15.455x Mathematical Methods of Quantitative Finance

Week 7: Linear algebra of asset pricing

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Where ingenuity drives results

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Simplest model of financial markets

State space: discrete and finite

Time: discrete and two-period

Securities

- Contracts to receive a future payoff
- Price now is known
- Future payoff is uncertain

Example:

- Probabilities and payoffs for 4 securities
- Stock, Bond, Calls (*K=1.5, K=1*)
- What is left out?

State	Probability	Bond	Stock	Call #1	Call #2
I	1/2	$1	$3	$1.5	$2
II	1/6	$1	$2	$0.5	$1
III	1/3	$1	$1	$0	$0

The payoff matrix

Let there be *n* different securities and *s* states of the world. Then represent the payoffs for each security as a column vector of length *s*, and collect them to form the **payoff matrix**

$$A : \mathbb{R}^n \to \mathbb{R}^s$$

In our example,

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

A **portfolio** is represented as a vector of quantities held for each security

The payoff matrix

A **portfolio** is represented as a vector of quantities held for each security.

The **portfolio payoff** is determined by action on the portfolio vector with the payoff matrix. For example if

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -2 \\ -1 \end{pmatrix}, \quad A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So it's simple to find the payoff of any portfolio. What about the reverse?

What if we had asked: what portfolio ***x*** gives the payoff (0 1 1)'?

The payoff matrix

In other words, we need to know how to solve the general equation $A\mathbf{x} = \mathbf{b}$

A: s * n (states * number of securities)

- If *A* is **non-singular**, there is a unique solution, $\mathbf{x} = A^{-1}\mathbf{b}$
  Requires:

  full rank matrix

  - A invertible
  - Columns of A are independent. (Rows, too)
  - Securities not redundant
  - #securities = #states, n = s

  target space is larger than the initial space

- If ***n* < *s***, then in general there is no solution, except for those $\mathbf{b} \in \mathrm{Image}(A)$
- If ***n* > *s***, then there might or might not be a solution. Solutions are not unique: one can add any element of the kernel, or null space, of *A*.

  target space is smaller than the initial space

Market completeness

A **complete market** is one in which **every** payoff can be generated by **some** portfolio

- Equivalently: Image(A) = the full space of payoffs
- Equivalently: rank(A) = s = # of independent future states of the world
- Equivalently: the linear transformation A is "onto," or "surjective."

covers the entire target space

vectors that can be reached by acting with A is equal to the dimension of the target space.

An **incomplete market** is one in which **some** payoffs cannot be generated by any portfolio.

- Equivalently: Image(A) = a proper subspace within the space of payoffs
- Equivalently: rank(A) < s = # independent future states of the world

The payoff matrix

In still other words, we want to know when it is possible to create a portfolio that matches a given or desired payoff.

Examples

- OTC derivatives
- Business risk management

One way to solve this business problem

- Compute the **perfect hedge** (...if exists...and if so, is it unique?)
- Add a markup (...if customers find this service worth paying for)
- Correct for deviations from idealized solution

Redundant securities

In both complete and incomplete markets, there could be **redundant securities**. This occurs when one or more sets of securities have payoffs that are linearly dependent.

- Equivalently: one or more securities have payoffs that can be **replicated** by a portfolio of other securities.
- Equivalently: the kernel (or null space) of A is non-empty; $\dim(\ker(A)) > 0$
- Equivalently: there exist "arbitrage portfolios" which are non-trivial portfolios that have **zero** payoff.

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Redundant securities

If redundant securities exist, the association of portfolios with payoffs is **not unique**.

- For a given portfolio, the payoff is unique, $A\mathbf{x} = \mathbf{b}$
- But for a given payoff, there can be **more than one portfolio x** that solves the equation.
  - Take a solution, add any multiple of any "arbitrage portfolio" (i.e., an element of ker(A)), and it will have the **same payoff**.

$$\mathbf{z} \in \ker A \implies A(\mathbf{x} + c\mathbf{z}) = A\mathbf{x} + c\,A(\mathbf{z}), \quad \forall c$$
$$= A\mathbf{x} = \mathbf{b}$$

  - (Of course, in an incomplete market the existence of solution depends on **b**. So there will either be an infinite number of solutions or none at all.)

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Replicating payoffs

- Example: can the *K=1* call be replicated from the other 3 securities?

$$A = \begin{array}{c}\begin{pmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{pmatrix} \\ \begin{array}{ccc} \text{bond} & \text{stock} & \text{K=1.5 call} \end{array}\end{array}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{K=1 call}$$

$$A^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ -1 & 3 & -2 \\ 2 & -4 & 2 \end{pmatrix}, \quad \text{so } \mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

long one share of stock and short one bond

- Therefore the security is **redundant**:
- Its payoff is identical to that of a portfolio of **basis assets**.

Replicating payoffs

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- Example: can the *K=1* call be replicated from **just two** other securities?

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

  - *A* cannot be invertible since it isn't even square.
  - But there is still a solution, we're taking two vectors into a three-dimensional space. There's no way that we can hit all possible three vectors and all possible payoffs, but we can hit some of them. And this is one of those lucky payoffs that we can hit.

$$A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \implies \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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Replicating payoffs

Example: can the *K=1.5* call be replicated from the other 3 securities?

redundant

$$A = \begin{pmatrix} 1 & 3 & \boxed{\begin{matrix} 2 \\ 1 \\ 0 \end{matrix}} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1.5 \\ 0.5 \\ 0 \end{pmatrix}$$

not full rank

The **rank** of *A* = 2, so the payoff matrix is singular. An inverse does not exist, and the option cannot be replicated. The market is **incomplete**.

Complete markets

A **complete market** is one in which any payout can be generated by a portfolio of basis assets.

In a complete market, the payoff matrix has **rank *s*.**

- If, in addition, the number of securities $n=s$, then the existence of an inverse means there is a unique solution for any **b**.

If $n > s$, select a complete basis and drop n-s redundant securities

just need a set of vectors that span the space

Note:

$$r(A) = r(A^\top) \leq \min(n, s),$$

$$r(AB) \leq \min(r(A), r(B))$$ linear transformations do not increase rank

$$r(AA^\top) = r(A)$$

Prices

So far we have talked only about future payoffs for securities.

- What are their prices now?
- What is the difference between payoffs and returns?

Present prices can be represented by a vector ***S*** (or just as often, its transpose)

- Market value of a portfolio is sum of (price) x (quantity):

$$MV = \sum_{i=1}^{n} S_i x_i = \begin{pmatrix} S_1 & S_2 & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \mathbf{S}^* \mathbf{x} = S[\mathbf{x}]$$

S acting on the vector x, which gives us a real number

- Note that ***S*** acts **linearly** on portfolio vectors, and its dimension is also *n.*

Arbitrage

Type I arbitrage

- Pay nothing now (i.e., negative or zero cost):

$$V = \mathbf{S}^*\mathbf{x} = \sum_{i=1}^{n} S_i x_i \leq 0$$

- Receive only non-negative payoffs later: $A\mathbf{x} \geq 0$
  and at least one payoff > 0 get something for nothing

- Note that the second inequality has a vector on the left-hand side.
  - The meaning here is that it holds for all the components.
  - More precisely, it means that every component of the vector is non-negative **and** that at least one component is positive.

Arbitrage

Type I arbitrage example: because the stock stochastically dominates the bond, so its price cannot be the same as the bond

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{S}^*\mathbf{x} = 0, \quad A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \geq 0$$

Arbitrage

Type I arbitrage example:

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{S}^*\mathbf{x} = 0, \quad A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \geq 0$$

Question: Can this arbitrage be avoided in this example?

- (a) Yes, if $S_2 > 1$
- (b) Yes, if $S_1 < 1$
- (c) No, arbitrage cannot be avoided

Arbitrage

Question: Can arbitrage be avoided in this market?

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{S}^*\mathbf{x} = 0, \quad A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \geq 0$$

Answer: Yes. Stock **stochastically dominates** bond. Change prices so that

$$\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \mathbf{S}^*\mathbf{x} > 0 \implies S_2 > S_1$$

Arbitrage

Type II arbitrage

get nothing for something

- There are redundant assets and non-trivial solutions to $A\mathbf{x} = 0$
- There is arbitrage if the portfolio has a non-zero price, $\mathbf{S}^*\mathbf{x} \neq 0$

get money in the present by long/short and no effect in future

- **Example:** $$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \implies A\mathbf{x} = 0, \quad \mathbf{S}^*\mathbf{x} = 4$$

Arbitrage

Type II arbitrage

- There are redundant assets and non-trivial solutions to $A\mathbf{x} = 0$
- There is arbitrage if the portfolio has a non-zero price, $\mathbf{S}^*\mathbf{x} \neq 0$

- **Example:** $$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \implies A\mathbf{x} = 0, \quad \mathbf{S}^*\mathbf{x} = 4$$

- **Question:** Can arbitrage be avoided in this market?

Arbitrage

Type II arbitrage

- There are redundant assets and non-trivial solutions to $A\mathbf{x} = 0$
- There is arbitrage if the portfolio has a non-zero price, $\mathbf{S}^*\mathbf{x} \neq 0$

- **Example:** $$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \implies A\mathbf{x} = 0, \quad \mathbf{S}^*\mathbf{x} = 4$$

- **Question:** Can arbitrage be avoided in this market? Prices required to obey

$$\mathbf{S}^*\mathbf{x} = S_1 - S_2 + S_4 = 0 \implies S_4 = S_2 - S_1$$

It's the law

Law of one price: different portfolios that have the same payoff have to have the same price.

- Absence of arbitrage
- Unique price (for securities in market)

Prices are linear

- Linear combination of assets
- Linear combination of prices

$$\text{If payoffs } \overbrace{A\mathbf{x}_1 = A\mathbf{x}_2}^{\text{payoff}}, \text{ then } \overbrace{\mathbf{S}^*\mathbf{x}_1 = \mathbf{S}^*\mathbf{x}_2}^{\text{price}}$$

$$\text{since } A(\mathbf{x}_1 - \mathbf{x}_2) = 0 \implies \mathbf{S}^*(\mathbf{x}_1 - \mathbf{x}_2) = 0$$

non–zero portfolios that have zero payoffs must have zero price

kernel space of A and S*

No-arbitrage pricing

Now we can use no-arbitrage among **basis** assets to price any asset (or portfolio) with any payoff

Relative pricing vs. absolute pricing

- No-arbitrage permits us to price assets **relative** to one another and to set **bounds** on allowed prices
- Absolute pricing models aim to fix prices of basis assets (or portfolios) in terms of external variables.

balance sheet, earnings report

No-arbitrage pricing

Implications of no-arbitrage principle

- Law of One Price
- Determine whether or not a given model is arbitrage-free
- Constrain / Design / Build models are guaranteed to be arbitrage-free
- Fundamental theorem of asset pricing

Concepts

- State prices
- State price density
- Risk-neutral probabilities
- Risk-neutral pricing

Arrow-Debreu securities

- Portfolios with the special property that they have **unit** payoff in a single state are called Arrow-Debreu (AD) securities.

- If a market has s such "securities," then the payoff matrix is the identity matrix, $A = I$.

Arrow-Debreu securities

In terms of a general payoff matrix, an elementary AD security can be replicated if there is a portfolio satisfying

$$A\mathbf{x} = \mathbf{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

If A has an inverse, then $\mathbf{x} = A^{-1}\mathbf{e}_j$ is a portfolio with unit payoff

The prices of AD securities are called **state prices:** $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_s \end{pmatrix}$

Arrow-Debreu securities

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Example: elementary securities for the payoff matrix

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 3 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = ?$$

$$A\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Question: what portfolio replicates the elementary AD security for state 3?

Arrow-Debreu securities

Question: what portfolio replicates the elementary AD security for state 3?

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 3 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = ?$$

$$(a)\ \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \quad (b)\ \mathbf{x}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad (c)\ \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \quad (d)\ \mathbf{x}_3 = \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

Arrow-Debreu securities

Question: what portfolio replicates the elementary AD security for state 3?

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 3 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = ?$$

Answer:

```
> A  <- matrix(c(1,1,1,3,2,1,1.5,.5,0,2,1,0),3)
> e3 <- matrix(c(0,0,1),ncol=1)
> x3 <- qr.solve(A,e3)
> x3
     [,1]
[1,]    3
[2,]   -2
[3,]    2
[4,]    0
```

Arrow-Debreu securities

Example: elementary securities for the payoff matrix

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 3 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix},$$

$$A\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Replicating portfolios **not unique.** Can add elements of ker *A*, $\tilde{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad A\tilde{\mathbf{x}} = 0$

No-arbitrage pricing

Example: Find the no-arbitrage price of the first AD security.

- First, we can use linearity to set the price of the bond = 1
- Next, use no-arbitrage to fix price of asset 4 (i.e., call #2)

$$\mathbf{S} = \begin{pmatrix} 1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}, \quad \tilde{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{S}^*\tilde{\mathbf{x}} = 0 = -1 + S_2 - S_4$$

$$\Longrightarrow \quad S_4 = S_2 - 1.$$

Summary:

- Given basis assets and payoff matrix *A*
- Given prices ***S***
- Given a redundant security
- Then its price is given by linear price of replicating portfolio

$$\mathbf{S} = \begin{pmatrix} 1 \\ S_2 \\ S_3 \\ S_2 - 1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathbf{S}^*\mathbf{x}_1 = 1 - S_2 + 2S_3.$$

Pricing bounds

What can we say about **non-redundant assets**?

Example: market with 3 states and only one security

payoff

price

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{a_1}, \quad S_1 = 2, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{Then } \mathbf{a}_1 \geq \mathbf{b} \implies S_1 > S_b.$$

$$\text{Also have } \frac{1}{2}\mathbf{a}_1 \leq \mathbf{b} \implies \frac{1}{2}S_1 < S_b.$$

$$\text{So } 1 < S_b < 2.$$

Pricing bounds

Example (cont): Somewhat more generally, consider

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{a}_1, \quad S_1 = 2, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{b} - \lambda \mathbf{a}_1 = \begin{pmatrix} 1-\lambda \\ 1-2\lambda \\ 2-3\lambda \end{pmatrix}$$

and find values of potential interest where entries change sign:

$$\lambda = 1 \implies \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda = \frac{1}{2} \implies \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \quad \lambda = \frac{2}{3} \implies \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}$$

State prices and complete markets

Arrow-Debreu security prices form the **state price vector** $\psi \in \mathbb{R}^s$

State prices must be **positive** since the elementary AD security payoffs are positive.

In a complete market, all payoffs can be replicated and have a **unique** price.

Positive state prices iff no arbitrage

- Can take AD securities as basis assets
- Payoff matrix *A* = *I*
- Price for **any** positive payoff **b** is just $\psi^* \mathbf{b} > 0$
- Therefore the price of original basis assets (i.e., columns of *A*) are given by

transpose

$$S_1 = \psi^* \mathbf{a}_1, \quad S_2 = \psi^* \mathbf{a}_2, \; etc. \implies \mathbf{S}^* = \psi^* A$$

price = state price * payoff

$$\implies \mathbf{S} = A^* \psi$$

Arbitrage theorem

Consider a market with *n* securities, *s* states of the world, payoff matrix

$$A \in \mathbb{R}^{s\times n}, \quad A : \mathbb{R}^n \to \mathbb{R}^s$$

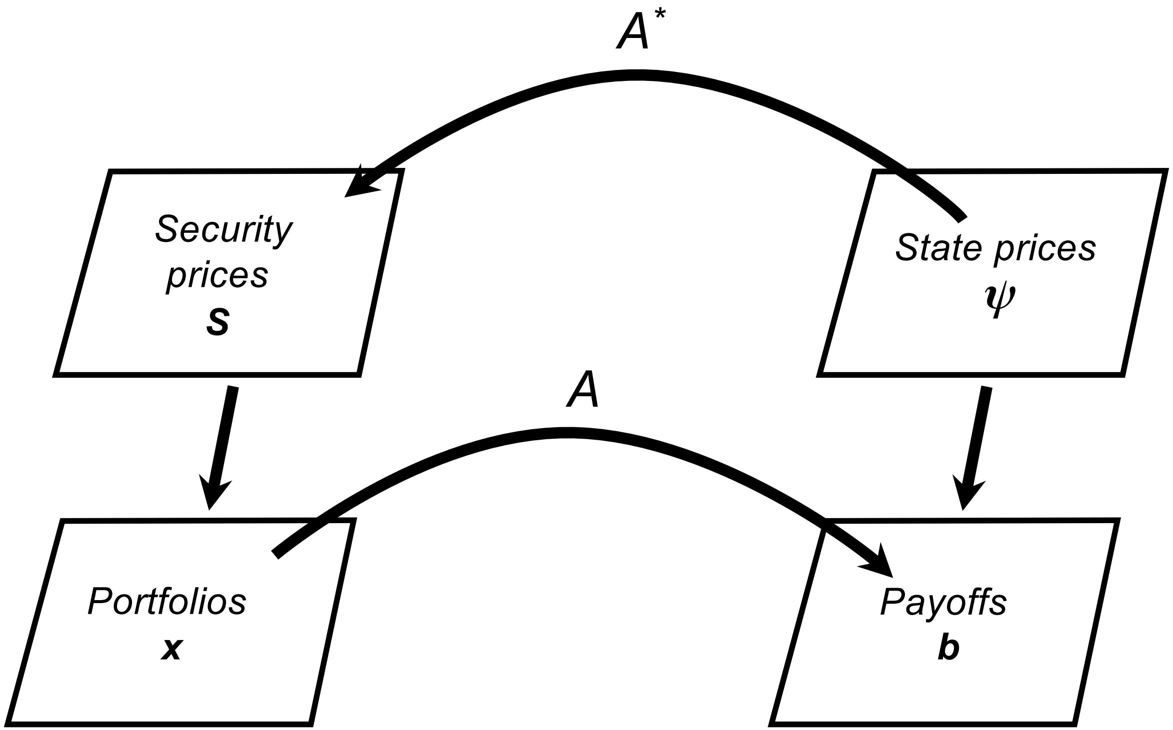
There is no arbitrage if and only if there exists a **strictly positive** state-price vector ψ **consistent with** the security-price vector,

$$\psi \in \mathbb{R}^s, \quad \mathbf{S} \in \mathbb{R}^n, \quad \mathbf{S} = A^*\psi$$

Already shown this for complete markets. For **incomplete markets**, there can be (infinitely) multiple solutions ψ

- If there is **at least one solution** where $\psi > 0$, then no arbitrage
- If **every solution** has $\psi \leq 0$, then there is arbitrage

Dual spaces and arbitrage



Arbitrage

Example: Determine if there is arbitrage in a market with payoffs and prices given by

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \\ 0.6 \\ 1 \end{pmatrix}$$

market is complete and there are redundant assets

Although *A* is not invertible, A^* has **pseudo-inverse** *M:*

$$M \equiv (AA^*)^{-1}A = \frac{1}{3}\begin{pmatrix} 1 & -1 & 6 & -2 \\ -3 & 3 & -12 & 6 \\ 4 & -1 & 6 & -5 \end{pmatrix}; \qquad \text{check: } MA^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{S} = A^*\psi \implies \psi = M\mathbf{S} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}; \qquad \text{check: } A^*\psi = \begin{pmatrix} 1 \\ 2 \\ 0.6 \\ 1 \end{pmatrix}$$

Price bounds

Example (*cont.*): What other (basis) prices are allowed? Suppose

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \\ x \\ 1 \end{pmatrix}, \quad \psi = M\mathbf{S} = \begin{pmatrix} 2x - 1 \\ -4x + 3 \\ 2x - 1 \end{pmatrix}$$

Imposing positive state prices,

$$\begin{aligned} 2x - 1 > 0 &\implies x > 1/2 \\ -4x + 3 > 0 &\implies x < 3/4 \\ &\implies \frac{1}{2} < x < \frac{3}{4} \\ \text{So for example } x = 0.7 &\implies \psi = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix} \end{aligned}$$

Price bounds

Example (*cont.*): What other (basis) prices are allowed? Suppose

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2+y \\ x \\ 1+y \end{pmatrix}, \quad \psi = M\mathbf{S} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} + x \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

Range of allowed prices, each gives unique solution for ψ.

Price bounds

In an **incomplete** market, there can be **multiple** solutions ψ.

Example:

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad AA^* = \begin{pmatrix} 10 & 7 & 4 \\ 7 & 5 & 3 \\ 4 & 3 & 2 \end{pmatrix}, \quad \ker(AA^*) = \ker A^* = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{Then if } \mathbf{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{S} = A^*\psi = \begin{pmatrix} 1 & \boxed{1} & 1 \\ 3 & \boxed{2} & 1 \end{pmatrix} \psi$$

$$\implies \psi = \boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + x \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

second security satisfies

$$\psi > 0 \implies 0 < x < \frac{1}{2}$$

Price bounds

Counterexamples:

If $\mathbf{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, there is arbitrage, as we saw earlier (stochastic dominance).

For $\mathbf{S} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\quad \mathbf{S} = A^* \psi = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \psi$

$$\implies \psi = \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

$\psi > 0 \implies$ no solutions for **any** values of x.

Arbitrage pricing theorem

Given payoffs *A*, prices ***S***, target asset with payoff **b**:

Find all $\psi > 0 \text{ such that } A^*\psi = \mathbf{S}$

- No solutions $\Rightarrow$ arbitrage
- 1 solution $\Rightarrow$ complete market
- Multiple solutions $\Rightarrow$ incomplete market

Price asset using **all** solutions

- 1 solution $\Rightarrow$ redundant asset $\psi > 0 : \{\psi^*\mathbf{b}\}$
- Otherwise find full set of allowed non-arbitrage prices

Arbitrage pricing

Example: incomplete market

So this is a new asset. It's not a redundant asset. There's no way to make it out of the existing securities.

$$\text{Suppose } A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 1.5 \\ 0.5 \\ 0 \end{pmatrix}.$$

$$\text{Then } \psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \implies A^*\psi = \mathbf{S} \text{ and } 0 < x < 1/2.$$

$$\begin{aligned} \text{Then price } S_b &= \psi^*\mathbf{b} = 0.5 + x(0.5) \\ &= \frac{1}{2}(1 + x), \\ \text{so } 1/2 &< S_b < 3/4. \end{aligned}$$

Asset pricing duality

Finance at MIT
Where ingenuity drives results

Algebra of arbitrage

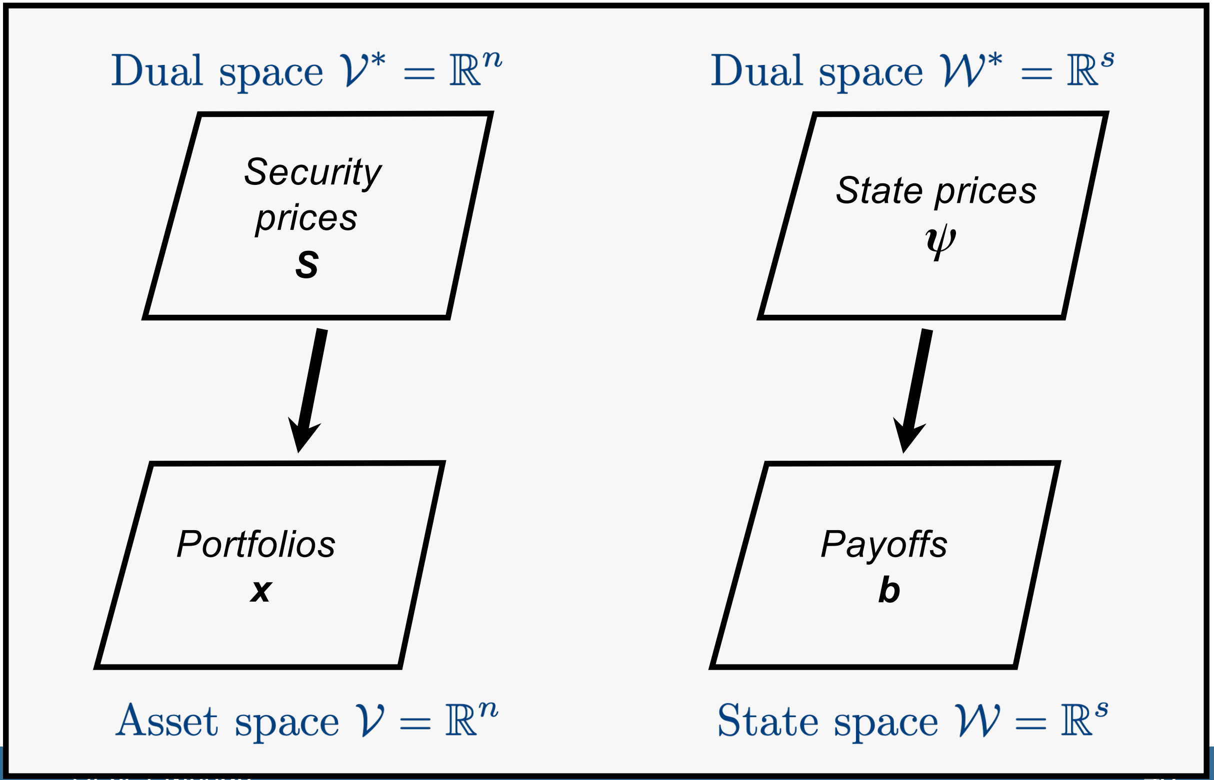
The **dual space** of a vector space consists of all **linear function(al)s** on vectors in the space.

- Example: price of a portfolio

$$S[\mathbf{x}] = \mathbf{S}^*\mathbf{x}$$

- Example: state price of a payoff

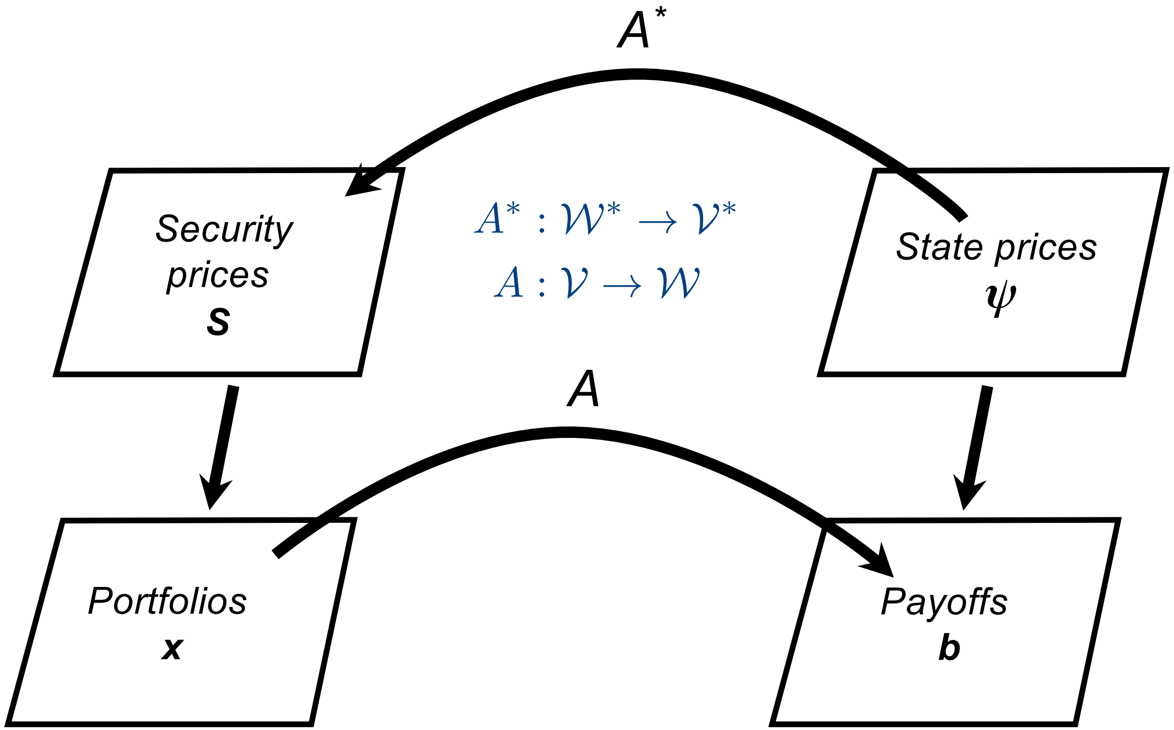
$$\psi[\mathbf{b}] = \psi^*\mathbf{b}$$



Algebra of arbitrage

The **adjoint transformation**, given by the **transpose** of a matrix, goes between dual spaces **in the opposite direction.**

$$
\begin{aligned}
S[\mathbf{x}] &= \psi[\mathbf{b}] = \psi[A\mathbf{x}] \\
\mathbf{S}^*\mathbf{x} &= \psi^* A\mathbf{x} = (A^*\psi)^*\mathbf{x} \\
&\boxed{\mathbf{S} = A^*\psi}
\end{aligned}
$$



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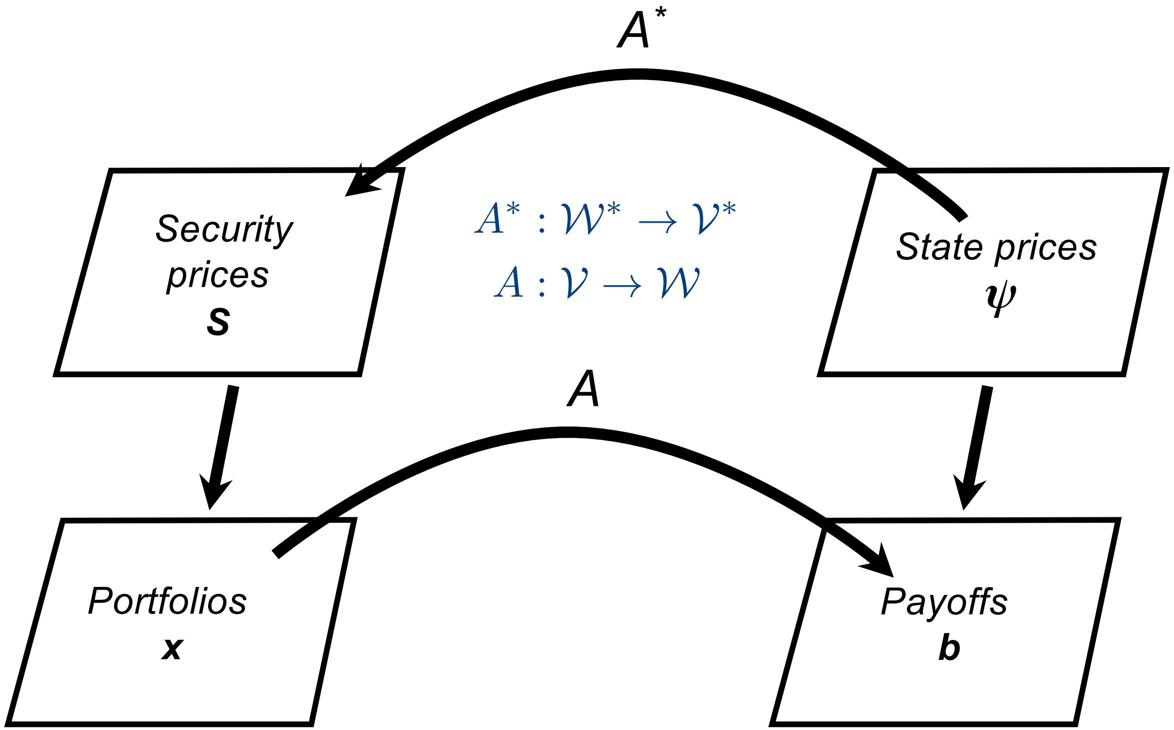
Algebra of arbitrage

The operators have special relationships among their subspaces

$$\operatorname{Ker} A^* \perp \operatorname{Im} A$$
$$\operatorname{Im} A^* \perp \operatorname{Ker} A$$

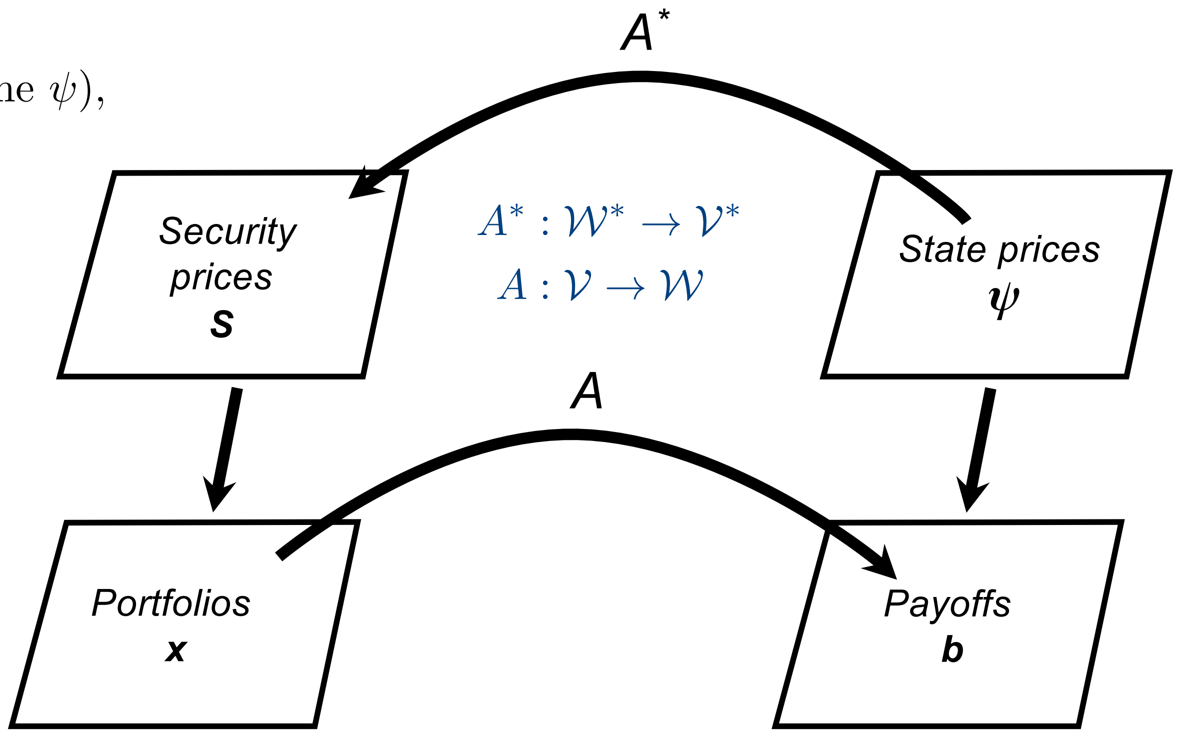


$$A^* : \mathcal{W}^* \to \mathcal{V}^*$$
$$A : \mathcal{V} \to \mathcal{W}$$

Algebra of arbitrage

For arbitrage portfolios,

$$
\begin{aligned}
&\text{If } \mathbf{x} \in \operatorname{Ker} A \quad (i.e., A\mathbf{x} = 0) \\
&\text{and } \mathbf{S} \in \operatorname{Im} A^* \ (i.e., \mathbf{S} = A^*\psi \text{ for some } \psi), \\
&\text{then } \mathbf{S}^*\mathbf{x} = (A^*\psi)^*\mathbf{x} \\
&\qquad\quad = (\psi^* A)\mathbf{x} \\
&\qquad\quad = \psi^*(A\mathbf{x}) \\
&\qquad\quad = 0.
\end{aligned}
$$



Algebra of arbitrage

Example: incomplete market $\quad \operatorname{Im} A \subset \mathbb{R}^s, \ \underset{2}{\operatorname{rank}(A)} < \underset{3}{s}$

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad (\operatorname{Ker} A^*) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

If $\psi = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = A\mathbf{x}$ for some portfolio $\mathbf{x}$,

then $\psi^* \mathbf{b} = \psi^*(A\mathbf{x}) = (\psi^* A)\mathbf{x} = (A^*\psi)^*\mathbf{x} = 0.$

Asset pricing duality

We have two approaches to pricing, and they are not just equivalent, they are **dual.** Given A, **S**, and payoff **b** for "focus" asset:

State pricing: compute price(s) of focus asset by applying all allowed $\boldsymbol{\psi}$

- The price is uniquely determined if **b** is the payoff of a redundant asset

$$S_b = \psi[\mathbf{b}] = \{\psi^* \mathbf{b} : A^* \psi = \mathbf{S}, \psi \in \mathcal{W}^*, \psi > 0\}$$

Replication pricing (no state prices involved): compute price(s) of replicating portfolio

- The price is uniquely determined if **b** is the payoff of a redundant asset

$$S_b = \mathbf{S}[\mathbf{x}] = \{\mathbf{S}^* \mathbf{x} : \mathbf{x} \in \mathcal{V}, A\mathbf{x} = \mathbf{b}\}$$

Asset pricing duality

Replication pricing (cont.): for non-redundant assets, can frame allowed prices as a **bound** on asset prices between $S_{\min} < S_b < S_{\max}$ where

- most expensive sub-replicating portfolio $S_{\min} = \max\{\mathbf{S}^*\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$,
- least expensive super-replicating portfolio $S_{\max} = \min\{\mathbf{S}^*\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\}$

Application: in this form, bounds identify (potential) arbitrage trade to execute *if* bound were ever violated (temporarily, for example).

Fundamental theorem of asset pricing (FTAP) revisited

There is no arbitrage if and only if there exists a **strictly positive** state-price vector ψ **consistent with** the security-price vector,

$$\mathbf{S} = A^*\psi, \quad \boxed{\psi > 0,} \qquad \text{where } \psi \in \mathcal{W}^*, \ \mathbf{S} \in \mathcal{V}^*, \ A^* : \mathcal{W}^* \to \mathcal{V}^*$$

Absence of type-I arbitrage:

$$\text{If } A\mathbf{x} = \boxed{\mathbf{b} \geq 0} \text{ then } \mathbf{S}^*\mathbf{x} > 0.$$
$$\textit{Proof:} \quad \mathbf{S}^*\mathbf{x} = (\psi^* A)\mathbf{x} = \psi^*(A\mathbf{x}) = \boxed{\psi^*\mathbf{b} > 0.}$$

Absence of type-II arbitrage:

$$\text{If } A\mathbf{x} = 0, \text{ then } \mathbf{S}^*\mathbf{x} = 0.$$
$$\textit{Proof:} \quad \mathbf{S}^*\mathbf{x} = \psi^*(A\mathbf{x}) = 0.$$

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Fundamental theorem of asset pricing (FTAP) revisited

The opposite direction of the FTAP follows from Farkas' lemma, which dictates strong alternatives:

Farkas' Lemma:

Either

$$(a)\ A^*\psi = \mathbf{S}, \quad \psi \geq 0 \text{ has a solution, or}$$

$$(b)\ A\mathbf{x} \geq 0, \quad \mathbf{S}^*\mathbf{x} < 0 \text{ has a solution}$$

but not both.

Geometric interpretation: the lemma says that a vector either lies within a convex cone (defined by A^*) or is separated from it by a hyperplane. (See Boyd & Vandenberghe (2004) Ch. 5 for proof and applications in linear optimization.)

References

- Books
  - Axler – "*Linear Algebra Done Right,*" Springer
  - Boyd and Vandenberghe (2004) - "*Convex Optimization,*" Cambridge
  - Capinski and Zastawniak (2003) – "*Mathematics for Finance,*" Springer
  - Cerny (2009) – "*Mathematical Techniques in Finance,*" Princeton
  - Lang – "*Introduction to Linear Algebra,*" Springer
  - Shreve (2004) – "*Stochastic Calculus for Finance I*," Springer

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