1 Induction

By Delta Direc funtion, we get

$$\delta_y^\sigma(x) \triangleq \frac{1}{\sqrt{2\pi\sigma}} exp(-\frac{1}{2}\frac{x^2}{\sigma^2})$$

Here if $\sigma = s - t$, then $\delta_y^\sigma(x) = \tau(x,t,y,s)$ From the previous induction, we get

$$z = \frac{y-x}{\sigma}$$

$$\lim_{\sigma\to 0}\int_R f(z)\delta_y^\sigma(z)dz = f(y)$$

Then we induce the backward and forward equation.

Backward Equation:

$$\begin{cases} \frac{\partial\tau}{\partial t} + \frac{1}{2}\frac{\partial^2\tau}{\partial x^2} = 0 & t < s \\ \lim_{t\to s}\tau(x,t,y,s) = \delta_y(x) & \text{the right terminal condition} \end{cases}$$

Forward Equation:

$$\begin{cases} \frac{\partial\tau}{\partial s} - \frac{1}{2}\frac{\partial^2\tau}{\partial y^2} = 0 & t < s \\ \lim_{s\to t}\tau(x,t,y,s) = \delta_x(y) & \text{the right initial condition} \end{cases}$$

Now if we have the backward equation as:

$$\frac{\partial\tau}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2\tau}{\partial x^2} = 0$$

We should insert σ as one parameter inside τ original function.

$$\tau(x,t,y,s) \triangleq \frac{1}{\sqrt{2\pi(s-t)}}\exp\left(-\frac{1}{2}\frac{(y-x)^2}{s-t}\right)$$

Here the final conclusion is

$$\tau(x,t,y,s) \triangleq \frac{1}{\sigma\sqrt{2\pi(s-t)}}\exp\left(-\frac{1}{2}\frac{(y-x)^2}{(s-t)\sigma^2}\right)$$

but I don't really get how can we induce it from the current backward equation and τ formula without σ parameter

After we get this the τ with σ parameter, we should deduce

$$\begin{aligned} \lim_{\sigma\to 0} -\sigma^2 log\tau &= \lim_{\sigma\to 0}[-\sigma^2(-\frac{1}{2}log(\sigma^2 2\pi(s-t)) - \frac{1}{2}\frac{(y-x)^2}{(s-t)\sigma^2})] \\ &= \lim_{\sigma\to 0}(\frac{\sigma^2}{2}log(\sigma^2 2\pi(s-t)) + \frac{1}{2}\frac{(y-x)^2}{s-t}) \end{aligned}$$

Here we notice that when $\sigma \to 0$, $\sigma^2 2\pi(s-t)$ goes to 0, and $log(\sigma^2 2\pi(s-t))$ goes to $-\infty$ while $\frac{\sigma^2}{2}$ goes to 0, since the speed to approach the limit of $\frac{\sigma^2}{2}$ is larger than $log(\sigma^2 2\pi(s-t))$, so $\lim_{\sigma\to 0}(\frac{\sigma^2}{2}log(\sigma^2 2\pi(s-t)))$ goes to 0, and the latter part $\frac{1}{2}\frac{(y-x)^2}{s-t}$ is constant.

Therefore,

$$\lim_{\sigma\to 0} -\sigma^2 log\tau = \frac{1}{2}\frac{(y-x)^2}{s-t}$$