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Topologically projective, injective and flat modules of harmonic analysis

N. T. Nemesh

We study homologically trivial modules of harmonic analysis on a locally compact group G. For $L_1(G)$- and $M(G)$-modules $C_0(G)$, $L_p(G)$ and $M(G)$ we give criteria of metric and topological projectivity, injectivity and flatness. In most cases, modules with these properties must be finite-dimensional.

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§ 1. Introduction

Banach homology has a long history dating back to the 1950s. One of the main questions of this discipline: whether a given Banach module is homologically trivial, i.e. projective, injective or flat? An example of a successful answer to this question is the work of Dales, Polyakov, Ramsden and Racher [1, 2, 3], where they gave criteria of homological triviality for classical modules of harmonic analysis. It is worth mentioning that all these studies were carried out for relative Banach homology. We answer the same questions but for two less explored versions of Banach homology — topological and metric ones. Metric Banach homology was introduced by Graven in [4], where he applied modern, at that moment, homological and Banach geometric techniques to modules of harmonic analysis. The notion of topological Banach homology appeared in the work of White [5]. Seemingly, the latter theory looks much less restrictive then the metric one, but as we shall see this is not the case.

§ 2. Preliminaries on Banach homology

In what follows, we give several versions of our statements in parallel, listing the respective options in angle brackets and separating them by slashes, like $\langle \ldots / \ldots \rangle$. For example, a real number x is called $\langle$ positive / non-negative $\rangle$ if $\langle\, x > 0 \,/\, x \geqslant 0 \,\rangle$.

All Banach spaces under consideration are over the field of complex numbers. Let E be a Banach space. By B_E we denote the closed unit ball of E. If F is another Banach space, then a bounded linear operator $T : E \to F$ is called $\langle$ *isometric* / *c-topologically injective* $\rangle$ if $\langle\, \|T(x)\| = \|x\| \,/\, c\|T(x)\| \geqslant \|x\| \,\rangle$ for all

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$x \in E$. Similarly, T is $\langle$ *strictly coisometric* / *strictly c-topologically surjective* $\rangle$ if $\langle$ $T(B_E) = B_F$ / $cT(B_E) \supset B_F$ $\rangle$. In some cases, the constant c is omitted. We use the symbol $\bigoplus_p$ for an ℓ_p-sum of Banach spaces, and $\widehat{\otimes}$ for a projective tensor product of Banach spaces.

By A we denote an arbitrary Banach algebra. The symbol A_+ stands for the standard unitization of A. In what follows we shall consider Banach modules with contractive outer action only. A Banach module A-module X is called $\langle$ essential / faithful $\rangle$ if $\langle$ the linear span of $A \cdot X$ is dense in X / $a \cdot X = \{0\}$ implies $a = 0$ $\rangle$. A bounded linear operator which is also a morphism of A-modules is called an A-morphism. The symbol $A-\mathbf{mod}$ stands for the category of left Banach A-modules with A-morphisms. By $A-\mathbf{mod}_1$ we denote the subcategory of $A-\mathbf{mod}$ with the same objects, but contractive A-morphisms only. The similar categories of right modules are denoted by $\mathbf{mod}-A$ and $\mathbf{mod}_1-A$, respectively. We use the symbol $\cong$ to denote an isomorphism of two objects in a category. By $\widehat{\otimes}_A$ we denote the functor of projective module tensor product and by Hom, the usual morphism functor. Now we can give our main definitions.

DEFINITION 1. A left Banach A-module P is $\langle$ *metrically* / *C-topologically* / *C-relatively* $\rangle$ *projective* if the morphism functor

$$\langle \ \mathrm{Hom}_{A-\mathbf{mod}_1}(P,-) \ / \ \mathrm{Hom}_{A-\mathbf{mod}}(P,-) \ / \ \mathrm{Hom}_{A-\mathbf{mod}}(P,-) \ \rangle$$

maps all $\langle$ strictly coisometric morphisms / strictly c-topologically surjective morphisms / morphisms with right inverse operator of norm at most c $\rangle$ to $\langle$ strictly coisometric / strictly cC-topologically surjective / strictly cC-topologically surjective $\rangle$ operators.

DEFINITION 2. A right Banach A-module J is $\langle$ *metrically* / *C-topologically* / *C-relatively* $\rangle$ *injective* if the morphism functor

$$\langle \ \mathrm{Hom}_{\mathbf{mod}_1-A}(-,J) \ / \ \mathrm{Hom}_{\mathbf{mod}-A}(-,J) \ / \ \mathrm{Hom}_{\mathbf{mod}-A}(-,J) \ \rangle$$

maps all $\langle$ strictly isometric morphisms / c-topologically injective morphisms / morphisms with left inverse operator of norm at most c $\rangle$ to $\langle$ strictly coisometric / strictly cC-topologically surjective / strictly cC-topologically surjective $\rangle$ operators.

DEFINITION 3. A left Banach A-module F is $\langle$ *metrically* / *C-topologically* / *C-relatively* $\rangle$ *flat* if the functor of module tensor product $-\widehat{\otimes}_A F$ maps all $\langle$ isometric morphisms / c-topologically injective morphisms / morphisms with left inverse operator of norm at most c $\rangle$ to $\langle$ isometric / cC-topologically injective / cC-topologically injective $\rangle$ operators.

We shall say that a Banach module is $\langle$ topologically / relatively $\rangle$ projective, injective or flat if it is $\langle$ C-topologically / C-relatively $\rangle$ projective, injective or flat for some $C > 0$.

These definitions were given in a slightly different form by Graven for metric theory [4], by White for topological theory [5] and by Helemskii for relative theory [6]. For topologically projective, injective and flat module White used the term strictly projective, injective and flat respectively. It is worth mentioning that initially strictly injective and flat modules where introduced by Helemskii in [8; section VII.1]. An overview of the basics of these theories is given in [7]. We shall heavily rely upon results of the latter paper.

§ 3. Preliminaries on harmonic analysis

Let G be a locally compact group with unit e_G. The left Haar measure of G is denoted by m_G and the symbol Δ_G stands for the modular function of G. For $\langle$ an infinite and discrete / a compact $\rangle$ group G we choose m_G as a $\langle$ counting / probability $\rangle$ measure. In what follows for $1 \leqslant p \leqslant +\infty$ we use the notation $L_p(G)$ to denote the Lebesgue space of functions that are p-integrable with respect to Haar measure.

We regard $L_1(G)$ as a Banach algebra with convolution operator in the role of multiplication. This Banach algebra has a contractive two-sided approximate identity [9; theorem 3.3.23]. Clearly, $L_1(G)$ is unital if and only if G is discrete. In this case, δ_{e_G}, the indicator function of e_G, is the identity of $L_1(G)$. Similarly, the space of complex finite Borel regular measures $M(G)$ endowed with convolution becomes a unital Banach algebra. The role of identity is played by Dirac delta measure δ_{e_G} supported on e_G. Moreover, $M(G)$ is a coproduct, in the sense of category theory, in $L_1(G)-\mathbf{mod}_1$ (but not in $M(G)-\mathbf{mod}_1$) of the two-sided ideal $M_a(G)$ of measures absolutely continuous with respect to m_G and the subalgebra $M_s(G)$ of measures singular with respect to m_G. Note that $M_a(G) \cong L_1(G)$ in $M(G)-\mathbf{mod}_1$ and $M_s(G)$ is an annihilator $L_1(G)$-module. Finally, $M(G) = M_a(G)$ if and only if G is discrete.

Now we proceed to discussion of the standard left and right modules over algebras $L_1(G)$ and $M(G)$. The Banach algebra $L_1(G)$ can be regarded as a two-sided ideal of $M(G)$ by means of isometric left and right $M(G)$-morphism $i : L_1(G) \to M(G) : f \mapsto fm_G$. Therefore it is enough to define all module structures over $M(G)$. For any $1 \leqslant p < +\infty$, $f \in L_p(G)$ and $\mu \in M(G)$ we define

$$(\mu *_p f)(s) = \int_G f(t^{-1}s)d\mu(t), \qquad (f *_p \mu)(s) = \int_G f(st^{-1})\Delta_G(t^{-1})^{1/p}d\mu(t).$$

These module actions turn all Banach spaces $L_p(G)$ for $1 \leqslant p < +\infty$ into left and right $M(G)$-modules. Note that for $p = 1$ and $\mu \in M_a(G)$ we get the usual definition of convolution. For $1 < p \leqslant +\infty$, $f \in L_p(G)$ and $\mu \in M(G)$ we define module actions as

$$(\mu \cdot_p f)(s) = \int_G \Delta_G(t)^{1/p} f(st)d\mu(t), \qquad (f \cdot_p \mu)(s) = \int_G f(ts)d\mu(t).$$

These module actions turn all Banach spaces $L_p(G)$ for $1 < p \leqslant +\infty$ into left and right $M(G)$-modules too. This special choice of module structure nicely interacts with duality. Indeed we have and $(L_p(G), *_p)^* \cong (L_{p^*}(G), \cdot_{p^*})$ in $\mathbf{mod}_1 - M(G)$ for all $1 \leqslant p < +\infty$. Here we set by definition $p^* = p/(p-1)$ for $1 < p < +\infty$ and $p^* = \infty$ for $p = 1$. Finally, the Banach space $C_0(G)$ also becomes a left and a right $M(G)$-module when endowed with $\cdot_\infty$ in the role of module action. Even more, $C_0(G)$ is a closed left and right $M(G)$-submodule of $L_\infty(G)$ such that $(C_0(G), \cdot_\infty)^* \cong (M(G), *)$ in $M(G) - \mathbf{mod}_1$.

By $\widehat{G}$ we shall denote the dual group of the group G. Any character $\gamma \in \widehat{G}$ gives rise to a continuous characters

$$\varkappa_\gamma^L : L_1(G) \to \mathbb{C} : f \mapsto \int_G f(s)\overline{\gamma(s)}dm_G(s), \quad \varkappa_\gamma^M : M(G) \to \mathbb{C} : \mu \mapsto \int_G \overline{\gamma(s)}d\mu(s).$$

on $L_1(G)$ and $M(G)$ respectively. By $\mathbb{C}_\gamma$ we denote left and right augmentation $L_1(G)$- or $M(G)$-module. Its module actions are defined by

$$f \cdot_\gamma z = z \cdot_\gamma f = \varkappa_\gamma^L(f) z, \qquad \mu \cdot_\gamma z = z \cdot_\gamma \mu = \varkappa_\gamma^M(\mu) z$$

for all $f \in L_1(G)$, $\mu \in M(G)$ and $z \in \mathbb{C}$.

One of the numerous definitions of amenable group says, that a locally compact group G is amenable if there exists an $L_1(G)$-morphism of right modules $M : L_\infty(G) \to \mathbb{C}_{e_{\widehat{G}}}$ such that $M(\chi_G) = 1$ [8; section 7.2.5]. We can even assume that M is contractive [8; remark 7.1.54].

Most of the results of this section that are not supported with references are presented in full detail in [9; section 3.3].

§ 4. $L_1(G)$-modules

Metric homological properties of $L_1(G)$-modules of harmonic analysis were first studied in [4]. We generalise these ideas for the case of topological Banach homology. To clarify the definitions we start from a general result on injectivity. It is instructive to prove it from the first principles.

Proposition 4.1. *Let A be a Banach algebra with a right contractive approximate identity, then the right A-module A^* is metrically injective.*

Proof. Let $\xi : Y \to X$ be an isometric A-morphism of right A-modules X and Y and an arbitrary contractive A-morphism $\phi : X \to A^*$. By assumption A has a contractive approximate identity, say $(e_\nu)_{\nu \in N}$. For each $\nu \in N$ we define a bounded linear functional $f_\nu : Y \to \mathbb{C} : y \to \phi(y)(e_\nu)$. By Hahn-Banach theorem there exists a bounded linear functional $g_\nu : X \to \mathbb{C}$ such that $g_\nu \xi = f_\nu$ and $\|g_\nu\| = \|f_\nu\|$. It is routine to check that $\psi_\nu : X \to A^* : x \mapsto (a \mapsto g_\nu(x \cdot a))$ is an A-morphism of right modules such that $\|\psi_\nu\| \leqslant \|\phi\|$ and $\psi_\nu(\xi(x))(a) = \phi(x)(a e_\nu)$ for all $x \in X$ and $a \in A$. Since the net $(\psi_\nu)_{\nu \in N}$ is norm bounded then there exists a subnet $(\psi_\mu)_{\mu \in M}$ with the same norm bound that converges in strong-to-weak* topology to some operator $\psi : X \to A^*$. Clearly, ψ is a morphism of right A-modules such that $\psi \xi = \phi$ and $\|\psi\| \leqslant \|\phi\|$. As ϕ is arbitrary, the map $\mathrm{Hom}_{\mathbf{mod}_1 - A}(\xi, A^*)$ is strictly coisometric. Hence A^* is metrically injective.

Proposition 4.2. *Let G be a locally compact group. Then $L_\infty(G)$ is a metrically and topologically injective $L_1(G)$-module. As the result, $L_1(G)$-module $L_1(G)$ is metrically and topologically flat.*

Proof. As $L_1(G)$ has a contractive approximate identity, then by proposition 4.1 the right $L_1(G)$-module $L_1(G)^*$ is metrically injective. As a consequence it is topologically injective [7; proposition 2.14]. Therefore it remains to recall that $L_\infty(G) \cong L_1(G)^*$ in $\mathbf{mod}_1 - L_1(G)$. The result on flatness of $L_1(G)$ follows from [7; proposition 2.21].

Proposition 4.3. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent:*

(i) G is compact;
(ii) $\mathbb{C}_\gamma$ is a metrically projective $L_1(G)$-module;
(iii) $\mathbb{C}_\gamma$ is a topologically projective $L_1(G)$-module.

PROOF. (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (i) The proof is similar to [4; theorem 4.2].
(ii) $\Longrightarrow$ (iii) Implication follows from [7; proposition 2.4].

PROPOSITION 4.4. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent:*

(i) G is amenable;
(ii) $\mathbb{C}_\gamma$ is a metrically injective $L_1(G)$-module;
(iii) $\mathbb{C}_\gamma$ is a topologically injective $L_1(G)$-module;
(iv) $\mathbb{C}_\gamma$ is a metrically flat $L_1(G)$-module;
(v) $\mathbb{C}_\gamma$ is a topologically flat $L_1(G)$-module.

PROOF. (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (i) The proof is similar to [4; theorem 4.5].
(ii) $\Longrightarrow$ (iii) This implication immediately follows from [7; proposition 2.14].
(ii) $\Longrightarrow$ (iv), (iii) $\Longrightarrow$ (v) Note that $\mathbb{C}_\gamma^* \cong \mathbb{C}_\gamma$ in $\mathbf{mod}_1 - L_1(G)$, so all equivalences follow from three previous paragraphs and the fact that flat modules are precisely the modules with injective dual [7; proposition 2.21].

In the following proposition we shall study specific ideals of the Banach algebra $L_1(G)$, namely the ideals of the form $L_1(G) * \mu$ for some idempotent measure μ. In fact, this class of ideals for the case of commutative compact groups coincides with those left ideals of $L_1(G)$ that admit a right bounded approximate identity.

THEOREM 4.1. *Let G be a locally compact group and $\mu \in M(G)$ be an idempotent measure, that is $\mu * \mu = \mu$. Assume that the left ideal $I = L_1(G) * \mu$ of the Banach algebra $L_1(G)$ is a topologically projective $L_1(G)$-module. Then $\mu = p m_G$, for some $p \in I$.*

PROOF. Let $\phi : I \to L_1(G)$ be an arbitrary morphism of left $L_1(G)$-modules. Consider $L_1(G)$-morphism $\phi' : L_1(G) \to L_1(G) : x \mapsto \phi(x * \mu)$. By Wendel's theorem [11; theorem 1], there exists a measure $\nu \in M(G)$ such that $\phi'(x) = x * \nu$ for all $x \in L_1(G)$. In particular, $\phi(x) = \phi(x * \mu) = \phi'(x) = x * \nu$ for all $x \in I$. It is clear now that $\psi : I \to I : x \mapsto \nu * x$ is a morphism of right I-modules satisfying $\phi(x)y = x\psi(y)$ for all $x, y \in I$. By paragraph (ii) of [10; lemma 2] the ideal I has a right identity, say $e \in I$. Then $x * \mu = x * \mu * e$ for all $x \in L_1(G)$. Two measures are equal if their convolutions with all functions of $L_1(G)$ coincide [9; corollary 3.3.24], so $\mu = \mu * e m_G$. Since $e \in I \subset L_1(G)$, then $\mu = \mu * e m_G \in M_a(G)$. Set $p = \mu * e \in I$, then $\mu = p m_G$.

We conjecture that a left ideal of the form $L_1(G) * \mu$ for an idempotent measure μ is a metrically projective $L_1(G)$-module if and only if $\mu = p m_G$ for $p \in I$ with $\|p\| = 1$. In [4; theorem 4.14], Graven gave a criterion of metric projectivity of $L_1(G)$-module $L_1(G)$. Now we can prove this fact as a mere corollary.

COROLLARY 4.1. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is discrete;
(ii) $L_1(G)$ is a metrically projective $L_1(G)$-module;
(iii) $L_1(G)$ is a topologically projective $L_1(G)$-module.

PROOF. (i) $\Longrightarrow$ (ii) If G is discrete, then $L_1(G)$ is unital with unit of norm 1. From [10; proposition 7] we conclude that $L_1(G)$ is metrically projective as $L_1(G)$-module.

(ii) $\Longrightarrow$ (iii) This implication is a direct corollary of [7; proposition 2.4].

(iii) $\Longrightarrow$ (i) Clearly, δ_{e_G} is an idempotent measure. Since $L_1(G) = L_1(G) * \delta_{e_G}$ is topologically projective, then by proposition 4.1 we have $\delta_{e_G} = f m_G$ for some $f \in L_1(G)$. This is possible only if G is discrete.

Note that $L_1(G)$-module $L_1(G)$ is relatively projective for any locally compact group G [8; exercise 7.1.17].

PROPOSITION 4.5. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is discrete;

(ii) $M(G)$ is a metrically projective $L_1(G)$-module;

(iii) $M(G)$ is a topologically projective $L_1(G)$-module;

(iv) $M(G)$ is a metrically flat $L_1(G)$-module.

PROOF. (i) $\Longrightarrow$ (ii) We have $M(G) \cong L_1(G)$ in $L_1(G) - \mathbf{mod}_1$ for discrete G, so the result follows from theorem 4.1.

(ii) $\Longrightarrow$ (iii) See [7; proposition 2.4].

(ii) $\Longrightarrow$ (iv) Implication follows from [7; proposition 2.26].

(iii) $\Longrightarrow$ (i) Recall that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G) - \mathbf{mod}_1$, so $M_s(G)$ is topologically projective as a retract of a topologically projective module [7; proposition 2.2]. Note that $M_s(G)$ is also an annihilator $L_1(G)$-module, therefore the algebra $L_1(G)$ has a right identity [7; proposition 3.3]. Recall that $L_1(G)$ also has a two-sided bounded approximate identity, so $L_1(G)$ is unital. The latter is equivalent to G being discrete.

(iv) $\Longrightarrow$ (i) Note that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G) - \mathbf{mod}_1$, so $M_s(G)$ is metrically flat as a retract of a metrically flat module [7; proposition 2.27]. Recall also that $M_s(G)$ is an annihilator module over a non-zero algebra $L_1(G)$, therefore $M_s(G)$ must be a zero module [7; proposition 3.6]. The latter is equivalent to G being discrete.

PROPOSITION 4.6. *Let G be a locally compact group. Then $M(G)$ is a topologically flat $L_1(G)$-module.*

PROOF. Since $M(G)$ is an L_1-space it is a fortiori an $\mathcal{L}_1^g$-space [12; paragraph 3.13, exercise 4.7(b)]. Since $M_s(G)$ is complemented in $M(G)$, then $M_s(G)$ is an $\mathcal{L}_1^g$-space too [12; corollary 23.2.1(2)]. Moreover, since $M_s(G)$ is an annihilator $L_1(G)$-module, hence it is a topologically flat $L_1(G)$-module [7; proposition 3.6]. The $L_1(G)$-module $L_1(G)$ is also topologically flat by proposition 4.2. Note that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G) - \mathbf{mod}_1$, so the $L_1(G)$-module $M(G)$ is topologically flat as a sum of topologically flat modules [7; proposition 2.27].

§ 5. $M(G)$-modules

We turn to study the standard $M(G)$-modules of harmonic analysis. As we shall see, most of the results can be derived from previous theorems and proposition on $L_1(G)$-modules.

PROPOSITION 5.1. *Let G be a locally compact group, and X be $\langle$ an essential / a faithful / an essential $\rangle$ $L_1(G)$-module. Then,*

(i) X is a metrically $\langle$ projective / injective / flat $\rangle$ $M(G)$-module if and only if it is a metrically $\langle$ projective / injective / flat $\rangle$ $L_1(G)$-module;

(ii) X is a topologically $\langle$ projective / injective / flat $\rangle$ $M(G)$-module if and only if it is a topologically $\langle$ projective / injective / flat $\rangle$ $L_1(G)$-module.

PROOF. Recall that $L_1(G)$ is a two-sided contractively complemented ideal of $M(G)$. Now (i) and (ii) follow from $\langle$ [7; proposition 2.6] / [7; proposition 2.16] / [7; proposition 2.24] $\rangle$.

It is worth mentioning here that $L_1(G)$-modules $C_0(G)$, $L_p(G)$ for $1 \leqslant p < \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are essential and $L_1(G)$-modules $C_0(G)$, $M(G)$, $L_p(G)$ for $1 \leqslant p \leqslant \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are faithful.

PROPOSITION 5.2. *Let G be a locally compact group. Then $M(G)$ is metrically and topologically projective $M(G)$-module. As the consequence it is metrically and topologically flat $M(G)$-module.*

PROOF. Since $M(G)$ is a unital algebra, then $\langle$ metric / topological $\rangle$ projectivity of $M(G)$ follows from [10; proposition 7], since one may regard $M(G)$ as a unital ideal of $M(G)$. It remains to recall that any $\langle$ metrically / topologically $\rangle$ projective module is $\langle$ metrically / topologically $\rangle$ flat [7; proposition 2.26].

§ 6. Banach geometric restrictions

In this section we shall show that many modules of harmonic analysis fail to be metrically or topologically projective, injective or flat for purely Banach geometric reasons. In metric theory for infinite dimensional $L_1(G)$-modules $L_p(G)$, $M(G)$ and $C_0(G)$ it was done in [4; theorems 4.12–4.14].

PROPOSITION 6.1. *Let G be an infinite locally compact group. Then*

(i) $L_1(G)$, $C_0(G)$, $M(G)$, $L_\infty(G)^*$ are not topologically injective Banach spaces;

(ii) $C_0(G)$, $L_\infty(G)$ are not complemented in any L_1-space.

PROOF. Since G is infinite all modules in question are infinite dimensional.

(i) If an infinite dimensional Banach space is topologically injective, then it contains a copy of $\ell_\infty(\mathbb{N})$ [13; corollary 1.1.4], and consequently a copy of $c_0(\mathbb{N})$. The Banach space $L_1(G)$ is weakly sequentially complete [14; corollary III.C.14], so by [15; corollary 5.2.11] it can't contain a copy of $c_0(\mathbb{N})$. Therefore, $L_1(G)$ is not a topologically injective Banach space. Assume, that $M(G)$ is topologically injective, then so is its complemented subspace $M_a(G)$, which is isometrically isomorphic to $L_1(G)$. By previous argument this is impossible, contradiction. By corollary 3 of [16] the Banach space $C_0(G)$ is not complemented in $L_\infty(G)$, hence it can't be topologically injective. Note that $L_1(G)$ is complemented in $L_\infty(G)^*$ which is isometrically isomorphic to $L_1(G)^{**}$ [12; proposition B10]. Therefore, if $L_\infty(G)^*$ is topologically injective as a Banach space, then so is its retract $L_1(G)$. By previous argument this is impossible, contradiction.

(ii) Suppose $C_0(G)$ is a retract of L_1-space, then $M(G)$, which is isometrically isomorphic to $C_0(G)^*$, is a retract of L_∞-space. Therefore $M(G)$ must be a topologically injective Banach space. This contradicts paragraph (i). Note that $\ell_\infty(\mathbb{N})$

embeds in $L_\infty(G)$, hence so does $c_0(\mathbb{N})$. If $L_\infty(G)$ is a retract of L_1-space, then there exists an L_1-space containing a copy of $c_0(\mathbb{N})$. This is impossible as already showed in paragraph (i).

From now on by A we denote either $L_1(G)$ or $M(G)$. Recall that $L_1(G)$ and $M(G)$ are both L_1-spaces.

PROPOSITION 6.2. *Let G be an infinite locally compact group. Then*

(i) $C_0(G)$, $L_\infty(G)$ are neither topologically nor metrically projective A-modules;

(ii) $L_1(G)$, $C_0(G)$, $M(G)$, $L_\infty(G)^*$ are neither topologically nor metrically injective A-modules;

(iii) $L_\infty(G)$, $C_0(G)$ are neither topologically nor metrically flat A-modules;

(iv) $L_p(G)$ for $1 < p < \infty$ are neither topologically nor metrically projective.

injective or flat A-modules.

PROOF. (i) Every metrically or topologically projective A-module is complemented in some L_1-space [7; proposition 3.8]. Now the result follows from proposition 6.1 paragraph (ii).

(ii) Every metrically or topologically injective A-module is topologically injective as a Banach space [7; proposition 3.8]. It remains to apply proposition 6.1 paragraph (i).

(iii) Note that $C_0(G)^* \cong M(G)$ in $\mathbf{mod}_1 - A$. Now the result follows from paragraph (i) and the fact that the adjoint module of a flat module is injective [7; proposition 2.21].

(iv) Since $L_p(G)$ is reflexive for $1 < p < \infty$ the result follows from [7; corollary 3.14].

Now we consider metric and topological homological properties of A-modules when G is finite.

PROPOSITION 6.3. *Let G be a non-trivial finite group and $1 \leqslant p \leqslant \infty$. Then the A-module $L_p(G)$ is metrically $\langle$ projective / injective $\rangle$ if and only if $\langle$ $p = 1$ / $p = \infty$ $\rangle$.*

PROOF. Assume, $L_p(G)$ is metrically $\langle$ projective / injective $\rangle$ as an A-module. As $L_p(G)$ is finite-dimensional, there exist isometric isomorphisms $\langle$ $L_p(G) \cong \ell_1(\mathbb{N}_n)$ / $L_p(G) \cong \ell_\infty(\mathbb{N}_n)$ $\rangle$ [7; proposition 3.8, paragraphs (i), (ii)], where $n = \mathrm{Card}(G) > 1$. Now we use the result of theorem 1 from [17] for Banach spaces over field $\mathbb{C}$: if for $2 \leqslant m \leqslant k$ and $1 \leqslant r, s \leqslant \infty$, there exists an isometric embedding from $\ell_r(\mathbb{N}_m)$ into $\ell_s(\mathbb{N}_k)$, then either $r = 2$, $s \in 2\mathbb{N}$ or $r = s$. Therefore $\langle$ $p = 1$ / $p = \infty$ $\rangle$. The converse easily follows from $\langle$ theorem 4.1 / proposition 4.2 $\rangle$.

PROPOSITION 6.4. *Let G be a finite group. Then*

(i) $C_0(G)$, $L_\infty(G)$ are metrically injective A-modules;

(ii) $C_0(G)$ and $L_p(G)$ for $1 < p \leqslant \infty$ are metrically projective A-modules if and only if G is trivial;

(iii) $M(G)$ and $L_p(G)$ for $1 \leqslant p < \infty$ are metrically injective A-modules if and only if G is trivial;

(iv) $C_0(G)$ and $L_p(G)$ for $1 < p \leqslant \infty$ are metrically flat A-modules if and only if G is trivial.

PROOF. (i) Since G is finite then $C_0(G) = L_\infty(G)$. The result follows from proposition 4.2.

(ii) If G is trivial, that is $G = \{e_G\}$, then $L_p(G) = C_0(G) = L_1(G)$ and the result follows from paragraph (i). If G is non trivial, then we recall that $C_0(G) = L_\infty(G)$ and use proposition 6.3.

(iii) If $G = \{e_G\}$, then $M(G) = L_p(G) = L_\infty(G)$ and the result follows from paragraph (i). If G is non-trivial, then we note that $M(G) = L_1(G)$ and use proposition 6.3.

(iv) From paragraph (iii) it follows that $L_p(G)$ for $1 \leqslant p < \infty$ is a metrically injective A-module if and only if G is trivial. Recall that a Banach module is flat if and only if its adjoint is injective [7; proposition 2.21]. Now the result for $L_p(G)$ follows from identifications $L_p(G)^* \cong L_{p^*}(G)$ in $\mathbf{mod}_1 - L_1(G)$ for $1 \leqslant p^* < \infty$. Similarly, using above characterisation of flat modules and isomorphisms $C_0(G)^* \cong M(G) \cong L_1(G)$ in $\mathbf{mod}_1 - L_1(G)$ we get a criterion of injectivity of $M(G)$.

It is worth mentioning here that if we consider all Banach spaces over the field of real numbers, then $L_\infty(G)$ and $L_1(G)$ will be metrically projective and injective respectively, for the group G consisting of two elements. The reason is that $L_\infty(\mathbb{Z}_2) \cong \mathbb{R}_{\gamma_0} \bigoplus_1 \mathbb{R}_{\gamma_1}$ in $L_1(\mathbb{Z}_2) - \mathbf{mod}_1$ and $L_1(\mathbb{Z}_2) \cong \mathbb{R}_{\gamma_0} \bigoplus_\infty \mathbb{R}_{\gamma_1}$ in $\mathbf{mod}_1 - L_1(\mathbb{Z}_2)$. Here, $\mathbb{Z}_2$ denotes the unique group of two elements and $\gamma_0, \gamma_1 \in \widehat{\mathbb{Z}_2}$ are defined by $\gamma_0(0) = \gamma_0(1) = \gamma_1(0) = -\gamma_1(1) = 1$.

PROPOSITION 6.5. *Let G be a finite group. Then the A-modules $C_0(G)$, $M(G)$, $L_p(G)$ for $1 \leqslant p \leqslant \infty$ are topologically projective, injective and flat.*

PROOF. For a finite group G we have $M(G) = L_1(G)$ and $C_0(G) = L_\infty(G)$, so modules $C_0(G)$ and $M(G)$ do not require special considerations. Since $M(G) = L_1(G)$, we can restrict our considerations to the case $A = L_1(G)$. The identity map $i : L_1(G) \to L_p(G) : f \mapsto f$ is a topological isomorphism of Banach spaces, because $L_1(G)$ and $L_p(G)$ for $1 \leqslant p < +\infty$ are of equal finite dimension. Since G is finite, it is unimodular. Therefore, the module actions in $(L_1(G), *)$ and $(L_p(G), *_p)$ coincide for $1 \leqslant p < +\infty$. Hence i is an isomorphism in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$. Similarly one can show that $(L_\infty(G), \cdot_\infty)$ and $(L_p(G), \cdot_p)$ for $1 < p \leqslant +\infty$ are isomorphic in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$. Finally, one can easily check that $(L_1(G), *)$ and $(L_\infty(G), \cdot_\infty)$ are isomorphic in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$ via the map $j : L_1(G) \to L_\infty(G) : f \mapsto (s \mapsto f(s^{-1}))$. Thus all the modules in question are pairwise isomorphic. It remains to recall that $L_1(G)$ is topologically projective and flat by theorem 4.1 and proposition 4.2, meanwhile $L_\infty(G)$ is topologically injective by proposition 4.2.

We summarise results on homological properties of modules of harmonic analysis into the table 1. Each cell of each table contains a condition under which the respective module has the respective property and references to the proofs. The arrow $\Longrightarrow$ indicates that only a necessary condition is known. We should mention that results for modules $L_p(G)$, where $1 < p < \infty$, are valid for both module actions $*_p$ and $\cdot_p$. Characterisations and proofs for homologically trivial modules $\mathbb{C}_\gamma$ in the case of relative theory are the same as in propositions 4.3 and 4.4, but this results are already well known. For example, projectivity of $\mathbb{C}_\gamma$ is characterized in [8; theorem IV.5.13], and the criterion of injectivity was given in [18; theorem 2.5]. For algebras $L_1(G)$ and $M(G)$ the notions of $\langle$ projectivity / injectivity / flatness $\rangle$ coincide for all three theories when one deals with modules ⟨ $M(G)$ and $\mathbb{C}_\gamma$ / $L_\infty(G)$, $C_0(G)$ and $\mathbb{C}_\gamma$ / $L_1(G)$ and $\mathbb{C}_\gamma$ ⟩. Finally, the $M(G)$-modules $M(G)$ also have the same characterization of flatness in metric topological and relative theory.

Таблица 1. Homologically trivial modules of harmonic analysis

	$L_1(G)$-modules			$M(G)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
			Metric theory			
$L_1(G)$	G is discrete 4.1	$G=\{e_G\}$ 6.2, 6.4	G is any 4.2	G is discrete 4.1, 5.1	$G=\{e_G\}$ 6.2, 6.4	G is any 4.2, 5.1
$L_p(G)$	$G=\{e_G\}$ 6.2, 6.3	$G=\{e_G\}$ 6.2, 6.3	$G=\{e_G\}$ 6.2, 6.4	$G=\{e_G\}$ 6.2, 6.3	$G=\{e_G\}$ 6.2, 6.3	$G=\{e_G\}$ 6.2, 6.4
$L_\infty(G)$	$G=\{e_G\}$ 6.2, 6.3	G is any 4.2	$G=\{e_G\}$ 6.2, 6.4	$G=\{e_G\}$ 6.2, 6.3	G is any 4.2, 5.1	$G=\{e_G\}$ 6.2, 6.4
$M(G)$	G is discrete 4.5	$G=\{e_G\}$ 6.2, 6.4	G is discrete 4.6	G is any 5.2	$G=\{e_G\}$ 6.2, 6.4	G is any 5.2
$C_0(G)$	$G=\{e_G\}$ 6.2, 6.4	G is finite 6.2, 6.4	$G=\{e_G\}$ 6.2, 6.4	$G=\{e_G\}$ 6.2, 6.4	G is finite 6.2, 6.4	$G=\{e_G\}$ 6.2, 6.4
$\mathbb{C}_\gamma$	G is compact 4.3	G is amenable 4.4	G is amenable 4.4	G is compact 4.3, 5.1	G is amenable 4.4, 5.1	G is amenable 4.4, 5.1
			Topological theory			
$L_1(G)$	G is discrete 4.1	G is finite 6.2, 6.5	G is any 4.2	G is discrete 4.1, 5.1	G is finite 6.2, 6.5	G is any 4.2, 5.1
$L_p(G)$	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5
$L_\infty(G)$	G is finite 6.2, 6.5	G is any 4.2	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is any 4.2, 5.1	G is finite 6.2, 6.5
$M(G)$	G is discrete 4.5	G is finite 6.2, 6.5	G is any 4.6	G is any 5.2	G is finite 6.2, 6.5	G is any 5.2
$C_0(G)$	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5	G is finite 6.2, 6.5
$\mathbb{C}_\gamma$	G is compact 4.3	G is amenable 4.4	G is amenable 4.4	G is compact 4.3, 5.1	G is amenable 4.4, 5.1	G is amenable 4.4, 5.1
			Relative theory			
$L_1(G)$	G is any [1], §6	G is amenable and discrete [1], §6	G is any [1], §6	G is any [2], §3.5	G is amenable and discrete [2], §3.5	G is any [2], §3.5
$L_p(G)$	G is compact [1], §6	G is amenable [3]	G is amenable [3]	G is compact [2], §3.5	G is amenable [2], §3.5, [3]	G is amenable [2], §3.5
$L_\infty(G)$	G is finite [1], §6	G is any [1], §6	G is amenable [1], §6	G is finite [2], §3.5	G is any [2], §3.5	G is amenable ($\Longrightarrow$)[2], §3.5
$M(G)$	G is discrete [1], §6	G is amenable [1], §6	G is any [2], §3.5	G is any [2], §3.5	G is amenable [2], §3.5	G is any [2], §3.5
$C_0(G)$	G is compact [1], §6	G is finite [1], §6	G is amenable [1], §6	G is compact [2], §3.5	G is finite [2], §3.5	G is amenable [2], §3.5
$\mathbb{C}_\gamma$	G is compact 4.3	G is amenable 4.4	G is amenable 4.4	G is compact 4.3, 5.1	G is amenable 4.4, 5.1	G is amenable 4.4, 5.1

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N. T. Nemesh (Norbert T. Nemesh)
Faculty of Mechanics and Mathematics,
Moscow State University
E-mail: nemeshnorbert@yandex.ru

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