The Dunford-Pettis property

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Abstract

This is a short note on the Dunford-Pettis property was written for self-educational purposes and future reference. A few additional subjects are discussed to give a firm introduction.

1 Weak topologies

Definition 1.1 *Let X be a Banach space, then the weakest topology making all functionals in X^* continuous is called the weak topology.*

If X is a Banach space, then the weak* topology on X^* is the weakest topology making all functionals $\iota_X(x) : X^* \to \mathbb{C} : f \mapsto f(x)$ continuous.

The sequences that converge in $\langle$ weak / weak* $\rangle$ topology to 0 we shall call $\langle$ weakly null / weakly* null $\rangle$. A subset A in $\langle$ X / X^* $\rangle$ is called $\langle$ weakly / weakly* $\rangle$ bounded if $\langle$ for all $f \in X^*$ the set $\{f(x) : x \in A\}$ / for all $x \in X$ the set $\{f(x) : f \in A\}$ $\rangle$ is bounded in $\mathbb{C}$.

We shall list the following well known properties of weak topologies. Let X be a Banach space.

(i) A subset A of X is weakly bounded iff it is norm bounded [[1], theorem 3.88].

(ii) If X is infinite dimensional, then every non empty weakly open subset of X is unbounded [[1], proposition 3.89].

(iii) If the weak topology of X is metriazable, then X is finite dimensional.

(iv) if A is a convex subset in X, then the norm and the weak closure of A coincide [[1], theorem 3.45].

Now we shall say a few words on weak* topologies. Let X be a Banach space.

(i) B_{X^*} is weak* compact [[1], theorem 3.37]

(ii) B_X is weak* dense in $B_{X^{**}}$ [[1], theorem 3.96]

We proceed to discussion of weak compactness in Banach spaces. A subset A of X is called $\langle$ weakly / relatively weakly $\rangle$ compact if $\langle$ A / the weak closure of A $\rangle$ is compact in the weak topology of X.

Again we list basic properties of weakly compact sets. Let X be a Banach space.

(i) Any weakly compact set in X is norm closed and norm bounded.

(ii) B_X is weakly compact iff X is reflexive [[1], theorem 3.111]

(iii) if X is reflexive, then any bounded set is relatively weakly compact

(iv) if X is reflexive, then for any bounded linear operator $T : X \to Y$ the set $T(B_X)$ is weakly compact.

Definition 1.2 *Let A be a subset of a topological space M, then A is called*

(i) $\langle$ sequentially / relatively sequentially $\rangle$ compact if every sequence in A has a subsequence convergent to a point $\langle$ in A / in M $\rangle$.

(ii) $\langle$ countably / relatively countably $\rangle$ compact if every sequence in A has a subnet convergent to a point $\langle$ in A / in M $\rangle$.

Countable compactness is implied by both compactness and sequential compactness. If M is metriazable, then all three concepts coincide, but the converse is not true. For example $B_{\ell_\infty^*}$ with weak* topology is compact by Banach-Alaoglu theorem, but not sequentially weak* compact, because the sequence of functionals $e_n^* : \ell_\infty \to \mathbb{C} : x \mapsto x(n)$ has no convergent subsequence. Though weak topology of Banach space is not metriazable in general its bounded subsets behave much like they are metriazable. They indeed metriazable if X is separable.

Theorem 1.3 (Eberlein-Smulian) *Let A be a subset of a Banach space X. The following are equivalent*

(i) A is $\langle$ weakly / relatively weakly $\rangle$ compact

(ii) A is $\langle$ weakly / relatively weakly $\rangle$ sequentially compact

(iii) A is $\langle$ weakly / relatively weakly $\rangle$ countably compact

$\triangleleft$ See [[2], theorem 1.6.3] $\triangleright$

Here are some examples of characterization of relatively weakly compact subsets:

Definition 1.4 *A subset of A of a Banach space $L_1(\Omega, \mu)$ is called equi-integrable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any measurable subset E of Ω with $\mu(E) < \delta$ and any $f \in A$ holds*

$$\int_E |f(\omega)| d\mu(\omega) < \varepsilon$$

Theorem 1.5 (Dunford, Pettis) *A bounded subset A of $L_1(\Omega, \mu)$ is relatively weakly compact iff it is equi-integrable.*

$\triangleleft$ See [[2], 5.2.9]. $\triangleright$

By $M(K)$ we denote the Banach space of complex Borel regular measures on a Hausdorff compact K.

Definition 1.6 *A subset A of a Banach space $M(K)$ is said to be uniformly regular if for any open subset U in K and any $\varepsilon > 0$, there is a compact set H in U such that $\sup_{\mu \in A} |\mu|(U \setminus H) < \varepsilon$.*

Theorem 1.7 (Grothendieck) *A bounded subset A of $M(K)$ is relatively weakly compact iff it is uniformly regular.*

$\triangleleft$ See [[2], 5.3.2] $\triangleright$

2 Classes of bounded linear operators

Note: by $\mathcal{B}(X,Y)$ we denote the Banach space of all bounded linear operators between Banach spaces X and Y.

Definition 2.1 *A bounded linear operator $T : X \to Y$ between Banach space X and Y is called*

(i) compact if $T(B_X)$ is relatively compact in Y

(ii) weakly compact if $T(B_X)$ is relatively weakly compact in Y

(iii) completely continuous if $T(W)$ is compact in Y for any weakly compact subset W of X

By $\langle$ $\mathcal{K}(X,Y)$ / $\mathcal{W}(X,Y)$ / $\mathcal{CC}(X,Y)$ $\rangle$ we denote the Banach space of $\langle$ compact / weakly compact / completely continuous $\rangle$ operators.

Proposition 2.2 *A bounded linear operator is completely continuous iff it is weak-to-norm sequentially continuous.*

$\triangleleft$ See [[2], 5.4.2]. $\triangleright$
Just for comparison we mention the following fact

Proposition 2.3 *Let X and Y be two Banach spaces. Then*

(i) a linear operator $T : X \to Y$ is bounded iff T is weak-to-weak continuous;

(ii) if $T : X \to Y$ is a bounded linear operator, then T^* is weak-to-weak* continuous;*

(iii) if $S : Y^* \to X^*$ is weak-to-weak* continuous then $S = T^*$ for some bounded linear operator $T : X \to Y$.*

(iv) if $T : X \to Y$ is weak-to-norm continuous, then it is a finite rank operator

$\triangleleft$ (i) See [[3], theorem 5.3.15].
(ii) Straightforward.
(iii) See [[1], exercise 3.60]
(iv) See [[1], exercise 15.3] $\triangleright$

Proposition 2.4 *For any Banach spaces X and Y we have isometric inclusions*

(i) $\mathcal{K}(X,Y) \subset \mathcal{W}(X,Y) \subset \mathcal{B}(X,Y)$

(ii) $\mathcal{K}(X,Y) \subset \mathcal{CC}(X,Y) \subset \mathcal{B}(X,Y)$

$\triangleleft$ The only non-trivial inclusion here is $\mathcal{K}(X,Y) \hookrightarrow \mathcal{CC}(X,Y)$. See [[1], exercise 1.77]. $\triangleright$

Theorem 2.5 (Schauder) *A bounded linear operator between Banach spaces is compact iff its adjoint is compact.*

$\triangleleft$ See theorem 15.3 in [1] $\triangleright$

Theorem 2.6 (Davis, Figel, Johnson, Pelczynski) *A bounded linear operator $T : X \to Y$ is weakly compact iff there exists a reflexive Banach space Z and bounded linear operators $R : X \to Z$, $Q : Z \to Y$ such that $T = QR$, that is T factors through a reflexive Banach space.*

$\lhd$ Necessity is proved in [[1], theorem 13.33]. The converse is obvious. Indeed, since X is reflexive, then Q is weakly compact and so does $T = QR$. $\rhd$

Theorem 2.7 (Gantmacher) *Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y. Then the following are equivalent*

(i) T *is weakly compact;*

(ii) $T^{**}(X^{**}) \subset Y$;

(iii) T^* *is weak***-to-weak continuous;*

(iv) T^* *is weakly compact.*

$\lhd$ $(i) \implies (ii)$ It is enough to check that all elements of $T^{**}(X^{**})$ are weak*-continuous.
$(ii) \implies (iii)$ Straightforward.
$(iii) \implies (iv)$ By Davis-Jhonson-Figel-Pelczynski theorem T factors through a reflexive space. Clearly, so does T^*.
$\rhd$

3 The Dunford-Pettis property

Definition 3.1 *We say that a Banach space X has the Dunford-Pettis property if $\mathcal{W}(X,Y) \subset \mathcal{CC}(X,Y)$ for any Banach space Y.*

Directly from definition it follows that the square of any weakly compact operator on a Banach space with Dunford-Pettis property is necessarily compact. If X is reflexive Banach space, then 1_X is weakly compact. Therefore X has the Dunford-Pettis property iff 1_X is compact. The latter is possible only for finite dimensional X. In other words, for infinite dimensional reflexive Banach space X the operator 1_X provides an example of weakly compact but not completely continuous operator.

Theorem 3.2 *A Banach space X has the Dunford-Pettis property iff for any weakly null sequences $(x_n)_{n\in\mathbb{N}} \subset X$, $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ holds $\lim_n x_n^*(x_n) = 0$.*

$\lhd$ Let Y be a Banach space and $T : X \to Y$ a weakly compact operator. Let us suppose that T is not Dunford-Pettis. Then there is weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for some $\delta > 0$ holds $\|T(x_n)\| \geq \delta$ for all $n \in \mathbb{N}$. Pick $(y_n^*)_{n\in\mathbb{N}} \subset Y^*$ such that $y^*(T(x_n)) = \|T(x_n)\|$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. By Gantmacher's theorem T^* is weakly compact hence $T^*(B_{Y^*})$ is a relatively weakly compact subset of X^*. By the Eberlein-Smulian theorem the sequence $T^*(y_n^*) \subset T^*(B_{Y^*})$ can be assumed weakly convergent to some $x^* \in X^*$. Then $(T^*(y_n^*) - x^*)_{n\in\mathbb{N}}$ is weakly null. Therefore, the Dunford-Pettis property of X gives that

$$(T^*(y_n^*) - x^*)(x_n) \to_n 0$$

when $n \to \infty$. Note that $x^*(x_n) \to_n 0$ since $(x_n)_{n\in\mathbb{N}}$ is weakly null. Therefore $\|T(x_n)\| = T^*(y_n)(x_n) \to_n 0$. Contradiction.

For the converse, let $(x_n)_{n\in\mathbb{N}} \subset X$ and $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ be weakly null sequences. Consider the operator

$$T : X \to c_0 : x \mapsto (x_n^*(x))_{n\in\mathbb{N}}$$

The adjoint operator T^* of T satisfies $T^*(e_k) = x_k^*$ for all $k \in \mathbb{N}$, where $(e_k)_{k\in\mathbb{N}}$ denotes the canonical basis of ℓ_1. This implies that $T^*(B_{\ell_1})$ is contained in the convex hull of the weakly null sequence $(x_n^*)_{n\in\mathbb{N}}$. Therefore T^* is weakly compact, hence by Gantmacher's theorem so is T. As T is weakly compact, T is also Dunford-Pettis by the hypothesis. Therefore $(T(x_n))_{n\in\mathbb{N}}$ is norm null sequence. Since

$$|x_n^*(x_n)| \leq \max_k |x_k^*(x_n)| = \|T(x_n)\|$$

then $(x_n^*(x_n))_{n\in\mathbb{N}}$ is norm null sequence too. $\triangleright$

Any Banach space X with the Schur property (that is any weakly null sequence is norm null sequence) has the Dunford-Pettis property. Indeed, let $(x_n)_{n\in\mathbb{N}} \subset X$ and $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ be weakly null sequences. Since $(x_n^*)_{n\in\mathbb{N}}$ is weakly null it is norm bounded by some constant C. The Schur property of X gives that $(x_n)_{n\in\mathbb{N}}$ is norm null, so $|x_n^*(x_n)| \leq C\|x_n\| \to_n 0$. In particular ℓ_1 has the Dunford-Pettis property.

Example 3.3 *Consider operator $T : L_1([0,1]) \to C([0,1]) : x \mapsto \left(t \mapsto \int_0^t x(s)ds\right)$. One can show that for the sequence $x_n(t) = 2n(\chi_{[1/2-1/n,1/2]} - \chi_{[1/2,1/2+1/n]})$ the sequence $T(x_n)$ has no weakly convergent subsequence. So T is not weakly compact. Using Arzela-Ascoli criterion of compactness in $C([0,1])$ and Dunford-Pettis criterion of weak compactness in $L_1([0,1])$ it is routine to check that T is completely continuous.*

Proposition 3.4 *If X^* has the Dunford-Pettis property, then so does X.*

$\triangleleft$ Let $(x_n)_{n\in\mathbb{N}} \subset X$ and $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ be weakly null sequences. Take arbitrary $x^{***} \in X^{***}$, then since $(x_n)_{n\in\mathbb{N}}$ is weakly null, then $x^{***}(\iota_X(x_n)) = \iota_X^*(x^{***})(x_n) \to_n 0$. Therefore $(\iota_X(x_n))_{n\in\mathbb{N}} \subset X^{**}$ is weakly null sequence. Since X^* has the Dunford-Pettis property, then $x_n^*(x_n) = \iota_X(x_n)(x_n^*) \to_n 0$. Therefore X has the Dunford-Pettis property. $\triangleright$

Proposition 3.5 *The Dunford-Pettis property is inherited by complemented subspaces of Banach spaces.*

$\triangleleft$ Let Y be a complemented subspace of X with bounded linear projection $P : X \to Y$. Let $(y_n)_{n\in\mathbb{N}} \subset Y$ and $(y_n^*)_{n\in\mathbb{N}} \subset Y^*$ be weakly null sequences. Then $(y_n)_{n\in\mathbb{N}}$ is a weakly null in X. Since P^* is bounded, then it is weak-to-weak continuous, so from assumption $(P^*(y_n))_{n\in\mathbb{N}}$ is weakly null in X^*. Since X has the Dunford-Pettis property, then

$$y_n^*(y_n) = y_n^*(P(y_n)) = P^*(y_n^*)(y_n) \to_n 0$$

Therefore Y has the Dunford-Pettis property. $\triangleright$

Proving a certain Banach space has the Dunford-Pettis property is always a challenge. Most proofs require understanding of weakly null sequences of dual spaces. It is known since the times of Grothendieck that L_1 spaces and $C(K)$-spaces have the Dunford-Pettis property.

References

[1] *Fabian M., Habala P.* Banach space theory. Springer, 2011.

[2] *Albiac F., Kalton N. J.* Topics in Banach space theory. Springer, 2006, vol 233.

[3] *Dunford N. Schwartz J.T.* Linear operators. Wiley-interscience New York, 1971, vol 1.