right) + \frac{\partial F}{\partial X}dX = \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right)dt + b\frac{\partial F}{\partial X}dB$$

Geometric Brownian **(lognormal)**: $dS_t = (\mu S_t)dt + (\sigma S_t)dB_t$;

$d(\log S) = (Ito: V = logS) = 0 + \frac{dS}{S} + \frac{(\sigma S)^2}{2}(\frac{-1}{S^2})dt = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB_t$;

$$S_t = S_0 * \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma(B_t - B_0)\right\} = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z\sqrt{t}};$$

$$\ln(S_T|S_t) \sim \mathcal{N}[lnS_t + (\mu - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)]$$

$E[S_T] = S_t e^{\mu(T-t)}; Var(S_T) = S_t^2 e^{2\mu(T-t)} e^{\sigma^2(T-t)-1}$

Black-Scholes Equation (BSE): $r = r_f; \pi = V(t,S) - \Delta S$; **RNP**: $d\pi = r_f \pi dt = d(V - \Delta S) = \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2}dt\right) + \left(\frac{\partial V}{\partial S} - \Delta\right)dS; Set\ \Delta = \frac{\partial V}{\partial S}, then$:

$(\boldsymbol{BSE})$: $\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$ **(S MUST ∈ log-form Geo Brow)**

Week-5: Itô Calculus (Stock price = Geo B; drift = μ , volatility = σ)

Self-financing condition for q share S and C bond M: $Sdq + MdC + dSdq + dMdC = 0; where: X_0 = q_0 S_0 + C_0 M_0 = 0 = X_t^{post} - X_t^{pre}$

$\pi = V(S,t) + qS + CM \to d\pi = dV + (qdS + CdM) + (Sdq + MdC + \cdots)$

$= dV + qdS + rCMdt = dV + qdS + r(\pi - V - qS)dt; dV = Ito\ (S) \ldots$

$Chose: q = -\frac{\partial V}{\partial S} = -\Delta$ → replicate V's payoff only stock & bonds ⟶ risk free ⟶ initial value π = 0, hence, dπ =0, ∀ t ⟶ BSE...

Black-Scholes: **commodity** cost of storage q, **dividend** D, foreign ir r*:

$\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} + (r + q - r^* - D)S\frac{\partial V}{\partial S} - rV = 0; dS = \mu S dt + \sigma S dB$

BSE for **futures option**: $\mathcal{F} = e^{r(T-t)}S_{spot} \to \frac{\partial V}{\partial t} + \frac{(\sigma \mathcal{F})^2}{2}\frac{\partial^2 V}{\partial \mathcal{F}^2} - rV = 0$

Bond pricing (ZCB): one-factor model $(V_i: value\ of\ bond; q_i: \#bond)$

$$\pi = q_1 V_1 + q_2 V_2; choose: q_1 = \frac{1}{\partial V_1/\partial y}, q_2 = -\frac{1}{\partial V_2/\partial y}$$

Δt= free short rate $dy_t = adt + bdB$, with: $d\pi = q_1 dV_1 + q_2 dV_2 = y\pi dt$

Equate → $\frac{\frac{\partial V_1}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_1}{\partial y^2} - yV_1}{\partial V_1/\partial y} = \frac{\frac{\partial V_2}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_2}{\partial y^2} - yV_2}{\partial V_2/\partial y} = f(t,y) \notin$ t,y

$$\frac{\partial V_i}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2} - yV_i - f(t,y)\frac{\partial V_i}{\partial y} = 0; with\ V_i(T,y) = 1$$

Excess return /unit risk: $\frac{dV - yVdt}{b\partial V/\partial y} = \frac{a+f}{b}dt + dB$; **Market price of risk** $\eta \equiv \frac{a+f}{b}; or\ f = b\eta - a$

Interest Rate 1-factor model: $dy = \alpha(\bar{y} - y)dt + \sigma dB \to RW\ trans: y = e^{-\alpha t}z \to dy = -\alpha y dt + e^{-\alpha t}dz = \alpha(\bar{y} - y)dt + \sigma dB$

$E[y(t)] = y_0 + (\bar{y} - y_0)(1 - e^{-\alpha t}); Var[y(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \to \sigma^2 t$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - yV + \alpha(\bar{y} - y)\frac{\partial V}{\partial y} = 0 \to \boldsymbol{Solution}: V(t,y) = e^{f(t) - yg(t)}$$

$\boldsymbol{Diffusion}$: $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial z^2} \to 1\ solution: p_0(z,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}} \to If\ \boldsymbol{p(z, t = 0)} = \boldsymbol{f(z)}\ then\ \boldsymbol{general}\ solution\ for\ \boldsymbol{diffusion}$:

$p(z,t)=\int_{-\infty}^{+\infty}p_0(z-w,t)f(w)dw=\frac{1}{\sqrt{2\pi t}}\int e^{-\frac{(z-w)^2}{2t}}f(w)dw$

$\boldsymbol{u}=\frac{\boldsymbol{w-z}}{\sqrt{t}}\rightarrow du=\frac{dw}{\sqrt{t}}\rightarrow p(z,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\boldsymbol{e^{-\frac{u^2}{2}}f(u\sqrt{t}+z)du}=$
$\boldsymbol{E[f(u\sqrt{t}+z)]}\rightarrow \boldsymbol{z\sim\mathcal{N}(0,1)}$: $E[z^m]$ & $\int\frac{1}{\sqrt{2\pi}}\boldsymbol{e^{-\frac{z^2}{2}}dz=1=E[z^0]}$

Brownian Integrals: $dB\sim\mathcal{N}(0,dt)$; $B_t\sim\mathcal{N}(0,t)$; $\boldsymbol{B_t-B_0=z\sqrt{t}}$; $\boldsymbol{z\sim\mathcal{N}(0,1)}\notin t$; $Eg: dX=\mu dt+\sigma dB$; $X_t-X_0=\mu t+\sigma(B_t-B_0)=\mu t+\sigma z\sqrt{t}$; $\boldsymbol{E[f(B_t-B_0)]=E[f(z\sqrt{t})]}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{z^2}{2}}f(z\sqrt{t})dz$

$\boldsymbol{E[e^{\alpha x+\beta}]}=\frac{1}{\sqrt{2\pi}}\int e^{-x^2/2}e^{\alpha x+\beta}dx=\frac{e^{\beta}}{\sqrt{2\pi}}\int e^{-(x-\alpha)^2/2}e^{\alpha^2/2}dx=\boldsymbol{e^{\alpha^2/2+\beta}}$;

$\boldsymbol{E[Z^m]}=\begin{cases}0 & ,if\ m\ is\ odd\\ 2^{\frac{-m}{2}}m!/(m/2)!, & if\ m\ is\ even\end{cases};\begin{cases}E[z^0]=E[Z^2]=1\\ E[Z^4]=3;E[Z^6]=15\end{cases}$

Solve by ↑↓ variable from Black-Scholes (BSE) to Diffusion Equation (DE):

$dS_t=(\mu S_t)dt+(\sigma S_t)dB_t$; $\boldsymbol{\frac{\partial V}{\partial t}+\frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0}$ **(BSE)**

(1) $V(S,t)=e^{-r(T-t)}U(S,t)\rightarrow\frac{\partial U}{\partial t}+\frac{(\sigma S)^2}{2}\frac{\partial^2 U}{\partial S^2}+rS\frac{\partial U}{\partial S}=0$

(2) $\tau=T-t$; $\boldsymbol{S=e^{\xi}}\rightarrow\frac{\partial U}{\partial\tau}-\frac{\sigma^2}{2}\frac{\partial^2 U}{\partial\xi^2}-(r-\frac{\sigma^2}{2})\frac{\partial U}{\partial\xi}=0$

(3) $x=\xi+\left(r-\frac{\sigma^2}{2}\right)\tau$, $chain\ rule\rightarrow\frac{\partial U}{\partial\tau}-\frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2}=0$

(4) $\eta=\tau\sigma^2\rightarrow\boldsymbol{\frac{\partial U}{\partial\eta}-\frac{1}{2}\frac{\partial^2 U}{\partial x^2}=0}$ **(DE)** $\rightarrow$ (5) $p_0=\frac{1}{\sqrt{2\pi\eta}}e^{-\frac{x^2}{2\eta}}$

PS5.3b: $p(z,T)=f(z)\rightarrow shift\ p(z,0)\rightarrow p(z,t): t\rightarrow t-T$

PS5.4c: $\tau=\sigma^2(T-t)$; $z=S\rightarrow u=\frac{w-S}{\sqrt{\tau}}:-\infty\ to\ u^*=(K-S)/\sqrt{\tau}$

Week-6: Continuous-time Finance

Probability of RW: $X\sim\mathcal{N}(\mu,\sigma^2)\rightarrow$ stochastic: $X_t\sim\mathcal{N}(\mu t,\sigma^2 t)$ has PDF:

$$p(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\rightarrow \boldsymbol{forward}\ PDE:\ \frac{\partial p}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}+\mu\frac{\partial p}{\partial x}=0$$

If we start at (x_0,t_0) & want to know PDF at (x_T,T) then:

$$p(x_T,T;x_0,t_0)=\frac{1}{\sqrt{2\pi\sigma^2(T-t_0)}}\exp\left(-\frac{[(x_T-x_0)-\mu(T-t_0)]^2}{2\sigma^2(T-t_0)}\right)$$

$$\rightarrow \boldsymbol{backward}\ PDE:\ \frac{\partial p}{\partial t_0}+\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x_0^2}+\mu\frac{\partial p}{\partial x_0}=0$$

Special function: **Call** = $f_1(S)=\max(S-K,0)=\frac{1}{2}(|S-K|+S-K)$; **Step**: $\frac{\partial f_1}{\partial S}\equiv\boldsymbol{\Theta}(S-K)=\begin{cases}1, & S>K\\ 0, & otherwise\end{cases}$;

$\boldsymbol{Dirac\ delta}$: $\frac{\partial^2 f_1}{\partial S^2}\equiv\boldsymbol{\delta}(S-K)=\begin{cases}0,S\neq K\\ \infty,S=K\end{cases}$; $\boldsymbol{\delta(x)}=\lim_{t\rightarrow 0}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}=\begin{cases}0,x\neq 0\\ \infty,x=0\end{cases}\rightarrow\int_{-\infty}^{+\infty}\delta(x)dx=1$; $\int_{-\infty}^{+\infty}\boldsymbol{\delta(x-y)f(x)dx=f(y)}$

Survival probabilities ($z_0\ to\ z\ without\ hit\ z^*\rightarrow$ mirror point $z_0^*=2z^*-z_0$) *– for every path hit z* there is equal imagine path start from z_0^* to reach z.*

$p_s(z,t)=p_0(z-z_0,t)-p_0(z-[2z^*-z_0],t)$; *= Probability without restriction to reach z of [Original start – Imagine/mirror start]*

$$p_s(z,t)=\begin{cases}\frac{1}{\sqrt{2\pi t}}\left(e^{\frac{-(z-z_0)^2}{2t}}-e^{\frac{-(z+z_0-2z^*)^2}{2t}}\right), & if\ z>z^*\\ 0, if\ z\leq z^*\end{cases};\ p_s(z^*,t)=0$$

Including **drift term** $\sim\mathcal{N}(\mu,\sigma^2)$ then:

$$p_s(z,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}[e^{\frac{-(z-\mu t-z_0)^2}{2\sigma^2 t}}-Ce^{\frac{-(z-\mu t+z_0-2z^*)^2}{2\sigma^2 t}}];C=e^{\frac{-2\mu(z_0-z^*)}{\sigma^2}}$$

$p_s(t)=\int_{z^*}^{\infty}p_s(z,t)dz=\Phi\left(\frac{\mu t+(z_0-z^*)}{\sigma\sqrt{t}}\right)-C\Phi\left(\frac{\mu t-(z_0-z^*)}{\sigma\sqrt{t}}\right)$;

Geometric: $dS=\mu Sdt+\sigma SdB\rightarrow\boldsymbol{probability}\ p(S_T,T;S,t)$ the satisfy: $\boldsymbol{\frac{\partial p}{\partial t}+\frac{(\sigma S)^2}{2}\frac{\partial^2 p}{\partial S^2}+\mu S\frac{\partial p}{\partial S}=0}$; $\boldsymbol{E_t[f(S_T)]}=\int p(S_T,T;S,t)f(S_T)dS_T=F(S,t)$; $\lim_{t\rightarrow T}F(S,t)=\int\delta(S_T-S)f(S_T)dS_T=f(S)$; $\boldsymbol{V(S,t)}=e^{-r(T-t)}F(S,t)=e^{-r(T-t)}E_t[f(S_T)]=\boldsymbol{e^{-r(T-t)}E_t[V(S_T,T)]}$ *[BSE, RNP: $\mu\rightarrow r$]*

The Black-Scholes Solution (Call option):

$\boldsymbol{V_{Call}(S,t)=S_t\Phi(d_+)-Ke^{-r(T-t)}\Phi(d_-)}$; $\boldsymbol{PCParity}$: $P+S=C+Ke^{-rT}$

$$d_{\pm}=\frac{\ln(S_t/K)+r(T-t)}{\sigma\sqrt{T-t}}\pm\frac{1}{2}\sigma\sqrt{T-t};\ \boldsymbol{\Phi(x)}\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-z^2/2}dz$$

The Greeks: $Delta\ \Delta\equiv\frac{\partial V}{\partial S}=\begin{cases}\Phi(d_+)=\Delta_{Call}\\ \Phi(-d_+)=\Delta_C-1=\Delta_{Put}\end{cases}$; $Gamma\ \Gamma\equiv\frac{\partial^2 V}{\partial S^2}=\frac{\Phi'(d_+)}{\sigma S\sqrt{T-t}}$; $Vega\ \mathcal{V}\equiv\frac{\partial V}{\partial\sigma}=\Phi'(d_+)S\sqrt{T-t}$; $\Phi'(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

Binary call option: $f(x')=g(S')=\theta(S'-K)=1\ (S'\geq K)\ \&\ 0\ (S'<K)$

$\rightarrow V(S,t)=e^{-r(T-t)}\Phi(d_-)$

Risk neutral x Gaussian: $Y\sim\mathcal{N}(\mu,\sigma^2)\rightarrow f(\lambda)=E[e^{\lambda Y}]=e^{\lambda\mu+\lambda^2\sigma^2/2}$ *(for RNP: replace μ with r everywhere)*

American Perpetual Put *(Satisfied BSE)*: $\notin t\rightarrow\frac{\partial V}{\partial t}=0\rightarrow\boldsymbol{V(S)=cS^{-2r/\sigma^2}}$

Boundary: $Option=Exercise: V(\hat{S})=K-\hat{S}\rightarrow\boldsymbol{V(S)}=(K-\hat{S})\left(\frac{S}{\hat{S}}\right)^{-\frac{2r}{\sigma^2}}$;

$Max\ V\ @\frac{\partial V}{\partial S}|_{S=\hat{S}}=0\rightarrow\hat{S}=\frac{K}{1+\sigma^2/2r}\rightarrow V(S)=\frac{K}{2r/\sigma^2+1}\left(\frac{S}{K}\left(1+\frac{\sigma^2}{2r}\right)\right)^{-2r/\sigma^2}$

Martingale Itô $\leftrightarrow drift=0: E_t[X_{t'}]=X_t, t<t'\rightarrow E[dX_t]=0\rightarrow a=0$

Trigger CALL option (Gap) (pay S-K if S>X): (= Call @X + Binary [X-K])

$V=[S\Phi(d_+)-Xe^{-rT}\Phi(d_-)]+[(X-K)e^{-rT}\Phi(d_-)]=S\Phi(d_+)-Ke^{-rT}\Phi(d_-)$; $where: d_{\pm}=\frac{\ln\left(\frac{S}{Xe^{-rT}}\right)}{\sigma\sqrt{T}}\pm\frac{1}{2}\sigma\sqrt{T}=\frac{\ln\left(\frac{S}{X}\right)+rT}{\sigma\sqrt{T}}\pm\frac{1}{2}\sigma\sqrt{T}$

EU Barrier Call: $C_{knock-IN}+C_{knock-OUT}=C_{vanilla}: e,g: C_{do}+C_{di}=C_v$

Solution for $C_{do}(X<K)\rightarrow assume: f=(S/X)^{\alpha}C(X^2/S,t)\rightarrow\boldsymbol{C_{do}(S,t)}=\begin{cases}C_v(S,t)-\left(\frac{S}{X}\right)C_v(X^2/S,t), S\geq X\\ 0, S\leq X\end{cases}$; $\boldsymbol{C_{di}=C_v-C_{do}}=\left(\frac{S}{X}\right)C_v(X^2/S,t)$

PDEs Solving Solution (DE): $\boldsymbol{\frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2 p}{\partial z^2}}$; $dS=\mu Sdt+\sigma SdB$

(1) $Assum: p(z,t)=f(z)*g(t)$ (2) $Sub\rightarrow\frac{g'}{g}(t)=\frac{f''}{2f}(z),\forall(\boldsymbol{t},\boldsymbol{z})$,

$=const=\lambda$ (3) $g(t)=c_0e^{\lambda t}; f(z)=c_1e^{t\sqrt{2\lambda}}+c_2e^{-t\sqrt{2\lambda}}\rightarrow p(z,t)$

CC6.4: $P(z_0\ wo\ hit\ z^*)=P(normal,z_0\rightarrow z)-P(imagine,z_0^*\rightarrow z)$

PS6.4: $V_0=e^{-rT}E_0^Q[V(T)]=e^{-rT}E\left[\left(S_0\exp(r-\sigma^2/2)t+\sigma z\sqrt{t}\right)^3\right]$...

Week-7: Linear Algebra of asset pricing: *(A: payoff matrix, x: port, price S, target payoff: b)* **Basis** = rank = dimension = **independent columns**

$\boldsymbol{Ax=b}\rightarrow x=A^{-1}b$: requires: A[s,n]: **s (row) =# states , n (column) = # security** → A: invertible (square), **n=s,** column & rows independent.

Complete market: every payoff can be generated ↔ rank(A) = s = # independent states ↔ linear trans.A = ea y exist x such that f(x)=y); if #securiry n>s → overcomplete (& can be reduce n-s redundant securities)

Kernel of A (null space): A$\vec{z}$ =0 (z, non-trivial case, # $\vec{0}$):: **Arbitrage portfolio** (∈ Ker(A)): $z \in \ker(A) \to A(x+cz) = Ax = b,\ \forall c\ (infinite\ or\ none)$

Redundant securities ↔ **payoffs** that are linearly **dependent**

Value of portfolio: $\boldsymbol{MV} = \Sigma S_i x_i = (S_1\, S_2 \ldots S_n)\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = \boldsymbol{S^* x} = S^T x = S[x]$

Type 1 Arbitrage: pay nothing now (V [0] ≤ 0) & get something later (V[T]>0)

$\mathrm{V} = \Sigma \mathrm{S_i x_i} = \mathbf{S^* x} \le \mathbf{0} \to$ Payoff (**only non negative**: $\mathbf{Ax} \ge \mathbf{0}$ (for **all** component $\mathrm{x_i}$) & **at least one** (component) payoff > 0

Type 2 Arbitrage: pay something now (V [0] ≠ 0) & get nothing later (V[T]=0)

Payoff: $\mathbf{Ax} = \mathbf{0}$, with non trivial $\mathbf{x}$, and : $\mathbf{S^* x} \ne \mathbf{0}$ (S: present price)

Arrow-Debreu (AD) – State Price: payoff (**$1**) in single state → **Ae ≡ I**

Elementary AD (state j) $e_j \leftrightarrow \boldsymbol{A x_j} = \boldsymbol{e_j} = (0\ 0 \ldots 1 \ldots 0)^*$ (1 @ j-post)

Price of any positive **payoff b** is $\boldsymbol{S_b} = \boldsymbol{\psi^* b} = \boldsymbol{\Sigma \phi_i X_i} > \mathbf{0} \to$ price of original basis assets (columns of A) is given by: $S_i = \psi^* a_1 \to \boldsymbol{S^*} = \boldsymbol{\psi^* A} \to \boldsymbol{S} = \boldsymbol{A^* \psi}$

Arbitrage check with pseudo-inverse *(only for n>s & rankA=s: market is complete & redundant assets)*: find non-negative state price ψ such that $S = A^*\psi \to \boldsymbol{M} = (\boldsymbol{AA^*})^{-1}\boldsymbol{A}$ & $MA^* = I$ **(Identity square matrix, all 0 except diagonal term=1)** → $\boldsymbol{Check}$: (1) $\boldsymbol{\psi = MS} \ge \mathbf{0}$? (2) $\boldsymbol{S = A^*\psi}$? → *both ans are* $\boldsymbol{YES}$ *if & only if* $\boldsymbol{No - arbitrage}$.

Arbitrage pricing theorem: find all ψ > 0 such that **A*ψ =S**:

(1) **No solution ≡ arbitrage** (2) 1 solution ≡ complete market (3) Multiple solution ≡ incomplete market

Dual space: Security price S → port x; State price ψ → payoffs b

$S^* x = S[x] = \psi[b] = \psi^* b = \psi[Ax] = \psi^*(Ax) = (A^*\psi)^* x; S = A^*\psi$

Fundamental Theory of Assets Pricing (FTAP): *no arbitrage* if & only if exist **strictly positive ψ** consistent with security-price vector:

$$\mathbf{S = A^*\psi \to no\ arbitrage \equiv \psi_j > 0\ \forall j}$$

For incomplete market: (1) **at least 1** solution ψ >0 → no arbitrage (2) If **every solution** ψ ≤ 0 → arbitrage

CC7.4.1: If $\psi \in Ker(A^*) \to \psi^* A = 0, S = A^*\psi = 0 \to$ ψ not valid state price

PS7.2: State price satisfy that $\boldsymbol{\psi_i = S^* x_i}$: x is the AD matrix such that $A * \boldsymbol{x} = \boldsymbol{I}$ *or* $Ax_i = e_i$. R_f payoff $1 at any state: $Ax_{rf} = I$ ([1]) → Price = $x_{rf} S_0$

No arbitrage for additional security still must both: $\boldsymbol{S = A^*\psi}$ & $\boldsymbol{\psi > 0}$

PS7.3b: construct arbitrage if ψ not ≥ 0 → reduce matrix to base (=rank), remove redundant → A1, solve for A1*x1 = e_i with i=1 at arbitrage portfolio → solution: x = (position of redundant security + x1)

Week-8: Optimization

Multi: $f(x) = f(x_0) + (\nabla f)^T (x - x_0) + \frac{1}{2}(x - x_0)^T Q (x - x_0) + \cdots$

Where: $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \end{pmatrix}; Q = Q^T = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$

So: $f(x) - f(x_0) \approx \frac{1}{2}(x - x_0)^T Q (x - x_0)$; Q = 2nd derivative, symmetric

Eigenvalues of Q, λ, such that: $\boldsymbol{Qx = \lambda x}; x = eigenvector \ne \vec{0}, \lambda: scalar$

$1. Find\ \boldsymbol{eigenvalue\ \lambda}: \det(Q - \lambda I) = 0;\quad 2. \lambda^k\ is\ eigenvalues\ of\ Q^k$

$3. Find\ \boldsymbol{eigenvectors\ x}\ by\ solve: (Q - \lambda I)x = 0$

Critical point by Q, λ : $(1)\forall\lambda > 0 \to \boldsymbol{min}; (2)\forall\lambda < 0 \to \boldsymbol{max}; (3)\ \lambda\ both + ve\ \&\ -ve \to both\ \max \&\ \min$: $\boldsymbol{saddle}$ point; $(4)\ Any\ \lambda = 0 \to \boldsymbol{flat}\ direction$ ***[Note: can double check by assume specific value]***

1 Var critical point: $f'(x) = 0 \to f'' < 0 = max, > 0 = min, = 0: saddle$

Lagrange Multipliers: extreme **h(x,y)** with constraint **g(x,y) = c** then:

$$\boldsymbol{L(x,y,\lambda) \equiv h(x,y) - \lambda[g(x,y) - c]}$$

Extreme @ all partial der **= 0:** $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$

Port-Optimization: $\mu_p = \mu^T w\ (return); \sigma_p^2 = w^T C w; \Sigma w_i = 1\ (budget)$

$\boldsymbol{\mu_p} = E[\Sigma w_i R_i] = \Sigma w_i E[R_i] = \boldsymbol{\Sigma w_i \mu_i = \mu^T w};\ \boldsymbol{Var(\Sigma w_i R_i) = \sigma_p^2} = \Sigma \boldsymbol{w_i^2 Var(R_i)} + \boldsymbol{2\Sigma w_i w_j Cov(R_i, R_j)} = \Sigma w_i^2 \sigma_i^2 + 2\Sigma w_i w_j \boldsymbol{\sigma_i \sigma_j \rho_{ij}} = \boldsymbol{w^T C w}$

$\boldsymbol{\sigma_p^2 = w^T C w} = \sum_j C_{j,j} w_j^2 + \sum_{j<k} C_{j,k} w_i w_j\ ; \boldsymbol{w^T \iota} = \Sigma w_i \iota_i$

$C = Cov(R_i, R_j) = \begin{pmatrix} Var(R_1) & \cdots & Cov(R_1, R_n) \\ \vdots & \ddots & \vdots \\ Cov(R_n, R_1) & \cdots & Var(R_n) \end{pmatrix}: w^T C w = (Cw)^T w'$

Eg: $C_{2x2} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

$\boldsymbol{iota\ function}: unit\ exposure\ vector; \iota^T = [1\ 1 \ldots]$

Minimum-Variance Portfolio: $\Sigma w_i = 1; \boldsymbol{\mathcal{L}(w,\ell) = \frac{1}{2} w^T C w + \ell(1 - \iota^T w)} = \frac{1}{2}\left(\Sigma C_{j,j} w_j^2 + 2\sum_{j<k} C_{j,k} w_j w_k\right) + \ell(1 - \Sigma \iota_j w_j)$;

Critical: $\frac{\partial \mathcal{L}}{\partial w_i} = \sum_j C_{ij} w_j - \ell \iota_i = 0 \to \boldsymbol{w_{min} = \ell C^{-1} \iota}; (Condi)\ \iota^T w = 1 \to$
Solution: $\boldsymbol{w_{min} = \ell C^{-1} \iota} = \frac{C^{-1}\iota}{\iota^T C^{-1} \iota}; \boldsymbol{\sigma_{min}^2 = \ell} = \frac{1}{\iota^T C^{-1} \iota}$

$\to \mathcal{L}_{min} = \frac{1}{2} w^T C w = \frac{1}{2(\iota^T C^{-1} \iota)} = \frac{1}{2}\ell = \frac{1}{2}\sigma_{min}^2$

Special case: $C_{ij} = diagonal\ (indepedent) \to w_i \propto 1/\sigma_i^2$

(Min-Var) Port Optimization: Risk & Returns (budget constraint $\boldsymbol{\iota^T w = w_p}$ + return constraint $\boldsymbol{\mu_p^T w = \mu_p}$) ($w_p$ is a constant, instead of sum=1)

$$\boldsymbol{\mathcal{L}(w,\ell,m) = \frac{1}{2} w^T C w + \ell(w_p - \iota^T w) + m(\mu_p - \mu^T w)}$$

$\frac{\partial \mathcal{L}}{\partial w_i} = 0 = \sum_j C_{ij} w_j - \ell \iota_i - m\mu_i \to \boldsymbol{w_{min} = C^{-1}(\ell\iota + m\mu)}$

$\begin{pmatrix} \boldsymbol{w_p} \\ \boldsymbol{\mu_p} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} \boldsymbol{l} \\ \boldsymbol{m} \end{pmatrix}; M = \begin{pmatrix} a & b \\ b & c \end{pmatrix};\ a = \iota^T C^{-1} \iota, b = \mu^T C^{-1} \iota, c = \mu^T C^{-1} \mu$

$$\ell = \frac{1}{ac - b^2}(cw_p - b\mu_p);\ m = \frac{1}{ac - b^2}(-bw_p + a\mu_p)$$

$$\sigma_p^2 = \boldsymbol{w^T C w} = (l\ m) M \begin{pmatrix} l \\ m \end{pmatrix} = \frac{1}{ac - b^2}(a\mu_p^2 - 2b\mu_p w_p + cw_p^2); \boldsymbol{\sigma_p \propto \mu_p}$$

PS8.1: Initial condition, if $\boldsymbol{\beta^T w = 1} \to then\ replace\ all\ \iota\ with\ \beta\ in\ w_{min} = \frac{C^{-1}\beta}{\beta^T C^{-1} \beta}$

PS8.3 (single variable): $f_{xx} < 0: max; f_{xx} > 0\ min; f_{xx} = 0: inflection\ point$ (change concave: up → down or vice versa)

Q5-Sample exam: $dB^2 \to dt$ **only dt is matter**, higher dt (eg dBdt) → 0; drift & volatility for GeoB are without S: $dV/V = (\boldsymbol{drift})dt + (\boldsymbol{volatility})dB$