Homological triviality of the category of modules L_p

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Abstract

We give complete characterisation of topologically injective (bounded below), topologically surjective (open mapping), isometric and coisometric (quotient mapping) multiplication operators between L_p spaces defined on different σ-finite measure spaces. We prove that all such operators invertible from the left or from the right. As the consequence we prove that all objects of the category of L_p spaces considered as left Banach modules over algebra of bounded measurable functions are metrically, extremelly and relatively projective, injective and flat.

1 Preliminaries

1.1 Measure theoretic facts

Let (Ω, Σ, μ) be a measure space with σ-additive real valued measure. We say that $\Omega' \in \Sigma$ is an atom if $\mu(\Omega') > 0$ and for every $E \in \Sigma$ such that $E \subset \Omega'$ either $\mu(E) = 0$ or $\mu(\Omega' \setminus E) = 0$. By $A(\Omega, \mu)$ we denote the set of atoms of (Ω, Σ, μ). Now we present several standard facts from measure theory.

Lemma 1.1.1 *If (Ω, Σ, μ) is a σ-finite measure space then all its atoms are of the finite measure.*

$\triangleleft$ Since (Ω, Σ, μ) is σ-finite we have representaion $\Omega = \bigcup_{n\in\mathbb{N}} F_n$ as disjoint union of measurable sets of finite measure. Assume we have $\Omega' \in A(\Omega, \mu)$ of infinite measure. For $n \in \mathbb{N}$ define $\widetilde{F}_n = F_n \cap \Omega' \in \Sigma$. Fix $n \in \mathbb{N}$, then $\widetilde{F}_n \subset \Omega'$ and either $\mu(\widetilde{F}_n) = 0$ or $\mu(\widetilde{F}_n) = +\infty$. Since $\mu(\widetilde{F}_n) \leq \mu(F_n) < +\infty$ we get $\mu(\widetilde{F}_n) = 0$. As $n \in \mathbb{N}$ is arbitrary we get $\mu(\Omega') = \mu(\bigcup_{n\in\mathbb{N}} \widetilde{F}_n) = \sum_{n\in\mathbb{N}} \mu(\widetilde{F}_n) = 0$. Contradiction, so $\mu(\Omega') < +\infty$. $\triangleright$

Lemma 1.1.2 *Let (Ω, Σ, μ) be a purely atomic measure space, then there exist pairwise disjoint family of atoms $\{\Omega_\lambda : \lambda \in \Lambda\}$ such that $\Omega = \bigcup_{\lambda\in\Lambda} \Omega_\lambda$. If (Ω, Σ, μ) is σ-finite, then the family $\{\Omega_\lambda : \lambda \in \Lambda\}$ is at most countable.*

$\triangleleft$ Let $\mathcal{F} = \{F \subset A(\Omega, \mu) : \Omega', \Omega'' \in F \implies \Omega' \cap \Omega'' = \varnothing\}$. For $F', F'' \in \mathcal{F}$ we take by definition $F' \leq F''$ if $F' \subset F''$. In this case $(\mathcal{F}, \leq)$ is a partially ordered set in which every totally ordered set have upper bound. By Zorn's lemma we have maximal element $\widetilde{F} = \{\widetilde{\Omega}_\lambda : \lambda \in \Lambda\}$. Define $\Omega_0 = \Omega \setminus (\bigcup_{\lambda\in\Lambda} \widetilde{\Omega}_\lambda)$. If $\mu(\Omega_0) > 0$ then since Ω is purely atomic there exist $\Omega_1 \in A(\Omega, \mu)$ such that $\Omega_1 \subset \Omega_0$. Consider $F = \widetilde{F} \cup \{\Omega_1\} \in \mathcal{F}$, then $\widetilde{F} \leq F$ and $\widetilde{F} \neq F$. This contradicts maximality of $\widetilde{F}$, hence $\mu(\Omega_0) = 0$. Now take any $\lambda_0 \in \Lambda$, then define

$$\Omega_\lambda = \begin{cases} \widetilde{\Omega}_{\lambda_0} \cup \Omega_0 & \text{if} \quad \lambda = \lambda_0 \\ \widetilde{\Omega}_\lambda & \text{if} \quad \lambda \neq \lambda_0 \end{cases}$$

Clearly $\widetilde{\Omega}_{\lambda_0} \cup \Omega_0$ is an atom disjoint from atoms $\widetilde{\Omega}_\lambda$ for $\lambda \neq \lambda_0$. Hence $\{\Omega_\lambda : \lambda \in \Lambda\}$ is the desired family.

If (Ω, Σ, μ) is σ-finite we have representation $\Omega = \bigcup_{n\in\mathbb{N}} E_n$ as disjoint union of measurable sets of finite measure. Define $\Omega_{\lambda,n} = \Omega_\lambda \cap E_n$, then for each $n \in \mathbb{N}$ the we have $E_n = \bigcup_{\lambda\in\Lambda} \Omega_{\lambda,n}$ and $\Omega_{\lambda',n} \cap \Omega_{\lambda'',n} = \varnothing$ for $\lambda' \neq \lambda''$. Since $\mu(E_n) < +\infty$, then the family $\{\lambda \in \Lambda : \mu(\Omega_{\lambda,n}) > k^{-1}\}$ is finite for every $k \in \mathbb{N}$. Thus the family $\Lambda_n = \{\lambda \in \Lambda : \mu(\Omega_{\lambda,n}) > 0\}$ is at most countable. Since for all $\lambda \in \Lambda$ we have a representaion $\Omega_\lambda = \bigcup_{n\in\mathbb{N}} \Omega_{\lambda,n}$ where $\mu(\Omega_\lambda) > 0$ and $\Omega_{\lambda,n} \cap \Omega_{\lambda,m} = \varnothing$, then $\mu(\Omega_{\lambda,n}) > 0$ for some $n \in \mathbb{N}$. In other words $\Lambda = \bigcup_{n\in\mathbb{N}} \Lambda_n$, so Λ is at most countable as union at most countable sets Λ_n. $\triangleright$

Theorem 1.1.3 ([1], 2.1) *Let (Ω, Σ, μ) be a σ-finite measure space, then there exist purely atomic measure $\mu_1 : \Sigma \to [0, +\infty]$ and non atomic measure $\mu_2 : \Sigma \to [0, +\infty]$ such that $\mu = \mu_1 + \mu_2$ and $\mu_1 \perp \mu_2$.*

Theorem 1.1.4 ([2]) *Let (Ω, Σ, μ) be nonatomic measure space. If $E \in \Sigma$ with $\mu(E) > 0$, then for all $t \in [0, \mu(E)]$ there exist $F \in \Sigma$ such that $F \subset E$ and $\mu(F) = t$*

Theorem 1.1.5 ([3], 2.1) *Let (Ω, Σ, μ), (Ω, Σ, ν) be σ-finite measure spaces, then there exist a measurable function $\rho_{\nu,\mu}$ a σ-finite measure $\nu_s : \Sigma \to [0, +\infty]$ and a set $\Omega_s \in \Sigma$ such that*

(i) $\nu = \rho_{\nu,\mu} \cdot \mu + \nu_s$

(ii) $\mu \perp \nu_s$ *i.e.* $\mu(\Omega_s) = \nu_s(\Omega_c) = 0$, *where* $\Omega_c = \Omega \setminus \Omega_s$

1.2 Decompostion of L_p spaces

All linear spaces in this article are considered over field $\mathbb{C}$. By $L_0(\Omega, \mu)$ we denote the linear space of measurable functions on Ω. If $p = \infty$ then we take by definition that $1/p = 0$. All equalities and inequalities about L_p functions are understood up to sets of measure zero.

Proposition 1.2.1 *Let (Ω, Σ, μ) be a measure space and $p \in [1, +\infty]$. Assume we have represetation $\Omega = \bigcup_{\lambda\in\Lambda} \Omega_\lambda$ where $\Omega_{\lambda'} \cap \Omega_{\lambda''} = \varnothing$ for $\lambda' \neq \lambda''$. Then the map*

$$I_p : L_p(\Omega, \mu) \to \bigoplus_p \left\{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) : \lambda \in \Lambda\right\}, f \mapsto (\lambda \mapsto f|_{\Omega_\lambda})$$

is an isometric isomorphism.

$\triangleleft$ If $f \in L_p(\Omega, \mu)$, then, of course, $f|_{\Omega_\lambda} \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$ for $\lambda \in \Lambda$. So I_p is well defined and, obviously, it is linear. Now it is remains to prove that I_p is surjective and isometric. Let $f_\lambda \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$ for $\lambda \in \Lambda$ then define $f(\omega) = f_\lambda(\omega)$ if $\omega \in \Omega_\lambda$. Clearly $I_p(f)_\lambda = f_\lambda$ so I_p is surjective. Then for $p \geq 1$ we have

$$\|I_p(f)\|_{\bigoplus_p \left\{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) : \lambda\in\Lambda\right\}} = \left(\sum_{\lambda\in\Lambda} \int_{\Omega_\lambda} |f|_{\Omega_\lambda}(\omega)|^p d\mu|_{\Omega_\lambda}(\omega)\right)^{1/p}$$

$$= \left(\int_\Omega |f(\omega)|^p d\mu(\omega)\right)^{1/p} = \|f\|_{L_p(\Omega,\mu)}$$

Similarly for $p = \infty$ we have

$$\|I_\infty(f)\|_{\bigoplus_p \left\{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) : \lambda\in\Lambda\right\}} = \sup_{\lambda\in\Lambda} \operatorname*{essup}_{\omega\in\Omega_\lambda} |f|_{\Omega_\lambda}(\omega)| = \operatorname*{essup}_{\omega\in\Omega} |f(\omega)| = \|f\|_{L_\infty(\Omega,\mu)}$$

In both cases I_p is isometric. $\triangleright$

Lemma 1.2.2 *Let (Ω, Σ, μ) be a σ-finite measure space with atom Ω'. Then*

(i) If $p \in [1, +\infty]$ and $f \in L_p(\Omega', \mu|_{\Omega'})$, then

$$f(\omega) = \mu(\Omega')^{-1} \int_{\Omega'} f(\omega') d\mu(\omega')$$

for $\omega \in \Omega'$.

(ii) If $p \in [1, +\infty]$ the map

$$J_p : L_p(\Omega', \mu|_{\Omega'}) \to \ell_p(\{1\}), f \mapsto \left(1 \mapsto \mu(\Omega')^{1/p-1} \int_{\Omega'} f(\omega') d\mu(\omega')\right)$$

is an isometric isomorphism.

$\triangleleft$ Since (Ω, Σ, μ) is σ-finite, by lemma 1.1.1 we have $\mu(\Omega') < +\infty$. (i) Since $\mu(\Omega') < +\infty$, then $f \in L_p(\Omega', \mu|_{\Omega'}) \subset L_1(\Omega', \mu|_{\Omega'})$. For the beginning assume that f is a real valued function. Denote $k = \mu(\Omega')^{-1} \int_{\Omega'} f(\omega') d\mu(\omega')$, then consider set $S_- = f^{-1}((k, +\infty])$. Since Ω' is an atom then $\mu(S_-) = \mu(\Omega')$ or $\mu(S_-) = 0$. In the first case we get

$$\int_{\Omega'} f(\omega')\mu(\omega') = \int_{S_-} f(\omega')\mu(\omega') < \int_{S_-} c\mu(\omega') = k\mu(S_-) = k\mu(\Omega') = \int_{\Omega'} f(\omega')\mu(\omega')$$

Contradiction, hence $\mu(S_-) = 0$. Similarly we get that $\mu(S_+) = 0$ for $S_+ = f^{-1}([-\infty, k))$. Hence $f(\omega) = k$ for μ-almost all $\omega \in \Omega'$. If f is complex valued we apply previous result to $\mathrm{Re}(f), \mathrm{Im}(f) \in L_1(\Omega', \mu|_{\Omega'})$ and get that

$$f(\omega) = \mathrm{Re}(f)(\omega) + i\,\mathrm{Im}(f)(\omega)$$

$$= \mu(\Omega')^{-1} \int_{\Omega'} \mathrm{Re}(f)(\omega') d\mu(\omega') + i\mu(\Omega')^{-1} \int_{\Omega'} \mathrm{Im}(f)(\omega') d\mu(\omega') = \mu(\Omega')^{-1} \int_{\Omega'} f(\omega') d\mu(\omega')$$

for μ-almost all $\omega \in \Omega$

(ii) Obviously J_p is linear. Take any $z \in \mathbb{C}$ and consider function $f = z\mu(\Omega_1)^{-1/p}\chi_{\Omega'}$, then $J_p(f)(1) = z$. Thus J_p is surjective. Now for $p \in [1, +\infty]$ and all $f \in L_p(\Omega', \mu|_{\Omega'})$ we have

$$\|J_p(f)\|_p = \left|\mu(\Omega')^{1/p-1} \int_{\Omega'} f(\omega') d\mu(\omega')\right| = \left|\mu(\Omega')^{1/p-1} k\mu(\Omega')\right| = \mu(\Omega')^{1/p}|k| = \|f\|_{L_p(\Omega', \mu|_{\Omega'})}$$

Thus, the map J_p is a surjective isometry, hence isometric isomorphism. $\triangleright$

Proposition 1.2.3 *Let (Ω, Σ, μ) be a σ-finite purely atomic measure space, then for $p \in [1, +\infty]$, the map*

$$\widetilde{I}_p : L_p(\Omega, \mu) \to \ell_p(\Lambda) : f \mapsto (\lambda \mapsto J_p(f|_{\Omega_\lambda})(1))$$

is an isometric isomorphism. Here $\{\Omega_\lambda : \lambda \in \Lambda\} \subset A(\Omega, \mu)$ is at most countable family of pairwise disjoint atoms such that $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$.

◁ By lemma 1.1.2 we have a family $\{\Omega_\lambda : \lambda \in \Lambda\} \subset A(\Omega, \mu)$ of pairwise disjoint atoms whose union is Ω. By proposition 1.2.1 we have an isometric isomorphism

$$L_p(\Omega, \mu) \cong_1 \bigoplus_p \{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) : \lambda \in \Lambda\}$$

via the map $I_p(f) = \oplus_p\{f|_{\Omega_\lambda} : \lambda \in \Lambda\}$. By lemma 1.2.2 we know that $L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) \cong_1 \ell_p(\{1\})$ via the map J_p. So we get

$$L_p(\Omega, \mu) \cong_1 \bigoplus_p \{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda}) : \lambda \in \Lambda\} \cong_1 \bigoplus_p \{\ell_p(\{1\}) : \lambda \in \Lambda\} = \ell_p(\Lambda)$$

via the map $\widetilde{I}_p$. Since (Ω, Σ, μ) is σ-finite by lemma 1.1.2 we get that Λ is at most countable. ▷

Proposition 1.2.4 *Let $p \in [1, +\infty]$. Let (Ω, Σ, μ) be σ-finite measure space, then there exist at most countable family of atoms $\{\Omega_\lambda : \lambda \in \Lambda\}$ and a set $\Omega_{na} = \Omega \setminus \left(\bigcup_{\lambda\in\Lambda} \Omega_\lambda\right) = \Omega \setminus \Omega_a$ such that the map*

$$\hat{I}_p : L_p(\Omega, \mu) \to L_p(\Omega_{na}, \mu|_{\Omega_{na}}) \bigoplus_p L_p(\Omega_a, \mu|_{\Omega_a}), f \mapsto (f|_{\Omega_{na}}, f|_{\Omega_a})$$

is isometric isomorphism and $\mu|_{\Omega_{na}}$ is nonatomic.

◁ By theorem 1.1.3 we have mutually singular purely atomic measure μ_1 and nonatomic measure μ_2 whose sum is μ. Since they are singular there exist $\Omega_a \in \Sigma$ such that $\mu_2(\Omega_a) = \mu_1(\Omega \setminus \Omega_a) = 0$. Thus $\mu|_{\Omega_a} = \mu_1$ is purely atomic and $\mu|_{\Omega_{na}} = \mu_2$ is nonatomic. Here $\Omega_{na} = \Omega \setminus \Omega_a$. By proposition 1.2.1

$$\hat{I}_p : L_p(\Omega, \mu) \to L_p(\Omega_{na}, \mu|_{\Omega_{na}}) \bigoplus_p L_p(\Omega_a, \mu|_{\Omega_a}), f \mapsto (f|_{\Omega_{na}}, f|_{\Omega_a})$$

is an isometric isomorphism. ▷

Proposition 1.2.5 *Let (Ω, Σ, μ) be a σ-finite measure space. Let $\rho \in L_0(\Omega, \mu)$ be a positive function. Then*

$$\bar{I}_p : L_p(\Omega, \mu) \to L_p(\Omega, \rho \cdot \mu), f \mapsto \rho^{-1/p} \cdot f$$

is an isometric isomorphism for all $p \in [1, +\infty]$.

◁ Obviously $\bar{I}_p$ is linear. Let $p \geq 1$, then for all $f \in L_p(\Omega, \mu)$ we have

$$\|\bar{I}_p(f)\|_{L_p(\Omega,\rho\cdot\mu)} = \left(\int_\Omega |\rho^{-1/p}(\omega) f(\omega)|^p \rho(\omega) d\mu(\omega)\right)^{1/p} = \left(\int_\Omega |f(\omega)|^p d\mu(\omega)\right)^{1/p} = \|f\|_{L_p(\Omega,\mu)}$$

so $\bar{I}_p$ is an isometry. Now for arbitrary $f \in L_p(\Omega, \rho \cdot \mu)$ consider $h = \rho^{1/p} \cdot f$, then

$$\|h\|_{L_p(\Omega,\mu)} = \left(\int_\Omega |\rho^{1/p}(\omega) f(\omega)|^p d\mu(\omega)\right)^{1/p} = \left(\int_\Omega |f(\omega)|^p \rho(\omega) d\mu(\omega)\right)^{1/p} = \|f\|_{L_p(\Omega,\rho\cdot\mu)}$$

So $h \in L_p(\Omega, \mu)$ and obviously $\bar{I}_p(h) = f$. Thus $\bar{I}_p$ is a surjective isometry, i.e. isometric isomorphism. Now let $p = +\infty$, then $\bar{I}_\infty$ is an identity map. Hence it is enough to show it is isometric. Let $E \in \Sigma$ with $\mu(E) = 0$, then $(\rho \cdot \mu)(E) = \int_E \rho(\omega) d\mu(\omega) = 0$. On the other hand if $(\rho \cdot \mu)(E) = \int_E \rho(\omega) d\mu(\omega) = 0$ then from positivity of ρ it follows that $\mu(E) = 0$. So for all $f \in L_\infty(\Omega, \mu)$ we have

$$\|\bar{I}_\infty(f)\|_{L_\infty(\Omega,\rho\cdot\mu)} = \inf\{C > 0 : (\rho \cdot \mu)(|f|^{-1}((C, +\infty))) = 0\}$$

$$= \inf\{C > 0 : \mu(|f|^{-1}((C, +\infty))) = 0\} = \|f\|_{L_\infty(\Omega,\mu)}$$

Thus, $\bar{I}_\infty$ is an isometry. $\triangleright$

Proposition 1.2.6 *Let Λ be a finite set and $p, q \in [1, +\infty]$, then there exist $C_{p,q}$ such that $\|x\|_{\ell_p(\Lambda)} \leq C_{p,q} \|x\|_{\ell_q(\Lambda)}$ for all $x \in \mathbb{C}^\Lambda$.*

$\triangleleft$ Since $(\mathbb{C}^\Lambda, \|\cdot\|_{\ell_r(\Lambda)})$ is a normed space of finite dimension equal to $\mathrm{Card}(\Lambda)$ for every $r \in [1, +\infty]$, then all norms $\left\{\|\cdot\|_{\ell_r(\Lambda)} : r \in [1, +\infty]\right\}$ on $\mathbb{C}^\Lambda$ are equivalent. So we get the desired inequality. $\triangleright$

1.3 Mutiplication operators

Let (Ω, Σ, μ) and (Ω, Σ, ν) be two measure spaces with the same σ-algebra of measurable sets. For a given $g \in L_0(\Omega, \mu)$ and $p, q \in [1, +\infty]$ we define the multiplication operator

$$M_g : L_p(\Omega, \mu) \to L_q(\Omega, \nu), f \mapsto g \cdot f$$

Of course certain restrictions on g, μ and ν are required for M_g to be well defined. For a given $E \in \Sigma$ by M_g^E we will denote the linear operator

$$M_g^E : L_p(E, \mu|_E) \to L_q(E, \nu|_E), f \mapsto g|_E \cdot f$$

It is well defined because $f|_{\Omega\setminus E} = 0$ implies $M_g(f)|_{\Omega\setminus E} = 0$.

Proposition 1.3.1 *Let (Ω, Σ, μ) be a measure space and $g \in L_0(\Omega, \mu)$. Denote $Z_g = g^{-1}(\{0\})$, then for the operator $M_g : L_p(\Omega, \mu) \to L_q(\Omega, \mu)$ we have*
(i) $\mathrm{Ker}(M_g) = \{f \in L_p(\Omega, \mu) : f|_{\Omega\setminus Z_g} = 0\}$, so M_g is injective if and only if $\mu(Z_g) = 0$
(ii) $\mathrm{Im}(M_g) \subset \{h \in L_q(\Omega, \mu) : h|_{Z_g} = 0\}$,so if M_g is surjective then $\mu(Z_g) = 0$.

$\triangleleft$ (i) We have the following chain of equivalences

$$f \in \mathrm{Ker}(M_g) \Longleftrightarrow g \cdot f = 0 \Longleftrightarrow f|_{\Omega\setminus Z_g} = 0$$

And we get the desired equality.
(ii) Since $g|_{Z_g} = 0$ then $M_g(f)|_{Z_g} = (g \cdot f)|_{Z_g} = 0$ for all $f \in L_p(\Omega, \mu)$, thus we get the inclusion. If M_g is surjective then, clearly, $\mu(Z_g) = 0$. $\triangleright$
We want to classify multiplication operators according to following definitions

Definition 1.3.2 *Let $T : E \to F$ be bounded linear operator between normed spaces E and F, then T is called*
(i) c-topologically injective, if there exist $c > 0$ such that for all $x \in E$ holds $\|x\|_E \leq c\|T(x)\|_F$.
(ii) (strictly) c-topologically surjective, if for all $c' > c$ and $y \in F$ there exist $x \in E$ such that $T(x) = y$ and $\|x\|_E < c'\|y\|_F$ ($\|x\|_E \leq c\|y\|_F$).
(iii) (strictly) coisometric, if it is contractive and (strictly) 1-topologically surjective.

Remark 1.3.3 *If the constant c is out of interest then we will simply say that operator is topologically injective or topologically surjective. In this case also there is no difference between topologically surjective and strictly topologically surjective operators.*

For a given $E \subset \Omega$ and $f \in L_0(E, \mu|_E)$ by $\widetilde{f}$ we will denote the function such that $\widetilde{f}(\omega) = f(\omega)$ if $\omega \in E$ and $\widetilde{f}(\omega) = 0$ otherwise.

Proposition 1.3.4 *Let (Ω, Σ, μ), (Ω, Σ, ν) be measure spaces and $p, q \in [1, +\infty]$. Assume we have a represetation $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ of Ω as finite disjoint union of measurable sets. Then*

(i) operator M_g be c-topologically injective for some $c > 0$ if and only if operators $M_g^{\Omega_\lambda}$ are c'-topologically injective for all $\lambda \in \Lambda$ and some $c' > 0$

(ii) operator M_g is c-topologically surjective for some $c > 0$ if and only if operators $M_g^{\Omega_\lambda}$ are c'-topologically surjective for all $\lambda \in \Lambda$ and some $c' > 0$

(iii) if operator M_g is isometric then so are $M_g^{\Omega_\lambda}$ for all $\lambda \in \Lambda$

(iv) if operator M_g is coisometric then so are $M_g^{\Omega_\lambda}$ for all $\lambda \in \Lambda$

$\triangleleft$ (i) Let M_g is c-topologically injective. Fix $\lambda \in \Lambda$ and $f \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$, then

$$\|M_g^{\Omega_\lambda}(f)\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})} = \|g \cdot \widetilde{f}\|_{L_q(\Omega, \nu)} \geq c^{-1}\|\widetilde{f}\|_{L_p(\Omega, \mu)} = c^{-1}\|f\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})}$$

So $M_g^{\Omega_\lambda}$ is c-topologically injective for all $\lambda \in \Lambda$.

Conversely, assume operators $\{M_g^{\Omega_\Lambda} : \lambda \in \Lambda\}$ are c'-topologically injective. Let $f \in L_p(\Omega, \mu)$. Using propositions 1.2.1, 1.2.6 we get

$$\|M_g(f)\|_{L_q(\Omega, \nu)} = \left\| \left(\|M_g^{\Omega_\lambda}(f|_{\Omega_\lambda})\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_q(\Lambda)}$$

$$\geq (c')^{-1} \left\| \left(\|f|_{\Omega_\lambda}\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_q(\Lambda)}$$

$$\geq (c')^{-1} C_{p,q}^{-1} \left\| \left(\|f|_{\Omega_\lambda}\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_p(\Lambda)} = (c')^{-1} C_{p,q}^{-1} \|f\|_{L_p(\Omega, \mu)}$$

Since f is arbitrary M_g is c-topologically injective for $c = c' C_{p,q} > 0$.

(ii) Let M_g is c-topologically surjective. Fix $\lambda \in \Lambda$ and $h \in L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})$. Then there exist $\widetilde{f} \in L_p(\Omega, \mu)$ such that $M_g(\widetilde{f}) = \widetilde{h}$ and $\|\widetilde{f}\|_{L_p(\Omega, \mu)} \leq c\|\widetilde{h}\|_{L_q(\Omega, \nu)}$. Consider $f = \widetilde{f}|_{\Omega_\lambda}$, then $M_g^{\Omega_\lambda}(f) = \widetilde{h}|_{\Omega_\lambda} = h$ and $\|f\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})} \leq \|\widetilde{f}\|_{L_p(\Omega, \mu)} \leq c\|\widetilde{h}\|_{L_q(\Omega, \nu)} = c\|h\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})}$. Since h is arbitrary then $M_g^{\Omega_\lambda}$ is c-topologically surjective for all $\lambda \in \Lambda$.

Conversely, assume operators $\{M_g^{\Omega_\Lambda} : \lambda \in \Lambda\}$ are c'-topologically surjective. Let $h \in L_q(\Omega, \nu)$. From assumption for each $\lambda \in \Lambda$ we have $f_\lambda \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$ such that $M_g^{\Omega_\lambda}(f_\lambda) = h|_{\Omega_\lambda}$ and $\|f_\lambda\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})} \leq c'\|h|_{\Omega_\lambda}\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})}$. Define $f \in L_0(\Omega, \mu)$ such that $f(\omega) = f_\lambda(\omega)$ if $\omega \in \Omega_\lambda$. Using propositions 1.2.1, 1.2.6 we get

$$\|f\|_{L_p(\Omega, \mu)} = \left\| \left(\|f_\lambda\|_{L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_p(\Lambda)} \leq c' \left\| \left(\|h|_{\Omega_\lambda}\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_p(\Lambda)}$$

$$\leq c' C_{p,q} \left\| \left(\|h|_{\Omega_\lambda}\|_{L_q(\Omega_\lambda, \nu|_{\Omega_\lambda})} : \lambda \in \Lambda \right) \right\|_{\ell_q(\Lambda)} = c' C_{p,q} \|h\|_{L_q(\Omega, \nu)}$$

Obviously, $M_g(f) = h$. Since h is arbitrary we get that M_g is c-topologiclly surjective for $c = c' C_{p,q} > 0$.

(iii) Fix $\lambda \in \Lambda$ and $f \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$, then

$$\|M_g^{\Omega_\lambda}(f)\|_{L_q(\Omega_\lambda,\nu|_{\Omega_\lambda})} = \|g \cdot \widetilde{f}\|_{L_q(\Omega,\nu)} = \|\widetilde{f}\|_{L_p(\Omega,\mu)} = \|f\|_{L_p(\Omega_\lambda,\mu|_{\Omega_\lambda})}$$

So $M_g^{\Omega_\lambda}$ is isometric for all $\lambda \in \Lambda$

(iv) Fix $\lambda \in \Lambda$. Since M_g is coisometric it is 1-topologically surjective and contractive. So from paragraph (ii) we see that $M_g^{\Omega_\lambda}$ is 1-topologically surjective. Let $f \in L_p(\Omega_\lambda, \mu|_{\Omega_\lambda})$. Since M_g is contractive we get

$$\|M_g^{\Omega_\lambda}(f)\|_{L_q(\Omega_\lambda,\nu|_{\Omega_\lambda})} = \|M_g(\widetilde{f})\chi_{\Omega_\lambda}\|_{L_q(\Omega,\nu)} = \|M_g(\widetilde{f}\chi_{\Omega_\lambda})\|_{L_q(\Omega,\nu)} \leq \|\widetilde{f}\chi_{\Omega_\lambda}\|_{L_p(\Omega,\mu)} = \|f\|_{L_p(\Omega_\lambda,\mu|_{\Omega_\lambda})}$$

Since $M_g^{\Omega_\lambda}$ is contractive and 1-topologically injective it is coisometric. $\triangleright$

Proposition 1.3.5 *Let (Ω, Σ, μ) and (Ω, Σ, ν) be two σ-finite measure spaces. Let $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$. If $\mu \perp \nu$ then $M_g : L_p(\Omega, \mu) \to L_q(\Omega, \nu_s)$ is zero operator.*

$\triangleleft$ Since $\mu \perp \nu$, then there exist $\Omega_s \in \Sigma$ such that $\mu(\Omega_s) = \nu(\Omega_c) = 0$, where $\Omega_c = \Omega \setminus \Omega_a$. Since $\mu(\Omega_s) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$ in $L_p(\Omega, \mu)$ and $\chi_{\Omega_c} = 0$ in $L_q(\Omega, \nu)$. Now for all $f \in L_p(\Omega, \mu)$ we have $M_g(f) = M_g(f \cdot \chi_\Omega) = M_g(f \cdot \chi_{\Omega_c}) = g \cdot f \cdot \chi_{\Omega_c} = 0$. Since f is arbitrary $M_g = 0$. $\triangleright$

2 Properties of multiplication operators

2.1 Topologically injective and isometric operators

This is the main section for subsequent study of multiplication operators. In the end we will show that topologically injective multiplication operators are isomorphisms or invertible from the left.

Proposition 2.1.1 *Let (Ω, Σ, μ) be a measure space and $g \in L_0(\Omega, \mu)$. Let $p = q$, then*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu))$ if and only if $g \in L_\infty(\Omega, \mu)$.

(ii) M_g is an isomorphism if and only if $C \geq |g| \geq c$ for some $C, c > 0$.

$\triangleleft$ (i) Assume $M_g \in \mathcal{B}(L_p(\Omega, \mu))$. Assume there exist $E \in \Sigma$ with $\mu(E) > 0$ such that $|g|_E| > \|M_g\|$, then

$$\|M_g(\chi_E)\|_{L_p(\Omega,\mu)} = \|g \cdot \chi_E\|_{L_p(\Omega,\mu)} > \|M_g\|\|\chi_E\|_{L_p(\Omega,\mu)}$$

Contradiction, hence for all $E \in \Sigma$ with $\mu(E) > 0$ we have $|g|_E| \leq \|M_g\|$ i.e. $|g| \leq \|M_g\|$. Thus $g \in L_\infty(\Omega, \mu)$

Conversely, let $g \in L_\infty(\Omega, \mu)$ then $|g| \leq C$ for some $C > 0$. Now for any $p \in [1, +\infty]$ we have and all $f \in L_p(\Omega, \mu)$ we have

$$\|M_g(f)\|_{L_p(\Omega,\mu)} = \|g \cdot f\|_{L_p(\Omega,\mu)} \leq C\|f\|_{L_p(\Omega,\mu)}$$

Hence $M_g \in \mathcal{B}(L_p(\Omega, \mu))$

(ii) Note that $M_g^{-1} = M_{1/g}$ as linear maps provided g is invertible. Now M_g is an isomorphism if and only if M_g and M_g^{-1} are bounded operators. From previous paragraph and equality $M_g^{-1} = M_{1/g}$ we see that it is equivalent to boundedness of g and $1/g$. This is equivalent to $C \geq |g| \geq c$ for some $C, c > 0$. $\triangleright$

Proposition 2.1.2 *Let (Ω, Σ, μ) be a σ-finite purely atomic measure space. Let $1 \leq p, q \leq +\infty$ and $g \in L_0(\Omega, \mu)$, then the operator $\widetilde{M_{\widetilde{g}}} := \widetilde{I}_q M_g \widetilde{I}_p^{-1} \in \mathcal{B}(\ell_p(\Lambda), \ell_q(\Lambda))$ is a multiplication operator by the function $\widetilde{g} : \Lambda \to \mathbb{C}, \lambda \mapsto \mu(\Omega_\lambda)^{1/q-1/p-1} \int_{\Omega_\lambda} f(\omega) d\mu(\omega)$ where $\{\Omega_\lambda : \lambda \in \Lambda\}$ is at most countable decomposition of Ω into pairwise disjoint atoms guranteed by proposition 1.2.3.*

$\triangleleft$ Let $p, q \in [1, +\infty]$. For any $x \in \ell_p(\Lambda)$ we have

$$\widetilde{M_{\widetilde{g}}}(x)(\lambda) = (\widetilde{I}_q((M_g \widetilde{I}_p^{-1})(x))(\lambda)) = J_q(M_g(\widetilde{I}_p^{-1}(x))|_{\Omega_\lambda})(1)$$

$$= J_q((g \cdot \widetilde{I}_p^{-1}(x))|_{\Omega_\lambda})(1) = \mu(\Omega_\lambda)^{1/q-1} \int_{\Omega_\lambda} (g|_{\Omega_\lambda} \cdot \widetilde{I}_p^{-1}(x)|_{\Omega_\lambda})(\omega) d\mu(\omega)$$

$$= \mu(\Omega_\lambda)^{1/q-1} \int_{\Omega_\lambda} (g \cdot \mu(\Omega)^{-1/p} x(\lambda) \chi_{\Omega_\lambda})(\omega) d\mu(\omega) = x(\lambda) \mu(\Omega_\lambda)^{1/q-1/p-1} \int_{\Omega_\lambda} g(\omega) d\mu(\omega)$$

Thus $\widetilde{M_{\widetilde{g}}}$ is a multiplication operator where $\widetilde{g}(\lambda) = \mu(\Omega_\lambda)^{1/q-1/p-1} \int_{\Omega_\lambda} g(\omega) d\mu(\omega)$ $\triangleright$

Since $\widetilde{I}_p$ and $\widetilde{I}_q$ are isometric isomorphisms then M_g is topologically injective if and only if $\widetilde{M_{\widetilde{g}}}$ is topologically injective.

Proposition 2.1.3 *Let (Ω, Σ, μ) be σ-finite purely atomic measure space, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$ then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$ *is topologically injective;*

(ii) $|g| \geq c$ *for some* $c > 0$ *and if* $p \neq q$ *the space* (Ω, Σ, μ) *have finitely many atoms.*

$\triangleleft$ $(i) \implies (ii)$ Assume M_g is topologically injective, then so is $\widetilde{M_{\widetilde{g}}}$, i.e. $\|\widetilde{M_{\widetilde{g}}}(x)\|_{\ell_q(\Lambda)} \geq c' \|x\|_{\ell_p(\Lambda)}$ for all $x \in \ell_p(\Lambda)$ and some $c' > 0$. Now we use at most countable decomposition $\{\Omega_\lambda : \lambda \in \Lambda\}$ of Ω into pairwise disjoint atoms of Ω given by proposition 1.2.3. Now we will consider two big cases.

(1) Let $p \neq q$. Assume Λ is countable.

(1.1) Consider subcase $p, q < +\infty$. If Λ is countable, then we get a contradiction, because by Pitt's theorem (see proposition 2.1.6 in [4]) there is no embeddings between $\ell_p(\Lambda)$ and $\ell_q(\Lambda)$ spaces for countable Λ and $1 \leq p, q < +\infty$, $p \neq q$.

(1.2) Consider subcase $q = +\infty$. Take any finite family $F \subset \Lambda$, then

$$\sup_{\lambda \in \Lambda} |\widetilde{g}(\lambda)| \geq \max_{\lambda \in F} |\widetilde{g}(\lambda)| = \left\| \widetilde{M_{\widetilde{g}}} \left(\sum_{\lambda \in F} e_\lambda \right) \right\|_{\ell_\infty(\Lambda)} \geq c' \left\| \sum_{\lambda \in F} e_\lambda \right\|_{\ell_p(\Lambda)} = c' \operatorname{Card}(F)$$

Since Λ is countable $\sup_{\lambda \in \Lambda} |\widetilde{g}(\lambda)| \geq c' \sup_{F \subset \Lambda} \operatorname{Card}(F) = +\infty$. On the other hand, since $\widetilde{M_{\widetilde{g}}}$ is bounded we have

$$\sup_{\lambda \in \Lambda} |\widetilde{g}(\lambda)| = \sup_{\lambda \in \Lambda} \|\widetilde{M_{\widetilde{g}}}(e_\lambda)\|_{\ell_\infty(\Lambda)} \leq \|\widetilde{M_{\widetilde{g}}}\| \|e_\lambda\|_{\ell_p(\Lambda)} = \|\widetilde{M_{\widetilde{g}}}\| < +\infty$$

Contradiction.

(1.3) Consider subcase $p = +\infty$. Since Λ is countable then $\ell_\infty(\Lambda)$ is non separable and $\ell_q(\Lambda)$ is separable. As $\widetilde{M_{\widetilde{g}}}$ is topologically injective, then $\operatorname{Im}(\widetilde{M_{\widetilde{g}}})$ is non separable subspace of $\ell_q(\Lambda)$. Contradiction.

In all subcases we got contradiction, hence Λ is finite i.e. (Ω, Σ, μ) have finitely many atoms. From lemma 1.2.2 we see that g is completely determined by its values $k_\lambda \in \mathbb{C}$ on atoms $\{\Omega_\lambda : \lambda \in \Lambda\}$. By proposition 1.3.1 the function g is zero only on sets of measre zero, so $k_\lambda \neq 0$ for all $\lambda \in \Lambda$. Since Λ is finite we conclude $|g| \geq c$ for $c = \min_{\lambda \in \Lambda} |k_\lambda|$.

(2) Let $p = q$. Fix $\lambda \in \Lambda$, then

$$|\widetilde{g}(\lambda)| = \|\widetilde{g} \cdot e_\lambda\|_{\ell_q(\Lambda)} = \|\widetilde{M_{\widetilde{g}}}(e_\lambda)\|_{\ell_q(\Lambda)} \geq c' \|e_\lambda\|_{\ell_p(\Lambda)} = c'$$

From lemma 1.2.2 for μ-almost all $\omega \in \Omega_\lambda$ we have

$$|g(\omega)| = \left| \mu(\Omega_\lambda)^{-1} \int_{\Omega_\lambda} g(\omega) d\mu(\omega) \right| = \left| \mu(\Omega_\lambda)^{-1} \mu(\Omega_\lambda)^{1+1/p-1/p} \widetilde{g}(\lambda) \right| = |\widetilde{g}(\lambda)| \geq c'$$

Since $\lambda \in \Lambda$ is arbitrary and $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$, then $|g| \geq c'$.

$(ii) \implies (i)$ Assume $|g| \geq c$ for $c > 0$. Then from proposition 2.1.2 we see that $|\widetilde{g}| \geq c$.

(1) Let $p \neq q$. Then we additionally assume that (Ω, Σ, μ) have finitely many atoms. Now from proposition 1.2.3 we get that $L_p(\Omega, \mu)$ is finite dimensional. From assumption on g we see it is have no zero values. Hence operator M_g is topologically injective.

(2) Let $p = q$, then for all $x \in \ell_p(\Lambda)$ we have

$$\|\widetilde{M_{\widetilde{g}}}\|_{\ell_p(\Lambda)} = \|g \cdot x\|_{\ell_p(\Lambda)} \geq c \|x\|_{\ell_p(\Lambda)}$$

so $\widetilde{M_{\widetilde{g}}}$ is topologically injective and so does M_g. $\triangleright$

Proposition 2.1.4 *Let (Ω, Σ, μ) be a nonatomic measure space, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$ then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$ is topologically injective

(ii) $|g| \geq c$ for some $c > 0$ and $p = q$.

$\triangleleft$ $(i) \implies (ii)$ Assume M_g is topologically injective i.e. $\|M_g(f)\|_{L_q(\Omega,\mu)} \geq c \|f\|_{L_p(\Omega,\mu)}$ for some $c > 0$ and all $f \in L_p(\Omega, \mu)$. We will consider three cases.

(1) Let $p > q$. There exist $C > 0$ and $E \in \Sigma$ with $\mu(E) > 0$ such that $|g|_E| \leq C$, otherwise M_g is not well defined. By theorem 1.1.4 we have a sequence $\{E_n : n \in \mathbb{N}\} \subset \Sigma$ of subsets of E such that $\mu(E_n) = 2^{-n}$. Then since $p > q$ we get

$$c \leq \frac{\|M_g(\chi_{E_n})\|_{L_q(\Omega,\mu)}}{\|\chi_{E_n}\|_{L_p(\Omega,\mu)}} \leq \frac{C\chi_{E_n}\|_{L_q(\Omega,\mu)}}{\|\chi_{E_n}\|_{L_p(\Omega,\mu)}} \leq C\mu(E_n)^{1/q-1/p}$$

$$c \leq \inf_{n \in \mathbb{N}} C\mu(E_n)^{1/q-1/p} = C \inf_{n \in \mathbb{N}} 2^{n(1/p-1/q)} = 0$$

Contradiction, so in this case M_g can not be topologically injective.

(2) Let $p = q$. Fix $c' < c$. Assume there exist $E \in \Sigma$ with $\mu(E) > 0$ and $|g|_E| < c'$, then

$$\|M_g(\chi_E)\|_{L_p(\Omega,\mu)} = \|g \cdot \chi_E\|_{L_p(\Omega,\mu)} \leq c' \|\chi_E\|_{L_p(\Omega,\mu)} < c \|\chi_E\|_{L_p(\Omega,\mu)}$$

Contradiction. Since $c' < c$ is arbitrary we conclude $|g|_E| \geq c$ for any $E \in \Sigma$ with $\mu(E) > 0$. Thus $|g| \geq c$.

(3) Let $p < q$. Assume we have some $c' > 0$ and $E \in \Sigma$ such that $\mu(E) > 0$, $|g|_E| > c'$. By theorem 1.1.4 we have a sequence $\{E_n : n \in \mathbb{N}\} \subset \Sigma$ of subsets of E such that $\mu(E_n) = 2^{-n}$. Then from inequality $p < q$ we get

$$\|M_g\| \geq \frac{\|M_g(\chi_{E_n})\|_{L_q(\Omega,\mu)}}{\|\chi_{E_n}\|_{L_p(\Omega,\mu)}} \geq \frac{c'\|\chi_{E_n}\|_{L_q(\Omega,\mu)}}{\|\chi_{E_n}\|_{L_p(\Omega,\mu)}} \geq c'\mu(E_n)^{1/q-1/p}$$

$$\|M_g\| \geq \sup_{n\in\mathbb{N}} c'\mu(E_n)^{1/q-1/p} \geq c' \sup_{n\in\mathbb{N}} 2^{n(1/p-1/q)} = +\infty$$

Contradiction, hence $g = 0$. In this case by proposition 1.3.1 operator M_g is not topologically injective.

$(ii) \implies (i)$ Conversely, assume $|g| \geq c$ for $c > 0$ and $p = q$. Then for all $f \in L_p(\Omega, \mu)$ we have

$$\|M_g(f)\|_{L_p(\Omega,\mu)} = \|g \cdot f\|_{L_p(\Omega,\mu)} \geq c\|f\|_{L_p(\Omega,\mu)}$$

So M_g is topologically injective. $\triangleright$

Theorem 2.1.5 *Let (Ω, Σ, μ) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \Sigma, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$ *is toplogically injective*

(ii) M_g *is an isomorphism*

(iii) $|g| \geq c$ *for some* $c > 0$, *if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms*

$\triangleleft$ $(i) \implies (iii)$ By proposition 1.2.4 we have decomposition $\Omega = \Omega_a \cup \Omega_{na}$, where $(\Omega_{na}, \Sigma|_{\Omega_{na}}, \mu|_{\Omega_{na}})$ is a nonatomic measure space and $(\Omega_a, \Sigma|_{\Omega_a}, \mu|_{\Omega_a})$ is a purely atomic measure space. By proposition 1.3.4 operator M_g is topologically injective if and only if so does $M_g^{\Omega_a}$ and $M_g^{\Omega_{na}}$. Propositions 2.1.3 and 2.1.4 give necessary and sufficient conditions for $M_g^{\Omega_a}$ and $M_g^{\Omega_{na}}$ to be topologically injective. So we get the result.

$(i) \implies (ii)$ Assume M_g is topologically injective. If $p = q$ from considerations above it follows that $|g| \geq c$ for some $c > 0$. Since M_g is bounded from proposition 2.1.1 we also have $C \geq |g|$ for some $C > 0$. Now from the same proposition we conclude that M_g is an isomorphism because $C \geq |g| \geq c$. Assume $p \neq q$, then from previous paragraph the space (Ω, Σ, μ) consist of finite amount of atoms and g is injective. Hence from proposition 1.2.3 we get $\dim(L_p(\Omega, \Sigma, \mu)) = \dim(\ell_p(\Lambda)) = \mathrm{Card}(\Lambda) < +\infty$. Similarly, $\dim(L_q(\Omega, \Sigma, \mu)) = \mathrm{Card}(\Lambda) < +\infty$ Since g is injective by proposition 1.3.1 operator M_g is injective. Thus M_g is an injective operator between finite dimensional spaces of equal dimension. Hence it is an isomorphism.

$(ii) \implies (i)$ Conversely, if M_g is an isomorphism, clearly, it is topologically injective. $\triangleright$

Proposition 2.1.6 *Let (Ω, Σ, μ) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g, \rho \in L_0(\Omega, \mu)$ and ρ is non negative then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \rho \cdot \mu))$ *is topologically injective*

(ii) M_g *is an isomorphism*

(iii) ρ *is positive,* $|g \cdot \rho^{1/q}| \geq c$ *for some* $c > 0$, *if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms.*

$\triangleleft$ $(i) \implies (iii)$ Consider set $E = \rho^{-1}(\{0\})$. Assume $\mu(E) > 0$ then $\chi_E \neq 0$ in $L_p(\Omega, \mu)$. On the other hand $(\rho \cdot \mu)(E) = \int_E \rho(\omega) d\mu(\omega) = 0$, so $\chi_E = 0$ in $L_q(\Omega, \rho \cdot \mu)$ and $M_g(\chi_E) = g \cdot \chi_E = 0$ in $L_q(\Omega, \rho \cdot \mu)$. Thus we see that M_g is not injective and as the consequence it is not topologically injective. Contradiction, so $\mu(E) = 0$ and ρ is positive. Hence by proposition 1.2.5 we have an isometric isomorphism $\bar{I}_q : L_q(\Omega, \mu) \to L_q(\Omega, \rho \cdot \mu), f \mapsto \rho^{-1/q} \cdot f$. Obviously $M_{g \cdot \rho^{1/q}} = \bar{I}_q^{-1} M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$. Since $\bar{I}_q$ is an isometric isomorphism and M_g is topologically injective, then $M_{g \cdot \rho^{1/q}}$ is topologically injective. From theorem 2.1.5 we get that $|g \cdot \rho^{1/q}| \geq c$ for some $c > 0$ and if $p \neq q$ the space is (Ω, Σ, μ) consist of finite amount of atoms.

$(iii) \implies (i)$ By theorem 2.1.5 operator $M_{g \cdot \rho^{1/q}}$ is topologically injective. Since ρ is positive by proposition 1.2.5 we have an isometric isomorohism $\bar{I}_q$. Then from equality $M_g = \bar{I}_q M_{g \cdot \rho^{1/q}}$ it follows that M_g is also topologically injective.

$(i) \implies (ii)$ As we proved above this implies that $M_{g \cdot \rho^{1/q}}$ is topologically injective and $\bar{I}_q$ is an isometric isomorphism. By theorem 2.1.5 $M_{g \cdot \rho^{1/q}}$ is an isomorphism. Since $M_g = \bar{I}_q M_{g \cdot \rho^{1/q}}$ and $\bar{I}_q$ is an isometric isomorphism, then M_g is also an isomorphism.

$(ii) \implies (i)$ If M_g is an isomorphism, then, obviously, it is topologically injective. $\triangleright$

Theorem 2.1.7 *Let (Ω, Σ, μ), (Ω, Σ, ν) be two σ-finite measure spaces, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are eqivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \nu))$ *is topologically injective*

(ii) $M_g^{\Omega_c}$ *is an isomorphism*

(iii) $\rho_{\nu,\mu}$ *is positive,* $|g \cdot \rho_{\nu,\mu}^{1/q}|_{\Omega_c}| \geq c$ *for some* $c > 0$*, if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms.*

$\triangleleft$ By proposition 1.3.4 operator M_g is topologically injective if and only if operators $M_g^{\Omega_c} : L_p(\Omega_c, \mu|_{\Omega_c}) \to L_q(\Omega_c, \rho_{\nu,\mu} \cdot \mu|_{\Omega_c})$ and $M_g^{\Omega_s} : L_p(\Omega_s, \mu|_{\Omega_s}) \to L_q(\Omega_s, \nu_s|_{\Omega_s})$ are topologically injective. By proposition 1.3.5 operator $M_g^{\Omega_s}$ is zero. Since $\mu(\Omega_s) = 0$, then $L_p(\Omega_s, \mu|_{\Omega_s}) = \{0\}$. From these two facts we conclude that $M_g^{\Omega_s}$ is topologically injective. Thus topological injectivity of M_g is equivalent to topological injectivity of $M_g^{\Omega_c}$. It is remains to apply proposition 2.1.6. $\triangleright$

Theorem 2.1.8 *Let (Ω, Σ, μ), (Ω, Σ, ν) be two σ-finite measure spaces, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \nu))$ *is topologically injective*

(ii) $M_{\chi_{\Omega_c}/g} \in \mathcal{B}(L_q(\Omega, \nu), L_p(\Omega, \mu))$ *its left inverse topologically surjective operator*

$\triangleleft$ $(i) \implies (ii)$ By proposition 1.3.4 $M_g^{\Omega_c}$ is topologically injective. By proposition 2.1.6 operator $M_g^{\Omega_c}$ is invertible and $(M_g^{\Omega_c})^{-1} = M_{1/g}^{\Omega_c}$. Then for all $h \in L_q(\Omega, \nu)$ we have

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} = \|M_{1/g}(h)\chi_{\Omega_c}\|_{L_p(\Omega,\mu)} = \|M_{1/g}^{\Omega_c}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})}$$

$$\leq \|M_{1/g}^{\Omega_c}\| \|h|_{\Omega_c}\|_{L_q(\Omega_c,\nu|_{\Omega_c})} \leq \|M_{1/g}^{\Omega_c}\| \|h\|_{L_q(\Omega,\nu)}$$

So $M_{\chi_{\Omega_c}/g}$ is bounded. Now note that for all $f \in L_p(\Omega, \mu)$ we have

$$M_{\chi_{\Omega_c}/g}(M_g(f)) = M_{\chi_{\Omega_c}/g}(g \cdot f) = (\chi_{\Omega_c}/g) \cdot g \cdot f = f \cdot \chi_{\Omega_c}$$

Since $\mu(\Omega \setminus \Omega_c) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$, so $M_{\chi_{\Omega_c}/g}(M_g(f)) = f \cdot \chi_{\Omega_c} = f \cdot \chi_\Omega = f$. This means that M_g have left inverse multiplication operator. Take any $f \in L_p(\Omega, \mu)$, then for $h = M_g(f)$ we have $M_{\chi_{\Omega_c}/g}(h) = f$ and $\|h\|_{L_q(\Omega,\nu)} \leq \|M_g\| \|f\|_{L_p(\Omega,\mu)}$. Since h is arbitrary $M_{\chi_{\Omega_c}/g}$ is topologically surjective.

Conversely if M_g have left inverse $M_{\chi_{\Omega_c}/g}$ then for all $f \in L_p(\Omega, \mu)$ we have

$$\|M_g(f)\|_{L_q(\Omega,\nu)} \geq \|M_{\chi_{\Omega_c}/g}\|^{-1} \|M_{\chi_{\Omega_c}/g}(M_g(f))|_{L_p(\Omega,\mu)} \geq \|M_{\chi_{\Omega_c}/g}\|^{-1} \|f\|_{L_p(\Omega,\mu)}$$

So M_g is topologically injective. $\triangleright$

Proposition 2.1.9 *Let (Ω, Σ, μ) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$ *is an isometry*

(ii) $|g| = \mu(\Omega)^{1/p-1/q}$, *if $p \neq q$, then (Ω, Σ, μ) consist of single atom.*

$\triangleleft$ $(i) \implies (ii)$ Let $p = q$. Assume there exist $E \in \Sigma$ with $\mu(E) > 0$ such that $|g|_E| < 1$, then

$$\|M_g(\chi_E)\|_{L_p(\Omega,\mu)} = \|g \cdot \chi_E\|_{L_p(\Omega,\mu)} < \|\chi_E\|_{L_p(\Omega,\mu)} = \|M_g(\chi_E)\|_{L_p(\Omega,\mu)}$$

Contradiction, hence for all $E \in \Sigma$ with $\mu(E) > 0$ we have $|g|_E| \geq 1$ i.e. $|g| \geq 1$. Assume there exist $E \in \Sigma$ with $\mu(E) > 0$ such that $|g|_E| > 1$, then

$$\|M_g(\chi_E)\|_{L_p(\Omega,\mu)} = \|g \cdot \chi_E\|_{L_p(\Omega,\mu)} > \|\chi_E\|_{L_p(\Omega,\mu)} = \|M_g(\chi_E)\|_{L_p(\Omega,\mu)}$$

Contradiction, hence for all $E \in \Sigma$ with $\mu(E) > 0$ we have $|g|_E| \leq 1$ i.e. $|g| \leq 1$. From both inequalities we get $|g| = 1 = \mu(\Omega)^{1/p-1/q}$. Let $p \neq q$, then since M_g is an isometry it is topologically injective. By theorem 2.1.5 the space (Ω, Σ, μ) consist of finitely many atoms. Assume there is at least two disjoint atoms, say Ω_1 and Ω_2. By lemma 1.1.1 they are of finite measure, so we can consider respective normalized functions $h_k = \|\chi_{\Omega_k}\|_{L_p(\Omega,\mu)}^{-1} \chi_{\Omega_k}$ where $k \in \{1, 2\}$. Since they these atoms are disjoint $h_1 h_2 = 0$ and as the result $M_g(h_1) M_g(h_2) = 0$. Note that for any $r \in [1, +\infty]$ and all $f_1, f_2 \in L_r(\Omega, \mu)$ such that $f_1 f_2 = 0$ we have

$$\|f_1 + f_2\|_{L_r(\Omega,\mu)} = \left\| \left(\|f_\lambda\|_{L_r(\Omega,\mu)} : \lambda \in \{1, 2\} \right) \right\|_{\ell_r(\{1,2\})}$$

Hence

$$\|M_g(h_1 + h_2)\|_{L_q(\Omega,\mu)} = \|h_1 + h_2\|_{L_p(\Omega,\mu)} = \|(1 : \lambda \in \{1, 2\})\|_{\ell_p(\{1,2\})} = 2^{1/p}$$

But on the other hand

$$\|M_g(h_1 + h_2)\|_{L_q(\Omega,\mu)} = \|M_g(h_1) + M_g(h_2)\|_{L_q(\Omega,\mu)} = \left\| \left(\|M_g(h_\lambda)\|_{L_q(\Omega,\mu)} : \lambda \in \{1, 2\} \right) \right\|_{\ell_q(\{1,2\})}$$

$$= \left\| \left(\|h_\lambda\|_{L_p(\Omega,\mu)} : \lambda \in \{1, 2\} \right) \right\|_{\ell_q(\{1,2\})} = \|(1 : \lambda \in \{1, 2\})\|_{\ell_q(\{1,2\})} = 2^{1/q}$$

Thus $2^{1/p} = 2^{1/q}$. Contradiction, so (Ω, Σ, μ) consist of single atom. In this case from lemma 1.2.2 it follows that for all $f \in L_p(\Omega, \mu)$ we have

$$\|M_g(f)\|_{L_q(\Omega,\mu)} = \|J_q(M_g(f))\|_{\ell_q(\{1\})} = \|J_q(g \cdot f)\|_{\ell_q(\{1\})} = \mu(\Omega)^{1/q-1} \left| \int_\Omega g(\omega) f(\omega) d\mu(\omega) \right|$$

$$\|f\|_{L_p(\Omega,\mu)} = \|J_p(f)\|_{\ell_p(\{1\})} = \mu(\Omega)^{1/p-1}\left|\int_\Omega f(\omega)d\mu(\omega)\right|$$

By c we denote the constant value of g, then

$$\|M_g(f)\|_{L_q(\Omega,\mu)} = \mu(\Omega)^{1/q-1}\left|\int_\Omega g(\omega)f(\omega)d\mu(\omega)\right| = \mu(\Omega)^{1/q-1}|c|\left|\int_\Omega f(\omega)d\mu(\omega)\right|$$

From this equality we conclude that in this case M_g is an isometry if

$$|g| = |c| = \mu(\Omega)^{1/p-1/q}$$

$(ii) \implies (i)$. Let $p = q$, then $|g| = 1$. So for all $f \in L_p(\Omega,\mu)$ we have

$$\|M_g(f)\|_{L_p(\Omega,\mu)} = \|g\cdot f\|_{L_p(\Omega,\mu)} = \||g|\cdot f\|_{L_p(\Omega,\mu)} = \|f\|_{L_p(\Omega,\mu)}$$

hence M_g is an isometry. Let $p \neq q$, then (Ω,Σ,μ) consist of single atom and we conclude

$$\|M_g(f)\|_{L_q(\Omega,\mu)} = \mu(\Omega)^{1/q-1}\left|\int_\Omega g(\omega)f(\omega)d\mu(\omega)\right| = \mu(\Omega)^{1/q-1}|c|\left|\int_\Omega f(\omega)d\mu(\omega)\right|$$

$$= \mu(\Omega)^{1/p-1}\left|\int_\Omega f(\omega)d\mu(\omega)\right| = \|f\|_{L_p(\Omega,\mu)}$$

hence M_g is isometric. $\triangleright$

Proposition 2.1.10 *Let (Ω,Σ,μ) be a σ-finite measure space, $p, q \in [1,+\infty]$ and $g, \rho \in L_0(\Omega,\mu)$, and ρ is non negative, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega,\mu), L_q(\Omega,\rho\cdot\mu))$ *is isometric*

(ii) M_g *is isometric isomorphism*

(iii) ρ *is positive,* $|g\cdot\rho^{1/q}| = \mu(\Omega)^{1/p-1/q}$ *and if* $p \neq q$ *the sapce* (Ω,Σ,μ) *consist of single atom.*

$\triangleleft$ $(i) \implies (iii)$ Since M_g is isometric, it is topologically injective and by theorem 2.1.7 we see that ρ is positive. Hence by proposition 1.2.5 we have an isometric isomorphism $\bar{I}_q : L_q(\Omega,\mu) \to L_q(\Omega,\rho\cdot\mu), f \mapsto \rho^{-1/q}\cdot f$. Obviously $M_{g\cdot\rho^{1/q}} = \bar{I}_q^{-1}M_g \in \mathcal{B}(L_p(\Omega,\mu), L_q(\Omega,\mu))$. Since $\bar{I}_q$ is an isometric isomorphism and M_g is isometric, then $M_{g\cdot\rho^{1/q}}$ is isometric too. It is remains to apply theorem 2.1.9.

$(iii) \implies (i)$ By theorem 2.1.9 operator $M_{g\cdot\rho^{1/q}}$ is isometric. Since ρ is positive by proposition 1.2.5 we have an isometric isomorohism $\bar{I}_q$. Then from equality $M_g = \bar{I}_q M_{g\cdot\rho^{1/q}}$ it follows that M_g is also isometric.

$(i) \implies (ii)$ Since M_g is isometric, it is topologically injective and by proposition 2.1.6 it is an isomorphism, which is isometric by assumption.

$(ii) \implies (i)$ Since M_g is an isometric isomorphism, trivially, it is isometric. $\triangleright$

Theorem 2.1.11 *Let (Ω,Σ,μ), (Ω,Σ,ν) be two σ-finite measure spaces, $p, q \in [1,+\infty]$ and $g \in L_0(\Omega,\mu)$, then the following are equivalent*

(i) M_g *is isometric*

(ii) $M_g^{\Omega_c}$ *is isometric*

(iii) $\rho_{\nu,\mu}$ *is positive,* $|g \cdot \rho_{\nu,\mu}^{1/q}|_{\Omega_c}| = \mu(\Omega_c)^{1/p-1/q}$ *and if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of single atom.*

$\triangleleft$ $(i) \implies (ii) \implies (iii)$ Since M_g is isometric, by proposition 1.3.4 operator $M_g^{\Omega_c}$, is isometric. It is remains to apply proposition 2.1.10.

$(iii) \implies (i)$ By proposition 2.1.10 operator $M_g^{\Omega_c}$ is isometric. Now take arbitrary $f \in L_p(\Omega, \mu)$. Since $\mu(\Omega \setminus \Omega_c) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$ in $L_p(\Omega, \mu)$. As the result $f = f\chi_\Omega = f\chi_{\Omega_c} = f\chi_{\Omega_c}\chi_{\Omega_c}$ in $L_p(\Omega, \mu)$ and $M_g(f) = M_g(f\chi_{\Omega_c})\chi_{\Omega_c}$. Thus using that $M_g^{\Omega_c}$ is isometric we get

$$\|M_g(f)\|_{L_q(\Omega,\nu)} = \|M_g(f\chi_{\Omega_c})\chi_{\Omega_c}\|_{L_q(\Omega,\nu)} = \|M_g(f\chi_{\Omega_c})\|_{L_q(\Omega_c,\nu|_{\Omega_c})}$$

$$= \|M_g^{\Omega_c}(f|_{\Omega_c})\|_{L_q(\Omega_c,\nu|_{\Omega_c})} = \|f|_{\Omega_c}\|_{L_p(\Omega_c,\mu|_{\Omega_c})}$$

Since $\mu(\Omega \setminus \Omega_c) = 0$ we have $\|f|_{\Omega_c}\|_{L_p(\Omega_c,\mu|_{\Omega_c})} = \|f\|_{L_p(\Omega,\mu)}$ so $\|M_g(f)\|_{L_q(\Omega,\nu)} = \|f\|_{L_p(\Omega,\mu)}$ i.e. M_g is isometric. $\triangleright$

Theorem 2.1.12 *Let* (Ω, Σ, μ), (Ω, Σ, ν) *be two* σ*-finite measure spaces,* $p, q \in [1, +\infty]$ *and* $g \in L_0(\Omega, \mu)$*, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \nu))$ *is isometric*

(ii) $M_{\chi_{\Omega_c}/g} \in \mathcal{B}(L_q(\Omega, \nu), L_p(\Omega, \mu))$ *its left inverse strictly coisometric operator.*

$\triangleleft$ $(i) \implies (ii)$ By proposition 1.3.4 operator $M_g^{\Omega_c}$ is isometric and by proposition 2.1.10 it is invertible with $(M_g^{\Omega_c})^{-1} = M_{1/g}^{\Omega_c}$. Since $M_g^{\Omega_c}$ is isometric then so does its inverse. Then for all $h \in L_q(\Omega, \nu)$ we have

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} = \|M_{1/g}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})} = \|M_{1/g}^{\Omega_c}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})}$$

$$= \|h|_{\Omega_c}\|_{L_q(\Omega_c,\nu|_{\Omega_c})} \leq \|h\|_{L_q(\Omega,\nu)}$$

So $M_{\chi_{\Omega_c}/g}$ is contractive. Now note that for all $f \in L_p(\Omega, \mu)$ we have

$$M_{\chi_{\Omega_c}/g}(M_g(f)) = M_{\chi_{\Omega_c}/g}(g \cdot f) = (\chi_{\Omega_c}/g) \cdot g \cdot f = f \cdot \chi_{\Omega_c}$$

Since $\mu(\Omega \setminus \Omega_c) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$, so $M_{\chi_{\Omega_c}/g}(M_g(f)) = f \cdot \chi_{\Omega_c} = f \cdot \chi_\Omega = f$. This means that M_g have left inverse multiplication operator. Take any $f \in L_p(\Omega, \mu)$, then for $h = M_g(f)$ we have $M_{\chi_{\Omega_c}/g}(h) = f$ and $\|h\|_{L_q(\Omega,\nu)} \leq \|f\|_{L_p(\Omega,\mu)}$ i.e. $M_{\chi_{\Omega_c}}/g$ is strictly 1-topologigically surjective. Since $M_{\chi_{\Omega_c}/g}$ is also contractive, it is strictly coisometric.

$(ii) \implies (i)$ Take any $f \in L_p(\Omega, \mu)$, then there exist $h \in L_q(\Omega, \nu)$ such that $M_{\chi_{\Omega_c}/g}(h) = f$ and $\|h\|_{L_q(\Omega,\nu)} \leq \|f\|_{L_p(\Omega,\mu)}$. Hence

$$\|M_g(f)\|_{L_q(\Omega,\nu)} = \|M_g(M_{\chi_{\Omega_c}/g}(h))\|_{L_q(\Omega,\nu)} = \|\chi_{\Omega_c} h\|_{L_q(\Omega,\nu)} \leq \|h\|_{L_q(\Omega,\nu|)} \leq \|f\|_{L_p(\Omega,\mu)}$$

Since $M_{\chi_{\Omega_c}/g}$ is contractive and left inverse to M_g then

$$\|f\|_{L_p(\Omega,\mu)} = \|M_{\chi_{\Omega_c}/g}(M_g(f))\|_{L_p(\Omega,\mu)} \leq \|M_g(f)\|_{L_q(\Omega,\nu)}$$

so $\|M_g(f)\|_{L_q(\Omega,\nu)} = \|f\|_{L_p(\Omega,\mu)}$. Since f is arbitrary M_g is isometric. $\triangleright$

2.2 Topologically surjective and coisometric operators

Description of topologically surjective operators is slightly easier to obtain. We will show that all such operators are isomorphisms or invertible from the right. Most of the proofs goes along the lines of previous sections.

Theorem 2.2.1 *Let (Ω, Σ, μ) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \mu))$ *is topologically surjective*

(ii) M_g *is an isomorphism*

(iii) $|g| \geq c$ *for some* $c > 0$, *if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms.*

$\triangleleft$ $(i) \implies (iii)$ Since M_g be topologically surjective, then it is surjective and by proposition 1.3.1 it is also injective. Thus M_g is bijective. Since L_p spaces are complete, from open mapping theorem we see that M_g is an isomorphism.

$(ii) \implies (i)$ If M_g is an isomorphism, obviously, it is topologically surjective.

$(i) \implies (iii)$ Follows from theorem 2.1.5 $\triangleright$

Proposition 2.2.2 *Let (Ω, Σ, ν) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g, \rho \in L_0(\Omega, \rho \cdot \nu)$ and ρ is non negative, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \rho \cdot \nu), L_q(\Omega, \nu))$ *is topologically surjective*

(ii) M_g *is an isomorphism*

(iii) ρ *is positive,* $|g \cdot \rho^{-1/p}| \geq c$ *for some* $c > 0$, *if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms.*

$\triangleleft$ $(i) \implies (iii)$ Consider set $E = \rho^{-1}(\{0\})$. Assume $\nu(E) > 0$ then $\chi_E \neq 0$ in $L_p(\Omega, \nu)$. On the other hand $(\rho \cdot \nu)(E) = \int_E \rho(\omega) d\nu(\omega) = 0$, so $\chi_E = 0$ in $L_q(\Omega, \rho \cdot \mu)$. Then for all $f \in L_p(\Omega, \rho \cdot \nu)$ holds $M_g(f)\chi_E = M_g(f \cdot \chi_E) = M_g(0) = 0$ in $L_q(\Omega, \nu)$. The last equality means that $\operatorname{Im}(M_g) \subset \{h \in L_q(\Omega, \mu) : h|_E = 0\}$. Since $\nu(E) \neq 0$ we see that M_g is not surjective and as the consequence it is not topologically surjective. Contradiction, so $\nu(E) = 0$ and ρ is positive. Hence by proposition 1.2.5 we have an isometric isomorphism $\bar{I}_p : L_p(\Omega, \nu) \to L_p(\Omega, \rho \cdot \nu), f \mapsto \rho^{-1/p} \cdot f$. Obviously $M_{g \cdot \rho^{-1/p}} = M_g \bar{I}_p \in \mathcal{B}(L_p(\Omega, \nu), L_q(\Omega, \nu))$. Since $\bar{I}_p$ is an isometric isomorphism and M_g is topologically surjective, then $M_{g \cdot \rho^{-1/p}}$ is topologically surjective. It is remains to apply theorem 2.2.1.

$(iii) \implies (i)$ By theorem 2.2.1 operator $M_{g \cdot \rho^{-1/p}}$ is topologically surjective. Since ρ is positive by proposition 1.2.5 we have an isometric isomorohism $\bar{I}_p$. Then from equality $M_g = M_{g \cdot \rho^{-1/p}} \bar{I}_p^{-1}$ it follows that M_g is also topologically surjective.

$(i) \implies (ii)$ As we proved above this operator $M_{g \cdot \rho^{1/q}}$ is topologically injective and $\bar{I}_q$ is an isometric isomorphism. By theorem 2.2.1 $M_{g \cdot \rho^{1/q}}$ is an isomorphism. Since $M_g = \bar{I}_q M_{g \cdot \rho^{1/q}}$ we see that M_g is also an isomorphism, as composition of such.

$(ii) \implies (i)$. If M_g is an isomorphism, obviously, it is topologically surjective. $\triangleright$

Theorem 2.2.3 *Let (Ω, Σ, μ), (Ω, Σ, ν) be two σ-finite measure spaces, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent* *(i)* $M_g \in \mathcal{B}(L_p(\Omega,\mu), L_q(\Omega,\nu))$ *is topologically surjective*

(ii) $M_g^{\Omega_c}$ *is topologically surjective*

(iii) $\rho_{\mu,\nu}$ *is positive,* $|g \cdot \rho_{\mu,\nu}^{-1/p}|_{\Omega_c}| \geq c$ *for some* $c > 0$*, if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of finitely many atoms.*

$\triangleleft$ By proposition 1.3.4 operator M_g is topologically surjective if and only if operators $M_g^{\Omega_c} : L_p(\Omega_c, \rho_{\mu,\nu} \cdot \nu|_{\Omega_c}) \to L_q(\Omega_c, \nu|_{\Omega_c})$ and $M_g^{\Omega_s} : L_p(\Omega_s, \mu_s|_{\Omega_s}) \to L_q(\Omega_s, \nu|_{\Omega_s})$ are topologically surjective. By proposition 1.3.5 operator $M_g^{\Omega_s}$ is zero. Since $\nu(\Omega_s) = 0$, then $L_p(\Omega_s, \nu|_{\Omega_s}) = \{0\}$. From these two facts we conclude that $M_g^{\Omega_s}$ is topologically surjective. Thus topological surjectivity of M_g is equivalent to topological injectivity of $M_g^{\Omega_c}$. It is remains to apply proposition 2.2.2. $\triangleright$

Theorem 2.2.4 *Let* (Ω, Σ, μ), (Ω, Σ, ν) *be two* σ*-finite measure spaces,* $p, q \in [1, +\infty]$ *and* $g \in L_0(\Omega, \mu)$*, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega,\mu), L_q(\Omega,\nu))$ *is topologically surjective*

(ii) $M_{\chi_{\Omega_c}/g} \in \mathcal{B}(L_q(\Omega,\nu), L_p(\Omega,\mu))$ *its right inverse topologically injective operator.*

$\triangleleft$ $(i) \implies (ii)$ By proposition 1.3.4 operator $M_g^{\Omega_c}$ is topologically surjective. By proposition 2.2.2 it is invertable and $(M_g^{\Omega_c})^{-1} = M_{1/g}^{\Omega_c}$. Then for all $h \in L_q(\Omega, \nu)$ we have

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} = \|M_{1/g}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})} = \|M_{1/g}^{\Omega_c}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})}$$

$$\leq \|M_{1/g}^{\Omega_c}\| \|h|_{\Omega_c}\|_{L_q(\Omega_c,\nu|_{\Omega_c})} \leq \|M_{1/g}^{\Omega_c}\| \|h\|_{L_q(\Omega,\nu)}$$

So $M_{\chi_{\Omega_c}/g}$ is bounded. Now note that for all $h \in L_q(\Omega, \nu)$ we have

$$M_g(M_{\chi_{\Omega_c}/g}(h)) = M_g(\chi_{\Omega_c}/g \cdot h) = g \cdot (\chi_{\Omega_c}/g) \cdot h = h \cdot \chi_{\Omega_c}$$

Since $\nu(\Omega \setminus \Omega_c) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$, so $M_g(M_{\chi_{\Omega_c}/g}(h)) = h \cdot \chi_{\Omega_c} = h \cdot \chi_\Omega = h$. This means that M_g have right inverse multiplication operator. Take any $h \in L_q(\Omega, \nu)$, then

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} \geq \|M_g\| \|M_g(M_{\chi_{\Omega_c}/g}(h))\|_{L_q(\Omega,\nu)} \geq \|M_g\| \|h\|_{L_q(\Omega,\nu)}$$

Since h is arbitrary $M_{\chi_{\Omega_c}/g}$ is topologically injective.

$(ii) \implies (i)$ Take arbitrary $h \in L_q(\Omega, \nu)$ and consider $f = M_{\chi_{\Omega_c}/g}(h)$. Then $M_g(f) = M_g(M_{\chi_{\Omega_c}/g}(h)) = h$ and $\|f\|_{L_p(\Omega,\mu)} \leq \|M_{\chi_{\Omega_c}/g}\| \|h\|_{L_q(\Omega,\nu)}$. Since h is arbitrary M_g is topologically surjective. $\triangleright$

Theorem 2.2.5 *Let* (Ω, Σ, μ) *be a* σ*-finite measure space,* $p, q \in [1, +\infty]$ *and* $g \in L_0(\Omega, \mu)$*, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega,\mu), L_q(\Omega,\mu))$ *is coisometric*

(ii) M_g *is an isometric isomorphism*

(iii) $|g| = \mu(\Omega)^{1/q - 1/p}$*, if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of single atom.*

$\triangleleft$ Since M_g is coisometric it is topologically injective so from theorem 2.2.1 we get that M_g is in fact isomorphism. As the consequence it is injective, but injective coisometric operator is an isometric isomorphisms. It is remains to note that every isometric isomorphism is a strict coisometry. Thus we conclude that M_g is coisometric if and only if it is strictly coisometric if and only if it is isometric isomorphism. Now we apply theorem 2.1.9. $\triangleright$

Proposition 2.2.6 *Let (Ω, Σ, ν) be a σ-finite measure space, $p, q \in [1, +\infty]$ and $g, \rho \in L_0(\Omega, \rho \cdot \nu)$ and ρ is non negative, then the following are eqivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \rho \cdot \nu), L_q(\Omega, \nu))$ *is coisometric*

(ii) M_g *is an isometric isomorphism*

(iii) ρ *is positive,* $|g \cdot \rho^{-1/p}| = \mu(\Omega)^{1/p - 1/q}$*, if* $p \neq q$ *the space* (Ω, Σ, μ) *consist single atom.*

$\triangleleft$ $(i) \implies (ii)$ Assume M_g is coisometric, then it is topologically surjective. By theorem 2.2.2 M_g is an isomorphism, hence bijective. It is remains to note that bijective coisometry is an isometric isomorphism.

$(ii) \implies (i)$ If M_g is an isometric isomorphism, of course, it is coisometry and even more a strict coisometry.

$(i) \implies (iii)$ Follows from proposition 2.1.10. $\triangleright$

Theorem 2.2.7 *Let (Ω, Σ, μ), (Ω, Σ, ν) be two σ-finite measure spaces, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \nu))$ *is coisometric*

(ii) $M_g^{\Omega_c}$ *is an isometric isomorphism*

(iii) $\rho_{\mu,\nu}$ *is positive,* $|g \cdot \rho_{\mu,\nu}^{-1/p}|_{\Omega_c}| = \mu(\Omega_c)^{1/p - 1/q}$*, if* $p \neq q$ *the space* (Ω, Σ, μ) *consist of single atom.*

$\triangleleft$ $(i) \implies (ii)$ Since M_g is coisimetric, then from proposition 1.3.4 we know that $M_g^{\Omega_c}$ is also coisometric. From proposition 2.2.6 we get that $M_g^{\Omega_c}$ is an isometric isomorphism.

$(ii) \implies (i)$ Take arbitrary $h \in L_q(\Omega, \nu)$, then there exist $f \in L_p(\Omega_c, \mu|_{\Omega_c})$ such that $M_g^{\Omega_c}(f) = h|_{\Omega_c}$. By proposition 1.3.5 operator $M_g^{\Omega_s} = 0$, so

$$M_g(\widetilde{f}) = \widetilde{M_g^{\Omega_c}(\widetilde{f}|_{\Omega_c})} + \widetilde{M_g^{\Omega_s}(\widetilde{f}|_{\Omega_s})} = \widetilde{h|_{\Omega_c}}$$

Since $\nu(\Omega_s) = 0$, then $\|h - \widetilde{h|_{\Omega_c}}\|_{L_q(\Omega,\nu)} = \|h\chi_{\Omega_s}\|_{L_q(\Omega,\nu)} = 0$ and we conclude $h = \widetilde{h|_{\Omega_c}}$. So we found $\widetilde{f} \in L_p(\Omega, \mu)$ such that $M_g(\widetilde{f}) = h$ and $\|\widetilde{f}\|_{L_p(\Omega,\mu)} = \|f\|_{L_p(\Omega_c,\mu|_{\Omega_c})} = \|h|_{\Omega_c}\|_{L_q(\Omega_c,\nu|_{\Omega_c})} \leq \|h\|_{L_q(\Omega,\nu)}$. Since h is arbitrary then M_g is 1-topologically surjective. For all $f \in L_p(\Omega, \mu)$ we have

$$\|M_g(f)\|_{L_q(\Omega,\nu)} = \|\widetilde{M_g^{\Omega_c}(f|_{\Omega_c})} + \widetilde{M_g^{\Omega_s}(f|_{\Omega_s})}\|_{L_q(\Omega,\nu)} = \|\widetilde{M_g^{\Omega_c}(f|_{\Omega_c})}\|_{L_q(\Omega,\nu)}$$

$$= \|M_g^{\Omega_c}(f|_{\Omega_c})\|_{L_q(\Omega_c,\nu|_{\Omega_c})} = \|f|_{\Omega_c}\|_{L_p(\Omega_c,\mu|_{\Omega_c})} \leq \|f\|_{L_p(\Omega,\mu)}$$

Since f is arbitrary, then M_g is contractive, but it is also 1-topologically injective. Thus M_g is coisometric.

$(i) \implies (iii)$ Follows from proposition 2.2.6 $\triangleright$

Theorem 2.2.8 *Let (Ω, Σ, μ), (Ω, Σ, ν) be two σ-finite measure spaces, $p, q \in [1, +\infty]$ and $g \in L_0(\Omega, \mu)$, then the following are equivalent*

(i) $M_g \in \mathcal{B}(L_p(\Omega, \mu), L_q(\Omega, \nu))$ *is coisometric*

(ii) $M_{\chi_{\Omega_c}/g} \in \mathcal{B}(L_q(\Omega, \nu), L_p(\Omega, \mu))$ *its right inverse isometric operator.*

$\triangleleft$ $(i) \implies (ii)$ By proposition 1.3.4 operator $M_g^{\Omega_c}$ is coisometric and by proposition 2.2.6 it is isometric, invertable and $(M_g^{\Omega_c})^{-1} = M_{1/g}^{\Omega_c}$. Then for all $h \in L_q(\Omega, \nu)$ we have

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} = \|M_{1/g}(h)\chi_{\Omega_c}\|_{L_p(\Omega,\mu)} = \|M_{1/g}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})} = \|M_{1/g}^{\Omega_c}(h|_{\Omega_c})\|_{L_p(\Omega_c,\mu|_{\Omega_c})}$$

$$= \|h|_{\Omega_c}\|_{L_q(\Omega_c,\nu|_{\Omega_c})} \leq \|h\|_{L_q(\Omega,\nu)}$$

So $M_{\chi_{\Omega_c}/g}$ is contractive. Now note that for all $h \in L_q(\Omega, \nu)$ we have

$$M_g(M_{\chi_{\Omega_c}/g}(h)) = M_g(\chi_{\Omega_c}/g \cdot h) = g \cdot (\chi_{\Omega_c}/g) \cdot h = h \cdot \chi_{\Omega_c}$$

Since $\nu(\Omega \setminus \Omega_c) = 0$, then $\chi_{\Omega_c} = \chi_\Omega$, so $M_g(M_{\chi_{\Omega_c}/g}(h)) = h \cdot \chi_{\Omega_c} = h \cdot \chi_\Omega = h$. This means that M_g have right inverse multiplication operator. Take any $h \in L_q(\Omega, \nu)$, then

$$\|M_{\chi_{\Omega_c}/g}(h)\|_{L_p(\Omega,\mu)} \geq \|M_g\|\|M_g(M_{\chi_{\Omega_c}/g}(h))\|_{L_q(\Omega,\nu)} \geq \|h\|_{L_q(\Omega,\nu)}$$

Since h is arbitrary $M_{\chi_{\Omega_c}/g}$ is 1-topologically injective, but it is contractive. Thus M_g is isometric.

$(ii) \implies (i)$ Take arbitrary $h \in L_q(\Omega, \nu)$ and consider $f = M_{\chi_{\Omega_c}/g}(h)$. Then $M_g(f) = M_g(M_{\chi_{\Omega_c}/g}(h)) = h$ and $\|f\|_{L_p(\Omega,\mu)} \leq \|h\|_{L_q(\Omega,\nu)}$. Since h is arbitrary M_g is strictly 1-topologically surjective. Let $f \in L_p(\Omega, \mu)$. By assumption $M_{\chi_{\Omega_c}/g}$ so

$$\|M_g(f)\|_{L_q(\Omega,\nu)} = \|M_{\chi_{\Omega_c}/g}(M_g(f))\|_{L_p(\Omega,\mu)} = \|f\chi_{\Omega_c}\|_{L_p(\Omega,\mu)} \leq \|f\|_{L_p(\Omega,\mu)}$$

Since f is arbitrary M_g is contractive, but it is also strictly 1-topologically surjective, hence strictly coisometric. $\triangleright$

3 Projective, injective and flat $B(\Omega)$-modules in the category of L_p spaces

3.1 Morphisms of $B(\Omega)$-modules L_p

By $B(\Omega)$ we will denote Banach algebra of bounded measurable functions on measurable space (Ω, Σ) with sup-norm. Obviously for any $b \in B(\Omega)$ and any $f \in L_p(\Omega, \mu)$ we have

$$\|b \cdot f\|_{L_p(\Omega,\mu)} \leq \|b\|_{B(\Omega)}\|f\|_{L_p(\Omega,\mu)}$$

Hence every L_p space is a left/right/two sided Banach $B(\Omega)$-module. Since for the same f and b we have $b \cdot f = f \cdot b$, and algebra $B(\Omega)$ is commutative we can restrict our considerations to the left modules.

By $M(\Omega)$ we will denote Banach space of finite complex valued σ-additive measures on Ω. By $\mathscr{L}$ we denote full subcategory of left Banach $B(\Omega)$ modules consisting of $L_p(\Omega, \mu)$ spaces for some $\mu \in M(\Omega)$. By $\mathscr{L}_1$ we will denote the category with the same objects but with contractive morphisms only. By $\mathscr{L}^{\mathrm{op}}$ we will denote the category of the right $B(\Omega)$ modules of the form $L_p(\Omega,\mu)$. In [5] Helemskii gave a complete characterisation of morphisms of $\mathscr{L}$, but only for for locally compact Ω, with Borel σ-algebra. Careful inspection of his proof shows that this characterization valid for all σ-finite measure spaces.

Let $p,q\in[1,+\infty]$ and $\mu,\nu\in M(\Omega)$. Denote $\Omega_+=\{\omega\in\Omega_c:\rho_{\nu,\mu}(\omega)>0\}$. Recall that by Ω_a we denote atomic part of measure space (Ω,Σ,μ) provided by proposition 1.2.4. Of course, we may assume that $\Omega_a\subset\Omega_c$. Introduce the notation

$$L_{p,q,\mu,\nu}(\Omega)=\begin{cases}\{g\in L_0(\Omega,\mu):g\in L_{pq/(p-q)}(\Omega,\rho_{\nu,\mu}^{p/(p-q)}\cdot\mu),\quad g|_{\Omega\setminus\Omega_+}=0\} & \text{if}\quad p>q\\ \{g\in L_0(\Omega,\mu):g\rho_{\nu,\mu}^{1/p}\in L_\infty(\Omega,\mu),\quad g|_{\Omega\setminus\Omega_+}=0\} & \text{if}\quad p=q\\ \{g\in L_0(\Omega,\mu):g\rho_{\nu,\mu}^{1/p}\mu^{pq/(p-q)}\in L_\infty(\Omega,\mu),\quad g|_{\Omega\setminus\Omega_a}=0\} & \text{if}\quad p<q\end{cases}$$

$$\|g\|_{L_{p,q,\mu,\nu}(\Omega)}=\begin{cases}\|g\|_{L_{pq/(p-q)}(\Omega,\rho^{p/(p-q)}\cdot\mu)} & \text{if}\quad p>q\\ \|g\rho_{\nu,\mu}^{1/p}\|_{L_\infty(\Omega,\mu)} & \text{if}\quad p=q\\ \|g\rho_{\nu,\mu}^{1/p}\mu^{pq/(p-q)}\|_{L_\infty(\Omega,\mu)} & \text{if}\quad p<q\end{cases}$$

Theorem 3.1.1 ([5], 4.1) *Let $p,q\in[1,+\infty]$ and $\mu,\nu\in M(\Omega)$,then there exist isometric isomorphism*

$$\mathcal{I}_{p,q,\mu,\nu}:L_{p,q,\mu,\nu}(\Omega)\to\mathrm{Hom}_{\mathscr{L}}(L_p(\Omega,\mu),L_q(\Omega,\nu)):g\mapsto M_g$$

Simply speaking all morphisms in $\mathscr{L}$ are multiplication operators. Now we need definitions for different types of “good” morphisms from the point of view of Banach homology theory to describe variants of projectivity, injectivity and flatness.

Definition 3.1.2 *Let $\mathscr{C}$ be a category of left Banach modules over algebra A. Let $X,Y\in\mathrm{Ob}(\mathscr{C})$, then we say that a morphism $\varphi\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a relatively/metrically/extremelly admissible epimorphism if it is topologically surjective/strictly coisometric/coisometric.*

Definition 3.1.3 *Let $\mathscr{C}$ be a category of left Banach modules over algebra A. Let $X,Y\in\mathrm{Ob}(\mathscr{C})$, then we say that morphism $\varphi\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a relatively/metrically/extremelly admissible monomorphism if it is topologically injective/isometric/isometric.*

All these notions are due to Helemskii ([6]). Now results of previous section may be reformulated as follows:

(i) All relatively/metrically/extremely admissible epimorphisms in $\mathscr{L}$ are retractions and vice versa.

(ii) All relatively/metrically/extremely admissible monomorphisms in $\mathscr{L}$ are coretractions and vice versa.

3.2 Injective modules L_p

Definition 3.2.1 *Let $\mathscr{C}$ be a category of left Banach modules over algebra A. We say that $I\in\mathrm{Ob}(\mathscr{C})$ is relatively/metrically/extremely injecive if the functor $\mathrm{Hom}_{\mathscr{C}}(-,I)$ from $\mathscr{C}/\mathscr{C}_1/\mathscr{C}_1$ to $\mathscr{B}an$ maps relatively/metricaly/extremely admissible monomorphisms to surjective/strictly coisometric/coisometric operators.*

Theorem 3.2.2 *Every $B(\Omega)$ module L_p is relatively/metrically/extremely injective in $\mathscr{L}$.*

$\triangleleft$ Relative injectivity. Let $I, X, Y \in \mathrm{Ob}(\mathscr{L})$. Take arbitrary $\varphi \in \mathrm{Hom}_{\mathscr{L}}(X, I)$ and relatively admissible monomorphism $i \in \mathrm{Hom}_{\mathscr{L}}(X, Y)$. By theorem 2.1.8 we have topologically surjective $\pi \in \mathrm{Hom}_{\mathscr{L}}(Y, X)$ such that $\pi i = 1_X$. Then for $\psi = \varphi\pi$ we have $\mathrm{Hom}_{\mathscr{L}}(i, I)(\psi) = \varphi$. Hence $\mathrm{Hom}_{\mathscr{L}}(i, I)$ is surjective. This means that I is relatively injective.

Metric/extreme injectivity. Let $I, X, Y \in \mathrm{Ob}(\mathscr{L}_1)$ Take arbitrary $\varphi \in \mathrm{Hom}_{\mathscr{L}_1}(X, I)$ and metrically/extremely admissible monomorphism $i \in \mathrm{Hom}_{\mathscr{L}_1}(X, Y)$. By theorem 2.1.12 we have coisometric $\pi \in \mathrm{Hom}_{\mathscr{L}_1}(Y, X)$ such that $\pi i = 1_X$. Then for $\psi = \varphi\pi$ we have $\mathrm{Hom}_{\mathscr{L}_1}(i, I)(\psi) = \varphi$ and what is more $\|\psi\| = \|\varphi\|$ because $\|\psi\| \leq \|\varphi\|\|\pi\| = \|\varphi\|$ and $\|\varphi\| \leq \|\psi\|\|i\| = \|\psi\|$. Hence $\mathrm{Hom}_{\mathscr{L}_1}(i, I)$ is strictly coisometric and a fortiori coisometric. This means that I is metrically/extremely injective. $\triangleright$

3.3 Projective modules L_p

Definition 3.3.1 *Let $\mathscr{C}$ be a category of left Banach modules over algebra A. We say that $P \in \mathrm{Ob}(\mathscr{C})$ is relatively/metrically/extremely projective if the functor $\mathrm{Hom}_{\mathscr{C}}(P, -)$ from $\mathscr{C}/\mathscr{C}_1/\mathscr{C}_1$ to $\mathscr{B}an$ maps relatively/metricaly/extremely admissible epimorphisms to surjective/strictly coisometric/coisometric operators.*

Theorem 3.3.2 *Every $B(\Omega)$ module L_p is relatively/metrically/extremely projective in $\mathscr{L}$.*

$\triangleleft$ Relative projectivity. Let $P, X, Y \in \mathrm{Ob}(\mathscr{L})$. Take arbitrary $\varphi \in \mathrm{Hom}_{\mathscr{L}}(P, X)$ and relatively admissible epimorphism $\pi \in \mathrm{Hom}_{\mathscr{L}}(Y, X)$. By theorem 2.2.4 we have topologically injective $i \in \mathrm{Hom}_{\mathscr{L}}(X, Y)$ such that $\pi i = 1_X$. Then for $\psi = i\varphi$ we have $\mathrm{Hom}_{\mathscr{L}}(P, \pi)(\psi) = \varphi$. Hence $\mathrm{Hom}_{\mathscr{L}}(P, \pi)$ is surjective. This means that P is relatively projective.

Metric/extreme projectivity. Let $P, X, Y \in \mathrm{Ob}(\mathscr{L}_1)$. Take arbitrary $\varphi \in \mathrm{Hom}_{\mathscr{L}_1}(P, X)$ and metrically/extremely admissible epimorphism $\pi \in \mathrm{Hom}_{\mathscr{L}_1}(Y, X)$. By theorem 2.2.8 we have isometric $i \in \mathrm{Hom}_{\mathscr{L}_1}(X, Y)$ such that $\pi i = 1_X$. Then for $\psi = i\varphi$ we have $\mathrm{Hom}_{\mathscr{L}_1}(P, \pi)(\psi) = \varphi$ and what is more $\|\psi\| = \|\varphi\|$ because i is isometric. Hence $\mathrm{Hom}_{\mathscr{L}_1}(P, \pi)$ is strictly coisometric and a fortiori coisometric. This means that P is metrically/extremely projective. $\triangleright$

3.4 Flat modules L_p

Definition 3.4.1 *Let $\mathscr{C}$ be a category of left Banach modules over algebra A. We say that $F \in \mathrm{Ob}(\mathscr{C})$ is relatively/metrically/extremely flat if the functor $- \otimes^A 1_F$ from $\mathscr{C}/\mathscr{C}_1/\mathscr{C}_1$ to $\mathscr{B}an$ maps relatively/metricaly/extremely admissible monomorphisms in $\mathscr{L}^{\mathrm{op}}$ to topologically injective/isometric/ isometric operators.*

Theorem 3.4.2 *Every $B(\Omega)$ module L_p is relatively/metrically/extremely flat in $\mathscr{L}$.*

$\triangleleft$ Relative flatness. Let $F, X, Y \in \mathrm{Ob}(\mathscr{L})$. Take arbitrary relatively admissible monomorphism $i \in \mathrm{Hom}_{\mathscr{L}^{\mathrm{op}}}(X, Y)$. By theorem 2.1.8 we have topologically surjective $\pi \in \mathrm{Hom}_{\mathscr{L}^{\mathrm{op}}}(Y, X)$ such that $\pi i = 1_X$. Then for arbitrary $u \in F \otimes^{B(\Omega)} X$ we have

$$\|(1_F \overset{B(\Omega)}{\otimes} \pi)\|\|(1_F \overset{B(\Omega)}{\otimes} i)(u)\|_{F \otimes^{B(\Omega)} Y} \geq \|(1_F \overset{B(\Omega)}{\otimes} \pi)(1_F \overset{B(\Omega)}{\otimes} i)(u)\|_{F \otimes^{B(\Omega)} X} = \|(1_F \overset{B(\Omega)}{\otimes} \pi i)(u)\|_{F \otimes^{B(\Omega)} X}$$

$$= \|(1_F \overset{B(\Omega)}{\otimes} 1_X)(u)\|_{F \otimes^{B(\Omega)} X} = \|u\|_{F \otimes^{B(\Omega)} X}$$

Also note that $\|(1_F \otimes^{B(\Omega)} \pi)\| \leq \|1_F\| \|\pi\|$, hence

$$\|(1_F \overset{B(\Omega)}{\otimes} i)(u)\|_{F \otimes^{B(\Omega)} Y} \geq \|\pi\|^{-1} \|u\|_{F \otimes^{B(\Omega)} X}$$

Thus $1_F \otimes^{B(\Omega)} i$ is topologically injective, so F is relatively flat.

Metric/extreme projectivity. Let $F, X, Y \in \mathrm{Ob}(\mathscr{L})$. Take arbitrary metrically/extremely admissible monomorphism $i \in \mathrm{Hom}_{\mathscr{L}^{\mathrm{op}}}(X, Y)$. By theorem 2.1.8 we have coisometric $\pi \in \mathrm{Hom}_{\mathscr{L}^{\mathrm{op}}}(Y, X)$ such that $\pi i = 1_X$. Fix $u \in F \otimes^{B(\Omega)} X$. Since π is cosiometric then from previous paragraph we get

$$\|(1_F \overset{B(\Omega)}{\otimes} i)(u)\|_{F \otimes^{B(\Omega)} Y} \geq \|\pi\|^{-1} \|u\|_{F \otimes^{B(\Omega)} X} \geq \|u\|_{F \otimes^{B(\Omega)} X}$$

On the other hand for the same u we have

$$\|(1_F \overset{B(\Omega)}{\otimes} i)(u)\|_{F \otimes^{B(\Omega)} Y} \leq \|1_F \overset{B(\Omega)}{\otimes} i\| \|u\|_{F \otimes^{B(\Omega)} X} \leq \|1_F\| \|i\| \|u\|_{F \otimes^{B(\Omega)} X} = \|u\|_{F \otimes^{B(\Omega)} X}$$

From these inequalities it follows that $1_F \otimes^{B(\Omega)} i$ is isometric. $\triangleright$

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