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15.455x Mathematical Methods of Quantitative Finance

Week 1: Probability

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Random variables, distributions, and moments

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Random variables

future outcomes or observations

- A **random variable** is a function which assigns a number to **events** in the **sample space**. (A better name might be "random-valued function on the sample space.")

sample space is a set of all events, or possible outcomes of experiments, or possible observations that we might have

$$X = \begin{cases} 1, & \text{heads} \\ -1, & \text{tails} \end{cases} \qquad Z = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$$

$$Y = \{\text{sum of two dice}\}$$

- We describe the **probability of an outcome** in terms of the probability of a random variable taking a given value:

$$P(X=1) = 1/2, \quad P(X^2=1) = 1,$$
$$P(Y=2) = 1/36, \quad P(Y=3) = 2/36, \ldots$$
$$P(Z=1) = p, \quad P(Z=0) = q, \quad (\text{where } p+q=1)$$

Continuous random variables

- Consider choosing a random number between 0 and 1, where all values are equally likely.

- Since there are an (uncountably) **infinite** number of values, the probability of any given value is zero:

$$\mathrm{Prob}(X = x_0) = 0$$

- Does **not** mean event is impossible. Points on a line have **"measure zero."**

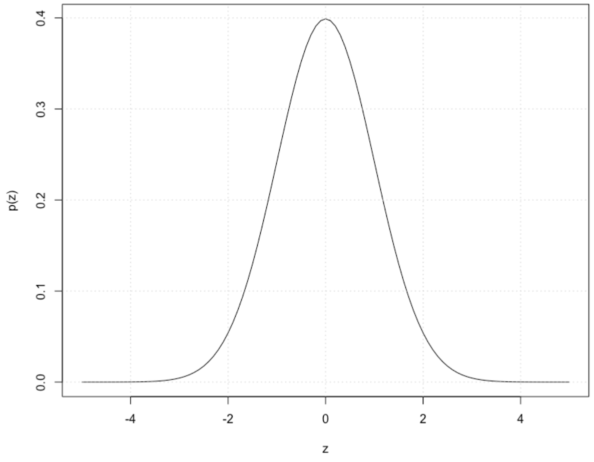
- Instead, ask for probability to lie within a given range, *e.g.*,

$$\mathrm{Prob}(a < X < b) = \int_a^b p(x)\mathrm{d}x$$

Probability distributions

- More generally, a **probability distribution** satisfies

$$
\begin{aligned}
p(x) &\geq 0, \\
\sum_k p(x_k) &= 1, \quad x_k \text{ discrete} \\
\int_{-\infty}^{\infty} p(x)\mathrm{d}x &= 1, \quad x \text{ continuous}
\end{aligned}
$$



Cumulative distribution function

- The **cumulative distribution function**, or CDF, gives the probability that *X* is less than or equal to a given value.

$$F(x) = \mathrm{Prob}(X < x) = \int_{-\infty}^{x} p(x')\mathrm{d}x'$$

- It contains (nearly) the same information as the probability density, since

$$p(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

- The CDF is often easier to approximate from empirical data and it is useful since

$$\mathrm{Prob}(a < X < b) = F(b) - F(a)$$

Change of variable

- Suppose we wish to change variables, shift the distribution, or consider functions of the random variable. If $x \to y = y(x)$ then the density in terms of the new variable is given by

$$p(x)\mathrm{d}x = g(y)\mathrm{d}y$$

which preserves the normalization condition. (This assumes *y* is an increasing function of *x.* If not, an absolute value is needed for positivity; and further care is needed if *y* has critical points.)

$$\begin{aligned} p(x)\mathrm{d}x &= g(y)\mathrm{d}y \\ g(y) &= \frac{p(x)}{|\mathrm{d}y/\mathrm{d}x|} \end{aligned}$$

Expectations and moments

- The probability distribution defines weighted averages over the sample space, where the weight of each event is equal to its probability. These are called **expected values**.

- For the discrete case,

$$\mathrm{E}\left[f(X)\right] = \sum_{k=1}^{n} f(x_k)p(x_k)$$

while for the continuous case,

$$\mathrm{E}\left[f\right] = \int_{-\infty}^{\infty} f(x)p(x)\mathrm{d}x$$

Mean of a distribution

- The **mean** of the distribution is simply the expectation of the random variable itself:

$$\mu \equiv \mathrm{E}\left[X\right] = \overline{X} = \langle X \rangle = \begin{cases} \sum_k x_k p(x_k) \\ \int x p(x) \mathrm{d}x \end{cases}$$

- In the case of an infinite sample space, whether continuous or discrete, the mean is not guaranteed to exist since the integral or the sum might not converge.

Moments of a distribution

- The **moments** of a distribution are the expectation of powers of the random variable itself.

$$\mu_\ell \equiv \mathrm{E}\left[X^\ell\right] \equiv \langle X^\ell \rangle = \begin{cases} \sum_k x_k^\ell p(x_k) \\ \int x^\ell p(x)\mathrm{d}x \end{cases}$$

might not exist in the case of a continuous distribution.

- If all the moments are known – and if they exist – they can be used to get the expectation of other functions using the **linearity** of the expectation operator

$$\mathrm{E}\left[cf(X)\right] = c\,\mathrm{E}\left[f(X)\right],$$
$$\mathrm{E}\left[f(X) + g(X)\right] = \mathrm{E}\left[f(X)\right] + \mathrm{E}\left[g(X)\right]$$

if we know all of the moments, then we can reconstruct the full probability distribution itself. (Taylor's theorem)

Variance and standard deviation

- Of particular interest is the second moment, in combination with the mean, defining the **variance**:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}(X) \equiv \mathrm{E}\left[(X-\mu)^2\right] &= \mathrm{E}\left[X^2 - 2\mu X + \mu^2\right] \\
&= \mathrm{E}\left[X^2\right] - 2\mu \mathrm{E}\left[X\right] + \mu^2 \\
&= \mathrm{E}\left[X^2\right] - \mu^2 \\
&= \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2
\end{aligned}
$$

volatility in financial processes

- The standard deviation, which is the square root of the variance, has the **same units** as the random variable (e.g., rate of return, dollars, etc.)

Higher moments characterize properties of a distribution

Variance – dispersion measure based on second moment

$$\sigma^2 \equiv \mathrm{E}\left[(X-\mu)^2\right] = \int (x-\mu)^2\, p(x)\mathrm{d}x$$

Skewness – asymmetry parameter based on 3rd moments; **dimensionless** – normalized cumulant a skewness of 0 is what we'll find if a probability distribution is symmetric.

$$s \equiv \frac{\mathrm{E}\left[(X-\mu)^3\right]}{\sigma^3} = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

Kurtosis – measure of tail "weights" in terms of 4th moments; zero for Gaussian, bounded below by -1.
normal distribution

$$\kappa \equiv \boxed{\frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4}} - 3 \qquad \text{excess kurtosis}$$

non–excess kurtosis

Covariance and correlation

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- For any **two** random variables, not necessarily independent **or** identically distributed, their covariance is defined as

not dimensionless
$$\mathrm{Cov}(X,Y) \equiv \mathrm{E}\left[(X-\mu_x)(Y-\mu_y)\right] = \mathrm{E}\left[XY\right] - \mu_x\mu_y$$

- The correlation is proportional to the covariance,

$$\rho(X,Y) = \mathrm{Corr}(X,Y) \equiv \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \mathrm{E}\left[\left(\frac{X-\mu_x}{\sigma_x}\right)\left(\frac{Y-\mu_y}{\sigma_y}\right)\right]$$

dimensionless

- Dividing the covariance by the standard deviations makes the correlation a pure number, and

$$-1 \leq \rho(X,Y) \leq +1$$

Covariance

- If the variables are independent, then the covariance will vanish. However the converse is not true.
  - Example: Let Y be a function of X, so it is completely <u>dependent</u>:

$$X = \pm a \text{ each with probability } p_1,\ \pm b \text{ with probability } p_2$$
$$Y = X^2 = a^2 \text{ with probability } 2p_1,\ b^2 \text{ with probability } 2p_2$$
$$\mu_x = 0, \quad \mu_y = \mathrm{E}[Y] = 2p_1 a^2 + 2p_2 b^2$$

$$\begin{aligned}\mathrm{Cov}(X,Y) &\equiv \mathrm{E}[(X-\mu_x)(Y-\mu_y)] = \mathrm{E}[XY] - \mu_x\mu_y = \mathrm{E}[XY] \\ &= p_1 \cdot a \cdot a^2 + p_1(-a)\cdot a^2 + p_2 \cdot b \cdot b^2 + p_2(-b)\cdot b^2 \\ &= 0\end{aligned}$$

  - Vanishing covariance does not imply independence.

Random variables

Summary

a number, not probability

- "**Random variables**" are functions that assign **a number** to events in the sample space. They can be discrete, continuous, or a mix of both.

- The probability distribution is positive and sums to one.

- Expected values, or expectations, are weighted averages that use the probabilities as the weights.

- The **moments** of a distribution, such as the mean, are expectations of various powers of a random variable. They are numbers, not functions.

- Variance, skewness, and kurtosis are simple functions of the moments that characterize the shape of the probability distribution.

- When there are multiple random variables, their **covariance** and **correlation** are also computed as expectations.

Common distributions

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Uniform distribution

- Standard form: $$p(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases},$$

$$\mathrm{Prob}(a < X < b) = \int_a^b p(x)\mathrm{d}x = (b-a)$$

$$F(x) = \int_{-\infty}^{x} p(x')\mathrm{d}x' = x$$

- Moments:

$$\mu = \int_{-\infty}^{\infty} x\, p(x)\mathrm{d}x = \int_0^1 x\mathrm{d}x = \frac{1}{2}$$

$$\mu_\ell = \int_0^1 x^\ell \mathrm{d}x = \frac{1}{\ell+1}$$

$$\sigma^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{1}{12}$$

Binomial distribution

- The **binomial distribution** provides a model for any observation or experiment that has two possible outcomes, "success" or "failure."

- Examples:
  - Coin toss: Heads vs. Tails
  - Roulette: 19 vs. {all other numbers}
  - Bond default prior to time *T* vs. no default
  - One-day trading loss > $100M vs. (P/L > -$100M)

Binomial distribution

- Identify events by total number of successes vs. failures, independent of the order in which they occur.

- **Two parameters**: probability of success *p*, number of trials *n.*

$$f(k; n, p) = \binom{n}{k} p^k q^{n-k}, \text{ where}$$

$$q = 1 - p \text{ is probability of ``failure,''}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Binomial distribution: moments

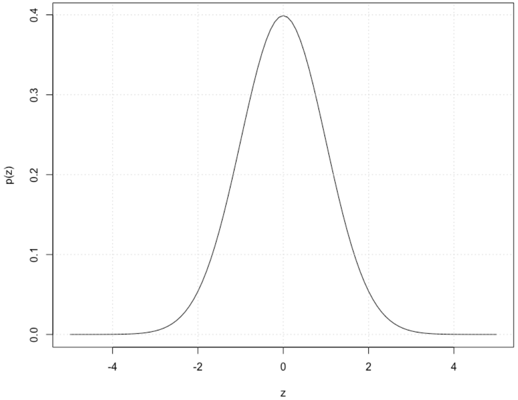
- Mean value (the hard way):

$$
\begin{aligned}
\mu = \mathrm{E}\left[X\right] &= \sum_{k=0}^{\infty} k\, f(k; n, p) \\
&= \sum_{k=1}^{n} \frac{k\, n!}{k!(n-k)!} p^k q^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{(n-1)-(k-1)} \\
&= np \sum_{k'=0}^{n'} f(k'; n', p), \;\; \text{where } k' = k-1, \;\; n' = n-1 \\
&= np
\end{aligned}
$$

Gaussian distribution

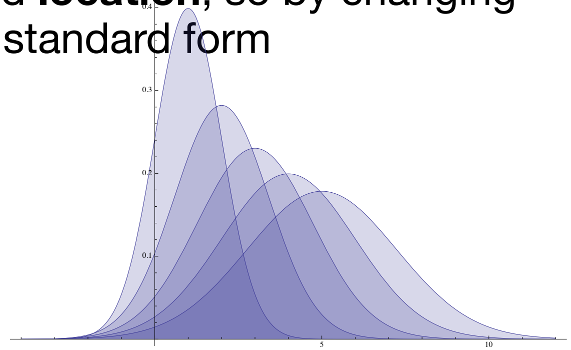
- The **normal** distribution – also known as **Gaussian** – is defined by the two-parameter probability density function

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



- Actually, the two parameters give its **scale** and **location**, so by changing to a rescaled variable, all Gaussians have the standard form

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \text{ where } z = \frac{x-\mu}{\sigma}$$



Gaussian distribution

- Check the normalization, mean, and variance with handy formulas

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}}$$

$$-\frac{\mathrm{d}I(a)}{\mathrm{d}a} = \int_{-\infty}^{\infty} x^2 e^{-ax^2} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

move the derivative inside the integral

- Probability in a range computed from CDF,

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z'^2/2} \mathrm{d}z' = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$F(b) - F(a) = \mathrm{Prob}(a < Z < b)$$

Lognormal distribution

- If we change variables and substitute, obtain the **lognormal** density function

$$X = \log Y, \quad x = \log y,$$
$$\text{then } g(y) = \frac{p(x)}{|\mathrm{d}y/\mathrm{d}x|} = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y - \mu)^2/(2\sigma^2)}$$

- The moments are more easily computed in original variables,

R: simple returns on an asset

$$\mathrm{E}[R] = \mathrm{E}[e^r - 1] = e^{\mu+\sigma^2/2} - 1 \approx \mu + \sigma^2/2 + \ldots,$$
$$\mathrm{Var}(R) = \mathrm{E}\left[(R-\bar{R})^2\right] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \approx \sigma^2(1+2\mu) + \ldots$$

Poisson distribution

- Depends on two parameters, one integer and one real:

$$p(k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

arrival rate

- Application: probability to find exactly *k* "arrivals" or "events" during interval of time *t*.

$$p(k; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Poisson distribution

- Mean

$$\mu = \mathrm{E}\left[X\right] = e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda\, e^{-\lambda} \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} = \lambda$$

- Second moment

$$\mathrm{E}\left[X^2\right] = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} = \lambda + \lambda^2,$$

$$\sigma^2 = \mathrm{E}\left[X^2\right] - \mu^2 = \lambda$$

Poisson distribution

- Limiting case of binomial distribution $n \to \infty, \quad np = \lambda$ fixed

$$p(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

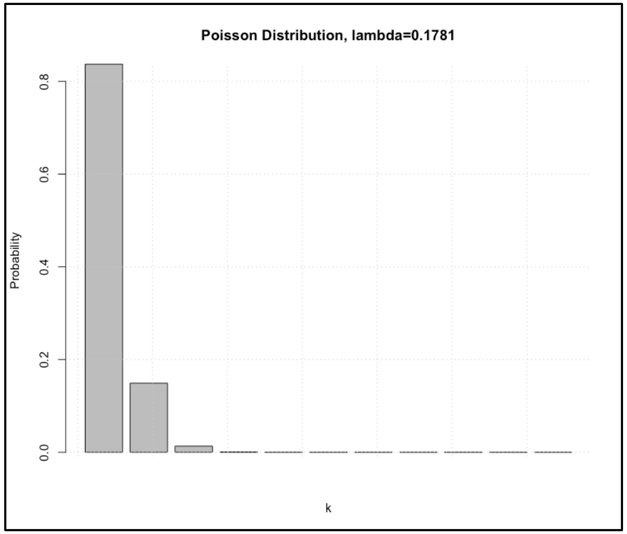
- Example: how many students in a room have a birthday today?

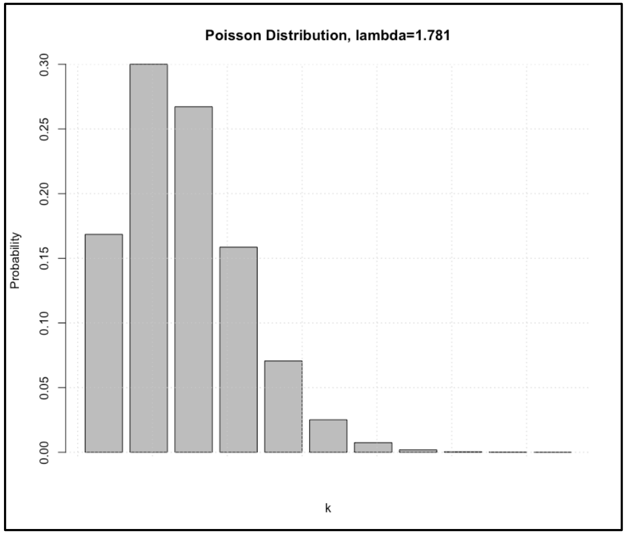
$$p = \frac{1}{365}, \; n = 65$$
$$\lambda = np \approx 0.1781$$

Number *k*	Binomial probability	Poisson probability
0	0.8367	0.8369
1	0.1494	0.1490
2	0.0131	0.0133

Poisson distribution

- For small lambda, values are always decreasing in *k*.
- For lambda > 1, there is a single maximum for *k* near lambda
- Distribution is **skewed** with a long right-hand tail.





Poisson distribution

- A few more handy formulas for evaluating moments:

$$
\begin{aligned}
\frac{k}{k!} &= \frac{k}{k(k-1)!} = \frac{1}{(k-1)!}, \\
\frac{k^2}{k!} &= \frac{k}{(k-1)!} = \frac{(k-1)+1}{(k-1)!} = \frac{1}{(k-2)!} + \frac{1}{(k-1)!}, \\
\frac{k^3}{k!} &= \frac{k^2}{(k-1)!} = \frac{(k^2-3k+2)+(3k-3)+1}{(k-1)!} = \frac{1}{(k-3)!} + \frac{3}{(k-2)!} + \frac{1}{(k-1)!}
\end{aligned}
$$

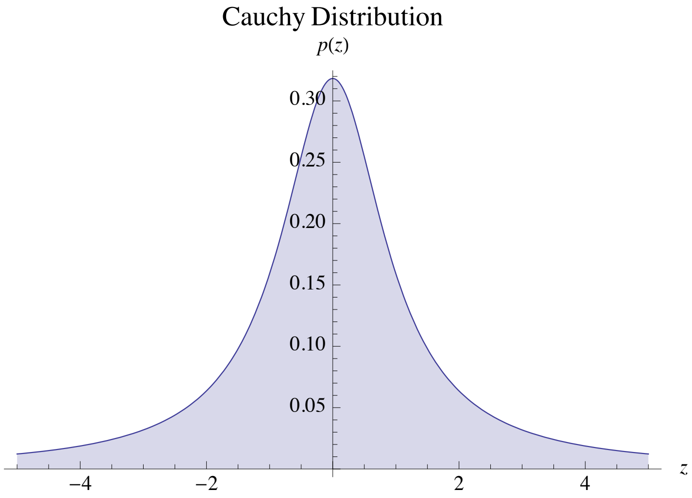
- To simplify these ratios, add and subtract terms terms in the numerators that help simplify the denominators, always starting from the highest power of *k* and working down

- Is there a more general way to generate **all the moments** of a distribution?

Fat tails

- The term "fat tails" is sometimes used to refer to distributions with a large kurtosis.
- It may also refer to distributions for which **higher moments diverge**, in which case the CLT may not hold.
- Example: power-law tails
  - Cauchy distribution

$$p(x) = \frac{A}{\pi^2 A^2 + x^2} \sim \frac{A}{x^2}$$



  - Variance and higher moments are infinite

none of the moments exist

Common distributions

Summary

- Common probability distributions used in finance include
  - Uniform
  - Binomial
  - Poisson
  - Normal (Gaussian)
  - Lognormal
- These have well-defined moments and interesting limiting cases
- Applications include option pricing, credit defaults, jump processes, and most models of asset pricing and forecasting.
- Choosing the right distribution depends on theory (sometimes) and on empirical observations (always).

Sums of random variables

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Multiple random variables

- When there are several random variables, it is often useful to ask about the **distribution of their sum.**

$$S = X_1 + X_2 + \cdots + X_n$$

- In simple cases, the full probability distribution can be derived.

- In more complex cases, we only try to compute some of the **moments** of the full distribution, such as the mean and the variance.
- Easy to do using **linearity** of the expectation operator

$$\mathrm{E}[S] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_n]$$

- Due to the CLT, this is often enough! If there are enough variables, and if the individual distributions in the sum are well-behaved, then the sum will be an (approximately) Gaussian variable…in which case the mean and variance describe it completely. regardless of what the X initial distributions are

Application: portfolio returns

- Financial modeling applications include sums over assets and sums across time. Their **risk** depends on the **covariance**.
  - Example: Let a portfolio consist of N assets and let
    - w_i = portfolio weight in asset i,
    - R_i be the return on asset i,
    - $\mu_i = \mathrm{E}[R_i], \quad \sigma_i^2 = \mathrm{Var}(R_i)$

$$R_p = \sum_{i=1}^{N} w_i R_i = w_1 R_1 + w_2 R_2 + \cdots + w_N R_N$$

$$\mu_p = \mathrm{E}[R_p] = \sum_{i=1}^{N} w_i \mathrm{E}[R_i] = \sum_{i=1}^{N} w_i \mu_i$$

Correlated random variables

- No assumptions made about the individual return distributions except for the existence of the mean and variance.

- The portfolio risk, measured by its standard deviation, does involve the **covariance of pairs** of asset returns

$$
\begin{aligned}
\sigma_p^2 = \mathrm{Var}(R_p) &= \mathrm{E}\left[(R_p - \mu_p)^2\right] = \mathrm{E}\left[\left(\sum_{i=1}^{N} w_i (R_i - \mu_i)\right)^2\right] \\
&= \sum_{i=1}^{N} w_i^2\, \mathrm{E}\left[(R_i - \mu_i)^2\right] + 2\sum_{i<j}^{N} w_i w_j\, \mathrm{E}\left[(R_i - \mu_i)(R_j - \mu_j)\right] \\
&= \sum_{i=1}^{N} w_i^2 \mathrm{Var}(R_i) + 2\sum_{i<j}^{N} w_i w_j \mathrm{Cov}(R_i, R_j) = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i<j}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}
\end{aligned}
$$

Correlated random variables: special cases

- Correlation = 0, variance equals sum of squares

$$\rho_{ij} = 0, \qquad \sigma_p^2 = \sum (w_i \sigma_i)^2$$

- Correlation = 1, variance equals square of sum

$$\rho_{ij} = 1, \qquad \sigma_p^2 = \left(\sum w_i \sigma_i\right)^2$$

- Correlation = 0, all weights and variances identical

$$\rho_{ij} = 0, w_i = \frac{1}{N}, \sigma_i = \sigma_0 \quad \Longrightarrow \quad \sigma_p^2 = \frac{\sigma_0^2}{N}$$

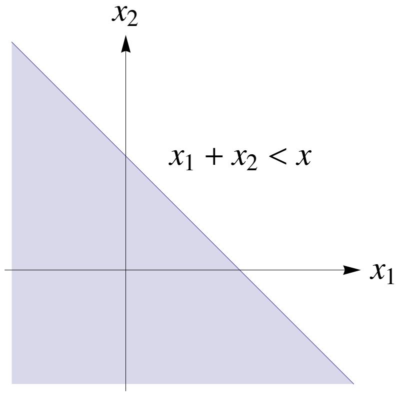
That means that as n gets large, the volatility in the portfolio goes down. And this is the basic idea behind portfolio diversification.

Two random variables

- Two, however, is not a large number.

Let $S = X_1 + X_2$ be the sum of two independent random variables with density function $p_1(x_1)$ and $p_2(x_2)$. Then

$$F(x) = \mathrm{Prob}(X_1 + X_2 < x) = \int\int_A p_1(x_1)p_2(x_2)\mathrm{d}x_1\mathrm{d}x_2$$
$$= \int_{x_1=-\infty}^{x_1=\infty}\int_{x_2=-\infty}^{x_2=x-x_1} p_1(x_1)p_2(x_2)\mathrm{d}x_1\mathrm{d}x_2$$

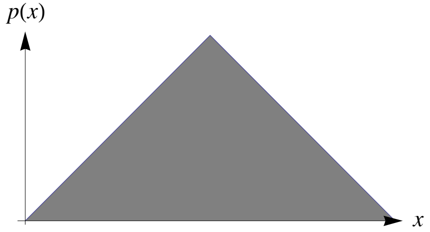


So

$$p(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} = \int_{-\infty}^{\infty} p_1(x_1)p_2(x - x_1)\mathrm{d}x_1,$$

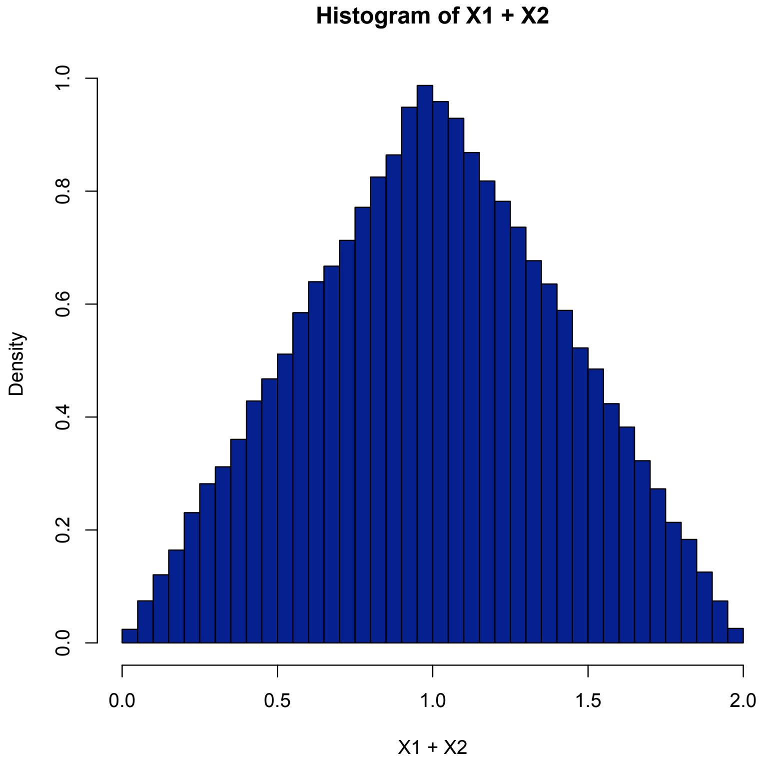
which defines the *convolution* of p_1 and p_2. In the case of the uniform distribution

$$p(x) = \begin{cases} x & x \in [0,1] \\ 2 - x & x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$



Two random variables: simulation

```
X1 <- runif(1e5)    # X1 and X2 drawn from uniform distribution on [0,1]
X2 <- runif(1e5)
hist(X1+X2, probability=TRUE, breaks=50)
```



Binomial distribution (the easy way)

- We can view the number of "successes" as a **sum of random variables**, one for each of the Bernoulli trials,

$$
\begin{aligned}
\text{Let } S &= X_1 + X_2 + \ldots + X_n. \\
\text{Then } \mathrm{E}[S] &= \mathrm{E}[X_1] + \mathrm{E}[X_2] + \ldots + \mathrm{E}[X_n] \\
&= np.
\end{aligned}
$$

- This approach builds up results, using the expectations of one or two of the random variables at a time and the fact that the *X*'s are **independent** and **identically distributed.**

$$
\begin{aligned}
\mathrm{E}[X_1] &= p \cdot 1 + q \cdot 0 = p, \\
\mathrm{E}\left[X_1^2\right] &= p \cdot 1^2 + q \cdot 0^2 = p, \\
\mathrm{E}[X_1 X_2] &= p^2 \cdot 1 \cdot 1 + pq \cdot 1 \cdot 0 + qp \cdot 0 \cdot 1 + q^2 \cdot 0 \cdot 0 = p^2
\end{aligned}
$$

Binomial distribution

- This approach makes the variance especially easy to compute:

$$
\begin{aligned}
\sigma^2 &= \mathrm{E}\left[(S-\mu)^2\right] \\
&= \mathrm{E}\left[(X_1 + X_2 + \ldots + X_n)^2\right] - \mu^2 \\
&= n\,\mathrm{E}\left[X_1^2\right] + n(n-1)\,\mathrm{E}\left[X_1 X_2\right] - \mu^2 \\
&= np + n(n-1)p^2 - (np)^2, \;\; \text{since } \mathrm{E}\left[X_i X_j\right] = \begin{cases} p, & \text{if } i = j \\ p^2, & \text{if } i \neq j \end{cases} \\
&= np(1-p) = npq,
\end{aligned}
$$

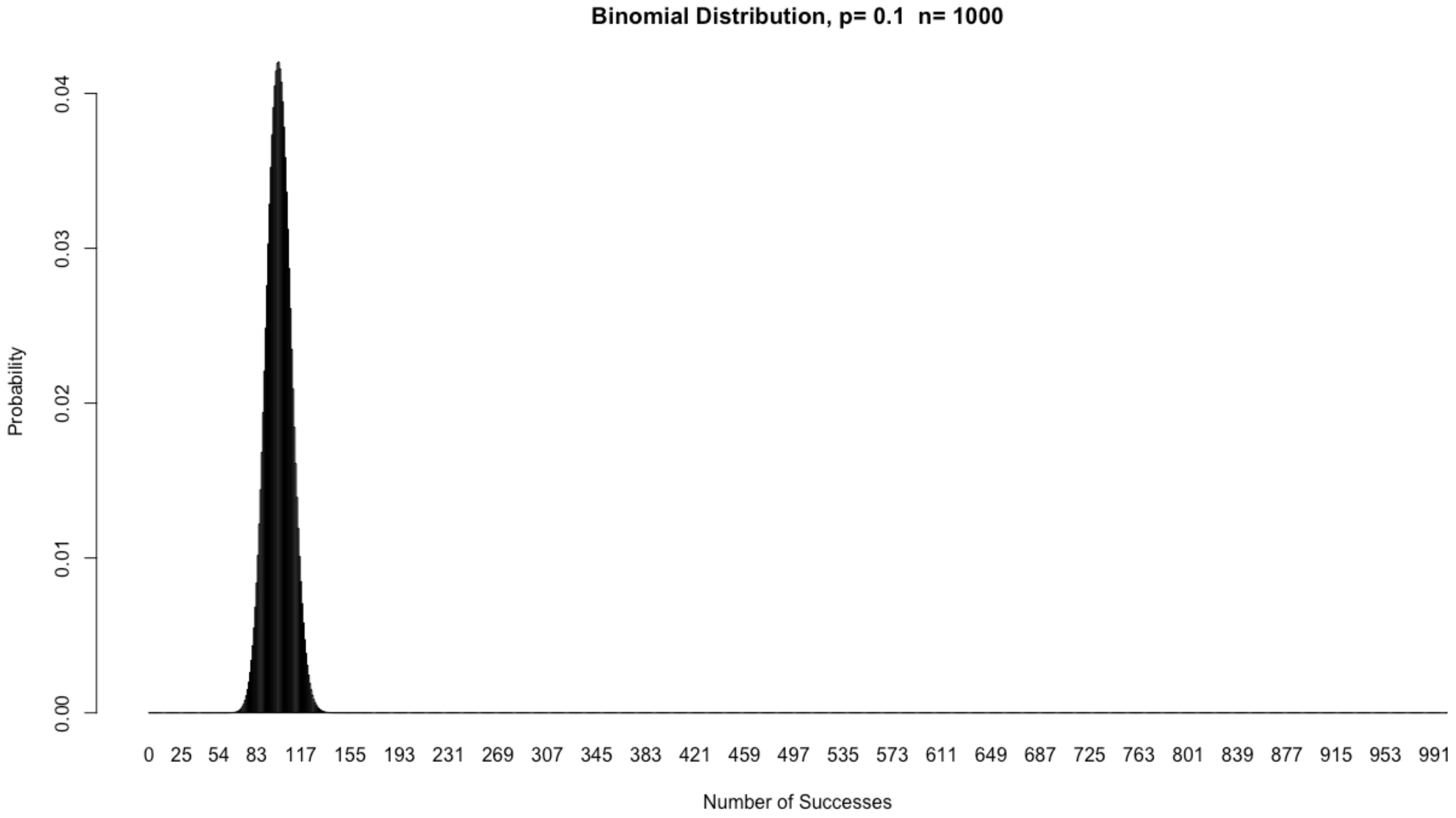
$$
\sigma = \sqrt{npq}.
$$

Binomial distribution

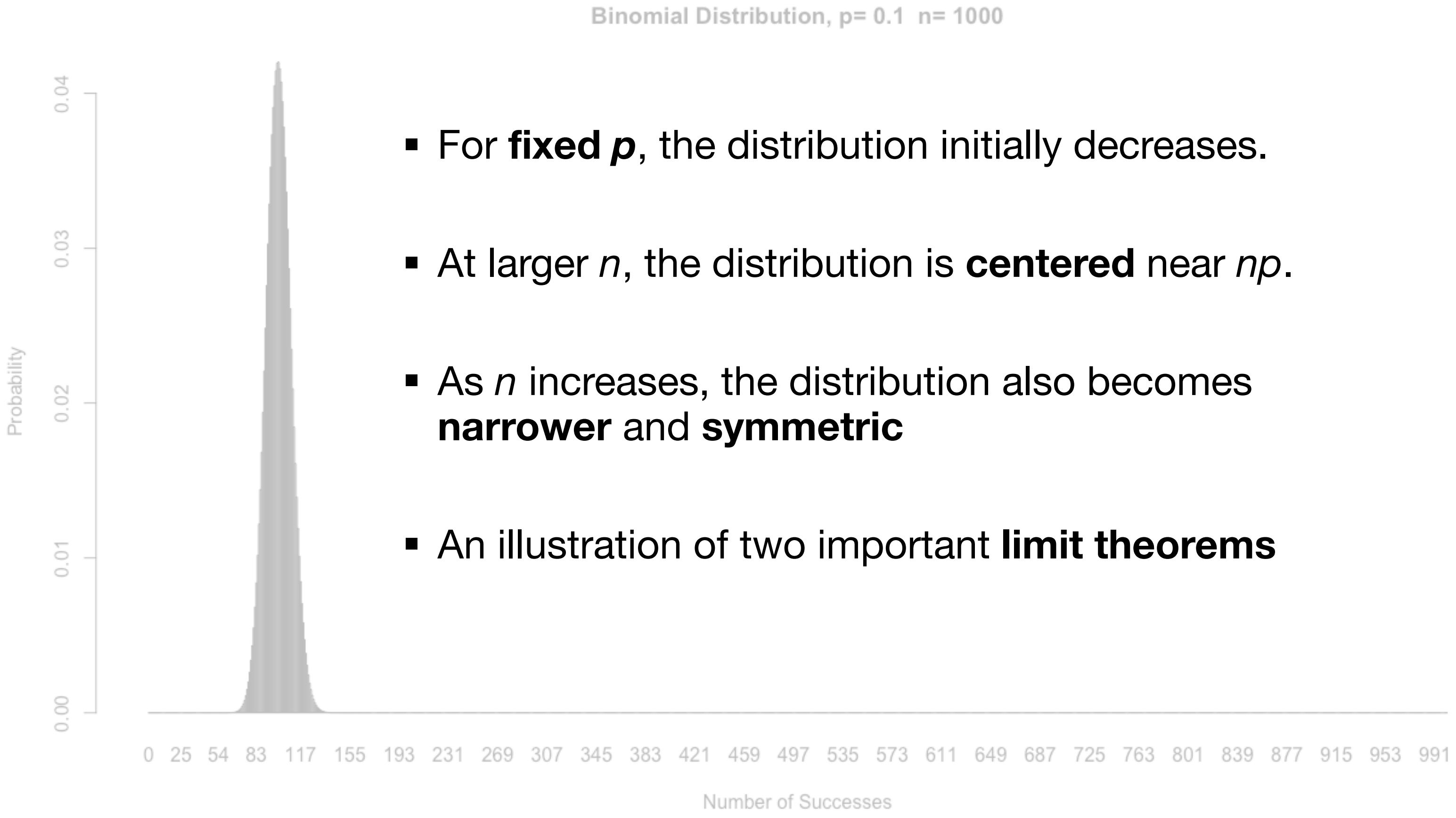
- In numerical calculations, factorial factors quickly overflow machine limits
- R functions:
  - choose(n,k)
  - dbinom(k,n,p)

```
nlist <- c(1,2,5,10,20,50,100,1000)
p <- 0.1
for (n in nlist) {
          k <- 0:n
          f <- dbinom(k,n,p)
          barplot(f,names=k,
          xlab="Number of Successes",ylab="Probability",
          main=paste("Binomial Distribution, p=",p," n=",n))
          readline()
}
```

Binomial distribution, p=10%, n=1,2,...,1000



Binomial distribution



- For **fixed *p***, the distribution initially decreases.
- At larger *n*, the distribution is **centered** near *np*.
- As *n* increases, the distribution also becomes **narrower** and **symmetric**
- An illustration of two important **limit theorems**

Binomial distribution

An illustration of two important **limit theorems**

- **Law of Large Numbers (LLN):**
  - As n increases, the probability that the mean deviates from np goes to zero.

that's why the probability distribution is narrowing around that mean value.

- **Central Limit Theorem (CLT):**
  - As n increases for fixed p, the distribution approaches Gaussian (or Normal)

a bunch of non– Gaussian random variables, when we add 1,000 of them together, we get something that looks Gaussian

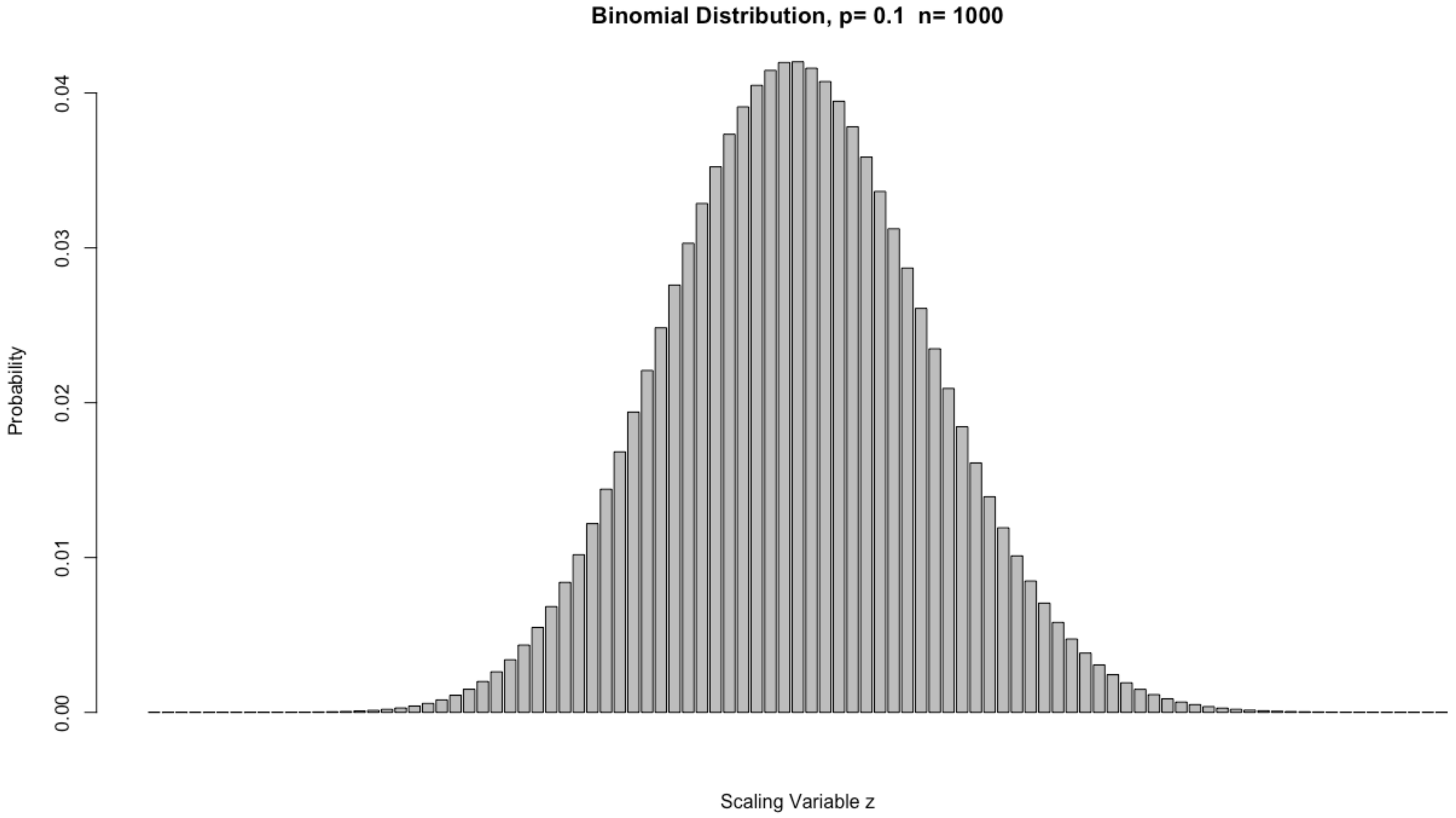
Binomial distribution

- Define the **scaling variable** $$z = z_k = \frac{k - np}{\sqrt{npq}}$$

```
nlist <- c(1,2,5,10,20,50,100,1000)
p     <- 0.1
zmax  <- 5

for (n in nlist) {
          k  <- 0:n
          z  <- (k - n*p)/sqrt(n*p*(1-p))
          zi <- (abs(z)<=zmax)
          f  <- dbinom(k,n,p)
          barplot(f[zi], xlab="Scaling Variable z", ylab="Probability",
          main=paste("Binomial Distribution, p=",p," n=",n))
          readline()
}
```

Scaled binomial distribution



Binomial distribution

- In terms of the scaling variable, the limiting distribution is **Gaussian**

$$z = z_k = \frac{k - np}{\sqrt{npq}}$$

$$\boxed{f(k; n, p) \approx f(z_k) = \frac{1}{\sqrt{2\pi}} e^{-z_k^2/2}}$$

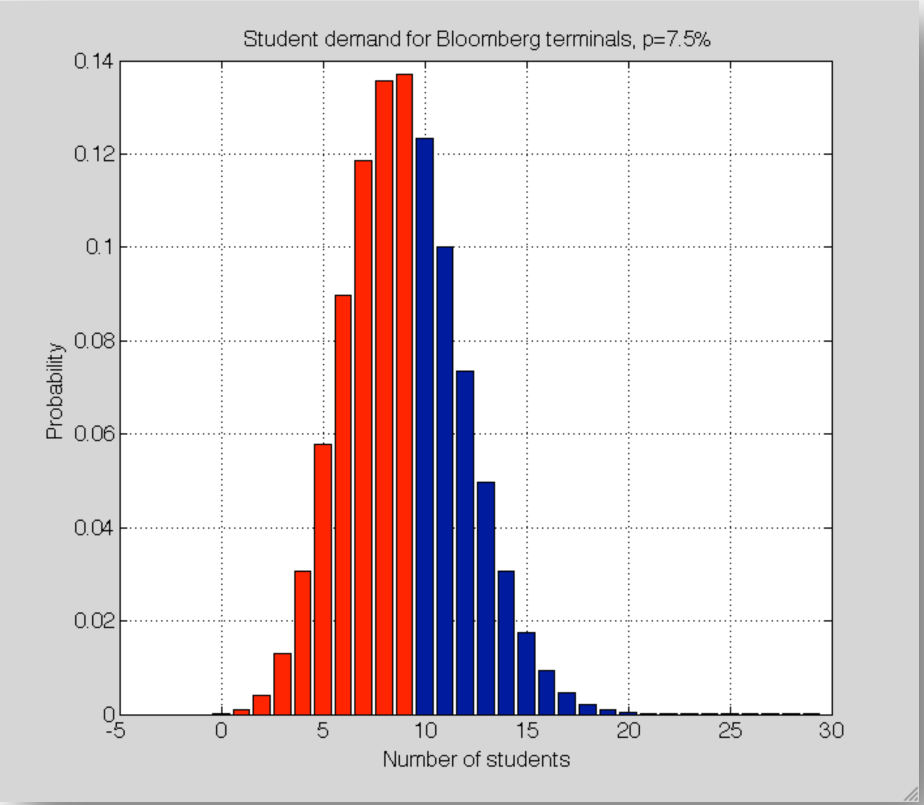
Example: Bloomberg terminals

- Parameters:
  - 120 students
  - 9 terminals
  - Demand independently with probability 7.5%

- Cumulative probability 58.75% that 9 or fewer students will want to use at same time.

```
n <- 120
k <- 0:9
p <- 0.075

sum(dbinom(k,n,p))
[1] 0.587475
```



Sums of random variables

Summary

- A set of individual random variables can be added to create a new random variable, their sum.

$$S_n = X_1 + X_2 + \cdots + X_n$$

- The full probability distribution can be computed using convolutions (which is hard), but we will usually need only the moments (which is easy).
- The **moments** of this new random variable can be easily computed using **linearity**.
- If the variables are **independent** and **identically distributed (IID)**, then the sum is **normally distributed** as $n \to \infty$.
- When the variables are ordered in **time**, the sequence of sums forms a **stochastic process**.

From moment generating functions to the CLT

Finance at MIT
Where ingenuity drives results

Characteristic function

- The **characteristic function** is defined so that its derivatives **generate** all the moments of the distribution

encoding all of the moments in one place

$$\tilde{f}(t) \equiv \mathrm{E}\left[e^{itX}\right], \quad \mathrm{E}\left[X^\ell\right] = (-i)^\ell \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} \tilde{f}(t)\bigg|_{t=0}$$

- It is useful whenever the function and its derivatives can be computed in closed form. For the binomial distribution,

$$\begin{aligned}\tilde{f}(t) \equiv \mathrm{E}\left[e^{itX}\right] &= \sum_{k=0}^{\infty} e^{itk}\,\mathrm{Prob}(X=k) \\ &= \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(pe^{it}\right)^k q^{n-k} \\ &= \left(pe^{it}+q\right)^n\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{E}\left[1\right] &= \tilde{f}(0) = (p+q)^n = 1, \\ \mathrm{E}\left[X\right] &= -i\tilde{f}'(0) = np, \\ \mathrm{E}\left[X^2\right] &= -\tilde{f}''(0) = npq + n^2p^2, \textit{ etc.}\end{aligned}$$

Characteristic function

"An example is the fat-tailed Cauchy distribution with a probability density function $p\left(x\right)=\frac{A}{\pi^2A^2+x^2}$*. Characteristic function of a Cauchy distribution is* $\tilde{p}\left(t\right)=e^{-\pi A|t|}$*. This is a generally well-behaved function, but its derivatives are not defined at* $t=0$*."*

- Continuous distributions: Fourier transform of density

$$\tilde{p}(t)\equiv\int_{-\infty}^{\infty}e^{itx}p(x)\,\mathrm{d}x=\sum_{\ell=0}^{\infty}\frac{(it)^\ell}{\ell!}\mathrm{E}\left[X^\ell\right],\qquad \mu_\ell=\mathrm{E}\left[X^\ell\right]=(-i)^\ell\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\tilde{p}(t)\bigg|_{t=0}$$

Especially useful for **sums of random variables**: $Y=X_1+X_2$,

- Density function of the sum = **convolution** of the individual functions
- Fourier transform of a convolution = **product** of the Fourier transforms

$$p(y)=\int p_1(x_1)p_2(y-x_1)\mathrm{d}x_1,$$

$$\tilde{p}(t)=\tilde{p}_1(t)\tilde{p}_2(t)$$

Normal distribution

- Gaussians are very special
  - The Fourier transform of a Gaussian is also a Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \qquad \tilde{p}(t) = e^{-\sigma^2 t^2/2 + i\mu t}$$

Normal distribution

- The sum of *N* independent Gaussian random variables is itself a Gaussian RV with its own mean and variance:

$$
\begin{aligned}
\tilde{p}(t) &= \tilde{p}_1(t)\tilde{p}_2(t)\cdots\tilde{p}_N(t) \\
&= \exp\left[-\frac{\sigma_1^2 t^2}{2} + i\mu_1 t\right]\exp\left[-\frac{\sigma_2^2 t^2}{2} + i\mu_2 t\right]\cdots \\
&= \exp\left[-\frac{(\sigma_1^2 + \cdots + \sigma_N^2)t^2}{2} + i(\mu_1 + \cdots + \mu_N)t\right] \\
&= \exp\left[-\frac{\hat{\sigma}^2 t^2}{2} + i\hat{\mu}t\right],
\end{aligned}
$$

where

$$
\hat{\sigma}^2 = \sum \sigma_i^2, \quad \hat{\mu} = \sum \mu_i
$$

Scaling for Gaussians

- The sum of *N* independent, **identical** Gaussian random variables is a Gaussian RV with

$$\mu \to \hat{\mu} = N\mu,$$
$$\sigma \to \hat{\sigma} = \sqrt{N}\sigma$$

Cumulant expansion

$$\tilde{p}(t) \equiv \int_{-\infty}^{\infty} e^{itx} p(x)\, \mathrm{d}x = \sum_{\ell=0}^{\infty} \frac{(it)^\ell}{\ell!} \mathrm{E}\left[X^\ell\right], \qquad \mu_\ell = \mathrm{E}\left[X^\ell\right] = (-i)^\ell \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} \tilde{p}(t)\bigg|_{t=0}$$

- Since Fourier transforms multiply, their logarithms **add**. Define the **cumulant** expansion,

$$\tilde{p}(t) \equiv \exp\left[\sum_{\ell=0}^{\infty} \frac{(it)^\ell}{\ell!} C_\ell\right], \qquad C_n = (-i)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \log \tilde{p}(t)\bigg|_{t=0}$$

- The first few terms are

$$\begin{aligned}
C_1 &= \langle X \rangle, \\
C_2 &= \langle X^2 \rangle - \langle X \rangle^2, \\
C_3 &= \langle X^3 \rangle - 3\langle X \rangle \langle X^2 \rangle + 2\langle X \rangle^3, \\
C_4 &= \langle X^4 \rangle - 3\langle X^2 \rangle^2 - 4\langle X \rangle \langle X^3 \rangle + 12\langle X \rangle^2 \langle X^2 \rangle - 6\langle X \rangle^4
\end{aligned}$$

- For a Gaussian, all cumulants are exactly zero for $n > 2$.

Central limit theorem

- Consider the cumulants for a **sum of *N* independent, identically-distributed** random variables. Since cumulants add,

$$\hat{C}_n = NC_n,$$

Since Fourier transforms multiply, their logarithms add

- Then the dimensionless, normalized cumulants have a power-law dependence on *N:*

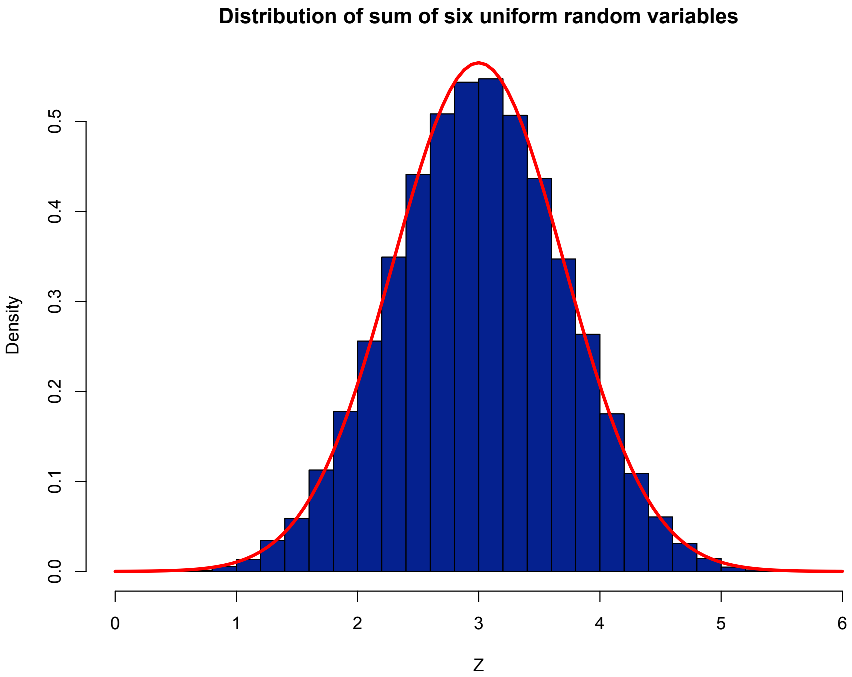
$$\frac{\hat{C}_n}{\hat{\sigma}^n} = \frac{1}{N^{n/2-1}}\left(\frac{C_n}{\sigma^n}\right)$$

- As $N \to \infty$, all cumulants vanish for $n > 2$, and the distribution for the sum **approaches a Gaussian.**

- However, this provides no information about the rate of convergence for different values of *x*; i.e., tails vs. center.

Six random variables: simulation

```
#Add 6 random variables drawn from uniform distribution on [0,1]
Z <- rowSums(matrix(runif(6e5),ncol=6))
hist(Z,probability=TRUE, breaks=30, col="#05218F",xlim=c(0,6),
 main="Distribution of sum of six uniform random variables")
curve(dnorm(x,mean=mean(Z),sd=sd(Z)),0,6,col="red",lwd=3,add=T)
```



Characteristic functions and the CLT

Summary

- The **characteristic function** of a probability distribution gives a compact formula for generating all of the moments.
- For continuous random variables, it is given by Fourier transforms, which have simple properties when applied to sums.
- Gaussians are special
  - The Fourier transform of a Gaussian is also a Gaussian.
  - A sum of Gaussian random variables is also a Gaussian random variable.
  - The central limit theorem (CLT) shows that Gaussians are **universal**: adding together a large number of **any** IID random variables approaches a Gaussian distribution*.

**Conditions apply. You mileage may vary.*