MITx – Foundation of Modern Finance 2 (Formulas)

Exam: Ans 2 dec - Unit – Delete EXCEL perm – GOAL seek

Forward, sport-rate: $\boldsymbol{f_t = \frac{B_t}{B_{t-1}} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1;}$ B-discount bond

EH: spot = predicted rate: $E_o[\widetilde{r_1}(t)] = f_{t+1}$

Fwd w/ **cont'**: $PV_0(F_t) = F_t * e^{-r_f T} = PV_0(\widetilde{S_T}) = S_0 * e^{-yT}$

→ Fwd$: $F_T = S_0 * e^{(r_f - y)T}$; ***Dis***: $F_T = (1 + r_f - y) * S_0$

Ft (C1) in **x-change** for 1-C2: $\boldsymbol{F_t = X_0 * e^{(r_1 - r_2)T}}$

Commodity price: $H_t = F_t = S_0 * e^{(r - \hat{y})T}$

Net conv. yield: $\hat{y} = y - c; c = storage\ cost$

Interest rate swap (fixed rate) $r_s = c$

$$\boldsymbol{r_s} = \Sigma w_t f_t = \Sigma\left(\frac{B_t}{\Sigma B_u}\right) f_t = \frac{1 - B_t}{\Sigma B_u} = \boldsymbol{c}; w_t = \frac{float}{fix} = \frac{B_t}{\Sigma B_u}$$

Call, long (**BUY**) payoff: $CF_t = \max[0, S_T - K]$

Put, short (**SELL**) payoff: $CF_t = \max[0, K - S_T]$

Put-call parity (PCP) (EU): $\boldsymbol{C + BK = P + S}; B = 1/(1 + r_f)^T$

Bull spread **Straddle**

Binomial **Risk Neutral Probability (always discount by Rf)**

$$q_u = \frac{(1 + r_f) - d}{u - d} = q; q_d = 1 - q = \frac{u - (1 + r_f)}{u - d};$$

$$C_0 = \frac{q_u C_u + q_d C_d}{1 + r_f} = \frac{E^Q[C_T]}{(1 + r_f)^T}; E^Q: expect\ under\ Q^{Prob}$$

State price: $\phi_u = \frac{q}{1+r}; \phi_d = \frac{1-q}{1+r}; \phi_{uu} = \phi_u \phi_u; \phi_{ud} = \phi_u \phi_d; ..$

American option pricing (APT model), option value:

$$P_t = \max\left\{Payoff_t, \frac{1}{1 + r_f} E_t^Q[P_{t+1}]\right\}; continuous\ value + 1$$

BSM Black Scholes Merton: $\boldsymbol{C_0 = S_0 N(x) - Ke^{-rT} N(x - \sigma\sqrt{T})}$

$where: x = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S_0}{Ke^{-rT}}\right) + \frac{1}{2}\sigma\sqrt{T};$ $N(.): norm.S.dist(1, true)$

w/ **PCP:** $\boldsymbol{P_0} = -S_0(1 - N(x)) + Ke^{-rT}(1 - N(x - \sigma\sqrt{T}))$

Portfolio **replica**: $\begin{cases} \delta(uS_o) + b(1 + r_f) = C_u \\ \delta(dS_o) + b(1 + r_f) = C_d \end{cases}$

Port with Rf asset: $\widetilde{r_p} = (1 - x) r_f + x\widetilde{r_q}; Var(r_p) = \sigma_p^2 = x^2 \sigma_q^2$

Sharpe ratio: $\boldsymbol{SR = \frac{r_p - r_f}{\sigma_p}}$;*SR=slope of **CAL**, higher=better*

SRmax *= SR(T)= same with all given Rp regardless of with or w/o Rf*

Tangency (only risky assets): $w'\bar{x} = \Sigma w_i \bar{x_i}; w_T' i = 1; sol\ for\ \boldsymbol{min}\Sigma\Sigma w_i w_j \sigma_{ij}$ with define $r_{p(T)} = \Sigma w_i r_i$ then:

$$W_T = \frac{1}{x^{-1}\Sigma^{-1} i}\Sigma^{-1}\bar{x} = \lambda\Sigma^{-1}\bar{x}; where: \lambda = 1/(\bar{x}'\Sigma^{-1} i)$$

Return: $r_p = r_f + \Sigma w_i(r_i - r_f) = (1 - \Sigma w_i) r_f + \Sigma w_i r_i$

Marginal contribution & return to risk ratio (**RRR**) (asset ***i*** to port ***p***)

$$\frac{\delta r_p}{\delta w_i} = r_i - r_f; \frac{\delta \sigma_p^2}{\delta w_i} = 2\Sigma w_j \sigma_{ij} = 2Cov(r_i, r_p); \frac{\delta \sigma_p}{\delta w_i} = \frac{Cov(r_i, r_p)}{\sigma_p} = \frac{\sigma_{ip}}{\sigma_p};$$

$$RRR_{ip} = \frac{r_i - r_f}{(\sigma_{ip}/\sigma_p)} = (\frac{\delta r_p}{\delta w_i})/(\frac{\delta \sigma_p}{\delta w_i}); \boldsymbol{Optimal\ T_{port}}: \boldsymbol{RRR_{iT} = SR_T} = \frac{r_T - r_f}{\sigma_T}$$

Regression: $r_i - r_f = \alpha_i + \beta_i(r_T - r_f) + \bar{\epsilon}_i; where: \beta_i = \sigma_{iT}/\sigma_T^2$

$$RRR_{iT} = \frac{r_i - r_f}{\beta_i \sigma_T} = \frac{r_T - r_f}{\sigma_T} \rightarrow \alpha_i = 0: \boldsymbol{optimal}\ T: \boldsymbol{RRR_i = RRR_j = SR_M}$$

CAPM: $\boldsymbol{r_i = r_f + \beta_i(r_M - r_f)}; where: \beta_{iM} = \frac{\sigma_{iM}}{\sigma_M^2}; \sigma_{iM} = Cov(r_i, r_M)$

APT: *(No arbitrage):* ***RPremium*** $= \boldsymbol{\pi = \widetilde{r_p} - r_{free}} = \sum_{i=1}^{k} \boldsymbol{\lambda_i b_{p,i}}$: Factor Risk premium= price of risk= $\boldsymbol{\lambda}$ (**linear**) = ***same*** $\forall$***p; ::: Risk Premium = Price of Risk x Quantity of*** *Risk (factor loading β)*

Single factor: $\tilde{r_i} = \bar{r_i} + \tilde{b_i}\tilde{f} + \tilde{\epsilon}_i \therefore Cov(\widetilde{r_i}, \tilde{r_j}) = b_i b_j \sigma_f^2$

$\boldsymbol{SecurityML} = f(\beta, return); \boldsymbol{CapitalML} = f(\sigma, return)$

Alpha choice: $r_i = r_f + \alpha_i + \beta_i(r_M - r_f) + \epsilon_i; where: SR_T = \sqrt{(SR_M^2 + SR_P^2)}; SR_p = \frac{\alpha_i}{\sigma_{\epsilon_i}}$; *with the tangency port consist of:*

$$w_M = \lambda\frac{r_M - r_f}{\sigma_M^2}; w_P = \lambda\frac{(r_P - r_f)}{\sigma_P^2} = \lambda\frac{\alpha_i}{\sigma_{\epsilon_i}^2}; \beta_P = \frac{Cov(r_P, r_M)}{Var(r_M)} = \frac{\rho_{P,M}\sigma_P}{\sigma_M}$$

APT for well-divert, COC: $\boldsymbol{E_P = r_f + \Sigma\lambda_i \beta_{P,i}}$

If Risk = totally idiosync + diversifiable ↔ $\boldsymbol{r_\pi = 0}$ $(no\ premium) = COC$

$\pi_{default}(\boldsymbol{DP}) = Promised_{YTM}(No\ Default) - Expected_{YTM}(prob)$

$\pi_{risk}(\boldsymbol{RP}) = Expected_{YTM} - Free_{YTM};$

$$B_{PromisedYield}: y = \frac{\bar{y} + p\lambda}{1 - p\lambda}, \lambda = loss\ rate: p = default\ probability$$

MM1 with tax *(X=terminal CF; delta =tax on debt **int**; pi=tax on equity; t=CIT):*

$$V_L = V_U + PVTS - PVDC = \frac{(1 - \tau)}{1 + r_A} X + \tau\left(\frac{r_D D}{1 + r_D}\right) - PVDC = APV$$

$V_L = V_U + PVTS = (1 - \boldsymbol{\pi_E})(1 - \tau)PV(X) + [(1 - \boldsymbol{\delta_D}) - (1 - \boldsymbol{\pi_E})(1 - \tau)]PV(r_D D)$

$\boldsymbol{MM2}$: **Leverge with constant ratio** $\boldsymbol{w_D}$ **&** $\boldsymbol{w_E}$ **& tax** $\boldsymbol{r_{TS} = r_A}$

$\boldsymbol{WACC}(r_A; COC) = w_D \boldsymbol{r_D} + w_E \boldsymbol{r_E}; \boldsymbol{r_E} = r_A + \frac{D}{E}(r_A - r_D)$

$\boldsymbol{\beta_A} = w_E \beta_E + w_D \beta_D = \Sigma w_i \beta_i; \boldsymbol{\beta_E} = \beta_A + \frac{D}{E}(\beta_A - \beta_D)$

$$\boldsymbol{WACC\_\tau} = r_L - w_D \tau r_D = (1 - \tau) w_D r_D + w_E r_E = \left(\frac{D}{D + E}\right) r_D + ..$$

$$V_L = E + D = V_U + PVTS = \sum_{s=1}^{\infty} \frac{(1 - \tau)}{(1 + r_A)^S} X_S + PVTS = APV$$

NPV by WACC/tax: $V_L = \sum_{s=1}^{\infty} \frac{(1-\tau)}{(1+WACC)^s} X_S = APV; \boldsymbol{V_L^g} = \frac{(1-\tau)EBIT}{WACC-g}$

Ex-dividend = on/after trade **NO** more receive dividend

Cum-dividend = on/before trade **YES** to receive dividend

Hedging: $\Delta V_{hedged} = \Delta V_{original} + (HedgeRatio) * \Delta V_{hedging}$

Perfect hedge (**no risk**): $(1) Corr(\Delta V_{org}; \Delta V_{hedging}) = 1; (2) HR: appropriate$ – Bond use **Modified Duration**

Interest hedge: $V_P = \Sigma V_i = \Sigma n_i B_i;\ \Delta V_P \approx (-\Delta y)(\Sigma V_i MD_i)$

$MD_P = \Sigma w_i MD_i$; **Hedge Ratio** = **Orig**inal / **Hedge Inst**rument

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*** Previous FMF-1 ***

Stock Price: Discount factor = expected return E(r):

$$P_0 = \frac{D_1 + P_1}{1 + r_1} + \cdots = \sum_{t=1}^{T} \frac{D_t}{(1+r_t)^t} + \frac{P_T}{(1+r_T)^T}$$

Gordon Model $\langle r = const; g < r: D_{t+1} = (1+g)D_t \rangle$; then DCF formula: $\boldsymbol{P_0} = \frac{\boldsymbol{D_1}}{\boldsymbol{r-g}}$

Payout: $p = \frac{DPS}{EPS} = \frac{Dividend}{Earnings} \to DPS = p * EPS$

Plow back ratio: $b = \frac{RE}{E} = 1 - p; \boldsymbol{g = ROE(1-p)}$

Book Value: $BVPS_{t+1} = BVPS_t + EPS_{t+1} \times b = BVPS_t + I_{t+1}$; $I_t = EPS_t \times b_t$; $EPS_{t+1} = EPS_t + ROI_t \times I_t$;**Growth** stocks = $opportunity: NPV(opp) > 0$:

$$P_0 = \frac{EPS_1}{r} + PVGO\ ; where: \frac{P}{E} = \frac{P_0}{EPS_1}\left(fwd\frac{P}{E}\right); PVGO = \frac{NPV_1}{r-g}$$

$\boldsymbol{No\ growth} \Leftrightarrow All\ by\ dividend: g = 0\ \&\ p = 1$

Annuity w/ grow g (**start @1**: $CF_1 = A$; $CF_2 = A(1+g)$...)

$$PV(0) = A \times \begin{cases} \frac{\boldsymbol{1}}{\boldsymbol{r-g}}\left[\boldsymbol{1} - \left(\frac{\boldsymbol{1+g}}{\boldsymbol{1+r}}\right)^{T}\right], if\ r \neq g \\ \frac{T}{1+r} \qquad , if\ r = g \end{cases}$$

Modified Duration: $MD = -\frac{1}{B}\frac{dB}{dy} = \frac{D}{1+y}$;

Discount bond, $B_t = (1+y)^{-t}$: $MD(Bt) = -\frac{1}{B_t}\frac{dB_t}{dy} = \frac{t}{1+y}$

Macaulay duration (weigted avg term to maurity) – D:

$$D = \sum_{t=1}^{T}\left[\frac{PV(CF_t)}{B}\right] t = \frac{1}{B}\sum_{t=1}^{T}\left[\frac{CF_t}{(1+y)^t}\right] t$$

Bond price: $(\Delta B) \approx [-MD \times (\Delta y) + CX \times (\Delta y)^2] \times B$; $B(P) = \sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}$:

Bond Convexity: $CX = \frac{1}{2}\frac{1}{B}\frac{d^2B}{dy^2}$;

$$CX = \frac{1}{2}\frac{1}{B(1+y)^2}\sum_{t=1}^{T}\frac{CF_t}{(1+y)^t} t(t+1)$$

$$\boldsymbol{VAR} = \sigma^2 = E[(\tilde{r} - \bar{r})^2];\ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$$

Std Deviation (**Volatility, Risk**): $\boldsymbol{\sigma} = \sqrt{\boldsymbol{VAR}}$

Correlation *(stand*ardized) & **Covariance**:

$$Cov(\tilde{r}_i, \tilde{r}_j) = \sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{1,t} - \hat{r}_1)(r_{2,t} - \hat{r}_2)$$

$$Corr(\tilde{r}_i, \tilde{r}_j) = \rho_{ij} = \frac{Cov}{\sigma_i \sigma_j} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{E[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)]}{\sigma_i \sigma_j}$$

Portfolio Management: $\widetilde{\boldsymbol{r_p}} = \overline{\boldsymbol{r_p}} + \sum_1^N \boldsymbol{b_{p,k} f_k} + \widetilde{\boldsymbol{\epsilon_p}}$

$E[r_p] = \bar{r}_p = \sum w_i \bar{r}_i;\ b_{p,k} = \Sigma w_i b_{ik}; \widetilde{\epsilon_p} = \Sigma w_i \epsilon_i;$

$$\sigma_p^2 = Var[\tilde{r}_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}\ ; \sigma_{ii} = \sigma_i^2$$

$Var(\widetilde{\epsilon_p}) = \Sigma w_i^2 Var(\tilde{\epsilon}_i)$; **risk-free:** $\boldsymbol{r_p \notin f_p \leftrightarrow b_p, \epsilon_p = 0}$

Portfolio return with risk-free assets:

$\tilde{r}_p = (1 - \Sigma w_i) r_f + \Sigma w_i \tilde{r}_i = r_f + \Sigma w_i(\tilde{r}_i - r_f)$

Equally weighted portfolio of n-assets:

$$\sigma_p^2 = \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2 - n}{n^2}\right)\left(\frac{1}{n^2 - n}\sum_{i=1}^{n}\sum_{j \neq i}^{n}\sigma_{ij}\right)$$

$$= \left(\frac{1}{n}\right)(average\ VARiance) + \left(1 - \frac{1}{n}\right)(average\ COVARiance)$$

Single factor: $\tilde{r}_i = \bar{r}_i + \tilde{b_i}\tilde{f} + \tilde{\epsilon}_i\ \therefore Cov(\tilde{r}_i, \tilde{r}_j) = b_i b_j \sigma_f^2$

Well-diversified portfolio: $Var(\tilde{\epsilon}_p) = 0 \therefore \widetilde{\epsilon_p} = E[\tilde{\epsilon}_i] = 0 \therefore$ $Only\ Sys. f: \tilde{r}_p = \bar{r}_p + \sum b_i f_{i,k} \therefore Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, i \neq j$

*** **Var[X]=Cov[X, X]=** $\boldsymbol{\sigma_X^2}$; **Var (X + a) = Var (X);** ***

X,Y **uncorrelated**: Cov(X,Y) = 0; E(X*Y) = E(X)*E(Y)

$$\boldsymbol{Var}(\boldsymbol{aX}) = \boldsymbol{a}^2 Var(X);\ \boldsymbol{Var}[\sum \boldsymbol{X_i}] = \sum Var[X_i] + \sum_{i=1}^{N}\sum_{i \neq j}^{N} Cov[X_i, X_j]$$

$Var[X_1 + X_2] = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

$Cov(X, Y) = E[(X - \bar{X})(Y - \bar{Y})] = E(X * Y) - E(X) * E(Y)$

$Cov(aX + bY, cW + dV) = acCov(X, W) + \cdots$

APT (util, factor structure, only for well-divert, multi idio factor)

CAPM (mean-var, not required, exact to all, market return)

Excel: solver (**Goal**-seek) for any equation