Norbert Nemesh

Metric and topological homology. Projectivity, injectivity and flatness

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Never trust any result proved after 11 PM.

Professional secret

Abstract

In this book we study metric and topological versions of projectivity injectivity and flatness of Banach modules over Banach algebras. These two non-standard versions of Banach homology theories are studied in parallel under unified approach.

Chapter 1 gives background required for our studies. In paragraph 1.1 we collect all necessary facts on category theory, topology and measure theory. Paragraph 1.2 contains a brief introduction into Banach structures. Here we discuss Banach spaces, Banach algebras and Banach modules. In paragraph 1.3 we give a short introduction into relative Banach homology.

In chapter 2 we establish general properties of projective injective and flat modules. Sometimes we give complete characterizations of such modules. Results of this chapter are extensively used later when dealing with specific modules of analysis. Let us discuss contents of this chapter in more detail. In paragraph 2.1 we derive basic properties of projective injective and flat modules. We also study different constructions that preserve homological triviality of Banach modules. These results are used to characterize projectivity and flatness of cyclic modules. We also give necessary conditions for projectivity of left ideals of Banach algebras. As a consequence we describe projective ideals of commutative Banach algebras that admit bounded approximate identities. Paragraph 2.2 is devoted to Banach geometric properties of homologically trivial modules. We characterize projective injective and flat annihilator modules and establish their strong relation to projective injective and flat Banach spaces. Then we give several examples confirming that homologically trivial module and its Banach algebra have similar Banach geometric properties. Examples include the property of being an $\mathscr{L}_1^g$-space, the Dunford-Pettis property and the l.u.st. property. Paragraph 2.3 is quite short. Here we list conditions under which projectivity injectivity and flatness are preserved under transition between modules over algebra to modules over ideal. Finally, we give a necessary and sufficient conditions of topological flatness of a Banach module and necessary condition of injectivity of two-sided ideals.

In chapter 3 we apply general results to specific modules of analysis. In paragraph 3.1 we investigate projectivity injectivity and flatness of ideals of C^*-algebras. We describe projective left ideals of C^*-algebras and give a criterion for injectivity of AW^*-algebras. These characterizations are indispensable in description of homologically trivial modules over algebras of bounded and compact operators on a Hilbert space. We perform similar research for commutative case of algebras of bounded and vanishing functions on discrete sets and locally compact Hausdorff spaces. In paragraph 3.2 we proceed to study

standard modules of harmonic analysis. Due to specific Banach geometric structure of convolution algebra and measure algebra we easily show that most of standard modules of harmonic analysis are homologically non-trivial. The most intriguing result of the paragraph is a lack of projectivity of convolution algebra of a non-discrete group.

Contents

Chapter 1

Preliminaries

In what follows, we present some parts in parallel fashion by listing the respective options in order, enclosed and separate like this: $\langle \ \dots \ / \ \dots \ \rangle$. For example: a real number x is $\langle$ positive / non-negative $\rangle$ if $\langle \ x > 0 \ / \ x \geq 0 \ \rangle$. Sometimes one of the parts might be empty. We use symbol $:=$ for equality by definition.

We use the following standard notation for some commonly used sets of numbers: $\mathbb{C}$ denotes the complex numbers, $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_n$ denotes the set of first n natural numbers, $\mathbb{R}_+$ denotes the set of non-negative real numbers, $\mathbb{T}$ denotes the set of complex numbers of modulus 1, finally, $\mathbb{D}$ denotes the set of complex numbers with modulus less than 1. For $z \in \mathbb{C}$ the symbol $\overline{z}$ stands for the complex conjugate number.

For a given map $f : M \to M'$ and subset $\langle \ N \subset M \ / \ N' \subset M'$ such that $\mathrm{Im}(f) \subset N' \ \rangle$ by $\langle \ f|_N \ / \ f|^{N'} \ \rangle$ we denote the $\langle$ restriction of f onto N / corestriction of f onto N' $\rangle$, that is $\langle \ f|_N : N \to M', \ x \mapsto f(x) \ / \ f|^{N'} : M \to N', \ x \mapsto f(x) \ \rangle$. The indicator function of a subset N of the set M is denoted by χ_N, so that $\chi_N(x) = 1$ for $x \in N$ and $\chi_N(x) = 0$ for $x \in M \setminus N$. We also use the shortcut $\delta_x = \chi_{\{x\}}$ where $x \in M$. By $\mathcal{P}(M)$ we denote the set of all subsets of M, and $\mathcal{P}_f(M)$ stands for the set of all finite subsets of M. The symbol M^N stands for the set of all functions from N to M. By $\mathrm{Card}(M)$ we denote the cardinality of M. By $\aleph_0$ we denote the cardinality of $\mathbb{N}$.

1.1 Broad foundations

1.1.1 Categorical language

Here we recall some basic facts and definitions from category theory and fix notation we shall use. We assume that our reader is familiar with such basics of category theory as category, functor, morphism. Otherwise, see [[1], chapter 0] for a quick introduction or [[2], chapter 1] for more details.

For a given category $\mathbf{C}$ by $\mathrm{Ob}(\mathbf{C})$ we denote the class of its objects. The symbol $\mathbf{C}^o$ stands for the opposite category. For given objects X and Y by $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ we denote the set of morphisms from X to Y. Often we shall write $\phi : X \to Y$ instead of $\phi \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$. A morphism $\phi : X \to Y$ is called $\langle$ retraction / coretraction $\rangle$ if it has a $\langle$ right / left $\rangle$ inverse morphism. A morphism is called an isomorphism if it is both a retraction and a coretraction. Usually we shall express existence of isomorphism between X and Y as $X \underset{\mathbf{C}}{\cong} Y$. We say that two morphisms $\phi : X_1 \to Y_1$ and $\psi : X_2 \to Y_2$ are equivalent in $\mathbf{C}$ if there exist isomorphisms $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ such that $\beta\phi = \psi\alpha$.

The first obvious example of a category that comes to mind is the category of all sets and all maps between them. We denote this category by $\mathbf{Set}$. Other examples will be given later. Two main examples of functors that any category has are functors of morphisms. For a fixed $X \in \mathrm{Ob}(\mathbf{C})$ we define covariant and contravariant functors

$$\mathrm{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Set},\ Y \mapsto \mathrm{Hom}_{\mathbf{C}}(X, Y), \phi \mapsto (\psi \mapsto \phi\psi),$$

$$\mathrm{Hom}_{\mathbf{C}}(-, X) : \mathbf{C} \to \mathbf{Set},\ Y \mapsto \mathrm{Hom}_{\mathbf{C}}(Y, X), \phi \mapsto (\psi \mapsto \psi\phi).$$

This construction has its reminiscent analogs in many categories of mathematics with slight modification of categories between which these functors act.

We say that two covariant functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{C} \to \mathbf{D}$ are isomorphic if there exists a class of isomorphisms $\{\eta_X : X \in \mathrm{Ob}(\mathbf{C})\}$ in $\mathbf{D}$ (called natural isomorphisms), such that $G(f)\eta_X = \eta_Y F(f)$ for all morphisms $f : X \to Y$. In this case we simply write $F \cong G$. A $\langle$ covariant / contravariant $\rangle$ functor $F : \mathbf{C} \to \mathbf{D}$ is called representable by object X if $\langle$ $F \cong \mathrm{Hom}_{\mathbf{C}}(X, -)$ / $F \cong \mathrm{Hom}_{\mathbf{C}}(-, X)$ $\rangle$. If a functor is representable, then its representing object is unique up to an isomorphism in $\mathbf{C}$.

Constructions of categorical product and coproduct shall play an important role in this book. We say that X is a $\langle$ product / coproduct $\rangle$ of the family of objects $\{X_\lambda : \lambda \in \Lambda\}$ if the functor $\langle$ $\prod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathbf{C}}(-, X_\lambda) : \mathbf{C} \to \mathbf{Set}$ / $\prod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathbf{C}}(X_\lambda, -) : \mathbf{C} \to \mathbf{Set}$ $\rangle$ is representable by object X. As a consequence we get that a $\langle$ product / coproduct $\rangle$, if it exists, is unique up to an isomorphism. Later we shall give examples of the $\langle$ products / coproducts $\rangle$ in different categories of functional analysis.

1.1.2 Topology

Let (S, τ) be a topological space. Elements of τ are called open sets, and their complements are called closed sets. Let E be an arbitrary subset of S. By $\mathrm{cl}_S(E)$ we denote the closure of E in S, that is the smallest closed set that contains E. Similarly, by $\mathrm{int}_S(E)$ we denote the interior of E, that is the largest open set that contained in E. We say that E is a neighborhood of point $s \in S$ if $s \in \mathrm{int}_S(E)$. We say that a set E is dense in a set $F \subset S$ if $F \subset \mathrm{cl}_S(E)$. Note that E can be regarded as a topological space, if we endow it with the subspace topology which equals $\{U \cap E : U \in \tau\}$. Topological space is called Hausdorff if any two distinct points have disjoint open neighborhoods. In this book we shall work with Hausdorff spaces only.

A map $f : X \to Y$ between topological spaces is called continuous if preimage under f of any open set is open. By **Top** we denote the category of topological spaces with continuous maps in the role of morphisms. Isomorphisms in **Top** are called homeomorphisms. The category **Top** admits products. For a given family of topological spaces $\{S_\lambda : \lambda \in \Lambda\}$ their product is the Tychonoff product $\prod_{\lambda\in\Lambda} S_\lambda$, that is the Cartesian product of the family $\{S_\lambda : \lambda \in \Lambda\}$ with the coarsest topology making all natural projections $p_\lambda : \prod_{\lambda\in\Lambda} S_\lambda \to S_\lambda$ continuous. The coproduct in **Top** is defined in a straightforward way. Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of topological spaces, then their coproduct $\coprod_{\lambda\in\Lambda} S_\lambda$ is the disjoint union $\bigsqcup_{\lambda\in\Lambda} S_\lambda$ with the strongest topology making all natural inclusions $i_\lambda : S_\lambda \to \coprod_{\lambda\in\Lambda} S_\lambda$ continuous.

Now we need to recall the notion of a cover. Let $\mathcal{E}$ be a family of subsets of topological space S. We say that $\mathcal{E}$ is a cover if its union equals S. We say that cover is open if all its elements are open sets. A cover is called locally finite if any point of S has a neighborhood that intersects with only finitely many elements of the cover. We say that cover $\mathcal{E}_1$ is inscribed into cover $\mathcal{E}_2$ if any element of $\mathcal{E}_1$ is a subset of some element of $\mathcal{E}_2$. A cover $\mathcal{E}_1$ is called a subcover of $\mathcal{E}_2$ if $\mathcal{E}_1 \subset \mathcal{E}_2$. Finally, a topological space is called

(i) compact if any open cover admits a finite open subcover;

(ii) paracompact if any open cover is inscribed into some locally finite open cover;

(iii) pseudocompact if any locally finite family of open sets is finite;

(iv) locally compact if any point has a compact neighborhood;

(v) extremely disconnected spaces if the closure of any open set is open;

(vi) Stonean if it is an extremely disconnected Hausdorff compact space.

The property of being $\langle$ compact / paracompact / locally compact $\rangle$ space is preserved by $\langle$ closed / closed / open and closed $\rangle$ subspaces. A topological space is pseudocompact iff any continuous function on this space is bounded. Any non-compact locally compact Hausdorff space S can be regarded as dense subspace of some compact Hausdorff space, which is called a compactification of S. There is the smallest and the largest such compactification. The smallest one is called the Alexandroff's compactification αS. By definition $\alpha S := S \cup \{S\}$. A subset of αS is called open if it is an open subset of S or has the form $\{S\} \cup S \setminus K$ for some compact set $K \subset S$. The largest compactification βS is called the Stone-Cech compactification. It can be represented as the image of the embedding $j : S \to \prod_{f \in C}[0,1] : s \mapsto \prod_{f \in C} f(s)$, where C is a set of all continuous maps from S to $[0,1]$. The Stone-Cech compactification is highly non-constructive. Even $\beta\mathbb{N}$ has no explicit description, though it is known that $\beta\mathbb{N}$ is an extremely disconnected Hausdorff compact.

Occasionally we shall apply the Urysohn's lemma to locally compact Hausdorff spaces. It states that for any compact subset K of an open set V in a locally compact Hausdorff space S there exists a continuous function $f : S \to [0,1]$ such that $f|_K = 1$ and $f|_{S \setminus V} = 0$.

For more details on topological spaces see comprehensive treatise [3].

1.1.3 Filters, nets and limits

We will use two generalizations of the notion of a sequence and a limit of a sequence.

Let M be an arbitrary set. A family $\mathfrak{F}$ of subsets of the set M is called a filter on M if $\mathfrak{F}$ doesn't contain the empty set, $\mathfrak{F}$ is closed under finite intersections and $\mathfrak{F}$ contains all supersets of its elements. In general filters are too large to be described explicitly. To overcome this difficulty we shall use filter bases. A non-empty family $\mathfrak{B}$ of subsets of the set M is called a filter base on a set M if $\mathfrak{B}$ doesn't contain the empty set and intersection of any two elements of $\mathfrak{B}$ contains some element of $\mathfrak{B}$. Given a filter base $\mathfrak{B}$ we can construct a filter by adding to $\mathfrak{B}$ all supersets of elements of $\mathfrak{B}$.

We say that filter $\mathfrak{F}_1$ dominates filter $\mathfrak{F}_2$ if $\mathfrak{F}_2 \subset \mathfrak{F}_1$. Therefore, the set of all filters on a given set a is partially ordered set. Filters that are maximal with respect to this order are called ultrafilters. An easy application of Zorn's lemma gives that any filter is dominated by some ultrafilter.

Let $\mathfrak{F}$ be a filter on the set M, and $\phi : M \to S$ be a map from M to the Hausdorff topological space S. We say that x is a limit of ϕ along $\mathfrak{F}$ and write $x = \lim_{\mathfrak{F}} \phi(m)$ if for every open neighborhood U of x holds $\phi^{-1}(U) \in \mathfrak{F}$. Directly from the definition it follows that if ϕ has a limit along $\mathfrak{F}$ then it has the same limit along any filter that dominates $\mathfrak{F}$.

Limit along filter preserve order structure of $\mathbb{R}$. More precisely: if two functions $\phi : M \to \mathbb{R}$ and $\psi : M \to \mathbb{R}$ have limits along filter $\mathfrak{F}$ and $\phi \leq \psi$, then $\lim_{\mathfrak{F}} \phi(m) \leq \lim_{\mathfrak{F}} \psi(m)$.

Limits along filters respect continuous functions. Rigorously this formulates as follows. Assume for each $\lambda \in \Lambda$ a function $\phi_\lambda : M \to S_\lambda$ has a limit along filter $\mathfrak{F}$, then for any continuous function $g : \prod_{\lambda \in \Lambda} S_\lambda \to Y$ holds

$$\lim_{\mathfrak{F}} g\left(\prod_{\lambda \in \Lambda} \phi_\lambda(m)\right) = g\left(\prod_{\lambda \in \Lambda} \lim_{\mathfrak{F}} \phi_\lambda(m)\right).$$

In particular, limits along filters are linear and multiplicative. Just like ordinary sequences.

The most important feature of filters and the reason of our interest is the following: if $\mathfrak{U}$ is an ultrafilter on the set M and $\phi : M \to K$ is a function with values in the compact Hausdorff space K, then $\lim_{\mathfrak{U}} \phi(m)$ exists. In particular, we always can speak of limits along ultrafilters of bounded scalar valued functions.

Another approach to the generalization of the notion of the limit is a limit of a net. A directed set is a partially ordered set $(N, \leq)$ in which any two elements have an upper bound. Every directed set gives rise to the so-called section filter, whose filter base consists of so-called sections $\{\nu' : \nu \leq \nu'\}$ for some $\nu \in N$. Any function $x : N \to X$ from a directed set $(N, \leq)$ into a topological space X is called a net. Usually it is denoted as $(x_\nu)_{\nu \in N}$ to allude to sequences. A limit of the net $x : N \to X$ is a limit of the function x along section filter of the directed set N. It is denoted $\lim_\nu x_\nu$. We shall exploit both notions of the limit.

More on this matters can be found in [[4], section 7].

1.1.4 Measure theory

A family Σ of subsets of the set Ω is called a σ-algebra if it contains an empty set, complements of all its elements and is closed under countable unions. If Σ is a σ-algebra of subsets of Ω we call (Ω, Σ) a measurable space. Elements of Σ are called measurable sets.

A function $\mu : \Sigma \to [0, +\infty]$ such that:

(i) $\mu(\varnothing) = 0$;

(ii) $\mu\left(\bigcup\limits_{n\in\mathbb{N}} E_n\right) = \sum\limits_{n\in\mathbb{N}} \mu(E_n)$ for any family of disjoint sets $(E_n)_{n\in\mathbb{N}}$ in Σ;

is called a measure. The triple (Ω, Σ, μ) is called a measure space. If μ attains only finite values we may drop the first condition. The second condition is essential and called the σ-additivity. The simplest example of measure space is an arbitrary set Λ with σ-algebra of all subsets and so-called counting measure $\mu_c : \mathcal{P}(\Lambda) \to [0, +\infty]$. By definition $\mu_c(E)$ equals $\mathrm{Card}(E)$ if E is finite and $+\infty$ otherwise. If E is a measurable set, by $\Sigma|_E$ we denote the σ-algebra $\{F \cap E : F \in \Sigma\}$ and by $\mu|_E$ we denote the restriction of μ to $\Sigma|_E$. A set E in Ω is called negligible if there exists a measurable set F of measure 0 that contains E. Similarly, a set E in Ω is conegligible if $\Omega \setminus E$ is negligible. Let P be some property that depends on points of Ω. We say that P holds almost everywhere if the set where P is violated is negligible. A measure space is called σ-finite if there exists a countable family of measurable sets of finite measure whose union is the whole space. The class of σ-finite spaces is enough for most applications, but we shall encounter a more generic measure spaces.

A measurable space (Ω, Σ, μ) is called strictly localizable if there exists a family of disjoint measurable sets $\{E_\lambda : \lambda \in \Lambda\}$ of finite measure such that:

(i) $\bigcup_{\lambda\in\Lambda} E_\lambda = \Omega$;

(ii) E is measurable iff $E \cap E_\lambda$ is measurable for all $\lambda \in \Lambda$;

(iii) for any measurable set E holds $\mu(E) = \sum_{\lambda\in\Lambda} \mu(E \cap E_\lambda)$.

The class of strictly localizable measure spaces is huge. It includes all σ-finite measure spaces, their arbitrary unions, Haar measures of locally compact groups, counting measures and much more. In what follows we shall consider only strictly localizable measure spaces.

We shall exploit a more detailed classification of measure spaces. We say that a measurable set E is an atom if $\mu(E) > 0$ and for any measurable subset F of E either F or $E \setminus F$ is negligible. Directly from the definition it follows that that all atoms of strictly localizable measure spaces are of finite measure. In general an atom may not be a mere singleton.

We say that a measure space is non-atomic if there are no atoms for its measure. A measure space is called purely atomic if every measurable set of positive measure contains an atom. A straightforward application of Zorn's lemma gives that a purely atomic measure space can be represented as disjoint union of some family of atoms. This family is countable if measure space is σ-finite. These facts allow us to say that the structure of purely atomic measure space is well understood. The structure of strictly localizable non-atomic measure spaces is given by Maharam's theorem [[5], 332B].

For completeness, we shall say a few words on constructions with measures. By $\mu_1 \times \mu_2$ we denote the product measure of two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. Formal definition of product measure for localizable measure spaces is rather involved [[5], definition 251F] and we don't give it here. For our purposes it is enough to know that the product of two strictly localizable measure spaces is again strictly localizable [[5], proposition 251N]. By direct sum of measure spaces $\{(\Omega_\lambda, \Sigma_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ we denote the disjoint union of sets $\{\Omega_\lambda : \lambda \in \Lambda\}$ with σ-algebra defined as $\Sigma = \{E \subset \Omega : E \cap E_\lambda \in \Sigma_\lambda \text{ for all } \lambda \in \Lambda\}$ and measure given by the formula $\mu(E) = \sum_{\lambda \in \Lambda} \mu_\lambda(E \cap E_\lambda)$. It is clear now that strictly localizable measure space are exactly direct sums of finite measure spaces.

Assume (Ω, Σ, μ) is a σ-finite measure space, then there exists a purely atomic measure $\mu_1 : \Sigma \to [0, +\infty]$ and a non-atomic measure $\mu_2 : \Sigma \to [0, +\infty]$ such that $\mu = \mu_1 + \mu_2$. Even more there exist measurable sets Ω_a^μ and $\Omega_{na}^\mu = \Omega \setminus \Omega_a^\mu$ such that $\mu_1(\Omega_{na}^\mu) = \mu_2(\Omega_a^\mu) = 0$. The sets Ω_a^μ and Ω_{na}^μ are called respectively the atomic and the non-atomic parts of the measure space (Ω, Σ, μ).

By measurable function we always mean a complex or real valued function on measurable space, with the property that preimage of every open set is measurable. We say that two measurable functions are equivalent if the set where they are different is negligible. If $f : \Omega \to \mathbb{R}$ is an integrable function on (Ω, Σ, μ), then we may define a new measure

$$f\mu : \Sigma \to [0, +\infty],\ E \mapsto \int_E f(\omega) d\mu(\omega).$$

The notion of measure can be extended by changing the range of values that a measure can attain. Any σ-additive function $\mu : \Sigma \to \mathbb{C}$ on a measurable space (Ω, Σ) is called a complex measure. Any complex measure μ can be represented as $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_1, \mu_2, \mu_3, \mu_4$ — are finite measures. As a consequence every complex measure is finite, and therefore we have a well-defined total variation measure:

$$|\mu| : \Sigma \to \mathbb{R}_+,\ E \mapsto \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(E_n)| : \{E_n : n \in \mathbb{N}\} \subset \Sigma,\ E = \bigsqcup_{n \in \mathbb{N}} E_n \right\}$$

Let μ and ν be two measures on a measurable space (Ω, Σ). We say that μ and ν are mutually singular and write $\mu \perp \nu$ if there exists a measurable set E such that $\mu(E) = \nu(\Omega \setminus E) = 0$. A somewhat opposite property is the absolute continuity. We say that ν is absolutely continuous with respect to μ and write $\nu \ll \mu$ if $\nu(E) = 0$ for every measurable set E with $\mu(E) = 0$. In general, two measures may neither be absolutely continuous nor singular with respect to each other. We have the Lebesgue decomposition theorem for this case. For a given two σ-finite measures μ and ν on a measurable space (Ω, Σ) there exists a measurable function $\rho_{\nu,\mu} : \Omega \to \mathbb{C}$, a σ-finite measure $\nu_s : \Sigma \to [0, +\infty]$ and two measurable sets $\Omega_s^{\nu,\mu}$, $\Omega_c^{\nu,\mu} = \Omega \setminus \Omega_s^{\nu,\mu}$ such that $\nu = \rho_{\nu,\mu}\mu + \nu_s$ and $\mu(\Omega_s^{\nu,\mu}) = \nu_s(\Omega_c^{\nu,\mu}) = 0$, i.e. $\mu \perp \nu_s$.

Finally, we shall say a few words on measures defined on topological spaces. We shall call these measures the topological measures. Given a topological space S we may consider the minimal σ-algebra containing all open subsets of S. It is called the Borel σ-algebra of S and denoted by $Bor(S)$. Measures and complex measures defined on Borel σ-algebras are supported with adjective Borel.

The support of a complex Borel measure μ is the set of all points $s \in S$ for which every open neighborhood of s has a positive measure. We denote the set of such points by $\mathrm{supp}(\mu)$. The support is always closed. A Borel measure μ is called

(i) strictly positive if $\mathrm{supp}(\mu) = S$;

(ii) a measure with full support if $\mu(S \setminus \mathrm{supp}(\mu)) = 0$;

(iii) locally finite if every point has an open neighborhood of finite measure;

(iv) inner regular with respect to a class $\mathcal{C}$ of Borel sets if for any Borel set E holds

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \in \mathcal{C}\};$$

(v) outer regular with respect to a class $\mathcal{C}$ of Borel sets if for any Borel set E holds

$$\mu(E) = \inf\{\mu(K) : K \subset E, K \in \mathcal{C}\};$$

(vi) residual if $\mu(E) = 0$ for any Borel nowhere dense set E;

(vii) normal if its residual and has full support.

Usually we consider regularity of measures with respect to classes of open and compact sets. Now recall a few facts on topological measures. All compact sets have finite measure if measure is locally finite. Any finite inner compact regular measure is outer open regular. Any finite inner compact regular and inner open regular measure is normal.

We say that a complex Borel measure μ defined on a locally compact Hausdorff space S is regular if it is inner compact regular.

Most of the results and definitions in this section can be found in the first, second and the fourth volumes of [5].

1.2 Banach structures

1.2.1 Banach spaces

We assume that our reader is familiar with fundamentals of functional analysis and its constructions, otherwise consult [1] or [6]. In this book we will highly rely on results about geometry of Banach spaces. See [7], [8] or [9] for a quick introduction. All Banach spaces are considered over the complex field, unless otherwise stated.

By $\langle\ B_E\ /\ B_E^\circ\ \rangle$ we denote the $\langle$ closed / open $\rangle$ unit ball of a Banach space E with center at zero. For a given set $S \subset E$ by $\operatorname{span}(S)$ we denote its linear span.

By E^{cc} we denote a Banach space with the same set of vectors as in E, the same addition but with new multiplication by conjugate scalars: $\alpha\overline{x} := \overline{\overline{\alpha}x}$ for $\alpha \in \mathbb{C}$ and $x \in E$. Note: elements of E^{cc} we denote by $\overline{x}$. Clearly, $(E^{cc})^{cc} = E$.

If F is a closed subspace of the Banach space E, then E/F stands for linear quotient Banach space. Its norm defined by equality

$$\|x + F\| = \inf\{\|x + y\| : y \in F\},$$

where $x + F \in E/F$.

Now fix two Banach spaces E and F. A map $T : E \to F$ is called conjugate linear if the respective map $T : E^{cc} \to F$ is linear. A linear operator $T : E \to F$ is called:

(i) bounded if its norm $\|T\| := \sup\{\|T(x)\| : x \in B_E\}$ is finite;

(ii) contractive if its norm is at most 1;

(iii) compact if $T(B_E)$ is relatively compact in F;

(iv) nuclear if it can be represented as an absolutely convergent series of rank one operators.

Any nuclear operator is compact. Any compact operator is bounded, any bounded operator is continuous. By $\langle\ \mathcal{B}(E,F)\ /\ \mathcal{K}(E,F)\ /\ \mathcal{N}(E,F)\ \rangle$ we denote the Banach space of $\langle$ bounded / compact / nuclear $\rangle$ linear operators from E to F. If $F=E$ we use the shortcut $\langle\ \mathcal{B}(E)\ /\ \mathcal{K}(E)\ /\ \mathcal{N}(E)\ \rangle$ for this space. The norms in $\mathcal{B}(E,F)$ and $\mathcal{K}(E,F)$ are just the usual operator norm. The norm of a nuclear operator T is defined by equality

$$\|T\|:=\inf\left\{\sum_{n=1}^{\infty}\|S_n\|:T=\sum_{n=1}^{\infty}S_n,\quad (S_n)_{n\in\mathbb{N}}-\ \text{rank one operators}\right\}.$$

By **Ban** we shall denote the category of Banach spaces with bounded linear operators in the role of morphisms, while $\mathbf{Ban}_1$ stands for the category of Banach spaces with contractive operators in the role of morphisms. As a consequence, $\mathrm{Hom}_{\mathbf{Ban}}(E,F)$ is just another name for $\mathcal{B}(E,F)$.

Two Banach spaces E and F are $\langle$ isometrically isomorphic / topologically isomorphic $\rangle$ as Banach spaces if there exists a bounded linear operator $T:E\to F$ which is both $\langle$ isometric and surjective / topologically injective and topologically surjective $\rangle$. The fact that E and F are $\langle$ isometrically isomorphic / topologically isomorphic $\rangle$ Banach spaces means that $\langle\ E\underset{\mathbf{Ban}_1}{\cong}F\ /\ E\underset{\mathbf{Ban}}{\cong}F\ \rangle$. The Banach-Mazur distance between E and F is defined by the formula

$$d_{BM}(E,F):=\inf\{\|T\|\|T^{-1}\|:T\in\mathcal{B}(E,F)\text{ — a topological isomorphism}\}.$$

If E and F are not topologically isomorphic the Banach-Mazur distance between them is infinite.

By E^* we denote the space of bounded linear functionals on E, that is $E^*=\mathcal{B}(E,\mathbb{C})$. It is called a dual space of E. Similarly, one can define the second and higher duals of E. An important corollary of the Hahn-Banach theorem says that the bounded linear operator

$$\iota_E:E\to E^{**},\ x\mapsto(f\mapsto f(x))$$

is isometric. We call this operator the natural embedding of E into its second dual E^{**}.

For a given bounded linear operator $T:E\to F$ its adjoint is bounded linear operator

$$T^*:F^*\to E^*,\ f\mapsto(x\mapsto f(T(x))).$$

Again from the Hahn Banach theorem it follows that $\|T^*\|=\|T\|$. One can easily check that $T^{**}\iota_E=\iota_F T$.

For a given subspace F of $\langle\ E\ /\ E^*\ \rangle$ we define its $\langle$ right / left $\rangle$ annihilator $\langle\ F^\perp\ /\ {}^\perp F\ \rangle$ as $\langle\ \{f \in E^* : f(x) = 0 \text{ for all } x \in F\}\ /\ \{x \in E : f(x) = 0 \text{ for all } f \in F\}\ \rangle$. Clearly, $E^\perp = \{0\}$, $0^\perp = E^*$ and ${}^\perp E^* = \{0\}$, ${}^\perp 0 = E$. Annihilators are closely related with quotient spaces. One can show that operators

$$i : F^* \to E^*/F^\perp,\ f \mapsto f|_F + F^\perp, \qquad q : (E/F)^* \to F^\perp,\ f \mapsto (x \mapsto f(x + F))$$

are isometric isomorphisms.

A few words on classification of bounded linear operators. A bounded linear operator $T : E \to F$ is called:

(i) topologically injective if it performs homeomorphism on its image;

(ii) topologically surjective if it is an open map;

(iii) coisometric if it maps open unit ball onto open unit ball;

(iv) strictly coisometric if it maps closed unit ball onto closed unit ball;

(v) c-topologically injective, if $\|x\| \leq c\|T(x)\|$ for all $x \in E$;

(vi) c-topologically surjective, if $cT(B_E^\circ) \supset B_F^\circ$;

(vii) strictly c-topologically surjective, if $cT(B_E) \supset B_F$.

Note that T is topologically $\langle$ injective / surjective $\rangle$ iff it is c-topologically $\langle$ injective / surjective $\rangle$ for some $c > 0$. Obviously $\langle$ coisometric / strictly coisometric $\rangle$ operators are exactly contractive $\langle$ 1-topologically surjective / strictly 1-topologically surjective $\rangle$ operators. These classes of operators behave nicely with respect to taking adjoints:

(i) if T is c-topologically surjective or strictly c-topologically surjective, then T^* is c-topologically injective;

(ii) if T c-topologically injective, then T^* is strictly c-topologically surjective;

(iii) if T^* c-topologically surjective or strictly c-topologically surjective, then T is c-topologically injective;

(iv) if T^* c-topologically injective and E is complete, then T is c-topologically surjective.

Let us discuss relations between Banach spaces and their subspaces. A bounded linear operator $T : E \to F$ is called a $\langle$ c-retraction / c-coretraction $\rangle$ if there exist a bounded linear operator $S : F \to E$ such that $\langle\ TS = 1_F\ /\ ST = 1_E\ \rangle$ and $\|T\|\|S\| \leq c$. In this case we say that $\langle\ F\ /\ E\ \rangle$ has a c-complemented copy in E via topologically $\langle$ injective / surjective $\rangle$ operator T. If the natural embedding of a subspace F into the ambient space E is a c-coretraction, then we say that F is c-complemented in E. Complemented subspaces can be described in terms of so-called projections. A bounded linear operator $P : E \to E$ is called a projection of E onto F if $P^2 = P$, $\mathrm{Im}(P) = F$. One can show that F is c-complemented in E iff there is a projection P from E onto F with $\|P\| \leq c$. We say that F is weakly c-complemented in E if F^* is c-complemented in E^* via the map i^*, which is the adjoint of the natural embedding. If F is c-complemented in E then it is weakly c-complemented in E. The term "contractively complemented" will be a synonym for 1-complemented. Sometimes we will omit a constant c and simply say that one Banach space $\langle$ has a complemented copy / is complemented $\rangle$ inside another. Then we can say that a subspace F of a Banach space E is complemented if there exists a closed subspace G in E such that $E \underset{\mathbf{Ban}}{\cong} F \bigoplus_1 G$.

All finite dimensional subspaces are complemented, but not necessarily contractively complemented. An interesting example of a contractively complemented subspace is the following: consider arbitrary Banach space E, then E^* is contractively complemented in E^{***} via Dixmier projection $P = \iota_{E^*}(\iota_E)^*$. A canonical example of uncomplemented subspace is $c_0(\mathbb{N})$ inside $\ell_\infty(\mathbb{N})$ [[8], theorem 2.5.5]. Nevertheless, $c_0(\mathbb{N})$ is weakly complemented in $\ell_\infty(\mathbb{N})$ by Dixmier projection.

Let E, F and G be three Banach spaces, then a bilinear operator $\Phi : E \times F \to G$ is called bounded if its norm

$$\|\Phi\| := \sup\{\|\Phi(x, y)\| : x \in B_E, y \in B_F\}$$

is finite. The Banach space of all bounded bilinear operators on $E \times F$ with values in G is denoted by $\mathcal{B}(E \times F, G)$.

Now consider the algebraic tensor product $E \otimes F$ of Banach spaces E and F. This linear space can be endowed with different norms, but the most important is the projective norm. For $u \in E \otimes F$ we define its projective norm as

$$\|u\| := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i, (x_i)_{i\in\mathbb{N}_n} \subset E, (y_i)_{i\in\mathbb{N}_n} \subset F\right\}.$$

It is indeed a norm, but not complete in general. The symbol $E \mathbin{\widehat{\otimes}} F$ stands for the completion of $E \otimes F$ under the projective norm. We call the resulting completion the projective tensor product of Banach spaces E and F. The resulting bilinear operator $\theta : E \times F \to E \mathbin{\widehat{\otimes}} F$, $(x, y) \mapsto x \otimes y$ is contractive. Let $T : E_1 \to E_2$ and $S : F_1 \to F_2$

be two bounded linear operators between Banach spaces, then there exists a unique bounded linear operator

$$T \widehat{\otimes} S : E_1 \widehat{\otimes} F_1 \to E_2 \widehat{\otimes} F_2$$

such that $(T \widehat{\otimes} S)(x \widehat{\otimes} y) = T(x) \widehat{\otimes} S(y)$ for all $x \in E_1$ and $y \in F_1$. Even more $\|T \widehat{\otimes} S\| = \|T\|\|S\|$. The main feature of projective tensor product which makes it so important is the following universal property: for any Banach spaces E, F and G there is a natural isometric isomorphism:

$$\mathcal{B}(E \widehat{\otimes} F, G) \underset{\mathbf{Ban}_1}{\cong} \mathcal{B}(E \times F, G)$$

In other words, projective tensor product linearizes bounded bilinear operators. Also, we have the following two (natural in E, F and G) isometric isomorphisms:

$$\mathcal{B}(E \widehat{\otimes} F, G) \underset{\mathbf{Ban}_1}{\cong} \mathcal{B}(E, \mathcal{B}(F, G)) \underset{\mathbf{Ban}_1}{\cong} \mathcal{B}(F, \mathcal{B}(E, G))$$

The last isomorphism is called the law of adjoint associativity. There are many other tensor norms on the algebraic tensor product of Banach spaces. Their thorough treatment can be found in [10].

Now we are able to craft four very important functors:

$$\mathcal{B}(-, E) : \mathbf{Ban} \to \mathbf{Ban} \qquad \mathcal{B}(E, -) : \mathbf{Ban} \to \mathbf{Ban}$$

$$- \widehat{\otimes} E : \mathbf{Ban} \to \mathbf{Ban} \qquad E \widehat{\otimes} - : \mathbf{Ban} \to \mathbf{Ban}.$$

We shall often encounter them. For example, the well known adjoint functor * is nothing more than $\mathcal{B}(-, \mathbb{C})$. All these functors have their obvious analogs on $\mathbf{Ban}_1$.

Now we proceed to classical examples of Banach spaces.

An important source of examples of Banach spaces are L_p-spaces, also known as Lebesgue spaces. A detailed discussion of basic properties of L_p-spaces can be found in [7]. Let (Ω, Σ, μ) be a measure space. For $1 \leq p < \infty$, as usually, the symbol $L_p(\Omega, \mu)$ stands for the Banach space of equivalence classes of functions $f : \Omega \to \mathbb{C}$ such that $|f|^p$ is Lebesgue integrable with respect to measure μ. The norm of such function is defined by

$$\|f\| := \left(\int_\Omega |f(\omega)|^p d\mu(\omega) \right)^{1/p}.$$

By $L_\infty(\Omega, \mu)$ we denote the Banach space of equivalence classes of bounded measurable functions with norm defined as

$$\|f\| := \inf\left\{\sup_{\omega\in\Omega\setminus N} |f(\omega)| : N \subset \Omega - \text{is negligible}\right\}.$$

For simplicity, we shall speak of functions in $L_p(\Omega, \mu)$ instead of their equivalence classes. All equalities and inequalities about functions of L_p-spaces are understood up to negligible sets. It is well-known that $L_p(\Omega, \mu)^* \underset{\mathbf{Ban}_1}{\cong} L_{p^*}(\Omega, \mu)$ for $1 \leq p < +\infty$ [[5], theorems 243G, 244K]. One more well known fact is that, L_p-spaces are reflexive for $1 < p < +\infty$. Here we exploited the standard notation $p^* = +\infty$ if $p = 1$ and $p^* = p/(p-1)$ if $1 < p < +\infty$. Clearly, $p^{**} = p$ for $1 < p < +\infty$.

The most well known classes of Banach spaces are related to continuous functions. Let S be a locally compact Hausdorff space. We say that a function $f : S \to \mathbb{C}$ vanishes at infinity if for any $\epsilon > 0$ there exists a compact $K \subset S$ such that $|f(s)| \leq \epsilon$ for all $s \in S \setminus K$. The linear space of continuous functions on S vanishing at infinity is denoted by $C_0(S)$. When endowed with sup-norm $C_0(S)$ becomes a Banach space. Any set Λ with discrete topology may be regarded as a locally compact space and following the traditional notation we shall write $c_0(\Lambda)$ instead of $C_0(\Lambda)$. If K is a compact Hausdorff space then all functions on K vanish at infinity. We use the notation $C(K)$ for $C_0(K)$ to indicate that K is compact. Some Banach spaces in fact are $C(K)$-spaces in disguise. For example, if we are given a measure space (Ω, Σ, μ), then $B(\Omega, \Sigma)$ — the space of bounded measurable functions with sup-norm or $L_\infty(\Omega, \mu)$ are $C(K)$ spaces for some compact Hausdorff space K [[8], remark 4.2.8]. By $M(S)$ we denote the Banach space of complex finite Borel regular measures on S. The norm of measure $\mu \in M(S)$ is defined by equality $\|\mu\| = |\mu|(S)$, where $|\mu|$ is a total variation measure of measure μ. By Riesz-Markov-Kakutani theorem [[6], section C.18] we have $C_0(S)^* \underset{\mathbf{Ban}_1}{\cong} M(S)$. In fact $M(S)$ is an L_1-space, see discussion after [[11], proposition 2.14].

We shall also mention one important specific case of L_p-spaces. For a given index set Λ and a counting measure $\mu_c : \mathcal{P}(\Lambda) \to [0, +\infty]$ the respective L_p-space is denoted by $\ell_p(\Lambda)$. For this type of measure spaces we have one more important isomorphism $c_0(\Lambda)^* \underset{\mathbf{Ban}_1}{\cong} \ell_1(\Lambda)$. By definition, we take $c_0(\varnothing) = \ell_p(\varnothing) = \{0\}$ for $1 \leq p \leq +\infty$. This example motivates the following construction.

Let $\{E_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. For each $x \in \prod_{\lambda\in\Lambda} E_\lambda$ we define $\|x\|_p = \|(\|x_\lambda\|)_{\lambda\in\Lambda}\|_{\ell_p(\Lambda)}$ for $1 \leq p \leq +\infty$ and $\|x\|_0 = \|(\|x_\lambda\|)_{\lambda\in\Lambda}\|_{c_0(\Lambda)}$. Then the Banach space $\left\{x \in \prod_{\lambda\in\Lambda} E_\lambda : \|x\|_p < +\infty\right\}$ with the norm $\|\cdot\|_p$ is denoted by $\bigoplus_p\{E_\lambda : \lambda \in \Lambda\}$. We call these objects $\bigoplus_p$-sums of Banach spaces $\{E_\lambda : \lambda \in \Lambda\}$. It is almost tautological that the Banach space $\ell_p(\Lambda)$ is the $\bigoplus_p$-sum of the family $\{\mathbb{C} : \lambda \in \Lambda\}$.

A nice property of $\bigoplus_p$-sums is their interrelation with duality:

$$\left(\bigoplus_p \{E_\lambda : \lambda \in \Lambda\}\right)^* \underset{\mathbf{Ban}_1}{\cong} \bigoplus_{p^*} \{E_\lambda^* : \lambda \in \Lambda\}$$

for all $1 \leq p < +\infty$ and

$$\left(\bigoplus_0 \{E_\lambda : \lambda \in \Lambda\}\right)^* \underset{\mathbf{Ban}_1}{\cong} \bigoplus_1 \{E_\lambda^* : \lambda \in \Lambda\}$$

If $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) : \lambda \in \Lambda\}$ is a family of bounded linear operators, then for all $1 \leq p \leq +\infty$ and $p = 0$ we have a well-defined linear operator

$$T : \bigoplus_p \{E_\lambda : \lambda \in \Lambda\} \to \bigoplus_p \{F_\lambda : \lambda \in \Lambda\} : x \mapsto \bigoplus_p \{T_\lambda(x_\lambda) : \lambda \in \Lambda\}$$

which we shall denote by $\bigoplus_p \{T_\lambda : \lambda \in \Lambda\}$. Its norm equals $\sup_{\lambda \in \Lambda} \|T_\lambda\|$.

Among different $\bigoplus_p$-sums the $\langle$ $\bigoplus_1$-sums / $\bigoplus_\infty$-sums $\rangle$ play a special role in Banach space theory. The reason is that any family of Banach spaces admits $\langle$ product / coproduct $\rangle$ in $\mathbf{Ban}_1$ which in fact is their $\langle$ $\bigoplus_1$-sum / $\bigoplus_\infty$-sum $\rangle$. The same statement holds for **Ban** if we restrict ourselves to finite families of objects [[1], chapter 2, section 5]. Obviously, any summand in a $\bigoplus_p$-sum is a 1-complemented subspace in the sum.

We proceed to advanced topics of Banach space theory. Below we shall discuss several geometric properties of Banach spaces such as the property of being an $\mathscr{L}_p^g$-space, weak sequential completeness, the Dunford-Pettis property, the l.u.st. property and the approximation property. In what follows, imitating Banach space geometers, we shall say that a Banach space E contains $\langle$ an isometric copy / a copy $\rangle$ of Banach space F if F is $\langle$ isometrically isomorphic / topologically isomorphic $\rangle$ to some closed subspace of E.

Let $1 \leq p \leq +\infty$. We say that E is an $\mathscr{L}_{p,C}^g$-space if for any $\epsilon > 0$ and any finite dimensional subspace F of E there exists a finite dimensional ℓ_p-space G and two bounded linear operators $S : F \to G$, $T : G \to E$ such that $TS|^F = 1_F$ and $\|T\|\|S\| \leq C + \epsilon$. If E is an $\mathscr{L}_{p,C}^g$-space for some $C \geq 1$ we simply say, that E is an $\mathscr{L}_p^g$-space. This definition [[12], definition 23.1] is an improvement of the definition of $\mathscr{L}_p$-spaces given by Lindenstrauss and Pelczynski in their pioneering work [13]. Clearly, any finite dimensional Banach space is an $\mathscr{L}_p^g$-space for all $1 \leq p \leq +\infty$. Any L_p-space is an $\mathscr{L}_{p,1}^g$-space [[12], exercise 4.7], but the converse is not true. Any c-complemented subspace of $\mathscr{L}_{p,C}^g$-space is an $\mathscr{L}_{p,cC}^g$-space [[12], corollary 23.2.1(2)]. A Banach space is an $\mathscr{L}_{p,C}^g$-space iff its dual is an $\mathscr{L}_{p^*,C}^g$-space [[12], corollary 23.2.1(1)]. All $C(K)$-spaces are $\mathscr{L}_{\infty,1}^g$-spaces [[12], lemma 4.4]. Note that, for a given locally compact Hausdorff space S the Banach space $C_0(S)$ is complemented in $C(\alpha S)$. Therefore, $C_0(S)$-spaces are $\mathscr{L}_\infty^g$-spaces too. We will mainly concern in $\mathscr{L}_1^g$- and $\mathscr{L}_\infty^g$-spaces.

We say that a Banach space E is weakly sequentially complete if for any sequence $(x_n)_{n\in\mathbb{N}} \subset E$ such that $(f(x_n))_{n\in\mathbb{N}} \subset \mathbb{C}$ is a Cauchy sequence for any $f \in E^*$ there exists a vector $x \in E$ such that $\lim_n f(x_n) = f(x)$ for all $f \in E^*$. That is any weakly Cauchy sequence converges in the weak topology. A typical example of a weakly sequentially complete Banach space is any L_1-space [[14], corollary III.C.14]. This property is preserved by closed subspaces. A typical example of a Banach space that is not weakly sequentially complete is $c_0(\mathbb{N})$, just consider the sequence $(\sum_{k=1}^{n} \delta_k)_{n\in\mathbb{N}}$.

Now we proceed to the discussion of the Dunford-Pettis property. A bounded linear operator $T : E \to F$ is called weakly compact if it maps the unit ball of E into a relatively weakly compact subset of F. A bounded linear operator is called completely continuous if the image of any weakly compact subset of E is norm compact in F. A Banach space E is said to have the Dunford-Pettis property if any weakly compact operator from E to any Banach space F is completely continuous. There is a simple internal characterization [[8], theorem 5.4.4]: a Banach space E has the Dunford-Pettis property iff $\lim_n f_n(x_n) = 0$ for all sequences $(x_n)_{n\in\mathbb{N}} \subset E$ and $(f_n)_{n\in\mathbb{N}} \subset E^*$, that both weakly converge to 0. Now it is easy to deduce, that if a Banach space E^* has the Dunford-Pettis property, then so does E. In his seminal work [15] Grothendieck showed that all L_1-spaces and $C(K)$-spaces have this property. The Dunford-Pettis property passes to complemented subspaces [[9], proposition 13.44]. This property behaves badly with reflexive spaces: since the unit ball of a reflexive space is weakly compact [[16], theorem 2.8.2], then reflexive Banach space with the Dunford-Pettis property has norm compact unit ball and therefore this space is finite dimensional.

To introduce the next Banach geometric property we need definitions of Banach lattice and unconditional Schauder basis.

A real Riesz space E is a vector space over $\mathbb{R}$ with the structure of partially ordered set such that $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in E$ and $ax \geq 0$ for every $x \geq 0$, $a \in \mathbb{R}_+$. A partially ordered set L is a lattice if any two elements $x, y \in L$ have the least upper bound $x \vee y \in L$ and the greatest lower bound $x \wedge y \in L$. A real vector lattice is real Riesz space which is a lattice as a partially ordered set. If E is a real vector lattice, then for every $x \in E$ we define its absolute value by equality $|x| := x \vee (-x)$. A complex vector lattice E is a vector space over $\mathbb{C}$ such that there exists a real vector subspace $\mathrm{Re}(E)$ called a real part of E, which is a real vector lattice and

(i) for any $x \in E$ there are unique $\mathrm{Re}(x), \mathrm{Im}(x) \in \mathrm{Re}(E)$ such that $x = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$;

(ii) for any $x \in E$ there exists an absolute value $|x| := \sup\{\mathrm{Re}(e^{i\theta}x) : \theta \in \mathbb{R}\}$.

A Banach lattice E is a Banach space with the structure of a complex vector lattice such that $\|x\| \leq \|y\|$ whenever $x, y \in E$ and $|x| \leq |y|$. A classical example of a Banach lattice is an L_p-space or a $C(K)$-space. In both cases their real part consists of real valued functions. If $\{E_\lambda : \lambda \in \Lambda\}$ is a family of Banach lattices then for any $1 \leq p \leq +\infty$ or $p = 0$ their $\bigoplus_p$-sum is a Banach lattice with lattice operation defined as $x \leq y$ if $x_\lambda \leq y_\lambda$ for all $\lambda \in \Lambda$, where $x, y \in \bigoplus_p \{E_\lambda : \lambda \in \Lambda\}$. The dual space E^* of a Banach lattice E is again a Banach lattice with lattice operation defined by $f \leq g$ if $f(x) \leq g(x)$ for all $x \geq 0$, where $f, g \in E^*$. A very nice account of Banach lattices can be found in [[17], section 1].

The property of being a Banach lattice puts some restrictions on the geometry of the space [18], [19]. To explain the Banach geometric reason of this phenomenon we need the definition of an unconditional Schauder basis. Let E be a Banach space. A collection of functionals $(f_\lambda)_{\lambda\in\Lambda} \subset E^*$ is called a biorthogonal system for vectors $(x_\lambda)_{\lambda\in\Lambda} \subset E$ if $f_\lambda(x_{\lambda'}) = 1$ whenever $\lambda = \lambda'$ and 0 otherwise. A collection $(x_\lambda)_{\lambda\in\Lambda} \subset E$ is called an unconditional Schauder basis if there exists a biorthogonal system $(f_\lambda)_{\lambda\in\Lambda} \subset E^*$ for it such that the series $\sum_{\lambda\in\Lambda} f_\lambda(x)x_\lambda$ unconditionally converges to x for any $x \in E$. All ℓ_p-spaces with $1 \leq p < +\infty$ have an unconditional Schauder basis, for example, it is $(\delta_\lambda)_{\lambda\in\Lambda}$. A typical example of a Banach space without unconditional basis is $C([0, 1])$. Even more this Banach space can not even be a subspace of the space with unconditional basis [[8], proposition 3.5.4]. Any unconditional Schauder basis $(x_\lambda)_{\lambda\in\Lambda}$ in E satisfy the following property [[20], proposition 1.6]: there exists a constant $\kappa \geq 1$ such that

$$\left\| \sum_{\lambda\in\Lambda} t_\lambda f_\lambda(x)x_\lambda \right\| \leq \kappa \left\| \sum_{\lambda\in\Lambda} f_\lambda(x)x_\lambda \right\|$$

for all $x \in E$ and $t \in \ell_\infty(\Lambda)$. The least such constant κ among all unconditional Schauder bases of E is denoted by $\kappa(E)$. Similar constant could be defined for Banach spaces without unconditional Schauder bases. The local unconditional constant $\kappa_u(E)$ of Banach space E is defined to be the infimum of all scalars c with the following property: given any finite dimensional subspace F of E there exists a Banach space G with unconditional Schauder basis and two bounded linear operators $S : F \to G$, $T : G \to E$ such that $TS|^F = 1_F$ and $\|T\|\|S\|\kappa(G) \leq c$. We say that a Banach space E has the local unconditional structure property (the l.u.st. property for short) if $\kappa_u(E)$ is finite. Clearly any Banach space E with unconditional Schauder basis has the l.u.st. property with $\kappa_u(E) = \kappa(E)$. In particular, all finite dimensional Banach spaces have the l.u.st. property. Though a general Banach lattice E may not have an unconditional Schauder basis it still has the l.u.st. property with $\kappa_u(E) = 1$ [[20], theorem 17.1]. Directly from the definition it follows that the l.u.st. property is preserved by complemented subspaces. More precisely: if F is a c-complemented subspace of E, then $\kappa_u(F) \leq c\kappa_u(E)$. Therefore, all complemented subspaces of Banach lattices have the l.u.st. property. This sufficient condition is not far from a criterion [[20], theorem 17.5]: a Banach space E has the l.u.st. property iff E^{**} is topologically isomorphic to a complemented subspace of some Banach lattice. As the corollary of this criterion we get that E has the l.u.st. property iff so does E^* [[20], corollary 17.6].

The last property we shall discuss is a well known approximation property introduced by Grothendieck in [21]. We say that a Banach space E has the approximation property if for any compact set $K \subset E$ and any $\epsilon > 0$ there exists a finite rank operator $T : E \to E$ such that $\|T(x) - x\| < \epsilon$ for all $x \in K$. If T can be chosen with $\|T\| \leq c$, then E is said to have the c-bounded approximation property. The metric approximation property is another name for 1-bounded approximation property. We say that E has the bounded approximation property if E has the c-bounded approximation property for some $c \geq 1$. None of these properties are preserved by subspaces or quotient spaces, but the approximation property and the bounded approximation properties are inherited by complemented subspaces [[12], exercise 5.5]. All L_p-spaces and $C(K)$-spaces have the metric approximation property [[12], section 5.2(3)], but their subspaces may fail the approximation property [[12], section 5.2(1)]. Any Banach space with unconditional Schauder basis has the approximation property [[22], example 4.4]. If E^* has the approximation property, then so does E [[12], corollary 5.7.2]. The reason why approximation property is so important is rather simple — it has a lot of equivalent reformulations that involve many nice properties of Banach spaces. For example, the following properties of Banach space E are equivalent [[12], sections 5.3, 5.6]:

(i) E has the approximation property;

(ii) the natural mapping $Gr : E^* \widehat{\otimes} E \to \mathcal{N}(E)$ is an isometric isomorphism;

(iii) for any Banach space F every compact operator $T : F \to E$ can be approximated in the operator norm by finite rank operators.

There is much more to list here, but we confine ourselves with these three properties.

1.2.2 Banach algebras and their modules

A thorough treatment of Banach algebras and Banach modules can be found in [23] or [24] or [25]. We shall describe only the bare minimum required for us.

A Banach algebra A is an associative algebra over $\mathbb{C}$ which is a Banach space and the multiplication bilinear operator $\cdot : A \times A \to A : (a, b) \mapsto ab$ is of the norm at most

1. A typical example of a commutative Banach algebra is the algebra of continuous functions on a compact Hausdorff space with pointwise multiplication. A typical non-commutative example is the algebra of bounded linear operators on a Hilbert space with composition in the role of multiplication. Both examples belong to a very important class of C^*-algebras to be discussed below. By $\langle$ left / right / two-sided $\rangle$ ideal I of a Banach algebra A we always mean a subalgebra of A such that $\langle$ ax / xa / ax and xa $\rangle$ belong to I for all $a \in A$ and $x \in I$.

We say that an element p of a Banach algebra A is a $\langle$ left / right $\rangle$ identity of A if $\langle$ $pa = a$ / $ap = a$ $\rangle$ for all $a \in A$. The element which is both left and right identity is called the identity and denoted by e_A. In general, we do not assume that Banach algebras are unital, i.e. has an identity. Even if a Banach algebra A is unital we do not require its identity to be of norm 1. We use notation $A_+ = A \bigoplus_1 \mathbb{C}$ for the standard unitization of Banach algebras. The multiplication in A_+ is defined as $(a \oplus_1 z)(b \oplus_1 w) = (ab + wa + zb) \oplus_1 zw$, for $a, b \in A$ and $z, w \in \mathbb{C}$. Clearly $0 \oplus_1 1$ is the identity of A_+. By $A_\times$ we denote the conditional unitization of A, i.e. $A_\times = A$ if A has identity of norm one and $A_\times = A_+$ otherwise. Even in the absence of identity in case of Banach algebras there are good substitutes for it which are called approximate identities. We say that a net $(e_\nu)_{\nu \in N} \subset A$ is a $\langle$ left / right / two-sided $\rangle$ approximate identity if $\langle$ $\lim_\nu e_\nu a = a$ / $\lim_\nu a e_\nu = a$ / $\lim_\nu e_\nu a = \lim_\nu a e_\nu = a$ $\rangle$ for all $a \in A$. Here convergence of nets is understood in the norm topology. If we replace norm topology with weak topology, we get definitions of left, right and two-sided weak approximate identities. We say that an approximate identity $(e_\nu)_{\nu \in N}$ is c-bounded if $\sup_\nu \|e_\nu\| \leq c$. An approximate identity $(e_\nu)_{\nu \in N}$ is called $\langle$ contractive / bounded $\rangle$ if it is $\langle$ 1-bounded / c-bounded for some $c \geq 1$ $\rangle$. Occasionally we will use the following simple fact: if A is a Banach algebra has a $\langle$ left / right $\rangle$ identity p and a $\langle$ right / left $\rangle$ approximate identity $(e_\nu)_{\nu \in N}$, then A is unital with identity p of norm $\lim_\nu \|e_\nu\|$.

If A is a unital Banach algebra we define the spectrum $\operatorname{sp}_A(a)$ of element $a \in A$ as the set of all complex numbers z such that $a - z e_A$ is not invertible in A. For Banach algebras the spectrum of any element is a non-empty compact subset of $\mathbb{C}$ [[23], corollary 2.1.16].

A character on a Banach algebra A is a non-zero linear homomorphism $\varkappa : A \to \mathbb{C}$. All characters are continuous and are contained in the unit ball of A^* [[23], theorem 1.2.6]. Therefore, we may consider the set of characters with the induced weak* topology. The resulting topological space is Hausdorff and locally compact. It is called the spectrum of the Banach algebra A and denoted by $\operatorname{Spec}(A)$. If A is unital then its spectrum is compact [[23], theorem 1.2.50]. Now for a given Banach algebra A with non-empty spectrum we can construct a contractive homomorphism $\Gamma_A : A \to C_0(\operatorname{Spec}(A)) : a \mapsto (\varkappa \mapsto \varkappa(a))$ called the Gelfand transform of A [[23], theorem 4.2.11]. The kernel of this homomorphism is called the Jacobson's radical and denoted by $\mathrm{Rad}(A)$. For a Banach algebra A with empty spectrum we define $\mathrm{Rad}(A) = A$. If $\mathrm{Rad}(A) = \{0\}$, then A is called semisimple. By Shilov's idempotent theorem [[26], section 3.5] any semisimple Banach algebra with compact spectrum is unital.

Most standard constructions for Banach spaces have their counterparts for Banach algebras. For example $\bigoplus_p$-sum of Banach algebras endowed with componentwise multiplication is a Banach algebra, the quotient of a given Banach algebra by its closed two-sided ideal is a Banach algebra. Even the projective tensor product of two Banach algebras is a Banach algebra with multiplication defined on elementary tensors the same way as in pure algebra.

We shall proceed to the discussion of the most important class of Banach algebras. Let A be an associative algebra over $\mathbb{C}$, then a conjugate linear operator $^* : A \to A$ is called an involution if $(ab)^* = b^*a^*$ and $a^{**} = a$ for all $a, b \in A$. Algebras with involution are called *-algebras. Homomorphisms between *-algebras that preserve involution are called *-homomorphisms. A Banach algebra with isometric involution is called a *-Banach algebra. An example of such algebra is the Banach algebra of bounded linear operators on a Hilbert space with operation of taking the Hilbert adjoint operator in the role of involution. In fact there is much more to this algebra than one could expect. We say that a *-Banach algebra A is a C^*-algebra if it satisfies $\|a^*a\| = \|a\|^2$ for all $a \in A$. One of the biggest advantages of C^*-algebras is their celebrated representation theorems by Gelfand and Naimark. The first representation theorem [[23], theorem 4.7.13] states that any commutative C^*-algebra A is isometrically isomorphic as *-algebra to $C_0(\mathrm{Spec}(A))$. The second theorem [[23], theorem 4.7.57] gives a description of generic C^*-algebras as closed *-Banach subalgebras of $\mathcal{B}(H)$ for some Hilbert space H. Such representation is not unique, but a norm (if it exists) that turn a *-algebra into a C^*-algebra is always unique. If a *-subalgebra of $\mathcal{B}(H)$ is weak* closed it is called a von Neumann algebra. If a C^*-algebra is isomorphic as *-algebra to a von Neumann algebra, then it is called a W^*-algebra. By well known Sakai's theorem [[27], theorem III.2.4.2] each C^*-algebra which is dual as Banach space is a W^*-algebra, but beware a W^*-algebra may be represented as non weak* closed *-subalgebra in $\mathcal{B}(H)$ for some Hilbert space H.

Till the end of this paragraph we assume that A is a unital C^*-algebra. An element $a \in A$ is called a projection (or an orthogonal projection) if $a = a^* = a^2$, self-adjoint if $a = a^*$, positive if $a = b^*b$ for some $b \in A$, unitary if $a^*a = aa^* = e_A$. The set A_{pos} of all positive elements of A is a closed cone in A. If an element $a \in A$ is $\langle$ self-adjoint / positive $\rangle$, then $\langle$ $\mathrm{sp}_A(a) \subset [-\|a\|, \|a\|]$ / $\mathrm{sp}_A(a) \subset [0, \|a\|]$ $\rangle$. For a given self-adjoint element $a \in A$, there always exists the isometric *-homomorphism $\mathrm{Cont}_a : C(\mathrm{sp}_A(a)) \to A$ such that $\mathrm{Cont}_a(f) = a$, where $f : \mathrm{sp}_A(a) \to \mathbb{C}, t \mapsto t$. It is called the continuous functional calculus [[23], theorem 4.7.24]. Loosely speaking it allows taking continuous functions of self-adjoint elements of C^*-algebras, so following standard convention we shall write $f(a)$ instead of $\mathrm{Cont}_a(f)$. Another related result called the spectral mapping theorem allows to compute the spectrum of elements given by continuous functional calculus: $\mathrm{sp}_A(f(a)) = f(\mathrm{sp}_A(a))$.

A lot of standard constructions pass to C^*-algebras from Banach algebras, but not all. For example a $\bigoplus_\infty$-sum of C^*-algebras is again a C^*-algebra. A quotient of C^*-algebra by a closed two-sided ideal is a C^*-algebra too. Meanwhile, the projective tensor product of C^*-algebras is rarely a C^*-algebra, though there are a lot of norms that may turn their algebraic tensor product into a C^*-algebra. In this book we shall exploit one specific and highly important construction for C^*-algebras — the matrix algebras. For a given C^*-algebra A by $M_n(A)$ we denote the linear space of $n \times n$ matrices with entries in A. In fact $M_n(A)$ is a *-algebra with involution and multiplication defined by equalities

$$(ab)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \qquad\qquad (a^*)_{i,j} = (a^*_{j,i})$$

for all $a, b \in M_n(A)$ and $i, j \in \mathbb{N}_n$. There is a unique norm on $M_n(A)$ that makes it a C^*-algebra [[28], theorem 3.4.2]. Obviously, $M_n(\mathbb{C})$ is isometrically isomorphic as *-algebra to $\mathcal{B}(\ell_2(\mathbb{N}_n))$. From [[28], remark 3.4.1] it follows that the natural embeddings $i_{k,l} : A \to M_n(A) : a \mapsto (a\delta_{i,k}\delta_{j,l})_{i,j\in\mathbb{N}_n}$ and projections $\pi_{k,l} : M_n(A) \to A : a \mapsto a_{k,l}$ are continuous. Therefore, for a given bounded linear operator $\phi : A \to B$ between C^*-algebras A and B the linear operator

$$M_n(\phi) : M_n(A) \to M_n(B),\, a \mapsto (\phi(a_{i,j}))_{i,j\in\mathbb{N}_n}$$

is also bounded. Even more if ϕ is an A-morphism or *-homomorphism, then so does $M_n(\phi)$. Finally, we shall mention two isometric isomorphisms that will be of use:

$$M_n\left(\bigoplus\nolimits_\infty \{A_\lambda : \lambda \in \Lambda\}\right) \underset{\mathbf{Ban}_1}{\cong} \bigoplus\nolimits_\infty \{M_n(A_\lambda) : \lambda \in \Lambda\},$$

$$M_n(C(K)) \underset{\mathbf{Ban}_1}{\cong} C(K, M_n(\mathbb{C}))$$

Now a few facts on approximate identities and identities of C^*-algebras and their ideals. Any closed two-sided ideal of a C^*-algebra has a two-sided contractive positive approximate identity [[23], theorem 4.7.79], and any closed left ideal has a right contractive positive approximate identity. In some cases even an approximate identity is not enough, so for this situation there is a procedure to endow a C^*-algebra with identity and preserve C^*-algebraic structure [[23], proposition 4.7.6]. This type of unitization we shall denote as $A_\#$.

We proceed to the discussion of more general objects — Banach modules. Let A be a Banach algebra, we say that X is a $\langle$ left / right $\rangle$ Banach A-module if X is a Banach space endowed with bilinear operator $\langle \cdot : A \times X \to X \ / \cdot : X \times A \to X \rangle$ of norm at most 1 (called a module action), such that $\langle\ a \cdot (b \cdot x) = ab \cdot x \ / \ (x \cdot a) \cdot b = x \cdot ab \ \rangle$ for all $a, b \in A$ and $x \in X$. Any Banach space E can be turned into a $\langle$ left / right $\rangle$ Banach A-module be defining $\langle\ a \cdot x = 0 \ / \ x \cdot a = 0\ \rangle$ for all $a \in A$ and $x \in E$. Any Banach algebra A can be regarded as a left and right Banach A-module — the module action coincides with algebra multiplication. Of course, there are more meaningful examples too. Usually we shall discuss only left Banach modules since for their right sided counterparts all definitions and results are similar. We call a left Banach module X over a unital Banach algebra A unital if $e_A \cdot x = x$ for all $x \in X$.

Let X and Y be two $\langle$ left / right $\rangle$ Banach A-modules. We say that a linear operator $\phi : X \to Y$ is an A-module map of $\langle$ left / right $\rangle$ modules if $\langle\ \phi(a \cdot x) = a \cdot \phi(x)$ / $\phi(x \cdot a) = \phi(x) \cdot a\ \rangle$ for all $a \in A$ and $x \in X$. A bounded A-module map is called an A-morphism. The set of A-morphisms between $\langle$ left / right $\rangle$ A-modules X and Y we denote as $\langle\ {}_A\mathcal{B}(X, Y) \ / \ \mathcal{B}_A(X, Y)\ \rangle$. Note that if X and Y are $\langle$ left / right $\rangle$ annihilator A-modules, then $\langle\ {}_A\mathcal{B}(X, Y) = \mathcal{B}(X, Y) \ / \ \mathcal{B}_A(X, Y) = \mathcal{B}(X, Y)\ \rangle$.

If X is a left Banach A-module and E is a Banach space, then $\langle\ \mathcal{B}(X, E) \ / \ \mathcal{B}(E, X)\ \rangle$ is a $\langle$ right / left $\rangle$ Banach A-module with module action defined by $\langle\ (T \cdot a)(x) = T(a \cdot x)$ for all $a \in A$, $x \in X$ and $T \in \mathcal{B}(X, E)$ / $(a \cdot T)(x) = a \cdot T(x)$ for all $a \in A$, $x \in E$ and $T \in \mathcal{B}(E, X)\ \rangle$. In particular, X^* is a right Banach A-module.

By $\langle A - \mathbf{mod} \ / \ \mathbf{mod} - A\rangle$ we shall denote the category of $\langle$ left / right $\rangle$ A-modules with continuous A-module maps in the role of morphisms. By $\langle\ A - \mathbf{mod}_1 \ / \ \mathbf{mod}_1 - A\ \rangle$ we denote its subcategory of $\langle\ A - \mathbf{mod} \ / \ \mathbf{mod} - A\ \rangle$ with the same objects and contractive morphisms only. Therefore, $\langle\ \mathrm{Hom}_{A-\mathbf{mod}}(X, Y) = {}_A\mathcal{B}(X, Y) \ / \ \mathrm{Hom}_{\mathbf{mod}-A}(X, Y) = \mathcal{B}_A(X, Y)\ \rangle$.

For a given left Banach A-module X and $S \subset A$, $M \subset X$ we define their products $S \cdot M = \{a \cdot x : a \in S, x \in M\}$, $SM = \mathrm{span}(S \cdot M)$ and algebraic annihilators

$$S^{\perp M} = \{a \in S : a \cdot M = \{0\}\}, \qquad {}^{S\perp}M = \{x \in M : S \cdot x = \{0\}\}.$$

The essential and annihilator parts of X are defined as $X_{ess} = \mathrm{cl}_X(AX)$, $X_{ann} = {}^{A\perp}X$. The module X is called $\langle$ faithful / annihilator / essential $\rangle$ if $\langle\ {}^{A\perp}X = \{0\} \ / \ X = X_{ann}$ / $X = X_{ess}\ \rangle$. Clearly, $(X^*)^{\perp S} = \mathrm{cl}_X(SX)^{\perp}$. Setting $S = A$ we get that X is an essential A-module iff X^* is a faithful A-module.

As in any category we can speak of retraction and coretractions in the category of Banach modules. But for this particular case we have several refinements for standard definitions. An A-morphism $\xi : X \to Y$ is called a $\langle$ c-retraction / c-coretraction $\rangle$ if there exists an A-morphism $\eta : Y \to X$ such that $\langle$ $\xi\eta = 1_Y$ / $\eta\xi = 1_X$ $\rangle$ and $\|\xi\|\|\eta\| \leq c$. From the definition it follows that composition of $\langle$ c_1- and c_2-retraction / c_1- and c_2-coretraction $\rangle$ gives a $\langle$ c_1c_2-retraction / c_1c_2-coretraction $\rangle$. Clearly, the adjoint of $\langle$ c-retraction / c-coretraction $\rangle$ is a $\langle$ c-coretraction / c-retraction $\rangle$. Finally, an A-morphism $\xi : X \to Y$ is called a c-isomorphism if there exists an A-morphism $\eta : Y \to X$ such that $\xi\eta = 1_Y$, $\eta\xi = 1_X$ and $\|\xi\|\|\eta\| \leq c$. In this case we say that A-modules X and Y are c-isomorphic.

Now we mention several constructions over Banach modules that we will encounter in this book. Any left Banach A-module can be regarded as a unital Banach module over A_+, and we put by definition $(a \oplus_1 z) \cdot x = a \cdot x + zx$ for all $a \in A$, $x \in X$ and $z \in \mathbb{C}$. Most constructions used for Banach spaces transfer to Banach modules. We say that a linear subspace Y of a left Banach A-module X is a left A-submodule of X if $A \cdot Y \subset Y$. For example, any closed left ideal I of a Banach algebra A is a left A-submodule of A. If Y is a closed left A-submodule of a left Banach A-module X, then the Banach space X/Y can be endowed with the structure of the left Banach A-module, just put by definition $a \cdot (x + Y) = a \cdot x + Y$ for all $a \in A$ and $x + Y \in X/Y$. This object is called the quotient A-module. Quotient modules of the form A/I, where I is a closed left ideal of A, are called cyclic modules. For motivation for this term see [[23], proposition 6.2.2]. Clearly, X/X_{ess} is an annihilator A-module. If $\{X_\lambda : \lambda \in \Lambda\}$ is a family of left Banach A-modules and $1 \leq p \leq +\infty$ or $p = 0$, then their $\bigoplus_p$-sum is a left Banach A-module with module action defined by $a \cdot x := \bigoplus_p \{a \cdot x_\lambda : \lambda \in \Lambda\}$, where $a \in A$, $x \in \bigoplus_p \{X_\lambda : \lambda \in \Lambda\}$. Again, as in Banach space theory, any family of A-modules admits the $\langle$ product / coproduct $\rangle$ in $A - \mathbf{mod}_1$ which in fact is their $\langle$ $\bigoplus_1$-sum / $\bigoplus_\infty$-sum $\rangle$. The category $A - \mathbf{mod}$ admits $\langle$ products / coproducts $\rangle$ only for finite families of objects. Similar statements are valid for $\mathbf{mod} - A$ and $\mathbf{mod}_1 - A$.

The projective tensor product of Banach spaces also has its module version, it is called the projective module tensor product. Assume X is a right and Y is a left Banach A-module. Their projective module tensor product $X \widehat{\otimes}_A Y$ is defined as a quotient space $X \widehat{\otimes} Y / N$ where $N = \mathrm{cl}_{X \widehat{\otimes} Y}(\mathrm{span}\{x \cdot a \widehat{\otimes} y - x \widehat{\otimes} a \cdot y : x \in X, y \in Y, a \in A\})$. Let $\phi \in \mathcal{B}_A(X_1, X_2)$ and $\psi \in {}_A\mathcal{B}(Y_1, Y_2)$ for right Banach A-modules X_1, X_2 and left Banach A-modules Y_1, Y_2, then there exists a unique bounded linear operator

$$\phi \widehat{\otimes}_A \psi : X_1 \widehat{\otimes}_A Y_1 \to X_2 \widehat{\otimes}_A Y_2$$

such that $(\phi \widehat{\otimes}_A \psi)(x \widehat{\otimes}_A y) = \phi(x) \widehat{\otimes}_A \psi(y)$ for all $x \in X_1$ and $y \in Y_1$. Even more $\|\phi \widehat{\otimes}_A \psi\| \leq \|\phi\|\|\psi\|$. The projective module tensor product has its own universal property: for any right Banach A-module X, any left Banach A-module Y and any

Banach space E there exists an isometric isomorphism:

$$\mathcal{B}(X \mathbin{\widehat{\otimes}}_A Y, E) \underset{\mathbf{Ban}_1}{\cong} \mathcal{B}_{bal}(X \times Y, E)$$

where $\mathcal{B}_{bal}(X \times Y, E)$ stands for the Banach space of bilinear operators $\Phi : X \times Y \to E$ satisfying $\Phi(x \cdot a, y) = \Phi(x, a \cdot y)$ for all $x \in X$, $y \in Y$ and $a \in A$. Such bilinear operators are called balanced. Furthermore, we have two (natural in X, Y and E) isometric isomorphisms:

$$\mathcal{B}(X \mathbin{\widehat{\otimes}}_A Y, E) \underset{\mathbf{Ban}_1}{\cong} {}_A\mathcal{B}(Y, \mathcal{B}(X, E)) \underset{\mathbf{Ban}_1}{\cong} \mathcal{B}_A(X, \mathcal{B}(Y, E))$$

Analogously to Banach space theory we may define the following functors:

$$\mathcal{B}(-, E) : A - \mathbf{mod} \to \mathbf{mod} - A \qquad \mathcal{B}(E, -) : \mathbf{mod} - A \to \mathbf{mod} - A$$

$$- \mathbin{\widehat{\otimes}}_A Y : \mathbf{mod} - A \to \mathbf{Ban} \qquad X \mathbin{\widehat{\otimes}}_A - : A - \mathbf{mod} \to \mathbf{Ban}$$

where E is a Banach space, X is a right A-module and Y is a left A-module. All these functors have their counterparts for categories $A - \mathbf{mod}_1$, $\mathbf{mod}_1 - A$.

In some cases it is possible to explicitly compute the projective module tensor product. For example [[23], proposition 6.3.24], if A is a Banach algebra and I is a closed left ideal of A_+ with left $\langle$ contractive / bounded $\rangle$ approximate identity, and X is a left Banach A-module then the linear operator

$$i_{I,X} : I \mathbin{\widehat{\otimes}}_A X \to \mathrm{cl}_X(IX) : a \mathbin{\widehat{\otimes}}_A x \mapsto a \cdot x$$

is $\langle$ a topological isomorphism / an isometric isomorphism $\rangle$ of Banach spaces. If I is a closed two-sided ideal, then $i_{I,X}$ is a morphism of left A-modules. We call reduced all left Banach modules of the form $A \mathbin{\widehat{\otimes}}_A X$.

Most of what have been said here can be generalized to Banach bimodules, but in this book we shall not exploit them much. In those rare case when we shall encounter bimodules, the respective definitions and results are easily recoverable from their one-sided counterparts.

1.3 Banach homology

1.3.1 Relative homology

Further we briefly discuss ABCs of so-called relative homology introduced and intensively studied by Helemskii. In this text we shall use a bit more involved definition given by White in [29], though it is equivalent to the one given by Helemskii. Fix an arbitrary Banach algebra A. We say that a morphism $\xi : X \to Y$ of left A-modules X and Y is a c-relatively admissible epimorphism if it admits a right inverse bounded linear operator of the norm at most c. A left A-module P is called C-relatively projective if for any c-relatively admissible epimorphism $\xi : X \to Y$ and for any A-morphism $\phi : P \to Y$ there exists an A-morphism $\psi : P \to X$ such that $\|\psi\| \leq cC\|\phi\|$ and the diagram

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ P \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"] & Y \end{tikzcd}$$

is commutative. Such A-morphism ψ is called a lifting of ϕ, and it is not unique in general. Similarly, we say that a morphism $\xi : Y \to X$ of right A-modules X and Y is a c-relatively admissible monomorphism if it admits a left inverse bounded linear operator of the norm at most c. A right A-module J is called C-relatively injective if for any c-relatively admissible monomorphism $\xi : Y \to X$ and for any A-morphism $\phi : Y \to J$ there exists an A-morphism $\psi : X \to J$ such that $\|\psi\| \leq cC\|\phi\|$ and the diagram

$$\begin{tikzcd} & X \arrow[dl, dashed, "\psi"'] \\ J & Y \arrow[l, "\phi"] \arrow[u, "\xi"'] \end{tikzcd}$$

is commutative. Such A-morphism ψ is called an extension of ϕ, and it is not unique in general.

Another closely related homological property is flatness. A left A-module F is called C-relatively flat if for each c-relatively admissible monomorphism $\xi : X \to Y$ the operator $\xi \mathbin{\widehat{\otimes}}_A 1_F : X \mathbin{\widehat{\otimes}}_A F \to Y \mathbin{\widehat{\otimes}}_A F$ is cC-topologically injective. A left A-module F is C-relatively flat iff F^* is C-relatively injective.

We say that an A-morphism is relatively admissible if it is c-relatively admissible for some $c \geq 1$. Similarly, we say that an A-module X is relatively $\langle$ projective / injective / flat $\rangle$ if it is C-relatively $\langle$ projective / injective / flat $\rangle$ for some $C \geq 1$.

The reason for considering relatively admissible morphisms in these definitions is the intention to separate Banach geometric and algebraic properties of a Banach module that may prevent it to be relatively projective, injective or flat. A straightforward check shows that any c-retract of a C-relatively $\langle$ projective / injective / flat $\rangle$ A-module is again cC-relatively $\langle$ projective / injective / flat $\rangle$. Obviously, any c-relatively admissible $\langle$ epimorphism / monomorphism $\rangle$ $\langle$ onto / from $\rangle$ a C-relatively $\langle$ projective / injective $\rangle$ A-module is a $\langle$ cC-retraction / cC-coretraction $\rangle$.

A special class of relatively $\langle$ projective / injective $\rangle$ A-modules is the class of so-called relatively $\langle$ free / cofree $\rangle$ modules. These are modules of the form $\langle$ $A_+ \widehat{\otimes} E$ / $\mathcal{B}(A_+, E)$ $\rangle$ for some Banach space E. Their main feature is the following: for any A-module X there exists a relatively $\langle$ free / cofree $\rangle$ A-module F, which in fact is $\langle$ $A_+ \widehat{\otimes} X$ / $\mathcal{B}(A_+, X)$ $\rangle$ and a 1-relatively admissible $\langle$ epimorphism $\xi : F \to X$ / monomorphism $\xi : X \to F$ $\rangle$. If X is C-relatively $\langle$ projective / injective $\rangle$ then we immediately get that ξ is a $\langle$ C-retraction / C-coretraction $\rangle$. Therefore, an A-module is C-relatively $\langle$ projective / injective $\rangle$ iff it is a C-retract of relatively $\langle$ free / cofree $\rangle$ A-module.

It is worth emphasizing one more time that the major nuance of relative Banach homology is a balance between algebra and topology in the choice of admissible morphisms. This choice allows one to build a homological theory with some interesting phenomena and with no analogs in pure algebra. We demonstrate an example related to Banach algebras. For a given Banach algebra define an A-morphism of A-bimodules $\Pi_A : A \widehat{\otimes} A \to A : a \widehat{\otimes} b \mapsto ab$. We say that a Banach algebra A is

(i) c-relatively biprojective if Π_A is a c-retraction of A-bimodules;

(ii) c-relatively biflat if Π_A^* is a c-coretraction of A-bimodules;

(iii) c-relatively contractible if Π_{A_+} is a c-retraction of A-bimodules;

(iv) c-relatively amenable if $\Pi_{A_+}^*$ is a c-coretraction of A-bimodules.

We say that A is relatively $\langle$ biprojective / biflat / contractive / amenable $\rangle$ if it is c-relatively $\langle$ biprojective / biflat / contractive / amenable $\rangle$ for some $c \geq 1$. The infimum of the constants c is called the $\langle$ biprojectivity / biflatness / contractivity / amenability $\rangle$ constant. With slight modifications of [[23], proposition 7.1.72] one can show that A is c-relatively $\langle$ contractible / amenable $\rangle$ iff there exists $\langle$ an element $d \in A \widehat{\otimes} A$ / a net $(d_\nu)_{\nu \in N} \subset A \widehat{\otimes} A$ $\rangle$ with norm not greater than c such that for all $a \in A$ holds $\langle$ $a \cdot d - d \cdot a = 0$ and $a\Pi_A(d) = a$ / $\lim_\nu (a \cdot d_\nu - d_\nu \cdot a) = 0$ and $\lim_\nu a\Pi_A(d_\nu) = a$ $\rangle$. Note that $\langle$ such element d / such net $(d_\nu)_{\nu \in N}$ $\rangle$ is called $\langle$ a diagonal / an approximate diagonal $\rangle$. From homological point of view, the main advantage of relatively $\langle$ biprojective / biflat

/ contractible / amenable $\rangle$ Banach algebras is that $\langle$ any reduced / any reduced / any / any $\rangle$ left and right Banach A-module is relatively $\langle$ projective / flat / projective / flat $\rangle$ [[23], theorem 7.1.60]. As for flatness such phenomena is typical for relative Banach homology, but not for the purely algebraic one.

Chapter 2

General theory

2.1 Projectivity, injectivity and flatness

2.1.1 Metric and topological projectivity

In what follows A denotes a not necessary unital Banach algebra. We immediately start from the most important definitions in this book.

Definition 2.1.1 ([30], definition 1.4; [29], definition 2.4)**.** An A-module P is called $\langle$ metrically / C-topologically $\rangle$ projective if for any strictly $\langle$ coisometric / c-topologically surjective $\rangle$ A-morphism $\xi : X \to Y$ and for any A-morphism $\phi : P \to Y$ there exists an A-morphism $\psi : P \to X$ such that $\xi\psi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq cC\|\phi\|$ $\rangle$. We say that an A-module P is topologically projective if it is C-topologically projective for some $C \geq 1$.

The task of constructing an A-morphism ψ for a given A-morphisms ϕ and ξ in the definition 2.1.1 is called a lifting problem and ψ is called a lifting of ϕ along ξ.

A short but more involved equivalent definition of $\langle$ metric / C-topological $\rangle$ projectivity is the following: an A-module P is called $\langle$ metrically / C-topologically $\rangle$ projective, if the functor $\langle$ $\operatorname{Hom}_{A-\mathbf{mod}_1}(P, -) : A - \mathbf{mod}_1 \to \mathbf{Ban}_1$ / $\operatorname{Hom}_{A-\mathbf{mod}}(P, -) : A - \mathbf{mod} \to \mathbf{Ban}$ $\rangle$ maps strictly $\langle$ coisometric / c-topologically surjective $\rangle$ A-morphisms into strictly $\langle$ coisometric / cC-topologically surjective $\rangle$ operators.

In category $\langle$ $A - \mathbf{mod}_1$ / $A - \mathbf{mod}$ $\rangle$ there is a special class of $\langle$ metrically / 1-topologically $\rangle$ projective modules of the form $A_+ \widehat{\otimes} \ell_1(\Lambda)$ for some set Λ. They are called free modules. These modules play a crucial role in our studies of projectivity.

Proposition 2.1.2 ([29], lemma 2.6)**.** *Let Λ be an arbitrary set, then the left A-modules $A_+ \widehat{\otimes} \ell_1(\Lambda)$ and $A_\times \widehat{\otimes} \ell_1(\Lambda)$ are $\langle$ metrically / 1-topologically $\rangle$ projective.*

Proof. By $A_\bullet$ we denote either A_+ or $A_\times$. Consider arbitrary A-morphism $\phi : A_\bullet \widehat{\otimes} \ell_1(\Lambda) \to X$ and a strictly $\langle$ coisometric / c-topologically surjective $\rangle$ A-morphism $\xi : X \to Y$. Fix an arbitrary $\lambda \in \Lambda$ and consider $y_\lambda = \phi(e_{A_\bullet} \widehat{\otimes} \delta_\lambda)$. Clearly $\|y_\lambda\| \leq \|\phi\|$. Since ξ is strictly $\langle$ coisometric / c-topologically surjective $\rangle$, then there exists $x_\lambda \in X$ such that $\xi(x_\lambda) = y_\lambda$ and $\|x_\lambda\| \leq K\|y_\lambda\|$ for $\langle$ $K = 1$ / $K = c$ $\rangle$. Now we can define a bounded linear operator $\psi : A_\bullet \widehat{\otimes} \ell_1(\Lambda) \to X$ such that $\psi(a \widehat{\otimes} \delta_\lambda) = a \cdot x_\lambda$ for $\lambda \in \Lambda$. It is routine to check that ψ is an A-morphism with $\xi\psi = \phi$ and $\|\psi\| \leq K\|\phi\|$. Thus, for a given ϕ and ξ we have constructed an A-morphism ψ such that $\xi\psi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq c\|\phi\|$ $\rangle$. Hence, the A-module $A_\bullet \widehat{\otimes} \ell_1(\Lambda)$ is $\langle$ metrically / 1-topologically $\rangle$ projective. □

Proposition 2.1.3. *The left A-module $A_\times$ is $\langle$ metrically / 1-topologically $\rangle$ projective.*

Proof. Consider set $\Lambda = \mathbb{N}_1$. By proposition 2.1.2 the A-module $A_\times \widehat{\otimes} \ell_1(\Lambda)$ is metrically and 1-topologically projective. Now it remains to note that $A_\times \widehat{\otimes} \ell_1(\Lambda) \underset{A-\mathbf{mod}_1}{\cong} A_\times \widehat{\otimes} \mathbb{C} \underset{A-\mathbf{mod}_1}{\cong} A_\times$. □

Proposition 2.1.4 ([29], lemma 2.7)**.** *Any $\langle$ 1-retract / c-retract $\rangle$ of a $\langle$ metrically / C-topologically $\rangle$ projective module is $\langle$ metrically / cC-topologically $\rangle$ projective.*

Proof. Suppose that P is a c-retract of a $\langle$ metrically / C-topologically $\rangle$ projective A-module P'. Then there exist A-morphisms $\eta : P \to P'$ and $\zeta : P' \to P$ such that $\zeta\eta = 1_P$ and $\|\zeta\|\|\eta\| \leq c$ for $\langle$ $c = 1$ / $c \geq 1$ $\rangle$. Consider arbitrary strictly $\langle$ coisometric / c'-topologically surjective $\rangle$ A-morphism $\xi : X \to Y$ and an arbitrary A-morphism $\phi : P \to Y$. Consider A-morphism $\phi' = \phi\zeta$. Since P' is $\langle$ metrically / C-topologically $\rangle$ projective, then there exists an A-morphism $\psi' : P' \to X$ such that $\phi' = \xi\psi'$ and $\|\psi'\| \leq K\|\phi'\|$ for $\langle$ $K = 1$ / $K = c'C$ $\rangle$. Now it is routine to check that for the A-morphism $\psi = \psi'\eta$ holds $\xi\psi = \phi$ and $\|\psi\| \leq cK\|\phi\|$. Thus, for a given ϕ and ξ we have constructed an A-morphism ψ such that $\xi\psi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq cC\|\phi\|$ $\rangle$. Hence, P is $\langle$ metrically / cC-topologically $\rangle$ projective. □

It is easy to show that, for any A-module X there exists a strictly $\langle$ coisometric / 1-topologically surjective $\rangle$ A-morphism

$$\pi_X^+ : A_+ \widehat{\otimes} \ell_1(B_X) \to X,\ a \widehat{\otimes} \delta_x \mapsto a \cdot x.$$

Proposition 2.1.5 ([29], proposition 2.10). *An A-module P is $\langle$ metrically / C-topologically $\rangle$ projective iff π_P^+ is a $\langle$ 1-retraction / C-retraction $\rangle$ in $A-\mathbf{mod}$.*

Proof. Suppose P is $\langle$ metrically / C-topologically $\rangle$ projective, then consider strictly $\langle$ coisometric / 1-topologically surjective $\rangle$ A-morphism π_P^+. Consider lifting problem with $\phi = 1_P$ and $\xi = \pi_P^+$, then from $\langle$ metric / C-topological $\rangle$ projectivity of P we get an A-morphism σ^+ such that $\pi_P^+\sigma^+ = 1_P$ and $\|\sigma^+\| \leq K$ for $\langle$ $K = 1$ / $K = C$ $\rangle$. Since $\|\pi_P^+\|\|\sigma^+\| \leq K$ we conclude that π_P^+ is a $\langle$ 1-retraction / C-retraction $\rangle$ in $A - \mathbf{mod}$.

Conversely, assume that π_P^+ is a $\langle$ 1-retraction / C-retraction $\rangle$. In other words P is a $\langle$ 1-retract / C-retract $\rangle$ of $A_+\widehat{\otimes}\ell_1(B_P)$. By proposition 2.1.2 the A-module $A_+\widehat{\otimes}\ell_1(B_P)$ is $\langle$ metrically / 1-topologically $\rangle$ projective. So from proposition 2.1.4 its $\langle$ 1-retract / C-retract $\rangle$ P is $\langle$ metrically / C-topologically $\rangle$ projective. □

Proposition 2.1.6 ([29], proposition 2.9). *Every metrically projective module is 1-topologically projective and every C-topologically projective module is C-relatively projective.*

Proof. By proposition 2.1.5 every metrically projective A-module P is a 1-retract of $A_+ \widehat{\otimes} \ell_1(B_P)$. Hence, by the same proposition P is 1-topologically projective. Again by proposition 2.1.5 every C-topologically projective A-module P is a C-retract of the A-module $A_+ \widehat{\otimes} \ell_1(B_P)$. In other words P is a C-retract of the module $A \widehat{\otimes} E$ for some Banach space E. Therefore, P is C-relatively projective. □

Clearly, every C-topologically projective module is C'-topologically projective for $C' \geq C$. But we can state even more.

Proposition 2.1.7. *An A-module is metrically projective iff it is 1-topologically projective.*

Proof. The result immediately follows from propositions 2.1.6 and 2.1.5. □

Let us proceed to examples. Note that the category of Banach spaces may be regarded as the category of left Banach modules over zero algebra. As a result we get the definition of $\langle$ metrically / topologically $\rangle$ projective Banach space. All the results mentioned above hold for this type of projectivity. Both types of projective objects are described by now. In [31] Köthe proved that all topologically projective Banach spaces are topologically isomorphic to $\ell_1(\Lambda)$ for some index set Λ. Using result of Grothendieck from [32] Helemskii showed that metrically projective Banach spaces are isometrically isomorphic to $\ell_1(\Lambda)$ for some index set Λ [[30], proposition 3.2]. Thus, the zoo of projective Banach spaces is wide but conformed.

Proposition 2.1.8. *Let P be an essential A-module. Then P is $\langle$ metrically / C-topologically $\rangle$ projective iff the map $\pi_P : A \widehat{\otimes} \ell_1(B_P)$, $a \widehat{\otimes} \delta_x \mapsto a \cdot x$ is a $\langle$ 1-retraction / C-retraction $\rangle$ in $A - \mathbf{mod}$.*

Proof. If P is $\langle$ metrically / C-topologically $\rangle$ projective, then by proposition 2.1.5 the morphism π_P^+ has a right inverse morphism σ^+ of norm $\langle$ at most 1 / at most C $\rangle$. Then

$$\sigma^+(P) = \sigma^+(\mathrm{cl}_{A_+ \widehat{\otimes} \ell_1(B_P)}(AP)) \subset \mathrm{cl}_{A_+ \widehat{\otimes} \ell_1(B_P)}(A \cdot \sigma(P)) =$$

$$\mathrm{cl}_{A_+ \widehat{\otimes} \ell_1(B_P)}(A \cdot (A_+ \widehat{\otimes} \ell_1(B_P))) = A \widehat{\otimes} \ell_1(B_P).$$

So we have well a defined corestriction $\sigma : P \to A \widehat{\otimes} \ell_1(B_P)$ which is also an A-morphism with norm $\langle$ at most 1 / at most C $\rangle$. Clearly, $\pi_P \sigma = 1_P$, so π_P is a $\langle$ 1-retraction / C-retraction $\rangle$ in $A - \mathbf{mod}$.

Conversely, assume π_P has a right inverse morphism σ of norm $\langle$ at most 1 / at most C $\rangle$. Then its coextension σ^+ also is a right inverse morphism to π_P^+ with the same norm. Again, by proposition 2.1.5 the module P is $\langle$ metrically / C-topologically $\rangle$ projective. □

Proposition 2.1.9. *Let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of A-modules. Then*

(i) $\bigoplus_1 \{P_\lambda : \lambda \in \Lambda\}$ *is metrically projective iff for all $\lambda \in \Lambda$ the A-module P_λ is metrically projective;*

(ii) $\bigoplus_1 \{P_\lambda : \lambda \in \Lambda\}$ *is C-topologically projective iff for all $\lambda \in \Lambda$ the A-module P_λ is C-topologically projective.*

Proof. Denote $P := \bigoplus_1 \{P_\lambda : \lambda \in \Lambda\}$.

(i) The proof is literally the same as in paragraph (ii).

(ii) Assume that P is C-topologically projective. Note that, for any $\lambda \in \Lambda$ the A-module P_λ is a 1-retract of P via natural projection $p_\lambda : P \to P_\lambda$. By proposition 2.1.4 the A-module P_λ is C-topologically projective.

Conversely, let each A-module P_λ be C-topologically projective. By proposition 2.1.5 we have a family of C-retractions $\pi_\lambda : A_+ \widehat{\otimes} \ell_1(S_\lambda) \to P_\lambda$. It follows that $\bigoplus_1 \{\pi_\lambda : \lambda \in \Lambda\}$ is a C-retraction in $A - \mathbf{mod}$. As a result P is a C-retract of

$$\bigoplus\nolimits_1 \left\{A_+ \widehat{\otimes} \ell_1(S_\lambda) : \lambda \in \Lambda\right\} \underset{A-\mathbf{mod}_1}{\cong} \bigoplus\nolimits_1 \left\{\bigoplus\nolimits_1 \{A_+ : s \in S_\lambda\} : \lambda \in \Lambda\right\}$$

$$\underset{A-\mathbf{mod}_1}{\cong} \bigoplus\nolimits_1 \{A_+ : s \in S\} \underset{A-\mathbf{mod}_1}{\cong} A_+ \widehat{\otimes} \ell_1(S)$$

in $A-\mathbf{mod}$ where $S = \bigsqcup_{\lambda\in\Lambda} S_\lambda$. Clearly, the latter module is 1-topologically projective, so by proposition 2.1.4 the A-module P is C-topologically projective. $\square$

Corollary 2.1.10. *Let P be an A-module and Λ be an arbitrary set. Then $P \widehat{\otimes} \ell_1(\Lambda)$ is $\langle$ metrically / C-topologically $\rangle$ projective iff P is $\langle$ metrically / C-topologically $\rangle$ projective.*

Proof. Note that $P \widehat{\otimes} \ell_1(\Lambda) \underset{A-\mathbf{mod}_1}{\cong} \bigoplus_1 \{P : \lambda \in \Lambda\}$. It remains to set $P_\lambda = P$ for all $\lambda \in \Lambda$ and apply proposition 2.1.9. $\square$

The property of being metrically, topologically or relatively projective module puts some restrictions on the Banach geometric structure of the module.

Proposition 2.1.11 ([33], proposition 2.1.1)**.** *Let P be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ projective A-module, and let I be a $\langle$ 1-complemented / c-complemented / c-complemented $\rangle$ closed right ideal of A. Then $\mathrm{cl}_P(IP)$ is $\langle$ 1-complemented / cC-complemented / cC-complemented $\rangle$ in P.*

Proof. Since A is 1-complemented in A_+, then I is $\langle$ 1-complemented / c-complemented / c-complemented $\rangle$ in A_+. Hence, there exists a bounded linear operator $r : A_+ \to I$ of $\langle$ norm 1 / norm c / norm c $\rangle$ such that $r|_I = 1_I$. Consider operator $R = r \widehat{\otimes} 1_P$, then $R|_{I \widehat{\otimes} P} = 1_{I \widehat{\otimes} P}$ and $\|R\| \leq \|r\|$. Since P is $\langle$ metrically / C-topologically / C-relatively $\rangle$ projective, by proposition 2.1.5 the A-morphism π_P^+ has a right inverse morphism σ^+ of norm $\langle$ at most 1 / at most C / at most C $\rangle$. Now consider a bounded linear operator $p = \pi_P^+ R \sigma^+$. Clearly, $\mathrm{Im}(p) = \pi_P^+(R(\sigma^+(P))) \subset \pi_P^+(R(A_+ \widehat{\otimes} P)) \subset \pi_P^+(I \widehat{\otimes} P) \subset \mathrm{cl}_P(IP)$. Since I is a right ideal of A and σ^+ is an A-morphism, then

$$\sigma^+(\mathrm{cl}_P(IP)) \subset \mathrm{cl}_{A_+ \widehat{\otimes} P}(\sigma^+(IP)) \subset \mathrm{cl}_{A_+ \widehat{\otimes} P}(I\sigma^+(P)) \subset \mathrm{cl}_{A_+ \widehat{\otimes} P}(I(A_+ \widehat{\otimes} P)) \subset I \widehat{\otimes} P.$$

In particular, for all $x \in \mathrm{cl}_P(IP)$, we have $p(x) = \pi_P^+(R(\sigma^+(x))) = \pi_P^+(1_{I \widehat{\otimes} P}(\sigma^+(x))) = \pi_P^+(\sigma^+(x)) = x$. Thus, p is a projection onto $\mathrm{cl}_P(IP)$ with norm at most $\|\sigma^+\|\|r\|$. Hence, $\mathrm{cl}_P(IP)$ is $\langle$ 1-complemented / cC-complemented / cC-complemented $\rangle$ in P. $\square$

Corollary 2.1.12. *Let P be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ projective A-moudle. Then P_{ess} is $\langle$ 1-complemented / C-complemented / C-complemented $\rangle$ in P.*

Proof. The result directly follows from proposition 2.1.11. $\square$

2.1.2 Metric and topological projectivity of ideals and cyclic modules

As we shall see idempotents play a significant role in the study of metric and topological projectivity, so we shall recall one of the corollaries of Shilov's idempotent theorem [[26], section 3.5]: every semisimple commutative Banach algebra with a compact spectrum admits an identity, but not necessarily of norm 1.

Proposition 2.1.13. *Let I be a closed left ideal of a Banach algebra A. Then*

(i) if $I = Ap$ for some $\langle$ norm one idempotent / idempotent $\rangle$ $p \in I$, then I is $\langle$ metrically / $\|p\|$-topologically $\rangle$ projective A-module;

(ii) if I is commutative, semisimple and $\operatorname{Spec}(I)$ is compact then I is topologically projective A-module.

Proof. (i) For A-module maps $\pi : A_\times \to I : x \mapsto xp$ and $\sigma : I \to A_\times : x \mapsto x$ we clearly have $\pi\sigma = 1_I$. Therefore, I is a $\langle$ 1-retract / $\|p\|$-retract $\rangle$ of $A_\times$. Now the result follows from propositions 2.1.4 and 2.1.3.

(ii) By Shilov's idempotent theorem the ideal I is unital. By paragraph (i) the ideal I is topologically projective. □

The assumption of semisimplicity in 2.1.13 is not necessary. From [[34], exercise 2.3.7] we know, that there exists a commutative non semisimple unital Banach algebra A. By proposition 2.1.13 it is topologically projective as A-module. To prove the main result of this section we need two preparatory lemmas.

Lemma 2.1.14. *Let I be a closed two-sided ideal of a Banach algebra A which is essential as left I-module and let $\phi : I \to A$ be an A-morphism. Then $\operatorname{Im}(\phi) \subset I$.*

Proof. Since I is a right ideal, then $\phi(ab) = a\phi(b) \in I$ for all $a, b \in I$. So $\phi(I \cdot I) \subset I$. Since I is an essential left I-module then $I = \operatorname{cl}_A(\operatorname{span}(I \cdot I))$ and $\operatorname{Im}(\phi) \subset \operatorname{cl}_A(\operatorname{span} \phi(I \cdot I)) = \operatorname{cl}_A(\operatorname{span} I) = I$. □

Lemma 2.1.15. *Let I be a closed left ideal of a Banach algebra A, which is $\langle$ metrically / C-topologically $\rangle$ projective as an A-module. Then the following holds:*

(i) Assume I has a left $\langle$ contractive / c-bounded $\rangle$ approximate identity and for each morphism $\phi : I \to A$ of left A-modules there exists a morphism $\psi : I \to I$ of right I-modules such that $\phi(x)y = x\psi(y)$ for all $x, y \in I$. Then I has the identity of norm $\langle$ at most 1 / at most c $\rangle$;

(*ii*) *Assume I has a right $\langle$ contractive / c-bounded $\rangle$ approximate identity and for $\langle$ $k = 1$ / some $k \geq 1$ $\rangle$ and each morphism $\phi : I \to A$ of left A-modules there exists a morphism $\psi : I \to I$ of right I-modules such that $\|\psi\| \leq k\|\phi\|$ and $\phi(x)y = x\psi(y)$ for all $x, y \in I$. Then I has a right identity of norm $\langle$ at most 1 / at most ckC $\rangle$.*

Proof. If either (*i*) or (*ii*) holds then I has a one-sided bounded approximate identity. So I is an essential left I-module, and a fortiori an essential A-module. By proposition 2.1.8 we have a right inverse A-morphism $\sigma : I \to A \widehat{\otimes} \ell_1(B_I)$ of π_I with norm $\langle$ at most 1 / at most C $\rangle$. For each $d \in B_I$ consider A-morphisms $p_d : A \widehat{\otimes} \ell_1(B_I) \to A : a \widehat{\otimes} \delta_x \mapsto \delta_x(d)a$ and $\sigma_d = p_d\sigma$. Then $\sigma(x) = \sum_{d\in B_I} \sigma_d(x) \widehat{\otimes} \delta_d$ for all $x \in I$. From identification $A \widehat{\otimes} \ell_1(B_I) \underset{\mathbf{Ban}_1}{\cong} \bigoplus_1\{A : d \in B_I\}$ we have $\|\sigma(x)\| = \sum_{d\in B_I} \|\sigma_d(x)\|$ for all $x \in I$. Since σ is a right inverse morphism of π_I, then $x = \pi_I(\sigma(x)) = \sum_{d\in B_I} \sigma_d(x)d$ for all $x \in I$.

Assume (*i*) holds. From assumption, for each $d \in B_I$ there exists a morphism of right I-modules $\tau_d : I \to I$ such that $\sigma_d(x)d = x\tau_d(d)$ for all $x \in I$. Let $(e_\nu)_{\nu\in N}$ be a left $\langle$ contractive / bounded $\rangle$ approximate identity of I bounded in norm by some constant $\langle$ $D = 1$ / $D = c$ $\rangle$. Since $\tau_d(d) \in I$ for all $d \in B_I$, then for all $S \in \mathcal{P}_f(B_I)$ holds

$$\sum_{d\in S} \|\tau_d(d)\| = \sum_{d\in S} \lim_\nu \|e_\nu\tau_d(d)\| = \lim_\nu \sum_{d\in S} \|e_\nu\tau_d(d)\| = \lim_\nu \sum_{d\in S} \|\sigma_d(e_\nu)d\|$$

$$\leq \liminf_\nu \sum_{d\in S} \|\sigma_d(e_\nu)\|\|d\| \leq \liminf_\nu \sum_{d\in S} \|\sigma_d(e_\nu)\| \leq \liminf_\nu \sum_{d\in B_I} \|\sigma_d(e_\nu)\|$$
$$= \liminf_\nu \|\sigma(e_\nu)\| \leq \|\sigma\| \liminf_\nu \|e_\nu\| \leq D\|\sigma\|$$

Since $S \in \mathcal{P}_f(B_I)$ is arbitrary we have a well-defined element $p = \sum_{d\in B_I} \tau_d(d)$ with norm $\langle$ at most 1 / at most cC $\rangle$. For all $x \in I$ we have $x = \sum_{d\in B_I} \sigma_d(x)d = \sum_{d\in B_I} x\tau_d(d) = xp$, i.e. p is a right identity for I. But I admits a left $\langle$ contractive / c-bounded $\rangle$ approximate identity, so p is the identity of I with $\|p\| = \lim_\nu \|e_\nu\|$. Therefore, the norm of p is $\langle$ at most 1 / at most c $\rangle$.

Assume (*ii*) holds. From assumption, for each $d \in B_I$ there exists a morphism of right I-modules $\tau_d : I \to I$ such that $\sigma_d(x)d = x\tau_d(d)$ for all $x \in I$ and $\|\tau_d\| \leq k\|\sigma_d\|$. Let $(e_\nu)_{\nu\in N}$ be a right $\langle$ contractive / bounded $\rangle$ approximate identity of I bounded in norm by some constant $\langle$ $D = 1$ / $D = c$ $\rangle$. For all $x \in I$ we have

$$\|\sigma_d(x)\| = \|\sigma_d(\lim_\nu xe_\nu)\| = \lim_\nu \|x\sigma_d(e_\nu)\| \leq \|x\| \liminf_\nu \|\sigma_d(e_\nu)\|$$

so $\|\sigma_d\| \leq \liminf_\nu \|\sigma_d(e_\nu)\|$. Then for all $S \in \mathcal{P}_f(B_I)$ holds

$$\sum_{d\in S} \|\tau_d(d)\| \leq \sum_{d\in S} \|\tau_d\|\|d\| \leq k\sum_{d\in S} \|\sigma_d\| \leq k\sum_{d\in S} \liminf_\nu \|\sigma_d(e_\nu)\| \leq k \liminf_\nu \sum_{d\in S} \|\sigma_d(e_\nu)\|$$

$$\leq k \liminf_{\nu} \sum_{d \in B_I} \|\sigma_d(e_\nu)\| = k \liminf_{\nu} \|\sigma(e_\nu)\| \leq k\|\sigma\| \liminf_{\nu} \|e_\nu\| \leq kD\|\sigma\|$$

Since $S \in \mathcal{P}_f(B_I)$ is arbitrary we have a well-defined element $p = \sum_{d \in B_I} \tau_d(d)$ with norm $\langle$ at most 1 / at most ckC $\rangle$. For all $x \in I$ we have $x = \sum_{d \in B_I} \sigma_d(x)d = \sum_{d \in B_I} x\tau_d(d) = xp$, i.e. p is a right identity for I. □

Theorem 2.1.16. *Let I be a closed ideal of a commutative Banach algebra A and I has a $\langle$ contractive / c-bounded $\rangle$ approximate identity. Then I is $\langle$ metrically / c-topologically $\rangle$ projective as an A-module iff I has the identity of norm $\langle$ at most 1 / at most c $\rangle$.*

Proof. Assume I is $\langle$ metrically / c-topologically $\rangle$ projective as A-module. Since A is commutative, then for any A-morphism $\phi : I \to A$ and $x, y \in I$ we have $\phi(x)y = x\phi(y)$. Since I has a bounded approximate identity and I is commutative we can apply lemma 2.1.14 to get that $\phi(y) \in I$. Now by paragraph (i) of lemma 2.1.15 we get that I has the identity of norm $\langle$ at most 1 / at most c $\rangle$.

The converse immediately follows from proposition 2.1.13. □

There is no analogous criterion of this theorem in relative theory. The most general result of this kind gives only a necessary condition: any closed ideal in a commutative Banach algebra A which is relatively projective as an A-module has a paracompact spectrum. This result is due to Helemskii [[24], theorem IV.3.6].

Note that existence of some bounded approximate identity is not necessary for a closed ideal of a commutative Banach algebra to be even topologically projective. Indeed, consider Banach algebra $A_0(\mathbb{D})$ — the ideal of the disk algebra consisting of functions vanishing at zero. By combination of propositions 4.3.5 and 4.3.13 paragraph (iii) from [25] we get that $A_0(\mathbb{D})$ has no bounded approximate identities. On the other hand, from [[23], example IV.2.2] we know that $A_0(\mathbb{D}) \underset{A_0(\mathbb{D})-\mathbf{mod}}{\cong} A_0(\mathbb{D})_+$, so $A_0(\mathbb{D})$ is topologically projective by proposition 2.1.3.

Next proposition is an obvious adaptation of purely algebraic argument for projective cyclic modules. It is almost identical to [[29], proposition 2.11].

Proposition 2.1.17. *Let I be a closed left ideal in $A_\times$. Assume the natural projection $\pi : A_\times \to A_\times/I$ is strictly $\langle$ coisometric / c-topologically surjective $\rangle$. Then the following holds:*

(i) *If $A_\times/I$ is $\langle$ metrically / C-topologically $\rangle$ projective as A-module, then there exists an idempotent $p \in I$ such that $I = Ap$ and $\|e_{A_\times} - p\|$ is $\langle$ at most 1 / at most cC $\rangle$;*

(ii) *If there exists an idempotent $p \in I$ such that $I = A_{\times}p$ and $\|e_{A_{\times}} - p\|$ is $\langle$ at most 1 / at most C $\rangle$, then $A_{\times}/I$ is $\langle$ metrically / C-topologically $\rangle$ projective.*

Proof. (i) Since the natural quotient map π is strictly $\langle$ coisometric / c-topologically surjective $\rangle$ and $A_{\times}/I$ is $\langle$ metrically / C-topologically $\rangle$ projective, then π has a right inverse A-morphism σ with norm $\langle$ at most 1 / at most cC $\rangle$. We set $e_{A_{\times}} - p = (\sigma\pi)(e_{A_{\times}})$, then $(\sigma\pi)(a) = a(e_{A_{\times}} - p)$. By construction, $\pi\sigma = 1_{A_{\times}}$, so

$$e_{A_{\times}} - p = (\sigma\pi)(e_{A_{\times}}) = (\sigma\pi)(\sigma\pi)(e_{A_{\times}}) = (\sigma\pi)(e_{A_{\times}} - p) = (e_{A_{\times}} - p)(\sigma\pi)(e_{A_{\times}}) = (e_{A_{\times}} - p)^2$$

This equality shows that $p^2 = p$. Therefore, $A_{\times}p = \mathrm{Ker}(\sigma\pi)$ because $(\sigma\pi)(a) = a - ap$. Since σ is injective this is equivalent to $A_{\times}p = \mathrm{Ker}(\pi)$ which equals to I. Finally, note that $\|e_{A_{\times}} - p\| = \|(\sigma\pi)(e_{A_{\times}})\| \leq \|\sigma\|\|\pi\|\|e_{A_{\times}}\| = \|\sigma\|$.

(ii) Since $p^2 = p$, then we have a well-defined closed left ideal $I = A_{\times}p$ and an A-module map $\sigma : A_{\times}/I \to A_{\times} : a + I \mapsto a - ap$. It is easy to check that $\pi\sigma = 1_{A_{\times}/I}$ and $\|\sigma\| \leq \|e_{A_{\times}} - p\|$. This means that $\pi : A_{\times} \to A_{\times}/I$ is a $\langle$ 1-retraction / C-retraction $\rangle$. From propositions 2.1.3 and 2.1.4 it follows that $A_{\times}/I$ is $\langle$ metrically / C-topologically $\rangle$ projective. □

In contrast with topological theory, there is no description of relatively projective cyclic modules. There are partial answers under additional assumptions. For example, if a closed left ideal I is complemented as Banach space in $A_{\times}$, then almost the same criterion as in previous proposition holds in relative theory [[23], proposition 7.1.29]. There are other characterizations of relatively projective cyclic modules under more mild assumptions on the Banach geometry of the cyclic module. For example, Selivanov proved that if I is a closed two-sided ideal and either A/I has the approximation property or all irreducible A-modules have the approximation property, then A/I is relatively projective iff $A_{\times} \underset{A-\mathbf{mod}}{\cong} I \bigoplus_1 I'$ for some closed left ideal I' of A. For details see [[24], chapter IV, §4].

2.1.3 Metric and topological injectivity

Unless otherwise stated we shall consider injectivity of right modules.

Definition 2.1.18 ([30], definition 4.3; [29], definition 3.4)**.** An A-module J is called $\langle$ metrically / C-topologically $\rangle$ injective if for any $\langle$ isometric / c-topologically injective $\rangle$ A-morphism $\xi : Y \to X$ and for any A-morphism $\phi : Y \to J$ there exists an A-morphism $\psi : X \to J$ such that $\psi\xi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq cC\|\phi\|$ $\rangle$. We say that an A-module J is topologically injective if it is C-topologically injective for some $C \geq 1$.

The task of constructing an A-morphism ψ for a given A-morphisms ϕ and ξ in the definition 2.1.18 is called an extension problem and ψ is called an extension of ϕ along ξ.

A short but more involved equivalent definition of $\langle$ metric / C-topological $\rangle$ injectivity is the following: an A-module J is called $\langle$ metrically / C-topologically $\rangle$ injective, if the functor $\langle$ $\operatorname{Hom}_{\mathbf{mod}_1-A}(-,J) : \mathbf{mod}_1 - A \to \mathbf{Ban}_1$ / $\operatorname{Hom}_{\mathbf{mod}-A}(-,J) : \mathbf{mod} - A \to \mathbf{Ban}$ $\rangle$ maps $\langle$ isometric / c-topologically injective $\rangle$ A-morphisms into strictly $\langle$ coisometric / cC-topologically surjective $\rangle$ operators.

In category $\langle$ $\mathbf{mod}_1 - A$ / $\mathbf{mod} - A$ $\rangle$ there is a special class of $\langle$ metrically / 1-topologically $\rangle$ injective modules of the form $\mathcal{B}(A_+, \ell_\infty(\Lambda))$ for some set Λ. They are called cofree modules. These modules play a crucial role in our studies of injectivity.

Proposition 2.1.19 ([29], lemma 3.6)**.** *Let Λ be an arbitrary set, then the left A-modules $\mathcal{B}(A_+, \ell_\infty(\Lambda))$ and $\mathcal{B}(A_\times, \ell_\infty(\Lambda))$ are $\langle$ metrically / 1-topologically $\rangle$ injective.*

Proof. By $A_\bullet$ we denote either A_+ or $A_\times$. Consider an arbitrary A-morphism $\phi : Y \to \mathcal{B}(A_\bullet, \ell_\infty(\Lambda))$ and $\langle$ an isometric / c-topologically injective $\rangle$ A-morphism $\xi : Y \to X$. Fix arbitrary $\lambda \in \Lambda$ and define a bounded linear functional $h_\lambda : Y \to \mathbb{C}$, $y \mapsto \phi(y)(e_{A_\times})(\lambda)$. Clearly, $\|h_\lambda\| \leq \|\phi\|$. Denote $X_0 = \operatorname{Im}(\xi)$ and consider A-morphism $\eta = \xi|^{X_0}$. Since ξ is $\langle$ an isometric / c-topologically injective $\rangle$, then X_0 is closed and η has a left inverse bounded linear operator $\zeta : X_0 \to Y$, such that ζ has norm at most $\langle$ $K = 1$ / $K = c$ $\rangle$. Now consider a bounded linear functional $f_\lambda = h_\lambda \zeta \in X_0^*$. By Hahn-Banach theorem we can extend f_λ to some bounded linear functional $g_\lambda : X \to \mathbb{C}$ with norm $\|g_\lambda\| = \|f_\lambda\| \leq \|h_\lambda\|\|\zeta\| \leq K\|\phi\|$. Consider A-morphism $\psi : X \to \mathcal{B}(A_\bullet, \ell_\infty(\Lambda)) : x \mapsto (a \mapsto (\lambda \mapsto g_\lambda(x \cdot a)))$. It is easy to check that $\psi\xi = \phi$ and $\|\psi\| \leq K\|\phi\|$. Thus, for a given ϕ and ξ we have constructed an A-morphism ψ such that $\psi\xi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq c\|\phi\|$ $\rangle$. Hence, the A-module $\mathcal{B}(A_\bullet, \ell_\infty(\Lambda))$ is $\langle$ metrically / 1-topologically $\rangle$ injective. ☐

Proposition 2.1.20. *The right A-module $A_\times^*$ is metrically and 1-topologically injective.*

Proof. Consider set $\Lambda = \mathbb{N}_1$. By proposition 2.1.19 the A-module $\mathcal{B}(A_\times, \ell_\infty(\Lambda))$ is metrically and 1-topologically injective. Now it remains to note that $\mathcal{B}(A_\times, \ell_\infty(\Lambda)) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{B}(A_\times, \mathbb{C}) \underset{\mathbf{mod}_1-A}{\cong} A_\times^*$. ☐

Proposition 2.1.21 ([29], lemma 3.7)**.** *Any $\langle$ 1-retract / c-retract $\rangle$ of a $\langle$ metrically / C-topologically $\rangle$ injective module is $\langle$ metrically / cC-topologically $\rangle$ injective.*

Proof. Suppose that J is a c-retract of a $\langle$ metrically / C-topologically $\rangle$ injective A-module J'. Then there exist A-morphisms $\eta : J \to J'$ and $\zeta : J' \to J$ such that $\zeta\eta = 1_J$ and $\|\zeta\|\|\eta\| \leq c$ for $\langle$ $c = 1$ / $c \geq 1$ $\rangle$. Consider arbitrary $\langle$ isometric / c'-topologically injective $\rangle$ A-morphism $\xi : Y \to X$ and an arbitrary A-morphism $\phi : Y \to J$. Consider A-morphism $\phi' = \eta\phi$. Since J' is $\langle$ metrically / C-topologically $\rangle$ injective, then there exists an A-morphism $\psi' : X \to J'$ such that $\phi' = \psi'\xi$ and $\|\psi'\| \leq K\|\phi'\|$ for $\langle$ $K = 1$ / $K = c'C$ $\rangle$. Now it is routine to check that for the A-morphism $\psi = \zeta\psi'$ holds $\psi\xi = \phi$ and $\|\psi\| \leq cK\|\phi\|$. Thus, for a given ϕ and ξ we have constructed an A-morphism ψ such that $\psi\xi = \phi$ and $\langle$ $\|\psi\| \leq \|\phi\|$ / $\|\psi\| \leq cC\|\phi\|$ $\rangle$. Hence, J is $\langle$ metrically / cC-topologically $\rangle$ injective. $\square$

It is easy to show that, for any A-module X there exists $\langle$ an isometric / a 1-topologically injective $\rangle$ A-morphism

$$\rho_X^+ : X \to \mathcal{B}(A_+, \ell_\infty(B_{X^*})),\ x \mapsto (a \mapsto (f \mapsto f(x \cdot a)))$$

Proposition 2.1.22 ([29], proposition 3.10)**.** *An A-module J is $\langle$ metrically / C-topologically $\rangle$ injective iff ρ_J^+ is a $\langle$ 1-retraction / C-retraction $\rangle$ in* $\mathbf{mod} - A$.

Proof. Suppose J is $\langle$ metrically / C-topologically $\rangle$ injective, then consider $\langle$ isometric / 1-topologically injective $\rangle$ A-morphism ρ_J^+. Consider extension problem with $\phi = 1_J$ and $\xi = \rho_J^+$, then from $\langle$ metric / C-topological $\rangle$ injectivity of J we get an A-morphism τ^+ such that $\tau^+\rho_J^+ = 1_J$ and $\|\tau^+\| \leq K$ for $\langle$ $K = 1$ / $K = C$ $\rangle$. Since $\|\rho_J^+\|\|\tau^+\| \leq K$ we conclude that ρ_J^+ is a $\langle$ 1-retraction / C-retraction $\rangle$ in $\mathbf{mod} - A$.

Conversely, assume that ρ_J^+ is a $\langle$ 1-retraction / C-retraction $\rangle$. In other words J is a $\langle$ 1-retract / C-retract $\rangle$ of $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$. By proposition 2.1.19 the A-module $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$ is $\langle$ metrically / 1-topologically $\rangle$ injective. So from proposition 2.1.21 its $\langle$ 1-retract / C-retract $\rangle$ J is $\langle$ metrically / C-topologically $\rangle$ injective. $\square$

Proposition 2.1.23 ([29], proposition 3.9)**.** *Every metrically injective module is 1-topologically injective and every C-topologically injective module is C-relatively injective.*

Proof. By proposition 2.1.22 every metrically injective A-module J is a 1-retract of $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$. Hence, by the same proposition J is 1-topologically injective. Again by proposition 2.1.22 every C-topologically injective A-module J is a C-retract of the A-module $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$. In other words J is a C-retract of the module $\mathcal{B}(A_+, E)$ for some Banach space E. Therefore, J is C-relatively injective. $\square$

Clearly, every C-topologically injective module is C'-topologically injective for $C' \geq C$. But we can state even more.

Proposition 2.1.24. *An A-module is metrically injective iff it is 1-topologically injective.*

Proof. The result immediately follows from propositions 2.1.23 and 2.1.22. $\square$

Let us proceed to examples. If we regard the category of Banach spaces as the category of right Banach modules over zero algebra, we may speak of $\langle$ metrically / topologically $\rangle$ injective Banach spaces. All results mentioned above hold for this type of injectivity. An equivalent definition says that a Banach space is $\langle$ metrically / topologically $\rangle$ injective if it is $\langle$ contractively complemented / complemented $\rangle$ in any ambient Banach space. Typical examples of metrically injective Banach spaces are L_∞-spaces. Only metrically injective Banach spaces are completely understood — these spaces are isometrically isomorphic to $C(K)$-space for some extremely disconnected compact Hausdorff space K [[17], theorem 3.11.6]. Usually such topological spaces are referred to as Stonean spaces. For the contemporary results on topologically injective Banach spaces see [[35], chapter 40].

Proposition 2.1.25. *Let J be a faithful A-module. Then J is $\langle$ metrically / C-topologically $\rangle$ injective iff the map $\rho_J : J \to \mathcal{B}(A, \ell_\infty(B_{J^*}))$, $x \mapsto (a \mapsto (f \mapsto f(x \cdot a)))$ is a $\langle$ 1-coretraction / C-coretraction $\rangle$ in* $\mathbf{mod} - A$.

Proof. If J is $\langle$ metrically / C-topologically $\rangle$ injective, then by proposition 2.1.22 the A-morphism ρ_J^+ has a right inverse morphism τ^+, with norm $\langle$ at most 1 / at most C $\rangle$. Assume we are given an operator $T \in \mathcal{B}(A_+, \ell_\infty(B_{J^*}))$, such that $T|_A = 0$. Fix $a \in A$, then $T \cdot a = 0$, and so $\tau^+(T) \cdot a = \tau^+(T \cdot a) = 0$. Since J is faithful and $a \in A$ is arbitrary, then $\tau^+(T) = 0$. Let $p : A_+ \to A$ be the natural projection from A_+ onto A, then define A-morphisms $j = \mathcal{B}(p, \ell_\infty(B_{J^*}))$ and $\tau = \tau^+ j$. For any $a \in A$ and $T \in \mathcal{B}(A, \ell_\infty(B_{J^*}))$ we have $\tau(T \cdot a) - \tau(T) \cdot a = \tau^+(j(T \cdot a) - j(T) \cdot a) = 0$, because $j(T \cdot a) - j(T) \cdot a|_A = 0$. Therefore, τ is an A-morphism. Note that $\|\tau\| \leq \|\tau^+\|\|j\| \leq \|\tau^+\|$. Therefore, τ has norm $\langle$ at most 1 / at most C $\rangle$. Obviously, for all $x \in J$ we have $\rho_J^+(x) - j(\rho_J(x))|_A = 0$, so $\tau^+(\rho_J^+(x) - j(\rho_J(x))) = 0$. As a consequence $\tau(\rho_J(x)) = \tau^+(j(\rho_J(x))) = \tau^+(\rho_J^+(x)) = x$ for all $x \in J$. Since $\tau \rho_J = 1_J$, then ρ_J is a $\langle$ 1-coretraction / C-coretraction $\rangle$ in $\mathbf{mod} - A$.

Conversely, assume ρ_J is a $\langle$ 1-coretraction / C-coretraction $\rangle$. In other words, ρ_J has a right inverse morphism τ with norm $\langle$ at most 1 / at most C $\rangle$. Define $i : A \to A_+$ to be the natural embedding of A into A_+ and define A-morphism $q = \mathcal{B}(i, \ell_\infty(B_{J^*}))$.

Obviously, $\rho_J = q\rho_J^+$. Consider A-morphism $\tau^+ = \tau q$. Note that $\|\tau^+\| \leq \|\tau\|\|q\| \leq \|\tau\|$. Therefore, τ^+ has norm $\langle$ at most 1 / at most C $\rangle$. Clearly $\tau^+\rho_J^+ = \tau q \rho_J^+ = \tau\rho_J = 1_J$. So ρ_J^+ is a $\langle$ 1-coretraction / C-coretraction $\rangle$ and by proposition 2.1.22 the A-module J is $\langle$ metrically / C-topologically $\rangle$ injective. □

Proposition 2.1.26. *Let $(J_\lambda)_{\lambda\in\Lambda}$ be a family of A-modules. Then*

(i) $\bigoplus_\infty\{J_\lambda : \lambda \in \Lambda\}$ is metrically injective iff for all $\lambda \in \Lambda$ the A-module J_λ is metrically injective;

(ii) $\bigoplus_\infty\{J_\lambda : \lambda \in \Lambda\}$ is C-topologically injective iff for all $\lambda \in \Lambda$ the A-module J_λ is a C-topologically injective.

Proof. Denote $J := \bigoplus_\infty\{J_\lambda : \lambda \in \Lambda\}$.

(i) The proof is literally the same as in paragraph (ii).

(ii) Assume that J is C-topologically injective. Note that, for any $\lambda \in \Lambda$ the A-module J_λ is a 1-retract of J via natural projection $p_\lambda : J \to J_\lambda$. By proposition 2.1.21 the A-module J_λ is C-topologically injective.

Conversely, let each A-module J_λ be C-topologically injective. By proposition 2.1.22 we have a family of C-coretractions $\rho_\lambda : J_\lambda \to \mathcal{B}(A_+, \ell_\infty(S_\lambda))$. It follows that $\bigoplus_\infty\{\rho_\lambda : \lambda \in \Lambda\}$ is a C-coretraction in $\mathbf{mod} - A$. As a result J is a C-retract of

$$\bigoplus_\infty\{\mathcal{B}(A_+, \ell_\infty(S_\lambda)) : \lambda \in \Lambda\} \underset{\mathbf{mod}_1 - A}{\cong} \bigoplus_\infty\left\{\bigoplus_\infty\{A_+^* : s \in S_\lambda\} : \lambda \in \Lambda\right\} \underset{\mathbf{mod}_1 - A}{\cong}$$

$$\bigoplus_\infty\{A_+^* : s \in S\} \underset{\mathbf{mod}_1 - A}{\cong} \mathcal{B}(A_+, \ell_\infty(S))$$

in $\mathbf{mod} - A$, where $S = \bigsqcup_{\lambda\in\Lambda} S_\lambda$. Clearly, the latter module is 1-topologically injective, so by proposition 2.1.21 the A-module J is C-topologically injective. □

Corollary 2.1.27. *Let J be an A-module and Λ be an arbitrary set. Then $\bigoplus_\infty\{J : \lambda \in \Lambda\}$ is $\langle$ metrically / C-topologically $\rangle$ injective iff J is $\langle$ metrically / C-topologically $\rangle$ injective.*

Proof. The result immediately follows from proposition 2.1.26 if one sets $J_\lambda = J$ for all $\lambda \in \Lambda$. □

Proposition 2.1.28. *Let J be an A-module and Λ be an arbitrary set. Then $\mathcal{B}(\ell_1(\Lambda), J)$ is $\langle$ metrically / C-topologically $\rangle$ injective iff J is $\langle$ metrically / C-topologically $\rangle$ injective.*

Proof. Assume $\mathcal{B}(\ell_1(\Lambda), J)$ is $\langle$ metrically / C-topologically $\rangle$ injective. Take any $\lambda \in \Lambda$ and consider contractive A-morphisms $i_\lambda : J \to \mathcal{B}(\ell_1(\Lambda), J) : x \mapsto (f \mapsto f(\lambda)x)$ and $p_\lambda : \mathcal{B}(\ell_1(\Lambda), J) \to J : T \mapsto T(\delta_\lambda)$. Clearly, $p_\lambda i_\lambda = 1_J$, so by proposition 2.1.21 the A-module J is $\langle$ metrically / C-topologically $\rangle$ injective as 1-retract of $\langle$ metrically / 1-topologically $\rangle$ injective A-module $\mathcal{B}(\ell_1(\Lambda), J)$.

Conversely, since J is $\langle$ metrically / C-topologically $\rangle$ injective, by proposition 2.1.22 the A-morphism ρ_J^+ is a $\langle$ 1-coretraction / C-coretraction $\rangle$. Now we apply the functor $\mathcal{B}(\ell_1(\Lambda), -)$ to this coretraction to get another $\langle$ 1-coretraction / C-coretraction $\rangle$ denoted by $\mathcal{B}(\ell_1(\Lambda), \rho_J^+)$. Note that

$$\mathcal{B}(\ell_1(\Lambda), \ell_\infty(B_{J^*})) \underset{\mathbf{Ban}_1}{\cong} (\ell_1(\Lambda) \mathbin{\widehat{\otimes}} \ell_1(B_{J^*}))^* \underset{\mathbf{Ban}_1}{\cong} \ell_1(\Lambda \times B_{J^*})^* \underset{\mathbf{Ban}_1}{\cong} \ell_\infty(\Lambda \times B_{J^*}),$$

so we have isometric isomorphisms of Banach modules

$$\mathcal{B}(\ell_1(\Lambda), \mathcal{B}(A_+, \ell_\infty(B_{J^*}))) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{B}(A_+, \mathcal{B}(\ell_1(\Lambda), \ell_\infty(B_{J^*})) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{B}(A_+, \ell_\infty(\Lambda \times B_{J^*})).$$

Therefore, $\mathcal{B}(\ell_1(\Lambda), J)$ is a $\langle$ 1-retract / C-retract $\rangle$ of the $\langle$ metrically / 1-topologically $\rangle$ injective A-module $\mathcal{B}(A_+, \ell_\infty(\Lambda \times B_{J^*}))$. By proposition 2.1.21 the A-module $\mathcal{B}(\ell_1(\Lambda), J)$ is $\langle$ metrically / C-topologically $\rangle$ injective. $\square$

The property of being metrically, topologically or relatively injective module puts some restrictions on the Banach geometric structure of the module.

Proposition 2.1.29 ([33], proposition 2.2.1)**.** *Let J be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ injective A-module, and let I be a $\langle$ 1-complemented / c-complemented / c-complemented $\rangle$ closed right ideal of A. Then $J^{\perp I}$ is $\langle$ 2-complemented / $1 + cC$-complemented / $1 + cC$-complemented $\rangle$ in J.*

Proof. Since A is 1-complemented in A_+, then I is $\langle$ 1-complemented / c-complemented / c-complemented $\rangle$ in A_+. Hence, there exists a bounded linear operator $r : A_+ \to I$ of $\langle$ norm 1 / norm c / norm c $\rangle$ such that $r|_I = 1_I$. Since J is $\langle$ metrically / C-topologically / C-relatively $\rangle$ injective, then by proposition 2.1.22 the morphism ρ_J^+ has a left inverse A-morphism τ^+ of norm $\langle$ at most 1 / at most C / at most C $\rangle$. Now consider a bounded linear operator $p : A \to A$, $x \mapsto x - \tau^+(\rho_J^+(x)r)$. Clearly, $\|p\| \leq 1 + \|\tau^+\|\|r\|$. Since $\operatorname{Im}(r) \subset I$, then for all $x \in J^{\perp I}$ we have $\rho_J^+(x)r = 0$. Hence, $p(x) = x$ for all $x \in J^{\perp I}$. Since I is a right ideal, then one can check that for all $x \in J$ and $a \in I$ holds $(\rho_J^+(x)r) \cdot a = \rho_J^+(x \cdot a)$. As a consequence

$$p(x) \cdot a = x \cdot a - \tau^+(\rho_J^+(x)r) \cdot a = x \cdot a - \tau^+((\rho_J^+(x)r) \cdot a) = x \cdot a - \tau^+(\rho_J^+(x \cdot a)) = 0.$$

In other words $\operatorname{Im}(p) \subset J^{\perp I}$. Thus, p is projection onto $J^{\perp I}$ with norm at most $1 + \|\tau^+\|\|r\|$. Hence, $J^{\perp I}$ is $\langle$ 2-complemented / $(1 + cC)$-complemented / $(1 + cC)$-complemented $\rangle$ complemented in J. $\square$

Corollary 2.1.30. *Let J be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ injective A-module. Then J_{ann} is $\langle$ 2-complemented / $(1 + C)$-complemented / $(1 + C)$-complemented $\rangle$ in J.*

Proof. The result directly follows from proposition 2.1.29. $\square$

2.1.4 Metric and topological flatness

The first definition of metrically flat modules was given by Helemeskii in [36] under the name of extremely flat modules. The notion of topological flatness was first studied by White in [29]. We use a somewhat different, but equivalent terminology. Unfortunately, White's definition was erroneous, meanwhile the results were correct. So we take responsibility to fix the definition.

Definition 2.1.31 ([36], definition I; [29], definition 4.4)**.** A left A-module F is called $\langle$ metrically / C-topologically $\rangle$ flat if for each $\langle$ isometric / c-topologically injective $\rangle$ A-morphism $\xi : X \to Y$ of right A-modules the operator $\xi \mathbin{\widehat{\otimes}}_A 1_F : X \mathbin{\widehat{\otimes}}_A F \to Y \mathbin{\widehat{\otimes}}_A F$ is $\langle$ isometric / cC-topologically injective $\rangle$. We say that an A-module F is topologically flat if it is C-topologically flat for some $C \geq 1$.

A short but more involved definition is the following: an A-module F is called $\langle$ metrically / C-topologically $\rangle$ flat, if the functor $\langle$ $- \mathbin{\widehat{\otimes}}_A F : A - \mathbf{mod}_1 \to \mathbf{Ban}_1$ / $- \mathbin{\widehat{\otimes}}_A F : A - \mathbf{mod} \to \mathbf{Ban}$ $\rangle$ maps $\langle$ isometric / c-topologically injective $\rangle$ A-morphisms into $\langle$ isometric / cC-topologically injective $\rangle$ operators.

The key result in the study of flatness is the following.

Proposition 2.1.32 ([29], lemma 4.10)**.** *An A-module F is $\langle$ metrically / C-topologically $\rangle$ flat iff F^* is $\langle$ metrically / C-topologically $\rangle$ injective.*

Proof. Consider any $\langle$ isometric / c-topologically injective $\rangle$ morphism of right A-modules, call it $\xi : X \to Y$. The operator $\xi \mathbin{\widehat{\otimes}}_A 1_F$ is $\langle$ isometric / cC-topologically injective $\rangle$ iff the adjoint operator $(\xi \mathbin{\widehat{\otimes}}_A 1_F)^*$ is strictly $\langle$ coisometric / cC-topologically surjective $\rangle$. Since operators $(\xi \mathbin{\widehat{\otimes}}_A 1_F)^*$ and $\mathcal{B}_A(\xi, F^*)$ are equivalent in $\mathbf{Ban}_1$ via universal property of projective module tensor product, then we get that $\xi \mathbin{\widehat{\otimes}}_A 1_F$ is $\langle$ isometric / cC-topologically injective $\rangle$ iff $\mathcal{B}_A(\xi, F^*)$ is strictly $\langle$ coisometric / cC-topologically surjective $\rangle$. Since ξ is arbitrary we conclude that F is $\langle$ metrically / C-topologically $\rangle$ flat iff F^* is $\langle$ metrically / C-topologically $\rangle$ injective. $\square$

This characterization allows one to prove many properties of flat modules by considering respective duals.

Proposition 2.1.33. *Any $\langle$ 1-retract / c-retract $\rangle$ of a $\langle$ metrically / C-topologically $\rangle$ flat module is $\langle$ metrically / cC-topologically $\rangle$ flat.*

Proof. The result follows from propositions 2.1.32 and 2.1.21 $\square$

Proposition 2.1.34. *Every metrically flat module is 1-topologically flat and every C-topologically flat module is C-relatively flat.*

Proof. The result follows from propositions 2.1.32 and 2.1.23. $\square$

Proposition 2.1.35. *An A-module is metrically flat iff it is 1-topologically flat.*

Proof. The result follows from propositions 2.1.32 and 2.1.24. $\square$

Loosely speaking flatness is "projectivity with respect to second duals". We can give this statement a precise meaning.

Proposition 2.1.36. *Let F be a left Banach A-module. Then the following are equivalent:*

(i) F is $\langle$ metrically / C-topologically $\rangle$ flat;

(ii) for any strictly $\langle$ coisometric / c-topologically surjective $\rangle$ A-morphism $\xi : X \to Y$ and for any A-morphism $\phi : F \to Y$ there exists an A-morphism $\psi : F \to X^{**}$ such that $\xi^{**}\psi = \iota_Y \phi$ and $\langle$ $\|\psi\| = \|\phi\|$ / $\|\psi\| \leq cC\|\phi\|$ $\rangle$;

(iii) there exists an A-morphism $\sigma : F \to (A_+ \widehat{\otimes} \ell_1(B_F))^{**}$ with norm $\langle$ at most 1 / at most C $\rangle$ such that $(\pi_F^+)^{**}\sigma = \iota_F$.

Proof. $(i) \implies (ii)$ Again, consider an arbitrary strictly $\langle$ coisometric / c-topologically surjective $\rangle$ A-morphism $\xi : X \to Y$ and an arbitrary A-morphism $\phi : F \to Y$. By [[1], exercise 4.4.6] we know that ξ^* is $\langle$ isometric / c-topologically injective $\rangle$. Since F is $\langle$ metrically / C-topologically $\rangle$ flat then $(\xi^* \widehat{\otimes}_A 1_F)$ is $\langle$ isometric / cC-topologically injective $\rangle$ too. Therefore, $(\xi^* \widehat{\otimes}_A 1_F)^*$ is $\langle$ strictly coisometric / strictly cC-topologically surjective $\rangle$ by [[1], exercise 4.4.7]. Note that $(\xi^* \widehat{\otimes}_A 1_F)^*$ and $\mathcal{B}_A(F, \xi^{**})$ are equivalent in $\mathbf{Ban}_1$ thanks to the law of adjoint associativity. So $\mathcal{B}_A(F, \xi^{**})$ is strictly $\langle$ coisometric / cC-topologically surjective $\rangle$ too. The latter implies that for the operator $\iota_Y \phi$ we can find an A-morphism $\psi : F \to X^{**}$ such that $\xi^{**}\psi = \iota_Y \phi$ and $\langle$ $\|\psi\| = \|\iota_Y \phi\| = \|\phi\|$ / $\|\psi\| \leq cC\|\iota_Y \phi\| = cC\|\phi\|$ $\rangle$

(ii) $\implies$ *(iii)* Set $\xi = \pi_F^+$ and $\phi = 1_F$. Since ξ is strictly $\langle$ coisometric/ 1-topologically surjective $\rangle$, then from assumption we get an A-morphism $\sigma : F \to (A_+ \widehat{\otimes} \ell_1(B_F))^{**}$ such that $(\pi_F^+)^{**}\sigma = \iota_F 1_F = \iota_F$ and $\langle$ $\|\sigma\| \leq \|\phi\| = 1$ / $\|\sigma\| \leq 1 \cdot C\|\phi\| = C$ $\rangle$.

(iii) $\implies$ *(i)* Let σ be a right inverse A-morphism for $(\pi_F^+)^{**}$ with norm $\langle$ at most 1 / at most C $\rangle$. Consider A-morphism $\tau = \sigma^* \iota_{(A_+ \widehat{\otimes} \ell_1(B_F))^*}$. Clearly, its norm is $\langle$ at most 1 / at most C $\rangle$. For any $f \in F^*$ and $x \in F$ we have

$$(\tau (\pi_F^+)^*)(f)(x) = \sigma^*(\iota_{(A_+ \widehat{\otimes} \ell_1(B_F))^*}((\pi_F^+)^*(f)))(x) = \iota_{(A_+ \widehat{\otimes} \ell_1(B_F))^*}((\pi_F^+)^*(f))(\sigma(x))$$

$$= \sigma(x)((\pi_F^+)^*(f)) = (\pi_F^+)^{**}(\sigma(x))(f) = \iota_F(x)(f) = f(x)$$

So $\tau(\pi_F^+)^* = 1_{F^*}$, which means F^* is a $\langle$ 1-retract / C-retract $\rangle$ of $(A_+ \widehat{\otimes} \ell_1(B_F))^*$. The latter module is $\langle$ metrically / 1-topologically $\rangle$ injective, because

$$(A_+ \widehat{\otimes} \ell_1(B_F))^* \underset{\mathbf{mod}_1 - A}{\cong} \mathcal{B}(A_+, \ell_\infty(B_F)).$$

By proposition 2.1.21 the A-module F^* is $\langle$ metrically / C-topologically $\rangle$ injective. By proposition 2.1.32 this is equivalent to $\langle$ metric / C-topological $\rangle$ flatness of F. $\square$

Let us proceed to examples. Consider the category of Banach spaces as the category of left Banach modules over zero algebra, then we get the definition of $\langle$ metrically / topologically $\rangle$ flat Banach space. From Grothendieck's paper [32] it follows that any metrically flat Banach space is isometrically isomorphic to $L_1(\Omega, \mu)$ for some measure space (Ω, Σ, μ). For topologically flat Banach spaces, in contrast with topologically injective ones, we also have a criterion [[12], corollary 23.5(1)]: a Banach space is topologically flat iff it is an $\mathscr{L}_1^g$-space.

Proposition 2.1.37. *Let $(F_\lambda)_{\lambda \in \Lambda}$ be family of A-modules. Then*

(i) *$\bigoplus_1 \{F_\lambda : \lambda \in \Lambda\}$ is metrically flat iff for all $\lambda \in \Lambda$ the A-module F_λ is metrically flat;*

(ii) *$\bigoplus_1 \{F_\lambda : \lambda \in \Lambda\}$ is C-topologically flat iff for all $\lambda \in \Lambda$ the A-module F_λ is C-topologically flat.*

Proof. By proposition 2.1.32 an A-module F is $\langle$ metrically / C-topologically $\rangle$ flat iff F^* is $\langle$ metrically / C-topologically $\rangle$ injective. It remains to apply proposition 2.1.26 with $J_\lambda = F_\lambda^*$ for all $\lambda \in \Lambda$ and recall that $(\bigoplus_1 \{F_\lambda : \lambda \in \Lambda\})^* \underset{\mathbf{mod}_1-A}{\cong} \bigoplus_\infty \{F_\lambda^* : \lambda \in \Lambda\}$. $\square$

The following propositions demonstrate a close relationship between flatness and projectivity.

Proposition 2.1.38. *Let P be a $\langle$ metrically / C-topologically $\rangle$ projective A-module, and Λ be an arbitrary set. Then $\mathcal{B}(P, \ell_\infty(\Lambda))$ is $\langle$ metrically / C-topologically $\rangle$ injective A-module. In particular, P^* is $\langle$ metrically / C-topologically $\rangle$ injective A-module.*

Proof. From proposition 2.1.5 we know that π_P^+ is a $\langle$ 1-retraction / C-retraction $\rangle$. Then the A-morphism $\rho^+ = \mathcal{B}(\pi_P^+, \ell_\infty(\Lambda))$ is a $\langle$ 1-coretraction / C-coretraction $\rangle$. Note that,

$$\mathcal{B}(A_+ \widehat{\otimes} \ell_1(B_P), \ell_\infty(\Lambda)) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{B}(A_+, \mathcal{B}(\ell_1(B_P), \ell_\infty(\Lambda))) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{B}(A_+, \ell_\infty(B_P \times \Lambda)).$$

Thus, we showed that ρ^+ is a $\langle$ 1-coretraction / C-coretraction $\rangle$ from $\mathcal{B}(P, \ell_\infty(\Lambda))$ into $\langle$ metrically / 1-topologically $\rangle$ injective A-module. By proposition 2.1.21 the A-module $\mathcal{B}(P, \ell_\infty(\Lambda))$ is $\langle$ metrically / C-topologically $\rangle$ injective. To prove the last claim, just set $\Lambda = \mathbb{N}_1$. $\square$

Proposition 2.1.39. *Every $\langle$ metrically / C-topologically $\rangle$ projective module is $\langle$ metrically / C-topologically $\rangle$ flat.*

Proof. The result follows from propositions 2.1.32 and 2.1.38. $\square$

The property of being metrically, topologically or relatively flat module puts some restrictions on the Banach geometric structure of the module.

Proposition 2.1.40 ([33], corollary 2.2.2)**.** *Let F be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ flat A-moudle, and let I be a $\langle$ 1-complemented / c-complemented / c-complemented $\rangle$ closed right ideal of A. Then $\mathrm{cl}_F(IF)$ is weakly $\langle$ 2-complemented / $(1 + cC)$-complemented / $(1 + cC)$-complemented $\rangle$ in F.*

Proof. From propositions 2.1.32, 2.1.29 it follows that $(F^*)^{\perp I}$ is complemented in F^*. It remains to recall that $(F^*)^{\perp I} = \mathrm{cl}_F(IF)$. $\square$

Corollary 2.1.41. *Let F be a $\langle$ metrically / C-topologically / C-relatively $\rangle$ flat A-moudle. Then F_{ess} is weakly $\langle$ 2-complemented / $(1 + C)$-complemented / $(1 + C)$-complemented $\rangle$ in F.*

Proof. The result directly follows from proposition 2.1.40. $\square$

2.1.5 Metric and topological flatness of ideals and cyclic modules

In this section we study conditions under which closed ideals and cyclic modules are metrically or topologically flat. The proofs are somewhat similar to ones used in the study of relative flatness of closed ideals and cyclic modules.

Proposition 2.1.42. *Let I be a closed left ideal of $A_\times$ and I has a right $\langle$ contractive / c-bounded $\rangle$ approximate identity. Then I is $\langle$ metrically / c-topologically $\rangle$ flat.*

Proof. Let $\mathfrak{F}$ be the section filter on N and let $\mathfrak{U}$ be an ultrafilter dominating $\mathfrak{F}$. For a fixed $f \in I^*$ and $a \in A_\times$ we have $|f(ae_\nu)| \leq \|f\|\|a\|\|e_\nu\| \leq c\|f\|\|a\|$ i.e. $(f(ae_\nu))_{\nu\in N}$ is a bounded net of complex numbers. Therefore, we have a well-defined limit $\lim_{\mathfrak{U}} f(ae_\nu)$ along ultrafilter $\mathfrak{U}$. Now it is routine to check that $\sigma : A_\times^* \to I^* : f \mapsto (a \mapsto \lim_{\mathfrak{U}} f(ae_\nu))$ is an A-morphism with norm $\langle$ at most 1 / at most c $\rangle$. Let $\rho : I \to A_\times$ be the natural embedding, then for all $f \in A_\times^*$ and $a \in I$ holds

$$\rho^*(\sigma(f))(a) = \sigma(f)(\rho(a)) = \sigma(f)(a) = \lim_{\mathfrak{U}} f(ae_\nu) = \lim_{\nu} f(ae_\nu) = f(\lim_{\nu} ae_\nu) = f(a)$$

i.e. $\sigma : I^* \to A_\times^*$ is a $\langle$ 1-coretraction / c-coretraction $\rangle$. The right A-module $A_\times^*$ is $\langle$ metrically / 1-topologically $\rangle$ injective by proposition 2.1.20, hence its $\langle$ 1-retract / c-retract $\rangle$ I^* is $\langle$ metrically / c-topologically $\rangle$ injective. Now from proposition 2.1.32 we conclude that the A-module I is $\langle$ metrically / c-topologically $\rangle$ flat. □

Note that the same sufficient condition holds for relative flatness for closed ideals of $A_\times$ [[23], proposition 7.1.45]. Now we are able to give an example of a metrically flat module which is not even topologically projective. Clearly $\ell_\infty(\mathbb{N})$-module $c_0(\mathbb{N})$ is not unital, as any proper ideal, but admits a contractive approximate identity. By theorem 2.1.16 it is not topologically projective, but it is metrically flat by proposition 2.1.42.

The "metric" part of the following proposition is a slight modification of [[29], proposition 4.11]. The case of topological flatness was solved by Helemskii in [[24], theorem VI.1.20].

Proposition 2.1.43. *Let I be a closed left proper ideal of $A_\times$. Then the following are equivalent:*

(i) $A_\times/I$ is $\langle$ metrically / C-topologically $\rangle$ flat A-module;

(ii) I has a right bounded approximate identity $(e_\nu)_{\nu\in N}$ with $\sup_{\nu\in N}\|e_{A_\times} - e_\nu\|$ $\langle$ at most 1 / at most C $\rangle$

Proof. $(i) \implies (ii)$ Since $A_\times/I$ is $\langle$ metrically / C-topologically $\rangle$ flat, then by proposition 2.1.32 the right A-module $(A_\times/I)^*$ is $\langle$ metrically / C-topologically $\rangle$ injective. Let $\pi : A_\times \to A_\times/I$ be the natural quotient map, then $\pi^* : (A_\times/I)^* \to A_\times^*$ is an isometry. Since $(A_\times/I)^*$ is $\langle$ metrically / C-topologically $\rangle$ injective, then π^* is a coretraction, i.e. there exists a $\langle$ strictly coisometric / topologically surjective $\rangle$ A-morphism $\tau : A_\times^* \to (A_\times/I)^*$ of norm $\langle$ at most 1 / at most C $\rangle$ such that $\tau\pi^* = 1_{(A_\times/I)^*}$. Consider $p \in A^{**}$ such that $\iota_{A_\times}(e_{A_\times}) - p = \tau^*(\pi^{**}(\iota_{A_\times}(e_{A_\times})))$. Fix $f \in I^\perp$. Since $I^\perp = \pi^*((A_\times/I)^*)$, then there exists a functional $g \in (A_\times/I)^*$ such that $f = \pi^*(g)$. Thus,

$$(\iota_{A_\times}(e_{A_\times}) - p)(f) = \tau^*(\pi^{**}(\iota_{A_\times}(e_{A_\times})))(\pi^*(g)) = \pi^{**}(\iota_{A_\times}(e_{A_\times}))(\tau(\pi^*(g)))$$

$$= \pi^{**}(\iota_{A_\times}(e_{A_\times}))(g) = \iota_{A_\times}(e_{A_\times})(\pi^*(g)) = \iota_{A_\times}(e_{A_\times})(f).$$

Therefore, $p(f) = 0$ for all $f \in I^\perp$, i.e. $p \in I^{\perp\perp}$. Recall that $I^{\perp\perp}$ is the weak* closure of I in A^{**}, so we can choose a net $(e''_\nu)_{\nu \in N''} \subset I$ such that $(\iota_I(e''_\nu))_{\nu \in N''}$ converges to p in the weak* topology. Clearly $(\iota_{A_\times}(e_{A_\times} - e''_\nu))_{\nu \in N''}$ converges to $\iota_{A_\times}(e_{A_\times}) - p$ in the same topology. By [[37], lemma 1.1] there exists a net in the convex hull $\operatorname{conv}(\iota_{A_\times}(e_{A_\times} - e''_\nu))_{\nu \in N''} = \iota_{A_\times}(e_{A_\times}) - \operatorname{conv}(\iota_{A_\times}(e''_\nu))_{\nu \in N''}$ that weak* converges to $\iota_{A_\times}(e_{A_\times}) - p$ and all elements of this net are bounded in norm by $\|\iota_{A_\times}(e_{A_\times}) - p\|$. Denote this net by $(\iota_{A_\times}(e_{A_\times}) - \iota_{A_\times}(e'_\nu))_{\nu \in N'}$, then $(\iota_{A_\times}(e'_\nu))_{\nu \in N'}$ weak* converges to p. For any $a \in I$ and $f \in I^*$ we have

$$\lim_\nu f(ae'_\nu) = \lim_\nu \iota_{A_\times}(e'_\nu)(f \cdot a) = p(f \cdot a) = \iota_{A_\times}(e_{A_\times})(f \cdot a) - \tau^*(\pi^{**}(\iota_{A_\times}(e_{A_\times})))(f \cdot a)$$

$$= f(a) - \iota_{A_\times}(e_{A_\times})(\pi^*(\tau(f \cdot a))) = f(a) - \pi^*(\tau(f) \cdot a)(e_{A_\times}) = f(a) - \tau(f)(\pi(a)) = f(a)$$

hence $(e'_\nu)_{\nu \in N'}$ is a weak right bounded approximate identity for I. By [[38], proposition 33.2] there is a net $(e_\nu)_{\nu \in N} \subset \operatorname{conv}(e'_\nu)_{\nu \in N'}$ which is a right bounded approximate identity for I. For any $\nu \in N$ we have $e_{A_\times} - e_\nu \in \operatorname{conv}(e_{A_\times} - e'_\nu)_{\nu \in N'}$, so taking into account the norm bound on $(\iota_{A_\times}(e_{A_\times} - e'_\nu))_{\nu \in N'}$ we get

$$\sup_{\nu \in N} \|e_{A_\times} - e_\nu\| \leq \|\iota_{A_\times}(e_{A_\times}) - p\| \leq \|\tau^*(\pi^{**}(\iota_{A_\times}(e_{A_\times})))\| \leq \|\tau^*\|\|\pi^{**}\|\|\iota_{A_\times}(e_{A_\times})\| = \|\tau\|$$

Since τ has norm $\langle$ at most 1 / at most C $\rangle$ we get the desired bound. By construction, $(e_\nu)_{\nu \in N}$ is a right bounded approximate identity for I.

$(ii) \implies (i)$ Denote $D = \sup_{\nu \in N} \|e_{A_\times} - e_\nu\|$. Let $\mathfrak{F}$ be the section filter on N and let $\mathfrak{U}$ be an ultrafilter dominating $\mathfrak{F}$. For a fixed $f \in A_\times^*$ and $a \in A_\times$ we have $|f(a - ae_\nu)| = |f(a(e_{A_\times} - e_\nu))| \leq \|f\|\|a\|\|e_{A_\times} - e_\nu\| \leq D\|f\|\|a\|$ i.e. $(f(a - ae_\nu))_{\nu \in N}$ is a bounded net of complex numbers. Therefore, we have a well-defined limit $\lim_{\mathfrak{U}} f(a - ae_\nu)$ along the ultrafilter $\mathfrak{U}$. Since $(e_\nu)_{\nu\in N}$ is a right approximate identity for I and $\mathfrak{U}$ contains section filter then for all $a \in I$ we have $\lim_{\mathfrak{U}} f(a - ae_\nu) = \lim_\nu f(a - ae_\nu) = 0$. Therefore, for each $f \in A_\times^*$ we have a well-defined map $\tau(f) : A_\times/I \to \mathbb{C}$, $a + I \mapsto \lim_{\mathfrak{U}} f(a - ae_\nu)$. Clearly, this is a linear functional and from inequalities above we see that its norm is bounded by $D\|f\|$. Now it is routine to check that $\tau : A_\times^* \to (A_\times/I)^*$, $f \mapsto \tau(f)$ is an A-morphism with norm $\langle$ at most 1 / at most C $\rangle$. For all $g \in (A_\times/I)^*$ and $a+I \in A_\times/I$ holds

$$\tau(\pi^*(g))(a + I) = \lim_{\mathfrak{U}} \pi^*(g)(a - ae_\nu) = \lim_{\mathfrak{U}} g(\pi(a - ae_\nu)) = \lim_{\mathfrak{U}} g(a + I) = g(a + I)$$

i.e. $\tau : A_\times^* \to (A_\times/I)^*$ is a retraction. The right A-module $A_\times^*$ is $\langle$ metrically / 1-topologically $\rangle$ injective by proposition 2.1.20, hence its $\langle$ 1-retract / C-retract $\rangle$ $(A_\times/I)^*$ is $\langle$ metrically / C-topologically $\rangle$ injective. Proposition 2.1.32 gives that the module $A_\times/I$ is $\langle$ metrically / C-topologically $\rangle$ flat. □

It is worth mentioning that every operator algebra A (not necessary self adjoint) with contractive approximate identity has a contractive approximate identity $(e_\nu)_{\nu\in N}$ such that $\sup_{\nu\in N} \|e_{A_\#} - e_\nu\| \leq 1$ and even $\sup_{\nu\in N} \|e_{A_\#} - 2e_\nu\| \leq 1$. Here $A_\#$ is a unitization of A as operator algebra. For details see [37], [39].

Again we shall compare our result on metric and topological flatness of cyclic modules with their relative counterparts. Helemeskii and Sheinberg showed [[24], theorem VII.1.20] that a cyclic module A/I is relatively flat if I admits a right bounded approximate identity. In case when $I^\perp$ is complemented in $A_\times^*$ the converse is also true. In topological theory we don't need this assumption, so we have a criterion. Metric flatness of cyclic modules is a much stronger property due to specific restriction on the norms of approximate identities. As we shall see in the next section, it is so restrictive that it doesn't allow one to construct any non-zero metrically projective, injective or flat annihilator module over a non-zero Banach algebra.

2.2 The impact of Banach geometry

2.2.1 Homologically trivial annihilator modules

In this section we concentrate on the study of metrically and topologically projective, injective and flat annihilator modules. Unless otherwise stated, all Banach spaces in this section are regarded as annihilator modules. Note the obvious fact that we shall often use in this section: any bounded linear operator between annihilator A-modules is an A-morphism.

Proposition 2.2.1. *Let X be a non-zero annihilator A-module. Then $\mathbb{C}$ is a 1-retract of X in $A-\mathbf{mod}_1$.*

Proof. Take any $x_0 \in X$ with $\|x_0\| = 1$ and using Hahn-Banach theorem choose $f_0 \in X^*$ such that $\|f_0\| = f_0(x_0) = 1$. Consider contractive linear operators $\pi : X \to \mathbb{C}$, $x \mapsto f_0(x)$, $\sigma : \mathbb{C} \to X$, $z \mapsto zx_0$. It is easy to check that π and σ are contractive A-morphisms and what is more $\pi\sigma = 1_{\mathbb{C}}$. In other words $\mathbb{C}$ is a 1-retract of X in $A-\mathbf{mod}_1$. $\square$

Now it is time to recall that any Banach algebra A can always be regarded as a proper closed maximal ideal of A_+, and what is more $\mathbb{C} \underset{A-\mathbf{mod}_1}{\cong} A_+/A$. If we regard $\mathbb{C}$ as a right annihilator A-module we also have $\mathbb{C} \underset{\mathbf{mod}_1-A}{\cong} (A_+/A)^*$.

Proposition 2.2.2. *An annihilator A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ projective iff $\langle$ $A = \{0\}$ / A has a right identity of norm at most $C - 1$ $\rangle$.*

Proof. It is enough to study $\langle$ metric / C-topological $\rangle$ projectivity of A_+/A. Since the natural quotient map $\pi : A_+ \to A_+/A$ is a strict coisometry, then by proposition 2.1.17 $\langle$ metric / C-topological $\rangle$ projectivity of A_+/A is equivalent to existence of an idempotent $p \in A$ such that $A = A_+p$ and $e_{A_+} - p$ has norm $\langle$ at most 1 / at most C $\rangle$. It remains to note that this norm bound holds iff $\langle$ $p = 0$ and therefore $A = A_+p = \{0\}$ / p has norm at most $C - 1$ $\rangle$. $\square$

Proposition 2.2.3. *Let P be a non-zero annihilator A-module. Then the following are equivalent:*

(i) *P is $\langle$ metrically / C-topologically $\rangle$ projective A-module;*

(ii) *$\langle$ $A = \{0\}$ / A has a right identity of norm at most $C-1$ $\rangle$ and P is a $\langle$ metrically / C-topologically $\rangle$ projective Banach space. As a consequence $\langle$ $P \underset{\mathbf{Ban}_1}{\cong} \ell_1(\Lambda)$ / $P \underset{\mathbf{Ban}}{\cong} \ell_1(\Lambda)$ $\rangle$ for some set Λ.*

Proof. $(i) \implies (ii)$ By propositions 2.1.4 and 2.2.1 the A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ projective as a 1-retract of $\langle$ metrically / C-topologically $\rangle$ projective module P. Proposition 2.2.2 gives that $\langle$ $A = \{0\}$ / A has right identity of norm at most $C-1$ $\rangle$. By corollary 2.1.10 the annihilator A-module $\mathbb{C}\widehat{\otimes}\ell_1(B_P) \underset{A-\mathbf{mod}_1}{\cong} \ell_1(B_P)$ is $\langle$ metrically / C-topologically $\rangle$ projective. Consider strict coisometry $\pi : \ell_1(B_P) \to P$ well-defined by equality $\pi(\delta_x) = x$ for all $x \in B_P$. Since P and $\ell_1(B_P)$ are annihilator modules, then π is also an A-module map. Since P is $\langle$ metrically / C-topologically $\rangle$ projective, then the A-morphism π has a right inverse morphism σ of norm $\langle$ at most

1 / at most $C-1$ $\rangle$. Therefore, P is a $\langle$ 1-retract / C-retract $\rangle$ of $\langle$ metrically / 1-topologically $\rangle$ projective Banach space $\ell_1(B_P)$. By proposition 2.1.4 the Banach space P is $\langle$ metrically / C-topologically $\rangle$ projective. Now from $\langle$ [[30], proposition 3.2] / results of [31] $\rangle$ we get that P is $\langle$ isometrically / topologically $\rangle$ isomorphic to $\ell_1(\Lambda)$ in for some set Λ.

$(ii) \implies (i)$ By proposition 2.2.2 the annihilator A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ projective. Therefore, by corollary 2.1.10 the annihilator A-module $\mathbb{C} \widehat{\otimes} \ell_1(\Lambda) \underset{A-\mathbf{mod}_1}{\cong} \ell_1(\Lambda)$ is $\langle$ metrically / C-topologically $\rangle$ projective too. □

Proposition 2.2.4. *The right annihilator A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ injective iff $\langle$ $A=\{0\}$ / A has a right $(C-1)$-bounded approximate identity $\rangle$.*

Proof. Because of proposition 2.1.32 it is enough to study $\langle$ metric / C-topological $\rangle$ flatness of A_+/A. By proposition 2.1.43 this is equivalent to existence of a right bounded approximate identity $(e_\nu)_{\nu\in N}$ in A with $\langle$ $\sup_{\nu\in N}\|e_{A_+}-e_\nu\|\leq 1$ / $\sup_{\nu\in N}\|e_{A_+}-e_\nu\|\leq C$ $\rangle$. It remains to note that the latter inequality holds iff $\langle$ $e_\nu=0$ and therefore $A=\{0\}$ / $(e_\nu)_{n\in N}$ is a right $(C-1)$-bounded approximate identity $\rangle$. □

Proposition 2.2.5. *Let J be a non-zero right annihilator A-module. Then the following are equivalent:*

(i) J is $\langle$ metrically / C-topologically $\rangle$ injective A-module;

(ii) $\langle$ $A=\{0\}$ / A has a right $(C-1)$-bounded approximate identity $\rangle$ and J is a $\langle$ metrically / C-topologically $\rangle$ injective Banach space. $\langle$ As a consequence $J \underset{\mathbf{Ban}_1}{\cong} C(K)$ for some Stonean space K / $\rangle$.

Proof. $(i) \implies (ii)$ By propositions 2.1.21 and 2.2.1 the A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ injective as a 1-retract of $\langle$ metrically / C-topologically $\rangle$ injective module J. Proposition 2.2.4 gives that $\langle$ $A=\{0\}$ / A has a right $(C-1)$-bounded approximate identity $\rangle$. By proposition 2.1.28 the annihilator A-module $\mathcal{B}(\ell_1(B_{J^*}),\mathbb{C}) \underset{\mathbf{mod}_1-A}{\cong} \ell_\infty(B_{J^*})$ is $\langle$ metrically / C-topologically $\rangle$ injective. Consider isometry $\rho: J\to\ell_\infty(B_{J^*})$ well-defined by $\rho(x)(f)=f(x)$ for all $x\in J$ and $f\in B_{J^*}$. Since J and $\ell_\infty(B_{J^*})$ are annihilator modules, then ρ is also an A-module map. Since J is $\langle$ metrically / C-topologically $\rangle$ injective, then the A-morphism ρ has a left inverse A-morphism τ with norm $\langle$ at most 1 / at most C $\rangle$. Therefore, J is a $\langle$ 1-retract / C-retract $\rangle$ of $\langle$ metrically / 1-topologically $\rangle$ injective Banach space $\ell_\infty(B_{J^*})$. By proposition $\langle$ 2.1.21 the Banach space J is $\langle$ metrically / C-topologically $\rangle$ injective. $\langle$ From

[[17], theorem 3.11.6] the Banach space J is isometrically isomorphic to $C(K)$ for some Stonean space K. / $\rangle$

$(ii) \implies (i)$ By proposition 2.2.4 the annihilator A-module $\mathbb{C}$ is $\langle$ metrically / C-topologically $\rangle$ injective. Even more, by proposition 2.1.28 the annihilator A-module $\mathcal{B}(\ell_1(B_{J^*}), \mathbb{C}) \underset{\mathbf{mod}_1-A}{\cong} \ell_\infty(B_{J^*})$ is $\langle$ metrically / C-topologically $\rangle$ injective too. Since J is a $\langle$ metrically / C-topologically $\rangle$ injective Banach space and there an isometric embedding $\rho : J \to \ell_\infty(B_{J^*})$, then J as a Banach space is a $\langle$ 1-retract / C-retract $\rangle$ of $\ell_\infty(B_{J^*})$. Recall, that J and $\ell_\infty(B_{J^*})$ are annihilator modules, so in fact we have a retraction in $\langle$ $\mathbf{mod}_1 - A$ / $\mathbf{mod} - A$ $\rangle$. By proposition 2.1.21 the A-module J is $\langle$ metrically / C-topologically $\rangle$ injective. □

Proposition 2.2.6. *Let F be a non-zero annihilator A-module. Then the following are equivalent:*

(i) F is $\langle$ metrically / C-topologically $\rangle$ flat A-module;

(ii) $\langle$ $A = \{0\}$ / A has a right $(C-1)$-bounded approximate identity $\rangle$ and F is a $\langle$ metrically / C-topologically $\rangle$ flat Banach space, that is $\langle$ $F \underset{\mathbf{Ban}_1}{\cong} L_1(\Omega, \mu)$ for some measure space (Ω, Σ, μ) / F is an $\mathscr{L}^g_{1,C}$-space $\rangle$.

Proof. By $\langle$ [[32], theorem 1] / [[12], corollary 23.5(1)] $\rangle$ the Banach space F is $\langle$ metrically / C-topologically $\rangle$ flat iff $\langle$ $F \underset{\mathbf{Ban}_1}{\cong} L_1(\Omega, \mu)$ for some measure space (Ω, Σ, μ) / F is an $\mathscr{L}^g_{1,C}$-space $\rangle$. Now the equivalence follows from propositions 2.2.5 and 2.1.32. □

We obliged to compare these results with similar ones in relative theory. From $\langle$ [[33], proposition 2.1.7] / [[33], proposition 2.1.10] $\rangle$ we know that an annihilator A-module over a Banach algebra A is relatively $\langle$ projective / flat $\rangle$ iff A has $\langle$ a right identity / a right bounded approximate identity $\rangle$. In metric and topological theory, in contrast with relative one, homological triviality of annihilator modules puts restrictions not only on the algebra itself but on the geometry of the module too. These geometric restrictions forbid existence of certain homologically excellent algebras. One of the most important properties of any relatively $\langle$ contractible / amenable $\rangle$ Banach algebra is relative $\langle$ projectivity / flatness $\rangle$ of all (and in particular of all annihilator) left Banach modules over it. In a sharp contrast in metric and topological theories such algebras cannot exist.

Proposition 2.2.7. *There is no Banach algebra A such that all A-modules are $\langle$ metrically / topologically $\rangle$ flat. A fortiori, there is no such Banach algebras that all A-modules are $\langle$ metrically / topologically $\rangle$ projective.*

Proof. Consider any infinite dimensional $\mathscr{L}_\infty^g$-space X (say $\ell_\infty(\mathbb{N})$) as an annihilator A-module. From remark right after [[12], corollary 23.3] we know that X is not an $\mathscr{L}_1^g$-space. Therefore, by proposition 2.2.6 the A-module X is not topologically flat. By proposition 2.1.34 it is not metrically flat. Now from proposition 2.1.39 we see that X is neither metrically nor topologically projective. □

2.2.2 Homologically trivial modules over Banach algebras with specific geometry

The purpose of this section is to convince our reader that homologically trivial modules over certain Banach algebras have similar geometric structure of those algebras. For the case of metric theory the following proposition was proved by Graven in [40].

Proposition 2.2.8. *Let A be a Banach algebra which is as Banach space isometrically isomorphic to $L_1(\Theta, \nu)$ for some measure space (Θ, Σ, ν). Then*

(i) if P is a $\langle$ metrically / topologically $\rangle$ projective A-module, then P is $\langle$ an L_1-space / complemented in some L_1-space $\rangle$.

(ii) if J is a $\langle$ metrically / topologically $\rangle$ injective A-module, then J is a $\langle$ $C(K)$-space for some Stonean space K / topologically injective Banach space $\rangle$.

(iii) if F is a $\langle$ metrically / topologically $\rangle$ flat A-module, then F is an $\langle$ L_1-space / $\mathscr{L}_1^g$-space $\rangle$.

Proof. Denote by (Θ', Σ', ν') the measure space (Θ, Σ, ν) with singleton atom adjoined, then $A_+ \underset{\mathbf{Ban}_1}{\cong} L_1(\Theta', \nu')$.

(i) Since P is a $\langle$ metrically / topologically $\rangle$ projective A-module, then by proposition 2.1.5 it is a retract of $A_+ \widehat{\otimes} \ell_1(B_P)$ in $\langle$ $A - \mathbf{mod}_1$ / $A - \mathbf{mod}$ $\rangle$. Let μ_c be the counting measure on B_P, then by Grothendieck's theorem [[1], theorem 2.7.5]

$$A_+ \widehat{\otimes} \ell_1(B_P) \underset{\mathbf{Ban}_1}{\cong} L_1(\Theta', \nu') \widehat{\otimes} L_1(B_P, \mu_c) \underset{\mathbf{Ban}_1}{\cong} L_1(\Theta' \times B_P, \nu' \times \mu_c)$$

Therefore P is $\langle$ 1-complemented / complemented $\rangle$ in some L_1-space. It remains to recall that any 1-complemented subspace of L_1-space is again an L_1-space [[17], theorem 6.17.3].

(ii) Since J is $\langle$ metrically / topologically $\rangle$ injective A-module, then by proposition 2.1.22 it is a retract of $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$ in $\langle$ $\mathbf{mod}_1 - A$ / $\mathbf{mod} - A$ $\rangle$. Let μ_c be the counting measure on B_{J^*}, then by Grothendieck's theorem [[1], theorem 2.7.5]

$$\mathcal{B}(A_+, \ell_\infty(B_{J^*})) \underset{\mathbf{Ban}_1}{\cong} (A_+ \widehat{\otimes} \ell_1(B_{J^*}))^* \underset{\mathbf{Ban}_1}{\cong} (L_1(\Theta', \nu') \widehat{\otimes} L_1(B_P, \mu_c))^*$$

$$\underset{\mathbf{Ban}_1}{\cong} L_1(\Theta' \times B_P, \nu' \times \mu_c)^* \underset{\mathbf{Ban}_1}{\cong} L_\infty(\Theta' \times B_P, \nu' \times \mu_c)$$

Therefore J is $\langle$ 1-complemented / complemented $\rangle$ in some L_∞-space. Since any L_∞-space is $\langle$ metrically / topologically $\rangle$ injective Banach space, then so is J. It remains to recall that every metrically injective Banach space is a $C(K)$-space for some Stonean space K [[17], theorem 3.11.6].

(iii) By $\langle$ [[32], theorem 1] / remark after [[12], corollary 23.5(1)] $\rangle$ the Banach space F^* is $\langle$ metrically / topologically $\rangle$ injective iff F is an $\langle$ L_1-space / $\mathscr{L}_1^g$-space $\rangle$. Now the implication follows from paragraph (ii) and proposition 2.1.32. □

Proposition 2.2.9. *Let A be a Banach algebra which is topologically isomorphic as Banach space to some $\mathscr{L}_1^g$-space. Then any topologically $\langle$ projective / injective / flat $\rangle$ A-module is an $\langle$ $\mathscr{L}_1^g$-space / $\mathscr{L}_\infty^g$-space / $\mathscr{L}_1^g$-space $\rangle$.*

Proof. If A is a $\mathscr{L}_1^g$-space, then so is A_+.

Let P be a topologically projective A-module. Then by proposition 2.1.5 it is a retract of $A_+ \widehat{\otimes} \ell_1(B_P)$ in $A - \mathbf{mod}$ and a fortiori in $\mathbf{Ban}$. Since $\ell_1(B_P)$ is an $\mathscr{L}_1^g$-space, then so is $A_+ \widehat{\otimes} \ell_1(B_P)$ as a projective tensor product of $\mathscr{L}_1^g$-spaces [[12], exercise 23.17(c)]. Therefore, P is an $\mathscr{L}_1^g$-space as a complemented subspace of $\mathscr{L}_1^g$-space [[12], corollary 23.2.1(2)].

Let J be a topologically injective A-module, then by proposition 2.1.22 it is a retract of $\mathcal{B}(A_+, \ell_\infty(B_{J^*})) \underset{\mathbf{mod}_1 - A}{\cong} (A_+ \widehat{\otimes} \ell_1(B_{J^*}))^*$ in $\mathbf{mod} - A$ and a fortiori in $\mathbf{Ban}$. As we showed in the previous paragraph $A_+ \widehat{\otimes} \ell_1(B_{J^*})$ is an $\mathscr{L}_1^g$-space, therefore its dual $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$ is an $\mathscr{L}_\infty^g$-space [[12], corollary 23.2.1(1)]. It remains to recall that any complemented subspace of $\mathscr{L}_\infty^g$-space is again an $\mathscr{L}_\infty^g$-space [[12], corollary 23.2.1(2)].

Finally, let F be a topologically flat A-module, then F^* is topologically injective A-module by proposition 2.1.32. From previous paragraph it follows that F^* is an $\mathscr{L}_\infty^g$-space. By [[12], corollary 23.5(1)] we get that F is an $\mathscr{L}_1^g$-space. □

We proceed to the discussion of the Dunford-Pettis property for homologically trivial modules.

Proposition 2.2.10. *Let (Ω, Σ, μ) be a measure spaces and Λ be an arbitrary set. Then the Banach space $\bigoplus_0 \{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ and all its duals has the Dunford-Pettis property. In particular, $\bigoplus_1 \{L_\infty(\Omega, \mu) : \lambda \in \Lambda\}$ and $\bigoplus_\infty \{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ have this property.*

Proof. Consider one point compactification $\alpha\Lambda := \Lambda \cup \{\Lambda\}$ of the set Λ with discrete topology. From [[41], corollary 7] we know that the Banach space $F := C(\alpha\Lambda, L_1(\Omega, \mu))$ and all its duals have the Dunford-Pettis property. Since $c_0(\Lambda)$ is complemented in $C(\alpha\Lambda)$ via projection $P : C(\alpha\Lambda) \to C(\alpha\Lambda) : x \mapsto x(\lambda) - x(\{\Lambda\})$, then $E := c_0(\Lambda, L_1(\Omega, \mu))$ is complemented in F. Similarly, any n-th dual of E is complemented in n-th dual of F, because we can take n-th adjoint of P in the role of projection. Now it remains to note that $E = \bigoplus_0\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ and that the Dunford-Pettis property is inherited by complemented subspaces [[9], proposition 13.44].

As a consequence of the previous paragraph the Banach spaces $E^* \underset{\mathbf{Ban}_1}{\cong} \bigoplus_1\{L_\infty(\Omega, \mu) : \lambda \in \Lambda\}$ and $E^{**} \underset{\mathbf{Ban}_1}{\cong} \bigoplus_\infty\{L_1(\Omega, \mu)^{**} : \lambda \in \Lambda\}$ have the Dunford-Pettis property. From [[12], proposition B10] we know that any L_1-space is contractively complemented in its second dual. By Q we denote the respective projection. Therefore, the Banach space $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ is contractively complemented in E^{**} via projection $\bigoplus_\infty\{Q : \lambda \in \Lambda\}$. Since E^{**} has the Dunford-Pettis property, then by [[9], proposition 13.44] so does its complemented subspace $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$. □

Proposition 2.2.11. *Let $\{(\Omega_\lambda, \Sigma_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a family of measure spaces. Then the Banach spaces $\bigoplus_0\{L_1(\Omega_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$, $\bigoplus_1\{L_\infty(\Omega_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ and $\bigoplus_\infty\{L_1(\Omega_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ have the Dunford-Pettis property.*

Proof. Let (Ω, Σ, μ) be a direct sum of $\{(\Omega_\lambda, \Sigma_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$. By construction each Banach space $L_1(\Omega_\lambda, \mu_\lambda)$ for $\lambda \in \Lambda$ is 1-complemented in $L_1(\Omega, \mu)$. Therefore, their $\bigoplus_0$-, $\bigoplus_1$- and $\bigoplus_\infty$-sums are contractively complemented in $\bigoplus_0\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$, $\bigoplus_1\{L_\infty(\Omega, \mu) : \lambda \in \Lambda\}$ and $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ respectively. It remains to combine proposition 2.2.10 and the fact that the Dunford-Pettis property is preserved by complemented subspaces [[9], proposition 13.44]. □

Proposition 2.2.12. *Let E be an $\mathscr{L}_\infty^g$-space and Λ be an arbitrary set. Then $\bigoplus_\infty\{E^* : \lambda \in \Lambda\}$ has the Dunford-Pettis property.*

Proof. Since E is an $\mathscr{L}_\infty^g$-space, then E^* is a $\mathscr{L}_1^g$-space [[12], corollary 23.2.1(1)]. Then from [[12], corollary 23.2.1(3)] it follows that E^{***} is a retract of L_1-space. Recall that E^* is complemented in E^{***} via Dixmier projection, so E^* is complemented in some L_1-space too. Thus, we have a bounded linear projection $P : L_1(\Omega, \mu) \to L_1(\Omega, \mu)$ with image topologically isomorphic to E. In this case $\bigoplus_\infty\{E^* : \lambda \in \Lambda\}$ is complemented in $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ via projection $\bigoplus_\infty\{P : \lambda \in \Lambda\}$. The space $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$ has the Dunford-Pettis property by proposition 2.2.11. By proposition 13.44 in [9] so does $\bigoplus_\infty\{E^* : \lambda \in \Lambda\}$ as a complemented subspace of $\bigoplus_\infty\{L_1(\Omega, \mu) : \lambda \in \Lambda\}$. □

Proposition 2.2.13. *Any $\langle$ $\mathscr{L}_1^g$-space / $\mathscr{L}_\infty^g$-space $\rangle$ admits the Dunford-Pettis property.*

Proof. Assume E is an $\langle$ $\mathscr{L}_1^g$-space / $\mathscr{L}_\infty^g$-space $\rangle$. Then E^{**} is complemented in some $\langle$ L_1-space / L_∞-space $\rangle$ [[12], corollary 23.2.1(3)]. Since any $\langle$ L_1-space / L_∞-space $\rangle$ admits the Dunford-Pettis property [15], then so does E^{**} as a complemented subspace [[9], proposition 13.44]. It remains to recall that a Banach space have the Dunford-Pettis property whenever so does its dual. □

Theorem 2.2.14. *Let A be a Banach algebra which is an $\mathscr{L}_1^g$-space or $\mathscr{L}_\infty^g$-space as Banach space. Then any topologically projective, injective or flat A-module has the Dunford-Pettis property.*

Proof. If A is an $\mathscr{L}_1^g$-space, we just need to combine propositions 2.2.13 and 2.2.9.

Assume A is an $\mathscr{L}_\infty^g$-space, then so does A_+. Let J be a topologically injective A-module, then by proposition 2.1.22 it is a retract of

$$\mathcal{B}(A_+, \ell_\infty(B_{J^*})) \underset{\mathbf{mod}_1 - A}{\cong} (A_+ \widehat{\otimes} \ell_1(B_{J^*}))^* \underset{\mathbf{mod}_1 - A}{\cong} \left(\bigoplus\nolimits_1 \{A_+ : \lambda \in B_{J^*}\}\right)^*$$

$$\underset{\mathbf{mod}_1 - A}{\cong} \bigoplus\nolimits_\infty \{A_+^* : \lambda \in B_{J^*}\}$$

in $\mathbf{mod} - A$ and a fortiori in $\mathbf{Ban}$. By proposition 2.2.12 this space has the Dunford-Pettis property. As J is a retract of $\mathcal{B}(A_+, \ell_\infty(B_{J^*}))$, then J also has the Dunford-Pettis property [[9], proposition 13.44].

If F is a topologically flat A-module, then F^* is a topologically injective A-module by proposition 2.1.32. By previous paragraph F^* has the Dunford-Pettis property and so does F.

If P is a topologically projective A-module, it is also topologically flat by proposition 2.1.39. From previous paragraph it follows that P has the Dunford-Pettis property. □

Corollary 2.2.15. *Let A be a Banach algebra which is an $\mathscr{L}_1^g$-space or an $\mathscr{L}_\infty^g$-space as Banach space. Then there is no topologically projective, injective or flat infinite dimensional reflexive A-modules. A fortiori there is no metrically projective, injective or flat infinite dimensional reflexive A-modules either.*

Proof. From theorem 2.2.14 we know that any topologically injective A-module has the Dunford-Pettis property. On the other hand there is no infinite dimensional reflexive Banach spaces with the Dunford-Pettis property. Thus, we get the desired result regarding topological injectivity. Since dual of reflexive module is reflexive, from proposition 2.1.32 we get the result for topological flatness. It remains to recall that by proposition 2.1.39 every topologically projective module is topologically flat. To prove the last claim note that by proposition $\langle$ 2.1.6 / 2.1.23 / 2.1.34 $\rangle$ metric $\langle$ projectivity / injectivity / flatness $\rangle$ implies topological $\langle$ projectivity / injectivity / flatness $\rangle$. $\square$

Note that in relative theory there are examples of infinite dimensional relatively projective injective and flat reflexive modules over Banach algebras that are $\mathscr{L}_1^g$- or $\mathscr{L}_\infty^g$-spaces. Here are two examples. The first one is about convolution algebra $L_1(G)$ on a locally compact group G with the left Haar measure. It is an $\mathscr{L}_1^g$-space. In [[42], §6] and [43] it was proved that for $1 < p < +\infty$ the $L_1(G)$-module $L_p(G)$ is relatively $\langle$ projective / injective / flat $\rangle$ iff G is $\langle$ compact / amenable / amenable $\rangle$. Note that any compact group is amenable [[44], proposition 3.12.1], so in case G is compact $L_p(G)$ is relatively projective injective and flat for all $1 < p < +\infty$. The second example is about $\mathscr{L}_\infty^g$-spaces $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$ for an infinite set Λ. In proposition 3.1.25 we shall show that $\ell_p(\Lambda)$ for $1 < p < \infty$ is relatively projective injective and flat as $c_0(\Lambda)$- or $\ell_\infty(\Lambda)$-module.

We finalize this lengthy section by short remark on the l.u.st. property of topologically projective, injective and flat modules.

Proposition 2.2.16. *Let A be a Banach algebra which as a Banach space has the l.u.st. property. Then any topologically projective, injective or flat A-module has the l.u.st. property.*

Proof. If J is a topologically injective A-module, then by proposition 2.1.22 it is a retract of $E := \mathcal{B}(A_+, \ell_\infty(B_{J^*})) \underset{\mathbf{mod}_1 - A}{\cong} \bigoplus_\infty \{A_+^* : \lambda \in B_{J^*}\}$ in $\mathbf{mod} - A$ and a fortiori in **Ban**. If A has the l.u.st. property, then A^{**} is complemented in some Banach lattice L [[20], theorem 17.5]. As a consequence A_+^{***} is complemented in the Banach lattice $M := L^* \bigoplus_1 \mathbb{C}$. Note that A_+^* is contractively complemented in A_+^{***} by the Dixmier projection. Therefore, A_+^* is complemented in M via some projection $P : M \to M$. Hence, E is complemented in the Banach lattice $K := \bigoplus_\infty \{M : \lambda \in B_{J^*}\}$ via projection $\bigoplus_\infty \{P : \lambda \in B_{J^*}\}$. Since J is a retract of E then J is also complemented in K. As any Banach lattice K has the l.u.st. property [[20], theorem 17.1]. This property is inherited by complemented subspaces, so J has the l.u.st. property too.

If F is a topologically flat A-module, then F^* is topologically injective by proposition 2.1.32. By previous paragraph F^* has the l.u.st. property. Corollary 17.6 from [20] gives that F has this property too.

If P is a topologically projective A-module, it is topologically flat by proposition 2.1.39. So P has the l.u.st. property by previous paragraph. $\square$

2.3 Further properties of projective injective and flat modules

2.3.1 Homological triviality of modules under the change of algebra

The following three propositions are metric and topological versions of propositions 2.3.2, 2.3.3 and 2.3.4 in [33].

Proposition 2.3.1. *Let X and Y be $\langle$ left / right $\rangle$ A-modules. Assume one of the following holds*

(i) I is a closed $\langle$ left / right $\rangle$ ideal of A and X is an essential I-module;

(ii) I is a closed $\langle$ right / left $\rangle$ ideal of A and Y is a faithful I-module.

Then $\langle$ ${}_A\mathcal{B}(X,Y) = {}_I\mathcal{B}(X,Y)$ / $\mathcal{B}_A(X,Y) = \mathcal{B}_I(X,Y)$ $\rangle$.

Proof. We shall prove both statements only for left modules, since their right counterparts can be proved with minimal adjustments. Fix $\phi \in {}_I\mathcal{B}(X,Y)$.

(i) Take $x \in I \cdot X$, then $x = a' \cdot x'$ for some $a' \in I$, $x' \in X$. For any $a \in A$ we have $\phi(a \cdot x) = \phi(aa' \cdot x') = aa' \cdot \phi(x') = a \cdot \phi(a' \cdot x') = a \cdot \phi(x)$. Therefore, $\phi(a \cdot x) = a \cdot \phi(x)$ for all $a \in A$ and $x \in \operatorname{cl}_X(IX) = X$. Hence, $\phi \in {}_A\mathcal{B}(X,Y)$.

(ii) For any $a \in I$ and $a' \in A$, $x \in X$ we have $a \cdot (\phi(a' \cdot x) - a' \cdot \phi(x)) = \phi(aa' \cdot x) - aa' \cdot \phi(x) = 0$. Since Y is faithful I-module we have $\phi(a' \cdot x) = a' \cdot \phi(x)$ for all $x \in X$, $a' \in A$. Hence, $\phi \in {}_A\mathcal{B}(X,Y)$.

In both cases we proved that $\phi \in {}_A\mathcal{B}(X,Y)$ for any $\phi \in {}_I\mathcal{B}(X,Y)$, therefore ${}_I\mathcal{B}(X,Y) \subset {}_A\mathcal{B}(X,Y)$. The reverse inclusion is obvious. $\square$

Proposition 2.3.2. *Let I be a closed subalgebra of A and P be an A-module which is essential as I-module. Then*

(i) if I is a closed left ideal of A and P is a $\langle$ metrically / C-topologically $\rangle$ projective I-module, then P is a $\langle$ metrically / C-topologically $\rangle$ projective A-module;

(ii) if I is a $\langle$ 1-complemented / c-complemented $\rangle$ closed right ideal of A and P is a $\langle$ metrically / C-topologically $\rangle$ projective A-module, then P is a $\langle$ metrically / cC-topologically $\rangle$ projective I-module.

Proof. By $\widetilde{\pi}_P : I \widehat{\otimes} \ell_1(B_P) \to P$ and $\pi_P : A \widehat{\otimes} \ell_1(B_P) \to P$ we shall denote the standard epimorphisms.

(i) By proposition 2.1.8 the morphism $\widetilde{\pi}_P$ has a right inverse morphism in $\langle$ $I - \mathbf{mod}_1$ / $I - \mathbf{mod}$ $\rangle$, say $\widetilde{\sigma}$, of norm $\langle$ at most 1 / at most C $\rangle$. Let $i : I \to A$ be the natural embedding, then consider a $\langle$ contractive / bounded $\rangle$ I-morphism $\sigma = (i \widehat{\otimes} 1_{\ell_1(B_P)})\widetilde{\sigma}$. By paragraph (i) of proposition 2.3.1 we have that σ is an A-morphism. Clearly, σ has norm $\langle$ at most 1 / at most C $\rangle$. For $\pi_P : A \widehat{\otimes} \ell_1(B_P) \to P$ we obviously have $\pi_P(i \widehat{\otimes} 1_{\ell_1(B_P)}) = \widetilde{\pi}_P$, hence $\pi_P\sigma = \pi_P(i \widehat{\otimes} 1_{\ell_1(B_P)})\widetilde{\sigma} = \widetilde{\pi}_P\widetilde{\sigma} = 1_P$. Thus, π_P is a $\langle$ 1-retraction / C-retraction $\rangle$ in $\langle$ $A - \mathbf{mod}_1$ / $A - \mathbf{mod}$ $\rangle$. So by proposition 2.1.8 the A-module P is $\langle$ metrically / C-topologically $\rangle$ projective.

(ii) Since P is an essential I-module it is a fortiori an essential A-module. By proposition 2.1.8 the morphism π_P has a right inverse morphism σ in $\langle$ $A - \mathbf{mod}_1$ / $A - \mathbf{mod}$ $\rangle$ with norm $\langle$ at most 1 / at most C $\rangle$. Obviously σ is a right inverse for π_P in $\langle$ $I - \mathbf{mod}_1$ / $I - \mathbf{mod}$ $\rangle$ too. By $i : I \to A$ we denote the natural embedding, and by $r : A \to I$ its $\langle$ contractive / bounded $\rangle$ left inverse. By assumption $\|r\| \leq c$. Consider a $\langle$ contractive / bounded $\rangle$ linear operator $\widetilde{\sigma} = (r \widehat{\otimes} 1_{\ell_1(B_P)})\sigma$. Clearly, its norm is $\langle$ at most 1 / at most cC $\rangle$. Since I is a right ideal of A and P is an essential I-module then $\sigma(P) = \sigma(\mathrm{cl}_P(IP)) = \mathrm{cl}_{A\widehat{\otimes}\ell_1(B_P)}(I \cdot (A \widehat{\otimes} \ell_1(B_P))) \subset I \widehat{\otimes} \ell_1(B_P)$, so $\sigma = (ir \widehat{\otimes} 1_{\ell_1(B_P)})\sigma$. Even more, since $\sigma(P) \subset I \widehat{\otimes} \ell_1(B_P)$ and $r|_I = 1_I$, then σ is an I-morphism. Clearly, $\pi_P(i \widehat{\otimes} 1_{\ell_1(B_P)}) = \widetilde{\pi}_P$, so

$$\widetilde{\pi}_P\widetilde{\sigma} = \pi_P(i \widehat{\otimes} 1_{\ell_1(B_P)})(r \widehat{\otimes} 1_{\ell_1(B_P)})\sigma = \pi_P(ir \widehat{\otimes} 1_{\ell_1(B_P)})\sigma = \pi_P\sigma = 1_P$$

Thus $\widetilde{\pi}_P$ is a $\langle$ 1-retraction / cC-retraction $\rangle$ in $\langle$ $I - \mathbf{mod}_1$ / $I - \mathbf{mod}$ $\rangle$, so by proposition 2.1.8 the I-module P is $\langle$ metrically / cC-topologically $\rangle$ projective. $\square$

Proposition 2.3.3. *Let I be a closed subalgebra of A and J be a right A-module which is faithful as I-module. Then*

(i) if I is a closed left ideal of A and J is a $\langle$ metrically / C-topologically $\rangle$ injective I-module, then J is a $\langle$ metrically / C-topologically $\rangle$ injective A-module;

(ii) if I is a weakly $\langle$ 1-complemented / c-complemented $\rangle$ closed right ideal of A and J is a $\langle$ metrically / C-topologically $\rangle$ injective A-module, then J is a $\langle$ metrically / cC-topologically $\rangle$ injective I-module.

Proof. By $\widetilde{\rho}_J : J \to \mathcal{B}(I, \ell_\infty(B_{J^*}))$ and $\rho_J : J \to \mathcal{B}(A, \ell_\infty(B_{J^*}))$ we will denote the standard monomorphisms.

(i) By proposition 2.1.25 the morphism $\widetilde{\rho}_J : J \to \mathcal{B}(I, \ell_\infty(B_{J^*}))$ has a left inverse morphism in $\langle$ $\mathbf{mod}_1 - I$ / $\mathbf{mod} - I$ $\rangle$, say $\widetilde{\tau}$ of norm $\langle$ at most 1 / at most C $\rangle$. Let $i : I \to A$ be the natural embedding, and define an I-morphism $q = \mathcal{B}(i, \ell_\infty(B_{J^*}))$. Obviously $\widetilde{\rho}_J = q\rho_J$. Consider I-morphism $\tau = \widetilde{\tau}q$. By paragraph (ii) of proposition 2.3.1 it is also an A-morphism. Note that $\|\tau\| \leq \|\widetilde{\tau}\|\|q\| \leq \|\widetilde{\tau}\|$, so τ has norm $\langle$ at most 1 / at most C $\rangle$. Clearly, $\tau\rho_J = \widetilde{\tau}q\rho_J = \widetilde{\tau}\widetilde{\rho}_J = 1_J$. Thus, ρ_J is a $\langle$ 1-coretraction / C-coretraction $\rangle$, so by proposition 2.1.25 the A-module J is $\langle$ metrically / C-topologically $\rangle$ injective.

(ii) If J is a $\langle$ metrically / C-topologically $\rangle$ injective as A-module, then by proposition 2.1.25 the A-morphism ρ_J has a left inverse in $\langle$ $\mathbf{mod}_1 - A$ / $\mathbf{mod} - A$ $\rangle$, say τ, of norm $\langle$ at most 1 / at most C $\rangle$. Assume we are given an operator $T \in \mathcal{B}(A, \ell_\infty(B_{J^*}))$, such that $T|_I = 0$. Fix $a \in I$, then $T \cdot a = 0$, and so $\tau(T) \cdot a = \tau(T \cdot a) = 0$. Since J is a faithful I-module and $a \in I$ is arbitrary, then $\tau(T) = 0$. Since I is weakly $\langle$ 1-complemented /c-complemented $\rangle$ in A, then i^* has a left inverse $r : I^* \to A^*$ with norm $\langle$ at most 1 / at most c $\rangle$. For a given $f \in B_{J^*}$ we define a bounded linear operator $g_f : \mathcal{B}(I, \ell_\infty(B_{J^*})) \to I^* : T \mapsto (x \mapsto T(x)(f))$. Now we can define two more bounded linear operators

$$j : \mathcal{B}(I, \ell_\infty(B_{J^*})) \to \mathcal{B}(A, \ell_\infty(B_{J^*})), T \mapsto (a \mapsto (f \mapsto r(g_f(T))(a)))$$

and $\widetilde{\tau} = \tau j$. Fix $a \in I$ and $T \in \mathcal{B}(I, \ell_\infty(B_{J^*}))$. Since r is a left inverse of i^* we have $r(h)(a) = h(a)$ for all $h \in I^*$. Now it is routine to check that $(j(T \cdot a) - j(T) \cdot a)|_I = 0$. As we have shown earlier this implies that $\widetilde{\tau}(T \cdot a) - \widetilde{\tau}(T) \cdot a = \tau(j(T \cdot a) - j(T) \cdot a) = 0$. Since $a \in I$ and $T \in \mathcal{B}(I, \ell_\infty(B_{J^*}))$ are arbitrary the map $\widetilde{\tau}$ is an I-morphism. Note that $\|\widetilde{\tau}\| \leq \|\tau\|\|j\| \leq \|\tau\|\|r\|$, so $\widetilde{\tau}$ has norm $\langle$ at most 1 / at most cC $\rangle$. In the same way one can show that, for all $x \in J$ holds $\rho_J(x) - j(\widetilde{\rho}_J(x))|_I = 0$, so $\tau(\rho_J(x) - j(\widetilde{\rho}_J(x))) = 0$. As a consequence, $\widetilde{\tau}(\widetilde{\rho}_J(x)) = \tau(j(\widetilde{\rho}_J(x))) = \tau(\rho_J(x)) = x$ for all $x \in J$. Since $\widetilde{\tau}\widetilde{\rho}_J = 1_J$, then $\widetilde{\rho}_J$ is a $\langle$ 1-coretraction / cC-coretraction $\rangle$ in $\langle$ $\mathbf{mod}_1 - I$ / $\mathbf{mod} - I$ $\rangle$, so by proposition 2.1.25 the I-module J is $\langle$ metrically / cC-topologically $\rangle$ injective. □

Proposition 2.3.4. *Let I be a closed subalgebra of A and F be an A-module which is essential as I-module. Then*

(i) if I is a closed left ideal of A and F is a $\langle$ metrically / C-topologically $\rangle$ flat I-module, then F is a $\langle$ metrically / C-topologically $\rangle$ flat A-module;

(ii) if I is a weakly $\langle$ 1-complemented / c-complemented $\rangle$ closed right ideal of A and F is a $\langle$ metrically / C-topologically $\rangle$ flat A-module, then F is a $\langle$ metrically / cC-topologically $\rangle$ flat I-module.

Proof. Note that the dual of essential module is faithful. Now the result follows from propositions 2.1.32 and 2.3.3. ☐

Proposition 2.3.5. *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of Banach algebras and for each $\lambda \in \Lambda$ let X_λ be $\langle$ an essential / a faithful / an essential $\rangle$ A_λ-module. Denote $A = \bigoplus_p \{A_\lambda : \lambda \in \Lambda\}$ for $1 \leq p \leq +\infty$ or $p = 0$. Let X denote $\langle$ $\bigoplus_1 \{X_\lambda : \lambda \in \Lambda\}$ / $\bigoplus_\infty \{X_\lambda : \lambda \in \Lambda\}$ / $\bigoplus_1 \{X_\lambda : \lambda \in \Lambda\}$ $\rangle$. Then*

(i) X is a metrically $\langle$ projective / injective / flat $\rangle$ A-module iff for all $\lambda \in \Lambda$ the A_λ-module X_λ is metrically $\langle$ projective / injective / flat $\rangle$;

(ii) X is a C-topologically $\langle$ projective / injective / flat $\rangle$ A-module iff for all $\lambda \in \Lambda$ the A_λ-module X_λ is C-topologically $\langle$ projective / injective / flat $\rangle$.

Proof. Note that for each $\lambda \in \Lambda$ the natural embedding $i_\lambda : A_\lambda \to A$ allows one regarding A_λ as a contractively complemented closed two-sided ideal of A.

(i) The proof is literally the same as in paragraph (ii).

(ii) Assume X_λ is a C-topologically $\langle$ projective / injective / flat $\rangle$ A_λ-module for all $\lambda \in \Lambda$, then by paragraph (i) of proposition $\langle$ 2.3.2 / 2.3.3 / 2.3.4 $\rangle$ it is C-topologically $\langle$ projective / injective / flat $\rangle$ as A-module. It remains to apply the proposition $\langle$ 2.1.9 / 2.1.26 / 2.1.37 $\rangle$.

Conversely, assume that X is C-topologically $\langle$ projective / injective / flat $\rangle$ as A-module. Fix arbitrary $\lambda \in \Lambda$. Clearly, we may regard X_λ as an A-module and even more X_λ is a 1-retract of X in $\langle$ $A - \mathbf{mod}_1$ / $\mathbf{mod}_1 - A$ / $A - \mathbf{mod}_1$ $\rangle$. By proposition $\langle$ 2.1.4 / 2.1.21 / 2.1.33 $\rangle$ we get that X_λ is C-topologically $\langle$ projective / injective / flat $\rangle$ as A-module. It remains to apply paragraph (ii) of proposition $\langle$ 2.3.2 / 2.3.3 / 2.3.4 $\rangle$. ☐

2.3.2 Characterizations of flat modules

Based on results obtained above, we collect more interesting facts on metric and topological injectivity and flatness of Banach modules.

Proposition 2.3.6. *Let B be a unital Banach algebra, A be its subalgebra with two-sided bounded approximate identity $(e_\nu)_{\nu\in N}$ and let X be a left B-module. Denote $c_1 = \sup_{\nu\in N} \|e_\nu\|$, $c_2 = \sup_{\nu\in N} \|e_B - e_\nu\|$ and $X_{ess} = \operatorname{cl}_X(AX)$. Then*

(i) X^* is $c_2(c_1+1)$-isomorphic as a right A-module to $X^*_{ess} \bigoplus_\infty (X/X_{ess})^*$;

(ii) $\langle$ X^*_{ess} / $(X/X_{ess})^*$ $\rangle$ is a $\langle$ c_1-retract / c_2-retract $\rangle$ of A-module X^*;

(iii) if X *is an* $\mathscr{L}^g_{1,C}$*-space, then* $\langle\, X_{ess} \,/\, X/X_{ess} \,\rangle$ *is an* $\langle\, \mathscr{L}^g_{1,c_1C}$*-space* / $\mathscr{L}^g_{1,c_2C}$*-space* $\rangle$.

Proof. (i) Define the natural embedding $\rho : X_{ess} \to X$, $x \mapsto x$ and the quotient map $\pi : X \to X/X_{ess}$, $x \mapsto x+X_{ess}$. Let $\mathfrak{F}$ be the section filter on N and let $\mathfrak{U}$ be an ultrafilter dominating $\mathfrak{F}$. For a fixed $f \in X^*$ and $x \in X$ we have $|f(x-e_\nu \cdot x)| \leq \|f\|\|e_B - e_\nu\|\|x\| \leq c_2\|f\|\|x\|$ i.e. $(f(x - e_\nu \cdot x))_{\nu\in N}$ is a bounded net of complex numbers. Therefore, we have a well-defined limit $\lim_{\mathfrak{U}} f(x-e_\nu \cdot x)$ along ultrafilter $\mathfrak{U}$. Since $(e_\nu)_{\nu\in N}$ is a two-sided approximate identity for A and $\mathfrak{U}$ contains the section filter then for all $x \in X_{ess}$ we have $\lim_{\mathfrak{U}} f(x - e_\nu \cdot x) = \lim_\nu f(x - e_\nu \cdot x) = 0$. Therefore, for each $f \in X^*$ we have a well-defined map $\tau(f) : X/X_{ess} \to \mathbb{C}$, $x + X_{ess} \mapsto \lim_{\mathfrak{U}} f(x - e_\nu \cdot x)$. Clearly this is a linear functional, and from inequalities above we see its norm is bounded by $c_2\|f\|$. Now it is routine to check that $\tau : X^* \to (X/X_{ess})^*$, $f \mapsto \tau(f)$ is an A-morphism with norm not greater than c_2. Similarly, one can show that $\sigma : X^*_{ess} \to X^* : h \mapsto (x \mapsto \lim_{\mathfrak{U}} h(e_\nu \cdot x))$ is an A-morphism with norm not greater than c_1. For any $f \in X^*$, $g \in (X/X_{ess})^*$, $h \in X^*_{ess}$ and $x \in X$, $y \in X_{ess}$ we have

$$\sigma(h)(y) = \lim_{\mathfrak{U}} h(e_\nu \cdot y) = \lim_{\nu} h(e_\nu \cdot y) = h(y), \qquad (\rho^*\sigma)(h)(y) = \sigma(h)(\rho(y))\sigma(h)(y) = h(y),$$

$$(\tau\pi^*)(g)(x + X_{ess}) = \lim_{\mathfrak{U}} \pi^*(g)(x - e_\nu \cdot x) = \lim_{\mathfrak{U}} g(x + X_{ess}) = g(x + X_{ess}),$$

$$(\tau\sigma)(h)(x+X_{ess}) = \lim_{\mathfrak{U}} \sigma(h)(x-e_\nu \cdot x) = \lim_{\mathfrak{U}}(\sigma(h)(x)-h(e_\nu \cdot x)) = \sigma(h)(x)-\lim_{\mathfrak{U}} h(e_\nu \cdot x) = 0,$$

$$(\pi^*\tau+\sigma\rho^*)(f)(x) = \tau(f)(x+X_{ess})+\lim_{\mathfrak{U}} \rho^*(f)(e_\nu \cdot x) = \lim_{\mathfrak{U}} f(x-e_\nu \cdot x)+\lim_{\mathfrak{U}} f(e_\nu \cdot x) = f(x).$$

Therefore, $\tau\pi^* = 1_{(X/X_{ess})^*}$, $\rho^*\sigma = 1_{X^*_{ess}}$ and $\pi^*\tau + \sigma\rho^* = 1_{X^*}$. Now it is easy to check that the linear maps

$$\xi : X^* \to X^*_{ess} \bigoplus{}_\infty (X/X_{ess})^*,\ f \mapsto \rho^*(f) \oplus_\infty \tau(f)$$

$$\eta : X^*_{ess} \bigoplus{}_\infty (X/X_{ess})^* \to X^*,\ h \oplus_\infty g \mapsto \pi^*(h) + \sigma(g)$$

are isomorphism of right A-modules with $\|\xi\| \leq c_2$ and $\|\eta\| \leq c_1 + 1$. Hence, X^* is $c_2(c_1 + 1)$-isomorphic in $\mathbf{mod} - A$ to $X^*_{ess} \bigoplus_\infty (X/X_{ess})^*$.

(ii) Both results immediately follow from equalities $\rho^*\sigma = 1_{X^*_{ess}}$, $\tau\pi^* = 1_{(X/X_{ess})^*}$ and estimates $\|\rho^*\|\|\sigma\| \leq c_1$, $\|\tau\|\|\pi^*\| \leq c_2$.

(iii) Now consider case when X is an $\mathscr{L}^g_{1,C}$-space. Then X^* is an $\mathscr{L}^g_{\infty,C}$-space [[12], corollary 23.2.1(1)]. As $\langle\, X^*_{ess} \,/\, (X/X_{ess})^* \,\rangle$ is $\langle\, c_1$-complemented / c_2-complemented $\rangle$ in X^* it is an $\langle\, \mathscr{L}^g_{\infty,c_1C}$-space / $\mathscr{L}^g_{\infty,c_2C}$-space $\rangle$ by [[12], corollary 23.2.1(1)]. Again we apply [[12], corollary 23.2.1(1)] to conclude that $\langle\, X_{ess} \,/\, X/X_{ess} \,\rangle$ is an $\langle\, \mathscr{L}^g_{1,c_1C}$-space / $\mathscr{L}^g_{1,c_2C}$-space $\rangle$. □

The following proposition is an analog of [[33], proposition 2.1.11].

Proposition 2.3.7. *Let A be a Banach algebra with two-sided c-bounded approximate identity, and F be a left A-module. Then*

(i) if F is a C-topologically flat A-module, then F_{ess} is a $(1+c)C$-topologically flat A-module and F/F_{ess} is an $\mathscr{L}^g_{1,(1+c)C}$-space;

(ii) if F_{ess} is a C_1-topologically flat A-module and F/F_{ess} is an $\mathscr{L}^g_{1,C_1}$-space, then F is a $(1+c)^2\max(C_1, C_2)$-topologically flat A-module.

(iii) F is a topologically flat A-module iff F_{ess} is a topologically flat A-module and F/F_{ess} is an $\mathscr{L}^g_1$-space.

Proof. We regard A as a closed subalgebra of unital Banach algebra $B := A_+$. Then F is a unital left B-module. Using notation of proposition 2.3.6 we may say that $c_1 = c$ and $c_2 = 1 + c$, so the right A-modules F^*_{ess} and $(F/F_{ess})^*$ are $(1 + c)$-retracts of F^*.

i) By proposition 2.1.32 the right A-module F^* is C-topologically injective. Therefore, from propositions 2.1.21, 2.1.32 the modules F_{ess} and F/F_{ess} are $(1+c)C$-topologically flat. It remains to note that F/F_{ess} is an annihilator A-module, so by proposition 2.2.6 it is an $\mathscr{L}^g_{1,(1+c)C}$-space.

(ii) Again, by proposition 2.1.32 the right A-modules F^*_{ess} and $(F/F_{ess})^*$ are C_1- and C_2-topologically injective respectively. So from proposition 2.1.26 their product is $\max(C_1, C_2)$-topologically injective. By proposition 2.3.6 this product is $(1+c)^2$-isomorphic to F^* in $\mathbf{mod} - A$. Therefore, F^* is a $(1 + c)^2 \max(C_1, C_2)$-topologically injective A-module. Now the result follows from proposition 2.1.32.

(iii) The result immediately follows from paragraphs (i) and (ii). □

Proposition 2.3.8. *Let A be a $\langle$ 1-relatively / c-relatively $\rangle$ amenable Banach algebra and F be an essential Banach A-module which is an $\langle$ L_1-space / $\mathscr{L}^g_{1,C}$-space $\rangle$. Then F is a $\langle$ metrically / c^2C-topologically $\rangle$ flat A-module.*

Proof. We may assume that A is c-relatively amenable for $\langle$ $c = 1$ / $c \geq 1$ $\rangle$. Let $(d_\nu)_{\nu\in N}$ be an approximate diagonal for A with norm bound at most c. Recall, that $(\Pi_A(d_\nu))_{\nu\in N}$ is a two-sided $\langle$ contractive / bounded $\rangle$ approximate identity for A. Since F is an essential left A-module, then $\lim_\nu \Pi_A(d_\nu)\cdot x = x$ for all $x \in F$ [[24], proposition 0.3.15]. As a consequence $c\pi_F(B_{A\widehat{\otimes}\ell_1(B_F)})$ is dense in B_F. Then for all $f \in F^*$ we have

$$\|\pi^*_F(f)\| = \sup\{|f(\pi_F(u))| : u \in B_{A\widehat{\otimes}\ell_1(B_F)}\} = \sup\{|f(x)| : x \in \mathrm{cl}_F(\pi_F(B_{A\widehat{\otimes}\ell_1(B_F)}))\}$$

$$\geq \sup\{c^{-1}|f(x)| : x \in B_F\} = c^{-1}\|f\|.$$

This means, that π_F^* is c-topologically injective. By assumption F is an $\langle$ L_1-space / $\mathscr{L}^g_{1,C}$-space $\rangle$, then by $\langle$ [[32], theorem 1] / remark after [[12], corollary 23.5(1)] $\rangle$ the Banach space F^* is $\langle$ metrically / C-topologically $\rangle$ injective. Since operator π_F^* is $\langle$ isometric / c-topologically injective $\rangle$, then there exists a linear operator $R : (A \mathbin{\widehat{\otimes}} \ell_1(B_F))^* \to F^*$ of norm $\langle$ at most 1 / at most cC $\rangle$ such that $R\pi_F^* = 1_{F^*}$.

Fix $h \in (A \mathbin{\widehat{\otimes}} \ell_1(B_F))^*$ and $x \in F$. Consider bilinear functional $M_{h,x} : A \times A \to \mathbb{C} : (a, b) \mapsto R(h \cdot a)(b \cdot x)$. Clearly, $\|M_{h,x}\| \leq \|R\|\|h\|\|x\|$. By universal property of projective tensor product we have a bounded linear functional $m_{h,x} : A \mathbin{\widehat{\otimes}} A \to \mathbb{C} : a \mathbin{\widehat{\otimes}} b \mapsto R(h \cdot a)(b \cdot x)$. Note that $m_{h,x}$ is linear in h and x. Even more, for any $u \in A \mathbin{\widehat{\otimes}} A$, $a \in A$ and $f \in F^*$ we have $m_{\pi_F^*(f),x}(u) = f(\Pi_A(u) \cdot x)$, $m_{h \cdot a,x}(u) = m_{h,x}(a \cdot u)$, $m_{h,a \cdot x}(u) = m_{h,x}(u \cdot a)$. It easily checked for elementary tensors. Then it is enough to recall that their linear span is dense in $A \mathbin{\widehat{\otimes}} A$.

Let $\mathfrak{F}$ be the section filter on N and let $\mathfrak{U}$ be an ultrafilter dominating $\mathfrak{F}$. For any $h \in (A \mathbin{\widehat{\otimes}} \ell_1(B_F))^*$ and $x \in F$ we have $|m_{h,x}(d_\nu)| \leq c\|R\|\|h\|\|x\|$, i.e. $(m_{h,x}(d_\nu))_{\nu \in N}$ is a bounded net of complex numbers. Therefore, we have a well-defined limit $\lim_{\mathfrak{U}} m_{h,x}(d_\nu)$ along the ultrafilter $\mathfrak{U}$. Consider a linear operator $\tau : (A \mathbin{\widehat{\otimes}} \ell_1(B_F))^* \to F^*$, $h \mapsto (x \mapsto \lim_{\mathfrak{U}} m_{h,x}(d_\nu))$. From norm estimates for $m_{h,x}$ it follows that τ is bounded with $\|\tau\| \leq c\|R\|$. For all $a \in A$, $x \in F$ and $h \in (A \mathbin{\widehat{\otimes}} \ell_1(B_F))^*$ we have

$$\tau(h \cdot a)(x) - (\tau(h) \cdot a)(x) = \tau(h \cdot a)(x) - \tau(h)(a \cdot x) = \lim_{\mathfrak{U}} m_{h \cdot a,x}(d_\nu) - \lim_{\mathfrak{U}} m_{h,a \cdot x}(d_\nu)$$

$$= \lim_{\mathfrak{U}} m_{h,x}(a \cdot d_\nu) - m_{h,x}(d_\nu \cdot a) = m_{h,x}\left(\lim_{\mathfrak{U}}(a \cdot d_\nu - d_\nu \cdot a)\right)$$

$$= m_{h,x}\left(\lim_{\nu}(a \cdot d_\nu - d_\nu \cdot a)\right) = m_{h,x}(0) = 0.$$

Therefore, τ is a morphism of right A-modules. Now for all $f \in F^*$ and $x \in F$ we have

$$(\tau(\pi_F^*)(f))(x) = \lim_{\mathfrak{U}} m_{\pi_F^*(f),x}(d_\nu) = \lim_{\mathfrak{U}} f(\Pi_A(d_\nu) \cdot x) = \lim_{\nu} f(\Pi_A(d_\nu) \cdot x)$$

$$= f\left(\lim_{\nu} \Pi_A(d_\nu) \cdot x\right) = f(x).$$

So $\tau\pi_F^* = 1_{F^*}$. This means that F^* is a $\langle$ 1-retract / c^2C-retract $\rangle$ of $(A \mathbin{\widehat{\otimes}} \ell_1(B_F))^*$ in $\langle$ $\mathbf{mod}_1 - A$ / $\mathbf{mod} - A$ $\rangle$. The latter A-module is $\langle$ metrically / topologically $\rangle$ injective, because $(A_+ \mathbin{\widehat{\otimes}} \ell_1(B_F))^* \underset{\mathbf{mod}_1 - A}{\cong} \mathcal{B}(A_+, \ell_\infty(B_F))$ and by proposition 2.1.21 so is its retract F^*. By proposition 2.1.32 this is equivalent to $\langle$ metric / c^2C-topological $\rangle$ flatness of the A-module F. $\square$

Theorem 2.3.9. *Let A be a c-relatively amenable Banach algebra and F be a left Banach A-module which as a Banach space is an $\mathscr{L}^g_{1,C}$-space. Then F is a $(1+c)^2 C \max(c^2, (1+c))$-topologically flat A-module.*

Proof. Since A is amenable, then it admits a two-sided c-bounded approximate identity. By proposition 2.3.6 the annihilator A-module F/F_{ess} is an $\mathscr{L}^g_{1,1+c}$-space. From proposition 2.3.8 we get that the essential A-module F_{ess} is c^2C-topologically flat. Now the result follows from proposition 2.3.7. □

We must point out here that in relative Banach homology any left Banach module over a relatively amenable Banach algebra is relatively flat [[23], theorem 7.1.60]. Even topological theory is so restrictive that in some cases, as the following proposition shows, we can obtain a complete characterization of all flat modules.

Proposition 2.3.10. *Let A be a relatively amenable Banach algebra which as a Banach space is an $\mathscr{L}^g_1$-space. Then for a Banach A-module F the following are equivalent:*

(i) F is topologically flat A-module;

(ii) F is an $\mathscr{L}^g_1$-space.

Proof. The equivalence follows from proposition 2.2.9 and theorem 2.3.9. □

Finally, we are able to give an example of relatively flat, but not topologically flat closed ideal of a Banach algebra. Consider $A = L_1(\mathbb{T})$. It is known, that A has a translation invariant infinite dimensional closed subspace I isomorphic to a Hilbert space [[45], p.52]. By [[26], proposition 1.4.7] we have that I is a two-sided ideal of A, as any translation invariant subspace of A. By [[12], section 23.3] this ideal is not an $\mathscr{L}^g_1$-space. So from proposition 2.3.10 we get that I is not topologically flat as A-module. We claim that it is still relatively flat. Since $\mathbb{T}$ is a compact group, then it is amenable [[44], proposition 3.12.1]. Thus, A is relatively amenable [[23], proposition VII.1.86], so all closed left ideals of A are relatively flat [[23], proposition VII.1.60(I)]. In particular, I is relatively flat.

2.3.3 Injectivity of ideals

Injective ideals are rare creatures, but we need to say a few words about them. Results of this section needed for the study of metric and topological injectivity of C^*-algebras.

Proposition 2.3.11. *Let I be a closed right ideal of a Banach algebra A. Assume I is $\langle$ metrically / C-topologically $\rangle$ injective A-module. Then I has a left identity of norm $\langle$ at most 1 / at most C $\rangle$ and is a $\langle$ 1-retract / C-retract $\rangle$ of A in* $\mathbf{mod} - A$.

Proof. Consider isometric embedding $\rho^+ : I \to A_+$ of I into A_+. Clearly, this is an A-morphism. Since I is $\langle$ metrically / C-topologically $\rangle$ injective, then ρ^+ has a left inverse A-morphism $\tau^+ : A_+ \to I$ with norm $\langle$ at most 1 / at most C $\rangle$. Now for all $x \in I$ we have $x = \tau^+(\rho^+(x)) = \tau^+(e_{A_+}\rho^+(x)) = \tau(e_{A_+})\rho^+(x) = \tau^+(e_{A_+})x$. In other words $p = \tau^+(e_{A_+}) \in I$ is a left unit for I. Clearly, $\|p\| \leq \|\tau^+\|\|e_{A_+}\| \leq \|\tau^+\|$. Consider maps $\rho : I \to A : x \mapsto x$ and $\tau : A \to I : x \mapsto px$. Clearly, they are morphisms of right A-modules and $\tau\rho = 1_I$. Hence, I is a $\langle$ 1-retract / C-retract $\rangle$ of A in $\mathbf{mod} - A$. □

Proposition 2.3.12. *Let I be a closed two-sided ideal of a Banach algebra A, which is faithful as a right I-module. Then*

(i) if I is a $\langle$ metrically / C-topologically $\rangle$ injective I-module, then I is a $\langle$ metrically / C-topologically $\rangle$ injective A-module;

(ii) if I is a $\langle$ metrically / C-topologically $\rangle$ injective A-module, then I is a $\langle$ metrically / C^2-topologically $\rangle$ injective I-module.

Proof. (i) The result immediately follows from paragraph (i) of proposition 2.3.3.

(ii) By proposition 2.3.11 the A-module I is $\langle$ 1-complemented / C-complemented $\rangle$ in A. By paragraph (ii) of proposition 2.3.3 the I-module I is $\langle$ metrically / C^2-topologically $\rangle$ injective. □

Chapter 3

Applications to algebras of analysis

Vaguely speaking there are three types of Banach modules depending on the type of module action: modules with pointwise multiplication, modules with composition of operators in the role of module action and modules with convolution. We shall investigate main examples of these types. Following the style of Dales and Polyakov from [42] we shall systematize all results on classical modules of analysis, but this time for metric and topological theory. We shall consider modules over operator algebras, sequence algebras, algebras of continuous functions and, finally, over classical modules of harmonic analysis.

3.1 Applications to modules over C^*-algebras

3.1.1 Spatial modules

We start from the simplest examples of modules over operator algebras — the spatial modules. By Gelfand-Naimark's theorem (see e.g. [[23], theorem 4.7.57]) for any C^*-algebra A there exists a Hilbert space H and an isometric *-homomorphism $\varrho : A \to \mathcal{B}(H)$. For Hilbert spaces that admit such homomorphism we may consider the left A-module H_ϱ with module action defined as $a \cdot x = \varrho(a)(x)$. Automatically we get the structure of right A-module on H^* which is by Riesz's theorem is isometrically isomorphic to H^{cc}. This isomorphism allows one to define a right A-module structure on H^{cc} by $\overline{x} \cdot a = \overline{\varrho(a^*)(x)}$. For a given $x_1, x_2 \in H$ we define a rank one operator $x_1 \bigcirc x_2 : H \to H : x \mapsto \langle x, x_2 \rangle x_1$.

Proposition 3.1.1. *Let A be a C^*-algebra and $\varrho : A \to \mathcal{B}(H)$ be an isometric *-homomorphism, such that its image contains a subspace of rank one operators of the form $\{x \bigcirc x_0 : x \in H\}$ for some non-zero $x_0 \in H$. Then the left A-module H_ϱ is metrically projective and flat, while the right A-module H_ϱ^{cc} is metrically injective.*

Proof. Without loss of generality we may assume that $\|x_0\| = 1$. Consider linear operators $\pi : A_+ \to H_\varrho : a \oplus_1 z \mapsto \varrho(a)(x_0) + zx_0$ and $\sigma : H_\varrho \to A_+ : x \mapsto \varrho^{-1}(x \bigcirc x_0)$. It is straightforward to check that π and σ are contractive A-morphisms such that $\pi\sigma = 1_{H_\varrho}$. Therefore, H_ϱ is a retract of A_+ in $A - \mathbf{mod}_1$. From propositions 2.1.3 and 2.1.4 it follows that H_ϱ is a metrically projective A-module. From proposition 2.1.39 it follows that H_ϱ is metrically flat too. Since $H_\varrho^{cc} \underset{\mathbf{mod}_1 - A}{\cong} H_\varrho^*$, proposition 2.1.38 gives that H_ϱ^{cc} is metrically injective. $\square$

In what follows we shall use the following simple application of the above result.

Proposition 3.1.2. *Let H be a finite dimensional Hilbert space. Then $\mathcal{N}(H)$ is topologically projective and hence flat as $\mathcal{B}(H)$-module.*

Proof. From [[23], proposition 0.3.38] we know that $\mathcal{N}(H) \underset{\mathbf{Ban}_1}{\cong} H \widehat{\otimes} H^*$. Let $\varrho = 1_{\mathcal{B}(H)}$, then we can claim a bit more: $\mathcal{N}(H) \underset{\mathcal{B}(H)-\mathbf{mod}_1}{\cong} H_\varrho \widehat{\otimes} H^*$. Since H^* is finite dimensional, then $H^* \underset{\mathbf{Ban}}{\cong} \ell_1(\mathbb{N}_n)$ for $n = \dim(H)$ and as a result $\mathcal{N}(H) \underset{\mathcal{B}(H)-\mathbf{mod}}{\cong} H_\varrho \widehat{\otimes} \ell_1(\mathbb{N}_n)$. By proposition 3.1.1 the module H_ϱ it topologically projective, so from corollary 2.1.10 we get that $\mathcal{N}(H)$ is topologically projective as $\mathcal{B}(H)$-module. The last claim of the theorem follows from proposition 2.1.39. $\square$

3.1.2 Projective ideals of C^*-algebras

We start the study of homologically trivial closed ideals of C^*-algebras from projectivity, but before stating the main result we need a preparatory lemma.

Lemma 3.1.3. *Let I be a closed left ideal of a unital C^*-algebra A. Assume $a \in I$ is a self-adjoint element and let E be the real subspace of real valued functions in $C(\operatorname{sp}_A(a))$ vanishing at zero. Then there is an isometric homomorphism $\operatorname{RCont}_a^0 : E \to I$ well-defined by $\operatorname{RCont}_a^0(f) = a$, where $f : \operatorname{sp}_A(a) \to \mathbb{C} : t \mapsto t$.*

Proof. By $\mathbb{R}_0[t]$ we denote the real linear subspace of E consisting of polynomials vanishing at zero. Since I is an ideal of A and any $p \in \mathbb{R}_0[t]$ has no constant term then $p(a) \in I$. Hence, we have a well-defined $\mathbb{R}$-linear homomorphism of algebras $\operatorname{RPol}_a^0 : \mathbb{R}_0[t] \to I : p \mapsto p(a)$. By continuous functional calculus for any polynomial

p holds $\|p(a)\| = \|p|_{\mathrm{sp}_A(a)}\|_\infty$, so $\|\operatorname{RPol}^0_a(p)\| = \|p|_{\mathrm{sp}_A(a)}\|_\infty$. Thus, RPol^0_a is isometric. As $\mathbb{R}_0[t]$ is dense in E and I is complete, then RPol^0_a has an isometric extension $\operatorname{RCont}^0_a : E \to I$ which is also an $\mathbb{R}$-linear homomorphism. $\square$

The following proof is inspired by ideas of Blecher and Kania. In [[46], lemma 2.1] they proved that any algebraically finitely generated left ideal of C^*-algebras is principal.

Theorem 3.1.4. *Let I be a closed left ideal of a C^*-algebra A. Then the following are equivalent:*

(i) $I = Ap$ for some self-adjoint idempotent $p \in I$;

(ii) I is a metrically projective A-module;

(iii) I is a topologically projective A-module.

Proof. $(i) \implies (ii)$ Since p is a self-adjoint idempotent, then $\|p\| = 1$, so by proposition 2.1.13 paragraph (i) the ideal I is metrically projective as A-module.

$(ii) \implies (iii)$ See proposition 2.1.6.

$(iii) \implies (i)$ Let $(e_\nu)_{\nu\in N}$ be a right contractive approximate identity of the ideal I [[23], theorem 4.7.79]. Since I admits a right approximate identity, then it is an essential left I-module, and a fortiori an essential A-module. By proposition 2.1.8 we have a right inverse A-morphism $\sigma : I \to A \mathbin{\widehat{\otimes}} \ell_1(B_I)$ of π_I in $A - \mathbf{mod}$. For each $d \in B_I$ consider A-morphisms $\sigma_d = p_d\sigma$, where $p_d : A \mathbin{\widehat{\otimes}} \ell_1(B_I) \to A$ is well-defined by $p_d(a \mathbin{\widehat{\otimes}} \delta_x) = \delta_x(d)a$ for all $a \in A$, $x \in B_I$. Then $\sigma(x) = \sum_{d\in B_I} \sigma_d(x) \mathbin{\widehat{\otimes}} \delta_d$ for all $x \in I$. From identification $A \mathbin{\widehat{\otimes}} \ell_1(B_I) \underset{\mathbf{Ban}_1}{\cong} \bigoplus_1 \{A : d \in B_I\}$, for all $x \in I$ we have $\|\sigma(x)\| = \sum_{d\in B_I} \|\sigma_d(x)\|$. Since σ is a right inverse morphism of π_I we have $x = \pi_I(\sigma(x)) = \sum_{d\in B_I} \sigma_d(x)d$ for all $x \in I$.

For all $x \in I$ we have $\|\sigma_d(x)\| = \|\sigma_d(\lim_\nu xe_\nu)\| = \lim_\nu \|x\sigma_d(e_\nu)\| \leq \|x\| \liminf_\nu \|\sigma_d(e_\nu)\|$, so $\|\sigma_d\| \leq \liminf_\nu \|\sigma_d(e_\nu)\|$. Then for all $S \in \mathcal{P}_f(B_I)$ holds

$$\sum_{d\in S} \|\sigma_d\| \leq \sum_{d\in S} \liminf_\nu \|\sigma_d(e_\nu)\| \leq \liminf_\nu \sum_{d\in S} \|\sigma_d(e_\nu)\| \leq \liminf_\nu \sum_{d\in B_I} \|\sigma_d(e_\nu)\|$$

$$= \liminf_\nu \|\sigma(e_\nu)\| \leq \|\sigma\| \liminf_\nu \|e_\nu\| \leq \|\sigma\|$$

Since $S \in \mathcal{P}_f(B_I)$ is arbitrary, then the sum $\sum_{d\in B_I} \|\sigma_d\|$ is finite. As a consequence, the sum $\sum_{d\in B_I} \|\sigma_d\|^2$ is finite too.

Now we regard A as an ideal in its unitization $A_\#$, then I is an ideal of $A_\#$ too. Fix a natural number $m \in \mathbb{N}$ and a real number $\epsilon > 0$. Then there exists a set $S \in \mathcal{P}_f(B_I)$

such that $\sum_{d\in B_I\setminus S}\|\sigma_d\|<\epsilon$. Denote its cardinality by N. Consider a positive element $b=\sum_{d\in B_I}\|\sigma_d\|^2d^*d\in I$. Now we perform a "power trick" by considering different powers $b^{1/m}$ of the positive element b, where $m\in\mathbb{N}$. By lemma 3.1.3 we have that $b^{1/m}\in I$, so $b^{1/m}=\sum_{d\in B_I}\sigma_d(b^{1/m})d$. By continuous functional calculus we have $\|b^{1/m}\|=\sup_{t\in \mathrm{sp}_{A_\#}(b)}t^{1/m}\le\|b\|^{1/m}$, then $\limsup_{m\to\infty}\|b^{1/m}\|\le 1$. Therefore, $\|b^{1/m}\|\le 2$ for sufficiently big m. Denote $\varsigma_d=\sigma_d(b^{1/m})$, $u=\sum_{d\in S}\varsigma_d d$ and $v=\sum_{d\in B_I\setminus S}\varsigma_d d$, so

$$b^{2/m}=\left(b^{1/m}\right)^*b^{1/m}=u^*u+u^*v+v^*u+v^*v$$

Clearly, $\varsigma_d^*\varsigma_d\le\|\varsigma_d\|^2e_{A_\#}\le\|\sigma_d\|^2\|b^{1/m}\|^2e_{A_\#}\le 4\|\sigma_d\|^2e_{A_\#}$. For any $x,y\in A$ we have $x^*x+y^*y-(x^*y+y^*x)=(x-y)^*(x-y)\ge 0$, therefore

$$d^*\varsigma_d^*\varsigma_c c+c^*\varsigma_c^*\varsigma_d d\le d^*\varsigma_d^*\varsigma_d d+c^*\varsigma_c^*\varsigma_c c\le 4\|\sigma_d\|^2d^*d+4\|\sigma_c\|^2c^*c$$

for all $c,d\in B_I$. We sum up these inequalities over $c\in S$ and $d\in S$, then

$$\begin{aligned}\sum_{c\in S}\sum_{d\in S}c^*\varsigma_c^*\varsigma_d d&=\frac{1}{2}\left(\sum_{c\in S}\sum_{d\in S}d^*\varsigma_d^*\varsigma_c c+\sum_{c\in S}\sum_{d\in S}c^*\varsigma_c^*\varsigma_d d\right)\\&\le\frac{1}{2}\left(4N\sum_{d\in S}\|\sigma_d\|^2d^*d+4N\sum_{c\in S}\|\sigma_c\|^2c^*c\right)\\&=4N\sum_{d\in S}\|\sigma_d\|^2d^*d.\end{aligned}$$

Therefore,

$$u^*u=\left(\sum_{c\in S}\varsigma_c c\right)^*\left(\sum_{d\in S}\varsigma_d d\right)=\sum_{c\in S}\sum_{d\in S}c^*\varsigma_c^*\varsigma_d d\le 4N\sum_{d\in S}\|\sigma_d\|^2d^*d=4Nb$$

Note that

$$\|u\|\le\sum_{d\in S}\|\varsigma_d\|\|d\|\le\sum_{d\in S}2\|\sigma_d\|\le 2\|\sigma\|,\qquad \|v\|\le\sum_{d\in B_I\setminus S}\|\varsigma_d\|\|d\|\le\sum_{d\in B_I\setminus S}2\|\sigma_d\|\le 2\epsilon,$$

so $\|u^*v+v^*u\|\le 8\|\sigma\|\epsilon$ and $\|v^*v\|\le 4\epsilon^2$. Since u^*v+v^*u and v^*v are self adjoint, then $u^*v+v^*u\le 8\|\sigma\|\epsilon e_{A_\#}$ and $v^*v\le 4\epsilon^2e_{A_\#}$ Therefore for any $\epsilon>0$ and sufficiently big m we have

$$b^{2/m}=u^*u+u^*v+v^*u+v^*v\le 4Nb+\epsilon(8\|\sigma\|+4\epsilon)e_{A_\#}.$$

In other words $g_m(b)\ge 0$ for a continuous function $g_m:\mathbb{R}_+\to\mathbb{R}:t\mapsto 4Nt+\epsilon(8\|\sigma\|+4\epsilon)-t^{2/m}$. Now choose $\epsilon>0$ such that $M:=\epsilon(8\|\sigma\|+4\epsilon)<1$. By spectral mapping theorem [[1], theorem 6.4.2] we get $g_m(\mathrm{sp}_{A_\#}(b))=\mathrm{sp}_{A_\#}(g_m(b))\subset\mathbb{R}_+$. It is routine to check that g_m has the only one extreme point $t_{0,m}=(2Nm)^{\frac{m}{2-m}}$ where the minimum of g_m is attained. Since $\lim_{m\to\infty} g_m(t_{0,m}) = M - 1 < 0$, $g_m(0) = M > 0$ and $\lim_{t\to\infty} g_m(t) = +\infty$, then for sufficiently big m the function g_m has exactly two zeros: $0 < t_{1,m} < t_{0,m}$ and $t_{2,m} > t_{0,m}$. Therefore, $g_m(t) \geq 0$ for $0 \leq t \leq t_{1,m}$ or $t \geq t_{2,m}$. Hence, for all sufficiently big m holds $\operatorname{sp}_{A_\#}(b) \subset T_{t_{1,m},t_{2,m}}$, where $T_{a,b} := \{t \in \mathbb{R} : 0 \leq t \leq a \vee t \geq b\}$. Since $\lim_{m\to\infty} t_{0,m} = 0$ then $\lim_{m\to\infty} t_{1,m} = 0$ too. Note that $g_m(1) = 4N + M - 1 > 0$, so for sufficiently big m we also have $t_{2,m} \leq 1$. Consequently, $\operatorname{sp}_{A_\#}(b) \subset T_{0,1}$.

Consider a continuous function $h : \mathbb{R}_+ \to \mathbb{R} : t \mapsto \min(1, t)$, then from lemma 3.1.3 we get an idempotent $p = h(b) = \operatorname{RCont}_b^0(h) \in I$ such that $\|p\| = \sup_{t\in \operatorname{sp}_{A_\#}(b)} |h(t)| \leq 1$. Therefore, p is a self-adjoint idempotent. Since $h(t)t = th(t) = t$ for all $t \in \operatorname{sp}_{A_\#}(b)$ we have $bp = pb = b$. The last equality implies

$$0 = (e_{A_\#} - p)b(e_{A_\#} - p) = \sum_{d\in B_I} (\|\sigma_d\| d(e_{A_\#} - p))^* \|\sigma_d\| d(e_{A_\#} - p).$$

Since the right-hand side of this equality is non-negative, then $d = dp$ for all $d \in B_I$ with $\sigma_d \neq 0$. Finally, for any $x \in I$ we obtain $xp = \sum_{d\in B_I} \sigma_d(x)dp = \sum_{d\in B_I} \sigma_d(x)d = x$, i.e. $I = Ap$, for self-adjoint idempotent $p \in I$. □

It is worth pointing out here that in relative theory there no such description for relative projectivity of closed left ideals of C^*-algebras. Though for the case of separable C^*-algebras (that is C^*-algebras that are separable as Banach spaces) all closed left ideals are relatively projective. See [[47], section 6] for a nice overview of the current state of the problem.

Corollary 3.1.5. *Let I be a closed two-sided ideal of a C^*-algebra A. Then the following are equivalent:*

(i) I is unital;

(ii) I is a metrically projective A-module;

(iii) I is a topologically projective A-module.

Proof. The ideal I has a contractive approximate identity [[23], theorem 4.7.79]. Therefore, I has a right identity iff I is unital. Now all equivalences follow from theorem 3.1.4. □

Corollary 3.1.6. *Let S be a locally compact Hausdorff space, and I be an ideal of $C_0(S)$. Then the following are equivalent:*

(i) $\operatorname{Spec}(I)$ *is compact;*

(ii) I *is a metrically projective* $C_0(S)$*-module;*

(iii) I *is a topologically projective* $C_0(S)$*-module.*

Proof. By Gelfand-Naimark's theorem $I \underset{\mathbf{Ban}_1}{\cong} C_0(\operatorname{Spec}(I))$, therefore I is semisimple. Now by Shilov's idempotent theorem I is unital iff $\operatorname{Spec}(I)$ is compact. It remains to apply corollary 3.1.5. $\square$

It is worth mentioning that the class of relatively projective closed ideals of $C_0(S)$ is much larger. In fact a closed ideal I of $C_0(S)$ is relatively projective iff $\operatorname{Spec}(I)$ is paracompact [[24], chapter IV,§§2–3].

3.1.3 Injective ideals of C^*-algebras

Now we proceed to injectivity of closed two-sided ideals of C^*-algebras. Unfortunately we don't have a complete characterization at hand, but some necessary conditions and several examples. The following proposition shows that we may restrict investigation of injective closed ideals to the case of C^*-algebras that are $\langle$ metrically / topologically $\rangle$ injective over themselves as right modules.

Proposition 3.1.7. *Let* I *be a closed two-sided ideal of a* C^**-algebra* A*, then* I *is* $\langle$ *metrically / topologically* $\rangle$ *injective as* A*-module iff* I *is* $\langle$ *metrically / topologically* $\rangle$ *injective as* I*-module.*

Proof. Note that any closed two-sided ideal I of a C^*-algebra A is again a C^*-algebra with a contractive approximate identity [[23], theorem 4.7.79]. Therefore, I is faithful as a right I-module. Now proposition 2.3.12 gives the desired equivalence. $\square$

We shall say a few words on so called AW^*-algebras, since they are key players here. In attempt to find a purely algebraic definition of W^*-algebras Kaplanski introduced this class of C^*-algebras in [48]. A C^*-algebra A is called an AW^*-algebra if for any subset $S \subset A$ the right algebraic annihilator $S^{\perp A}$ has the form pA for some self-adjoint idempotent $p \in A$. This class contains all W^*-algebras, but strictly larger. Note that for the case of commutative C^*-algebras the property of being an AW^*-algebra is equivalent to $\operatorname{Spec}(A)$ being a Stonean space [[49], theorem 1.7.1]. The main reference to AW^*-algebras and more general Baer *-rings is [49].

The following proposition is a combination of results by Hamana and Takesaki.

Proposition 3.1.8 (Hamana, Takesaki)**.** *Let A be a C^*-algebra, then it is a metrically injective right A-module iff it is a commutative AW^*-algebra, that is* $\operatorname{Spec}(A)$ *is a Stonean space.*

Proof. If A is metrically injective as A-module, then it has norm one left identity by proposition 2.3.11. But A also has a contractive approximate identity [[23], theorem 4.7.79], therefore A is unital. Now by result of Hamana [[50], proposition 2] the C^*-algebra A is a commutative AW^*-algebra. Hamana's argument was for left modules, but one can easily modify his proof to get the result for right modules.

The converse was proved by Takesaki in [[51], theorem 2]. Although only two-sided Banach modules were treated there, the reasoning is essentially the same for right modules. □

It remains to consider topological injectivity. As the following proposition shows topologically injective C^*-algebras are not so far from commutative ones. This proposition exploits the l.u.st. property. For its definition see section 2.2.2.

Proposition 3.1.9. *Let A be a C^*-algebra which is topologically injective as an A-module. Then A has the l.u.st. property and as a consequence it cannot contain $\mathcal{B}(\ell_2(\mathbb{N}_n))$ as *-subalgebra for arbitrarily big $n \in \mathbb{N}$.*

Proof. By Gelfand-Naimark's theorem [[23], theorem 4.7.57] there exists a Hilbert space H and an isometric *-homomorphism $\varrho : A \to \mathcal{B}(H)$. Denote $\Lambda := B_{H_\varrho^{cc}}$. It is easy to check that

$$\rho : A \to \bigoplus\nolimits_\infty \{H_\varrho^{cc} : \overline{x} \in \Lambda\},\, a \mapsto \bigoplus\nolimits_\infty \{\overline{x} \cdot a : \overline{x} \in \Lambda\}$$

is an isometric A-morphism of right A-modules. Since A is a topologically injective A-module, then ρ has a left inverse A-morphism τ. Therefore, A is complemented in $E := \bigoplus_\infty \{H_\varrho^{cc} : \overline{x} \in \Lambda\}$ via projection $\rho\tau$. Note that H_ϱ^{cc} is a Banach lattice as any Hilbert space, then so is E. As any Banach lattice E has the l.u.st. property [[20], theorem 17.1], then so does A, because the l.u.st. property is inherited by complemented subspaces.

Assume A contains $\mathcal{B}(\ell_2(\mathbb{N}_n))$ as *-subalgebra for arbitrarily big $n \in \mathbb{N}$. In fact this copy of $\mathcal{B}(\ell_2(\mathbb{N}_n))$ is 1-complemented in A [[52], lemma 2.1]. Therefore, we have an inequality for local unconditional constants $\kappa_u(\mathcal{B}(\ell_2(\mathbb{N}_n))) \leq \kappa_u(A)$. By theorem 5.1 in [53] we know that $\lim_n \kappa_u(\mathcal{B}(\ell_2(\mathbb{N}_n))) = +\infty$, so $\kappa_u(A) = +\infty$. This contradicts the l.u.st. property of A. Hence, A cannot contain $\mathcal{B}(\ell_2(\mathbb{N}_n))$ as *-subalgebra for arbitrarily big $n \in \mathbb{N}$. □

As the proposition 3.1.9 shows C^*-algebras that are topologically injective over themselves cannot contain $\mathcal{B}(\ell_2(\mathbb{N}_n))$ as a *-subalgebra for arbitrarily big $n \in \mathbb{N}$. Such C^*-algebras are called subhomogeneous, and in fact [[27], proposition IV.1.4.3] they can be treated as closed *-subalgebras of $M_n(C(K))$ for some compact Hausdorff space K and some natural number n. For more on subhomogeneous C^*-algebras see [[27], section IV.1.4].

We shall give two important examples of non-commutative C^*-algebras that are topologically injective over themselves.

Proposition 3.1.10. *Let H be a finite dimensional Hilbert space. Then $\mathcal{B}(H)$ is topologically injective as $\mathcal{B}(H)$-module.*

Proof. Note that $\mathcal{B}(H) \underset{\mathbf{mod}_1-\mathcal{B}(H)}{\cong} \mathcal{N}(H)^*$, and the result immediately follows from propositions 3.1.2 and 2.1.38. □

Proposition 3.1.11. *Let K be a Stonean space and $n \in \mathbb{N}$, then $M_n(C(K))$ is topologically injective $M_n(C(K))$-module.*

Proof. For a fixed $s \in K$ by $\mathbb{C}_s$ we denote the right $C(K)$-module $\mathbb{C}$ with module action defined by $z \cdot a = a(s)z$ for all $a \in C(K)$ and $z \in \mathbb{C}$. By $M_n(\mathbb{C}_s)$ we denote the right Banach $M_n(C(K))$-module $M_n(\mathbb{C})$ with module action defined by $(x \cdot a)_{i,j} = \sum_{k=1}^{n} x_{i,k} a_{k,j}(s)$ for $a \in M_n(C(K))$ and $x \in M_n(\mathbb{C}_s)$. The C^*-algebra $M_n(C(K))$ is nuclear [[54], corollary 2.4.4], then by [[55], theorem 3.1] this algebra is relatively amenable and even 1-relatively amenable [[56], example 2]. Since $M_n(\mathbb{C}_s)$ is finite dimensional, it is an $\mathscr{L}^g_{1,C}$-space for some constant $C \geq 1$ independent of s. Thus, by proposition 2.3.8 the $M_n(C(K))$-module $M_n(\mathbb{C}_s)^*$ is C-topologically flat. Since the latter module is essential, by proposition 2.1.32 the right $M_n(C(K))$-module $M_n(\mathbb{C}_s)^{**}$ is C-topologically injective. Note that $M_n(\mathbb{C}_s)^{**}$ is isometrically isomorphic to $M_n(\mathbb{C}_s)$ as a right $M_n(C(K))$-module, so from proposition 2.1.26 we get that $\bigoplus_\infty \{M_n(\mathbb{C}_s) : s \in K\}$ is topologically injective as $M_n(C(K))$-module.

Note that by proposition 3.1.8 the $C(K)$-module $C(K)$ is metrically injective, therefore an isometric $C(K)$-morphism $\widetilde{\rho} : C(K) \to \bigoplus_\infty \{\mathbb{C}_s : s \in K\}$, $x \mapsto \bigoplus_\infty \{x(s) : s \in K\}$ admits a left inverse contractive $C(K)$-morphism $\widetilde{\tau} : \bigoplus_\infty \{\mathbb{C}_s : s \in K\} \to C(K)$. It is routine to check now that linear operators

$$\rho : M_n(C(K)) \to \bigoplus\nolimits_\infty \{M_n(\mathbb{C}_s) : s \in K\},\ x \mapsto \bigoplus\nolimits_\infty \{(x_{i,j}(s))_{i,j \in \mathbb{N}_n} : s \in K\}$$

$$\tau : \bigoplus\nolimits_\infty \{M_n(\mathbb{C}_s) : s \in K\} \to M_n(C(K)) : y \mapsto \left(\widetilde{\tau}\left(\bigoplus\nolimits_\infty \{y_{s,i,j} : s \in K\}\right)\right)_{i,j \in \mathbb{N}_n}$$

are bounded $M_n(C(K))$-morphisms such that $\tau\rho = 1_{M_n(C(K))}$. That is $M_n(C(K))$ is a retract of topologically injective $M_n(C(K))$-module $\bigoplus_\infty\{M_n(\mathbb{C}_s) : s \in K\}$ in $\mathbf{mod}_1 - M_n(C(K))$. Finally, from proposition 2.1.21 we conclude that $M_n(C(K))$ is topologically injective $M_n(C(K))$-module. $\square$

Theorem 3.1.12. *Let A be a C^*-algebra. Then the following are equivalent:*

(i) A is an AW^*-algebra which is topologically injective as A-module;

(ii) $A = \bigoplus_\infty\{M_{n_\lambda}(C(K_\lambda)) : \lambda \in \Lambda\}$ for some finite families of natural numbers $(n_\lambda)_{\lambda\in\Lambda}$ and Stonean spaces $(K_\lambda)_{\lambda\in\Lambda}$.

Proof. $(i) \implies (ii)$ From proposition 6.6 in [57] we know that an AW^*-algebra is either isomorphic as C^*-algebra to $\bigoplus_\infty\{M_{n_\lambda}(C(K_\lambda)) : \lambda \in \Lambda\}$ for some finite families of natural numbers $(n_\lambda)_{\lambda\in\Lambda}$ and Stonean spaces $(K_\lambda)_{\lambda\in\Lambda}$ or contains a *-subalgebra $\bigoplus_\infty\{\mathcal{B}(\ell_2(\mathbb{N}_n)) : n \in \mathbb{N}\}$. The second option is canceled out by proposition 3.1.9.

$(ii) \implies (i)$ For each $\lambda \in \Lambda$ the algebra $M_{n_\lambda}(C(K_\lambda))$ is unital because K_λ is compact. Therefore, $M_{n_\lambda}(C(K_\lambda))$ is faithful as $M_{n_\lambda}(C(K_\lambda))$-module. It is also topologically injective as $M_{n_\lambda}(C(K_\lambda))$-module by proposition 3.1.11. Now the topological injectivity of A-module A follows from paragraph (ii) of proposition 2.3.5 with $p = \infty$ and $X_\lambda = A_\lambda = M_{n_\lambda}(C(K_\lambda))$ for all $\lambda \in \Lambda$.

For all $\lambda \in \Lambda$ the algebra $C(K_\lambda)$ is an AW^*-algebra, because K_λ is a Stonean space [[49], theorem 1.7.1]. Therefore, $M_{n_\lambda}(C(K_\lambda))$ is an AW^*-algebra too [[49], corollary 9.62.1]. Finally, A is an AW^*-algebra as $\bigoplus_\infty$-sum of such algebras [[49], proposition 1.10.1]. $\square$

It is desirable to prove that any C^*-algebra which is topologically injective over itself is an AW^*-algebra, but it seems to be a challenge even in the commutative case.

3.1.4 Flat ideals of C^*-algebras

By considering flatness we finalize this lengthy investigations of closed ideals of C^*-algebras.

Proposition 3.1.13. *Let I be a closed left ideal of a C^*-algebra A. Then I is metrically and topologically flat as A-module.*

Proof. From [[23], proposition 4.7.78] it follows that I has a right contractive identity. It remains to apply proposition 2.1.42. $\square$

Proposition 3.1.14. *Let A be a C^*-algebra, then A is an $\langle$ L_1-space / $\mathscr{L}_1^g$-space $\rangle$ iff $\langle$ $\dim(A) \leq 1$ / A is finite dimensional $\rangle$.*

Proof. Assume A is an $\mathscr{L}_1^g$-space, then A^{**} is complemented in some L_1-space [[12], corollary 23.2.1(2)]. Since A isometrically embeds in its second dual via ι_A we may regard A as closed subspace of some L_1-space. The latter space is weakly sequentially complete [[14], corollary III.C.14]. The property of being weakly sequentially complete is preserved by closed subspaces, therefore A is weakly sequentially complete too. By proposition 2 in [58] every weakly sequentially complete C^*-algebra is finite dimensional, hence A is finite dimensional. Conversely, if A is finite dimensional it is an $\mathscr{L}_1^g$-space as any finite dimensional Banach space.

Assume A is an L_1-space and, a fortiori, an $\mathscr{L}_1^g$-space. As was noted above A is a finite dimensional, so $A \underset{\mathbf{Ban}_1}{\cong} \ell_1(\mathbb{N}_n)$ for $n = \dim(A)$. On the other hand, A is a finite dimensional C^*-algebra, so it is isometrically isomorphic to $\bigoplus_\infty \{\mathcal{B}(\ell_2(\mathbb{N}_{n_k})) : k \in \mathbb{N}_m\}$ for some natural numbers $(n_k)_{k\in\mathbb{N}_m}$ [[59], theorem III.1.1]. Assume $\dim(A) > 1$, then A contains an isometric copy of $\ell_\infty(\mathbb{N}_2)$. Therefore, we have an isometric embedding of $\ell_\infty(\mathbb{N}_2)$ into $\ell_1(\mathbb{N}_n)$. This is impossible by theorem 1 from [60]. Therefore, $\dim(A) \leq 1$. $\square$

Proposition 3.1.15. *Let I be a proper closed two-sided ideal of a C^*-algebra A. Then the following are equivalent:*

(i) A is a $\langle$ metrically / topologically $\rangle$ flat I-module;

(ii) $\langle$ $\dim(A) = 1$, $I = \{0\}$ / A/I is finite dimensional $\rangle$.

Proof. We may regard I as a closed ideal of the unitazation $A_\#$ of A. Since I is a two-sided ideal, then it has a contractive approximate identity $(e_\nu)_{\nu\in N}$ such that $0 \leq e_\nu \leq e_{A_\#}$ [[23], proposition 4.7.79]. As a corollary $\sup_{\nu\in N} \|e_{A_\#} - e_\nu\| \leq 1$. Since I has an approximate identity we also have $A_{ess} := \mathrm{cl}_A(IA) = I$. Since I is a two-sided ideal then A/I is a C^*-algebra [[23], theorem 4.7.81].

Assume, A is a metrically flat I-module. Since $\sup_{\nu\in N} \|e_{A_\#} - e_\nu\| \leq 1$, then paragraph (ii) of proposition 2.3.6 tells us that $(A/A_{ess})^* = (A/I)^*$ is a retract of A^* in $\mathbf{mod}_1 - I$. Now from propositions 2.1.32 and 2.1.21 it follows that A/I is metrically flat I-module. Since this is an annihilator module, then from proposition 2.2.6 it follows that $I = \{0\}$ and A/I is an L_1-space. Now from proposition 3.1.14 we get that $\dim(A/I) \leq 1$. Since A contains a proper ideal $I = \{0\}$, then $\dim(A) = 1$. Conversely, if $I = \{0\}$ and $\dim(A) = 1$, then we have an annihilator I-module A which is isometrically isomorphic to $\ell_1(\mathbb{N}_1)$. By proposition 2.2.6 it is metrically flat.

By proposition 2.3.7 the I-module A is topologically flat iff $A_{ess} = I$ and $A/A_{ess} = A/I$ are topologically flat I-modules. Note that by proposition 3.1.13 the ideal I is a topologically flat I-module. From proposition 2.2.6 we get that the annihilator I-module A/I is topologically flat iff it is an $\mathscr{L}_1^g$-space. The latter is equivalent to A/I being finite dimensional, see proposition 3.1.14. □

3.1.5 $\mathcal{K}(H)$- and $\mathcal{B}(H)$-modules

In this section we apply general results on closed ideals obtained above to classical modules over C^*-algebras. For a given Hilbert space H we consider $\mathcal{B}(H)$, $\mathcal{K}(H)$ and $\mathcal{N}(H)$ as left and right Banach modules over $\mathcal{B}(H)$ and $\mathcal{K}(H)$. For all these modules the module action is just the composition of operators. The Schatten-von Neumann isomorphisms $\mathcal{N}(H) \underset{\mathbf{Ban}_1}{\cong} \mathcal{K}(H)^*$, $\mathcal{B}(H) \underset{\mathbf{Ban}_1}{\cong} \mathcal{N}(H)^*$ (see [[61], theorems II.1.6, II.1.8]) will be of use here. In fact, these identifications are isomorphisms of left and right $\mathcal{B}(H)$-modules and a fortiori of $\mathcal{K}(H)$-modules.

Proposition 3.1.16. *Let H be a Hilbert space. Then*

(i) $\mathcal{B}(H)$ *is metrically and topologically projective and flat as* $\mathcal{B}(H)$*-module;*

(ii) $\mathcal{B}(H)$ *is metrically or topologically projective or flat as* $\mathcal{K}(H)$*-module iff* H *is finite dimensional;*

(iii) $\mathcal{B}(H)$ *is topologically injective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff* H *is finite dimensional;*

(iv) $\mathcal{B}(H)$ *is metrically injective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff* $\dim(H) \leq 1$.

Proof. (i) Since $\mathcal{B}(H)$ is a unital algebra it is metrically and topologically projective as $\mathcal{B}(H)$-module by proposition 2.1.3. Both results regarding flatness follow from proposition 2.1.39.

(ii) For infinite dimensional H the Banach space $\mathcal{B}(H)/\mathcal{K}(H)$ is of infinite dimension, so by proposition 3.1.15 the module $\mathcal{B}(H)$ neither topologically nor metrically flat as $\mathcal{K}(H)$-module. Both claims regarding projectivity follow from proposition 2.1.39. If H is finite dimensional, then $\mathcal{K}(H) = \mathcal{B}(H)$, so the result follows from paragraph (i).

(iii) If H is infinite dimensional, then $\mathcal{B}(H)$ contains $\mathcal{B}(\ell_2(\mathbb{N}_n))$ as *-subalgebra for all $n \in \mathbb{N}$. By proposition 3.1.9 we get that $\mathcal{B}(H)$ is not topologically injective as $\mathcal{B}(H)$-module. The rest follows from paragraph (i) of proposition 2.3.3. If H is finite dimensional, then $\mathcal{K}(H) = \mathcal{B}(H)$, so the result follows from proposition 3.1.10.

(iv) If $\dim(H) > 1$, then the C^*-algebra $\mathcal{B}(H)$ is not commutative. By proposition 3.1.8 we get that it is not metrically injective as $\mathcal{B}(H)$-module. Now from paragraph (i) of 2.3.3 we get that $\mathcal{B}(H)$ is not metrically injective as $\mathcal{K}(H)$-module. If $\dim(H) \leq 1$ both claims obviously follow from 3.1.8. $\square$

Proposition 3.1.17. *Let H be a Hilbert space. Then*

(i) $\mathcal{K}(H)$ *is metrically and topologically flat as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module;*

(ii) $\mathcal{K}(H)$ *is metrically or topologically projective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff H is finite dimensional;*

(iii) $\mathcal{K}(H)$ *is topologically injective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff H is finite dimensional;*

(iv) $\mathcal{K}(H)$ *is metrically injective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff* $\dim(H) \leq 1$.

Proof. Let A be either $\mathcal{B}(H)$ or $\mathcal{K}(H)$. Note that $\mathcal{K}(H)$ is a two-sided ideal of A.

(i) Recall that $\mathcal{K}(H)$ has a contractive approximate identity consisting of orthogonal projections onto all finite-dimensional subspaces of H. Since $\mathcal{K}(H)$ is a two-sided ideal of A, then the result follows from proposition 3.1.13.

(ii), (iii), (iv) If H is infinite dimensional, then $\mathcal{K}(H)$ is not unital as a Banach algebra. From corollary 3.1.5 and proposition 2.3.11 the A-module $\mathcal{K}(H)$ is neither metrically nor topologically projective or injective. If H is finite dimensional, then $\mathcal{K}(H) = \mathcal{B}(H)$, so both results follow from paragraphs (i), (iii) and (iv) of proposition 3.1.16. $\square$

Proposition 3.1.18. *Let H be a Hilbert space. Then*

(i) $\mathcal{N}(H)$ *is metrically and topologically injective as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module;*

(ii) $\mathcal{N}(H)$ *is topologically projective or flat as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff H is finite dimensional;*

(iii) $\mathcal{N}(H)$ *is metrically projective or flat as* $\mathcal{B}(H)$*- or* $\mathcal{K}(H)$*-module iff* $\dim(H) \leq 1$.

Proof. Let A be either $\mathcal{B}(H)$ or $\mathcal{K}(H)$.

(i) Note that $\mathcal{N}(H) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{K}(H)^*$, so the result follows from proposition 2.1.32 and paragraph (i) of proposition 3.1.17.

(ii) Assume H is infinite dimensional. Note that $\mathcal{B}(H) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{N}(H)^*$, so from proposition 2.1.38 and paragraph (iii) of proposition 3.1.16 we get that $\mathcal{N}(H)$ is not topologically projective as A-module. Both results regarding flatness follow from proposition 2.1.39. If H is finite dimensional we use proposition 3.1.2.

(iii) Assume $\dim(H) > 1$, then by paragraph (iv) of proposition 3.1.16 the A-module $\mathcal{B}(H)$ is not metrically injective. Since $\mathcal{B}(H) \underset{\mathbf{mod}_1-A}{\cong} \mathcal{N}(H)^*$, then from proposition 2.1.32 we get that $\mathcal{N}(H)$ is not metrically flat as A-module. By proposition 2.1.39, it is not metrically projective as A-module. If $\dim(H) \leq 1$, then $\mathcal{N}(H) = \mathcal{K}(H) = \mathcal{B}(H)$, so both results follow from paragraph (i) of proposition 3.1.16. $\square$

Proposition 3.1.19. *Let H be a Hilbert space. Then*

(i) as $\mathcal{K}(H)$-modules $\mathcal{N}(H)$ is relatively projective injective and flat, $\mathcal{K}(H)$ is relatively projective and flat, but relatively injective only for finite dimensional H, $\mathcal{B}(H)$ is relatively injective and flat, but relatively projective only for finite dimensional H;

(ii) as $\mathcal{B}(H)$-modules $\mathcal{N}(H)$ is relatively projective injective and flat, $\mathcal{K}(H)$ is relatively projective and flat, $\mathcal{B}(H)$ is relatively projective, injective and flat.

Proof. (i) Note that H is relatively projective as $\mathcal{K}(H)$-module [[23], theorem 7.1.27], so from proposition 7.1.13 in [23] we get that $\mathcal{N}(H) \underset{\mathcal{K}(H)-\mathbf{mod}_1}{\cong} H \widehat{\otimes} H^*$ is also relatively projective as $\mathcal{K}(H)$-module. By theorem IV.2.16 in [24] the $\mathcal{K}(H)$-module $\mathcal{K}(H)$ is relatively projective. A fortiori $\mathcal{N}(H)$ and $\mathcal{K}(H)$ are relatively flat $\mathcal{K}(H)$-modules [[23], proposition 7.1.40], so $\mathcal{N}(H) \underset{\mathbf{mod}_1-\mathcal{K}(H)}{\cong} \mathcal{K}(H)^*$ and $\mathcal{B}(H) \underset{\mathbf{mod}_1-\mathcal{K}(H)}{\cong} \mathcal{N}(H)^*$ are relatively injective $\mathcal{K}(H)$-modules. From [[33], proposition 2.2.8 (i)] we know that a Banach algebra relatively injective over itself as a right module, necessarily has a left identity. Therefore, $\mathcal{K}(H)$ is not relatively injective $\mathcal{K}(H)$-module for infinite dimensional H. If H is finite dimensional, then $\mathcal{K}(H)$-module $\mathcal{K}(H)$ is relatively injective because $\mathcal{K}(H) = \mathcal{B}(H)$ and $\mathcal{B}(H)$ is relatively injective $\mathcal{K}(H)$-module as was shown above. By corollary 5.5.64 from [25] the algebra $\mathcal{K}(H)$ is relatively amenable, so all its left modules are relatively flat [[23], theorem 7.1.60]. In particular $\mathcal{B}(H)$ is relatively flat $\mathcal{K}(H)$-module. From [[24], exercise V.2.20] we know that $\mathcal{B}(H)$ is not relatively projective as $\mathcal{K}(H)$-module when H is infinite dimensional. If H is finite dimensional then $\mathcal{B}(H)$ is relatively projective $\mathcal{K}(H)$-module because $\mathcal{B}(H) = \mathcal{K}(H)$ and $\mathcal{K}(H)$ is relatively projective $\mathcal{K}(H)$-module as was shown above.

(ii) From proposition 3.1.16 paragraph (i) and proposition 2.1.6 it follows that $\mathcal{B}(H)$ is relatively projective $\mathcal{B}(H)$-module. From [[33], propositions 2.3.3, 2.3.4] we know that $\langle$ an essential relatively projective / a faithful relatively injective $\rangle$ module over an ideal of a Banach algebra is $\langle$ relatively projective / relative injective $\rangle$ over an algebra itself. Since $\mathcal{K}(H)$ and $\mathcal{N}(H)$ are essential and faithful $\mathcal{K}(H)$-modules, then from results of previous paragraph $\mathcal{N}(H)$ is relatively projective and injective, while $\mathcal{K}(H)$ is relatively projective as $\mathcal{B}(H)$-modules. Now, by [[23], proposition 7.1.40] all aforementioned modules are relatively flat $\mathcal{B}(H)$-modules. In particular $\mathcal{B}(H) \underset{\mathbf{mod}_1-\mathcal{B}(H)}{\cong} \mathcal{N}(H)^*$ is relatively injective $\mathcal{B}(H)$-module. $\square$

Results of this section are summarized in the following three tables. Each cell contains a condition under which the respective module has the respective property and propositions where this result is proved. We use "?" symbol to indicate open problems. These tables confirm that the property of being homologically trivial in metric and topological theory is too restrictive. It is easier to mention cases where metric and topological properties coincide with relative ones: flatness of $\mathcal{K}(H)$ as $\mathcal{B}(H)$- or $\mathcal{K}(H)$-module, injectivity of $\mathcal{N}(H)$ as $\mathcal{B}(H)$- or $\mathcal{K}(H)$-module, projectivity and flatness of $\mathcal{B}(H)$-module $\mathcal{B}(H)$. In the remaining cases H needs to be at least finite dimensional in order for these properties be equivalent in metric, topological and relative theory.

Homologically trivial $\mathcal{K}(H)$- and $\mathcal{B}(H)$-modules in metric theory

	$\mathcal{K}(H)$-modules			$\mathcal{B}(H)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\mathcal{N}(H)$	$\dim(H) \leq 1$ 3.1.18	H is any 3.1.18	$\dim(H) \leq 1$ 3.1.18	$\dim(H) \leq 1$ 3.1.18	H is any 3.1.18	$\dim(H) \leq 1$ 3.1.18
$\mathcal{B}(H)$	$\dim(H) < \aleph_0$ 3.1.16	$\dim(H) \leq 1$ 3.1.16	$\dim(H) < \aleph_0$ 3.1.16	H is any 3.1.16	$\dim(H) \leq 1$ 3.1.16	H is any 3.1.16
$\mathcal{K}(H)$	$\dim(H) < \aleph_0$ 3.1.17	$\dim(H) \leq 1$ 3.1.17	H is any 3.1.17	$\dim(H) < \aleph_0$ 3.1.17	$\dim(H) \leq 1$ 3.1.17	H is any 3.1.17

Homologically trivial $\mathcal{K}(H)$- and $\mathcal{B}(H)$-modules in topological theory

	$\mathcal{K}(H)$-modules			$\mathcal{B}(H)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\mathcal{N}(H)$	$\dim(H) < \aleph_0$ 3.1.18	H is any 3.1.18	$\dim(H) < \aleph_0$ 3.1.18	$\dim(H) < \aleph_0$ 3.1.18	H is any 3.1.18	$\dim(H) < \aleph_0$ 3.1.18
$\mathcal{B}(H)$	$\dim(H) < \aleph_0$ 3.1.16	$\dim(H) < \aleph_0$ 3.1.16	$\dim(H) < \aleph_0$ 3.1.16	H is any 3.1.16	$\dim(H) < \aleph_0$ 3.1.16	H is any 3.1.16
$\mathcal{K}(H)$	$\dim(H) < \aleph_0$ 3.1.17	$\dim(H) < \aleph_0$ 3.1.17	H is any 3.1.17	$\dim(H) < \aleph_0$ 3.1.17	$\dim(H) < \aleph_0$ 3.1.17	H is any 3.1.17

Homologically trivial $\mathcal{K}(H)$- and $\mathcal{B}(H)$-modules in relative theory

	$\mathcal{K}(H)$-modules			$\mathcal{B}(H)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\mathcal{N}(H)$	H is any 3.1.19, (i)	H is any 3.1.19, (i)	H is any 3.1.19, (i)	H is any 3.1.19, (ii)	H is any 3.1.19, (ii)	H is any 3.1.19, (ii)
$\mathcal{B}(H)$	$\dim(H) < \aleph_0$ 3.1.19, (i)	H is any 3.1.19, (i)	H is any 3.1.19, (i)	H is any 3.1.19, (ii)	H is any 3.1.19, (ii)	H is any 3.1.19, (ii)
$\mathcal{K}(H)$	H is any 3.1.19, (i)	$\dim(H) < \aleph_0$ 3.1.19, (i)	H is any 3.1.19, (i)	H is any 3.1.19, (ii)	?	H is any 3.1.19, (ii)

3.1.6 $c_0(\Lambda)$- and $\ell_\infty(\Lambda)$-modules

We continue our study of modules over C^*-algebras and move to commutative examples. For a given index set Λ we consider spaces $c_0(\Lambda)$ and $\ell_p(\Lambda)$ for $1 \leq p \leq +\infty$ as left and right modules over algebras $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$. For all these modules the module action is just the pointwise multiplication. It is well known that $c_0(\Lambda)^* \underset{\mathbf{Ban}_1}{\cong} \ell_1(\Lambda)$ and $\ell_p(\Lambda)^* \underset{\mathbf{Ban}_1}{\cong} \ell_{p^*}(\Lambda)$ for $1 \leq p < +\infty$. In fact these isomorphisms are isomorphisms of $\ell_\infty(\Lambda)$- and $c_0(\Lambda)$-modules.

For a given $\lambda \in \Lambda$ we define $\mathbb{C}_\lambda$ as a left or right $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module $\mathbb{C}$ with module action defined by

$$a \cdot_\lambda z = a(\lambda)z, \qquad z \cdot_\lambda a = a(\lambda)z$$

for $a \in \ell_\infty(\Lambda)$ or $a \in c_0(\Lambda)$ and $z \in \mathbb{C}_s$.

Proposition 3.1.20. *Let Λ be a set and $\lambda \in \Lambda$. Then $\mathbb{C}_\lambda$ is a metrically and topologically projective, injective and flat $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module.*

Proof. Let A be either $\ell_\infty(\Lambda)$ or $c_0(\Lambda)$. One can easily check that the maps $\pi : A_+ \to \mathbb{C}_\lambda, a \oplus_1 z \mapsto a(\lambda) + z$ and $\sigma : \mathbb{C}_\lambda \to A_+, z \mapsto z\delta_\lambda \oplus_1 0$ are contractive A-morphisms of left A-modules. Since $\pi\sigma = 1_{\mathbb{C}_\lambda}$, then $\mathbb{C}_\lambda$ is a retract of A_+ in $A - \mathbf{mod}_1$. From proposition 2.1.3 and 2.1.4 it follows that $\mathbb{C}_\lambda$ is a metrically and topologically projective left A-module and a fortiori metrically and topologically flat by proposition 2.1.39. By proposition 2.1.38 we have that $\mathbb{C}_\lambda^*$ is metrically and topologically injective as A-module. Now metric and topological injectivity of $\mathbb{C}_\lambda$ follow from isomorphism $\mathbb{C}_\lambda \underset{\mathbf{mod}_1 - A}{\cong} \mathbb{C}_\lambda^*$. □

Proposition 3.1.21. *Let Λ be a set. Then*

(i) $\ell_\infty(\Lambda)$ is metrically and topologically projective and flat as $\ell_\infty(\Lambda)$-module;

(ii) $\ell_\infty(\Lambda)$ is metrically or topologically projective or flat as $c_0(\Lambda)$-module iff Λ is finite;

(iii) $\ell_\infty(\Lambda)$ is metrically and topologically injective as $\ell_\infty(\Lambda)$- and $c_0(\Lambda)$-module.

Proof. (i) Since $\ell_\infty(\Lambda)$ is a unital algebra, then it is metrically and topologically projective as $\ell_\infty(\Lambda)$-module by proposition 2.1.3. Results on flatness follow from proposition 2.1.39.

(ii) For infinite Λ the Banach space $\ell_\infty(\Lambda)/c_0(\Lambda)$ is of infinite dimension, so by proposition 3.1.15 the module $\ell_\infty(\Lambda)$ is neither topologically nor metrically flat as $c_0(\Lambda)$-module. Both claims regarding projectivity follow from proposition 2.1.39. If Λ is finite, then $c_0(\Lambda) = \ell_\infty(\Lambda)$, so the result follows from paragraph (i).

(iii) Let A be either $\ell_\infty(\Lambda)$ or $c_0(\Lambda)$. Note that $\ell_\infty(\Lambda) \underset{A-\mathbf{mod}_1}{\cong} \bigoplus_\infty\{\mathbb{C}_\lambda : \lambda \in \Lambda\}$, then from propositions 3.1.20 and 2.1.26 it follows that $\ell_\infty(\Lambda)$ is metrically injective as A-module. Topological injectivity follows from proposition 2.1.23. □

Proposition 3.1.22. *Let Λ be a set. Then*

(i) $c_0(\Lambda)$ is metrically and topologically flat as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module;

(ii) $c_0(\Lambda)$ is metrically or topologically projective as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module iff Λ is finite;

(iii) $c_0(\Lambda)$ is metrically or topologically injective as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module iff Λ is finite.

Proof. Let A be either $\ell_\infty(\Lambda)$ or $c_0(\Lambda)$. Note that $c_0(\Lambda)$ is a two-sided ideal of A.

(i) Recall that $c_0(\Lambda)$ has a contractive approximate identity of the form $\left(\sum_{\lambda\in S}\delta_\lambda\right)_{S\in\mathcal{P}_f(\Lambda)}$. Since $c_0(\Lambda)$ is a two-sided ideal of A, then the result follows from proposition 3.1.13.

(ii), (iii) If Λ is infinite, then $c_0(\Lambda)$ is not unital as a Banach algebra. From corollary 3.1.5 and proposition 2.3.11 the A-module $c_0(\Lambda)$ is neither metrically nor topologically projective or injective. If Λ is finite, then $c_0(\Lambda) = \ell_\infty(\Lambda)$, so both results follow from paragraphs (i) and (iii) of proposition 3.1.21. □

Proposition 3.1.23. *Let Λ be a set. Then*

(i) $\ell_1(\Lambda)$ is metrically and topologically injective as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module;

(ii) $\ell_1(\Lambda)$ is metrically and topologically projective and flat as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module;

Proof. Let A be either $\ell_\infty(\Lambda)$ or $c_0(\Lambda)$.

(i) Note that $\ell_1(\Lambda) \underset{\mathbf{mod}_1-A}{\cong} c_0(\Lambda)^*$, so the result follows from proposition 2.1.32 and paragraph (i) of proposition 3.1.22.

(ii) Note that $\ell_1(\Lambda) \underset{A-\mathbf{mod}_1}{\cong} \bigoplus_1\{\mathbb{C}_\lambda : \lambda \in \Lambda\}$, then from propositions 3.1.20 and 2.1.9 it follows that $\ell_1(\Lambda)$ is metrically projective as A-module. Topological projectivity follows from proposition 2.1.6. Metric and topological flatness follow from proposition 2.1.39. □

Proposition 3.1.24. *Let Λ be a set and $1 < p < +\infty$. Then $\ell_p(\Lambda)$ is metrically or topologically projective, injective or flat as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module iff Λ is finite.*

(i) $\ell_p(\Lambda)$ *is topologically projective, injective or flat as* $\ell_\infty(\Lambda)$*- or* $c_0(\Lambda)$*-module iff* Λ *is finite;*

(ii) if $\ell_p(\Lambda)$ *is metrically projective, injective or flat as* $\ell_\infty(\Lambda)$*- or* $c_0(\Lambda)$*-module then* Λ *is finite;*

Proof. (i), (ii) Let A be either $\ell_\infty(\Lambda)$ or $c_0(\Lambda)$, then A is an $\mathscr{L}_\infty^g$-space. Since $\ell_p(\Lambda)$ is reflexive for $1 < p < +\infty$, then from corollary 2.2.15 it follows that $\ell_p(\Lambda)$ is necessarily finite dimensional if it is metrically or topologically projective injective or flat. This is equivalent to Λ being finite. If Λ is finite then $\ell_p(\Lambda) \underset{A-\mathbf{mod}}{\cong} \ell_1(\Lambda)$ and $\ell_p(\Lambda) \underset{\mathbf{mod}-A}{\cong} \ell_1(\Lambda)$, so topological projectivity injectivity and flatness follow from proposition 3.1.23. □

Proposition 3.1.25. *Let* Λ *be a set. Then*

(i) as $c_0(\Lambda)$*-modules* $\ell_p(\Lambda)$ *for* $1 \leq p < +\infty$ *and* $\mathbb{C}_\lambda$ *for* $\lambda \in \Lambda$ *are relatively projective, injective and flat,* $c_0(\Lambda)$ *is relatively projective and flat, but relatively injective only for finite* Λ*,* $\ell_\infty(\Lambda)$ *is relatively injective and flat, but relatively projective only for finite* Λ*;*

(ii) as $\ell_\infty(\Lambda)$*-modules* $\ell_p(\Lambda)$ *for* $1 \leq p \leq +\infty$ *and* $\mathbb{C}_\lambda$ *for* $\lambda \in \Lambda$ *are relatively projective, injective and flat,* $c_0(\Lambda)$ *is relatively projective, injective and flat.*

Proof. (i) The algebra $c_0(\Lambda)$ is relatively biprojective [[24], theorem IV.5.26] and admits a contractive approximate identity, so by [[23], theorem 7.1.60] all essential $c_0(\Lambda)$-modules are projective. Thus, $c_0(\Lambda)$ and $\ell_p(\Lambda)$ for $1 \leq p < +\infty$ are relatively projective $c_0(\Lambda)$-modules. A fortiori they are relatively flat as $c_0(\Lambda)$-modules [[23], proposition 7.1.40]. By the same proposition $\ell_1(\Lambda) \underset{\mathbf{mod}_1-c_0(\Lambda)}{\cong} c_0(\Lambda)^*$ and $\ell_{p^*}(\Lambda) \underset{\mathbf{mod}_1-c_0(\Lambda)}{\cong} \ell_p(\Lambda)^*$ for $1 \leq p < +\infty$ are relatively injective $c_0(\Lambda)$-modules. From [[33], proposition 2.2.8 (i)] we know that a Banach algebra which is relatively injective over itself as a right module, necessarily has a left identity. Therefore, $c_0(\Lambda)$ is not relatively injective $c_0(\Lambda)$-module for infinite Λ. If Λ is finite, then $c_0(\Lambda)$-module $c_0(\Lambda)$ is relatively injective because $c_0(\Lambda) = \ell_\infty(\Lambda)$ and $\ell_\infty(\Lambda)$ is relatively injective $c_0(\Lambda)$-module as was shown above. From [[24], corollary V.2.16(II)] we know that $\ell_\infty(\Lambda)$ is not relatively projective as $c_0(\Lambda)$-module provided Λ is infinite. If Λ is finite then $\ell_\infty(\Lambda)$ is relatively projective $c_0(\Lambda)$-module because $\ell_\infty(\Lambda) = c_0(\Lambda)$ and $c_0(\Lambda)$ is relatively projective $c_0(\Lambda)$-module as was shown above. Propositions 3.1.20, 2.1.6, 2.1.23 and 2.1.34 give the result for modules $\mathbb{C}_\lambda$, where $\lambda \in \Lambda$.

(ii) From proposition 3.1.21 paragraph (i) and proposition 2.1.6 it follows that $\ell_\infty(\Lambda)$ is a relatively projective $\ell_\infty(\Lambda)$-module. In [[62], theorem 4.4] it was shown that $\ell_\infty(\Lambda)$-module $c_0(\Lambda)$ is relatively injective. From [[33], propositions 2.3.3, 2.3.4] we know that

$\langle$ an essential relatively projective / a faithful relatively injective $\rangle$ module over an ideal of a Banach algebra is $\langle$ relatively projective / relatively injective $\rangle$ over the algebra itself. For each $1 \leq p < +\infty$ the $c_0(\Lambda)$-module $\ell_p(\Lambda)$ is essential and faithful, hence by results of the previous paragraph $\ell_p(\Lambda)$ is relatively projective and injective as $\ell_\infty(\Lambda)$-module. Since $c_0(\Lambda)$-module $c_0(\Lambda)$ is essential and relatively projective, then analogously the $\ell_\infty(\Lambda)$-module $c_0(\Lambda)$ is relatively projective. Therefore, all these $\ell_\infty(\Lambda)$-modules are relatively flat [[23], proposition 7.1.40]. As a consequence $\ell_\infty(\Lambda) \underset{\mathbf{mod}_1-\ell_\infty(\Lambda)}{\cong} \ell_1(\Lambda)^*$ is relatively injective $\ell_\infty(\Lambda)$-module. Propositions 3.1.20, 2.1.6, 2.1.23 and 2.1.34 give the result for modules $\mathbb{C}_\lambda$, where $\lambda \in \Lambda$. $\square$

Results of this section are summarized in the following three tables. Each cell contains a condition under which the respective module has the respective property and propositions where this result is proved. For the case of $\ell_\infty(\Lambda)$- and $c_0(\Lambda)$-modules $\ell_p(\Lambda)$ for $1 < p < +\infty$ we don't have a criterion of homological triviality in metric theory, just a necessary condition. We indicate this fact via symbol *. From these tables one can easily see that for modules over commutative C^*-algebras, there is much more in common between relative, metric and topological theory. For example $\ell_1(\Lambda)$ is projective injective and flat as $\ell_\infty(\Lambda)$- or $c_0(\Lambda)$-module in all three theories.

Homologically trivial $c_0(\Lambda)$- and $\ell_\infty(\Lambda)$-modules in metric theory

	$c_0(\Lambda)$-modules			$\ell_\infty(\Lambda)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\ell_1(\Lambda)$	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23
$\ell_p(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24*
$\ell_\infty(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.21	Λ is any 3.1.21	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.21	Λ is any 3.1.21	Λ is any 3.1.21	Λ is any 3.1.21
$c_0(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	Λ is any 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	Λ is any 3.1.22
$\mathbb{C}_\lambda$	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20

Homologically trivial $c_0(\Lambda)$- and $\ell_\infty(\Lambda)$-modules in topological theory

	$c_0(\Lambda)$-modules			$\ell_\infty(\Lambda)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\ell_1(\Lambda)$	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23	Λ is any 3.1.23
$\ell_p(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.24
$\ell_\infty(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.21	Λ is any 3.1.21	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.21	Λ is any 3.1.21	Λ is any 3.1.21	Λ is any 3.1.21
$c_0(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	Λ is any 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.22	Λ is any 3.1.22
$\mathbb{C}_\lambda$	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20	λ is any 3.1.20

Homologically trivial $c_0(\Lambda)$- and $\ell_\infty(\Lambda)$-modules in relative theory

	$c_0(\Lambda)$-modules			$\ell_\infty(\Lambda)$-modules		

	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$\ell_1(\Lambda)$	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (ii)	Λ is any 3.1.25, (ii)	Λ is any 3.1.21, (ii)
$\ell_p(\Lambda)$	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (ii)	Λ is any 3.1.25, (ii)	Λ is any 3.1.21, (ii)
$\ell_\infty(\Lambda)$	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (ii)	Λ is any 3.1.25, (ii)	Λ is any 3.1.21, (ii)
$c_0(\Lambda)$	Λ is any 3.1.25, (i)	$\mathrm{Card}(\Lambda) < \aleph_0$ 3.1.25, (i)	Λ is any 3.1.25, (i)	Λ is any 3.1.25, (ii)	Λ is any 3.1.25, (ii)	Λ is any 3.1.21, (ii)
$\mathbb{C}_\lambda$	λ is any 3.1.25, (i)	λ is any 3.1.25, (i)	λ is any 3.1.25, (i)	λ is any 3.1.25, (ii)	λ is any 3.1.25, (ii)	λ is any 3.1.21, (ii)

3.1.7 $C_0(S)$-modules

This section is devoted to the study of homological triviality of classical modules over the algebra $C_0(S)$, where S is a locally compact Hausdorff space. By classic we mean modules $C_0(S)$, $M(S)$ and $L_p(S,\mu)$ for positive measures $\mu \in M(S)$. The pointwise multiplication plays the role of module action for these modules.

We shall give short preliminaries on these modules. Recall that $C_0(S)^* \underset{\mathbf{Ban}_1}{\cong} M(S)$ and $L_p(S,\mu)^* \underset{\mathbf{Ban}_1}{\cong} L_{p^*}(S,\mu)$ for $1 \leq p < +\infty$. In fact these identifications are isomorphisms of left and right $C_0(S)$-modules. For a given positive measure $\mu \in M(S)$ by $M_s(S,\mu)$ we shall denote the closed $C_0(S)$-submodule of $M(S)$ consisting of measures strictly singular with respect to μ. Then the well known Lebesgue decomposition theorem can be stated as $M(S) \underset{\mathbf{Ban}_1}{\cong} L_1(S,\mu) \bigoplus_1 M_s(S,\mu)$. Even more, this identification is an isomorphism of left and right $C_0(S)$-modules.

We obliged to emphasize here that we consider only finite Borel regular positive measures. This shall simplify many considerations. For example, any atom of a regular measure on a locally compact Hausdorff spaces is a point [[63], chapter 5, §5, exercise 7]. Since we consider only finite measures, one cannot say that this section simply generalizes results of the previous one. Strictly speaking these sections are different, though their methods have much in common.

For a fixed point $s \in S$ by $\mathbb{C}_s$ we denote the left or right Banach $C_0(S)$-module $\mathbb{C}$ with module action defined by

$$a \cdot_s z = a(s)z, \qquad z \cdot_s a = a(s)z$$

Proposition 3.1.26. *Let S be a locally compact Hausdorff space and let $s \in S$. Then*

(i) $\mathbb{C}_s$ is a metrically, topologically or relatively projective $C_0(S)$-module iff s is an isolated point of S;

(ii) $\mathbb{C}_s$ *is a metrically, topologically and relatively flat* $C_0(S)$*-module.*

(iii) $\mathbb{C}_s$ *is a metrically, topologically and relatively injective* $C_0(S)$*-module;*

Proof. (i) If $\mathbb{C}_s$ is metrically or topologically or relatively projective, then by proposition 2.1.6 it is at least relatively projective. Now from [[23], proposition 7.1.31] we know that the latter forces s to be an isolated point of S. Conversely, assume that s is an isolated point of S. One can easily check that the maps $\pi : C_0(S)_+ \to \mathbb{C}_s$, $a \oplus_1 z \mapsto a(s) + z$ and $\sigma : \mathbb{C}_s \to C_0(S)_+$, $z \mapsto z\delta_s \oplus_1 0$ are contractive $C_0(S)$-morphisms. Since $\pi\sigma = 1_{\mathbb{C}_s}$, then $\mathbb{C}_s$ is a retract of $C_0(S)_+$ in $C_0(S) - \mathbf{mod}_1$. From propositions 2.1.4 and 2.1.3 it follows that $\mathbb{C}_s$ is a metrically and topologically projective left $C_0(S)$-module. From 2.1.6 we conclude that $\mathbb{C}_s$ is also relatively projective $C_0(S)$-module.

(ii) By [[23], theorem 7.1.87] the algebra $C_0(S)$ is relatively amenable. Since this algebra is a C^*-algebra it is 1-relatively amenable [[56], example 3]. Clearly, $\mathbb{C}_s$ is a 1-dimensional L_1-space and an essential $C_0(S)$-module. Therefore, by proposition 2.3.8 this module is metrically flat. Now the result follows from proposition 2.1.34.

(iii) From paragraph (ii) and proposition 2.1.32 it follows that the $C_0(S)$-module $\mathbb{C}_s^*$ is metrically injective. By proposition 2.1.34 it is topologically and relatively injective too. It remains to note that $\mathbb{C}_s \underset{\mathbf{mod}_1 - A}{\cong} \mathbb{C}_s^*$. □

Proposition 3.1.27. *Let* S *be a locally compact Hausdorff space and let* $s \in S$*. Then*

(i) $C_0(S)$ *is a* $\langle$ *metrically / topologically / relatively* $\rangle$ *projective* $C_0(S)$*-module iff* S *is* $\langle$ *compact / compact / paracompact* $\rangle$*;*

(ii) $C_0(S)$ *is a metrically injective as* $C_0(S)$*-module iff* S *is a Stonean space; if* $C_0(S)$ *is topologically or relatively injective then* S *is compact and* $S = \beta(S \setminus \{s\})$ *for any limit point* $s \in S$*;*

(iii) $C_0(S)$ *is a metrically, topologically and relatively flat* $C_0(S)$*-module.*

Proof. We regard $C_0(S)$ as a two-sided ideal of $C_0(S)$. Recall that $\operatorname{Spec}(C_0(S))$ is homeomorphic to S [[24], corollary 3.1.6].

(i) It is enough to note that by $\langle$ proposition 3.1.6 / proposition 3.1.6 / [[24], chapter IV,§§2–3] $\rangle$ the spectrum of $C_0(S)$ is $\langle$ compact / compact / paracompact $\rangle$.

(ii) The result on metric injectivity is a weakened version of proposition 3.1.8. The result on relatively injectivity follows from [[62], theorem 4.4]. It remains to recall that by proposition 2.1.23 any metrically injective module is relatively injective.

(iii) From proposition 3.1.13 it immediately follows that $C_0(S)$-module $C_0(S)$ is metrically and topologically flat. By proposition 2.1.34 it is also relatively flat. □

Proposition 3.1.28. *Let S be a locally compact Hausdorff space, μ be a finite positive Borel regular measure on S. Assume that $1 \leq p \leq +\infty$ and $C_0(S)$-module $L_p(S,\mu)$ is relatively projective. Then any atom of μ is an isolated point in S.*

Proof. Assume μ has at least one atom, otherwise there is nothing to prove. From [[63], chapter 5, §5, exercise 7] we know that any atom of μ is a point. Call it $s \in S$. Consider well-defined linear maps $\pi : L_p(\Omega,\mu) \to \mathbb{C}_s : f \mapsto f(s)$ and $\sigma : \mathbb{C}_s \to L_p(\Omega,\mu) : z \mapsto z\delta_s$. One can easily check that these maps are $C_0(S)$-morphisms and $\pi\sigma = 1_{\mathbb{C}_s}$. Therefore, $\mathbb{C}_s$ is a retract of $L_p(S,\mu)$ in $C_0(S) - \mathbf{mod}$. By assumption, the latter module is relatively projective, so by [[23], proposition 7.1.6] the $C_0(S)$-module $\mathbb{C}_s$ is relatively projective. By paragraph (i) of proposition 3.1.27 we see that s is an isolated point of S. □

Proposition 3.1.29. *Let S be a locally compact Hausdorff space and μ be a finite Borel regular positive measure on S. Then*

(i) $L_1(S,\mu)$ is a metrically or topologically or relatively projective $C_0(S)$-module iff μ is purely atomic and all its atoms are isolated points in S;

(ii) $L_1(S,\mu)$ is a metrically, topologically and relatively injective $C_0(S)$-module;

(iii) $L_1(S,\mu)$ is a metrically, topologically and relatively flat $C_0(S)$-module.

Proof. (i) If $L_1(S,\mu)$ is metrically or topologically or relatively projective, then by proposition 2.1.6 it is at least relatively projective. Now from [[64], theorem 2] the measure μ is purely atomic, and all atoms are isolated points. Conversely, assume that μ is purely atomic, and all atoms are isolated points. By S_a^μ we denote the set of these atoms. Now one can easily show that the linear map $i : L_1(S,\mu) \to \bigoplus_1 \{\mathbb{C}_s : s \in S_a^\mu\} : f \mapsto \bigoplus_1 \{\mu(\{s\})f(s) : s \in S_a^\mu\}$ is an isometric isomorphism of $C_0(S)$-modules. By paragraphs (i) of propositions 2.1.9 and 3.1.26 the $C_0(S)$-module $\bigoplus_1 \{\mathbb{C}_s : s \in S_a^\mu\}$ is metrically projective. Therefore, so is $L_1(S,\mu)$. By proposition 2.1.6 it is also topologically and relatively projective.

(ii) By paragraph (iii) of proposition 3.1.27 the $C_0(S)$-module $C_0(S)$ is metrically flat. From proposition 2.1.38 we get that $M(S) \underset{\mathbf{mod}_1 - C_0(S)}{\cong} C_0(S)^*$ is metrically injective. Since $M(S) \underset{\mathbf{mod}_1 - C_0(S)}{\cong} L_1(S,\mu) \bigoplus_1 M_s(S,\mu)$, then $L_1(S,\mu)$ is a retract of $M(S)$ in $\mathbf{mod}_1 - C_0(S)$. So by proposition 2.1.21 the $C_0(S)$-module $L_1(S,\mu)$ is metrically injective. Relative and topological injectivity of $L_1(S,\mu)$ follows from proposition 2.1.23.

(iii) By [[23], theorem 7.1.87] the algebra $C_0(S)$ is relatively amenable. Since this algebra is a C^*-algebra it is 1-relatively amenable [[56], example 3]. Since $L_1(S,\mu)$ is an essential $C_0(S)$-module which tautologically an L_1-space, then by proposition 2.3.8 this module is metrically flat. From proposition 2.1.34 the $C_0(S)$-module $L_1(S,\mu)$ is also topologically and relatively flat. □

Proposition 3.1.30. *Let S be a locally compact Hausdorff space and μ be a finite Borel regular positive measure on S. Assume $1 < p < +\infty$, then*

(i) $L_p(S,\mu)$ is a relatively injective and flat, but relatively projective $C_0(S)$-module iff μ is purely atomic, and all atoms are isolated points;

(ii) $L_p(S,\mu)$ is a topologically projective or injective or flat $C_0(S)$-module iff μ is purely atomic with finitely many atoms, and all atoms are isolated points;

(iii) if $L_p(S,\mu)$ is a metrically projective or injective or flat $C_0(S)$-module, then μ is purely atomic with finitely many atoms, and all atoms are isolated points.

Proof. (i) By [[23], theorem 7.1.87] the algebra $C_0(S)$ is relatively amenable. Now from [[23], theorem 7.1.60] it follows that $L_p(S,\mu)$ is relatively flat for all $1 < p < +\infty$. Note that $L_p(S,\mu) \underset{\mathbf{mod}_1 - C_0(S)}{\cong} L_a p^*(S,\mu)^*$. Then from [[23], proposition 7.1.42] we get that $L_p(S,\mu)$ is relatively injective for any $1 < p < +\infty$. Now assume that $L_p(S,\mu)$ is relatively projective, then by [[64], theorem 2] the measure μ is purely atomic and all its atoms are isolated points. Conversely, let μ be purely atomic with all atoms isolated. Denote by S_a^μ the set of these atoms. Since S_a^μ is discrete, then $C_0(S_a^\mu)$ is relatively biprojective [[24], theorem 4.5.26]. Then $L_p(S,\mu)$ is a relatively projective $C_0(S_a^\mu)$-module because it is an essential module over a relatively biprojective algebra with a two-sided bounded approximate identity. Clearly $C_0(S_a^\mu)$ is a closed two-sided ideal of $C_0(S)$, so from [[33], proposition 2.3.2(i)] we get that $L_p(S,\mu)$ is relatively projective as $C_0(S)$-module.

$(ii), (iii)$ Assume that $L_p(S,\mu)$ is a metrically or topologically projective or injective or flat $C_0(S)$-module. Since $L_p(S,\mu)$ is reflexive and $C_0(S)$ is an $\mathscr{L}_\infty^g$-space, then $L_p(S,\mu)$ is finite dimensional by corollary 2.2.15. The latter is equivalent to measure μ being purely atomic with finitely many atoms, and all atoms are isolated points. On the other hand, if μ is purely atomic with finitely many atoms, then $L_p(S,\mu)$ is topologically isomorphic to $L_1(S,\mu)$ as a left or right $C_0(S)$-module. The latter module is topologically projective, injective and flat by proposition 3.1.29. Hence, so is $L_p(S,\mu)$. □

Proposition 3.1.31. *Let S be a locally compact Hausdorff space and μ be a finite positive Borel regular measure on S. Then* *(i) if $L_\infty(S,\mu)$ is a metrically, topologically or relatively projective $C_0(S)$-module, then μ is a normal measure with pseudocompact support;*

(ii) $L_\infty(S,\mu)$ is a metrically, topologically and relatively injective $C_0(S)$-module;

(iii) $L_\infty(S,\mu)$ is a relatively flat $C_0(S)$-module.

Proof. (i) From [[64], theorem 3] it follows that $\operatorname{supp}(\mu)$ is pseudocompact and μ is inner open regular. Since μ is regular and finite it is normal [[64], proposition 9].

(ii) Since $L_\infty(S,\mu) \underset{\mathbf{mod}_1-C_0(S)}{\cong} L_1(S,\mu)^*$, then the result immediately follows from proposition 2.1.38 and paragraph (iii) of proposition 3.1.29.

(iii) By [[23], theorem 7.1.87] the algebra $C_0(S)$ is relatively amenable. Any left Banach module over a relatively amenable Banach algebra is relatively flat [[23], theorem 7.1.60]. In particular $L_\infty(S,\mu)$ is a relatively flat $C_0(S)$-module. □

Proposition 3.1.32. *Let S be a locally compact Hausdorff space and μ be a finite positive Borel regular measure on S. Then*

(i) $M(S)$ is a metrically or topologically or relatively projective $C_0(S)$-module iff S is discrete;

(ii) $M(S)$ is a metrically, topologically and relatively injective $C_0(S)$-module;

(iii) $M(S)$ is a metrically, topologically and relatively flat $C_0(S)$-module.

Proof. (i) If the $C_0(S)$-module $M(S)$ is metrically or topologically or relatively projective, then by proposition 2.1.6 it is at least relatively projective. For arbitrary $s \in S$ consider measure $\mu = \delta_s$ and recall the decomposition $M(S) \underset{C_0(S)-\mathbf{mod}_1}{\cong} L_1(S,\mu) \bigoplus_1 M_s(S,\mu)$. Then $L_1(S,\mu)$ is a retract of $M(S)$ in $C_0(S) - \mathbf{mod}_1$. So from [[23], proposition 7.1.6] we get that $L_1(S,\mu)$ is relatively projective $C_0(S)$-module. Since s is the only atom of μ, then from proposition 3.1.29 it follows that s is an isolated point in S. Since $s \in S$ is arbitrary, then S is discrete. Conversely, assume S is discrete. Then $C_0(S) = c_0(S)$, and $M(S) \underset{C_0(S)-\mathbf{mod}_1}{\cong} C_0(S)^* \underset{C_0(S)-\mathbf{mod}_1}{\cong} \ell_1(S) \underset{C_0(S)-\mathbf{mod}_1}{\cong} \bigoplus_1 \{\mathbb{C}_s : s \in S\}$. The latter $C_0(S)$-module is metrically projective by paragraphs (i) of propositions 2.1.9 and 3.1.26. Therefore, $M(S)$ is a metrically projective $C_0(S)$-module too. By proposition 2.1.6 it is also topologically and relatively projective.

(ii) Since $M(S) \underset{\mathbf{mod}_1-C_0(S)}{\cong} C_0(S)^*$, then the result immediately follows from proposition 2.1.38 and paragraph (iii) of proposition 3.1.27.

(iii) By [[23], theorem 7.1.87] the algebra $C_0(S)$ is relatively amenable. Since this algebra is a C^*-algebra it is 1-relatively amenable [[56], example 2]. Note that $M(S)$ is an essential $C_0(S)$-module which as Banach space is an L_1-space [[11], discussion after proposition 2.14]. Then by proposition 2.3.8 this module is metrically flat. From proposition 2.1.34 it is also topologically and relatively flat. □

Results of this section are summarized in the following three tables. Each cell contains a condition under which the respective module has the respective property and propositions where this result is proved. We use "?" symbol to indicate open problems, and "*" symbol for partial results.

Below is a short overview of unsolved problems of this section. We don't have a complete description of relatively and topologically injective $C_0(S)$-modules $C_0(S)$. It seems quite a challenge for one simple reason — to this day there is no standard category of functional analysis where topologically injective objects were fully understood. The question of relative projectivity of $C_0(S)$-module $L_\infty(S, \mu)$ is rather old. It seems that even relative projectivity of $L_\infty(S, \mu)$ is a rare property. Our conjecture that μ must be normal and S must be hyper-Stonean. For metric and topological flatness of $C_0(S)$-module $L_\infty(S, \mu)$ we conjecture that μ must have a compact support. Neither do we have a criterion of homological triviality of $C_0(S)$-modules $L_p(S, \mu)$ in metric theory for $1 < p < +\infty$. Using advanced Banach geometric techniques on factorization constants through finite dimensional Hilbert spaces one may show that atoms count must not exceed some universal constant. It seems that $L_p(S, \mu)$ is homologically trivial $C_0(S)$-module in metric theory only for purely atomic measures with a unique atom.

Homologically trivial $C_0(S)$-modules in metric theory

	Projectivity	Injectivity	Flatness
$L_1(S, \mu)$	μ is purely atomic, all atoms are isolated points 3.1.29 (i)	μ is any 3.1.29 (ii)	μ is any 3.1.29 (iii)
$L_p(S, \mu)$	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (iii)*	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (iii)*	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (iii)*
$L_\infty(S, \mu)$	μ is normal, with pseudocompact support 3.1.31 (i)*	μ is any 3.1.31 (ii)	?
$M(S)$	S is discrete 3.1.32	S is any 3.1.32	S is any 3.1.32
$C_0(S)$	S is compact 3.1.27 (i)	S is Stonean 3.1.27 (ii)	S is any 3.1.27 (iii)
$\mathbb{C}_s$	s is an isolated point 3.1.26	s is any 3.1.26	s is any 3.1.26

Homologically trivial $C_0(S)$-modules in topological theory

	Projectivity	Injectivity	Flatness

$L_1(S,\mu)$	μ is purely atomic, all atoms are isolated points 3.1.29	μ is any 3.1.29	μ is any 3.1.29
$L_p(S,\mu)$	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (ii)	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (ii)	μ is purely atomic with finitely many atoms, all atoms are isolated points 3.1.30 (ii)
$L_\infty(S,\mu)$	μ is normal, with pseudocompact support 3.1.31 (i)*	μ is any 3.1.31 (ii)	?
$M(S)$	S is discrete 3.1.32 (i)	S is any 3.1.32 (ii)	S is any 3.1.32 (iii)
$C_0(S)$	S is compact 3.1.27 (i)	$S = \beta(S \setminus \{s\})$ for any limit point s 3.1.27 (ii)*	S is any 3.1.27 (iii)
$\mathbb{C}_s$	s is an isolated point 3.1.26	s is any 3.1.26	s is any 3.1.26

Homologically trivial $C_0(S)$-modules in relative theory

	Projectivity	Injectivity	Flatness
$L_1(S,\mu)$	μ is purely atomic, all atoms are isolated points 3.1.29	μ is any 3.1.29	μ is any 3.1.29
$L_p(S,\mu)$	μ is purely atomic, all atoms are isolated points 3.1.30 (i)	μ is any 3.1.30 (i)	μ is any 3.1.30 (i)
$L_\infty(S,\mu)$	μ is normal, with pseudocompact support 3.1.31 (i)*	μ is any 3.1.31 (ii)	μ is any 3.1.31 (iii)
$M(S)$	S is discrete 3.1.32 (i)	S is any 3.1.32 (ii)	S is any 3.1.32 (iii)
$C_0(S)$	S is paracompact 3.1.27 (i)	$S = \beta(S \setminus \{s\})$ for any limit point s 3.1.27 (ii)*	S is any 3.1.27 (iii)
$\mathbb{C}_s$	s is an isolated point 3.1.26	s is any 3.1.26	s is any 3.1.26

3.2 Applications to modules of harmonic analysis

3.2.1 Preliminaries on harmonic analysis

Let G be a locally compact group. Its identity we shall denote by e_G. By well known Haar's theorem [[65],section 15.8] there exists a unique up to a positive constant Borel regular measure m_G which is finite on all compact sets, positive on all open sets and left translation invariant, that is $m_G(sE) = m_G(E)$ for all $s \in G$ and $E \in Bor(G)$. It is called the left Haar measure of group G. If G is compact we assume that $m_G(G) = 1$. If G is infinite and discrete we choose m_G as the counting measure. For each $s \in G$ the map $m : Bor(G) \to [0, +\infty] : E \mapsto m_G(Es)$ is also a left Haar measure, so from uniqueness we infer that $m(E) = \Delta_G(s)m_G(E)$ for some $\Delta_G(s) > 0$. The function $\Delta_G : G \to (0, +\infty)$ is called the modular function of the group G. It is clear that $\Delta_G(st) = \Delta_G(s)\Delta_G(t)$ for all $s, t \in G$. Groups with modular function equal to one are called unimodular. Examples of unimodular groups include compact groups, commutative groups and discrete groups. In what follows we use the notation $L_p(G)$ instead of $L_p(G, m_G)$ for $1 \leq p \leq +\infty$. For a fixed $s \in G$ we define the left shift operator $L_s : L_1(G) \to L_1(G)$, $f \mapsto (t \mapsto f(s^{-1}t))$ and the right shift operator $R_s : L_1(G) \to L_1(G)$, $f \mapsto (t \mapsto f(ts))$.

Group structure of G allows us to introduce the Banach algebra structure on $L_1(G)$. For a given $f, g \in L_1(G)$ we define their convolution as

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)dm_G(t) = \int_G f(st)g(t^{-1})dm_G(t)$$

$$= \int_G f(st^{-1})g(t)\Delta_G(t^{-1})dm_G(t)$$

for almost all $s \in G$. In this case the Banach space $L_1(G)$ endowed with convolution product becomes a Banach algebra. The Banach algebra $L_1(G)$ has a contractive two-sided approximate identity consisting of positive compactly supported continuous functions. The algebra $L_1(G)$ is unital iff G is discrete, and in this case δ_{e_G} is the identity of $L_1(G)$. Group structure of G allows us to turn the Banach space of complex finite Borel regular measures $M(G)$ into a Banach algebra too. We define the convolution of two measures $\mu, \nu \in M(G)$ as

$$(\mu * \nu)(E) = \int_G \nu(s^{-1}E)d\mu(s) = \int_G \mu(Es^{-1})d\nu(s)$$

for all $E \in Bor(G)$. The Banach space $M(G)$ along with this convolution is a unital Banach algebra. The role of identity is played by Dirac's delta measure δ_{e_G} supported on e_G. In fact $M(G)$ is a coproduct in $L_1(G) - \mathbf{mod}_1$ (but not in $M(G) - \mathbf{mod}_1$) of the closed two-sided ideal $M_a(G)$ of measures absolutely continuous with respect to m_G and subalgebra $M_s(G)$ of measures singular with respect to m_G. Note that $M_a(G) \underset{M(G)-\mathbf{mod}_1}{\cong} L_1(G)$ and $M_s(G)$ is an annihilator $L_1(G)$-module. Finally, $M(G) = M_a(G)$ iff G is discrete.

Now we proceed to the discussion of standard left and right modules over $L_1(G)$ and $M(G)$. Since the Banach algebra $L_1(G)$ can be regarded as a closed two-sided ideal of $M(G)$ because of isometric left and right $M(G)$-morphism $i : L_1(G) \to M(G) : f \mapsto fm_G$. Therefore, it is enough to define all module structures over $M(G)$. For $1 \leq p < +\infty$ and any $f \in L_p(G)$, $\mu \in M(G)$ we define

$$(\mu *_p f)(s) = \int_G f(t^{-1}s)d\mu(t), \qquad (f *_p \mu)(s) = \int_G f(st^{-1})\Delta_G(t^{-1})^{1/p}d\mu(t)$$

These module actions turn any Banach space $L_p(G)$ for $1 \leq p < +\infty$ into left and right $M(G)$-module. Note that for $p = 1$ and $\mu \in M_a(G)$ we get the usual definition of convolution. For $1 < p \leq +\infty$ and any $f \in L_p(G)$, $\mu \in M(G)$ we define module actions

$$(\mu \cdot_p f)(s) = \int_G \Delta_G(t)^{1/p} f(st) d\mu(t), \qquad (f \cdot_p \mu)(s) = \int_G f(ts) d\mu(t)$$

These module actions turn any Banach space $L_p(G)$ for $1 < p \leq +\infty$ into left and right $M(G)$-module too. This special choice of module structure nicely interacts with duality. Indeed, we have and $(L_p(G), *_p)^* \underset{\mathbf{mod}_1 - M(G)}{\cong} (L_{p^*}(G), \cdot_{p^*})$ for all $1 \leq p < +\infty$. Finally, the Banach space $C_0(G)$ also becomes left and right $M(G)$-module when endowed with $\cdot_\infty$ in the role of module action. Even more, $C_0(G)$ is a closed left and right $M(G)$-submodule of $L_\infty(G)$ and $(C_0(G), \cdot_\infty)^* \underset{M(G) - \mathbf{mod}_1}{\cong} (M(G), *)$.

A character on a locally compact group G is by definition a continuous homomorphism from G to $\mathbb{T}$. The set of all characters on G forms a group denoted by $\widehat{G}$. It becomes a locally compact group when considered with compact open topology. Any character $\gamma \in \widehat{G}$ gives rise to a continuous character $\varkappa_\gamma^L : L_1(G) \to \mathbb{C}$, $f \mapsto \int_G f(s)\overline{\gamma(s)} dm_G(s)$ on $L_1(G)$. In fact all characters of $L_1(G)$ arise this way. This result is due to Gelfand [[26], theorems 2.7.2, 2.7.5]. Similarly, for each $\gamma \in \widehat{G}$ we have a character on $M(G)$ defined by $\varkappa_\gamma^M : M(G) \to \mathbb{C}$, $\mu \mapsto \int_G \overline{\gamma(s)} d\mu(s)$. By $\mathbb{C}_\gamma$ we denote the respective augmentation left and right $L_1(G)$- or $M(G)$-module. Their module actions are defined by

$$f \cdot_\gamma z = z \cdot_\gamma f = \varkappa_\gamma^L(f) z \qquad \mu \cdot_\gamma z = z \cdot_\gamma \mu = \varkappa_\gamma^M(\mu) z$$

for all $f \in L_1(G)$, $\mu \in M(G)$ and $z \in \mathbb{C}$.

One of the numerous definitions of amenable group says, that a locally compact group G is amenable if there exists an $L_1(G)$-morphism of right modules $M : L_\infty(G) \to \mathbb{C}_{e_{\widehat{G}}}$ such that $M(\chi_G) = 1$ [[23], section VII.2.5]. We can even assume that M is contractive [[23], remark 7.1.54].

Most results of this section that are not supported with references are presented in a full detail in [[25], section 3.3].

3.2.2 $L_1(G)$-modules

Metric homological properties of most of the standard $L_1(G)$-modules of harmonic analysis are studied in [40]. We borrow these ideas to unify approaches to metric and topological homological properties of modules over group algebras.

Proposition 3.2.1. *Let G be a locally compact group. Then $L_1(G)$ is metrically and topologically flat $L_1(G)$-module, i.e. the $L_1(G)$-module $L_\infty(G)$ is metrically and topologically injective.*

Proof. Since $L_1(G)$ has a contractive approximate identity, then $L_1(G)$ is metrically and topologically flat $L_1(G)$-module by proposition 2.1.42. Since $L_\infty(G) \underset{\mathbf{mod}_1 - L_1(G)}{\cong} L_1(G)^*$, then by proposition 2.1.32 this module is metrically and topologically injective. □

Proposition 3.2.2. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent:*

(i) G is compact;

(ii) $\mathbb{C}_\gamma$ is a metrically projective $L_1(G)$-module;

(iii) $\mathbb{C}_\gamma$ is a topologically projective $L_1(G)$-module.

Proof. $(i) \implies (ii)$ Consider $L_1(G)$-morphisms $\sigma^+ : \mathbb{C}_\gamma \to L_1(G)_+$, $z \mapsto z\gamma \oplus_1 0$ and $\pi^+ : L_1(G)_+ \to \mathbb{C}_\gamma$, $f \oplus_1 w \to f \cdot_\gamma 1 + w$. One can easily check that $\|\pi^+\| = \|\sigma^+\| = 1$ and $\pi^+\sigma^+ = 1_{\mathbb{C}_\gamma}$. Therefore, $\mathbb{C}_\gamma$ is a retract of $L_1(G)_+$ in $L_1(G) - \mathbf{mod}_1$. From propositions 2.1.3 and 2.1.4 it follows that $\mathbb{C}_\gamma$ is metrically projective.

$(ii) \implies (iii)$ See proposition 2.1.6.

$(iii) \implies (i)$ Consider $L_1(G)$-morphism $\pi : L_1(G) \to \mathbb{C}_\gamma$, $f \mapsto f \cdot_\gamma 1$. It is easy to see that π is strictly coisometric. Since $\mathbb{C}_\gamma$ is topologically projective, then there exists an $L_1(G)$-morphism $\sigma : \mathbb{C}_\gamma \to L_1(G)$ such that $\pi\sigma = 1_{\mathbb{C}_\gamma}$. Let $f = \sigma(1) \in L_1(G)$ and $(e_\nu)_{\nu \in N}$ be a standard approximate identity of $L_1(G)$. Since σ is an $L_1(G)$-morphism, then for all $s, t \in G$ we have

$$f(s^{-1}t) = L_s(f)(t) = \lim_\nu L_s(e_\nu * \sigma(1))(t) = \lim_\nu((\delta_s * e_\nu) * \sigma(1))(t) = \lim_\nu \sigma((\delta_s * e_\nu) \cdot_\gamma 1)(t)$$

$$= \lim_\nu \sigma(\varkappa_\gamma^L(\delta_s * e_\nu))(t) = \lim_\nu \varkappa_\gamma^L(\delta_s * e_\nu)\sigma(1)(t) = \lim_\nu(e_\nu * \gamma)(s^{-1})f(t) = \gamma(s^{-1})f(t).$$

Therefore, for the function $g(t) := \gamma(t^{-1})f(t)$ in $L_1(G)$ we have $g(st) = g(t)$ for all $s, t \in G$. Thus, g is a constant function in $L_1(G)$, which is possible only when G is compact. □

Proposition 3.2.3. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent:*

(i) G is amenable;

(ii) $\mathbb{C}_\gamma$ *is a metrically injective* $L_1(G)$*-module;*

(iii) $\mathbb{C}_\gamma$ *is a topologically injective* $L_1(G)$*-module.*

(iv) $\mathbb{C}_\gamma$ *is a metrically flat* $L_1(G)$*-module;*

(v) $\mathbb{C}_\gamma$ *is a topologically flat* $L_1(G)$*-module.*

Proof. *(i)* $\Longrightarrow$ *(ii)* Since G is amenable, then we have a contractive $L_1(G)$-morphism $M : L_\infty(G) \to \mathbb{C}_{e_{\widehat{G}}}$ with $M(\chi_G) = 1$. Consider linear operators $\rho : \mathbb{C}_\gamma \to L_\infty(G)$, $z \mapsto z\overline{\gamma}$ and $\tau : L_\infty(G) \to \mathbb{C}_\gamma$, $f \mapsto M(f\gamma)$. These are $L_1(G)$-morphisms of right $L_1(G)$-modules. We shall check this for operator τ: for all $f \in L_\infty(G)$ and $g \in L_1(G)$ we have

$$\tau(f \cdot_\infty g) = M((f \cdot_\infty g)\gamma) = M(f\gamma \cdot_\infty g\overline{\gamma}) = M(f\gamma) \cdot_{e_{\widehat{G}}} g\overline{\gamma} = M(f\gamma)\varkappa_\gamma^L(g) = \tau(f) \cdot_\gamma g.$$

It is easy to check that ρ and τ are contractive and $\tau\rho = 1_{\mathbb{C}_\gamma}$. Therefore, $\mathbb{C}_\gamma$ is a retract of $L_\infty(G)$ in $\mathbf{mod}_1 - L_1(G)$. From propositions 3.2.1 and 2.1.21 it follows that $\mathbb{C}_\gamma$ is metrically injective as $L_1(G)$-module.

(ii) $\Longrightarrow$ *(iii)* See proposition 2.1.23.

(iii) $\Longrightarrow$ *(i)* Since ρ is an isometric $L_1(G)$-morphism of right $L_1(G)$-modules and $\mathbb{C}_\gamma$ is topologically injective as $L_1(G)$-module, then ρ is a coretraction in $\mathbf{mod} - L_1(G)$. Denote its left inverse morphism by π, then $\pi(\overline{\gamma}) = \pi(\rho(1)) = 1$. Consider a bounded linear functional $M : L_\infty(G) \to \mathbb{C}_\gamma : f \mapsto \pi(f\overline{\gamma})$. For all $f \in L_\infty(G)$ and $g \in L_1(G)$ we have

$$M(f \cdot_\infty g) = \pi((f \cdot_\infty g)\overline{\gamma}) = \pi(f\overline{\gamma} \cdot_\infty g\gamma) = \pi(f\overline{\gamma}) \cdot_\gamma g\gamma = M(f)\varkappa_\gamma^L(g\gamma) = M(f) \cdot_{e_{\widehat{G}}} g.$$

Therefore, M is an $L_1(G)$-morphism, but we also have $M(\chi_G) = \pi(\overline{\gamma}) = 1$. Therefore, G is amenable.

(ii) $\Longleftrightarrow$ *(iv)*, *(iii)* $\Longleftrightarrow$ *(v)* Note that $\mathbb{C}_\gamma^* \underset{\mathbf{mod}_1 - L_1(G)}{\cong} \mathbb{C}_\gamma$, so all equivalences follow from three previous paragraphs and proposition 2.1.32. $\square$

In the next proposition we shall study specific ideals of the Banach algebra $L_1(G)$. They have the form $L_1(G) * \mu$ for some idempotent measure $\mu \in M(G)$. In fact, this class of ideals in case of commutative compact groups G coincides with those closed left ideals of $L_1(G)$ that admit a right bounded approximate identity.

Proposition 3.2.4. *Let* G *be a locally compact group and* $\mu \in M(G)$ *be an idempotent measure, that is* $\mu * \mu = \mu$. *If the left ideal* $I = L_1(G) * \mu$ *of the Banach algebra* $L_1(G)$ *is a topologically projective* $L_1(G)$*-module, then* $\mu = pm_G$*, for some* $p \in I$.

Proof. Let $\phi : I \to L_1(G)$ be arbitrary morphism of left $L_1(G)$-modules. Consider $L_1(G)$-morphism $\phi' : L_1(G) \to L_1(G) : x \mapsto \phi(x * \mu)$. By Wendel's theorem [[66], theorem 1], there exists a measure $\nu \in M(G)$ such that $\phi'(x) = x * \nu$ for all $x \in L_1(G)$. In particular, $\phi(x) = \phi(x * \mu) = \phi'(x) = x * \nu$ for all $x \in I$. It is now clear that $\psi : I \to I : x \mapsto \nu * x$ is a morphism of right I-modules satisfying $\phi(x)y = x\psi(y)$ for all $x, y \in I$. By paragraph (ii) of lemma 2.1.15 the ideal I has a right identity, say $e \in I$. Then $x * \mu = x * \mu * e$ for all $x \in L_1(G)$. Two measures are equal if their convolutions with all functions of $L_1(G)$ coincide [[25], corollary 3.3.24], so $\mu = \mu * em_G$. Since $e \in I \subset L_1(G)$, then $\mu = \mu * em_G \in M_a(G)$. Set $p = \mu * e \in I$, then $\mu = pm_G$. □

We conjecture that the left ideal $L_1(G) * \mu$ for an idempotent measure $\mu \in M(G)$ is metrically projective $L_1(G)$-module iff $\mu = pm_G$ where $p \in I$ and $\|p\| = 1$.

Theorem 3.2.5. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is discrete;

(ii) $L_1(G)$ is a metrically projective $L_1(G)$-module;

(iii) $L_1(G)$ is a topologically projective $L_1(G)$-module.

Proof. $(i) \implies (ii)$ If G is discrete, then the algebra $L_1(G)$ is unital with unit of norm 1. By proposition 2.1.13 we see that $L_1(G)$ is metrically projective as $L_1(G)$-module.

$(ii) \implies (iii)$ See proposition 2.1.6.

$(iii) \implies (i)$ Clearly, δ_{e_G} is an idempotent measure. Since $L_1(G) = L_1(G) * \delta_{e_G}$ is topologically projective, then by proposition 3.2.4 we have $\delta_{e_G} = fm_G$ for some $f \in L_1(G)$. This is possible only if G is discrete. □

Note that $L_1(G)$-module $L_1(G)$ is relatively projective for any locally compact group G [[23], exercise 7.1.17].

Proposition 3.2.6. *Let G be a locally compact group. Then the following are equivalent:*

(i) G is discrete;

(ii) $M(G)$ is a metrically projective $L_1(G)$-module;

(iii) $M(G)$ is a topologically projective $L_1(G)$-module;

(iv) $M(G)$ is a metrically flat $L_1(G)$-module.

Proof. (*i*) $\implies$ (*ii*) We have $M(G) \underset{L_1(G)-\mathbf{mod}_1}{\cong} L_1(G)$ for discrete G, so the result follows from theorem 3.2.5.

(*ii*) $\implies$ (*iii*) See proposition 2.1.6.

(*ii*) $\implies$ (*iv*) See proposition 2.1.39.

(*iii*) $\implies$ (*i*) Note that $M(G) \underset{L_1(G)-\mathbf{mod}_1}{\cong} L_1(G) \bigoplus_1 M_s(G)$, so $M_s(G)$ is topologically projective by proposition 2.1.9. Note that $M_s(G)$ is an annihilator $L_1(G)$-module, then by proposition 2.2.3 the algebra $L_1(G)$ has a right identity. Recall that $L_1(G)$ also has a two-sided bounded approximate identity, so $L_1(G)$ is unital. The last is equivalent to G being discrete.

(*iv*) $\implies$ (*i*) Note that $M(G) \underset{L_1(G)-\mathbf{mod}_1}{\cong} L_1(G) \bigoplus_1 M_s(G)$, so $M_s(G)$ is metrically flat by proposition 2.1.37. Note that $M_s(G)$ is an annihilator $L_1(G)$-module, then by proposition 2.2.6 it is equal to zero. The last is equivalent to G being discrete. □

Proposition 3.2.7. *Let G be a locally compact group. Then $M(G)$ is a topologically flat $L_1(G)$-module.*

Proof. Since $M(G)$ is an L_1-space it is a fortiori an $\mathscr{L}_1^g$-space. Since $M_s(G)$ is complemented in $M(G)$, then $M_s(G)$ is an $\mathscr{L}_1^g$-space too [[12], corollary 23.2.1(2)]. As $M_s(G)$ is an annihilator $L_1(G)$-module, then from proposition 2.2.6 we have that $M_s(G)$ is topologically flat $L_1(G)$-module. The $L_1(G)$-module $L_1(G)$ is also topologically flat by proposition 3.2.1. Since $M(G) \underset{L_1(G)-\mathbf{mod}_1}{\cong} L_1(G) \bigoplus_1 M_s(G)$, then $M(G)$ is topologically flat $L_1(G)$-module by proposition 2.1.37. □

3.2.3 $M(G)$-modules

We turn to the study of standard $M(G)$-modules of harmonic analysis. As we shall see most of the results can be derived from results on $L_1(G)$-modules.

Proposition 3.2.8. *Let G be a locally compact group, and X be $\langle$ essential / faithful / essential $\rangle$ $L_1(G)$-module. Then*

(*i*) *X is a metrically $\langle$ projective / injective / flat $\rangle$ $M(G)$-module iff it is a metrically $\langle$ projective / injective / flat $\rangle$ $L_1(G)$-module;*

(*ii*) *X is a topologically $\langle$ projective / injective / flat $\rangle$ $M(G)$-module iff it is a topologically $\langle$ projective / injective / flat $\rangle$ $L_1(G)$-module.*

Proof. Recall that $L_1(G) \underset{L_1(G)-\mathbf{mod}_1}{\cong} M_a(G)$ is a closed two-sided contractively complemented ideal of $M(G)$. Now (i) and (ii) follow from proposition $\langle$ 2.3.2 / 2.3.3 / 2.3.4 $\rangle$. □

It is worth mentioning here that the $L_1(G)$-modules $C_0(G)$, $L_p(G)$ for $1 \leq p < \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are essential and $L_1(G)$-modules $C_0(G)$, $M(G)$, $L_p(G)$ for $1 \leq p \leq \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are faithful.

Proposition 3.2.9. *Let G be a locally compact group. Then $M(G)$ is metrically and topologically projective $M(G)$-module. As a consequence it is metrically and topologically flat $M(G)$-module.*

Proof. Since $M(G)$ is a unital algebra, then metric and topological projectivity of $M(G)$ follow from proposition 2.1.3. It remains to apply proposition 2.1.39. □

3.2.4 Banach geometric constraints

In this section we shall show that many modules of harmonic analysis are fail to be metrically or topologically projective, injective or flat for purely Banach geometric reasons.

Proposition 3.2.10. *Let G be an infinite locally compact group. Then*

(i) $L_1(G)$, $C_0(G)$, $M(G)$, $L_\infty(G)^*$ are not topologically injective Banach spaces;

(ii) $C_0(G)$, $L_\infty(G)$ are not complemented in any L_1-space.

Proof. Since G is infinite all modules in question are infinite dimensional.

(i) If an infinite dimensional Banach space is topologically injective, then it contains a copy of $\ell_\infty(\mathbb{N})$ [[67], corollary 1.1.4], and consequently a copy of $c_0(\mathbb{N})$. The Banach space $L_1(G)$ is weakly sequentially complete [[14], corollary III.C.14], so by corollary 5.2.11 in [8] it cannot contain a copy of $c_0(\mathbb{N})$. Therefore, $L_1(G)$ is not a topologically injective Banach space. If $M(G)$ is a topologically injective Banach space, then so is its complemented subspace $M_a(G) \underset{\mathbf{Ban}_1}{\cong} L_1(G)$. By previous argument this is impossible. So $M(G)$ is not topologically injective as Banach space. By corollary 3 of [68] the space $C_0(G)$ is not complemented in $L_\infty(G)$. Then $C_0(G)$ cannot be topologically injective either. The Banach space $L_1(G)$ is complemented in $L_\infty(G)^* \underset{\mathbf{Ban}_1}{\cong} L_1(G)^{**}$ [[12], proposition B10]. Therefore, if $L_\infty(G)^*$ is topologically injective as Banach space, then so is its retract $L_1(G)$. By previous arguments this is impossible, so $L_\infty(G)^*$ is not topologically injective Banach space.

(ii) If $C_0(G)$ is a retract of L_1-space, then $M(G) \underset{\mathbf{Ban}_1}{\cong} C_0(G)^*$ is a retract of L_∞-space, so it must be a topologically injective Banach space. This contradicts paragraph (i), so $C_0(G)$ is not a retract of L_1-space. Note that $\ell_\infty(\mathbb{N})$ embeds in $L_\infty(G)$, hence so is $c_0(\mathbb{N})$. Therefore, if $L_\infty(G)$ is a retract of L_1-space, then there would exist an L_1-space containing a copy of $c_0(\mathbb{N})$. This is impossible as already showed in paragraph (i). $\square$

From now on by A we denote either $L_1(G)$ or $M(G)$. Recall that $L_1(G)$ and $M(G)$ are both L_1-spaces.

Proposition 3.2.11. *Let G be an infinite locally compact group. Then*

(i) $C_0(G)$, $L_\infty(G)$ are neither topologically nor metrically projective A-modules;

(ii) $L_1(G)$, $C_0(G)$, $M(G)$, $L_\infty(G)^*$ are neither topologically nor metrically injective A-modules;

(iii) $L_\infty(G)$, $C_0(G)$ are neither topologically nor metrically flat A-modules.

(iv) $L_p(G)$ for $1 < p < \infty$ are neither topologically nor metrically projective, injective or flat A-flat.

Proof. (i) The result follows from propositions 2.2.8 paragraph (i) and 3.2.10 paragraph (ii).

(ii) The result follows from propositions 2.2.8 paragraph (ii) and 3.2.10.

(iii) Note that $C_0(G)^* \underset{\mathbf{mod}_1-A}{\cong} M(G)$. Now the result follows from paragraph (i) and proposition 2.1.32.

(iv) Since $L_p(G)$ is reflexive for $1 < p < \infty$ the result follows from proposition 2.2.15. $\square$

It remains to consider metric and topological homological properties of A-modules when G is finite.

Proposition 3.2.12. *Let G be a non-trivial finite group and $1 \leq p \leq \infty$. Then the A-module $L_p(G)$ is metrically $\langle$ projective / injective $\rangle$ iff $\langle$ $p = 1$ / $p = \infty$ $\rangle$*

Proof. Assume $L_p(G)$ is metrically $\langle$ projective / injective $\rangle$ as A-module. Since $L_p(G)$ is finite dimensional, then by paragraphs (i) and (ii) of proposition 2.2.8 we have identifications $\langle$ $L_p(G) \underset{\mathbf{Ban}_1}{\cong} \ell_1(\mathbb{N}_n)$ / $L_p(G) \underset{\mathbf{Ban}_1}{\cong} C(\mathbb{N}_n) \underset{\mathbf{Ban}_1}{\cong} \ell_\infty(\mathbb{N}_n)$ $\rangle$, where $n = \operatorname{Card}(G) > 1$. Now we use the result of theorem 1 from [60] for Banach spaces over field $\mathbb{C}$: if for $2 \leq m \leq k$ and $1 \leq r, s \leq \infty$, there exists an isometric embedding from $\ell_r(\mathbb{N}_m)$ into

$\ell_s(\mathbb{N}_k)$, then either $r = 2$, $s \in 2\mathbb{N}$ or $r = s$. Therefore, $\langle\ p = 1\ /\ p = \infty\ \rangle$. The converse easily follows from $\langle$ theorem 3.2.5 / proposition 3.2.1 $\rangle$ □

Proposition 3.2.13. *Let G be a finite group. Then*

(i) $C_0(G)$, $L_\infty(G)$ are metrically injective A-modules;

(ii) $C_0(G)$, $L_p(G)$ for $1 < p \leq \infty$ are metrically projective A-modules iff G is trivial;

(iii) $M(G)$, $L_p(G)$ for $1 \leq p < \infty$ are metrically injective A-modules iff G is trivial;

(iv) $C_0(G)$, $L_p(G)$ for $1 < p \leq \infty$ are metrically flat A-modules iff G is trivial.

Proof. (i) Since G is finite then $C_0(G) = L_\infty(G)$. The result follows from proposition 3.2.1.

(ii) If G is trivial, that is $G = \{e_G\}$, then $L_p(G) = C_0(G) = L_1(G)$ and the result follows from paragraph (i). If G is non-trivial, then we recall that $C_0(G) = L_\infty(G)$ and use proposition 3.2.12.

(iii) If G is trivial, then $M(G) = L_p(G) = L_\infty(G)$ and the result follows from paragraph (i). If G is non-trivial, then we note that $M(G) = L_1(G)$ and use proposition 3.2.12.

(iv) From paragraph (iii) it follows that $L_p(G)$ for $1 \leq p < \infty$ is a metrically injective A-module iff G is trivial. Now the result follows from proposition 2.1.32 and the facts that $C_0(G)^* \underset{\mathbf{mod}_1-L_1(G)}{\cong} M(G) \underset{\mathbf{mod}_1-L_1(G)}{\cong} L_1(G)$, $L_p(G)^* \underset{\mathbf{mod}_1-L_1(G)}{\cong} L_{p^*}(G)$ for $1 \leq p^* < \infty$. □

It is worth mentioning here that if we would consider all Banach spaces over the field of real numbers, then the $L_1(G)$-modules $L_\infty(G)$ and $L_1(G)$ would be metrically projective and injective respectively, additionally for G consisting of two elements, because

$$L_\infty(\mathbb{Z}_2) \underset{L_1(\mathbb{Z}_2)-\mathbf{mod}_1}{\cong} \mathbb{R}_{\gamma_0} \bigoplus\nolimits_1 \mathbb{R}_{\gamma_1}, \qquad L_1(\mathbb{Z}_2) \underset{L_1(\mathbb{Z}_2)-\mathbf{mod}_1}{\cong} \mathbb{R}_{\gamma_0} \bigoplus\nolimits_\infty \mathbb{R}_{\gamma_1}$$

for $\gamma_0, \gamma_1 \in \widehat{\mathbb{Z}_2}$ defined by $\gamma_0(0) = \gamma_0(1) = \gamma_1(0) = -\gamma_1(1) = 1$. Here $\mathbb{Z}_2$ denotes the unique group of two elements.

Proposition 3.2.14. *Let G be a finite group. Then the A-modules $C_0(G)$, $M(G)$, $L_p(G)$ for $1 \leq p \leq \infty$ are both topologically projective, injective and flat.*

Proof. Since the group G is finite, we have $M(G) = L_1(G)$ and $C_0(G) = L_\infty(G)$, so these modules do not require special considerations. As $M(G) = L_1(G)$, we can restrict our considerations to the case $A = L_1(G)$. The identity map $i : L_1(G) \to L_p(G) : f \mapsto f$ is a topological isomorphism of Banach spaces, because $L_1(G)$ and $L_p(G)$ for $1 \leq p < +\infty$ are of equal finite dimension. Since G is finite, it is unimodular. Therefore, the module actions in $(L_1(G), *)$ and $(L_p(G), *_p)$ coincide for $1 \leq p < +\infty$. Therefore, i is an isomorphism in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$. Similarly, one can show that $(L_\infty(G), \cdot_\infty)$ and $(L_p(G), \cdot_p)$ for $1 < p \leq +\infty$ are isomorphic in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$. Finally, one can easily check that $(L_1(G), *)$ and $(L_\infty(G), \cdot_\infty)$ are isomorphic in $L_1(G) - \mathbf{mod}$ and $\mathbf{mod} - L_1(G)$ via the map $j : L_1(G) \to L_\infty(G) : f \mapsto (s \mapsto f(s^{-1}))$. Therefore, all modules discussed above are pairwise isomorphic. It remains to recall that $L_1(G)$ is topologically projective and flat by theorem 3.2.5 and proposition 3.2.1, while $L_\infty(G)$ is topologically injective by proposition 3.2.1. □

Now we can summarize results on homological properties of modules of harmonic analysis into three tables. Each cell of the table contains a condition under which the respective module has the respective property and propositions where this result is proved. We shall mention that results for modules $L_p(G)$ are valid for both module actions $*_p$ and $\cdot_p$. Characterization and proofs for homologically trivial modules $\mathbb{C}_\gamma$ in case of relative theory is the same as in propositions 3.2.2, 3.2.3 and 3.2.3. As usually, we use * to indicate that only a necessary conditions is known. As we showed above even topological theory is too restrictive for $L_1(G)$ to be projective as $L_1(G)$-module. Similarly, a Banach space is topologically projective iff it is an L_1-space, and the underlying measure space is atomic. This analogy confirms important role of Banach geometry in metric and topological Banach homology.

Homologically trivial $L_1(G)$- and $M(G)$-modules in metric theory

	$L_1(G)$-modules			$M(G)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$L_1(G)$	G is discrete 3.2.5	$G = \{e_G\}$ 3.2.11, 3.2.13	G is any 3.2.1	G is discrete 3.2.5, 3.2.8	$G = \{e_G\}$ 3.2.11, 3.2.13	G is any 3.2.1, 3.2.8
$L_p(G)$	$G = \{e_G\}$ 3.2.11, 3.2.12	$G = \{e_G\}$ 3.2.11, 3.2.12	$G = \{e_G\}$ 3.2.11, 3.2.13	$G = \{e_G\}$ 3.2.11, 3.2.12	$G = \{e_G\}$ 3.2.11, 3.2.12	$G = \{e_G\}$ 3.2.11, 3.2.13
$L_\infty(G)$	$G = \{e_G\}$ 3.2.11, 3.2.12	G is any 3.2.1	$G = \{e_G\}$ 3.2.11, 3.2.13	$G = \{e_G\}$ 3.2.11, 3.2.12	G is any 3.2.1, 3.2.8	$G = \{e_G\}$ 3.2.11, 3.2.13
$M(G)$	G is discrete 3.2.6	$G = \{e_G\}$ 3.2.11, 3.2.13	G is discrete 3.2.7	G is any 3.2.9	$G = \{e_G\}$ 3.2.11, 3.2.13	G is any 3.2.9
$C_0(G)$	$G = \{e_G\}$ 3.2.11, 3.2.13	G is finite 3.2.11, 3.2.13	$G = \{e_G\}$ 3.2.11, 3.2.13	$G = \{e_G\}$ 3.2.11, 3.2.13	G is finite 3.2.11, 3.2.13	$G = \{e_G\}$ 3.2.11, 3.2.13
$\mathbb{C}_\gamma$	G is compact 3.2.2	G is amenable 3.2.3	G is amenable 3.2.3	G is compact 3.2.2, 3.2.8	G is amenable 3.2.3, 3.2.8	G is amenable 3.2.3, 3.2.8

Homologically trivial $L_1(G)$- and $M(G)$-modules in topological theory

	$L_1(G)$-modules			$M(G)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$L_1(G)$	G is discrete 3.2.5	G is finite 3.2.11, 3.2.14	G is any 3.2.1	G is discrete 3.2.5, 3.2.8	G is finite 3.2.11, 3.2.14	G is any 3.2.1, 3.2.8

$L_p(G)$	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14
$L_\infty(G)$	G is finite 3.2.11, 3.2.14	G is any 3.2.1	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is any 3.2.1, 3.2.8	G is finite 3.2.11, 3.2.14
$M(G)$	G is discrete 3.2.6	G is finite 3.2.11, 3.2.14	G is any 3.2.7	G is any 3.2.9	G is finite 3.2.11, 3.2.14	G is any 3.2.9
$C_0(G)$	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14	G is finite 3.2.11, 3.2.14
$\mathbb{C}_\gamma$	G is compact 3.2.2	G is amenable 3.2.3	G is amenable 3.2.3	G is compact 3.2.2, 3.2.8	G is amenable 3.2.3, 3.2.8	G is amenable 3.2.3, 3.2.8

Homologically trivial $L_1(G)$- and $M(G)$-modules in relative theory

	$L_1(G)$-modules			$M(G)$-modules		
	Projectivity	Injectivity	Flatness	Projectivity	Injectivity	Flatness
$L_1(G)$	G is any [42], §6	G is amenable and discrete [42], §6	G is any [42], §6	G is any [33], §3.5	G is amenable and discrete [33], §3.5	G is any [33], §3.5
$L_p(G)$	G is compact [42], §6	G is amenable [43]	G is amenable [43]	G is compact [33], §3.5	G is amenable [33], §3.5, [43]	G is amenable [33], §3.5
$L_\infty(G)$	G is finite [42], §6	G is any [42], §6	G is amenable [42], §6	G is finite [33], §3.5	G is any [33], §3.5	G is amenable [33], §3.5*
$M(G)$	G is discrete [42], §6	G is amenable [42], §6	G is any [33], §3.5	G is any [33], §3.5	G is amenable [33], §3.5	G is any [33], §3.5
$C_0(G)$	G is compact [42], §6	G is finite [42], §6	G is amenable [42], §6	G is compact [33], §3.5	G is finite [33], §3.5	G is amenable [33], §3.5
$\mathbb{C}_\gamma$	G is compact 3.2.2	G is amenable 3.2.3	G is amenable 3.2.3	G is compact 3.2.2, 3.2.8	G is amenable 3.2.3, 3.2.8	G is amenable 3.2.3, 3.2.8

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