1 Question

We know that

$$\tau(x,t,y,s) \triangleq \frac{1}{\sqrt{2\pi(s-t)}} \exp(-\frac{1}{2}\frac{(y-x)^2}{s-t})$$

y, s are fixed parameters, f is any arbitrary continuous and bounded function

prove that

$$\lim_{t\to s}[\int_R \tau(x,t,y,s)f(x)\,dx]$$

is finite

[note: using change of variables]

2 Solution

let $a = s - t$ Therefore,

$$\lim_{t\to s}[\int_R \tau(x,t,y,s)f(x)\,dx] = \lim_{a\to 0}[\int_R \frac{1}{\sqrt{2\pi}\sqrt{a}} \exp(-\frac{1}{2}(\frac{y-x}{\sqrt{a}})^2 f(x)\,dx]$$

then we let

$$z = \frac{x-y}{\sqrt{a}}$$

therefore.

$$dz = \frac{dx}{\sqrt{a}}$$

So we replace z with x and y, we get

$$\lim_{a\to 0}[\int_R \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)f(y+\sqrt{a}z)\,dz]$$

From the integral of τ function, when $a \to 0$, we can get the $f(y+\sqrt{a}z)$ would approach $f(y)$ which is a constant number. We let c = f(y) Therefore,

$$\lim_{a\to 0}[\int_R \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)f(y+\sqrt{a}z)\,dz] = c\int_R \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)\,dz$$

From this final formula, we can get the normal distribution integral which should be 1 when $x \in (-\infty, +\infty)$.

Therefore, $\lim_{t\to s}[\int_R \tau(x,t,y,s)f(x)\,dx]$ is finite and approaches to f(y) when t approaches s.