Lack of metric projectivity, injectivity, and flatness for modules L_p

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Abstract: In this paper we demonstrate that for a locally compact Hausdorff space S and a decomposable Borel measure μ metric projectivity, injectivity, or flatness of the $C_0(S)$-module $L_p(S,\mu)$ implies that μ is purely atomic with at most one atom.

Keywords: metric projectivity, metric injectivity, metric flatness, L_p-space.

1 Introduction

This paper finalizes the author's research on the homological properties of modules L_p. In [1], it was shown that modules L_p are relatively projective for a small class of underlying measure spaces, namely purely atomic measure spaces, with atoms being isolated points. This paper aims to solve the same problem for metric projectivity, injectivity, and flatness. It was expected that for the metric theory, the class of measure spaces would need to be even narrower. As demonstrated in this paper, this class is very small, as it includes only purely atomic measure spaces with at most one atom. Hence, it is safe to conclude that L_p-spaces are almost never metrically projective, injective, or flat.

Before delving into the main topic, we shall provide a few definitions. For any natural number $n \in \mathbb{N}$, we denote the set of the first n natural numbers by $\mathbb{N}_n$. Let M be a subset of a set N, then χ_M denotes the indicator function of M. The symbol 1_N denotes the identity map on N. If $k, l \in N$, then δ_k^l stands for their Kronecker symbol.

All Banach spaces discussed in this paper are considered over the complex numbers. We will actively employ the following Banach space constructions: for a given family of Banach spaces $\{E_\lambda : \lambda \in \Lambda\}$, by $\bigoplus_p \{E_\lambda : \lambda \in \Lambda\}$ we denote their ℓ_p-sum (see [[2], proposition 1.1.7]). Similarly, for a family of linear operators $T_\lambda : E_\lambda \to F_\lambda$ where $\lambda \in \Lambda$, their ℓ_p-sum is denoted as $\bigoplus_p \{T_\lambda : \lambda \in \Lambda\}$ (see [[2], proposition 1.1.7]). We use the symbol $E \widehat{\otimes} F$ to denote the projective tensor product of Banach spaces E and F [[2], theorem 2.7.4].

Let $T : E \to F$ be a bounded linear operator between Banach spaces. Now, we provide a quantitative version of the definition of embedding. If there exists a constant $c > 0$ such that $c\|T(x)\| \geq \|x\|$ for all $x \in E$, then T is called c-topologically injective. Similarly, a quantitative definition of the open map is as follows: if there exists a constant $c > 0$ such that for any $y \in F$, we can find an $x \in E$ such that $T(x) = y$ and $c\|y\| \geq \|x\|$, then T is called c-topologically surjective. Finally, the operator T is called contractive if its norm is not greater than 1.

Consider a Banach algebra A. We consider both left and right Banach A-modules with contractive outer action $\cdot : A \times X \to X$. Let X and Y be two Banach A-modules, then a map $\phi : X \to Y$ is called an A-morphism if it is a continuous A-module map. All left Banach A-modules and their A-morphisms form a category denoted by $A - \mathbf{mod}$. Similarly, one can define the category of right A-modules $\mathbf{mod} - A$. Finally, $X \widehat{\otimes}_A Y$ denotes the projective module tensor product of a left A-module X and a right A-module Y (see [[3], definition VI.3.18]).

Now, we proceed to define metric projectivity, injectivity, and flatness. The first paper on this subject was written in 1978 by Graven [4]. These notions were later rediscovered by White [5] and

Helemeskii [6, 7].

A left Banach A-module P is termed metrically projective if, for any c-topologically surjective A-morphism $\xi : X \to Y$ and any A-morphism $\phi : P \to Y$, there exists an A-morphism $\psi : P \to X$ such that $\|\psi\| \leq c$, and the diagram

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ P \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"'] & Y \end{tikzcd}$$

commutes. The original definition was somewhat different [[4], definition 2.4], but it is still equivalent to the one above. The simplest example of a metrically projective A-module is the algebra A itself, provided it is unital [[4], theorem 2.5].

A right Banach A-module J is termed metrically injective if, for any c-topologically injective A-morphism $\xi : Y \to X$ and any A-morphism $\phi : Y \to J$, there exists an A-morphism $\psi : X \to J$ such that $\|\psi\| \leq c$, and the diagram

$$\begin{tikzcd} & X \arrow[dl, dashed, "\psi"'] \\ J & Y \arrow[l, "\phi"] \arrow[u, "\xi"'] \end{tikzcd}$$

commutes. Our definition is equivalent to the original one [[4], definition 3.1]. Note that P^* is a metrically injective A-module whenever P is metrically injective [[4], theorem 3.2]. Therefore, the right Banach module A^* is metrically injective whenever A is unital.

A left A-module F is termed metrically flat if, for each c-topologically injective A-morphism $\xi : X \to Y$ of right A-modules, the operator $\xi \widehat{\otimes}_A 1_F : X \widehat{\otimes}_A F \to Y \widehat{\otimes}_A F$ is c-topologically injective. This definition was implicitly given in the theorem [[4], theorem 3.10]. From this theorem and remarks above, we conclude that any metrically projective module is metrically flat. In particular, a unital Banach algebra A is a metrically flat A-module.

2 Metric injectivity of finite-dimensional $\ell_\infty(\Lambda)$-module $\ell_p(\Lambda)$

For an index set Λ and $1 \leq p \leq +\infty$, $\ell_p(\Lambda)$ denotes the standard ℓ_p-space. Its norm is denoted by $\|\cdot\|_p$, and its natural basis is $(e_\lambda)_{\lambda\in\Lambda}$. For $1 \leq p < +\infty$, we often use the standard identification $\ell_p(\Lambda)^* = \ell_{p^*}(\Lambda)$, where $p^* = p/(p-1)$. By convention $1/0 = +\infty$, so $1^* = +\infty$. The space $\ell_p(\Lambda)$ can function as both left and right Banach module over the Banach algebra $\ell_\infty(\Lambda)$. In this section, we demonstrate that for finite Λ, the right $\ell_\infty(\Lambda)$-module $\ell_p(\Lambda)$ is metrically injective only if Λ has at most one element.

Definition 2.1. *Let Λ be a set, $1 < p < +\infty$, and $\mathcal{F}$ be a bounded subset of $\ell_{p^*}(\Lambda)$. We define a linear operator*

$$\xi_{\mathcal{F}} : \ell_p(\Lambda) \to \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\},\ x \mapsto \bigoplus_\infty \{x \cdot f : f \in \mathcal{F}\}.$$

Clearly $\xi_{\mathcal{F}}$ is an $\ell_\infty(\Lambda)$-morphism with norm $\sup\{\|f\|_{p^*} : f \in \mathcal{F}\}$.

Definition 2.2. *Let Λ be a set, $1 < p < +\infty$, and $\mathcal{F}$ be a bounded subset of $\ell_{p^*}(\Lambda)$. We define the coercivity constant for the operator $\xi_{\mathcal{F}}$ as*

$$\gamma_{\mathcal{F}} = \sup\{\|x\|_p : x \in \ell_p(\Lambda),\ \|\xi_{\mathcal{F}}(x)\| \leq 1\}.$$

Note that $\xi_\mathcal{F}$ is $\gamma_\mathcal{F}$-topologically injective whenever $\gamma_\mathcal{F}$ is finite.

Proposition 2.3. *Let Λ be a set, $1 \leq p, q \leq +\infty$, and $\phi : \ell_p(\Lambda) \to \ell_q(\Lambda)$ be an $\ell_\infty(\Lambda)$-morphism of right modules. Then there exists a vector $\eta \in \ell_\infty(\Lambda)$ such that $\phi(x) = \eta \cdot x$ for all $x \in \ell_p(\Lambda)$.*

Proof. Denote $\eta_\lambda = \phi(e_\lambda)_\lambda$ for $\lambda \in \Lambda$. For any $x \in \ell_p(\Lambda)$ and $\lambda \in \Lambda$, we have

$$\phi(x)_\lambda = (\phi(x) \cdot e_\lambda)_\lambda = \phi(x \cdot e_\lambda)_\lambda = \phi(x_\lambda e_\lambda)_\lambda = x_\lambda \phi(e_\lambda)_\lambda = x_\lambda \eta_\lambda = (\eta \cdot x)_\lambda.$$

Therefore, $\phi(x) = \eta \cdot x$. By construction $\|\eta\|_\infty \leq \|\phi\|$, so $\eta \in \ell_\infty(\Lambda)$. □

Proposition 2.4. *Let Λ be a set, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set. Then for any morphism of right $\ell_\infty(\Lambda)$-modules $\psi : \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\} \to \ell_p(\Lambda)$ there exists a family of vectors $\eta \in \ell_\infty(\Lambda)^\mathcal{F}$ such that*

$$\psi(t) = \sum_{f \in \mathcal{F}} \eta_f \cdot t_f$$

for all $t \in \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\}$.

Proof. For each $f \in \mathcal{F}$ we define a natural embedding $\mathrm{in}_f : \ell_1(\Lambda) \to \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\}$ which is a morphism of right $\ell_\infty(\Lambda)$-modules. Then we define an $\ell_\infty(\Lambda)$-morphism $\psi_f = \psi \circ \mathrm{in}_f$. By proposition 2.3 there exists a vector $\eta_f \in \ell_\infty(\Lambda)$ such that $\psi_f(x) = \eta_f \cdot x$ for all $x \in \ell_1(\Lambda)$. Since $\mathcal{F}$ is finite, then for all $t \in \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\}$ we have

$$\psi(t) = \psi\left(\bigoplus_\infty \{t_f : f \in \mathcal{F}\}\right) = \psi\left(\sum_{f \in \mathcal{F}} \mathrm{in}_f(t_f)\right) = \sum_{f \in \mathcal{F}} \psi_f(t_f) = \sum_{f \in \mathcal{F}} \eta_f \cdot t_f.$$

□

Definition 2.5. *Let Λ be a set, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set. For a given family $\eta \in \ell_\infty(\Lambda)^\mathcal{F}$ we define*

$$\psi_\eta : \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\} \to \ell_p(\Lambda),\ t \mapsto \sum_{f \in \mathcal{F}} \eta_f \cdot t_f.$$

Definition 2.6. *Let Λ be a set, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set, then we define*

$$\mathcal{N}_\mathcal{F} = \left\{\eta \in \ell_\infty(\Lambda)^\mathcal{F} : \sum_{f \in \mathcal{F}} \eta_{f,\lambda} f_\lambda = 1,\ \ \lambda \in \Lambda\right\}.$$

Proposition 2.7. *Let Λ be a set, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set. Then ψ_η is a left inverse $\ell_\infty(\Lambda)$-morphism to $\xi_\mathcal{F}$ iff $\eta \in \mathcal{N}_\mathcal{F}$.*

Proof. Suppose ψ_η is a left inverse to $\xi_\mathcal{F}$, then for any $\lambda \in \Lambda$ we have

$$1 = (e_\lambda)_\lambda = \psi_\eta(\xi_\mathcal{F}(e_\lambda))_\lambda = \psi_\eta\left(\bigoplus_\infty \{e_\lambda \cdot f : f \in \mathcal{F}\}\right)_\lambda = \left(\sum_{f \in \mathcal{F}} \eta_f \cdot e_\lambda \cdot f\right)_\lambda = \sum_{f \in \mathcal{F}} \eta_{f,\lambda} f_\lambda.$$

Hence, $\eta \in \mathcal{N}_\mathcal{F}$. Conversely, let $\eta \in \mathcal{N}_\mathcal{F}$, then for any $x \in \ell_p(\Lambda)$ holds

$$\psi_\eta(\xi_\mathcal{F}(x)) = \sum_{f \in \mathcal{F}} \eta_f \cdot x \cdot f = \sum_{f \in \mathcal{F}} \sum_{\lambda \in \Lambda} (\eta_f \cdot x \cdot f)_\lambda e_\lambda = \sum_{\lambda \in \Lambda} \left(\sum_{f \in \mathcal{F}} \eta_{f,\lambda} f_\lambda\right) x_\lambda e_\lambda = \sum_{\lambda \in \Lambda} x_\lambda e_\lambda = x.$$

Therefore, ψ_η is a left inverse for $\xi_\mathcal{F}$. □

Proposition 2.8. *Let Λ be a finite set, $1 < p < +\infty$, and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set. Suppose $\eta \in \ell_\infty(\Lambda)^{\mathcal{F}}$, then*

$$\|\psi_\eta\| = \max\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}} |\eta_{f,\lambda}| \delta_\lambda^{d(f)} \right|^p \right)^{1/p} : d \in \Lambda^{\mathcal{F}} \right\}.$$

Proof. By definition of operator norm,

$$\begin{aligned}
\|\psi_\eta\| &= \sup\left\{ \|\psi_\eta(t)\|_p : t \in \bigoplus_\infty \{\ell_1(\Lambda) : f \in \mathcal{F}\},\ \|t\| \le 1 \right\} \\
&= \sup\left\{ \left\| \sum_{f\in\mathcal{F}} \eta_f \cdot t_f \right\|_p : t_f \in \ell_1(\Lambda),\ f \in \mathcal{F},\ \max\{\|t_f\| : f \in \mathcal{F}\} \le 1 \right\} \\
&= \sup\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}} \eta_{f,\lambda} t_{f,\lambda} \right|^p \right)^{1/p} : \sum_{\lambda\in\Lambda} |t_{f,\lambda}| \le 1,\ t_{f,\lambda} \in \mathbb{C},\ f \in \mathcal{F}, \lambda \in \Lambda \right\}.
\end{aligned}$$

For each $\lambda \in \Lambda$ and $f \in \mathcal{F}$ we denote $r_{f,\lambda} = |t_{f,\lambda}|$ and $\alpha_{f,\lambda} = \arg(t_{f,\lambda})$. Then $t_{f,\lambda} = r_{f,\lambda} e^{i\alpha_{f,\lambda}}$. So,

$$\|\psi_\eta\| = \sup\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}} \eta_{f,\lambda} r_{f,\lambda} e^{i\alpha_{f,\lambda}} \right|^p \right)^{1/p} : \sum_{\lambda\in\Lambda} r_{f,\lambda} \le 1,\ r_{f,\lambda} \in \mathbb{R}_+,\ \alpha_{f,\lambda} \in \mathbb{R},\ f \in \mathcal{F},\ \lambda \in \Lambda \right\}.$$

For each $\lambda \in \Lambda$ consider vectors $a_\lambda = (\eta_{f,\lambda} r_{f,\lambda})_{f\in\mathcal{F}}$ and $b_\lambda = (e^{-i\alpha_{f,\lambda}})_{f\in\mathcal{F}}$ in $\ell_2(\mathcal{F})$. By Cauchy-Schwartz inequality the scalar product of a_λ and b_λ has the maximum absolute value only if $a_\lambda = k b_\lambda$ for some $k \in \mathbb{C}$. The latter is possible only if $\arg(\eta_{f,\lambda} r_{f,\lambda}) = -\alpha_{f,\lambda} + \arg(k)$ for all $f \in \mathcal{F}$. As a corollary, for all $\lambda \in \Lambda$ the maximum of the expression $\left|\sum_{f\in\mathcal{F}} \eta_{f,\lambda} r_{f,\lambda} e^{i\alpha_{f,\lambda}}\right|$ is attained if $\alpha_{f,\lambda} = \arg(k) - \arg(\eta_{f,\lambda})$ for all $f \in \mathcal{F}$. In this case,

$$\|\psi_\eta\| = \sup\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}} |\eta_{f,\lambda}| r_{f,\lambda} \right|^p \right)^{1/p} : \sum_{\lambda\in\Lambda} r_{f,\lambda} \le 1,\ r_{f,\lambda} \in \mathbb{R}_+,\ f \in \mathcal{F},\ \lambda \in \Lambda \right\}.$$

Consider linear operators $\tau_f : \mathbb{R}^\Lambda \to \ell_p(\Lambda) : r \mapsto \eta_f \cdot r$ for $f \in \mathcal{F}$. Then,

$$\|\psi_\eta\| = \sup\left\{ \left\| \sum_{f\in\mathcal{F}} \tau_f(r_f) \right\|_p : \sum_{\lambda\in\Lambda} r_{f,\lambda} \le 1,\ r_{f,\lambda} \in \mathbb{R}_+,\ f \in \mathcal{F},\ \lambda \in \Lambda \right\}.$$

Since linear operators $(\tau_f)_{f\in\mathcal{F}}$ attain their values in $\ell_p(\Lambda)$, whose norm is strictly convex, then the function $F : (\mathbb{R}^\Lambda)^{\mathcal{F}} \to \mathbb{R}_+,\ r \mapsto \left\|\sum_{f\in\mathcal{F}} \tau_f(r_f)\right\|_p$ is strictly convex. Since the set $C = \left\{r \in (\mathbb{R}^\Lambda)^{\mathcal{F}} : \sum_{\lambda\in\Lambda} r_{f,\lambda} \le 1,\ r_f \in \mathbb{R}_+^\Lambda,\ f \in \mathcal{F},\ \lambda \in \Lambda\right\}$ is a convex polytope in a finite-dimensional space, then F attains its maximum on $\operatorname{ext}(C)$ — the set of extreme points of C. So,

$$\|\psi_\eta\| = \max\{F(r) : r \in \operatorname{ext}(C)\},$$

Clearly, $r \in \operatorname{ext}(C)$ iff $r = 0$ or for some function $d : \mathcal{F} \to \Lambda$ and all $\lambda \in \Lambda$, $f \in \mathcal{F}$ holds $r_{f,\lambda} = \delta_\lambda^{d(f)}$. So,

$$\|\psi_\eta\| = \max\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}} |\eta_{f,\lambda}| \delta_\lambda^{d(f)} \right|^p \right)^{1/p} : d \in \Lambda^{\mathcal{F}} \right\}.$$

$\square$

Definition 2.9. *Let Λ be a set, $1 < p < +\infty$, and $\mathcal{F} \subset \ell_{p^*}(\Lambda)$ be a finite set. Then we define*

$$\nu_{\mathcal{F}} = \inf\{\|\psi_\eta\| : \eta \in \mathcal{N}_{\mathcal{F}}\}.$$

Definition 2.10. *Let Λ be a finite set and $\kappa \in \mathbb{R}$. We set $f_\lambda = e_\lambda$ and $f_\star = \kappa \sum_{\lambda\in\Lambda} e_\lambda$. Now we define*

$$\mathcal{F}_\kappa(\Lambda) = \{f_\lambda : \lambda \in \Lambda\} \cup \{f_\star\}.$$

Remark 2.11. *Let Λ be a finite set, $1 < p < +\infty$ and $\kappa \in \mathbb{R}$. Then we can regard $\mathcal{F}_\kappa(\Lambda)$ as a bounded subset of $\ell_{p^*}(\Lambda)$. In this case for all $x \in \ell_p(\Lambda)$ holds*

$$\xi_{\mathcal{F}_\kappa(\Lambda)}(x) = \left(\bigoplus_\infty \{x \cdot e_\lambda : \lambda \in \Lambda\}\right) \bigoplus_\infty (\kappa x), \qquad \|\xi_{\mathcal{F}_\kappa(\Lambda)}(x)\| = \max\{\|x\|_\infty, \|\kappa x\|_1\}.$$

Proposition 2.12. *Assume Λ is a finite set with $n > 1$ elements, $1 < p < +\infty$, and $n^{-1} < \kappa < (n-1)^{-1}$. Then*

$$\gamma_{\mathcal{F}_\kappa(\Lambda)} = (n - 1 + (\kappa^{-1} - (n-1))^p)^{1/p}.$$

Proof. By the definition of coercivity constant,

$$\begin{aligned}
\gamma_{\mathcal{F}_\kappa(\Lambda)} &= \sup\{\|x\|_p : x \in \ell_p(\Lambda),\ \|\xi_{\mathcal{F}_\kappa(\Lambda)}(x)\| \le 1\} \\
&= \sup\left\{ \left(\sum_{\lambda\in\Lambda} |x_\lambda|^p\right)^{1/p} : x \in \mathbb{C}^\Lambda,\ \max\left\{\max\{|x_\lambda| : \lambda \in \Lambda\}, \kappa \sum_{\lambda\in\Lambda} |x_\lambda|\right\} \le 1 \right\} \\
&= \sup\left\{ \left(\sum_{\lambda\in\Lambda} |t_\lambda|^p\right)^{1/p} : t \in \mathbb{R}^\Lambda,\ \sum_{\lambda\in\Lambda} |t_\lambda| \le \kappa^{-1},\ |t_\lambda| \le 1,\ \lambda \in \Lambda \right\}.
\end{aligned}$$

Consider strictly convex function $F : \mathbb{R}^\Lambda \to \mathbb{R}$, $t \mapsto \left(\sum_{\lambda\in\Lambda} |t_\lambda|^p\right)^{1/p}$ and the convex polytope $C = \left\{t \in \mathbb{R}^\Lambda : \sum_{\lambda\in\Lambda} |t_\lambda| \le \kappa^{-1},\ |t_\lambda| \le 1,\ \lambda \in \Lambda\right\}$ in a finite-dimensional space. Since F is strictly convex, then F attains its maximum on $\mathrm{ext}(C)$ — the set of extreme points of C. Therefore,

$$\gamma_{\mathcal{F}_\kappa(\Lambda)} = \max\{F(t) : t \in \mathrm{ext}(C)\}.$$

Geometrically C is an n-dimensional cube $[-1, 1]^n$ with vertices chopped off by hyperplanes of the form $\pm t_1 \pm t_2 \pm \ldots \pm t_n = \kappa^{-1}$. Clearly, any point $t \in \mathrm{ext}(C)$ has all coordinates but one equal to 1 or -1. Therefore, $\mathrm{ext}(C) = \left\{t \in \mathbb{R}^\Lambda : \exists \lambda' \in \Lambda \quad |t_{\lambda'}| = \kappa^{-1} - (n-1) \wedge \forall \lambda \in \Lambda \setminus \{\lambda'\} \quad |t_\lambda| = 1\right\}$. As a consequence, $\gamma_{\mathcal{F}_\kappa(\Lambda)} = (n - 1 + (\kappa^{-1} - (n-1))^p)^{1/p}$. $\square$

Proposition 2.13. *Assume Λ is a finite set with $n > 1$ elements, $1 < p < +\infty$, and $n^{-1} < \kappa < (n-1)^{-1}$. Then*

$$\nu_{\mathcal{F}_\kappa(\Lambda)} \ge \kappa^{-1} \left(\left(\frac{n-1}{\kappa^{-1} - 1}\right)^{\frac{p}{p-1}} + n - 1 \right)^{1/p} \left(\left(\frac{n-1}{\kappa^{-1} - 1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1} \right)^{-1}.$$

Proof. By construction $\mathcal{N}_{\mathcal{F}_\kappa(\Lambda)} = \{\eta \in \ell_\infty(\Lambda)^{\mathcal{F}_\kappa(\Lambda)} : \eta_{f_\lambda,\lambda} + \kappa\eta_{f_\star,\lambda} = 1,\, \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$ consider the function

$$d_\lambda : \mathcal{F}_\kappa(\Lambda) \to \Lambda,\ f_i \mapsto \begin{cases} i & \text{if } i \neq \star, \\ \lambda & \text{if } i = \star. \end{cases}$$

Then from proposition 2.8, for any $\eta \in \ell_\infty(\Lambda)^{\mathcal{F}_\kappa(\Lambda)}$ we would get

$$\begin{aligned} \|\psi_\eta\| &\geq \max\left\{ \left(\sum_{\lambda\in\Lambda} \left| \sum_{f\in\mathcal{F}_\kappa(\Lambda)} |\eta_{f,\lambda}| \delta_\lambda^{d_{\lambda'}(f)} \right|^p \right)^{1/p} : \lambda' \in \Lambda \right\} \\ &= \max\left\{ \left((|\eta_{f_{\lambda'},\lambda'}| + |\eta_{f_\star,\lambda'}|)^p + \sum_{\lambda\in\Lambda,\lambda\neq\lambda'} |\eta_{f_\lambda,\lambda}|^p \right)^{1/p} : \lambda' \in \Lambda \right\}. \end{aligned}$$

For any $\eta \in \mathcal{N}_{\mathcal{F}_\kappa(\Lambda)}$ and $\lambda \in \Lambda$ we have $\eta_{f_\lambda,\lambda} + \kappa\eta_{f_\star,\lambda} = 1$. Therefore, by reverse triangle inequality

$$\begin{aligned} \|\psi_\eta\| &\geq \max\left\{ \left((|\eta_{f_{\lambda'},\lambda'}| + |\kappa^{-1}(1-\eta_{f_{\lambda'},\lambda'})|)^p + \sum_{\lambda\in\Lambda,\lambda\neq\lambda'} |\eta_{f_\lambda,\lambda}|^p \right)^{1/p} : \lambda' \in \Lambda \right\} \\ &\geq \max\left\{ \left((|\eta_{f_{\lambda'},\lambda'}| + \kappa^{-1}|1-|\eta_{f_{\lambda'},\lambda'}||)^p + \sum_{\lambda\in\Lambda,\lambda\neq\lambda'} |\eta_{f_\lambda,\lambda}|^p \right)^{1/p} : \lambda' \in \Lambda \right\} \end{aligned}$$

for any $\eta \in \mathcal{N}_{\mathcal{F}_\kappa(\Lambda)}$. Denote

$$\begin{aligned} &\alpha_i : \mathbb{R}^n \to \mathbb{R},\ t \mapsto \left((|t_i| + \kappa^{-1}|1-|t_i||)^p + \sum_{k=1,k\neq i}^n |t_k|^p \right)^{1/p} \quad \text{for} \quad i \in \mathbb{N}_n \\ &\alpha : \mathbb{R}^n \to \mathbb{R},\ t \mapsto \max\{\alpha_i(t) : i \in \mathbb{N}_n\}. \end{aligned}$$

Then for any enumeration $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, we get

$$\nu_{\mathcal{F}_\kappa(\Lambda)} \geq \inf\{\|\psi_\eta\| : \eta \in \mathcal{N}_{\mathcal{F}_\kappa(\Lambda)}\} = \inf\{\alpha(|\eta_{f_{\lambda_1},\lambda_1}|, \ldots, |\eta_{f_{\lambda_n},\lambda_n}|) : \eta \in \mathcal{N}_{\mathcal{F}_\kappa(\Lambda)}\} = \inf\{\alpha(t) : t \in \mathbb{R}_+^n\}.$$

Consider functions

$$F_1 : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto t, \quad F_2 : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto t + \kappa^{-1}|1-t|, \quad F_3 : \mathbb{R}^n \to \mathbb{R},\ t \mapsto \left(\sum_{k=1}^n |t_k|^p \right)^{1/p}.$$

Clearly, for each $i \in \mathbb{N}_n$ the function α_i is a composition of F_1, F_2, and F_3. Since F_1 and F_2 are convex and F_3 is strictly convex, then all functions $(\alpha_i)_{i\in\mathbb{N}_n}$ are strictly convex on $\mathbb{R}_+^n$. Hence, so is their maximum α. Note that α is continuous, strictly convex, and $\lim_{\|t\|\to+\infty} \alpha(t) = +\infty$. Therefore, α has a unique global minimum at some point $t_0 \in \mathbb{R}_+^n$. Observe that α is invariant under permutation of its arguments. Then from the uniqueness of the global minimum at t_0, we can conclude that all coordinates of t_0 are equal. As a corollary

$$\begin{aligned} \nu_{\mathcal{F}_\kappa(\Lambda)} &\geq \inf\{\alpha(t) : t \in \mathbb{R}_+^n\} \\ &= \inf\{\alpha(s, \ldots, s) : s \in \mathbb{R}_+\} \\ &= \inf\{((s + \kappa^{-1}|1-s|)^p + (n-1)s^p)^{1/p} : s \in \mathbb{R}_+\}. \end{aligned}$$

Consider the function $F : \mathbb{R}_+ \to \mathbb{R}$, $s \mapsto ((n-1)s^p + (s + \kappa^{-1}|1-s|)^p)^{1/p}$. We have

$$F(s) = \begin{cases} ((s+\kappa^{-1}(1-s))^p + (n-1)s^p)^{1/p} & \text{if} \quad 0 \le s \le 1, \\ \left((\kappa^{-1}+1)^p \left(s - \frac{\kappa^{-1}}{\kappa^{-1}+1}\right)^p + (n-1)s^p\right)^{1/p} & \text{if} \quad s > 1. \end{cases}$$

Since F is continuous and clearly increasing on $(1, +\infty)$, then F attains its minimum on $[0,1]$. Let us find stationary points of F on $[0,1]$. For $s \in [0,1]$, we have

$$F'(s) = ((s+\kappa^{-1}(1-s))^p + (n-1)s^p)^{1/p-1}((\kappa^{-1} + (1-\kappa^{-1})s)^{p-1}(1-\kappa^{-1}) + (n-1)s^{p-1}).$$

The stationary point can be found from the equation $F'(s) = 0$. The solution is

$$s_0 = \kappa^{-1}\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}\right)^{-1}.$$

By assumption $n^{-1} < \kappa$, so $\frac{n-1}{\kappa^{-1}-1} < 1$ and therefore $0 < s_0 < 1$. Since F is convex, then s_0 is the point of minimum on $[0,1]$. The minimum equals

$$F(s_0) = \kappa^{-1}\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{p}{p-1}} + n - 1\right)^{1/p}\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}\right)^{-1}.$$

This gives the desired lower bound for $\nu_{\mathcal{F}_\kappa(\Lambda)}$. □

Proposition 2.14. *Let $n \in \mathbb{N}$, $r > 1$, and $x \in \mathbb{C}^n$. Then*

$$\|x\|_r \le \|x\|_1^{1/r}\|x\|_\infty^{1-1/r}.$$

The equality is attained if and only if all non-zero entries of x have the same absolute value.

Proof. For any $r > 1$ and any $x \in \mathbb{C}^n$, we have

$$\|x\|_r = \left(\sum_{k=1}^n |x_k|^r\right)^{1/r} = \left(\sum_{k=1}^n |x_k||x_k|^{r-1}\right)^{1/r} \le \left(\sum_{k=1}^n |x_k|\right)^{1/r}\left(\max_{k\in\mathbb{N}_n} |x_k|^{r-1}\right)^{1/r} = \|x\|_1^{1/r}\|x\|_\infty^{1-1/r}.$$

□

Proposition 2.15. *Let Λ be a finite set with $n > 1$ elements, $1 < p < +\infty$, and $n^{-1} < \kappa < (n-1)^{-1}$. Then $\nu_{\mathcal{F}_\kappa(\Lambda)} > \gamma_{\mathcal{F}_\kappa(\Lambda)}$.*

Proof. Using results of propositions 2.12 and 2.13, it is enough to show that

$$\kappa^{-1}\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{p}{p-1}} + n - 1\right)^{1/p}\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}\right)^{-1} > (n-1+(\kappa^{-1}-(n-1))^p)^{1/p}.$$

Let us make a substitution $m = n-1$ and $\rho = \kappa^{-1}$. Then $m \in \mathbb{N}$ and $m < \rho < m+1$. Then the last inequality is equivalent to

$$\rho\left(\left(\frac{m}{\rho-1}\right)^{\frac{p}{p-1}} + m\right)^{1/p}\left(\left(\frac{m}{\rho-1}\right)^{\frac{1}{p-1}} - 1 + \rho\right)^{-1} > (m+(\rho-m)^p)^{1/p}.$$

After simplifications, we arrive at inequality

$$\frac{m\rho}{\rho-1} > (m+(\rho-m)^p)^{1/p}\left(m+\left(\frac{m}{\rho-1}\right)^{p^*}\right)^{1/p^*}.$$

To prove this inequality, we apply proposition 2.14 to the vector $x = (1,\dots,1,\rho-m)^T \in \mathbb{C}^{m+1}$ with $r = p$ and to the vector $x = (1,\dots,1,\frac{m}{\rho-1}) \in \mathbb{C}^{m+1}$ with $r = p^*$. Since $m < \rho < m+1$, then the components of these vectors are not all equal, so the inequalities are strict:

$$(m+(\rho-m)^p)^{1/p} < (m+(\rho-m))^{1/p}1^{1-1/p},$$

$$\left(m+\left(\frac{m}{\rho-1}\right)^{p^*}\right)^{1/p^*} < \left(m+\frac{m}{\rho-1}\right)^{1/p^*}\left(\frac{m}{\rho-1}\right)^{1-1/p^*}.$$

By multiplying these inequalities, we get the desired result. $\square$

Proposition 2.16. *Let Λ be a finite set with $n > 1$ elements, $1 < p < +\infty$, and $n^{-1} < \kappa < (n-1)^{-1}$. Then for any $\ell_\infty(\Lambda)$-morphism ψ that is a left inverse to $\xi_{\mathcal{F}_\kappa(\Lambda)}$, holds $\|\psi\| > \gamma_{\mathcal{F}_\kappa(\Lambda)}$.*

Proof. By proposition 2.4, there exists a family of vectors $\eta \in \ell_\infty(\Lambda)^{\mathcal{F}_\kappa(\Lambda)}$ such that $\psi = \psi_\eta$. From definition 2.9, we have $\|\psi_\eta\| \geq \nu_{\mathcal{F}_\kappa(\Lambda)}$. Now, from proposition 2.15, we get $\|\psi\| \geq \nu_{\mathcal{F}_\kappa(\Lambda)} > \gamma_{\mathcal{F}_\kappa(\Lambda)}$. $\square$

Proposition 2.17. *Let Λ be a finite set. Then the right $\ell_\infty(\Lambda)$-module $\ell_p(\Lambda)$ is metrically injective if and only if Λ has at most 1 element.*

Proof. Suppose that the right $\ell_\infty(\Lambda)$-module $\ell_p(\Lambda)$ is metrically injective and Λ has n elements. Assume that $n > 1$. Pick any real number $\kappa \in (n^{-1},(n-1)^{-1})$. By proposition 2.12, we have $\gamma_{\mathcal{F}_\kappa(\Lambda)}$ is finite, therefore $\xi_{\mathcal{F}_\kappa(\Lambda)}$ is $\gamma_{\mathcal{F}_\kappa(\Lambda)}$-topologically injective. From metric injectivity of $\ell_p(\Lambda)$, it follows that there exists an $\ell_\infty(\Lambda)$-morphism ψ which is a left inverse to $\xi_{\mathcal{F}_\kappa(\Lambda)}$, such that $\|\psi\| \leq \gamma_{\mathcal{F}_\kappa(\Lambda)}$. This contradicts proposition 2.17, therefore $n \leq 1$.

Now assume that Λ has at most 1 element. If Λ is empty, then $\ell_p(\Lambda) = \{0\}$. The zero module is always injective. If Λ has one element, then $\ell_p(\Lambda)$ is isometrically isomorphic to $\ell_\infty(\Lambda)^*$ as an $\ell_\infty(\Lambda)$-module. The dual of the unital algebra is always metrically injective. $\square$

3 Measure theory preliminaries

In this section, we set the stage for the main theorem. Although it's stated for Borel measures on locally compact spaces, we shall prove all propositions of this section for general measure spaces. A comprehensive study of general measure spaces can be found in [8].

Let Ω be a set. By measure, we mean a countably additive set function with values in $[0,+\infty]$, defined on a σ-algebra Σ of measurable subsets of a set Ω. A pair (Ω,μ) is called a measure space. A measurable set A is called an atom if $\mu(A) > 0$ and for every measurable subset $B \subset A$, either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. A measure μ is called purely atomic if every measurable set of positive measure has an atom. A measure μ is semi-finite if for any measurable set E of infinite measure, there exists a measurable subset of E with finite positive measure. A family $\mathcal{D}$ of measurable subsets of finite measure is called a decomposition of Ω if for any measurable set E holds $\mu(E) = \sum_{D\in\mathcal{D}} \mu(E \cap D)$ and a set F is measurable whenever $F \cap D$ is measurable for all

$D \in \mathcal{D}$. Finally, a measure μ is called decomposable if it is semi-finite and admits a decomposition of Ω. Most measures encountered in functional analysis are decomposable.

We shall define a few Banach spaces constructed from measure spaces. Let (Ω, μ) be a measure space. By $B(\Sigma)$, we denote the algebra of bounded measurable functions with the sup norm. For $1 \le p \le +\infty$, by $L_p(\Omega, \mu)$, we denote the Banach space of equivalence classes of p-integrable (or essentially bounded if $p = +\infty$) functions on Ω. Elements of $L_p(\Omega, \mu)$ are denoted by $[f]$.

Definition 3.1. *Let (Ω, μ) be a measure space, E be a measurable set of finite positive measure, and $f : \Omega \to \mathbb{C}$ be a measurable function. For any $r \in \mathbb{R}$, we define a linear map*

$$m_{E,r}(f) = \mu(E)^{\frac{1}{r}-1} \int_E f(\omega) d\mu(\omega).$$

Note that, $m_{E,r}(f) = \mu(E)^{1/r} m_{E,\infty}(f)$ and $m_{E,\infty}(f)$ is nothing more than a mean value of f over E.

Proposition 3.2. *Let E be a finite measure subset of a measure space (Ω, μ) and $r \neq 0$. Then for any measurable functions $f : \Omega \to \mathbb{C}$, $g : \Omega \to \mathbb{C}$, the following hold:*

(i) $m_{E,r}(\chi_E) = \mu(E)^{1/r}$;

(ii) $m_{E,r}(f) = m_{E,r}(f\chi_E)$;

(iii) If E is an atom, then $m_{E,\infty}(f)$ is a finite number;

(iv) If E is an atom, then $f = m_{E,\infty}(f)$ almost everywhere on E;

(v) If E is an atom, then $m_{E,\infty}(f) m_{E,\infty}(g) = m_{E,\infty}(f \cdot g)$;

(vi) If E is an atom, then $m_{E,r}(f) m_{E,s}(g) = m_{E,\frac{rs}{r+s}}(f \cdot g)$.

Proof. Paragraphs (i) and (ii) are obvious.

(iii) Without loss of generality, we assume that f is real-valued. For each $n \in \mathbb{N}$ consider measurable set $A_n = \{\omega \in E : f(\omega) \le n\}$. Clearly, $(A_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence and $E = \bigcup_{n\in\mathbb{N}} A_n$, so $\mu(E) = \sup\{\mu(A_n) : n \in \mathbb{N}\}$. On the other hand, for each $n \in \mathbb{N}$ the set A_n is a measurable subset of the atom E, so either $\mu(A_n) = 0$ or $\mu(A_n) = \mu(E)$. Therefore, $\mu(A_N) = \mu(E)$ for some $N \in \mathbb{N}$. In other words, $f \le N$ almost everywhere on E. Hence, $m_{E,\infty}(f) \le N < +\infty$. Similarly, one can show that $m_{E,\infty}(f) > -\infty$.

(iv) Without loss of generality, we assume that f is real-valued. Denote $k = m_{E,\infty}(f)$. By paragraph (iii) we know that k is finite. Consider set $A_+ = \{\omega \in E : f(\omega) > k\}$. Since A_+ is a measurable subset of the atom E of finite measure, either $\mu(A_+) = 0$ or $\mu(A_+) = \mu(E) > 0$. In the latter case, we get

$$\int_E f(\omega) d\mu(\omega) = \int_{A_+} f(\omega) d\mu(\omega) > k\mu(A_+) = k\mu(E) = \int_E f(\omega) d\mu(\omega).$$

Contradiction, so $\mu(A_+) = 0$. Similarly, one can show that the set $A_- = \{\omega \in E : f(\omega) < k\}$ also has measure zero. Thus, $f = k$ almost everywhere on E.

Paragraph (vi) immediately follows from (v), which in turn is an easy consequence of (iv). □

Proposition 3.3. *Let (Ω, μ) be a purely atomic measure space. Let $\mathcal{A}$ be a decomposition of Ω into atoms of finite measure. Then for any $1 \leq p < +\infty$, the linear maps*

$$I_p : L_p(\Omega, \mu) \to \ell_p(\mathcal{A}),\ [f] \mapsto \sum_{A \in \mathcal{A}} m_{A,p}(f) e_A, \quad J_p : \ell_p(\mathcal{A}) \to L_p(\Omega, \mu),\ x \mapsto \sum_{A \in \mathcal{A}} x_A m_{A,-p}(\chi_A)[\chi_A]$$

are isometric isomorphisms that are inverse to each other.

Proof. Clearly, I_p and J_p are linear operators. Since $\mathcal{A}$ is a decomposition of Ω into atoms, then $[f] = \sum_{A \in \mathcal{A}} m_{A,\infty}(f)[\chi_A]$ for any $[f] \in L_p(\Omega, \mu)$. Using proposition 3.2 for each $[f] \in L_p(\Omega, \mu)$ and $A \in \mathcal{A}$, we get

$$\int_A |f(\omega)|^p d\mu(\omega) = \int_A |m_{A,\infty}(f)|^p \, d\mu(\omega) = \mu(A) \, |m_{A,\infty}(f)|^p = |m_{A,p}(\chi_A) m_{A,\infty}(f)|^p = |m_{A,p}(f)|^p.$$

So for any $[f] \in L_p(\Omega, \mu)$, we have

$$\|I_p([f])\| = \left(\sum_{A \in \mathcal{A}} |m_{A,p}(f)|^p \right)^{1/p} = \left(\sum_{A \in \mathcal{A}} \int_A |f(\omega)|^p d\mu(\omega) \right)^{1/p} = \left(\int_\Omega |f(\omega)|^p d\mu(\omega) \right)^{1/p} = \|[f]\|,$$

hence I_p is isometric. Note that for any $x \in \ell_p(\mathcal{A})$ and $A \in \mathcal{A}$ holds $m_{A,p}(J_p(x)) = m_{A,p}(J_p(x)\chi_A) = m_{A,p}(x_A m_{A,-p}(\chi_A)[\chi_A]) = x_A$. Hence, for any $x \in \ell_p(\mathcal{A})$, we have

$$I_p(J_p(x)) = \sum_{A \in \mathcal{A}} m_{A,p}(J_p(x)) e_A = \sum_{A \in \mathcal{A}} x_A e_A = x.$$

In other words $I_p \circ J_p = 1_{\ell_p(\mathcal{A})}$. Note that $I_p \circ (1_{L_p(\Omega,\mu)} - J_p \circ I_p) = I_p - I_p \circ J_p \circ I_p = I_p - I_p = 0$. Since I_p is isometric and a fortiori injective, then $1_{L_p(\Omega,\mu)} - J_p \circ I_p = 0$, i.e. $J_p \circ I_p = 1_{L_p(\Omega,\mu)}$. Thus, J_p and I_p are inverse to each other. Then J_p is isometric as the inverse of the isometry I_p. $\square$

Proposition 3.4. *Let (Ω, μ) be a purely atomic measure space and $\mathcal{A}$ be a decomposition of Ω into atoms of finite measure. Suppose $1 \leq p, q < +\infty$, then*

(i) If $\Phi : L_p(\Omega, \mu) \to L_q(\Omega, \mu)$ is a $B(\Sigma)$-morphism, the map $I_q \circ \Phi \circ J_p$ is an $\ell_\infty(\mathcal{A})$-morphism of the same norm;

(ii) If $\phi : \ell_p(\mathcal{A}) \to \ell_q(\mathcal{A})$ is an $\ell_\infty(\mathcal{A})$-morphism, the map $J_q \circ \phi \circ I_p$ is a $B(\Sigma)$-morphism of the same norm;

Proof. (i) Denote $\phi = I_q \circ \Phi \circ J_p$, then by proposition 3.2 for any atom $A \in \mathcal{A}$ holds $\phi(e_A) = I_q(\Phi(J_p(e_A))) = I_q(\Phi(m_{A,-p}(\chi_A)[\chi_A])) = m_{A,-p}(\chi_A) I_q(\Phi([\chi_A]))$. Since A is an atom, then we get $\Phi([\chi_A]) = \Phi([\chi_A] \cdot \chi_A) = \Phi([\chi_A]) \cdot \chi_A = m_{A,\infty}(\Phi([\chi_A]))[\chi_A]$, so

$$\begin{aligned} \phi(e_A) &= m_{A,-p}(\chi_A) I_q(m_{A,\infty}(\Phi([\chi_A]))[\chi_A]) = m_{A,\infty}(\Phi([\chi_A])) m_{A,-p}(\chi_A) I_q([\chi_A]) \\ &= m_{A,\infty}(\Phi([\chi_A])) m_{A,-p}(\chi_A) m_{A,q}(\chi_A) e_A = \mu(A)^{1/q-1/p} m_{A,\infty}(\Phi([\chi_A])) e_A. \end{aligned}$$

Now for any $x \in \ell_p(\mathcal{A})$ and $a \in \ell_\infty(A)$, we have

$$\phi(x \cdot a) = \sum_{A \in \mathcal{A}} x_A a_A \phi(e_A) = \sum_{A \in \mathcal{A}} x_A a_A \mu(A)^{1/q-1/p} m_{A,\infty}(\Phi([\chi_A])) e_A = \sum_{A \in \mathcal{A}} (x_A \phi(e_A)) \cdot a = \phi(x) \cdot a.$$

Therefore, ϕ is an $\ell_\infty(\mathcal{A})$-morphism. By proposition 3.3, the maps I_q and J_p are isometric isomorphisms; hence, ϕ and Φ have the same norm.

(ii) Denote $\Phi = J_q \circ \phi \circ I_p$, then by proposition 3.2 for any atom $A \in \mathcal{A}$ holds $\Phi([\chi_A]) = J_q(\phi(I_p([\chi_A]))) = J_q(\phi(m_{A,p}(\chi_A)e_A)) = m_{A,p}(\chi_A)J_q(\phi(e_A))$. Moreover, $\phi(e_A) = \phi(e_A \cdot e_A) = \phi(e_A) \cdot e_A = \phi(e_A)_A e_A$, so

$$\begin{aligned}\Phi([\chi_A]) &= m_{A,p}(\chi_A)J_q(\phi(e_A)_A e_A) = \phi(e_A)_A m_{A,p}(\chi_A)J_q(e_A)\\ &= \phi(e_A)_A m_{A,p}(\chi_A)m_{A,-q}(\chi_A)[\chi_A] = \mu(A)^{1/p-1/q}\phi(e_A)_A[\chi_A].\end{aligned}$$

Now for any $[f] \in L_p(\Omega, \mu)$ and $a \in B(\Sigma)$, we have

$$\begin{aligned}\Phi([f]\cdot a) &= \Phi\left(\sum_{A\in\mathcal{A}} m_{A,\infty}(f\cdot a)[\chi_A]\right) = \sum_{A\in\mathcal{A}} m_{A,\infty}(f\cdot a)\Phi([\chi_A])\\ &= \sum_{A\in\mathcal{A}} m_{A,\infty}(f)m_{A,\infty}(a)\mu(A)^{1/p-1/q}\phi(e_A)_A[\chi_A] = \sum_{A\in\mathcal{A}}(m_{A,\infty}(f)\mu(A)^{1/p-1/q}\phi(e_A)_A[\chi_A])\cdot a\\ &= \sum_{A\in\mathcal{A}} m_{A,\infty}(f)\Phi([\chi_A])\cdot a = \Phi\left(\sum_{A\in\mathcal{A}} m_{A,\infty}(f)[\chi_A]\right)\cdot a = \Phi([f])\cdot a.\end{aligned}$$

Therefore, Φ is a $B(\Sigma)$-morphism. By proposition 3.3, the maps J_q and I_p are isometric isomorphisms; hence, Φ and ϕ have the same norm. □

Proposition 3.5. *Let (Ω, μ) be a decomposable measure space and $1 \leq p \leq +\infty$. If $L_p(\Omega, \mu)$ is finite-dimensional, then (Ω, μ) is purely atomic with finitely many atoms of finite measure.*

Proof. Suppose (Ω, μ) is not purely atomic, then there exists an atomless measurable set $E \subset \Omega$. By [[8], proposition 215D] we may assume that E has a positive finite measure. By [[8], exercise 215X(e)], there is a countable family $\mathcal{E}$ of disjoint sets of positive finite measure. In this case, $([\chi_E])_{E\in\mathcal{E}}$ is a countable linearly independent set in $L_p(\Omega, \mu)$, hence $L_p(\Omega, \mu)$ is infinite-dimensional. Contradiction, so (Ω, μ) is purely atomic. Let $\mathcal{A}$ be a family of atoms whose union is Ω. Since Ω is decomposable, then all these atoms have a finite measure. Therefore, $([\chi_A])_{A\in\mathcal{A}}$ is a linearly independent set. Since $L_p(\Omega, \mu)$ is finite-dimensional, then $\mathcal{A}$ is finite. □

4 Metric projectivity, injectivity and flatness of $C_0(S)$-modules $L_p(S, \mu)$

Let S be a locally compact Hausdorff space. By $\mathrm{Bor}(S)$ we denote the σ-algebra generated by open subsets of S. In this section, we shall consider only decomposable Borel measures. Let μ be such a measure on S. We shall show that for $1 < p < +\infty$, Banach $C_0(S)$-modules $L_p(S, \mu)$ are almost never metrically projective, injective, or flat.

Proposition 4.1. *Let S be a locally compact Hausdorff space. Then*

(i) Any reflexive $C_0(S)$-module (left or right) can be endowed with the structure of a $B(\mathrm{Bor}(S))$-module, and the outer action is consistent with multiplication by elements of the algebra $C_0(S)$;

(ii) Any $C_0(S)$-morphism between reflexive modules is a $B(\mathrm{Bor}(S))$-morphism;

Proof. (i) Suppose Z is a $C_0(S)$-module, then via Arens product Z^{**} is a $C_0(S)^{**}$-module [[9], proposition 2.6.15(iii)]. If Z is reflexive, then the natural embedding $\iota_Z : Z \to Z^{**}$ is an isometric isomorphism. Recall that $B(\mathrm{Bor}(S))$ is a subalgebra of $C_0(S)^{**}$ [[9], proposition 4.2.30], therefore we can endow Z with the structure of $B(\mathrm{Bor}(S))$-module via the formula $z \cdot b = \iota_Z^{-1}(\iota_Z(z) \cdot b)$ for $z \in Z$ and $b \in B(\mathrm{Bor}(S))$.

(ii) Let $\phi : X \to Y$ be a morphism of right reflexive $C_0(S)$-modules. Then ϕ^{**} is a $C_0(S)^{**}$-morphism [[9], proposition A.3.53]. As we have noted above, X and Y are $B(\mathrm{Bor}(S))$-modules, and ι_X, ι_Y are isometric isomorphisms. Since $\phi = \iota_Y^{-1} \circ \phi^{**} \circ \iota_X$, then ϕ is a $B(\mathrm{Bor}(S))$-morphism. □

Remark 4.2. *Let S be a locally compact Hausdorff space and μ be a finite Borel measure on S. Using proposition 4.1 for any $1 < p < +\infty$ we can endow the reflexive $C_0(S)$-module $L_p(S, \mu)$ with the structure of $B(\mathrm{Bor}(S))$-module. According to [[10], proposition 2.2] this new outer action coincides with the pointwise multiplication.*

Proposition 4.3. *Let S be a locally compact Hausdorff space and μ be a purely atomic Borel measure on S with finitely many atoms of finite measure. Suppose $1 < p < +\infty$ and the $C_0(S)$-module $L_p(S, \mu)$ is metrically injective, then μ has at most one atom.*

Proof. Let $\mathcal{A}$ be a decomposition of S into n atoms of finite measure. Suppose $n > 1$, then pick any $\kappa \in (n^{-1}, (n-1)^{-1})$. Now we set $\mathcal{F} = \mathcal{F}_\kappa(\mathcal{A})$. For each $f \in \mathcal{F}$, we define an $\ell_\infty(\mathcal{A})$-morphism $m_f : \ell_p(\mathcal{A}) \to \ell_1(\mathcal{A})$, $x \mapsto x \cdot f$. We shall also use a natural embedding $\mathrm{in}_f : L_1(S, \mu) \to \bigoplus_\infty \{L_1(S, \mu) : f' \in \mathcal{F}\}$, and a natural projection $\mathrm{pr}_f : \bigoplus_\infty \{\ell_1(\mathcal{A}) : f' \in \mathcal{F}\} \to \ell_1(\mathcal{A})$, which are $B(\mathrm{Bor}(S))$-morphisms and $\ell_\infty(\mathcal{A})$-morphisms respectively. Now consider $B(\mathrm{Bor}(S))$-morphisms $I_p^\infty = \bigoplus_\infty \{I_p : f \in \mathcal{F}\}$, and $J_p^\infty = \bigoplus_\infty \{J_p : f \in \mathcal{F}\}$. These are isometric isomorphisms, and one can easily check that $I_p^\infty \circ \mathrm{in}_f = \mathrm{in}_f \circ I_p$, and $\mathrm{pr}_f \circ I_p^\infty = I_p \circ \mathrm{pr}_f$. By proposition 3.4, the map $\Xi_f = J_1 \circ m_f \circ I_p$ is a $B(\mathrm{Bor}(S))$-morphism. Hence, the map $\Xi_\mathcal{F} = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ \Xi_f$ is a $B(\mathrm{Bor}(S))$-morphism and a fortiori a $C_0(S)$-morphism. Note that $I_1^\infty \circ \Xi_\mathcal{F} \circ J_p = \sum_{f \in \mathcal{F}} I_1^\infty \circ \mathrm{in}_f \circ \Xi_f \circ J_p = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ I_1 \circ J_1 \circ m_f \circ I_p \circ J_p = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ m_f = \xi_\mathcal{F}$. By proposition 2.12, the coercivity constant $\gamma_\mathcal{F}$ is finite and positive, so $\xi_\mathcal{F}$ is $\gamma_\mathcal{F}$-topologically injective. As I_1 and J_p are isometric isomorphisms, the map $\Xi_\mathcal{F}$ is also $\gamma_\mathcal{F}$-topologically injective. By assumption, the $C_0(S)$-module $L_p(S, \mu)$ is metrically injective. Hence, there exists a $C_0(S)$-morphism $\Psi : \bigoplus_\infty \{L_1(S, \mu) : f \in \mathcal{F}\} \to L_p(S, \mu)$ such that $\Psi \circ \Xi_\mathcal{F} = 1_{L_p(S,\mu)}$ and $\|\Psi\| \leq \gamma_\mathcal{F}$.

Again, for each $f \in \mathcal{F}$, we define a $C_0(S)$-morphism $\Psi_f = \Psi \circ \mathrm{in}_f$. Since $\mathcal{A}$ is finite, then $L_p(S, \mu)$ and $L_1(S, \mu)$ are finite-dimensional and therefore reflexive. Hence, by proposition 4.1 and remark 4.2, for any $f \in \mathcal{F}$, the map Ψ_f is a $B(\mathrm{Bor}(S))$-morphism. Note that $\Psi = \sum_{f \in \mathcal{F}} \Psi_f \circ \mathrm{pr}_f$. Now we define a bounded linear operator $\psi = I_p \circ \Psi \circ J_1^\infty = I_p \circ \left(\sum_{f \in \mathcal{F}} \Psi_f \circ \mathrm{pr}_f\right) \circ J_1^\infty = \sum_{f \in \mathcal{F}} I_p \circ \Psi_f \circ \mathrm{pr}_f \circ J_1^\infty = \sum_{f \in \mathcal{F}} I_p \circ \Psi_f \circ J_1 \circ \mathrm{pr}_f$. By proposition 3.4, for each $f \in \mathcal{F}$, the map $I_p \circ \Psi_f \circ J_1$ is an $\ell_\infty(\mathcal{A})$-morphism. Therefore, ψ is an $\ell_\infty(\mathcal{A})$-morphism too. As I_p and J_1^∞ are isometric isomorphisms, we have $\|\psi\| = \|\Psi\|$. Moreover, $\psi \circ \xi_\mathcal{F} = I_p \circ \Psi \circ J_1^\infty \circ I_1^\infty \circ \Xi_\mathcal{F} \circ J_p = I_p \circ \Psi \circ \Xi_\mathcal{F} \circ J_p = I_p \circ J_p = 1_{\ell_p(\mathcal{A})}$. Thus, we have constructed an $\ell_\infty(\mathcal{A})$-morphism ψ such that $\psi \circ \xi_\mathcal{F} = 1_{\ell_p(\mathcal{A})}$ and $\|\psi\| \leq \gamma_\mathcal{F}$. Since we assumed that $n > 1$, we arrive at a contradiction with proposition 2.16. Hence, $n \leq 1$, i.e., $\mathcal{A}$ has at most one atom. □

In the following proposition we shall mention the so called $\mathscr{L}_\infty^g$-spaces. Their definition is quite involved [[11], definition 3.13] and we shall not give it here. It suffices to say that these spaces are Banach space whose finite-dimensional subspaces are "very similar" to finite-dimensional ℓ_∞-spaces.

Proposition 4.4. *Let S be a locally compact Hausdorff space, and μ be a decomposable Borel measure on S. Suppose $1 < p < +\infty$, and the $C_0(S)$-module $L_p(S, \mu)$ is metrically injective, then μ is purely atomic with at most one atom.*

Proof. Let K be the Alexandroff compactification of S. Then $C_0(S)$ is complemented in $C(K)$. By [[11], lemma 4.4], the space $C(K)$ is an $\mathscr{L}_\infty^g$-space, and so is $C_0(S)$ as its complemented subspace [[11], corollary 23.1.2(1)]. Thus, $L_p(S, \mu)$ is a reflexive [[8], theorem 244K] metrically injective module over the algebra $C_0(S)$, which is an $\mathscr{L}_\infty^g$-space. By [[12], corollary 3.14], this module must be finite-dimensional. From proposition 3.5, we deduce that μ is purely atomic with finitely many atoms of finite measure. Finally, proposition 4.3 concludes that μ has at most one atom. □

Theorem 4.5. *Let S be a locally compact Hausdorff space, and μ be a decomposable Borel measure on S. Suppose $1 < p < +\infty$, and the $C_0(S)$-module $L_p(S, \mu)$ is metrically projective, injective, or flat. Then μ has at most one atom.*

Proof. If $L_p(S, \mu)$ is a metrically injective $C_0(S)$-module, the result follows from proposition 4.3.

Suppose $L_p(S, \mu)$ is a metrically flat $C_0(S)$-module. Then by [[12], proposition 2.21], the dual $C_0(S)$-module $L_p(S, \mu)^*$ is metrically injective. Note that $C_0(S)$-modules $L_p(S, \mu)^*$ and $L_{p^*}(S, \mu)$ are isometrically isomorphic, where $1 < p^* < +\infty$ for $1 < p < +\infty$. Thus, the result follows from the previous paragraph.

If $L_p(S, \mu)$ is a metrically projective $C_0(S)$-module, then by [[12], proposition 2.26], it is metrically flat. Hence, the result follows from the previous paragraph. □

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