Lack of metric projectivity, injectivity, and flatness for modules L_p

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Abstract: In this paper we show that for a locally compact Hausdorff space S and a decomposable Borel measure μ metric projectivity, injectivity or flatness of $C_0(S)$-module $L_p(S, \mu)$ implies that μ is purely atomic with at most one atom.

Keywords: metric projectivity, metric injectivity, metric flatness, L_p-space.

1 Introduction

This paper finalizes authors research on homological properties of modules L_p. In [1] it was shown that modules L_p are relatively projective for a small class of underlying measure spaces, namely purely atomic measure spaces, with atoms being isolated points. In this paper we solve the same problem for metric projectivity, injectivity and flatness. It was expected that for metric theory the class of measure spaces has to be even narrower. As we show in this paper it is extremely poor. It includes only purely atomic measure spaces with at most one atom. It is safe to say that L_p-spaces are almost never metrically projective, injective or flat.

Before we proceed to the main topic we shall give a few definitions. For any natural number $n \in \mathbb{N}$ by $\mathbb{N}_n$ we denote the set of first n natural numbers. Let M be a subset of a set N, then χ_M denotes the indicator function of M. The symbol 1_N denotes the identity map on N. If $n', n'' \in N$, then $\delta_{n'}^{n''}$ stands for their Kronecker symbol.

All Banach spaces in this paper are considered over the complex numbers. We shall actively use the following Banach space constructions. For given family of Banach spaces $\{E_\lambda : \lambda \in \Lambda\}$ by $\bigoplus_p \{E_\lambda : \lambda \in \Lambda\}$ we denote their ℓ_p-sum (see [[2], proposition 1.1.7]). Similarly, for a family of linear operators $T_\lambda : E_\lambda \to F_\lambda$ where $\lambda \in \Lambda$ we denote their ℓ_p-sum as $\bigoplus_p \{T_\lambda : \lambda \in \Lambda\}$ (see [[2], proposition 1.1.7]). We use symbol $E \widehat{\otimes} F$ do denote the projective tensor product of Banach spaces E and F [[2], theorem 2.7.4].

Let $T : E \to F$ be a bounded linear operator between Banach spaces. Now we give quantitative version of the definition of embedding. If for some constant $c > 0$ and all $x \in E$ holds $c\|T(x)\| \geq \|x\|$, then T is called c-topologically injective. Similarly, a quantitative definition of the open map is as follows: if there exists a constant $c > 0$ such that for any $y \in F$ we can find an $x \in E$ such that $T(x) = y$ and $c\|y\| \geq \|x\|$, then T is called c-topologically surjective. Finally, the operator T is called contractive if its norm is not greater than 1.

Let A be a Banach algebra. We shall consider both left and right Banach A-modules with contractive outer action $\cdot : A \times X \to X$. Let X and Y be two Banach A-modules, then a map $\phi : X \to Y$ is called an A-morphism if it is a continuous A-module map. All left Banach A-modules and their A-morphisms form a category which we denote by $A - \mathbf{mod}$. Similarly, one can define the category of right A-modules $\mathbf{mod} - A$. By $X \otimes_A Y$ we denote the projective module tensor product of a left A-module X and a right A-module Y (see [[3], definition VI.3.18])

Now we can give definitions of metric projectivity, injectivity and flatness. The first paper on this subject was written in 1978 by Graven [4]. These notions were later rediscovered by White [5] and Helemeskii [6, 7].

A left Banach A-module P is called metrically projective if for any c-topologically surjective A-morphism $\xi : X \to Y$ and any A-morphism $\phi : P \to Y$ there exists an A-morphism $\psi : P \to X$ such that $\|\psi\| \leq c$ and the diagram

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ P \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"'] & Y \end{tikzcd}$$

is commutative. The original definition, was somewhat different [[4], definition 2.4], but it is still equivalent to the one above. The simplest example of a metrically projective A-module is the algebra A itself, provided it is unital [[4], theorem 2.5].

A right Banach A-module J is called metrically injective if for any c-topologically injective A-morphism $\xi : Y \to X$ and any A-morphism $\phi : Y \to J$ there exists an A-morphism $\psi : X \to J$ such that $\|\psi\| \leq c$ and the diagram

$$\begin{tikzcd} & X \arrow[dl, dashed, "\psi"'] \\ J & Y \arrow[l, "\phi"] \arrow[u, "\xi"'] \end{tikzcd}$$

is commutative. Our definition is equivalent to the original one [[4], definition 3.1]. Note that P^* is a metrically injective A-module, whenever P is metrically injective [[4], theorem 3.2]. Therefore, the right Banach module A^* is metrically injective, whenever A is unital.

A left A-module F is called metrically flat if for each c-topologically injective A-morphism $\xi : X \to Y$ of right A-modules the operator $\xi \widehat{\otimes}_A 1_F : X \widehat{\otimes}_A F \to Y \widehat{\otimes}_A F$ is c-topologically injective. This definition was implicitly given in the theorem [[4], theorem 3.10]. From this theorem and remarks above we conclude that any metrically projective module is metrically flat. In particular, a unital Banach algebra A is a metrically flat A-module.

2 Metric injectivity of ℓ_∞^n-module ℓ_p^n

For a given $1 \leq p \leq +\infty$ by ℓ_p^n we denote the standard n-dimensional ℓ_p-space. We denote its norm by $\|\cdot\|_p$ and its natural basis by $(e_i)_{i \in \mathbb{N}_n}$. For $1 \leq p < +\infty$ we shall often exploit the standard identification $(\ell_p^n)^* = \ell_{p^*}^n$, where $p^* = p/(p-1)$. By convention, we set $1^* = +\infty$. The space ℓ_p^n can be regarded both as left and right Banach module over the Banach algebra ℓ_∞^n. In this section we shall show that the right ℓ_∞^n-module ℓ_p^n is metrically injective only if $n = 1$.

Definition 2.1. *Let $n \in \mathbb{N}$ and $\mathcal{F} \subset \ell_{p^*}^n$, then we define a linear operator*

$$\xi_{\mathcal{F}} : \ell_p^n \to \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\},\ x \mapsto \oplus_\infty \{x \cdot f : f \in \mathcal{F}\}.$$

Definition 2.2. *Let $n \in \mathbb{N}$, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}^n$, then we define the coercivity constant for the operator $\xi_{\mathcal{F}}$ as*

$$\gamma_{\mathcal{F}} = \sup\{\|x\| : x \in \ell_p^n, \quad \|\xi_{\mathcal{F}}(x)\| \leq 1\}.$$

Note that $\xi_{\mathcal{F}}$ is $\gamma_{\mathcal{F}}$-topologically injective whenever $\gamma_{\mathcal{F}}$ is finite.

Proposition 2.3. *Suppose $n \in \mathbb{N}$, $1 \leq p, q \leq +\infty$ and $\phi : \ell_p^n \to \ell_q^n$ is an ℓ_∞^n-morphism of right modules. Then there exists a vector $\eta \in \ell_\infty^n$ such that $\phi(x) = \eta \cdot x$ for all $x \in \ell_p^n$.*

Proof. Denote $\eta_i = \phi(e_i)_i$ for $i \in \mathbb{N}_n$, then for any $x \in \ell_p^n$ and $i \in \mathbb{N}_n$ we have

$$\phi(x)_i = (\phi(x) \cdot e_i)_i = \phi(x \cdot e_i)_i = \phi(x_i e_i)_i = x_i \phi(e_i)_i = x_i \eta_i = (\eta \cdot x)_i.$$

Therefore, $\phi(x) = \eta \cdot x$. $\square$

Proposition 2.4. *Let $n \in \mathbb{N}$, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}^n$ be a finite set. Then for any morphism of right ℓ_∞^n-modules $\psi : \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\} \to \ell_p^n$ there exists a family of vectors $\eta \in (\ell_\infty^n)^{\mathcal{F}}$ such that*

$$\psi(t) = \sum_{f \in \mathcal{F}} \eta_f \cdot t_f$$

for all $t \in \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\}$.

Proof. For each $f \in \mathcal{F}$ we define a natural embedding

$$\mathrm{in}_f : \ell_1^n \to \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\}$$

which is a morphism of right ℓ_∞^n-modules. Then we define an ℓ_∞^n-morphism $\psi_f = \psi \circ \mathrm{in}_f$. By proposition 2.3 there exists a vector $\eta_f \in \ell_\infty^n$ such that $\psi_f(x) = \eta_f \cdot x$ for all $x \in \ell_1^n$. Since $\mathcal{F}$ is finite, then for all $t \in \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\}$ we have

$$\psi(t) = \psi(\oplus_\infty \{t_f : f \in \mathcal{F}\}) = \psi\left(\sum_{f \in \mathcal{F}} \mathrm{in}_f(t_f)\right) = \sum_{f \in \mathcal{F}} \psi_f(t_f) = \sum_{f \in \mathcal{F}} \eta_f \cdot t_f.$$

$\square$

Definition 2.5. *Let $n \in \mathbb{N}$, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}^n$ be a finite set. For a given family $\eta \in (\ell_\infty^n)^{\mathcal{F}}$ we define*

$$\psi_\eta : \bigoplus_\infty \{\ell_1^n : f \in \mathcal{F}\} \to \ell_p^n,\ t \mapsto \sum_{f \in \mathcal{F}} \eta_f \cdot t_f.$$

Definition 2.6. *Let $n \in \mathbb{N}$, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}^n$ be a finite set, then we define*

$$\mathcal{N}_{\mathcal{F}} = \left\{ \eta \in (\ell_\infty^n)^{\mathcal{F}} : \sum_{f \in \mathcal{F}} \eta_{f,k} f_k = 1,\ \ k \in \mathbb{N}_n \right\}.$$

Proposition 2.7. *Let $n \in \mathbb{N}$, $1 < p < +\infty$ and $\mathcal{F} \subset \ell_{p^*}^n$ be a finite set. Then ψ_η is a left inverse ℓ_∞^n-morphism to $\xi_{\mathcal{F}}$ iff $\eta \in \mathcal{N}_{\mathcal{F}}$.*

Proof. Suppose ψ_η is a left inverse of $\xi_{\mathcal{F}}$, then for any $k \in \mathbb{N}_n$ we have

$$1 = e_k = \psi_\eta(\xi_{\mathcal{F}}(e_k)) = \psi_\eta\left(\oplus_\infty \{e_k \cdot f : f \in \mathcal{F}\}\right) = \sum_{f \in \mathcal{F}} \eta_f \cdot e_k \cdot f.$$

At k-th coordinate we get

$$1 = (e_k)_k = \left(\sum_{f \in \mathcal{F}} \eta_f \cdot e_k \cdot f\right)_k = \sum_{f \in \mathcal{F}} \eta_{f,k} f_k.$$

Hence, $\eta \in \mathcal{N}_{\mathcal{F}}$. Conversely, let $\eta \in \mathcal{N}_{\mathcal{F}}$, then for any $x \in \ell_p^n$ holds

$$\psi_\eta(\xi_{\mathcal{F}}(x)) = \psi_\eta(\oplus_\infty x \cdot f : f \in \mathcal{F}) = \sum_{f\in\mathcal{F}} \eta_f \cdot x \cdot f = \sum_{f\in\mathcal{F}} \sum_{k=1}^{n} (\eta_f \cdot x \cdot f)_k e_k = \sum_{f\in\mathcal{F}} \sum_{k=1}^{n} \eta_{f,k} x_k f_k e_k$$
$$= \sum_{k=1}^{n} \left(\sum_{f\in\mathcal{F}} \eta_{f,k} f_k \right) x_k e_k = \sum_{k=1}^{n} x_k e_k = x.$$

Therefore, ψ_η is a left inverse for $\xi_{\mathcal{F}}$. □

Proposition 2.8. *Let $n, m \in \mathbb{N}$, $1 < p < +\infty$ and $\eta \in (\ell_\infty^n)^{\mathbb{N}_m}$, then*

$$\|\psi_\eta\| = \max\left\{ \left(\sum_{k=1}^{n} \left| \sum_{l=1}^{m} |\eta_{l,k}| \delta_k^{d(l)} \right|^p \right)^{1/p} : d \in \mathbb{N}_n^{\mathbb{N}_m} \right\}.$$

Proof. By definition of operator norm

$$\|\psi_\eta\| = \sup\left\{ \|\psi_\eta(t)\| : t \in \bigoplus_\infty \{\ell_1^n : l \in \mathbb{N}_m\},\ \|t\| \le 1 \right\}$$
$$= \sup\{\|\eta_l \cdot t_l\| : t_l \in \ell_1^n,\ l \in \mathbb{N}_m,\ \max\{\|t_l\| : l \in \mathbb{N}_m\} \le 1\}$$
$$= \sup\left\{ \left(\sum_{k=1}^{n} \left| \sum_{l=1}^{m} \eta_{l,k} t_{l,k} \right|^p \right)^{1/p} : \sum_{k=1}^{n} |t_{l,k}| \le 1,\ t_{l,k} \in \mathbb{C},\ l \in \mathbb{N}_m \right\}.$$

For each $k \in \mathbb{N}_m$ and $l \in \mathbb{N}_m$ we denote $r_{l,k} = |t_{l,k}|$ and $\alpha_{l,k} = arg(t_{l,k})$. Then $t_{l,k} = r_{l,k} e^{i\alpha_{l,k}}$. So

$$\|\psi_\eta\| = \sup\left\{ \left(\sum_{k=1}^{n} \left| \sum_{l=1}^{m} \eta_{l,k} r_{l,k} e^{i\alpha_{l,k}} \right|^p \right)^{1/p} : \sum_{k=1}^{n} r_{l,k} \le 1,\ r_{l,k} \in \mathbb{R}_+,\ \alpha_{l,k} \in \mathbb{R},\ l \in \mathbb{N}_m \right\}.$$

For any $k \in \mathbb{N}_n$ the maximum of the expression

$$\left| \sum_{l=1}^{m} \eta_{l,k} \cdot r_{l,k} e^{i\alpha_{l,k}} \right|$$

is attained if $e^{i\alpha_{l,k}} = \mathrm{sgn}(\eta_{l,k})$ for all $l \in \mathbb{N}_m$. In this case

$$\|\psi_\eta\| = \sup\left\{ \left(\sum_{k=1}^{n} \left| \sum_{l=1}^{m} |\eta_{l,k}| r_{l,k} \right|^p \right)^{1/p} : \sum_{k=1}^{n} r_{l,k} \le 1,\ r_{l,k} \in \mathbb{R}_+,\ l \in \mathbb{N}_m \right\}.$$

Consider linear operators

$$\omega_l : \mathbb{R}^n \to \ell_p^n : r \mapsto \eta_l \cdot r$$

for $l \in \mathbb{N}_m$. Then

$$\|\psi_\eta\| = \sup\left\{ \left\| \sum_{l=1}^{m} \omega_l(r_l) \right\| : \sum_{k=1}^{n} r_{l,k} \le 1,\ r_{l,k} \in \mathbb{R}_+,\ l \in \mathbb{N}_m \right\}.$$

Since operators $(\omega_l)_{l\in\mathbb{N}_m}$ attain their values in ℓ_p^n, whose norm is strictly convex, then the function

$$F:(\mathbb{R}^n)^m\to\mathbb{R}_+,\ r\mapsto\left\|\sum_{l=1}^m\omega_l(r_l)\right\|$$

is strictly convex. Since the set

$$C=\left\{r\in(\mathbb{R}^n)^m:\sum_{k=1}^n r_{l,k}\leq 1,\ \ r_l\in\mathbb{R}_+^n,\ \ l\in\mathbb{N}_m\right\}$$

is a convex polytope, then F attains its maximum on $\operatorname{ext}(C)$ — the set of extreme points of C. So

$$\|\psi_\eta\|=\max\{F(r):r\in\operatorname{ext}(C)\},$$

Clearly, $r\in\operatorname{ext}(C)$ iff for some function $d:\mathbb{N}_m\to\mathbb{N}_n$ and all $k\in\mathbb{N}_n$, $l\in\mathbb{N}_m$ holds $r_{l,k}=\delta_k^{d(l)}$. So

$$\|\psi_\eta\|=\max\left\{\left(\sum_{k=1}^n\left|\sum_{l=1}^m|\eta_{l,k}|\delta_k^{d(l)}\right|^p\right)^{1/p}:d\in(\mathbb{N}_n)^{\mathbb{N}_m}\right\}.$$

□

Definition 2.9. *Let $n\in\mathbb{N}$, $1<p<+\infty$ and $\mathcal{F}\subset\ell_{p^*}^n$ is a finite set, then we define*

$$\nu_{\mathcal{F}}=\inf\{\|\psi_\eta\|:\eta\in\mathcal{N}_{\mathcal{F}}\}.$$

Definition 2.10. *Let $n\in\mathbb{N}$ and $\kappa\in\mathbb{R}$, then we define*

$$\mathcal{F}_\kappa=\left\{e_1,\ldots,e_n,\kappa\sum_{l=1}^n e_l\right\}\subset\mathbb{C}^{n+1}.$$

Proposition 2.11. *Assume $n\in\mathbb{N}$, $n>1$, $1<p<+\infty$ and $n^{-1}<\kappa<(n-1)^{-1}$. Then*

$$\gamma_{\mathcal{F}_\kappa}=(n-1+(\kappa^{-1}-(n-1))^p)^{1/p}.$$

Proof. By definition of coercivity constant

$$\begin{aligned}\gamma_{\mathcal{F}_\kappa}&=\sup\{\|x\|:x\in\ell_p^n,\ \|\xi_{\mathcal{F}_\kappa}(x)\|\leq 1\}\\&=\sup\left\{\left(\sum_{k=1}^n|x_k|^p\right)^{1/p}:x\in\mathbb{C}^n,\ \max\left\{|x_1|,\ldots,|x_n|,\kappa\sum_{k=1}^n|x_k|\right\}\leq 1\right\}\\&=\sup\left\{\left(\sum_{k=1}^n|t_k|^p\right)^{1/p}:t\in\mathbb{R}^n,\ |t_1|\leq 1,\ldots,|t_n|\leq 1,\ \sum_{k=1}^n|t_k|\leq\kappa^{-1}\right\}.\end{aligned}$$

Consider function

$$F:\mathbb{R}^n\to\mathbb{R},\ t\mapsto\left(\sum_{k=1}^n|t_k|^p\right)^{1/p}$$

and a convex polytope

$$C = \left\{ t \in \mathbb{R}^n : |t_1| \leq 1, \ldots, |t_n| \leq 1, \sum_{k=1}^{n} |t_k| \leq \kappa^{-1} \right\}.$$

Since F is strictly convex, then F attains its maximum on $\mathrm{ext}(C)$ — the set of extreme points of C. Therefore,

$$\gamma_{\mathcal{F}_\kappa} = \max\{F(t) : t \in \mathrm{ext}(C)\},$$

Clearly, any point $t \in \mathrm{ext}(C)$ has all coordinates but equal to 1 or -1. Therefore,

$$\mathrm{ext}(C) = \left\{ t \in \mathbb{R}^n : \exists l \in \mathbb{N}_n \quad |t_l| = \kappa^{-1} - (n-1) \wedge \forall k \in \mathbb{N}_n \setminus \{l\} \quad |t_k| = 1 \right\}.$$

As a consequence

$$\gamma_{\mathcal{F}_\kappa} = (n - 1 + (\kappa^{-1} - (n-1))^p)^{1/p}.$$

□

Proposition 2.12. *Assume $n \in \mathbb{N}$, $n > 1$, $1 < p < +\infty$ and $n^{-1} < \kappa < (n-1)^{-1}$, then*

$$\nu_{\mathcal{F}_\kappa} \geq \kappa^{-1} \frac{\left(\left(\frac{n-1}{\kappa^{-1}-1} \right)^{\frac{p}{p-1}} + n - 1 \right)^{1/p}}{\left(\frac{n-1}{\kappa^{-1}-1} \right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}}.$$

Proof. We denote elements of $\mathcal{F}_\kappa$ as $f_1, \ldots, f_{n+1}$. For $\eta \in (\mathbb{R}^n)^{\mathcal{F}_\kappa}$ we also denote $\eta_l = \eta_{f_l}$, where $l \in \mathbb{N}_{n+1}$. Then

$$\mathcal{N}_{\mathcal{F}_\kappa} = \{\eta \in (\mathbb{C}^n)^{\mathcal{F}_\kappa} : \eta_{k,k} + \kappa \eta_{n+1,k} = 1,\ k \in \mathbb{N}_n\}.$$

For each $i \in \mathbb{N}_n$ consider function

$$d_i : \mathbb{N}_{n+1} \to \mathbb{N}_n,\ l \mapsto \begin{cases} l & \text{if } l \neq n+1, \\ i & \text{otherwise.} \end{cases}$$

Then from proposition 2.8 for any $\eta \in (\mathbb{C}^n)^{\mathcal{F}_\kappa}$ we would get

$$\begin{aligned} \|\psi_\eta\| &\geq \max \left\{ \left(\sum_{k=1}^{n} \left| \sum_{l=1}^{n+1} |\eta_{l,k}| \delta_k^{d_i(l)} \right|^p \right)^{1/p} : i \in \mathbb{N}_n \right\} \\ &= \max \left\{ \left(\sum_{k=1, k\neq i}^{n} |\eta_{k,k}|^p + (|\eta_{i,i}| + |\eta_{n+1,i}|)^p \right)^{1/p} : i \in \mathbb{N}_n \right\}. \end{aligned}$$

For any $\eta \in \mathcal{N}_{\mathcal{F}_\kappa}$ and $k \in \mathbb{N}_n$ we have $\eta_{k,k} + \kappa \eta_{n+1,k} = 1$. Therefore,

$$\begin{aligned} \|\psi_\eta\| &\geq \max \left\{ \left(\sum_{k=1, k\neq i}^{n} |\eta_{k,k}|^p + (|\eta_{i,i}| + |\kappa^{-1}(1 - \eta_{i,i})|)^p \right)^{1/p} : i \in \mathbb{N}_n \right\} \\ &\geq \max \left\{ \left(\sum_{k=1, k\neq i}^{n} |\eta_{k,k}|^p + (|\eta_{i,i}| + \kappa^{-1}|1 - |\eta_{i,i}||)^p \right)^{1/p} : i \in \mathbb{N}_n \right\} \end{aligned}$$

for any $\eta \in \mathcal{N}_{\mathcal{F}_\kappa}$. Denote

$$\begin{aligned} \alpha_i &: \mathbb{R}^n \to \mathbb{R},\ t \mapsto \left(\sum_{k=1, k\neq i}^{n} |t_k|^p + (|t_i| + \kappa^{-1}|1-|t_i||)^p \right)^{1/p} \qquad \text{for} \quad i \in \mathbb{N}_n \\ \alpha &: \mathbb{R}^n \to \mathbb{R},\ t \mapsto \max\{\alpha_i(t) : i \in \mathbb{N}_n\}. \end{aligned}$$

Then

$$\nu_{\mathcal{F}_\kappa} \geq \inf\{\|\psi_\eta\| : \eta \in \mathcal{N}_{\mathcal{F}_\kappa}\} = \inf\{\alpha(|\eta_{1,1}|, \dots, |\eta_{n,n}|) : \eta \in \mathcal{N}_{\mathcal{F}_\kappa}\} = \inf\{\alpha(t) : t \in \mathbb{R}_+^n\}.$$

Consider functions

$$\begin{aligned} &F_1 : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto t \\ &F_2 : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto t + \kappa^{-1}|1-t| \\ &F_3 : \mathbb{R}^n \to \mathbb{R},\ t \mapsto \left(\sum_{k=1}^{n} |t_k|^p \right)^{1/p} \end{aligned}$$

Clearly, for each $i \in \mathbb{N}_n$ the function α_i is a composition of F_1, F_2 and F_3. Since F_1 and F_2 are convex and F_3 is strictly convex, then all functions $(\alpha_i)_{i\in\mathbb{N}_n}$ are strictly convex on $\mathbb{R}_+^n$. Hence, so is their maximum α. Note that α is continuous, strictly convex and $\lim\limits_{\|t\|\to+\infty} \alpha(t) = +\infty$. Therefore, α has a unique global minimum at some point $t_0 \in \mathbb{R}_+^n$. Observe, that α is invariant under permutation of its arguments. Then from the uniqueness of the global minimum at t_0 we can conclude that all coordinates of t_0 are equal. As a corollary

$$\begin{aligned} \nu_{\mathcal{F}_\kappa} &\geq \inf\{\alpha(t) : t \in \mathbb{R}_+^n\} \\ &= \inf\{\alpha(s, \dots, s) : s \in \mathbb{R}_+\} \\ &= \inf\{((n-1)s^p + (s + \kappa^{-1}|1-s|)^p)^{1/p} : s \in \mathbb{R}_+\}. \end{aligned}$$

Consider function

$$F : \mathbb{R}_+ \to \mathbb{R},\ s \mapsto ((n-1)s^p + (s + \kappa^{-1}|1-s|)^p)^{1/p}$$

We have

$$F(s) = \begin{cases} ((s + \kappa^{-1}(1-s)) + (n-1)s^p)^{1/p} & \text{if} \quad 0 \leq s \leq 1, \\ \left((\kappa^{-1}+1)^p \left(s - \frac{\kappa^{-1}}{\kappa^{-1}+1} \right) + (n-1)s^p \right)^{1/p} & \text{if} \quad s > 1. \end{cases}$$

Since F is continuous and clearly increasing on $(1, +\infty)$, then F attains its minimum on $[0, 1]$. Let us find stationary points of F on $[0, 1]$. For $s \in [0, 1]$ we have

$$F'(s) = ((s + \kappa^{-1}(1-s)) + (n-1)s^p)^{1/p-1}((\kappa^{-1} + (1-\kappa^{-1})s)^{p-1} + (n-1)s^{p-1}).$$

The stationary point can be found from the equation $F'(s) = 0$. The solution is

$$s_0 = \frac{\kappa^{-1}}{\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}}.$$

By assumption $n^{-1} < \kappa$, so $\frac{n-1}{\kappa^{-1}-1} < 1$ and therefore $0 < s_0 < 1$. Since F is convex, then s_0 is the point of minimum on $[0, 1]$. The minimum equals

$$F(s_0) = \kappa^{-1} \frac{\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{p}{p-1}} - 1 + \kappa^{-1} \right)^{1/p}}{\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}}.$$

This gives the desired lower bound for $\nu_{\mathcal{F}_\kappa}$. $\square$

Proposition 2.13. *Let $n \in \mathbb{N}$, $r > 1$ and $x \in \mathbb{C}^n$. Then*

$$\|x\|_p \leq \|x\|_1^{1/r} \|x\|_\infty^{1-1/r}.$$

The equality holds iff $|x_1| = \ldots = |x_n|$.

Proof. For any $r > 1$ and any $x \in \mathbb{C}^n$ we have

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^r\right)^{1/r} = \left(\sum_{k=1}^{n} |x_k||x_k|^{r-1}\right)^{1/r} \leq \left(\sum_{k=1}^{n} |x_k|\right)^{1/r} \left(\max_{k\in\mathbb{N}_n} |x_k|^{r-1}\right)^{1/r} = \|x\|_1^{1/r}\|x\|_\infty^{1-1/r}.$$

Clearly, the equality holds iff $|x_k| = \max\limits_{k\in\mathbb{N}_n} |x_k|$ for all $k \in \mathbb{N}_n$. □

Proposition 2.14. *Let $n \in \mathbb{N}$, $n > 1$ and $n^{-1} < \kappa < (n-1)^{-1}$, then $\nu_{\mathcal{F}_\kappa} > \gamma_{\mathcal{F}_\kappa}$.*

Proof. Using results of propositions 2.11 and 2.12 it is enough to show that

$$\kappa^{-1} \frac{\left(\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{p}{p-1}} - 1 + \kappa^{-1}\right)^{1/p}}{\left(\frac{n-1}{\kappa^{-1}-1}\right)^{\frac{1}{p-1}} - 1 + \kappa^{-1}} > (n - 1 + (\kappa^{-1} - (n-1))^p)^{1/p}.$$

Let us make a substitution $m = n - 1$ and $\rho = \kappa^{-1}$. Then $m \in \mathbb{N}$ and $m < \rho < m + 1$. Then the last inequality is equivalent to

$$\rho \frac{\left(\left(\frac{m}{\rho-1}\right)^{\frac{p}{p-1}} - 1 + \rho\right)^{1/p}}{\left(\frac{m}{\rho-1}\right)^{\frac{1}{p-1}} - 1 + \rho} > (m + (\rho - m)^p)^{1/p}.$$

After simplifications, we arrive at the inequality

$$\frac{m\rho}{\rho - 1} > (m + (\rho - m)^p)^{1/p} \left(m + \left(\frac{m}{\rho - 1}\right)^{p^*}\right)^{1/p^*}.$$

To prove this inequality we apply proposition 2.13 to the vector $x = (1, \ldots, 1, \rho - m)^T \in \mathbb{C}^{m+1}$ with $r = p$ and to the vector $x = (1, \ldots, 1, \frac{m}{\rho-1}) \in \mathbb{C}^{m+1}$ with $r = p^*$. Since $m < \rho < m + 1$, then the components of these vectors are not all equal, so the inequalities are strict:

$$(m + (\rho - m)^p)^{1/p} < (m + (\rho - m))^{1/p} 1^{1-1/p}$$

$$\left(m + \left(\frac{m}{\rho - 1}\right)^{p^*}\right)^{1/p^*} < \left(m + \frac{m}{\rho - 1}\right)^{1/p^*} \left(\frac{m}{\rho - 1}\right)^{1-1/p^*}.$$

By multiplying these inequalities we get the desired result. □

Proposition 2.15. *Let $n \in \mathbb{N}$, $n > 1$, $1 < p < +\infty$ and $n^{-1} < \kappa < (n-1)^{-1}$. Then for any ℓ_∞^n-morphism ψ which is a left inverse to $\xi_{\mathcal{F}_\kappa}$ holds $\|\psi\| > \gamma_{\mathcal{F}_\kappa}$.*

Proof. By proposition 2.4 there exists a family of vectors $\eta \in (\ell_\infty^n)^{\mathcal{F}_\kappa}$ such that $\psi = \psi_\eta$. From the definition 2.9 we have $\|\psi_\eta\| \geq \nu_{\mathcal{F}_\kappa}$. Now from proposition 2.14 we get $\|\psi\| \geq \nu_{\mathcal{F}_\kappa} > \gamma_{\mathcal{F}_\kappa}$. □

Proposition 2.16. *Let $n \in \mathbb{N}$, then the right ℓ_∞^n-module ℓ_p^n is metrically injective iff $n = 1$.*

Proof. Suppose that the right ℓ_∞^n-module ℓ_p^n is metrically injective. Assume that $n > 1$. Pick any real number $\kappa \in (n^{-1}, (n-1)^{-1})$. By proposition 2.11 we have $\gamma_{\mathcal{F}_\kappa}$ is finite, therefore $\xi_{\mathcal{F}_\kappa}$ is $\gamma_{\mathcal{F}_\kappa}$-topologically injective. From assumption, it follows that there exists an ℓ_∞^n-morphism ψ which is a left inverse to $\xi_{\mathcal{F}_\kappa}$ such that $\|\psi\| \leq \gamma_{\mathcal{F}_\kappa}$. This contradicts proposition 2.16, therefore $n = 1$.

Now assume that $n = 1$. Then ℓ_p^n is isometrically isomorphic to $(\ell_\infty^n)^*$ as an ℓ_∞^n-module, hence it is metrically injective. $\square$

3 Measure theory preliminaries

In this section we set the stage for the main theorem. Even though it is stated for Borel measures on locally compact spaces we shall prove propositions of this section for general measure spaces. A comprehensive study of general measure spaces can be found in [8].

Let Ω be a set. By measure we mean a countably additive set function with values in $[0, +\infty]$ defined on a σ-algebra Σ of measurable subsets of a set Ω. A measurable set E is called an atom if $\mu(A) > 0$ and for every measurable subset $B \subset A$ holds either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. A measure μ is called purely atomic if every measurable set of positive measure has an atom. A measure μ is semi-finite if for any measurable set A of infinite measure there exists a measurable subset of A with finite positive measure. A family $\mathcal{D}$ of measurable subsets of finite measure is called a decomposition of Ω if for any measurable set E $\mu(E) = \sum_{D \in \mathcal{D}} \mu(E \cap D)$ and a set F is measurable whenever $F \cap D$ is measurable for all $D \in \mathcal{D}$. Finally, a measure μ is called decomposable if it is semi-finite and admits a decomposition of Ω. Most measures encountered in functional analysis are decomposable. A pair (Ω, μ) is called a measure space.

We shall define a few Banach spaces constructed from measure spaces. Let (Ω, μ) be a measure space. By $B(\Sigma)$ we denote the algebra of bounded measurable functions with the sup norm. For $1 \leq p \leq +\infty$ by $L_p(\Omega, \mu)$ we denote the Banach space of equivalence classes of p-integrable (or essentially bounded if $p = +\infty$) functions on Ω. Elements of $L_p(\Omega, \mu)$ are denoted by $[f]$.

Definition 3.1. *Let (Ω, μ) be a measure space, E be a measurable set of finite positive measure and $f : \Omega \to \mathbb{C}$ be a measurable function. For any real $r \neq 0$ we define a linear map*

$$m_{E,r}(f) = \mu(E)^{\frac{1}{r}-1} \int_E f(\omega) d\mu(\omega).$$

Proposition 3.2. *Let E be a finite measure subset of a measure space (Ω, μ) and $r \neq 0$. Then for any measurable functions $f : \Omega \to \mathbb{C}$, $g : \Omega \to \mathbb{C}$ holds*

(i) $m_{E,r}(\chi_E) = \mu(E)^{1/r}$;

(ii) $m_{E,r}(f) = m_{E,r}(f\chi_E)$;

(iii) If E is an atom, then $f = m_{E,\infty}(f)$ almost everywhere on E;

(iv) If E is an atom, then $m_{E,\infty}(f) m_{E,\infty}(g) = m_{E,\infty}(f \cdot g)$.

(v) If E is an atom and $s \neq 0$, then $m_{E,r}(f) m_{E,s}(g) = m_{E,\frac{rs}{r+s}}(f \cdot g)$.

Proof. Paragraphs (i) and (ii) are obvious.

(iii) Without loss of generality we assume, that f is real-valued. Denote $k = m_{E,\infty}(f)$. Clearly, k is a mean value of f on E. Consider set $A_+ = \{\omega \in E : f(\omega) > k\}$. Since A_+ is a subset of an atom E of finite measure, then either $\mu(A_+) = 0$ or $\mu(A_+) = \mu(E) > 0$. In the latter case we get

$$\int_E f(\omega)d\mu(\omega) = \int_{A_+} f(\omega)d\mu(\omega) > k\mu(A_+) = k\mu(E) = \int_E f(\omega)d\mu(\omega).$$

Contradiction, so $\mu(A_+) = 0$. Similarly, one can show that the set $A_- = \{\omega \in E : f(\omega) < k\}$ also has measure zero. Thus, $f = k$ almost everywhere on E.

Paragraph (v) immediately follows from (iv), which in turn an easy consequence of (iii). □

Proposition 3.3. *Let (Ω, μ) be a purely atomic measure space. Let $\mathcal{A}$ be a decomposition of Ω into atoms of finite measure. Then for any $1 \leq p < +\infty$ the linear maps*

$$I_p : L_p(\Omega, \mu) \to \ell_p(\mathcal{A}),\ [f] \mapsto \sum_{A \in \mathcal{A}} m_{A,p}(f)e_A,$$

$$J_p : \ell_p(\mathcal{A}) \to L_p(\Omega, \mu),\ \omega \mapsto \sum_{A \in \mathcal{A}} \omega_A m_{A,-p}(\chi_A)[\chi_A]$$

are isometric isomorphisms which are inverse to each other.

Proof. Since $\mathcal{A}$ is a decomposition of X into atoms, then

$$[f] = \sum_{A \in \mathcal{A}} m_{A,\infty}(f)[\chi_A]$$

for any $f \in L_p(\Omega, \mu)$. Using proposition 3.2 for each $f \in L_p(\Omega, \mu)$ and $A \in \mathcal{A}$ we get

$$\int_A |f(\omega)|^p d\mu(\omega) = \int_A |m_{A,\infty}(f)|^p\, d\mu(\omega) = \mu(A)\, |m_{A,\infty}(f)|^p = |m_{A,p}(\chi_A)m_{A,\infty}(f)|^p = |m_{A,p}(f)|^p.$$

So for any $f \in L_p(\Omega, \mu)$ we have

$$\|I_p([f])\| = \left(\sum_{A \in \mathcal{A}} |m_{A,p}(f)|^p\right)^{1/p} = \left(\sum_{A \in \mathcal{A}} \int_A |f(\omega)|^p d\mu(\omega)\right)^{1/p} = \left(\int_\Omega |f(\omega)|^p d\mu(\omega)\right)^{1/p} = \|[f]\|.$$

Again, using proposition 3.2 for each $A \in \mathcal{A}$ we get

$$|m_{A,-p}(\chi_A)|^p \int_\Omega |\chi_A(\omega)|^p d\mu(\omega) = (\mu(A)^{-1/p})^p \mu(A) = 1.$$

So for any $x \in \ell_p(\mathcal{A})$ we have

$$\|J_p(x)\| = \left(\int_\Omega \sum_{A \in \mathcal{A}} |x_A|^p |m_{A,-p}(\chi_A)|^p |\chi_A(\omega)|^p d\mu(\omega)\right)^{1/p} = \left(\sum_{A \in \mathcal{A}} |x_A|^p\right)^{1/p} = \|x\|.$$

Thus, I_p and J_p are isometric maps. Obviously these maps are linear. Note that, any $x \in \ell_p(\mathcal{A})$ and $A \in \mathcal{A}$ holds

$$m_{A,p}(J_p(x)) = m_{A,p}(J_p(x)\chi_A) = m_{A,p}(x_A m_{A,-p}[\chi_A]) = x_A m_{A,-p}(\chi_A) m_{A,p}(\chi_A) = x_A.$$

Hence, for any $x \in \ell_p(\mathcal{A})$ we have

$$I_p(J_p(x)) = \sum_{A \in \mathcal{A}} m_{A,p}(J_p(x)) e_A = \sum_{A \in \mathcal{A}} x_A e_A = x.$$

Note that for any $A \in \mathcal{A}$ holds

$$J_p(e_A) = \sum_{A' \in \mathcal{A}} (e_A)_{A'} m_{A',-p}(\chi_{A'})[\chi_{A'}] = \sum_{A' \in \mathcal{A}} \delta_A^{A'} m_{A',-p}(\chi_{A'})[\chi_{A'}] = m_{A,-p}(\chi_A)[\chi_A].$$

Therefore, for any $f \in L_p(\Omega, \mu)$ we have

$$J_p(I_p(f)) = \sum_{A \in \mathcal{A}} m_{A,p}(f) J_p(e_A) = \sum_{A \in \mathcal{A}} m_{A,p}(f) m_{A,-p}(\chi_A)[\chi_A] = \sum_{A \in \mathcal{A}} m_{A,\infty}(f)[\chi_A] = [f].$$

Thus, J_p and I_p are inverse to each other. $\square$

Proposition 3.4. *Let (Ω, μ) be a purely atomic measure space and $\mathcal{A}$ be a decomposition of Ω into atoms of finite measure. Suppose $1 \leq p, q < +\infty$, then*

(i) If $\Phi : L_p(\Omega, \mu) \to L_q(\Omega, \mu)$ is a $B(\Sigma)$-morphism the map $I_q \circ \Phi \circ J_p$ is an $\ell_\infty(\mathcal{A})$-morphism of the same norm;

(ii) If $\phi : \ell_p(\mathcal{A}) \to \ell_q(\mathcal{A})$ is an $\ell_\infty(\mathcal{A})$-morphism the map $J_q \circ \phi \circ I_p$ is a $B(\Sigma)$-morphism of the same norm;

Proof. (i) Denote $\phi = I_q \circ \Phi \circ J_p$, then by proposition 3.2 for any atom $A \in \mathcal{A}$ holds

$$\phi(e_A) = I_q(\Phi(J_p(e_A))) = I_q(\Phi(m_{A,-p}(\chi_A)[\chi_A])) = m_{A,-p}(\chi_A) I_q(\Phi([\chi_A])).$$

Since A is an atom, then we get

$$\Phi([\chi_A]) = \Phi([\chi_A] \cdot \chi_A) = \Phi([\chi_A]) \cdot \chi_A = m_{A,\infty}(\Phi([\chi_A]))[\chi_A],$$

so

$$\begin{aligned}\phi(e_A) &= m_{A,-p}(\chi_A) I_q(m_{A,\infty}(\Phi([\chi_A]))[\chi_A]) \\ &= m_{A,\infty}(\Phi([\chi_A])) m_{A,-p}(\chi_A) I_q([\chi_A]) \\ &= m_{A,\infty}(\Phi([\chi_A])) m_{A,-p}(\chi_A) m_{A,p}(\chi_A) e_A \\ &= m_{A,\infty}(\Phi([\chi_A])) e_A.\end{aligned}$$

Now for any $x \in \ell_p(\mathcal{A})$ and $a \in \ell_\infty(A)$ we have

$$\phi(x \cdot a) = \sum_{A \in \mathcal{A}} x_A a_A \phi(e_A) = \sum_{A \in \mathcal{A}} x_A a_A m_{A,\infty}(\Phi([\chi_A])) e_A = \sum_{A \in \mathcal{A}} (x_A \phi(e_A)) \cdot a = \phi(x) \cdot a.$$

Therefore, ϕ is an $\ell_\infty(\mathcal{A})$-morphism. By proposition 3.3 the maps I_q and J_p are isometric isomorphisms, hence ϕ and Φ have the same norm.

(ii) Denote $\Phi = J_q \circ \phi \circ I_p$, then by proposition 3.2 for any atom $A \in \mathcal{A}$ holds

$$\Phi([\chi_A]) = J_q(\phi(I_p([\chi_A]))) = J_q(\phi(m_{A,p}(\chi_A) e_A)) = m_{A,p}(\chi_A) J_q(\phi(e_A)).$$

Moreover,

$$\phi(e_A) = \phi(e_A \cdot e_A) = \phi(e_A) \cdot e_A = \phi(e_A)_A e_A,$$

so

$$\begin{aligned}
\Phi([\chi_A]) &= m_{A,p}(\chi_A) J_q(\phi(e_A)_A e_A) \\
&= \phi(e_A)_A m_{A,p}(\chi_A) J_q(e_A) \\
&= \phi(e_A)_A m_{A,p}(\chi_A) m_{A,-p}(\chi_A)[\chi_A] \\
&= \phi(e_A)_A [\chi_A].
\end{aligned}$$

Now for any $f \in L_p(\Omega, \mu)$ and $a \in B(\Sigma)$ we have

$$\begin{aligned}
\Phi([f] \cdot a) &= \Phi\left(\sum_{A \in \mathcal{A}} m_{A,\infty}(f \cdot a)[\chi_A]\right) \\
&= \sum_{A \in \mathcal{A}} m_{A,\infty}(f \cdot a)\Phi([\chi_A]) \\
&= \sum_{A \in \mathcal{A}} m_{A,\infty}(f) m_{A,\infty}(a) \phi(e_A)_A [\chi_A] \\
&= \sum_{A \in \mathcal{A}} (m_{A,\infty}(f)\phi(e_A)_A[\chi_A]) \cdot a \\
&= \sum_{A \in \mathcal{A}} m_{A,\infty}(f)\Phi([\chi_A]) \cdot a \\
&= \Phi\left(\sum_{A \in \mathcal{A}} m_{A,\infty}(f)[\chi_A]\right) \cdot a \\
&= \Phi([f]) \cdot a.
\end{aligned}$$

Therefore, Φ is a $B(\Sigma)$-morphism. By proposition 3.3 the maps J_q and I_p are isometric isomorphisms, hence Φ and ϕ have the same norm. □

Proposition 3.5. *Let (Ω, μ) be a decomposable measure space and $1 \leq p \leq +\infty$. If $L_p(\Omega, \mu)$ is finite-dimensional, then (Ω, μ) is purely atomic with finitely many atoms of finite measure.*

Proof. Suppose (Ω, μ) is not purely atomic, then there exists an atomless measurable set $E \subset \Omega$. By [[8], proposition 215D] we may assume that E has a positive finite measure. By [[8], exercise 215X(e)] there is a countable family $(E_n)_{n \in \mathbb{N}}$ of disjoint sets of positive finite measures. In this case $(\chi_{E_n})_{n \in \mathbb{N}}$ is a countable linearly independent set in $L_p(\Omega, \mu)$, hence $L_p(\Omega, \mu)$ is infinite-dimensional. Contradiction, so (Ω, μ) is purely atomic. Let $\mathcal{A}$ be a family of atoms whose union is Ω. Since Ω is decomposable, then all these atoms have a finite measure. Therefore, $(\chi_A)_{A \in \mathcal{A}}$ is a linearly independent set. Since $L_p(\Omega, \mu)$ is finite-dimensional, then $\mathcal{A}$ is finite. □

4 Metric projectivity, injectivity and flatness of $C_0(S)$-modules $L_p(S, \mu)$

Let S be a locally compact Hausdorff space. By $\mathrm{Bor}(S)$ we denote the σ-algebra generated by open subsets of S. In this section we shall consider only decomposable Borel measures. Let μ be such measure on S. We shall show that for $1 < p < +\infty$ Banach $C_0(S)$-modules $L_p(S, \mu)$ are almost never metrically projective, injective or flat.

Proposition 4.1. *Let S be a locally compact Hausdorff space and μ be a decomposable Borel measure on S. Then any $C_0(S)$-morphism between right reflexive modules is a $B(\mathrm{Bor}(S))$-morphism.*

Proof. Suppose Z is a right $C_0(S)$-module, then Z^{**} is a right $C_0(S)^{**}$-module [[9], proposition 2.6.15(iii)]. If Z is reflexive, then the natural embedding $\iota_Z : Z \to Z^{**}$ is an isometric isomorphism. Recall that $B(\mathrm{Bor}(S))$ is a subalgebra of $C_0(S)^{**}$ [[9], proposition 4.2.30], therefore we can endow Z with the structure of $B(\mathrm{Bor}(S))$-module via formula $z \cdot b = \iota_Z^{-1}(\iota_Z(z) \cdot b)$ for $z \in Z$ and $b \in B(\mathrm{Bor}(S))$.

Let $\phi : X \to Y$ be a morphism of right reflexive $C_0(S)$-modules. Then ϕ^{**} is a $C_0(S)^{**}$-morphism [[9], proposition A.3.53]. As we have noted above, X and Y are $B(\mathrm{Bor}(S))$-modules and ι_X, ι_Y — are isometric isomorphisms. Since $\phi = \iota_Y^{-1} \circ \phi^{**} \circ \iota_X$, then ϕ is a $B(\mathrm{Bor}(S))$-morphism. □

Proposition 4.2. *Let S be a locally compact Hausdorff space and μ be a purely atomic Borel measure on S with finitely many atoms of finite measure. Suppose $1 < p < +\infty$ and the $C_0(S)$-module $L_p(S, \mu)$ is metrically injective, then μ has at most 1 atom.*

Proof. Let $\mathcal{A}$ be a decomposition of S into n atoms of finite measure. Suppose $n > 1$, then pick any $\kappa \in (n^{-1}, (n-1)^{-1})$. By identifying $\mathcal{A}$ and $\mathbb{N}_n$ we set $\mathcal{F} = \mathcal{F}_\kappa$. For each $f \in \mathcal{F}$ we define an $\ell_\infty(\mathcal{A})$-morphism

$$m_f : \ell_p(\mathcal{A}) \to \ell_1(\mathcal{A}),\ x \mapsto x \cdot f.$$

We shall also use a natural embedding and a natural projection

$$\mathrm{in}_f : L_1(S, \mu) \to \bigoplus_\infty \{L_1(S, \mu) : f' \in \mathcal{F}\}, \qquad \mathrm{pr}_f : \bigoplus_\infty \{\ell_1(\mathcal{A}) : f' \in \mathcal{F}\} \to \ell_1(\mathcal{A}),$$

which are $B(\mathrm{Bor}(S))$-morphism and $\ell_\infty(A)$-morphism respectively. Now consider $B(\mathrm{Bor}(S))$-morphisms

$$I_p^\infty = \bigoplus_\infty \{I_p : f \in \mathcal{F}\}, \qquad J_p^\infty = \bigoplus_\infty \{J_p : f \in \mathcal{F}\}.$$

These are isometric isomorphisms and one can easily check that

$$I_p^\infty \circ \mathrm{in}_f = \mathrm{in}_f \circ I_p, \qquad \mathrm{pr}_f \circ I_p^\infty = I_p \circ \mathrm{pr}_f.$$

By proposition 3.4 the map $\Xi_f = J_1 \circ m_f \circ I_p$ is a $B(\mathrm{Bor}(S))$-morphism. Hence, the map

$$\Xi_\mathcal{F} = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ \Xi_f$$

is a $B(\mathrm{Bor}(S))$-morphism and a fortiori a $C_0(S)$-morphism. Note that

$$I_1^\infty \circ \Xi_\mathcal{F} \circ J_p = \sum_{f \in \mathcal{F}} I_1^\infty \circ \mathrm{in}_f \circ \Xi_f \circ J_p = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ I_1 \circ J_1 \circ m_f \circ I_p \circ J_p = \sum_{f \in \mathcal{F}} \mathrm{in}_f \circ m_f = \xi_\mathcal{F}.$$

By proposition 2.11 the coercivity constant $\gamma_\mathcal{F}$ is finite and positive, so $\xi_\mathcal{F}$ is $\gamma_\mathcal{F}$-topologically injective. As I_1 and J_p are isometric isomorphisms the map $\Xi_\mathcal{F}$ is also $\gamma_\mathcal{F}$-topologically injective. By assumption, the $C_0(S)$-module $L_p(S, \mu)$ is metrically injective, hence there exists a $C_0(S)$-morphism

$$\Psi : \bigoplus_\infty \{L_1(S, \mu) : f \in \mathcal{F}\} \to L_p(S, \mu)$$

such that $\Psi \circ \Xi_\mathcal{F} = 1_{L_p(S,\mu)}$ and $\|\Psi\| \leq \gamma_\mathcal{F}$.

Again, for each $f \in \mathcal{F}$ we define a $C_0(S)$-morphism $\Psi_f = \Psi \circ \mathrm{in}_f$. Since $\mathcal{A}$ is finite, then $L_p(S,\mu)$ and $L_1(S,\mu)$ are finite-dimensional and therefore reflexive. Hence, by proposition 4.1 for any $f \in \mathcal{F}$ the map Ψ_f is a $B(\mathrm{Bor}(S))$-morphism. Note that

$$\Psi = \sum_{f\in\mathcal{F}} \Psi_f \circ \mathrm{pr}_f \,.$$

Now we define a bounded linear operator

$$\psi = I_p \circ \Psi \circ J_1^\infty = I_p \circ \left(\sum_{f\in\mathcal{F}} \Psi_f \circ \mathrm{pr}_f \right) \circ J_1^\infty = \sum_{f\in\mathcal{F}} I_p \circ \Psi_f \circ \mathrm{pr}_f \circ J_1^\infty = \sum_{f\in\mathcal{F}} I_p \circ \Psi_f \circ J_1 \circ \mathrm{pr}_f \,.$$

By proposition 3.4 for each $f \in \mathcal{F}$ the map $I_p \circ \Psi_f \circ J_1$ is an $\ell_\infty(\mathcal{A})$-morphism. Therefore, ψ is an $\ell_\infty(\mathcal{A})$-morphism too. As I_p and J_1^∞ are isometric isomorphisms we have $\|\psi\| = \|\Psi\|$. Moreover,

$$\psi \circ \xi_{\mathcal{F}} = I_p \circ \Psi \circ J_1^\infty \circ I_1^\infty \circ \Xi_{\mathcal{F}} \circ J_p = I_p \circ \Psi \circ \Xi_{\mathcal{F}} \circ J_p = I_p \circ J_p = 1_{\ell_p(\mathcal{A})}.$$

Thus, we have constructed an $\ell_\infty(\mathcal{A})$-morphism ψ such that $\psi \circ \xi_{\mathcal{F}} = 1_{\ell_p(\mathcal{A})}$ and $\|\psi\| \leq \gamma_{\mathcal{F}}$. Since we assumed that $n > 1$ we arrive at contradiction with proposition 2.15. Hence, $n \leq 1$, i.e. $\mathcal{A}$ has at most one atom. □

Proposition 4.3. *Let S be a locally compact Hausdorff space and μ be a decomposable Borel measure on S. Suppose $1 < p < +\infty$ and the $C_0(S)$-module $L_p(S,\mu)$ is metrically injective, then μ has at most 1 atom.*

Proof. Let K be the Alexandroff's compactification of S, then $C_0(S)$ is complemented in $C(K)$. By [[10], lemma 4.4] the space $C(K)$ is an $\mathscr{L}_\infty^g$-space, hence so is $C_0(S)$ as its complemented subspace. [[10], corollary 23.1.2(1)]. Since $L_p(S,\mu)$ is reflexive [[8], theorem 244K], then by [[11], corollary 3.14] this $C_0(S)$-module must be finite-dimensional. Now from proposition 3.5 we get that μ is purely atomic with finitely many atoms of finite measure. Finally, from proposition 4.2 we conclude that μ has at most one atom. □

Theorem 4.4. *Let S be a locally compact Hausdorff space, μ be a decomposable Borel measure on S. Suppose $1 < p < +\infty$ and the $C_0(S)$-module $L_p(S,\mu)$ is metrically projective, injective or flat then μ has at most 1 atom.*

Proof. If $L_p(S,\mu)$ is a metrically injective $C_0(S)$-module, then the result follows from proposition 4.2.

Suppose $L_p(S,\mu)$ is a metrically flat $C_0(S)$-module. Then by [[11], proposition 2.21] the dual $C_0(S)$-module $L_p(S,\mu)^*$ is metrically injective. It remains to note that $C_0(S)$-modules $L_p(S,\mu)^*$ and $L_{p^*}(S,\mu)$ are isometrically isomorphic and that $1 < p^* < +\infty$ for $1 < p < +\infty$. Now the result follows from the previous paragraph.

If $L_p(S,\mu)$ is a metrically projective $C_0(S)$-module, then by [[11], proposition 2.26] it is metrically flat. Now the result follows from the previous paragraph. □

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