

The Dunford-Pettis property

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Abstract

This is a short note on the Dunford-Pettis property was written for self-educational purposes and future reference. A few additional subjects are discussed to give a firm introduction.

1 Weak topologies

Definition 1.1 *Let X be a Banach space, then the weakest topology making all functionals in X^* continuous is called the weak topology.*

If X is a Banach space, then the weak topology on X^* is the weakest topology making all functionals $\iota_X(x) : X^* \rightarrow \mathbb{C} : f \mapsto f(x)$ continuous.*

The sequences that converge in $\langle \text{weak} / \text{weak}^* \rangle$ topology to 0 we shall call $\langle \text{weakly null} / \text{weakly}^* \text{ null} \rangle$. A subset A in $\langle X / X^* \rangle$ is called $\langle \text{weakly} / \text{weakly}^* \rangle$ bounded if $\langle \text{for all } f \in X^* \text{ the set } \{f(x) : x \in A\} / \text{for all } x \in X \text{ the set } \{f(x) : f \in A\} \rangle$ is bounded in \mathbb{C} .

We shall list the following well known properties of weak topologies. Let X be a Banach space.

- (i) A subset A of X is weakly bounded iff it is norm bounded [[1], theorem 3.88].
- (ii) If X is infinite dimensional, then every non empty weakly open subset of X is unbounded [[1], proposition 3.89].
- (iii) If the weak topology of X is metrizable, then X is finite dimensional.
- (iv) if A is a convex subset in X , then the norm and the weak closure of A coincide [[1], theorem 3.45].

Now we shall say a few words on weak* topologies. Let X be a Banach space.

- (i) B_{X^*} is weak* compact [[1], theorem 3.37]
- (ii) B_X is weak* dense in $B_{X^{**}}$ [[1], theorem 3.96]

We proceed to discussion of weak compactness in Banach spaces. A subset A of X is called $\langle \text{weakly} / \text{relatively weakly} \rangle$ compact if $\langle A / \text{the weak closure of } A \rangle$ is compact in the weak topology of X .

Again we list basic properties of weakly compact sets. Let X be a Banach space.

- (i) Any weakly compact set in X is norm closed and norm bounded;
- (ii) B_X is weakly compact iff X is reflexive [[1], theorem 3.111];
- (iii) if X is reflexive, then any bounded set is relatively weakly compact;
- (iv) if X is reflexive, then for any bounded linear operator $T : X \rightarrow Y$ the set $T(B_X)$ is weakly compact.

Definition 1.2 Let A be a subset of a topological space M , then A is called

- (i) \langle sequentially / relatively sequentially \rangle compact if every sequence in A has a subsequence convergent to a point \langle in A / in M \rangle .
- (ii) \langle countably / relatively countably \rangle compact if every sequence in A has a subnet convergent to a point \langle in A / in M \rangle .

Countable compactness is implied by both compactness and sequential compactness. If M is metrizabel, then all three concepts coincide, but the converse is not true. For example $B_{\ell_\infty^*}$ with weak* topology is compact by Banach-Alaoglu theorem, but not sequentially weak* compact, because the sequence of functionals $e_n^* : \ell_\infty \rightarrow \mathbb{C} : x \mapsto x(n)$ has no convergent subsequence. Though weak topology of Banach space is not metrizabel in general its bounded subsets behave much like they are metrizabel. They indeed metrizabel if X is separable.

Theorem 1.3 (Eberlein-Smulian) Let A be a subset of a Banach space X . The following are equivalent

- (i) A is \langle weakly / relatively weakly \rangle compact
- (ii) A is \langle weakly / relatively weakly \rangle sequentially compact
- (iii) A is \langle weakly / relatively weakly \rangle countably compact

◁ See [[2], theorem 1.6.3] ▷

Here are some examples of characterization of relatively weakly compact subsets:

Definition 1.4 A subset of A of a Banach space $L_1(\Omega, \mu)$ is called equi-integrable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any measurable subset E of Ω with $\mu(E) < \delta$ and any $f \in A$ holds

$$\int_E |f(\omega)| d\mu(\omega) < \varepsilon$$

Theorem 1.5 (Dunford, Pettis) A bounded subset A of $L_1(\Omega, \mu)$ is relatively weakly compact iff it is equi-integrable.

◁ See [[2], 5.2.9]. ▷

By $M(K)$ we denote the Banach space of complex Borel regular measures on a Hausdorff compact K .

Definition 1.6 A subset A of a Banach space $M(K)$ is said to be uniformly regular if for any open subset U in K and any $\varepsilon > 0$, there is a compact set H in U such that $\sup_{\mu \in A} |\mu|(U \setminus H) < \varepsilon$.

Theorem 1.7 (Grothendieck) A bounded subset A of $M(K)$ is relatively weakly compact iff it is uniformly regular.

◁ See [[2], 5.3.2] ▷

2 Classes of bounded linear operators

Note: by $\mathcal{B}(X, Y)$ we denote the Banach space of all bounded linear operators between Banach spaces X and Y .

Definition 2.1 A bounded linear operator $T : X \rightarrow Y$ between Banach space X and Y is called

- (i) compact if $T(B_X)$ is relatively compact in Y
- (ii) weakly compact if $T(B_X)$ is relatively weakly compact in Y
- (iii) completely continuous if $T(W)$ is compact in Y for any weakly compact subset W of X

By $\langle \mathcal{K}(X, Y) / \mathcal{W}(X, Y) / \mathcal{CC}(X, Y) \rangle$ we denote the Banach space of \langle compact / weakly compact / completely continuous \rangle operators.

Proposition 2.2 A bounded linear operator is completely continuous iff it is weak-to-norm sequentially continuous.

◁ See [[2], 5.4.2]. ▷

Just for comparison we mention the following fact

Proposition 2.3 Let X and Y be two Banach spaces. Then

- (i) a linear operator $T : X \rightarrow Y$ is bounded iff T is weak-to-weak continuous;
- (ii) if $T : X \rightarrow Y$ is a bounded linear operator, then T^* is weak*-to-weak* continuous;
- (iii) if $S : Y^* \rightarrow X^*$ is weak*-to-weak* continuous then $S = T^*$ for some bounded linear operator $T : X \rightarrow Y$.
- (iv) if $T : X \rightarrow Y$ is weak-to-norm continuous, then it is a finite rank operator

◁ (i) See [[3], theorem 5.3.15].

(ii) Straightforward.

(iii) See [[1], exercise 3.60]

(iv) See [[1], exercise 15.3] ▷

Proposition 2.4 For any Banach spaces X and Y we have isometric inclusions

- (i) $\mathcal{K}(X, Y) \subset \mathcal{W}(X, Y) \subset \mathcal{B}(X, Y)$
- (ii) $\mathcal{K}(X, Y) \subset \mathcal{CC}(X, Y) \subset \mathcal{B}(X, Y)$

◁ The only non-trivial inclusion here is $\mathcal{K}(X, Y) \hookrightarrow \mathcal{CC}(X, Y)$. See [[1], exercise 1.77]. ▷

Theorem 2.5 (Schauder) A bounded linear operator between Banach spaces is compact iff its adjoint is compact.

◁ See theorem 15.3 in [1] ▷

Theorem 2.6 (Davis, Figel, Johnson, Pelczynski) *A bounded linear operator $T : X \rightarrow Y$ is weakly compact iff there exists a reflexive Banach space Z and bounded linear operators $R : X \rightarrow Z$, $Q : Z \rightarrow Y$ such that $T = QR$, that is T factors through a reflexive Banach space.*

◁ Necessity is proved in [[1], theorem 13.33]. The converse is obvious. Indeed, since X is reflexive, then Q is weakly compact and so does $T = QR$. ▷

Theorem 2.7 (Gantmacher) *Let $T : X \rightarrow Y$ be a bounded linear operator between Banach spaces X and Y . Then the following are equivalent*

- (i) T is weakly compact;
- (ii) $T^{**}(X^{**}) \subset Y$;
- (iii) T^* is weak*-to-weak continuous;
- (iv) T^* is weakly compact.

◁ (i) \implies (ii) It is enough to check that all elements of $T^{**}(X^{**})$ are weak*-continuous.

(ii) \implies (iii) Straightforward.

(iii) \implies (iv) By Davis-Jhonson-Figel-Pelczynski theorem T factors through a reflexive space. Clearly, so does T^* .

▷

3 The Dunford-Pettis property

Definition 3.1 *We say that a Banach space X has the Dunford-Pettis property if $\mathcal{W}(X, Y) \subset \mathcal{CC}(X, Y)$ for any Banach space Y .*

Directly from definition it follows that the square of any weakly compact operator on a Banach space with Dunford-Pettis property is necessarily compact. If X is reflexive Banach space, then 1_X is weakly compact. Therefore X has the Dunford-Pettis property iff 1_X is compact. The latter is possible only for finite dimensional X . In other words, for infinite dimensional reflexive Banach space X the operator 1_X provides an example of weakly compact but not completely continuous operator.

Theorem 3.2 *A Banach space X has the Dunford-Pettis property iff for any weakly null sequences $(x_n)_{n \in \mathbb{N}} \subset X$, $(x_n^*)_{n \in \mathbb{N}} \subset X^*$ holds $\lim_n x_n^*(x_n) = 0$.*

◁ Let Y be a Banach space and $T : X \rightarrow Y$ a weakly compact operator. Let us suppose that T is not Dunford-Pettis. Then there is weakly null sequence $(x_n)_{n \in \mathbb{N}}$ in X such that for some $\delta > 0$ holds $\|T(x_n)\| \geq \delta$ for all $n \in \mathbb{N}$. Pick $(y_n^*)_{n \in \mathbb{N}} \subset Y^*$ such that $y_n^*(T(x_n)) = \|T(x_n)\|$ and $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$. By Gantmacher's theorem T^* is weakly compact hence $T^*(B_{Y^*})$ is a relatively weakly compact subset of X^* . By the Eberlein-Smulian theorem the sequence $T^*(y_n^*) \subset T^*(B_{Y^*})$ can be assumed weakly convergent to some $x^* \in X^*$. Then $(T^*(y_n^*) - x^*)_{n \in \mathbb{N}}$ is weakly null. Therefore, the Dunford-Pettis property of X gives that

$$(T^*(y_n^*) - x^*)(x_n) \rightarrow_n 0$$

when $n \rightarrow \infty$. Note that $x^*(x_n) \rightarrow_n 0$ since $(x_n)_{n \in \mathbb{N}}$ is weakly null. Therefore $\|T(x_n)\| = T^*(y_n)(x_n) \rightarrow_n 0$. Contradiction.

For the converse, let $(x_n)_{n \in \mathbb{N}} \subset X$ and $(x_n^*)_{n \in \mathbb{N}} \subset X^*$ be weakly null sequences. Consider the operator

$$T : X \rightarrow c_0 : x \mapsto (x_n^*(x))_{n \in \mathbb{N}}$$

The adjoint operator T^* of T satisfies $T^*(e_k) = x_k^*$ for all $k \in \mathbb{N}$, where $(e_k)_{k \in \mathbb{N}}$ denotes the canonical basis of ℓ_1 . This implies that $T^*(B_{\ell_1})$ is contained in the convex hull of the weakly null sequence $(x_n^*)_{n \in \mathbb{N}}$. Therefore T^* is weakly compact, hence by Gantmacher's theorem so is T . As T is weakly compact, T is also Dunford-Pettis by the hypothesis. Therefore $(T(x_n))_{n \in \mathbb{N}}$ is norm null sequence. Since

$$|x_n^*(x_n)| \leq \max_k |x_k^*(x_n)| = \|T(x_n)\|$$

then $(x_n^*(x_n))_{n \in \mathbb{N}}$ is norm null sequence too. \triangleright

Any Banach space X with the Schur property (that is any weakly null sequence is norm null sequence) has the Dunford-Pettis property. Indeed, let $(x_n)_{n \in \mathbb{N}} \subset X$ and $(x_n^*)_{n \in \mathbb{N}} \subset X^*$ be weakly null sequences. Since $(x_n^*)_{n \in \mathbb{N}}$ is weakly null it is norm bounded by some constant C . The Schur property of X gives that $(x_n)_{n \in \mathbb{N}}$ is norm null, so $|x_n^*(x_n)| \leq C\|x_n\| \rightarrow_n 0$. In particular ℓ_1 has the Dunford-Pettis property.

Example 3.3 Consider operator $T : L_1([0, 1]) \rightarrow C([0, 1]) : x \mapsto \left(t \mapsto \int_0^t x(s)ds\right)$. One can show that for the sequence $x_n(t) = 2n(\chi_{[1/2-1/n, 1/2]} - \chi_{[1/2, 1/2+1/n]})$ the sequence $T(x_n)$ has no weakly convergent subsequence. So T is not weakly compact. Using Arzela-Ascoli criterion of compactness in $C([0, 1])$ and Dunford-Pettis criterion of weak compactness in $L_1([0, 1])$ it is routine to check that T is completely continuous.

Proposition 3.4 If X^* has the Dunford-Pettis property, then so does X .

\triangleleft Let $(x_n)_{n \in \mathbb{N}} \subset X$ and $(x_n^*)_{n \in \mathbb{N}} \subset X^*$ be weakly null sequences. Take arbitrary $x^{***} \in X^{***}$, then since $(x_n)_{n \in \mathbb{N}}$ is weakly null, then $x^{***}(\iota_X(x_n)) = \iota_X^*(x^{***})(x_n) \rightarrow_n 0$. Therefore $(\iota_X(x_n))_{n \in \mathbb{N}} \subset X^{**}$ is weakly null sequence. Since X^* has the Dunford-Pettis property, then $x_n^*(x_n) = \iota_X(x_n)(x_n^*) \rightarrow_n 0$. Therefore X has the Dunford-Pettis property. \triangleright

Proposition 3.5 The Dunford-Pettis property is inherited by complemented subspaces of Banach spaces.

\triangleleft Let Y be a complemented subspace of X with bounded linear projection $P : X \rightarrow Y$. Let $(y_n)_{n \in \mathbb{N}} \subset Y$ and $(y_n^*)_{n \in \mathbb{N}} \subset Y^*$ be weakly null sequences. Then $(y_n)_{n \in \mathbb{N}}$ is a weakly null in X . Since P^* is bounded, then it is weak-to-weak continuous, so from assumption $(P^*(y_n))_{n \in \mathbb{N}}$ is weakly null in X^* . Since X has the Dunford-Pettis property, then

$$y_n^*(y_n) = y_n^*(P(y_n)) = P^*(y_n^*)(y_n) \rightarrow_n 0$$

Therefore Y has the Dunford-Pettis property. \triangleright

Proving a certain Banach space has the Dunford-Pettis property is always a challenge. Most proofs require understanding of weakly null sequences of dual spaces. It is known since the times of Grothendieck that L_1 spaces and $C(K)$ -spaces have the Dunford-Pettis property.

References

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