15.415x Foundations of Modern Finance

Leonid Kogan and Jiang Wang
MIT Sloan School of Management

Lecture 11: Forwards and Futures

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Ancient derivatives

Financial contracts, with many features found in modern derivatives such as forwards, futures, and options, date back to early periods of human history.

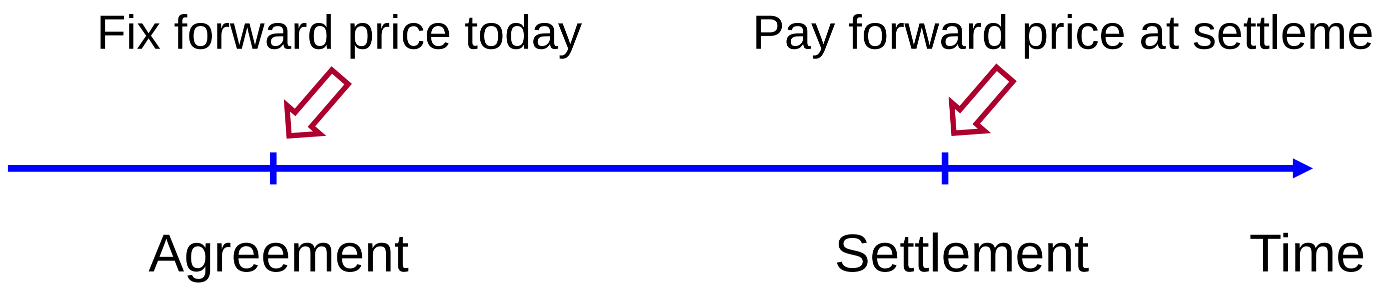
In Ancient Mesopotamia, … Some types of contracts were arrangements on the future delivery of grain that stipulated for instance before planting that a seller would deliver a certain quantity of grain for a price paid at the time of contracting. Such types of contracts not only dealt with grain but also with all sorts of commodities. …

These types of contracts had the features of today's forwards and were used across borders. By about 1,400 BC, cuneiform script in the Babylonian language was even used in Egypt to record transactions with Crete, Cyprus, the Aegean Islands, Assyria and the Hittites.

Source: S. Kummer and C. Pauletto, 2012, "The History of Derivatives: A Few Milestones"

Forward contracts

- A **forward contract** is a commitment to buy (sell) at a future date a given amount of a commodity or an asset at a price agreed on today.



- The price fixed now for future exchange is the forward price.
- The buyer obtains a “long position” in the asset/commodity.

An example: forward contract

- A tofu manufacturer needs 100,000 bushels of soybeans in 3 months.
- Current price of soybeans is $12.50/bu but may go up.
- Wants to make sure that 100,000 bushels will be available.
- Enter 3-month forward contract for 100,000 bushels of soybeans at $13.50/bu.
- Long side buys 100,000 bushels from short side at $13.50/bu in 3 months.

Features of forward contracts

direct trade centralized

- Traded over the counter (not on exchanges);
- Custom tailored;
- No money changes hands until maturity.
- Advantages of forward contracts:
  - Full flexibility;
  - No payments prior to contract maturity.
- Disadvantages of forward contracts:
  - Illiquidity; over-the-counter customized nature makes it costly for one of the counterparties in the forward to get out of the contract prior to its settlement
  - Counterparty risk; the losing side of the contract may not fulfill their obligations at the settlement date.
  - High collateral requirements (to mitigate default risk).

To prevent risk, participants need to post collateral, which they would lose in the event of failing to honor the terms of the contract.

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Forward interest rates

- So far, we have focused on spot interest rates: rates for a transaction between today, date 0, and a future date t, denoted r_t.
- Now, we study forward interest rates: rates for a transaction between two future dates, for instance, $t = 1$ and $t = 2$.
- For a forward transaction to borrow money at $t = 1$:
  - Terms of the transaction are agreed on today, $t = 0$;
  - Loan is received on a future date $t = 1$;
  - Repayment of the loan occurs on date $t = 2$.
- Note: Future spot rates are random, they can be different from current corresponding forward rates.

Example: forward interest rate

- As the CFO of a U.S. multinational, you expect to repatriate $10M from a foreign subsidiary in 1 year, which will be used to pay dividends 1 year later.

- Not knowing the interest rates in 1 year, you would like to lock into a lending rate one year from now for a period of one year.

- The current interest rates are as follows.

Time to maturity t (years)	1	2
Spot interest rate r_t	0.05	0.07

- What should you do?

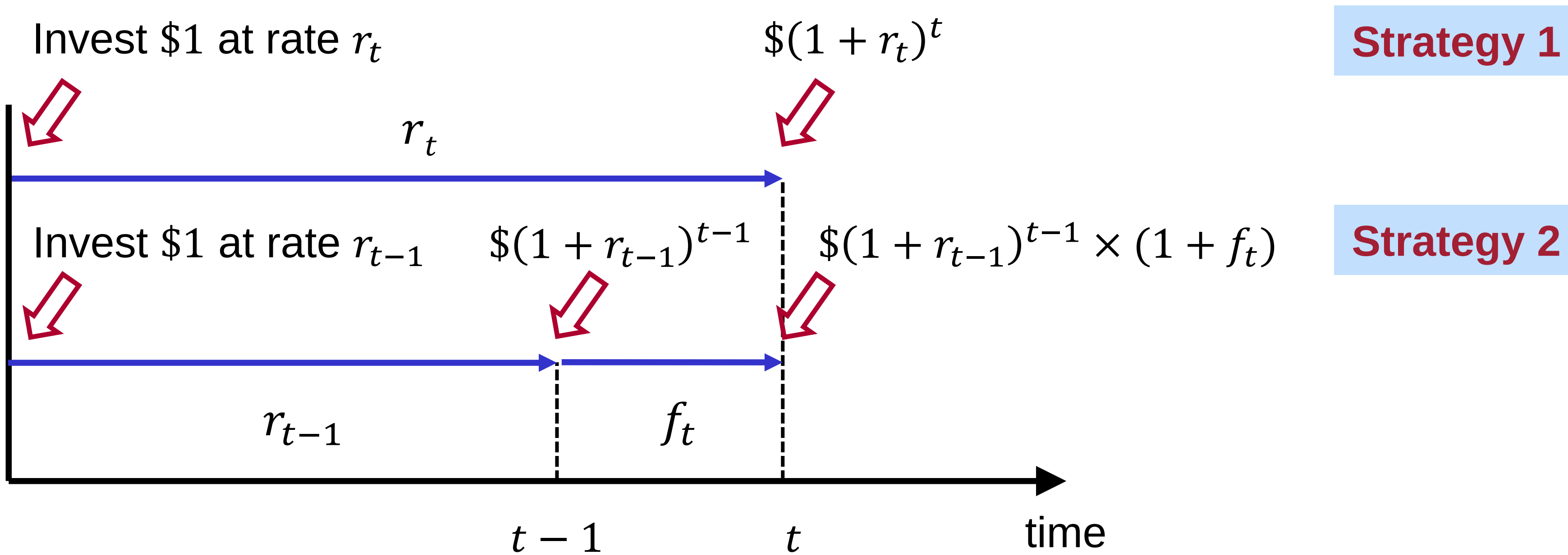
Example: forward interest rate

- Strategy:
  - Borrow $9.524M now for one year at 5%;
  - Invest the proceeds $9.524M for two years at 7%.
- Outcome (in million dollars):

Year	0	1	2
1-yr borrowing	9.524	-10,000	0
2-yr investment	-9.524	0	10,904
Repatriation	0	10,000	0
Net	0	0	10.904

- The locked-in 1-year lending rate 1 year from now is 9.04%.

Forward interest rates vs spot rates



- The forward interest rate between time $t-1$ and t satisfies:

$$(1+r_t)^t = (1+r_{t-1})^{t-1}(1+f_t)$$

Strategy 1 Strategy 2

or

$$f_t = \frac{B_{t-1}}{B_t} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1$$

Example: forward interest rates

- Suppose that discount bond prices are as follows:

t	1	2	3	4
B_t	0.9524	0.8900	0.8278	0.7629
r_t	0.05	0.06	0.065	0.07

- A customer wants a forward contract to borrow $20M for one year in three years from now. Can you (a bank) quote a rate?

- Answer:

$$f_4 = \frac{(1+r_4)^4}{(1+r_3)^3} - 1 = \frac{(1+0.07)^4}{(1+0.065)^3} - 1 = 8.51\%$$

Example: forward interest rates

- What should you do today to lock-in these cash flows?

1. Buy 20,000,000 of 3-year discount bonds, costing

$$(\$20{,}000{,}000)(0.8278) = \$16{,}556{,}000$$

2. Finance this by selling 4-year discount bonds with face value of

$$\frac{\$16{,}556{,}000}{0.7629} = \$21{,}701{,}403$$

3. This creates a liability in year 4 in the amount $21,701,403.

Example: forward interest rates

- Cash flows from this strategy (in million dollars):

Year	0	1-2	3	4
Purchase of 3-year bonds	−16.556	0	20.000	0
Sale of 4-year bonds	16.556	0	0	−21.701
Total	0	0	20.000	−21.701

- The interest for this future investment is given by:

$$\frac{21{,}701{,}403}{20{,}000{,}000} - 1 = 8.51\%$$

Forward rates and the expectations hypothesis

- We can re-state the expectations hypothesis (EH) in terms of the relation between spot and forward rates.

- Under the EH, expected returns on all bonds are the same, and

$$\mathrm{E}_0[\tilde{r}_1(t)] = \underbrace{\frac{\left(1+r_{t+1}(0)\right)^{t+1}}{\left(1+r_t(0)\right)^t}}_{1+f_{t+1}} - 1 = f_{t+1}$$

As of time 0

- Under the EH, forward rates are unbiased predictors of future spot rates.

- Empirically, forwards rates over-predict future spot rates on average: forward rate reflects a risk premium in addition to the expectations of the future spot rates.

Investors regard long bonds as riskier than short bonds, earn a premium λ_t -- "risk premium", or "liquidity premium" invest in (t+1)-bond is less liquid and more risky than t-bond plus 1-bond, t+1 should earn more

$$\mathrm{E}_0[\tilde{r}_1(t)] + \lambda_t = \left\{ \frac{\left(1+r_{t+1}(0)\right)^{t+1}}{\left(1+r_t(0)\right)^t} - 1 \right\}$$

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Financial forwards

- Stock index forwards, e.g., S&P 500, Nikkei,…
  - Underlying: baskets of stocks.
- Fixed income forwards.
  - Underlying: fixed income instruments (T-bonds,…).
- Currency forwards.

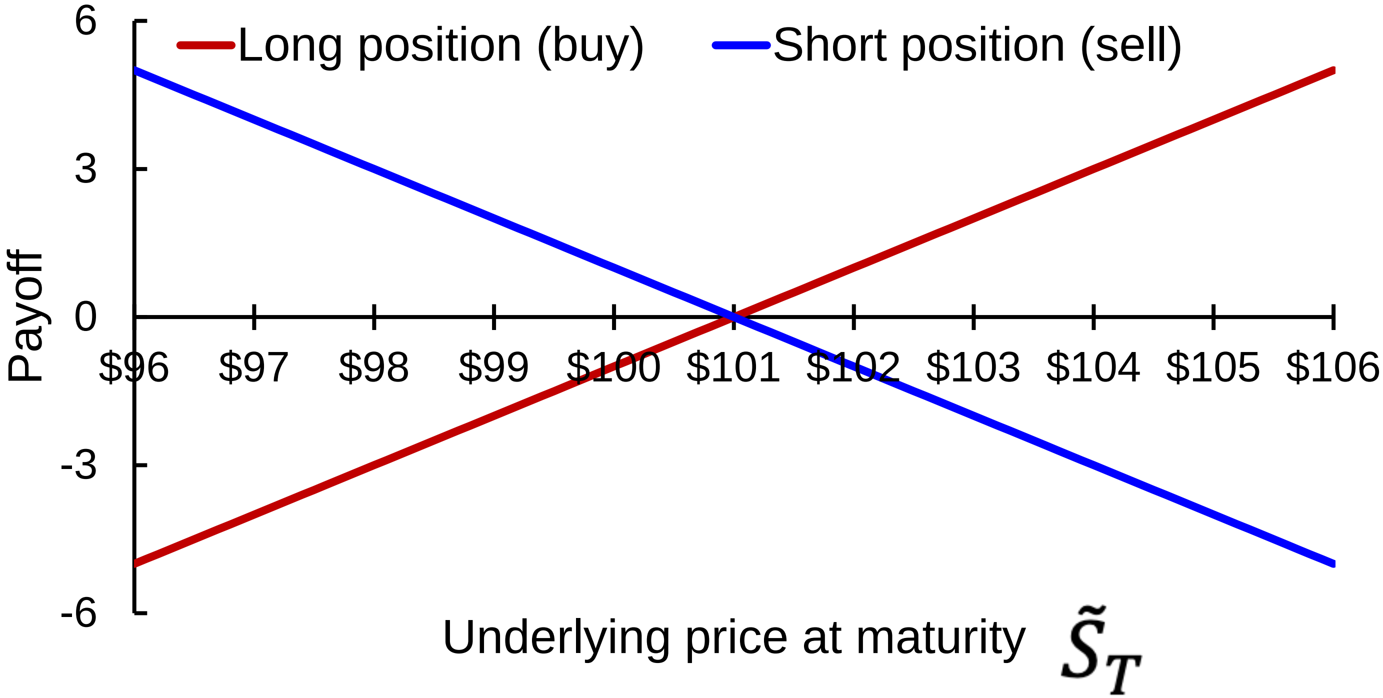
Notation

Contract	Spot now	Spot at T	Forward
Price	S_0	$\tilde{S}_T$	F_T

- The current spot price of the underlying asset is S_0.
- The spot price at maturity $\tilde{S}_T$ is random. T: settlement date of the forward (maturity date)
- The forward price F_T is fixed at time zero so that the market value of the forward contract equals zero. no money changes hands initially
- Risk-free rate is constant. Denote continuously compounded interest rate by r.

Payoff diagrams of forwards

- Forwards are derivative securities.
  - Payoffs tied to prices of underlying assets/commodities;
- Payoffs are linear in underlying asset price: $\tilde{S}_T - F_T$.



Forward prices

replication strategy

- Forward prices are linked to spot prices.

two ways should have no arbitrage

- Two ways to buy the underlying asset for date-T delivery:

  1. Buy a forward contract with maturity date T,
  2. Buy the underlying asset today and hold it until T.

holding the asset until T
without influence of final price

A model of payout

dividend yield: related
expected change in spot price: unrelated

dividend is proportional to spot price, whatever price will be,
reinvest of all dividend will grow the size exponentially

- Consider a financial asset paying dividends (or coupons).
- We assume that the asset pays a continuous flow of dividends proportional to the asset price: dividend yield is constant, y.
- Reinvest dividends: number of units of the asset grows exponentially at rate y.
- Start with one share of the asset; by time T hold e^{yT} shares.
- To accumulate one share by T, need to start with e^{-yT} and continuously reinvest dividends.
- We conclude that the time-0 present value of $\tilde{S}_T$ ($\tilde{S}_T$ is the price of one share at time T) equals the time-0 value of e^{-yT} shares:

how to obtain one share at time T:
buy e^(-yT) share at time 0

$$PV_0\left(\tilde{S}_T\right) = e^{-yT} S_0$$

because of the reinvest of dividend, size will grow exponentially to 1

Forward price

- The payoff of the forward contract at maturity (long position) is $\tilde{S}_T - F_T$.
- The forward price is set so that the market value of this cash flow at time $t = 0$ is zero:

$$PV_0\left(\tilde{S}_T - F_T\right) = 0$$

- Therefore,

$$PV_0(F_T) = e^{-rT} F_T = PV_0\left(\tilde{S}_T\right) = e^{-yT} S_0$$

- We find the forward price:

The forward price does not depend on the expected rate of return

$$F_T = e^{(r-y)T} S_0$$

Forward price is not equal to the expected spot price at maturity!

if asset is very profitable (i.e. y >> r), why should buyer wait until time T to buy it? The wise buyer will enter the contract only if the quote forward price is very low

if this growth rate is high, the forward price (guaranteed price of 1 share in the future) should be low because if its not, you can achieve the same target in a cheaper manner by buying fewer shares now and continuously reinvesting dividends such that you get to 1 share in the future.

Replicating a forward

market price forward price

- The payoff of the forward contract at maturity (long position) is $\tilde{S}_T - F_T$.
- To replicate the forward contract, we replicate two components of its payoff:
  - We buy e^{-yT} units of the underlying asset at $t = 0$ and reinvest the dividends back into the asset continuously – receive $\tilde{S}_T$ at time T (see slide 21);
  - We borrow the present value F_T, which is $e^{-rT}F_T = e^{-yT}S_0$ -- receive $-F_T$ at time T.
- This transaction has zero initial value, and produces a payoff equal to the payoff of the forward at maturity. thus replicate the cash flow of a forward

Example: a forward on a stock index

- The underlying asset (basket of stocks) pays dividends.
- Data:
  - S&P 500 closed at the end of June at 3,000.00;
  - S&P 500 forward with settlement at the end of September has a forward price of 2,995.00;
  - The 3-month interest rate 1.5% (annualized, continuously compounded).

0.25 = a quarter

$$F_T = e^{0.25\times(r-y)} S_0$$

$$\Rightarrow y = r - 4\ln\left(\frac{F_T}{S_0}\right) = 1.5\% - 4\ln\left(\frac{2{,}995.00}{3{,}000.00}\right) = 2.17\%$$

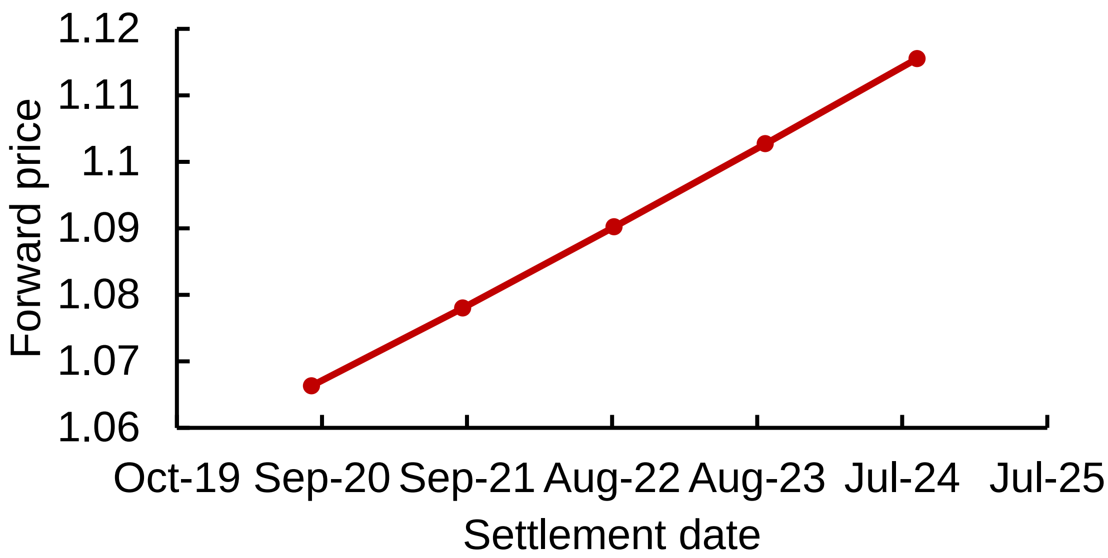
- Dividend yield implied by the market prices (spot and forward) is $y = 2.17\%$.

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Currency forwards

- A forward contract to exchange a unit of one currency for a specified number of units of another currency.
- Example: on 7/20/2020, a forward contract to exchange one Swiss Franc for 1.10 US Dollars in July of 2023.
  - Forward prices differ among contracts with different settlement dates:



Data source: CME Globex. We use Swiss Franc Futures Quotes on 07/20/2020 to approximate forward prices.

Pricing of currency forwards

considered: price of Swiss Francs

- Suppose the forward price is $\$F_T$ for ₣1; contract matures at time T.
- Let X_t denote the spot exchange rate: the price of ₣1 in USD.
- Let r_{USD} and r_{CHF} denote continuously-compounded spot interest rates for tenor T, in US Dollars and Swiss Francs, respectively.
- When invested at the risk-free rate, the number of Swiss Francs grows exponentially at the rate of r_{CHF}.
  - The Swiss Franc position is effectively a financial asset with the dividend yield $y = r_{CHF}$.
- We conclude that the forward exchange rate is given by

the future spot price is still unknown, thus the payoff is unknown

$$F_T = X_0 e^{(r_{USD} - r_{CHF})T} \qquad \boldsymbol{F_T = e^{(r-y)T} S_0}$$

- The relation $F_T = X_0 e^{(r_{USD} - r_{CHF})T}$ is called the covered interest rate parity – it is a no-arbitrage condition.

Replication of currency forwards

- At $t = 0$:

  1. Borrow $\$F_T e^{-r_{USD}T} = \$X_0 e^{-r_{CHF}T}$ at the interest rate r_{USD};
  2. Convert the borrowed amount into ₣$e^{-r_{CHF}T}$;
  3. Invest the proceeds (₣$e^{-r_{CHF}T}$) at the interest rate r_{CHF} (buy a CHF-denominated discount bond).

- At time $t = T$:

$$\text{₣}1 - \$F_T$$

Position (3) Repay loan (1)

- The payoff is identical to the forward. replicated!

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Futures contracts: main characteristics

more liquid

- A futures contract is an exchange-traded, standardized, forward-like contract that is marked to market daily.

The main distinction from the forwards is that futures contracts are marked to market daily.

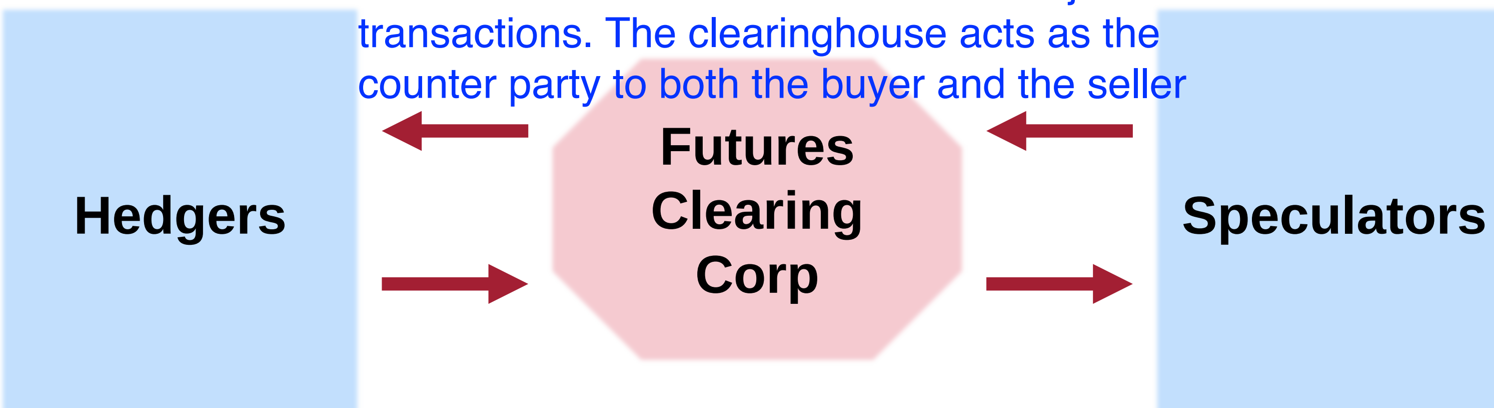
- Standardized contracts:
  - Underlying commodity or asset,
  - Quantity,
  - Maturity.
- Settlement: physical delivery or cash.

such as deliver cash instead of basket of stocks

Trading of futures contracts

- Traded on exchanges:
  - CME Group, National Stock Exchange of India, Intercontinental Exchange, CBOE, Eurex, NASDAQ, Shanghai Futures Exchange, etc.
- Guaranteed by the clearing house – little counter-party risk.

each transaction is broken into two major transactions. The clearinghouse acts as the counter party to both the buyer and the seller



recognized

- Gains/losses settled daily – marked to market.
- Margin account required as collateral to cover losses.

Margin account

- Example: NYMEX crude oil (light sweet) futures with delivery in Oct. 2019 were traded at a price of $58.83/barrel on July 1, 2019.
- Each contract is for 1,000 barrels.

both the buyer and the seller must deposit this amount into their margin accounts.

- Initial margin: $3,960.

leverage: this amount represents a small fraction of the value of oil covered by a single contract

- Maintenance margin: $3,600.

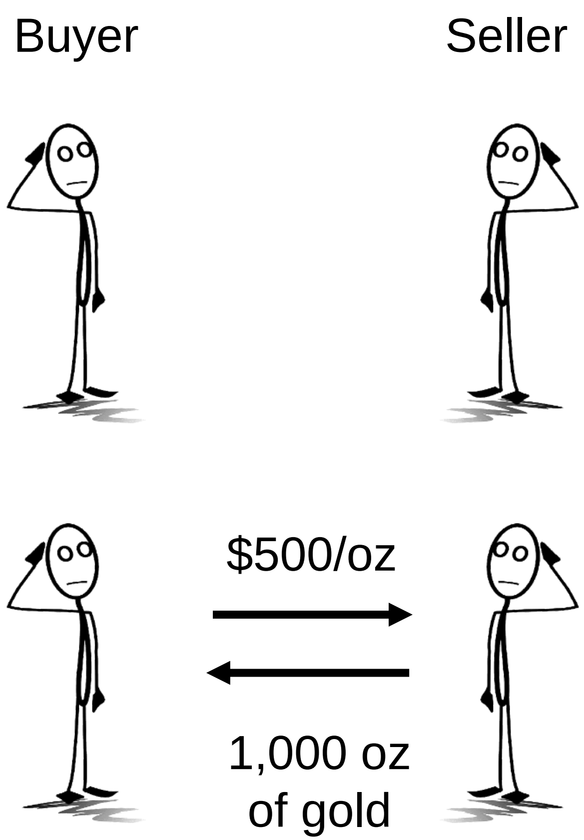
Once the balance in the margin account falls below the maintenance margin, traders are required to restore the margin balance to the level of the initial margin. Alternatively, their position may be reduced or liquidated.

- No cash changes hands today (contract price is $0).

When traders enter into this contract, no cash changes hands. The total market value of the contract is zero.

- Buyer has a "long" position (wins if prices go up).
- Seller has a "short" position (wins if prices go down).

Mark to market: a forward contract



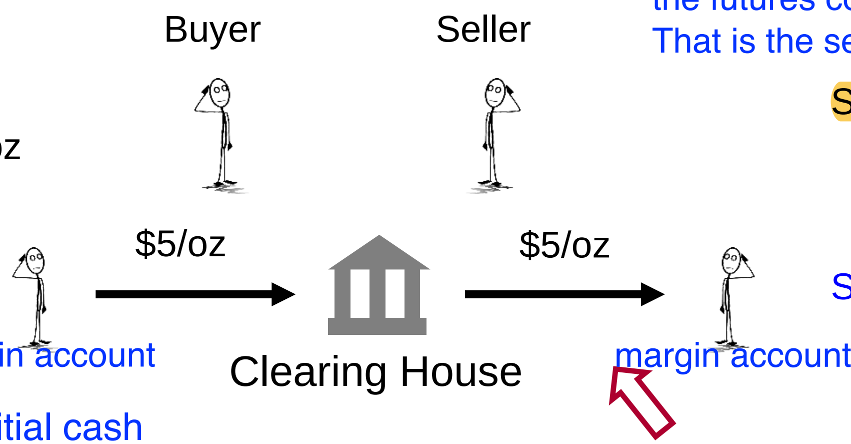
December 12: enter into a forward contract on gold.
Forward price: $500/oz

December 15: settlement

Mark to market: a futures contract

the futures contract closes at $495 per ounce.
That is the settlement price for that day.

Enter a forward contract at $500/oz



Settle at $495/oz December 12

Settle at $491/oz December 13

future do not need initial cash exchange, why there are loss and gain during period?

Futures contracts can be traded purely for profit, as long as the trade is closed before expiration. Closing a position refers to executing a security transaction that is the exact opposite of an open position, thereby nullifying it and eliminating the initial exposure. Closing a long position in a security would entail selling it, while closing a short position in a security would involve buying it back.

$5 = $500 - $495: settle gains/losses for 12/12. This is the timing convention we use in our example. It is also common to assume that $5 gain/loss occurs immediately when market closes on 12/12 – that's an alternative timing assumption.

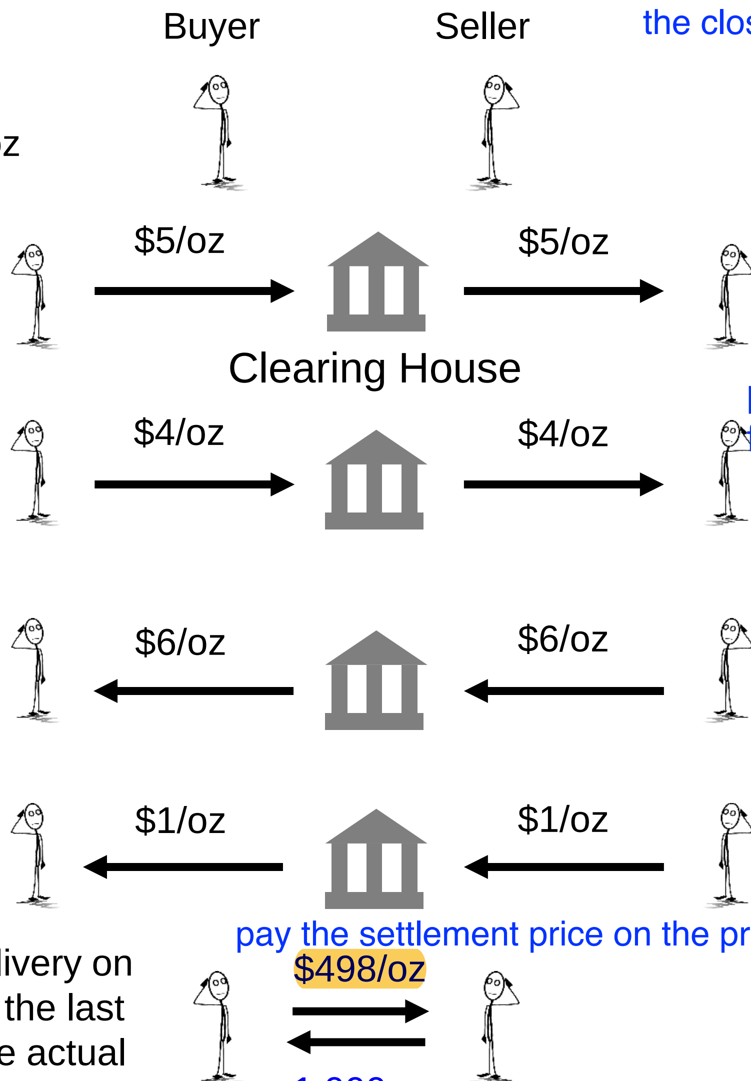
buyer can nullify the future by sell at price 495, which means that buyer can pay 500 and re-sell at 495 at maturity, loss is 5

seller can nullify the future by buy at price 495, which means that seller can pay 495 and re-sell at 500 at maturity, gain is 5

Mark to market: a futures contract

the cumulative gain or loss on the futures contract is the same as on the forward. It is \$2 per ounce, based on the difference between the initial futures price of \$500 and the closing price of \$498 on December 15.

Enter a forward contract at \$500/oz



Settle at \$495/oz December 12

cash settle
Settle at \$491/oz December 13

price of the futures contract moves (price of the future is the price of the maturity transaction)

Settle at \$497/oz December 14

Settle at \$498/oz December 15

December 16

pay the settlement price on the previous day

Delivery December 16

Here we show delivery on the next day after the last day of trading. The actual time window for delivery may be wider and differs across futures contracts.

Forwards prices vs futures prices

- Both forward and futures prices are linked to spot prices.

timing of cash flow

- Differences have to do with the mark-to-market process for futures.

Contract	Spot now	Spot at T	Forward	Futures
Price	S_0	$\tilde{S}_T$	F_T	H_T

Under the forward contract, the total gain or loss are realized at contract maturity. Under the futures contract, gains and losses are realized incrementally, with daily mark-to-market throughout the life of the contract.

- Ignore differences between forward and futures prices for now:

$$F_T \approx H_T$$

- Futures are different from forwards under stochastic interest rates.

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Commodity forward and futures prices

- Can price commodity futures using the same arbitrage argument as for financial futures.
- When holding commodities, net payout must reflect storage costs and any effective convenience yield from holding the physical commodity, called convenience yield.
- Continuous compounding, assume storage costs flow in proportion to commodity value:

$$Cost_t = cS_t$$

- Valuation formula is

$$H_T \approx F_T = e^{(r-\hat{y})T} S_0$$

where $\hat{y}$ denotes the net convenience yield

$$\hat{y} = y - c$$

the benefits of storing the commodity, net of storage costs.

Example: gold futures

rather than industrial production

- Gold is often held for long-term investments. produces no convenience yield
- Easy to store — negligible cost of storage.
- No dividends or benefits: therefore zero net convenience yield.
- Futures price:

$$H_T \approx F_T = S_0 e^{rT}$$

Example: gold futures

$$H_T \approx F_T = S_0 e^{rT}$$

- Prices on 2019.07.01:
  - Spot price of Gold: $1,387.45/oz;
  - 2019 October futures (CME): $1,397.80/oz.

The futures price for the contracts maturing in October

- Implied continuously-compounded interest rate is r = 2.23%, relative to the 3-month T-bill rate of 2.05%.

Example: oil futures

- Unlike gold, held for future use and not for long-term investment.
- Costly to store.
- Additional benefits (convenience yield) for holding physical commodity (over holding futures). liquidity premium investors can demand in times of sudden scarcity/demand spikes
- Valuation equation:

$$H_T \approx F_T = e^{(r-\hat{y})T} S_0$$

Example: oil futures

$$H_T \approx F_T = e^{(r-\hat{y})T} S_0$$

- Prices on 2019.07.19:
  - Spot oil price 55.68/barrel (light sweet);
  - October oil futures price 55.83/barrel (NYMEX);
  - 3-month continuously-compounded interest rate is 2.3%.
- 3.5 months to expiration.
- Annualized net convenience yield: solve $55.83 = e^{(0.023-\hat{y})\times\frac{3.5}{12}} 55.68$ to find

$$\hat{y} = 1.37\%.$$

Prices of commodity futures

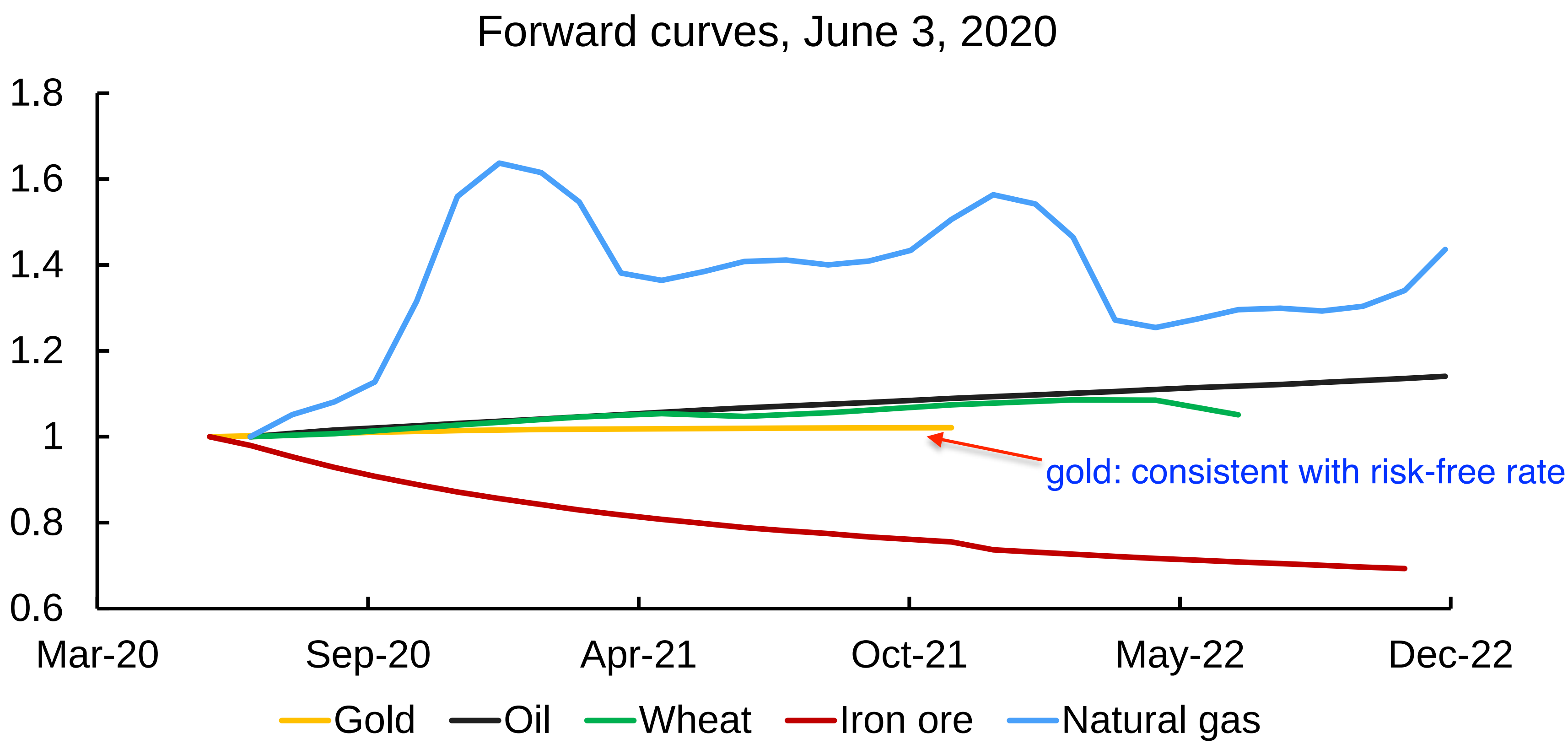
- For commodity futures:
  - Contango: Futures prices increase with maturity;
  - Backwardation: Futures prices decrease with maturity.
- Backwardation occurs if net convenience yield exceeds the interest rate:

$$\hat{y} - r = y - c - r > 0$$

- Another definition adjusts for the time-value of money:
  - Contango: $H_T > S_0 e^{rT}$; increase more than risk-free
  - Backwardation: $H_T < S_0 e^{rT}$.

Various shapes of commodity forward curves

- Forward curves plot futures prices across contracts with different maturity dates.



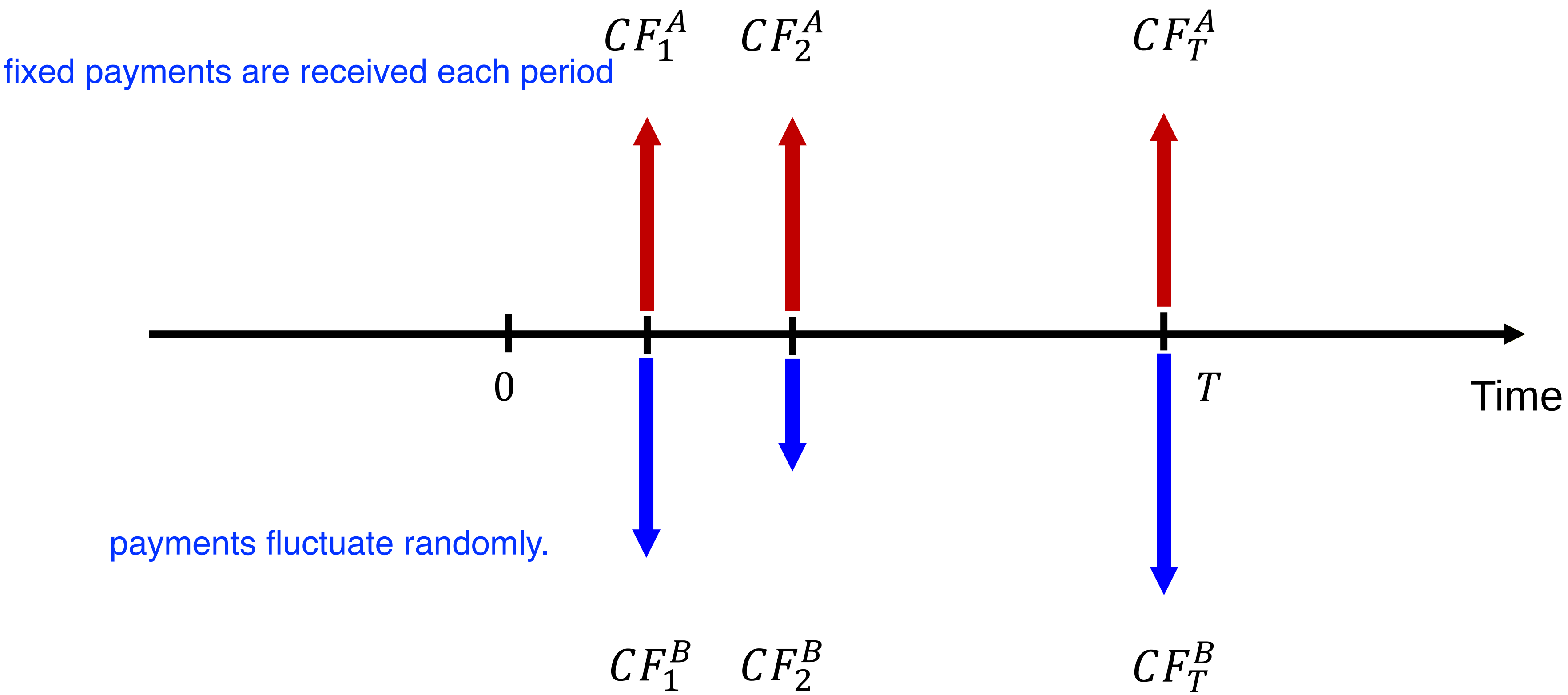
Storage-based models are also more suitable for short-dated agricultural contracts, such as sweet futures than for longer dated contracts, which are affected by supply from future harvests.

Key concepts

- Introduction: forward contracts
- Forward interest rates
- Pricing of forwards on financial assets
- Currency contracts
- Futures: introduction
- Commodity futures
- Swaps

Swaps

- Swap: A contract in which two counterparties agree to exchange specified amounts of assets (e.g., cash, financial assets or commodities) at a set of given future dates.



Example: LIBOR swap

- Interest rate swap: a fixed rate of interest is exchanged for a reference floating rate.
- For example, the London Interbank Offered Rate (LIBOR) could be used as the reference floating rate.
- Payments are made periodically, e.g., at the end of each 6-month sub-period.
- Example: assume the current 1-month LIBOR is 0.5%. You enter into a 5-year fixed-for-floating swap with the fixed rate of 0.7%.
  - If LIBOR rises in the future, you receive higher payments on the floating leg, continue making fixed payments on the fixed leg.
  - The swap represents a bet on higher future values of LIBOR.

pay fixed rate, get real rate, hope that real rate is higher

Valuation of an interest rate swap

- Suppose that at the end of each period t, the floating leg of the swap pays the spot risk-free rate over that period, $\tilde{r}_1(t-1)$.
- The fixed leg pays a fixed rate, r_S.
- The fixed rate is chosen so that the swap contract has zero market value initially: no money changes hands.
- The swap is over T periods. with payments made at the end of each period
- What is the swap rate r_S?
- First, need to establish a relation between the forward rates and the future spot rates.

Forward rates and future spot rates

- Consider 2 strategies, both start with a $\$1$ initial investment.
- Strategy 1:
  - At $t = 0$, invest $\$1$ at the risk-free rate $r_T(0)$ up to time T.
  - At time T, re-invest $\left(1 + r_T(0)\right)^T$ for one more period at the spot rate $\tilde{r}_1(T)$.
- Strategy 2:
  - At $t = 0$, enter into a forward contract to invest the amount of $\left(1 + r_T(0)\right)^T$ at time T for one period.
  - At $t = 0$, invest $\$1$ at the risk-free rate up to time T.
  - At time T, re-invest $\left(1 + r_T(0)\right)^T$ for one more period at the forward rate f_{T+1}.

Forward rates and future spot rates

- Payoff of Strategy 1 at time $T+1$ (random):

$$\left(1+r_T(0)\right)^T \times \left(1+\tilde{r}_1(T)\right)$$

- Payoff of Strategy 2 at time $T+1$ (non-random):

$$\left(1+r_T(0)\right)^T \times \left(1+f_{T+1}\right)$$

- Both payoffs have the same PV at $t=0$: $\$1$.
- Conclusion:

$$\mathrm{PV}_0[\tilde{r}_1(T) \text{ at } T+1] = \mathrm{PV}_0[f_{T+1} \text{ at } T+1]$$

- Recall that

$$\mathrm{E}_0[\tilde{r}_1(T) \text{ at } T+1] \neq f_{T+1}$$

because f_{T+1} is fixed while $\tilde{r}_1(T)$ is random, may earn a risk premium.

forward rate reflects a risk premium in addition
to the expectations of the future spot rate.

Valuation of an interest rate swap

- Let B_t denote the time-0 price of a discount bond paying $1 at time t.
- The present value of the fixed leg of the swap is $r_S \times \sum_{u=1}^{T} B_u$.
- The present value of the floating leg of the swap is $\sum_{t=1}^{T} \mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t]$.
- We impose that no money should change hands initially:

no money should change hands initially.

$$r_S \times \underbrace{\sum_{u=1}^{T} B_u}_{\text{PV of cash flows on the fixed leg}} = \underbrace{\sum_{t=1}^{T} \mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t]}_{\text{PV of cash flows on the floating leg}} = \sum_{t=1}^{T} \mathrm{PV}_0[f_t \text{ at } t] = \sum_{t=1}^{T} B_t f_t$$

  - Recall that $\mathrm{PV}_0[\tilde{r}_1(t-1) \text{ at } t] = \mathrm{PV}_0[f_t \text{ at } t] = B_t f_t$.
- Conclude that the swap rate is a weighted average of forward rates:

$$r_S = \frac{\sum_{t=1}^{T} B_t f_t}{\sum_{u=1}^{T} B_u} = \sum_{t=1}^{T} w_t \times f_t \,, \qquad \text{with the weights } w_t = \frac{B_t}{\sum_{u=1}^{T} B_u}$$

Valuation of an interest rate swap

first strategy: invest 1 dollar continously to time t yield $\frac{1}{B_t}$

- Start with $r_S = \frac{\sum_{t=1}^{T} B_t f_t}{\sum_{u=1}^{T} B_u}$. second strategy: invest 1 dollar to time t-1 and reinvest at forward rate yield $\frac{1}{B_{t-1}} * (1 + f_t)$

two strategies should yield same value: $\frac{1}{B_t} = \frac{1}{B_{t-1}} * (1 + f_t)$

- Recall that $f_t = \frac{B_{t-1}}{B_t} - 1$ to obtain an alternative expression:

$$r_S = \frac{\sum_{t=1}^{T} B_t \left(\frac{B_{t-1}}{B_t} - 1 \right)}{\sum_{u=1}^{T} B_u} = \frac{\sum_{t=1}^{T} B_{t-1} - B_t}{\sum_{u=1}^{T} B_u} = \frac{1 - B_T}{\sum_{u=1}^{T} B_u}$$

- Suppose that the bond with coupon rate c trades at par. Then,

$$\sum_{u=1}^{T} B_u c + B_T = 1 \;\Rightarrow\; c = \frac{1 - B_T}{\sum_{u=1}^{T} B_u} = r_S$$

coupon rate bond trades at par

- We conclude that the swap rate equals the coupon rate on the coupon bond trading at par: $r_S = c$.