Relative projectivity of modules L_p

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Abstract: In this paper we give criteria of relative projectivity of L_p-spaces regarded as left Banach modules over the algebras of bounded measurable functions ($1 \leq p \leq +\infty$) and the algebra of continuous functions vanishing at infinity ($1 \leq p < +\infty$). The main result is as follows: for a locally compact Hausdorff space S and a locally finite inner compact regular Borel measure μ relative projectivity of $C_0(S)$-module $L_\infty(S,\mu)$ forces μ to be inner open regular with pseudocompact support.

Keywords: projective module, L_p-space, atom, normal measure, pseudocompact space.

1 Introduction

The main question of Banach homology sounds like this: what is the homological dimension of a given Banach algebra A? To solve this problem one needs to answer another question: is a given Banach A-module X projective? For many modules of analysis the answers are known. Still there are examples of classical modules of analysis for which this question is not addressed, for example the L_p-spaces. We shall regard Lebesgue's spaces as left Banach modules over the algebra of vanishing at infinity continuous functions defined on a locally compact Hausdorff space S and as modules over the algebra of bounded measurable functions. We shall give necessary and sufficient conditions for relative projectivity of these spaces. Special attention must be paid to the case of projective L_∞ modules. The reason is that one of the main results of Banach homology — the global dimension theorem [[1], proposition V.2.21] is based on the fact that the homological dimension of the module of bounded sequences over the algebra of vanishing sequences equals 2. In particular this module is not projective. As we shall see this behaviour is typical for most modules L_∞. Before we proceed to the main topic we shall give a few definitions.

Let M be a subset of a set N, then χ_M denotes the indicator function of M. If $f : N \to L$ is an arbitrary function, then $f|_M$ denotes its restriction to M. The symbol 1_M denotes the identity map on M.

Let S be an arbitrary topological space and M be its subset. Then $\mathrm{cl}_S(M)$ and $\mathrm{int}_S(M)$ denote the closure and the interior of M in S.

All Banach spaces in this paper are considered over the complex field. For a given Banach spaces X and Y by $X \oplus_1 Y$ we denote their ℓ_1-sum and by $X \widehat{\otimes} Y$ their projective tensor product. We say that a Banach space X is complemented in Y if X is a subspace of Y and there exists a bounded linear operator $P : Y \to Y$ such that $P|_X = 1_X$ and $\mathrm{Im}(P) = X$. For $1 \leq p \leq +\infty$ and a given measure space (X,μ) by $L_p(X,\mu)$ we shall denote the Banach space of equivalence classes of p-integrable (or essentially bounded if $p = +\infty$) functions on X. Elements of $L_p(X,\mu)$ are denoted by $[f]$. Note that all L_p-spaces have the approximation property.

For a given Banach algebra A by $A_+ := A \oplus_1 \mathbb{C}$ we denote its standard unitization. We shall consider only left Banach modules with contractive outer action $\cdot : A \times X \to X$. If A is a Banach algebra with the unit e, then a Banach A-module X is called unital if $e \cdot x = x$ for all $x \in X$. For a given Banach A-module X its essential part X_{ess} is the closed linear span of the set $A \cdot X$. We say that the module X is essential if $X = X_{ess}$. Clearly, any unital Banach module is essential. Let X and Y be two Banach A-modules, then a map $\phi : X \to Y$ is an A-morphism if it is a continuous A-module map. Banach A-modules and A-morphisms form a category which we denote by $A - \mathbf{mod}$.

The category A-mod has its own notion of projectivity. An A-morphism $\xi : X \to Y$ is called admissible if there exists a right inverse bounded linear operator $\eta : Y \to X$, i.e if $\xi\eta = 1_Y$. A Banach A-module P is called relatively projective if for any admissible A-morphism $\xi : X \to Y$ and any A-morphism $\phi : P \to Y$ there exists an A-morphism $\psi : P \to X$ making the diagram

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ P \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"] & Y \end{tikzcd}$$

commutative. Instead of checking by definition one may show that a Banach module P is relatively projective by constructing an A-morphism $\sigma : P \to A_+ \widehat{\otimes} P$ which is a right inverse for the canonical A-morphism $\pi_P^+ : A_+ \widehat{\otimes} P \to P : (a \oplus_1 z) \widehat{\otimes} x \mapsto a \cdot x + zx$ [[1], proposition IV.1.1]. If a Banach module P is essential then it is projective if and only if the canonical A-morphism $\pi_P : A \widehat{\otimes} P \to P : a \widehat{\otimes} x \mapsto a \cdot x$ has a right inverse A-morphism [[1], proposition IV.1.2].

2 Necessary conditions for relative projectivity

In this section we shall show that for a relatively projective A-module X its essential part is complemented and A-valued A-morphisms separate points of the essential part. These necessary conditions will play a key role in this paper.

Proposition 2.1. *Let X be a Banach A-module and E be a Banach space. Let $j_E : A_+ \widehat{\otimes} E \to (A \widehat{\otimes} E) \oplus_1 E$ denote the natural isomorphism. Then for any A-morphism $\sigma : X \to A_+ \widehat{\otimes} E$ there exist bounded linear operators $\sigma_1 : X \to A \widehat{\otimes} E$, $\sigma_2 : X \to E$ such that*

(i) $j_E(\sigma(x)) = \sigma_1(x) \oplus_1 \sigma_2(x)$ *for all* $x \in X$*;*

(ii) $\sigma_1(a \cdot x) = a \cdot \sigma_1(x) + a \widehat{\otimes} \sigma_2(x)$ *for all* $x \in X$ *and* $a \in A$*;*

(iii) $\sigma_2(a \cdot x) = 0$ *for all* $x \in X$ *and* $a \in$*.*

As a consequence, $\sigma_1|_{X_{ess}}$ is an A-morphism, $\sigma_2|_{X_{ess}} = 0$.

Proof. Consider bounded linear operators $q_1 : A_+ \widehat{\otimes} X \to A \widehat{\otimes} X : (a \oplus_1 z) \widehat{\otimes} x \mapsto a \widehat{\otimes} x$ and $q_2 : A_+ \widehat{\otimes} X \to X : (a \oplus_1 z) \widehat{\otimes} x \mapsto zx$. Now define $\sigma_1 = q_1\sigma$, $\sigma_2 = q_2\sigma$. Clearly, $j_E = q_1 \oplus_1 q_2$, hence $j_E(\sigma(x)) = \sigma_1(x) \oplus_1 \sigma_2(x)$ for all $x \in X$. Note that $a \cdot u = a \cdot q_1(u) + a \widehat{\otimes} q_2(u)$ for all $a \in A$ and $u \in A_+ \widehat{\otimes} X$. As σ is an A-morphism, it is routine to check that $\sigma_1(a \cdot x) = a \cdot \sigma_1(x) + a \widehat{\otimes} \sigma_2(x)$ and $\sigma_2(a \cdot x) = 0$ for all $a \in A$, $x \in X$. □

The following proposition is a slight generalization of [[2], lemma 1.4].

Proposition 2.2. *Let X be a relatively projective Banach A-module. Then X_{ess} is complemented in X as Banach space.*

Proof. Since X is relatively projective there exists an A-morphism $\sigma : X \to A_+ \widehat{\otimes} X$ such that $\pi_X^+\sigma = 1_X$. Let σ_1 and σ_2 be bounded linear operators given by proposition 2.1. Now consider A-morphism $\pi_X : A \widehat{\otimes} X \to X : a \widehat{\otimes} x \mapsto a \cdot x$, then for all $x \in X$ we have $x = \pi_X^+(\sigma(x)) = \pi_X(\sigma_1(x)) + \sigma_2(x)$. Consider a bounded linear operator $\eta = \pi_X\sigma_1$. Since $\sigma_2|_{X_{ess}} = 0$ we have $\eta|_{X_{ess}} = 1_X$. Moreover $\mathrm{Im}(\eta) \subset \mathrm{Im}(\pi_X) = X_{ess}$, therefore η is a bounded linear projection of X onto X_{ess}. □

Proposition 2.3. *Let A be a Banach algebra and X be a relatively projective Banach A-module. Suppose, that either A or X possess the approximation property. Then*

(i) for any non-zero $x \in X$ there exists an A-morphism $\phi : X \to A_+$

(ii) for any non-zero $x \in X_{ess}$ there exists an A-morphism $\psi : X_{ess} \to A$ such that $\psi(x) \neq 0$;

Proof. Let $i_E : E \widehat{\otimes} \mathbb{C} \to E$ be the natural isomorphism.

(i) Fix non-zero $x \in X$. Since X is relatively projective there exists an A-morphism $\sigma : X \to A_+ \widehat{\otimes} X$ such that $\pi_X^+ \sigma = 1_X$. Consider $u := \sigma(x) \in A_+ \widehat{\otimes} X$. Since $\pi_X^+(u) = x \neq 0$, then $u \neq 0$. Recall that either A or X has the approximation property, so there exist $f \in A_+^*$ and $g \in X^*$ such that $(f \widehat{\otimes} g)(u) \neq 0$ [[3], corollary I.5.1, p. 168]. Consider $a := ((1_{A_+} \widehat{\otimes} g)(u)) \in A_+ \widehat{\otimes} \mathbb{C}$ and $F := (f \widehat{\otimes} 1_{\mathbb{C}}) \in A_+^* \widehat{\otimes} \mathbb{C}$. Since $F(a) = (f \widehat{\otimes} g)(u) \neq 0$, then $a \neq 0$. Now it is routine to check that the linear operator $\xi := (1_{A_+} \widehat{\otimes} g)\sigma$ is an A-morphism. Clearly, $\xi(x) = a \neq 0$. It remains to set $\phi = i_{A_+}\xi$.

(ii) Fix a non-zero $x \in X_{ess}$. Let ξ be the morphism constructed in paragraph (i). Consider morphisms ξ_1 and ξ_2 given by proposition 2.1. Since $\xi(x) = j_{\mathbb{C}}(\xi_1(x) \oplus_1 \xi_2(x)) \neq 0$ and $\xi_2(x) = 0$, then $\xi_1(x) \neq 0$. By the same proposition $\xi_1|_{X_{ess}}$ is an A-morphism, so it remains to set $\psi = i_A \xi_1|_{X_{ess}}$.

□

3 Preliminaries on general measure theory

A comprehensive study of general measure spaces can be found in [4]. We follow its definitions.

Let X be a set. By measure we mean a countably additive set function with values in $[0, +\infty]$ defined on a σ-algebra Σ of measurable subsets of a set X. If F is a measurable set, then we have a well defined measures $\mu^F : \Sigma \to [0, +\infty] : E \mapsto \mu(E \cap F)$ and $\mu_F : \Sigma_F \to [0, +\infty] : E \mapsto \mu(E)$, where $\Sigma_F = \{E \in \Sigma : E \subset F\}$. A measurable set E is called an atom if $\mu(A) > 0$ and for every measurable subset $B \subset A$ holds either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. A measure μ is called purely atomic if every measurable set of positive measure has an atom. A measure μ is semi-finite if for any measurable set A of infinite measure there exists a measurable subset of A with finite positive measure. A family $\mathcal{D}$ of measurable subsets of finite measure is called a decomposition of X if for any measurable set E $\mu(E) = \sum_{D \in \mathcal{D}} \mu(E \cap D)$ and a set F is measurable whenever $F \cap D$ is measurable for all $D \in \mathcal{D}$. Finally, a measure μ is called decomposable if it is semi-finite and admits a decomposition of X. In fact a measure space is decomposable if and only if it is a disjoint union of finite measure spaces [[4], exercise 214X (i)]. Most measures encountered in functional analysis are decomposable.

Definition 3.1. *Let A be an atom of a measure space (X, μ). Then a measurable set $C \subset A$ is called a core of A if C is an atom and the only measurable subsets of C are $\varnothing$ and C. An atom A is called hard if it has a core. Clearly, if core exists it is unique and in this case the core is denoted by $A^\bullet$*

Proposition 3.2. *Let (X, μ) be a non-empty finite measure space such that the only set of measure zero in X is the empty set. Then (X, μ) is purely atomic and each atom is hard.*

Proof. Let E be a measurable set of positive measure. Peek any $x \in E$ and consider value $c := \inf\{\mu(F) : x \in F \in \Sigma,\ F \subset E\}$. For any $n \in \mathbb{N}$ there exists $E_n \in \Sigma$ such that $x \in E_n \subset E$

and $\mu(E_n) < c + 2^{-n}$. Define $A = \bigcap\{E_n : n \in \mathbb{N}\} \subset E$, then $x \in A \in \Sigma$ and $\mu(A) = c$. By construction A is not empty, so $\mu(A) > 0$. Suppose B is a measurable subset of A. If $x \in A \setminus B$, then $c \leq \mu(A \setminus B) \leq \mu(A) = c$, i.e. $\mu(B) = 0$. Similarly, if $x \in B$ we get $\mu(A \setminus B) = 0$. Thus $A \subset E$ is an atom. Since E is arbitrary, (X, μ) is purely atomic.

Now let A be an atom of (X, μ). If $B \in \Sigma$ and $B \subset A$, then either $\mu(B)$ or $\mu(A \setminus B) = 0$. From assumption on (X, μ) we get either B or $A \setminus B$ is empty. Thus $A^\bullet = A$. $\square$

4 Relative projectivity of $B(\Sigma)$-modules $L_p(X, \mu)$

Let (X, μ) be a measure space. By $B(\Sigma)$ we shall denote the algebra of bounded measurable functions with the sup norm. In this section we give a criterion of projectivity of $B(\Sigma)$-modules $L_p(X, \mu)$. Speaking informally all these modules look like $\ell_\infty(\Lambda)$-modules $\ell_p(\Lambda)$ for some index set Λ.

Proposition 4.1. *Let (X, μ) be a measure space. Let $1 \leq p \leq +\infty$ and $L_p(X, \mu)$ be a relatively projective $B(\Sigma)$-module. Then for any measurable set $B \subset X$ the $B(\Sigma)$-module $L_p(X, \mu^B)$ is relatively projective.*

Proof. It is easy to check that for $B(\Sigma)$-morphisms $\pi : L_p(X, \mu) \to L_p(X, \mu^B) : [f] \mapsto [f]\chi_B$ and $\sigma : L_p(X, \mu^B) \to L_p(X, \mu) : [f] \mapsto [f]$ holds $\pi\sigma = 1_{L_p(X, \mu^B)}$. In other words $L_p(X, \mu^B)$ is a retract of $L_p(X, \mu)$ in $B(\Sigma) - \mathbf{mod}$. Now the result follows from [[7], proposition VII.1.6]. $\square$

Proposition 4.2. *Let (X, μ) be a decomposable measure space and $L_p(X, \mu)$ be a relatively projective $B(\Sigma)$-module. Then (X, μ) is a disjoint union of hard atoms of finite measure.*

Proof. Let $\mathcal{D}$ be a decomposition of X onto measurable subsets of finite measure. Fix $D \in \mathcal{D}$ and denote $\nu := \mu^D$. By proposition 4.1 the $B(\Sigma)$-module $L_p(X, \nu)$ is relatively projective. Peek any $E \in \Sigma$ of positive measure ν. Since ν is finite, so is $\nu(E)$. Then $[f] := [\chi_E]$ is well defined and non-zero in $L_p(X, \nu)$. As $B(\Sigma)$ is a unital algebra the module $L_p(X, \nu)$ is essential. Now from proposition 2.3 we get a $B(\Sigma)$-morphism $\psi : L_p(X, \nu) \to B(\Sigma)$ such that $\psi([f]) \neq 0$. Therefore the set $F := a^{-1}(\mathbb{C} \setminus \{0\}) \in \Sigma$ is not empty. Note that $[f] = [f]\chi_E$, so $a = \psi([f]\chi_E) = \psi([f])\chi_E = a\chi_E$. Hence $a|_{X \setminus E} = 0$ and $F \subset E$. Consider arbitrary measurable set $A \subset F$ with ν measure zero. Then $[\chi_A]$ is zero in $L_p(X, \nu)$ and $[\chi_A] = [\chi_E]\chi_A$. Therefore $a\chi_A = \psi([\chi_E])\chi_A = \psi([\chi_E]\chi_A) = \psi([\chi_A]) = 0$. As $A \subset F$ and a is not zero at any point of F we get $A = \varnothing$. Since $F \neq \varnothing$, then from proposition 3.2 we get that the measure space (F, ν_F) has a hard atom. Thus we have shown that any measurable set E of positive ν measure has a hard atom, hence by a standard application of Zorn's lemma we get that (X, μ^D) is a disjoint union of hard atoms. The same conclusion holds for (X, μ_D). Since μ_D is finite, then so is every atom. Since D is arbitrary the result follows from [[4], exercise 214X (i)]. $\square$

Let (X, μ) be a measure space and A be a measurable set of finite positive measure. Then we have a well defined bounded linear functional $m_A : B(\Sigma) \to \mathbb{C} : a \mapsto \mu(A)^{-1} \int_A f(x) d\mu(x)$ of norm 1.

Proposition 4.3. *Let (X, μ) be a disjoint union of the family $\mathcal{A}$ of finite hard atoms. Then*

(i) the set $X^\bullet := \bigcup\{A^\bullet : A \in \mathcal{A}\}$ is measurable and $\mu(X \setminus X^\bullet) = 0$;

(ii) for any $A \in \mathcal{A}$ and any functions $a, b \in B(\Sigma)$ holds $a|_{A^\bullet} = m_{A^\bullet}(a)$ and $m_{A^\bullet}(ab) = m_{A^\bullet}(a) m_{A^\bullet}(b)$;

(iii) *for any $a \in B(\Sigma)$ there is a function $b \in B(\Sigma)$ such that $b|_{X^\bullet} = 0$ and a pointwise equality holds $a = \sum_{A\in\mathcal{A}} m_{A^\bullet}(a)\chi_{A^\bullet} + b$;*

(iv) *for any $[f] \in L_p(X,\mu)$ holds $[f] = [\sum_{A\in\mathcal{A}} m_{A^\bullet}(f)\chi_{A^\bullet}]$.*

Proof. (i) Since $\mathcal{A}$ is a decomposition of X, then (X,μ) is decomposable and $X^\bullet$ is measurable. Note that disjoint sets $A \setminus A^\bullet$ for $A \in \mathcal{A}$ are of measure zero, hence so is their union $X \setminus X^\bullet$.

(iii) Fix $a \in B(\Sigma)$ and $A \in \mathcal{A}$. Since $A^\bullet$ has only two measurable subsets, then a is constant on $A^\bullet$. Therefore $a|_{A^\bullet} = m_{A^\bullet}(a)$. As a consequence for the measurable function $b = a - \sum_{A\in\mathcal{A}} m_{A^\bullet}(a)\chi_{A^\bullet}$ we have $b|_{X^\bullet} = 0$.

(iv) The result immediately follows from paragraph (iii). □

Proposition 4.4. *Let $1 \leq p \leq +\infty$ and (X,μ) be a disjoint union of finite hard atoms. Then the $B(\Sigma)$-module $L_p(X,\mu)$ is relatively projective.*

Proof. Let $\mathcal{A}$ denote the set of hard atoms of X.

Consider the case $p = +\infty$. Define a bounded linear operator

$$\rho : L_\infty(X,\mu) \to B(\Sigma) : [f] \mapsto \sum_{A\in\mathcal{A}} m_{A^\bullet}(f)\chi_{A^\bullet}.$$

From paragraph (ii) of proposition 4.3 it follows that ρ is a $B(\Sigma)$-morphism. Therefore $\sigma = \rho \widehat{\otimes} 1_{L_\infty(X,\mu)}$ is a $B(\Sigma)$-morphism too. From paragraph (iv) of proposition 4.3 we get that $\pi_{L_\infty(X,\mu)}\sigma = 1_{L_\infty(X,\mu)}$. Since $L_\infty(X,\mu)$ is a unital $B(\Sigma)$-module, then by [[1], proposition IV.1.2] it is relatively projective.

Consider the case $1 \leq p < +\infty$. Let $[f] \in L_p(X,\mu)$, then from paragraph (iv) of proposition 4.3 we get $[f] = [\sum_{A\in\mathcal{A}} m_{A^\bullet}(f)\chi_{A^\bullet}]$. Even more, since $p < +\infty$ we have $[f] = \sum_{A\in\mathcal{A}} m_{A^\bullet}(f)[\chi_{A^\bullet}]$ in $L_p(X,\mu)$. Note that the latter sum contains only countably many non-zero summands. We denote indices of these summands as $\mathcal{A}_f$. Consider arbitrary finite subset $\mathcal{F} = \{A_1,\ldots,A_n\} \subset \mathcal{A}_f$ and denote $x_k = \chi_{A_k^\bullet}$, $y_k = m_{A_k^\bullet}(f)[\chi_{A_k^\bullet}]$ for $k \in \{1,\ldots,n\}$. Let $\omega \in \mathbb{C}$ be any n-th root of 1. Since $\mathcal{F}$ is a disjoint family $\|\sum_{k=1}^n \omega^k x_k\|_{B(\Sigma)} \leq 1$ and $\|\sum_{k=1}^n \omega^k y_k\|_{L_p(X,\mu)} \leq \|f\|_{L_p(X,\mu)}$. Therefore by [[1], proposition II.2.44] for any $f \in L_p(X,\mu)$ we have a well defined element $\sigma_f = \sum_{A\in\mathcal{A}_f} x_k \widehat{\otimes} y_k = \sum_{A\in\mathcal{A}} x_k \widehat{\otimes} y_k \in B(\Sigma) \widehat{\otimes} L_p(X,\mu)$ of norm not greater than $\|f\|_{L_p(X,\mu)}$. Now using paragraph (ii) of proposition 4.3 it is easy to check that the mapping

$$\sigma : L_p(X,\mu) \to B(\Sigma) \widehat{\otimes} L_p(X,\mu) : [f] \mapsto \sum_{A\in\mathcal{A}} m_{A^\bullet}(f)\chi_{A^\bullet} \widehat{\otimes} [\chi_{A^\bullet}]$$

is a well defined $B(\Sigma)$-morphism of norm at most 1. From paragraph (iv) of proposition 4.3 we get that $\pi_{L_p(X,\mu)}\sigma = 1_{L_p(X,\mu)}$. Since $L_p(X,\mu)$ is a unital $B(\Sigma)$-module, then by [[1], proposition IV.1.2] it is relatively projective. □

Theorem 4.5. *Let (X,μ) be a decomposable measure space and $1 \leq p \leq +\infty$. Then the following are equivalent:*

(i) *$L_p(X,\mu)$ is a relatively projective $B(\Sigma)$-module;*

(ii) *(X,μ) is a disjoint union of hard atoms of finite measure.*

Proof. The result follows from propositions 4.2 and 4.4. □

5 Preliminaries on topological measure theory

A detailed discussion of topological measures can be found in [5]. From now we shall consider measures μ defined on the σ-algebra $Bor(S)$ of Borel sets of a topological space S. By $\mathrm{supp}(\mu)$ we shall denote the support of μ. The measure μ is called:

(i) strictly positive if $\mathrm{supp}(\mu) = S$;

(ii) fully supported if $\mu(S \setminus \mathrm{supp}(\mu)) = 0$;

(iii) locally finite if every point in S has an open neighbourhood of finite measure;

(iv) inner compact regular if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}$ for any $E \in Bor(S)$;

(v) outer open regular if $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ is open}\}$ for any $E \in Bor(S)$;

(vi) inner open regular if $\mu(E) = \sup\{\mu(U) : U \subset E, U \text{ is open}\}$ for any $E \in Bor(S)$;

(vii) residual if $\mu(E) = 0$ for every Borel nowhere dense set E;

(viii) normal if it is residual and fully supported.

If μ is locally finite, then all compact sets have finite measure [[5], proposition 411G (a)]. Any finite inner compact regular measure is outer open regular [[5], proposition 411X (a)]. Clearly, μ^B is inner compact regular for any $B \in Bor(S)$ whenever μ is inner compact regular.

Proposition 5.1. *Let S be a locally compact Hausdorff space and μ be a Borel measure on S. Then*

(i) μ is inner open regular if and only if $\mu(E) = \mu(\mathrm{int}_S(E))$ for $E \in Bor(S)$;

(ii) if μ is finite and inner open regular, then $\mu(E) = \mu(\mathrm{int}_S(E)) = \mu(\mathrm{cl}_S(E))$ and μ is residual;

(iii) if μ is finite, inner compact regular and inner open regular, then μ is normal;

Proof. (i) It is enough to note that the supremum in the definition of inner open regular measure is attained at maximal open subset of E, which is $\mathrm{int}_S(E)$.

(ii) The first equality was proved in the previous paragraph. Since μ is finite for all $E \in Bor(S)$ we have $\mu(E) = \mu(S) - \mu(\mathrm{int}_S(S \setminus E)) = \mu(S) - \mu(S \setminus \mathrm{cl}_S(E)) = \mu(\mathrm{cl}_S(E))$. Now consider a nowhere dense Borel set $E \subset S$, then $\mu(E) = \mu(\mathrm{cl}_S(E)) = \mu(\mathrm{cl}_S(\mathrm{int}_S(E))) = \mu(\varnothing) = 0$. Since E is arbitrary, then μ is residual.

(iii) Any inner compact regular measure has support and is fully supported [[5], propositions 411C, 411N (d)]. The rest follows from paragraphs (i) and (ii). □

Proposition 5.2. *Let S be a locally compact Hausdorff space and μ be a Borel measure on S. Suppose that for any compact set $K \subset S$ with $\mu(K) > 0$ there exists an open set $U \subset K$ with $\mu(U) > 0$. Then*

(i) $\mu(K) = \mu(\mathrm{int}_S(K))$ for any compact set $K \subset S$;

(ii) if μ is inner compact regular then μ is inner open regular.

Proof. Denote $K' = K \setminus \mathrm{int}_S(K)$. This is a closed subset of the compact set K, hence it is compact. Suppose $\mu(K') > 0$, then there exists an open set $U \subset K'$ with $\mu(U) > 0$. As a consequence $U \subset K$ is a non-empty open set disjoint from $\mathrm{int}_S(K)$. Contradiction, hence $\mu(K') = 0$ and $\mu(K) = \mu(\mathrm{int}_S(K))$.

(ii) Fix $c < \mu(B)$. Since μ is inner compact regular, then there exists a compact set $K \subset B$ with $c < \mu(K)$. From previous paragraph we get $c < \mu(K) = \mu(\mathrm{int}_S(K)) \leq \mu(\mathrm{int}_S(B))$. As $c < \mu(B)$ is arbitrary we conclude that $\mu(B) \leq \mu(\mathrm{int}_S(B))$. The inverse inequality is obvious. □

Proposition 5.3. *Let S be a locally compact Hausdorff space and μ be a locally finite inner compact regular Borel measure on S. Let A be an atom for μ. Then*

(i) $\mu(A)$ is finite;

(ii) there exists a point $s \in A$ such that $\mu(A) = \mu(\{s\})$.

Proof. (i) Since μ is inner compact regular there exists a compact set $K \subset A$ of positive measure. Since μ is locally finite $\mu(K)$ is finite. As A is an atom and $\mu(K) > 0$, we get $\mu(A) = \mu(K) < +\infty$.

(ii) By previous paragraph $0 < \mu(A) < +\infty$. Let $\mathcal{K}$ denote the compact subsets of A that have the same measure as A. Since μ is inner compact regular there exists a compact set $K \subset A$ of positive measure. Since A is an atom we immediately get $\mu(K) = \mu(A)$, that is $K \in \mathcal{K}$. Thus $\mathcal{K}$ is not empty. Now consider two arbitrary sets $K', K'' \in \mathcal{K}$. Clearly, $C := K' \cap K''$ is a compact subset of A. Suppose that $\mu(C) = 0$, and consider $L' = K' \setminus C$, $L'' = K'' \setminus C$. These are two disjoint subsets of A, such that $\mu(L') = \mu(L'') = \mu(A)$, so $\mu(A) \geq \mu(L' \cup L'') = 2\mu(A)$. Contradiction, hence $\mu(C) > 0$ and therefore $C \in \mathcal{K}$. As $K', K'' \in \mathcal{K}$ are arbitrary we have shown that $\mathcal{K}$ is a family of compact sets with the finite intersection property. Therefore $K^* = \bigcap \mathcal{K}$ is not empty. Clearly, K^* is compact as intersection of compact sets. Suppose K^* has two distinct points s' and s''. Consider singletons $C' = \{s'\}$ and $C'' = \{s''\}$. Suppose $\mu(C') > 0$, then $\mu(C') = \mu(A)$ and $C' \in \mathcal{K}$ as A is an atom. This contradicts minimality of K^* as C' is a proper subset of K^*, so $\mu(C') = 0$. Similarly, $\mu(C'') = 0$. Consider $L = K^* \setminus (C' \cup C'')$, then $\mu(L) = \mu(K^*) = \mu(A)$. As μ is inner compact regular there exists a compact set $\hat{K} \subset L \subset A$ of positive measure, therefore $\mu(\hat{K}) = \mu(A)$. By construction $\hat{K} \in \mathcal{K}$ is a proper subset of $\mathcal{K}$. This contradicts minimality of K^*, therefore K^* is a non-empty set without two distinct points, hence a singleton. Thus $\mu(A) = \mu(K^*) = \mu(\{s\})$ for some $s \in A$. □

6 Relative projectivity of $C_0(S)$-modules $L_p(S, \mu)$

Results of this section are somewhat similar to the case of modules over the algebra of bounded measurable functions, but the case $p = +\infty$ doesn't seem to have a simple criterion.

Proposition 6.1. *Let S be a locally compact Hausdorff space, μ be a locally finite Borel measure on S and $1 \leq p \leq +\infty$. Then*

(i) $[f] \in L_p(S, \mu)_{ess}$ if and only if for any $\varepsilon > 0$ there exists a compact set $K \subset S$ such that $\|[f\chi_{S \setminus K}]\|_{L_p(S,\mu)} < \varepsilon$;

(ii) if $p < +\infty$ and μ is inner compact regular, then $L_p(S, \mu)_{ess} = L_p(S, \mu)$.

In particular, for any compact set $K \subset S$ and $[f] \in L_p(S, \mu)$ holds $[f]\chi_K \in L_p(S, \mu)_{ess}$.

Proof. The proof is a standard density argument. □

Proposition 6.2. *Let S be a locally compact Hausdorff space and μ be a locally finite Borel measure on S. Assume we are given a $C_0(S)$-morphism $\psi : L_p(S,\mu)_{ess} \to C_0(S)$ with $1 \leq p \leq +\infty$, a function $[f] \in L_p(S,\mu)$ and a compact set $K \subset S$. Then*

(i) if $[f] = [f]\chi_K$, then $\psi(f)|_{S\setminus K} = 0$;

(ii) if $[f] = [f]\chi_K$ and $\psi(f) \neq 0$, then there is an open set $U \subset K$ of positive measure.

Proof. (i) By paragraph (i) of proposition 6.1 we have $[f] = [f]\chi_K \in L_p(S,\mu)_{ess}$, so we can speak of the function $a = \psi(f) \in C_0(S)$. Let V be an open set containing K, then there exists a continuous function $b \in C_0(S)$ such that $b|_K = 1$ and $b|_{S\setminus V} = 0$ [[6], theorem 1.4.25]. By construction $\chi_K = b\chi_K$, so $a = \psi([f]) = \psi([f]\chi_K) = \psi(b[f]\chi_K) = b\psi([f]\chi_K) = b\psi([f]) = ba$. As $b|_{S\setminus V} = 0$ we get $a|_{S\setminus V} = 0$. Since S is Hausdorff and V is an arbitrary open set containing K, then $a|_{S\setminus K} = 0$.

(ii) Using notation of the previous paragraph we have $a \neq 0$ and $a|_{S\setminus K} = 0$. Consider non-negative continuous function $c = |a|$, then $c \neq 0$ and $c|_{S\setminus K} = 0$. Since $c \neq 0$, then the open set $U = c^{-1}((0,+\infty))$ is non-empty. Moreover, $U \subset K$ as $c|_{S\setminus K} = 0$. Now peek any $s \in U$. By construction $a(s) \neq 0$. Since $\{s\}$ is a compact set there exists a continuous function $e \in C_0(S)$ such that $e(s) = 1$ and $e|_{S\setminus U} = 0$ [[6], theorem 1.4.25]. Consider function $[g] = e[f] \in L_p(S,\mu)_{ess}$, then $\psi([g]) \neq 0$ as $\psi([g])(s) = \psi(e[f])(s) = (e\psi([f]))(s) = e(s)\psi([f])(s) = a(s) \neq 0$. Since $\psi([g]) \neq 0$, we have $[g] \neq 0$ in $L_p(S,\mu)_{ess}$. Therefore $\mu(U) > 0$ because by construction $[g]\chi_{S\setminus U} = 0$. □

Proposition 6.3. *Let S be a locally compact Hausdorff space and μ be a locally finite inner compact regular Borel measure on S. Let $1 \leq p \leq +\infty$ and $L_p(S,\mu)$ is a relatively projective $C_0(S)$-module. Then,*

(i) μ is inner open regular;
(ii) any atom of μ is an isolated point in S;
(iii) if $p < +\infty$ and μ is outer open regular then μ is purely atomic.

Proof. (i) Let $K \subset S$ be a compact set with of positive measure. Then by paragraph (i) of proposition 6.1 the function $[f] := [\chi_K]$ is non-zero in $L_p(S,\mu)_{ess}$. Since $C_0(S)$-module $L_p(S,\mu)$ is relatively projective, then by paragraph (ii) of proposition 2.3 there exists a $C_0(S)$-morphism $\psi : L_p(S,\mu) \to C_0(S)$ such that $\psi([f]) \neq 0$. Now from paragraph (ii) of proposition 6.2 we get that there exists an open set $U \subset K$ with $\mu(U) > 0$. Since $K \subset S$ is arbitrary we are in position to apply paragraph (ii) of proposition 5.2. Hence $\mu(B) = \mu(\mathrm{int}_S(B))$ for any Borel set $B \subset S$. It remains to apply proposition 5.1.

(ii) Let A be an atom of μ. By paragraph (ii) of proposition 5.3 there exists a point $s \in A$ such that $\mu(\{s\}) = \mu(A) > 0$. From paragraph (i) it follows that $\mu(\mathrm{int}_S(\{s\})) = \mu(\{s\}) > 0$. Therefore $\{s\}$ is an open set, i.e. s is an isolated point.

(iii) Let S_a^μ be the set of singleton atoms of μ and $S_c^\mu = S \setminus S_a^\mu$. By paragraph (ii) all atoms are isolated, so S_c^μ is closed and hence Borel. Consider arbitrary compact subset $K \subset S_c^\mu$. Suppose $\mu(K) > 0$, then by paragraph (i) of proposition 6.1 the function $[f] := [\chi_K]$ is non-zero in $L_p(S,\mu)_{ess}$. As $C_0(S)$-module $L_p(S,\mu)$ is relatively projective by paragraph (ii) of propositions 2.3 and proposition 6.2 we have a $C_0(S)$-morphism $\psi : L_p(S,\mu)_{ess} \to C_0(S)$ such that $\psi([f]) \neq 0$ and $\psi([f])|_{S\setminus K} = 0$. Denote $a := \psi([f]) \neq 0$. Since $a|_{S\setminus K} = 0$, there exists a point $s \in K$ such that $a(s) \neq 0$. Fix $\varepsilon > 0$. Note that s is not an atom because $s \in K \subset S_c^\mu$, hence from outer open regularity of μ we get an open set $W \subset S$ such that $s \in W$ and $\mu(W) < \varepsilon$. Since $\{s\}$ is compact, then there exists a continuous function $b \in C_0(S)$ such that $b(s) = 1$, $0 \leq b \leq 1$ and $b|_{S\setminus W} = 0$ [[6], theorem 1.4.25]. As $p < +\infty$ we get $\|b[f]\|_{L_p(S,\mu)} \leq \mu(W \cap K)^{1/p} < \varepsilon^{1/p}$. Finally,

$$|a(s)| = |a(s)b(s)| = |(ba)(s)| = |(b\psi([f]))(s)| = |\psi(b[f])(s)| \leq \|\psi(b[f])\|_{C_0(S)} \leq$$

$$\leq \|\psi\|\|b[f]\|_{L_p(S,\mu)} \leq \|\psi\|\varepsilon^{1/p}$$

Since $\varepsilon > 0$ is arbitrary $|a(s)| = 0$, but $a(s) \neq 0$ by choice of s. Contradiction, hence $\mu(K) = 0$. As $K \subset S_c^\mu$ is arbitrary from inner compact regularity of μ we get $\mu(S_c^\mu) = 0$. In other words μ is purely atomic. $\square$

Proposition 6.4. *Let S be a locally compact Hausdorff space and μ be a Borel measure on S. Let $1 \leq p \leq +\infty$ and $L_p(S,\mu)$ be a relatively projective $C_0(S)$-module. Then for any Borel set $B \subset S$ the $C_0(S)$-module $L_p(S,\mu^B)$ is relatively projective.*

Proof. The proof is the same as in proposition 4.1. $\square$

Theorem 6.5. *Let S be a locally compact Hausdorff space and μ be a decomposable inner compact regular Borel measure on S. Let $1 \leq p < +\infty$. Then the following are equivalent:*

(i) $L_p(S,\mu)$ is a relatively projective $C_0(S)$-module;

(ii) μ is purely atomic and all atoms are isolated points.

Proof. $(i) \implies (ii)$ Let $\mathcal{D}$ be a decomposition of S onto Borel subsets of finite measure. Fix $D \in \mathcal{D}$ and consider $C_0(S)$-module $L_p(S,\mu^D)$. Since D has finite measure then μ^D is finite, inner compact regular and outer open regular. By proposition 6.4 the $C_0(S)$-module $L_p(S,\mu^D)$ is relatively projective. So from paragraph (iii) of proposition 6.3 we get that μ^D (and a fortiori μ_D) is purely atomic and all atoms are isolated points. Since $D \in \mathcal{D}$ is arbitrary, by proposition [[4], exercise 214X (i)] the measure μ is purely atomic and all atoms are isolated points.

$(ii) \implies (i)$ Let S_a^μ denote the set of singleton atoms of μ. Since all points in S_a^μ are isolated, then S_a^μ is discrete and $C_0(S_a^\mu)$ is biprojective [[1], theorem 4.5.26]. Since $p < +\infty$, μ is purely atomic and all atoms are isolated points the $C_0(S_a^\mu)$-module $L_p(S,\mu)$ is essential. Bearing all this in mind from [[7], proposition VII.1.60(II)] we get that the $C_0(S_a^\mu)$-module $L_p(S,\mu)$ is relatively projective. Since S_a^μ is open in S, then $C_0(S_a^\mu)$ is a two-sided closed ideal of $C_0(S)$. Now by [[8], proposition 2.3.3(i)] the $C_0(S)$-module $L_p(S,\mu)$ is relatively projective. $\square$

The case of $C_0(S)$-module $L_\infty(S,\mu)$ is much harder. We shall only give two necessary but quite restrictive conditions for relative projectivity.

Definition 6.6. *Let S be a locally compact Hausdorff space and μ be a Borel measure on S. A family $\mathcal{F}$ of Borel subsets of S is called wide if*

(i) each element of $\mathcal{F}$ has finite positive measure and is contained in some compact subset;

(ii) any compact subset of S intersects only finitely many sets of $\mathcal{F}$;

(iii) any two distinct sets in $\mathcal{F}$ do not intersect.

Proposition 6.7. *Let S be a locally compact Hausdorff space and μ be a Borel measure on S. If S admits an infinite wide family $\mathcal{F}$ then the essential part of $C_0(S)$-module $L_\infty(S,\mu)$ is not complemented in $L_\infty(S,\mu)$.*

Proof. Assume that $L_\infty(S,\mu)_{ess}$ is complemented in $L_\infty(S,\mu)$, then there exists a bounded linear operator $P : L_\infty(S,\mu) \to L_\infty(S,\mu)_{ess}$ such that $P([f]) = [f]$ for all $[f] \in L_\infty(S,\mu)_{ess}$. Now given a wide family $\mathcal{F} = (F_\lambda)_{\lambda\in\Lambda}$ we define a bounded linear operator

$$I : \ell_\infty(\Lambda) \to L_\infty(S,\mu) : x \mapsto \left[\sum_{\lambda\in\Lambda} x_\lambda \chi_{F_\lambda}\right]$$

which is a well because $\mathcal{F}$ is a disjoint famliy. Consider $x \in c_0(\Lambda)$. Fix $\varepsilon > 0$, then there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $|x_\lambda| < \varepsilon$ for all $\lambda \in \Lambda \setminus \Lambda_0$. Let K_λ denote the compact set containing F_λ for $\lambda \in \Lambda$. Then $K_0 = \bigcup_{\lambda\in\Lambda_0} K_\lambda$ is a compact set. If $s \in S \setminus K$, then $\chi_{F_\lambda}(s) = 0$ for all $\lambda \in \Lambda \setminus \Lambda_0$. Therefore $\|I(x)\chi_{S\setminus K}\|_{L_\infty(S,\mu)} = \|[\sum_{\lambda\in\Lambda\setminus\Lambda_0} x_\lambda \chi_{F_\lambda}]\|_{L_\infty(S,\mu)} = \sup_{\lambda\in\Lambda\setminus\Lambda_0} |x_\lambda| < \varepsilon$. Now by paragraph (i) of proposition 6.1 we get that $I(x) \in L_\infty(S,\mu)_{ess}$. Next we define a bounded linear operator

$$R : L_\infty(S,\mu) \to c_0(\Lambda) : [f] \mapsto \left(\lambda \mapsto \mu(F_\lambda)^{-1} \int_{F_\lambda} f(s)d\mu(s)\right)$$

The only thing that needs clarification is the fact that R has range in $c_0(\Lambda)$. Fix $[f] \in L_\infty(S,\mu)_{ess}$. Let $\varepsilon > 0$. By paragraph (i) of proposition 6.1 there exists a compact set $K \subset S$ such that $\|[f]\chi_K\|_{L_\infty(S,\mu)} < \varepsilon$. Consider set $\Lambda_K = \{\lambda \in \Lambda : K \cap F_\lambda \neq \varnothing\}$. By definition of $\mathcal{F}$ the set Λ_K is finite. For any $\lambda \in \Lambda \setminus \Lambda_K$ holds $F_\lambda \cap K = \varnothing$, so $|R(f)_\lambda| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary $R([f]) \in c_0(\Lambda)$. Now we define a bounded linear operator $Q = RPI$. Recall that $P([f]) = [f]$ for all $[f] \in L_\infty(S,\mu)_{ess}$. Then it is easy to check that for all $x \in c_0(\Lambda)$ and $\lambda \in \Lambda$ holds $Q(x)_\lambda = x_\lambda$. Thus $Q : \ell_\infty(\Lambda) \to c_0(\Lambda)$ is a bounded linear operator such that $Q(x) = x$ for all $x \in c_0(\Lambda)$. Since Λ is infinite we get a contradiction with Phillips' theorem [9]. Therefore, $L_\infty(S,\mu)_{ess}$ is not complemented in $L_\infty(S,\mu)$. □

Now we need to remind some notions from general topology. A family $\mathcal{F}$ of subsets of a topological space S is called locally finite if every point of S has an open neighbourhood that intersects only finitely many sets in $\mathcal{F}$. A topological space S is called pseudocompact if every locally finite family of non-empty open sets is finite.

Proposition 6.8. *Let S be an locally compact Hausdorff space and μ be a locally finite Borel measure on S. If $L_\infty(S,\mu)$ is relatively projective as $C_0(S)$-module then* $\operatorname{supp}(\mu)$ *is pseudocompact.*

Proof. Denote $M := \operatorname{supp}(\mu)$. Assume that M is not pseudocompact, then there is an infinite disjoint family $\mathcal{U}$ of non-empty open sets in M which is locally finite. Since S is locally compact for each $U \in \mathcal{U}$ we can choose a non-empty open set V_U and a compact set K_U such that $V_U \subset K_U \subset U$. We may choose V such that $\mu(V)$ is finite as μ is locally finite. Even more, $\mu(V) > 0$ since V is an open subset of M. Clearly, the family $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$ is infinite disjoint and locally finite. Hence for any $s \in S$ there exists on open set W_s such that $s \in W_s$ and the set $\{V \in \mathcal{V} : V \cap W_s \neq \varnothing\}$ is finite.

By construction $\mathcal{V}$ is a disjoint family of sets of positive finite measure each contained in some compact set. Let $K \subset S$ be an arbitrary compact set. Then $\{W_s : s \in K\}$ is an open cover of K. Since K is compact there exists a finite set S_0 such that $\{W_s : s \in S_0\}$ is a cover for K. Since each W_s intersects only finitely many sets of $\mathcal{V}$ then so does $\bigcup_{s\in S_0} W_s$ and a fortiori so does K. Thus $\mathcal{V}$ is a wide family. By proposition 6.7 the essential part of the $C_0(S)$-module $L_\infty(S,\mu)$ is not complemented in $L_\infty(S,\mu)$. Now from proposition 2.2 it follows that $L_\infty(S,\mu)$ is not a relatively projective $C_0(S)$-module. Contradiction, so M is pseudocompact. □

Theorem 6.9. *Let S be a locally compact Hausdorff space and μ be a locally finite inner compact regular Borel measure on S. If $L_\infty(S,\mu)$ is relatively projective as $C_0(S)$-module then μ is inner open regular and* $\operatorname{supp}(\mu)$ *is pseudocompact.*

Proof. The result follows from propositions 6.3 and 6.8. □

Even though the last theorem is not a criterion we shall say a few words on how that hypothetic criterion may look like. The last theorem puts restrictions on the topology of underlying space S, but it cannot determine it completely. Indeed, consider arbitrary locally compact Hausdorff space S with at least one isolated points $\{s\}$. Let μ be a point mass measure at $\{s\}$. It is easy to check that the resulting $C_0(S)$-module $L_\infty(S, \mu)$ is relatively projective. Thus we shall restrict our attention to strictly positive measures.

If μ is a strictly positive measure, then under assumptions of proposition 6.9 the space S is pseudocompact. Recall that every continuous function on a pseudocompact space is bounded [[10], theorem 1.1.3(3)]. Now note that any finite inner open regular measure is residual, then by result of [[11], corollary 2.7] every measurable function on S is continuous at open dense set. These facts suggest that S must have a peculiar topology. Indeed, if a space S has no isolated points and admits a non-zero finite normal measure then S cannot be separable locally compact Hausdorff [[6], proposition 4.7.20], locally connected locally compact Hausdorff [[6], proposition 4.7.23], connected locally compact Hausdorff F-space [[6], proposition 4.7.24], separable metrizable [[12], example 1].

From previous discussion it is tempting to say that $C_0(S)$ "looks like" $L_\infty(S, \mu)$ whenever μ is strictly positive and $L_\infty(S, \mu)$ is relatively projective. In this direction we have the following result.

Proposition 6.10. *Let S be a hyper-Stonean space and μ be a finite strictly positive normal inner compact regular Borel measure on S. Then $C_0(S)$-module $L_\infty(S, \mu)$ is relatively projective.*

Proof. By [[6], corollary 4.7.6] spaces $L_\infty(S, \mu)$ and $C_0(S)$ are isomorphic as C^*-algebras. In particular $L_\infty(S, \mu)$ is isomorphic to $C_0(S)$ as $C_0(S)$-module. Since S is compact then $C_0(S)$ is a unital algebra, so it is relatively projective as $C_0(S)$-module [[7], example VII.1.1]. $\square$

For strictly positive measures previous proposition is the only known example of relatively projective $C_0(S)$-module $L_\infty(S, \mu)$.

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