The Australian National University
Research School of Computer Science

Semester 2, 2020
Theory Assignment 1 of 5

COMP3670: Introduction to Machine Learning

Release Date. Aug 4th, 2020
Due Date. 23:59pm, Aug 23th, 2020
Maximum credit. 100

Exercise 1 **Solving Linear Systems** (5+5 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & -5 \\ 1 & -1 & 3 \\ 3 & -3 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 9 \\ 5 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & -10 \\ -4 & 2 & 3 \\ 10 & -3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 2 **Inverses** (5 credits)

For what values of λ does the inverse of the following matrix exist?

$$\begin{bmatrix} \lambda & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 3 **Subspaces** (4×2.5 credits)

Which of the following sets are subspaces of $\mathbb{R}^3$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y, 1) : x, y, \in \mathbb{R}\}$

(b) $B = \{(x, y, z) : x + 4y - 3z = t\}$, where t is some real number. (Your answer may depend on the value of t.)

(c) $C = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$

(d) $D = \{(x, y, z) : x, y \in \mathbb{R}, z \in \mathbb{Q}\}$.

Exercise 4 **Linear Independence** (10 credits)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^2$. Prove that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependant.

Exercise 5 **Inner Products** (10+5 credits)

(a) Is the following function $\langle \cdot, \cdot \rangle$ defined for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = y_1(x_1 - x_2) + y_2(x_2 - x_1)$$

an inner product? Which of the three inner product axioms are satisfied?

(b) Prove that $\langle \cdot, \cdot \rangle$ defined for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$$

is **not** an inner product.

Exercise 6 **Properties of Norms** (8+5+10+12 credits)

Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ and an induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

(a) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependant vectors, then $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\| \|\mathbf{y}\|$.

(b) Show that we can retrieve the inner product from the norm via the following expression:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2} \left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 \right)$$

Given a vector space V with two norms $\| \cdot \|_a : V \to \mathbb{R}_{\geq 0}$ and $\| \cdot \|_b : V \to \mathbb{R}_{\geq 0}$, we say that the two norms $\| \cdot \|_a$ and $\| \cdot \|_b$ are *equivalent* if there exists $M_1 > 0, M_2 > 0$ such that for any $\mathbf{v} \in V$, we have that

$$M_1 \|\mathbf{v}\|_a \leq \|\mathbf{v}\|_b \leq M_2 \|\mathbf{v}\|_a.$$

(c) Show that norm equivalence is an equivalence relation, that is, that norm equivalence is reflexive, symmetric and transitive.

(d) Assuming that $V = \mathbb{R}^2$, show that $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent norms.

Exercise 7 **Projections** (5+7+3 credits)

Consider the Euclidean vector space $\mathbb{R}^3$ with the dot product. A subspace $U \subset \mathbb{R}^3$ and vector $\mathbf{x} \in \mathbb{R}^3$ are given by

$$U = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

(a) Show that $\mathbf{x} \notin U$.

(b) Determine the orthogonal projection $\pi_U(\mathbf{x})$ of $\mathbf{x}$ onto U. Show that $\pi_U(\mathbf{x})$ can be written as a linear combination of $[1, 1, 1]^T$ and $[2, 2, 3]^T$.

(c) Determine the distance $d(\mathbf{x}, U)$.