dynamic system & literature review

yz6201

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1 Question

$$u_1(t) = e^{-t}cos(t)$$

$$u_2(t) = \mathbb{1}_{[0,1}(t) - 2\mathbb{1}_{[1,3)} + \frac{1}{2}\mathbb{1}_{[3,10)}$$

The deterministic system is

$$\dot{x} = cos(x(t)) + u(t), x(0) = 0$$

$$\dot{y} = x(t) + v(t)$$

y is produced from x and a disturbance

compute the following number:

$$n_1 = \frac{1}{2}\int_0^{10}(u_1(s))^2 ds + \frac{1}{2}\int_0^{10}|x_1(s) - \dot{\eta}(s)|^2 ds$$

$$n_2 = \frac{1}{2}\int_0^{10}(u_2(s))^2 ds + \frac{1}{2}\int_0^{10}|x_2(s) - \dot{\eta}(s)|^2 ds$$

(notation: we know that $|x(s) - \dot{\eta}(s)|^2$ is the norm of v)

$\eta(s)$ is a solution of

$$\dot{\eta}(t) = cos(\eta(t)) + e^{-t}$$

$$\eta(0) = 0$$

2 Solution

we open the latter norm and get

$$J = \frac{1}{2}\int_0^{10}(u(s))^2 ds + \frac{1}{2}\int_0^{10}(x(s)^2 - x(s)\dot{\eta}(s))ds$$

(We omit $\eta(s)^2$ which is the same approach as the paper "Nonlinear filtering and large deviations: A PDE-control theoretic approach" did)

Then for the first part $\frac{1}{2}\int_0^{10}(u(s))^2 ds$, we can simply use staircase functions to simulate it.

```
dt = 0.0001;
N = 100000;
% integral for u_1(s)^2 between 1 and 10
u = zeros(N,1); t = u;
s = zeros(N,1);
for k = 1:N
    t(k) = k*dt;
    u(k) = exp(-t(k))*cos(t(k));
    s(k) =u(k).^2*dt;
end
total1 = 1/2 * sum(s,"all");
```

Meanwhile, if $u_2(t)$ is an indicator function, we got

```
    dt = 0.0001;
N = 100000;
% integral for u(s)^2 between 1 and 10
u = zeros(N,1); t = u;
s = zeros(N,1);
for k = 1:N
    t(k) = k*dt;
    % indicator functions expression
    if t<1
        u(k) = 1;
    elseif t<3
        u(k) = -2;
    elseif t<10
        u(k) = 1/2;
    end
    s(k) =u(k).^2*dt;
end
total1 = 1/2 * sum(s,"all")
```

For the second part and the third part, we need to use **Trapezoidal rule** to approximate the ODE solution. In this case, the code looks like this:

```
    % TRAPEZOIDAL RULE for x(t) & \eta(t)
x = zeros(N,1); t = x;
x(1) = 0;
eta = zeros(N,1); t = eta;
eta_der = zeros(N,1);
eta(1) = 0;
eta_der(1) = 2;
for k = 2:N
    t(k) = (k-1)*dt;
    x_tmp = x(k-1);
    x_tmp_old = x_tmp;
```

```
    x_tmp = x(k-1)+dt/2*(cos(x(k-1))+cos(t(k-1))*exp(-t(k-1))+
            cos(x_tmp)+cos(t(k))*exp(-t(k)));
    x(k) = x_tmp;
    ss(k) = x(k).^2*dt;
    eta_tmp = eta(k-1);
    eta_tmp_old = eta_tmp;
    eta_tmp = eta(k-1)+dt/2*(cos(eta(k-1))+
            exp(-t(k-1))+cos(eta_tmp)+exp(-t(k)));
    eta(k) = eta_tmp;
    eta_der(k) = cos(eta(k))+exp(-t(k));
    result(k) = x(k).*eta_der(k)*dt;
end
total2 = 1/2 * sum(ss,"all");
total3 = sum(result,"all");
```

We don't show the $u_2(t)$ here since they follows the same logic.

After these calculations, we got the final result $n_1 = 9.8651$, and $n_2 = 15.5661$. The trajectories of $x(t), u(t), \eta(t), \dot{\eta}(t)$ are as follows.

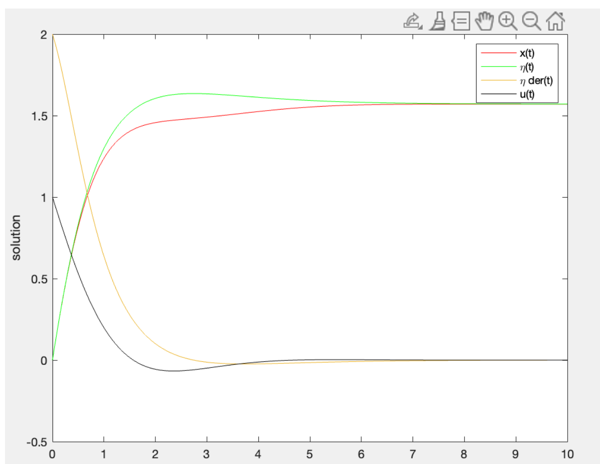

Figure 1: This is graph for $u_1(t) = e^{-t}cos(t)$

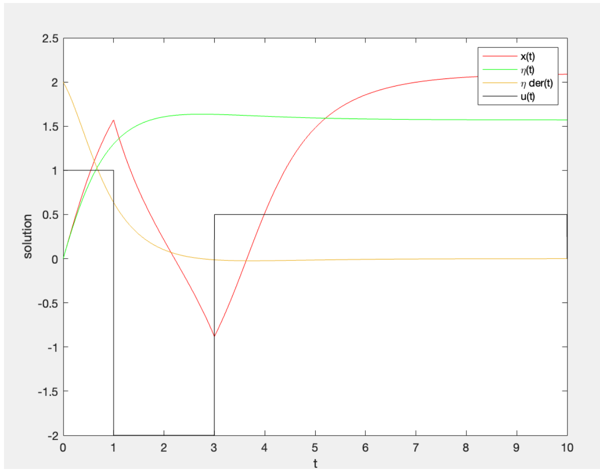

Figure 2: This is graph for $u_2(t) = \mathbb{1}_{[0,1}(t) - 2\mathbb{1}_{[1,3)} + \frac{1}{2}\mathbb{1}_{[3,10)}$

3 discussion

After I got the answer, I try to verify it by hands, and the numbers seem rational. However, I'm thinking about the error analysis in terms of Trapezoidal rule. Due to the time limit, I haven't tried other numerical methods such as Simpson's method, Backward or Forward Euler's equation. But if we have further discuss, we can talk more about different quadrature rules and see their error comparison.

4 literature review

In this research, we first focus on the deterministic estimation system by Mortensen. The systems are as follows.

$$\dot{x} = f(x(t)) + u(t), x(0) = x_0$$

$$\dot{y} = h(x(t)) + v(t), y(0) = 0$$

This is a system that the initial condition x_0 is unknown And the payoff function is

$$J_t(x_0, u) = S_0(x_0) + \frac{1}{2}\int_0^t (|u(s)|^2 + v(s)^2)ds$$

First, we replace $v(s)$ by using $v(t) = \dot{y}(s) - h(x(s))$ and omitting the $\dot{y}(s)^2$. The purpose to replace $v(s)$ is to formulate an unconstrained optimal control problem. Then we have

$$J_t(x_0, u, v) = S_0(x_0) + \frac{1}{2}\int_0^t (|u(s)|^2 + h(x(s))^2 - 2\dot{y}(s)h(x(s)))ds$$

So the deterministic energy estimate $\hat{x}(t)$ given Y_t is defined to be the endpoint of the optimal trajectory $s \to x^*(s)$ corresponding to a minimum energy pair $(x_0^*, u^*) := \hat{x}(t) = x^*(t)$. After that, we use dynamic programming to study the problem. (Bellman's equation). Define a class of admission pairs (x_0, u)

$$U_{x,t} = \{(x_0, u) : x_u(0) = x_0, x_u(t) = x\}$$

which defines as pairs for which the corresponding trajectory passes through a specific point x at time t.

Then we define a value function

$$W(x,t) = \inf_{(x_0, u \in U_{x,t}} J_t(x_0, u)$$

This $W(x,t)$ is continuous and satisfies **the Bellman equation:**

$$\frac{\partial W(x,t}{\partial t} + \hat{H}(x,t,DW(x,t)) = 0$$

$$W(x,0) = S_0(x)$$

where we have

$$\hat{H}(x,t,\lambda) = \max_{u \in U}\{\lambda(f(x)+u) - L(x,u,t)\}$$

Before we use dynamic system to solve the problem, we know that $(x_0^*, u^*) := \hat{x}(t) = x^*(t)$ which satisfies the minimum energy estimate. Therefore, we have the conclusion that

$W(\hat{x}(t), t) \leq W(x,t)$ for all $x \in R^n$

After the author uses a series of theorem and lemmas (the basic idea is to prove W is both a viscosity subsolution and supersolution. So $W(x,t)$ is the unique viscosity solution of HJB Equation.

5 Implication

The purpose for the author to practice the deterministic estimation is to get the logic for an unkown inital position's dynamic system. And finally, we can use the same logic to get the HJB Equation and say the conclusion that S(x,t) is also a unique viscosity solution for the filtering problem with $\epsilon = 1$ and the logarithmic form

$$S^\epsilon(x,t) = -\epsilon log^\epsilon(x,t)$$

the *robust* form of the Zakai equation is

$$\tfrac{\partial}{\partial t}p^\epsilon(x,t) - \frac{\epsilon}{2}\Delta p^\epsilon(x,t) + Dp^\epsilon(x,t)g^\epsilon(x,t) + \frac{1}{\epsilon}V^\epsilon(x,t)p^\epsilon(x,0) = 0, \tag{7}$$

$$p^\epsilon(x,t) \;=\; q_0^\epsilon(x),$$

where

$$g^\epsilon(x,t) \;=\; f(x) - y(t)Dh(x)', \tag{8}$$

$$\begin{aligned} V^\epsilon(x,t) \;&=\; \frac{1}{2}h(x)^2 \;+\; y(t)A_\epsilon h(x) \\ &\quad -\frac{1}{2}y(t)^2 \mid Dh(x) \mid^2 \;+\; \epsilon \mathrm{div}\,(f(x) - y(t)Dh(x)')\,. \end{aligned} \tag{9}$$

Note that (7) is a linear parabolic PDE and the coefficient V^ϵ depends on the observation path $t \mapsto y(t)$. We shall omit the ϵ–dependence of y, and view (7) as a functional of the observation path $y \in \Omega_0 = C([0,T], I\!R^n;\ y(0) = 0)$. This transformation provides a convenient choice of a version of the conditional density, and under our assumptions we can recover the unnormalised density $q^\epsilon(x,t)$ from solutions of (7); see for example Pardoux [14].

Following Fleming and Mitter [6], who considered filtering problems with $\epsilon = 1$, we apply the logarithmic transformation

$$S^\epsilon(x,t) \;=\; -\epsilon \log p^\epsilon(x,t). \tag{10}$$

Then $S^\epsilon(x,t)$ satisfies

$$\tfrac{\partial}{\partial t}S^\epsilon(x,t) - \frac{\epsilon}{2}\Delta S^\epsilon(x,t) + H^\epsilon(x,t,DS^\epsilon(x,t)) \;= 0, \tag{11}$$

$$S^\epsilon(x,0) \;=\; S_0(x),$$

where

$$H^\epsilon(x,t,\lambda) \;=\; \lambda g^\epsilon(x,t) + \frac{1}{2} \mid \lambda \mid^2 - V^\epsilon(x,t). \tag{12}$$

Equation (11) is a nonlinear parabolic PDE, which can be interpreted as the Bellman equation for a stochastic control problem [6].

Formally letting $\epsilon \to 0$ we obtain a Hamilton–Jacobi equation

$$\tfrac{\partial}{\partial t}S(x,t) + H(x,t,DS(x,t)) \;=\; 0, \tag{13}$$