Galerkin Method

March 2, 2022

[1]:
```julia
using Symbolics, FastGaussQuadrature, Plots, QuadGK, LinearAlgebra, Revise

#Define Piecewise Linear Functions
function Na(x, mesh, index, delta)
    if x < 0 #Handle Gauss-legendre
        return 0
    end
    if index == 1 #Edge case handling
        if x < mesh[index+1]
            u = (mesh[index+1] - x)/ delta
            return u
        else
            return 0
        end
    elseif x < mesh[index-1] #Note: Index must be greated than 2
        return 0
    elseif x < mesh[index] # Piece from NA-1 to NA
        return (x - mesh[index-1]) / (delta)
    elseif x < mesh[index+1]
        return (mesh[index+1] - x) / (delta)
    else
        return 0
    end
end

function Naprime(x, mesh, index, delta)
    if x < 0 #Handle Gauss-legendre
        return 0
    end
    if index == 1 #Edge case handling
        if x < mesh[index+1]
            return (-1)/ delta
        else
            return 0
        end
    elseif x < mesh[index-1] #Note: Index must be greated than 2
        return 0
```

```
    elseif x < mesh[index] # Piece from NA-1 to NA
        return (1) / (delta)
    elseif x < mesh[index+1]
        return (-1) / (delta)
    else
        return 0
    end
end

function Nn1(x, mesh, delta)
    if x < 0#Handle Gauss-legendre
        return 0
    elseif x < mesh[length(mesh)-1]
        return 0
    else
        u = (x - mesh[length(mesh)-1])/delta
        return u
    end
end

function construct_K_elem(a, k, mesh, row, col, delta)
    #Using sym
    x, w = gausslegendre(100); #integrates from -1 to 1 but all basis functions are 0 for x < 0
    f(x) = k * Naprime(x, mesh, row, delta) * Naprime(x, mesh, col, delta) - a * Na(x, mesh, row, delta) * Naprime(x, mesh, col, delta)
    I = dot(w, f.(x));
    return I
end

function construct_F_elem(a, k, mesh, row, delta, boundaries)
    x, w = gausslegendre(100)
    if row == 1
        p(x) = Na(x, mesh, row, delta) * 12*x^2
        I = dot(w, p.(x)) + boundaries[1]
        return I
    elseif row == length(mesh)-1
        q(x) = Na(x, mesh, row, delta) * 12*x^2 + Naprime(x, mesh, row, delta)*Naprime(x, mesh, row, delta)*boundaries[2]
        I = dot(w, q.(x))
        return I
    else
        f(x) = Na(x, mesh, row, delta) * 12*x^2
        I = dot(w, f.(x))
        return I
    end
end
```

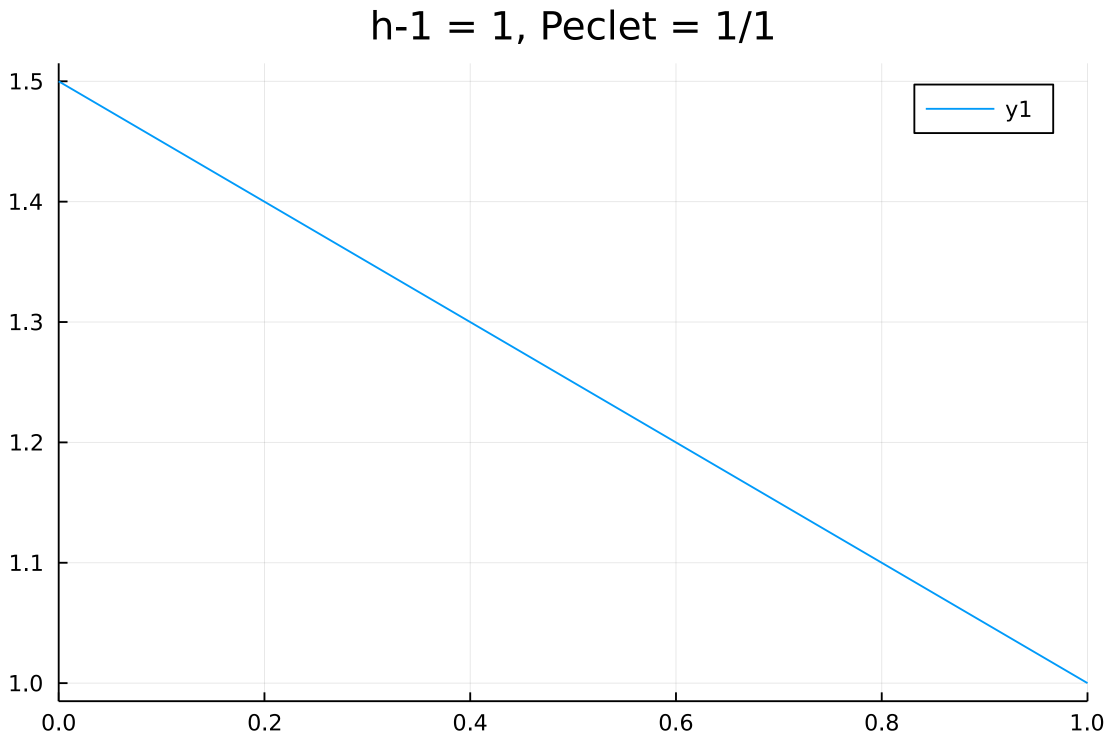
```julia
function construct_K(a, k, mesh, numrow, numcol, delta)
    KMat = zeros(numrow, numcol)
    for i = 1:numrow
        for j = 1:numcol
            KMat[i, j] = construct_K_elem(a, k, mesh, i, j, delta)
        end
    end
    return KMat
end

function construct_F(a, k, mesh, numrow, delta)
    FVec = zeros(numrow)
    for i = 1:numrow
        FVec[i] = construct_F_elem(a, k, mesh, i, delta, [0,1])
    end
    return FVec
end

function ADR_galerkin(n::Int, a::Int, k::Int, boundaries)
    @variables x
    delta = 1. / (n)
    mesh = collect(0:(1. / (n)):1) #n+1 points for n subintervals; x1 = 0;␣
 ↪x_n+1 = 1
    d = construct_F(a, k, mesh, n, delta) \ construct_K(a,k, mesh, n, n, delta)
    prog = "uh(x) = "
    for i in 1:length(d)
        coeff = d[i]
        prog = prog * "+ $coeff*Na(x, $mesh, $i, $delta)"
    end
    prog = prog * "+ (x - $mesh[$n])/$delta"
    exp = Meta.parse(prog)
    eval(exp)
    print("\n1D Uniform Mesh: ", mesh, " with spacing: ", delta, "\n")
    plot(uh, xlim=(0,1), title="h-1 = $n, Peclet = $a/$k")
end
```

[1]:
```
ADR_galerkin (generic function with 1 method)
```

[3]:
```julia
ADR_galerkin(1, 1, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 1.0] with spacing: 1.0
```

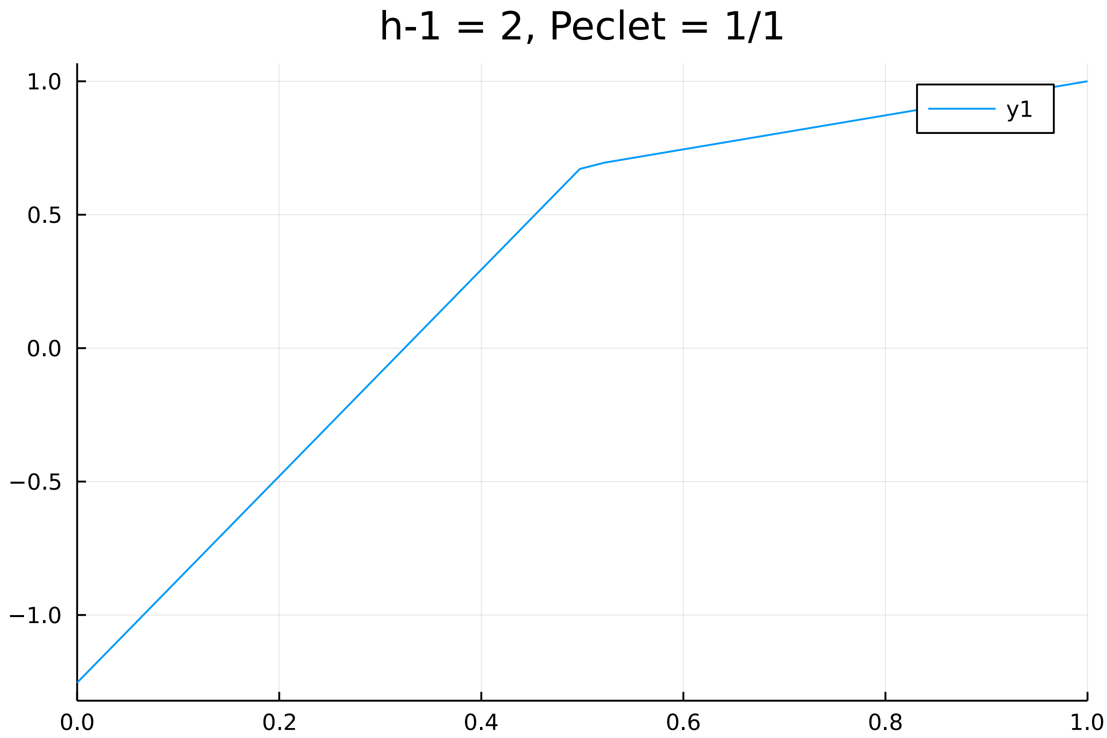
[3]:



[5]:
```
ADR_galerkin(2, 1, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.5, 1.0] with spacing: 0.5
```

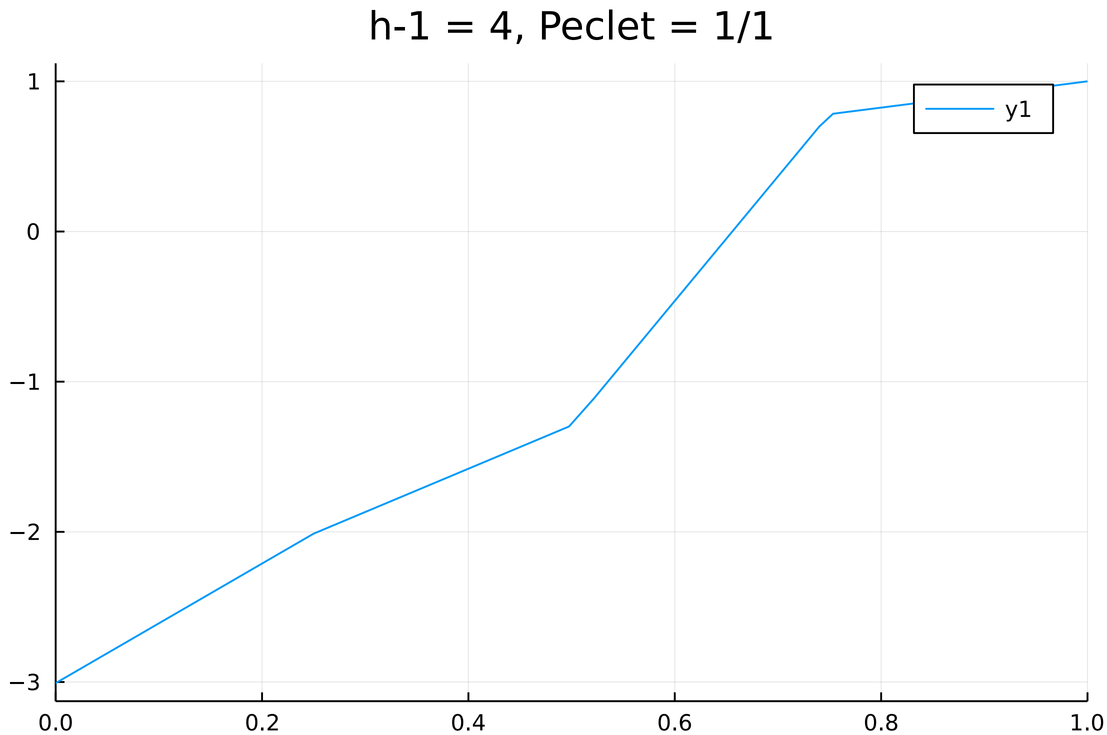
[5]:



[7]:
```
ADR_galerkin(4, 1, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.25, 0.5, 0.75, 1.0] with spacing: 0.25
```

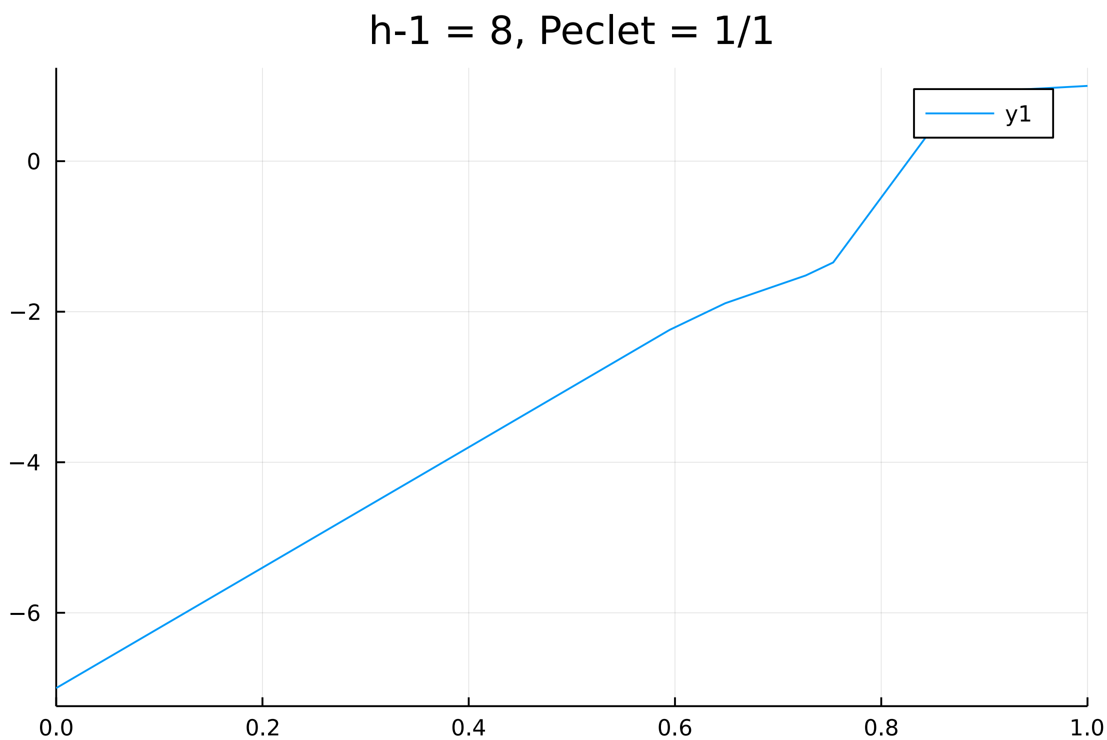
[7]:



[9]:
```
ADR_galerkin(8, 1, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1.0] with
spacing: 0.125
```

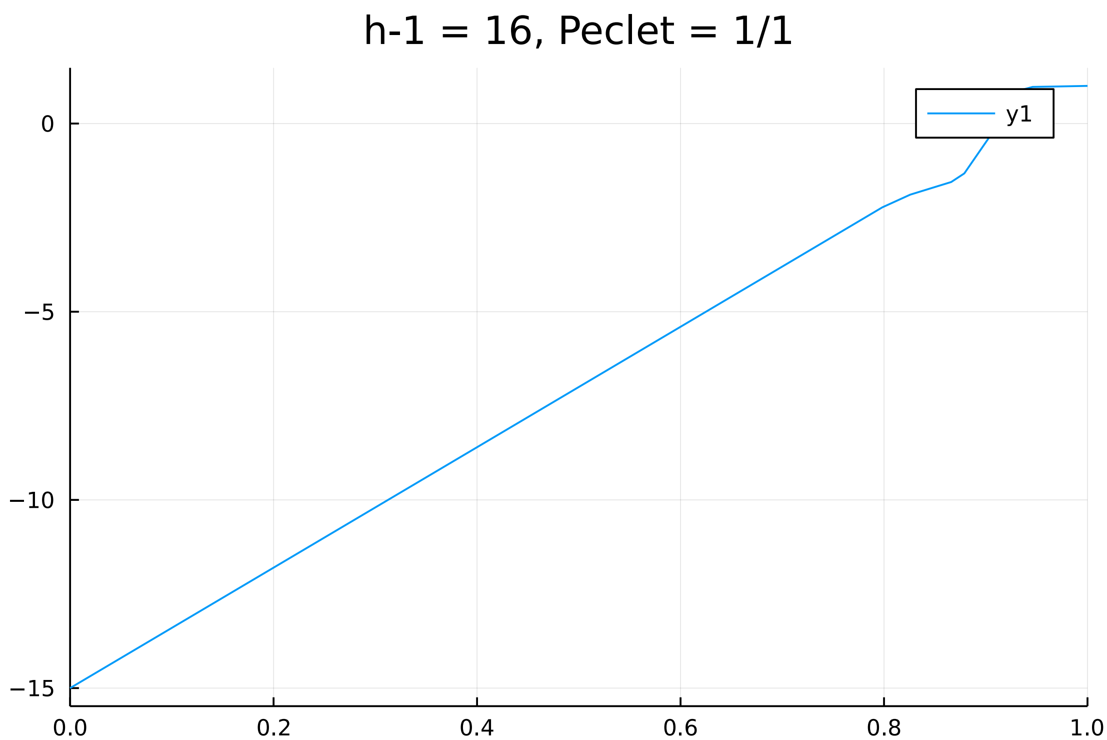
[9]:



[11]:
```
ADR_galerkin(16, 1, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.0625, 0.125, 0.1875, 0.25, 0.3125, 0.375, 0.4375, 0.5,
0.5625, 0.625, 0.6875, 0.75, 0.8125, 0.875, 0.9375, 1.0] with spacing: 0.0625
```

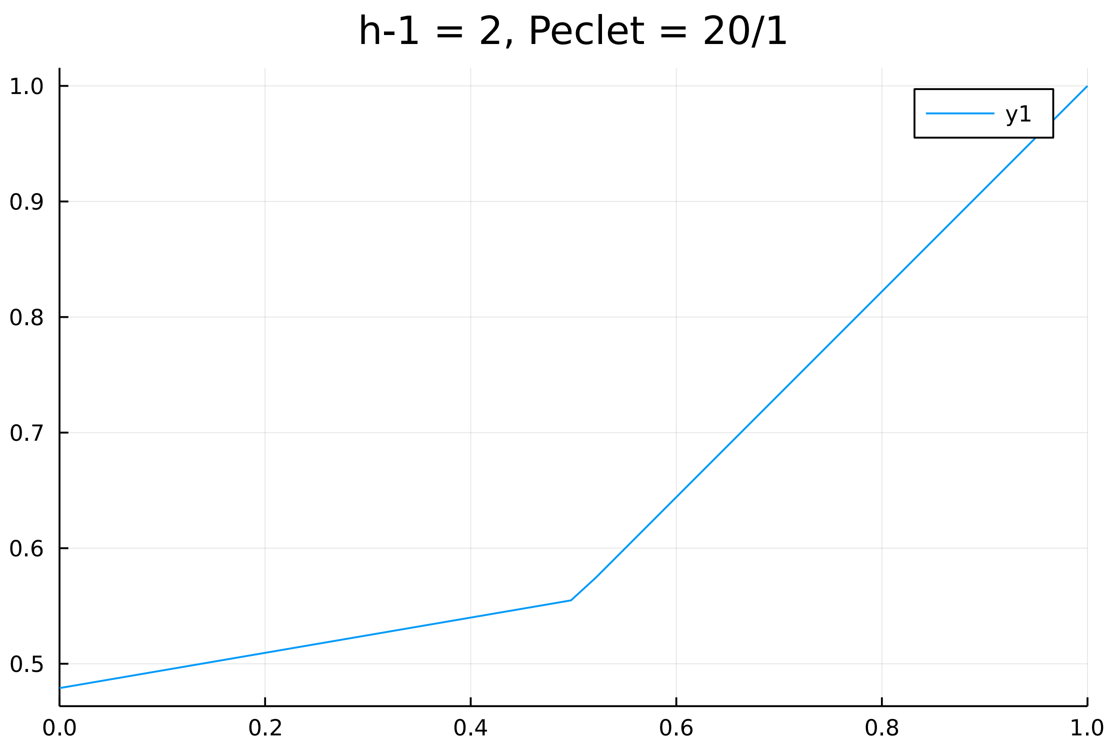
[11]:



[13]:
```
ADR_galerkin(2, 20, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.5, 1.0] with spacing: 0.5
```

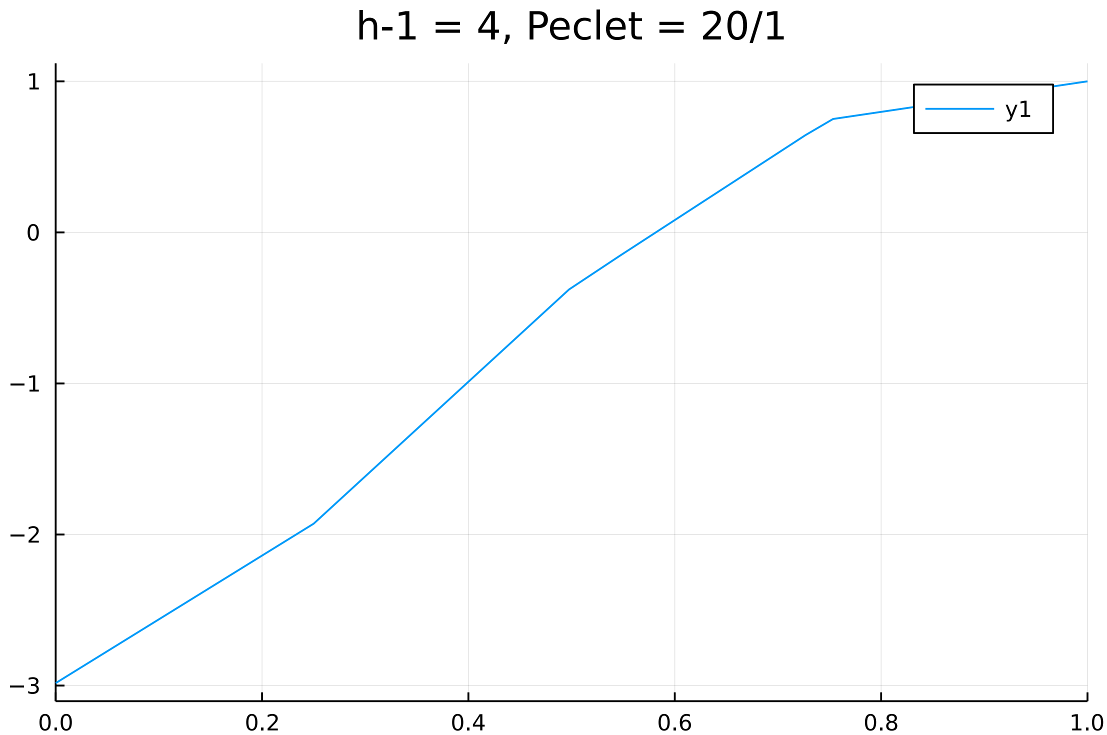
[13]:



[15]:
```
ADR_galerkin(4, 20, 1, [0,1])
```

```
1D Uniform Mesh: [0.0, 0.25, 0.5, 0.75, 1.0] with spacing: 0.25
```

[15]:


h-1 = 4, Peclet = 20/1
1
0
−1
−2
−3
0.0
0.2
0.4
0.6
0.8
1.0
y1

```
[ ]:
```

Exact Solutions

$a = 1$, $k = 1$, $a u_{,x} - k u_{,xx} = 12x^2$

$a u(0) - k u_{,x}(0) = 0$, $u(1) = 1$

Homogeneous: $u_{,xx} - a/k \, u_{,x} = 0$

$\rightarrow \frac{a/k + a/k}{2} \rightarrow a/k \rightarrow u(t) = c_1 e^{a/k \, t}$

B.C.'s: $u(1) = 1 \rightarrow c_1 e^{\frac{at}{k}} + \overset{0}{\cancel{c_2}} = 1 \rightarrow c_1 = \frac{1}{e^{a/k}}$

$a u(0) - k u'(0) = 0$: $a(c_1 + c_2) - k(\frac{a}{k} c_1) \Rightarrow c_2 = 0$

Particular: Guess $y_p = c_1 x^3 + c_2 x^2 + c_3 x$

$a(3c_1 x^3 + 2c_2 x^2 + c_3) - k(6c_1 x + c_2) = 12x^2$

$\rightarrow 2ac_2 = 6kc_1$, $c_2 = \frac{12k}{a^2}$

$ac_3 = 2kc_2$, $c_3 = 24k^2/a^2$

$c_1 = 4/a$

$$y = -\frac{24k^2 x}{a^3} + \frac{12kx^2}{a^2} + \frac{4kx^3}{a} - \frac{24k^3 e^{\frac{a}{k}x}}{a^4}$$

$a, k = 1 \rightarrow y = -24x + 12x^2 + 4x^3 - 24(e^x - 1)$

$$a=20,\ k=1 \Rightarrow y = \frac{4x^3}{20} + \frac{12x^2}{400} - \frac{24e^{\frac{20x}{4}}}{20^4} - \frac{24x}{8000}$$

Galerkin Method / Weak form

$\mathcal{L}u: \; 12x^2 = a u_{,x} - k u_{,xx}$, $\quad u(0)=0$, $u(1)=1$

$$w^n = \sum_1^n c_A N_A, \quad N_A(1)=0, \quad N_{n+1}(1)=1 \quad \text{(Shape/Basis)}$$

$$\int_0^1 w(12x^2 - a u_{,x} + k u_{,xx})\,dx = 0$$

$$\rightarrow \int_0^1 \sum c_A N_A \left(12x^2 - a\frac{d}{dx}\sum_1^n d_B N_B + k\frac{d}{dx^2}\left(\sum_1^n d_B N_B\right)\right)$$

IBP

$$\rightarrow \int_0^1 \frac{d}{dx}\left(k\sum_1^n d_B N_B \cdot \sum_1^n c_A N_A\right)dx - \int_0^1 \sum c_A N_A \cdot a\frac{d}{dx}\sum d_B N_B\,dx$$

$$- \left[\sum c_A N_A \cdot k\frac{d}{dx}\sum d_B N_B\right]_0^1$$

$$= \int_0^1 \sum c_A N_A \cdot 12x^2\,dx$$

let $n=1$. then $N_1 = 1-x$, $N_2 = x$

$u^n = d_1 N_1 + N_2$, $u^n(0)=0$, $u^n(1)=1$ ✓

$K = \int_0^1 [K \cdot N \cdot N \quad - a N \cdot N \quad]\,dx$