MITx – Foundation of Modern Finance 1 (Formulas)

ExamNote: Round **ans to 2 decimals - Unit - Per-delete**

LoP: *in absence of arbitrage*, same payoffs = same price

Gross Return = $\frac{Future}{Present}$; Net Return = $\frac{Future-Present}{Present}$

*State-space model: risk-free bond

Payoff **X(t=1)** = $[(X_1..., X_N);(p_1..., p_n)]_{t=1} = \sum_{i=1}^{n} X_i\, p_i$

A-D: pay $1 in single state, state price **(t=0)** $\phi_j > 0, \forall j$

Price today: **P (t=0)** = PV = $\sum_{i=1}^{n} \phi_i X_i$

Expected return: $\bar{r} = \frac{E[X]-P}{P}$; PV = $\sum_{i=1}^{T} \frac{CF_i}{(1+r)^i}$

$RealCF_t = \frac{NominalCF_t}{(1+i)^t}$; $\mathbf{1 + r_{nominal} = (1 + r_{real}) * (1 + i)}$

Return: $r = \frac{(D1+P1)-P0}{P0}$; Expected return $\bar{r} = E[r] - r_F$

Expected Excess Return = Risk premium: $\boldsymbol{\pi = \bar{r} - r_f}$

Annuity (t=0) w/ grow g (**start @1**: $CF_1 = A$; $CF_2 = A(1+g)$...)

$$PV_{t=0}(Ag) = A \times \begin{cases} \frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^T\right], if\ r \neq g \\ \frac{T}{1+r} \qquad\qquad\qquad\quad , if\ r = g \end{cases}$$

$PV_{t=0}\ (Perpetuity\ with\ g) = \frac{A}{r-g}$, *r>g (start @1)*

APR, k-period/year → Effective Annual Rate (**EAR**):

$(1 + r_{EAR}) = (1 + \frac{r_{APR}}{k})^k$; all CF must follow k-period

$\sum_{t=1}^{T} \frac{M}{(1+y)^t} = \frac{M}{y}\left[1 - \frac{1}{(1+y)^T}\right]$: **monthly pay** $T = kn, y = \frac{APR}{k}$

Spot interest rate (r_t) is the current (annualized) interest rate (r_t) for maturity date t (payment only @t) **– Price @SR r_t = no arbitrage.**

Yield Curve (*term structure* of r_t) $= f(r_t, t_{maturity})$

Discount Bond = **1-off**; **Coupon** = **period** payment:

LoP: P(c-bond) = P (replica of d-bond)

$$B = \sum_{t=1}^{T}(C_t \times B_t) + (P \times B_T) = \frac{C_1}{1+r_1} + \cdots + \frac{C_{T-1}}{(1+r_{T-1})^{T-1}} + \frac{C_T + P}{(1+r_T)^T}$$

Yield to Maturity (**YTM**) of a bond, YTM ≠ $E(B)$

$$B = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} + \frac{P}{(1+y)^T}$$

Sell **@par** (coupon rate, c=YTM); @**discount** (c<y); @**premium** (c>y)

Modified Duration: $MD = -\frac{1}{B}\frac{dB}{dy} = \frac{D}{1+y}$;

Discount bond, $B_t = (1+y)^{-t}$: $MD(Bt) = -\frac{1}{B_t}\frac{dB_t}{dy} = \frac{t}{1+y}$

Macaulay duration (weigted avg term to maurity) – D:

$$D = \sum_{t=1}^{T}\left[\frac{PV(CF_t)}{B}\right]t = \frac{1}{B}\sum_{t=1}^{T}\left[\frac{CF_t}{(1+y)^t}\right]t$$

Bond price: $(\Delta B) \approx [-MD \times (\Delta y) + CX \times (\Delta y)^2] \times B$

Convexity: $CX = \frac{1}{2}\frac{1}{B}\frac{d^2B}{dy^2}$ (curvature of BP ∈ y)

Bond Convexity: $B(P) = \sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}$

$$CX = \frac{1}{2}\frac{1}{B(1+y)^2}\sum_{t=1}^{T}\frac{CF_t}{(1+y)^t}t(t+1)$$

Price Elasticity: $PE = \frac{1}{B}\frac{dB}{dy} = -\frac{1}{B}\sum_{t=1}^{T}\frac{CF_t}{(1+y)^{t+1}} * t$

Stock Price: Discount factor = expected return E(r):

$$P_0 = \frac{D_1 + P_1}{1 + r_1} + \cdots = \sum_{t=1}^{T}\frac{D_t}{(1+r_t)^t} + \frac{P_T}{(1+r_T)^T}$$

Gordon Model $\langle r = const; g < r: D_{t+1} = (1+g)D_t \rangle$; then DCF formula: $\boldsymbol{P_0 = \frac{D_1}{r-g}}$

Payout Ratio: $p = \frac{DPS}{EPS} = \frac{Dividend}{Earnings} \rightarrow DPS = p * EPS$

Retaine-E = E-Div → **Plow back ratio**: $b = \frac{RE}{E} = 1 - p$

Book Value: $BVPS_{t+1} = BVPS_t + EPS_{t+1} \times b = BVPS_t + I_{t+1}$

$I_t = EPS_t \times b_t$; $EPS_{t+1} = EPS_t + ROI_t \times I_t$;

Growth stocks = $opportunity$: $NPV(opp) > 0$:

$$P_0 = \frac{EPS_1}{r} + PVGO\ ; where: \frac{P}{E} = \frac{P_0}{EPS_1}\left(fwd\frac{P}{E}\right); PVGO = \frac{NPV_1}{r-g}$$

$\boldsymbol{No\ growth} \Leftrightarrow All\ by\ dividend: g = 0\ \&\ p = 1$

Stock Valuation = CF + **Horizon Value PV_T** *(P/E; P/B; DCF):*

$$PV = \sum_{t=1}^{T}\frac{FCF_t}{(1+r)^t} + \frac{PV_T}{(1+r)^T}$$

$$PV_T = \frac{FCF_{T+1}(No\ Growth)}{r} = \left(\frac{P}{E}\right)^* \times E = \left(\frac{P}{B}\right)^* \times B$$

Expected Utility: $x > y \Leftrightarrow E[u(x)] > E[u(y)]$

*Prefer **more** to less*: $x + \epsilon \geq x, u'(x) \geq 0; \forall x, \epsilon \geq 0$

Aversion *to risk*: $u(E[x]) \geq E[u(x)] \Leftrightarrow u''(x) \leq 0$

Risk premium $\boldsymbol{\pi}$: $E[u\{\boldsymbol{W}(\boldsymbol{1}+\boldsymbol{x})\}] = u\{\boldsymbol{W}(\boldsymbol{1}-\boldsymbol{\pi})\}; E(x) = 0$

$$\boldsymbol{\pi} = -\frac{1}{2}\frac{\boldsymbol{Wu''(W)}}{\boldsymbol{u'(W)}}\sigma_x^2;\ \boldsymbol{RRA(W)} = -\frac{Wu''(W)}{u'(W)};\ \sigma_x^2 = E(x^2)$$

Mean: $\bar{\boldsymbol{r}} = E[\tilde{r}]$; $\quad \hat{\boldsymbol{r}} = \frac{1}{T}\sum_{t=1}^{T} r_t$

$$\boldsymbol{VAR} = \sigma^2 = E[(\tilde{r}-\bar{r})^2];\ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$$

Std Deviation (**Volatility**): $\sigma = \sqrt{VAR}$

Correlation *(standardized)* & **Covariance**: $\bar{r} = mean, expected\ return$

$$Cov(\tilde{r}_i,\tilde{r}_j) = \sigma_{ij} = E[(\tilde{r}_i-\bar{r}_i)(\tilde{r}_j-\bar{r}_j)] = \frac{1}{T-1}\sum_{t=1}^{T}(r_{1,t}-\hat{r}_1)(r_{2,t}-\hat{r}_2)$$

$$Corr(\tilde{r}_i,\tilde{r}_j) = \rho_{ij} = \frac{Cov}{\sigma_i\sigma_j} = \frac{\sigma_{ij}}{\sigma_i\sigma_j} = \frac{E[(\tilde{r}_i-\bar{r}_i)(\tilde{r}_j-\bar{r}_j)]}{\sigma_i\sigma_j}$$

$\beta_{ij} = \frac{\sigma_{ij}}{\sigma_j^2}$ [Beta] ∴ **Leverage Ratio**: $LR = \frac{Asset\ Value}{Net\ Investment}$

$E[r_p] = \bar{r}_p = \sum_{i=1}^{n} w_i\bar{r}_i;$

$w_i = \frac{N_iP_i}{V}; V = \Sigma N_iP_i(V = 0\ arbitrage\ portfolio)$

$$\sigma_p^2 = Var[\tilde{r}_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_iW_j\sigma_{ij};\ \sigma_{ii} = \sigma_i^2$$

Equally weighted portfolio of n-assets:

$$\sigma_p^2 = \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right)$$

$$= \left(\frac{1}{n}\right)(average\ VARiance) + \left(1-\frac{1}{n}\right)(average\ COVARiance)$$

Certainty Equivalent (CE): *smallest riskless pay-off (indiff: taking risk & accept certainty)*: $u(CE) = E[u(x)]$

Portfolio Return: $\widetilde{\boldsymbol{r_p}} = \overline{\boldsymbol{r_p}} + \sum_1^N \boldsymbol{b_{p,k}f_k} + \widetilde{\boldsymbol{\epsilon_p}}$

$\bar{r}_p = \sum w_i\bar{r}_i;\ b_{p,k} = \Sigma w_ib_{ik}; \widetilde{\epsilon_p} = \Sigma w_i\epsilon_i;$

$Var(\widetilde{\epsilon_p}) = \Sigma w_i^2 Var(\widetilde{\epsilon_i})$; **risk-free: $\boldsymbol{r_p \notin f_p \leftrightarrow b_p, \epsilon_p = 0}$**

APT Model *(No arbitrage)*: ***RPremium*** = $\boldsymbol{\pi} = \widetilde{\boldsymbol{r_p}} - \boldsymbol{r_{free}} = \sum_{i=1}^{k}\boldsymbol{\lambda_i b_{p,i}}$: Factor Risk premium= price of risk= $\boldsymbol{\lambda}$ **(linear)** = ***same*** $\forall \boldsymbol{p}$*;*
::: Risk ***Premium = Price of*** *Risk* ***x Quantity of*** *Risk (factor loading,b)*

Single factor: $\tilde{r}_i = \bar{r}_i + \tilde{b}_i\tilde{f} + \tilde{\epsilon}_i \therefore Cov(\tilde{r}_i,\tilde{r}_j) = b_ib_j\sigma_f^2$

Factor mimicking (b=1): $\boldsymbol{\pi_{factor}(i) = \pi_{mimic}(i) = \lambda_i}$;

$where \sum_1^k \boldsymbol{W_ib_{pi,j}} = \begin{cases}\boldsymbol{0, i \neq j}\\ \boldsymbol{1, i = j}\end{cases}; \boldsymbol{b_j = 1}$

Well-diversified portfolio: $Var(\tilde{\epsilon}_p) = 0 \therefore \widetilde{\epsilon_p} = E[\tilde{\epsilon}_i] = 0 \therefore$
$Only\ Sys.f: \tilde{r_p} = \bar{r_p} + \sum b_if_{i,k} \therefore Cov(\tilde{\epsilon}_i,\tilde{\epsilon}_j) = 0, i \neq j$

Key assumption for Abitrage (APT): (payoff X, FV = 0, well-D)

$(1)\ E(r_p) = FV = 0 \therefore (2)\ SystemRisk\ (b_{p,i}) = 0 \therefore (3)\ \epsilon_{risk} = 0$

EMH – Abnormal Return: $AR_{it} = R_{it} - R_{mt} :: Average: AAR_t = \frac{1}{N}\Sigma AR_{it}; Cumulate\ AAR: CAAR_T = \Sigma AAR_t$

Corporate Finance: <u>*using after-tax CF (Dep=Dep.value/yr)*</u>

$\boldsymbol{CF} = (1-\tau)Oper.Profit + \tau * DepValue - CAPEX - \Delta WC$

$\boldsymbol{WC} = Inventory + AR - AP; \boldsymbol{OP} = OR - OE\ w/o\ Dep$

$$\boldsymbol{NPV} = CF_0 + \sum_1^T \frac{CF_i}{(1+r_i)^i} = CF_{in} - CF_{out};\ CF_0 < 0$$

$$\boldsymbol{IRR}: 0 = CF_0 + \sum_1^T \frac{CF_t}{(1+IRR)^t};\ \ \boldsymbol{PI} = \frac{\Sigma PV(CF_t)}{CF_0}$$

IRR >COC? PI>1?: only **1 CF_0 <0** ; from year 1; 1 prj; scaling factor

$\boldsymbol{Net\ Investment} = CAPEX - Depreciation;$

$\boldsymbol{FCF} = Earning - NI; Profit = Rev - Expense$

$[e^{f(x)}]' = f'(x) \times e^{f(x)}$ $\quad [\ln(f(x))]' = \frac{1}{x}f'(x)$

Var[X]=Cov[X, X]= $\boldsymbol{\sigma_X^2}$; Var (X + a) = Var (X);

X,Y **uncorrelated**: Cov(X,Y) = 0; E(X*Y) = E(X)*E(Y)

$\boldsymbol{Var}\ (\boldsymbol{aX}) = \boldsymbol{a}^2Var(X);\ \boldsymbol{Var}[\sum \boldsymbol{X_i}] = \sum Var[X_i] + \sum_{i=1}^{N}\sum_{i\neq j} Cov[X_i, X_j]$

$Var[X_1 + X_2] = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

$\boldsymbol{Cov(X,Y) = E[(X-\bar{X})(Y-\bar{Y})] = E(X*Y) - E(X)*E(Y)}$

$Cov(X,a) = 0; Cov(X,X) = Var(X) = \sigma_X^2$

$Cov(aX,bY) = ab \times Cov(X,Y);$

$Cov(aX+bY, cW+dV) = acCov(X,W) + \cdots$

Excel Operation Note: ***Out****flow* = -ve, ***In****flow* = +ve

NPV, **IRR/RATE (compute YTM, COC**…), Minreserve, VAR/CORRE **.S (sample, T-1)**

Linest (y, x1:x2, 1,1) – **Shift Ctrl Enter** → b2, b1, alpha (reverse)