$$H = \Delta a^{+}a + \frac{1}{2} i\left[\varepsilon (a^{+})^{2} - \varepsilon^{*} a^{2}\right]$$ *degenerate case*

$$\dot{a} = -i\Delta a + \varepsilon a^{+} - \kappa a - \sqrt{2\kappa}\, a_{in} - e^{i\varphi} g\, a(t-\tau)$$

$$\left(-i(\nu-\Delta) + \kappa + g e^{i(\varphi+\nu\tau)}\right)\tilde{a} - \varepsilon \tilde{a}^{+}(-\nu) = -\sqrt{2\kappa}\,\tilde{a}_{in}$$

$$\left(-i(\nu+\Delta) + \kappa + g e^{-i(\varphi-\nu\tau)}\right)\tilde{a}^{+}(-\nu) - \varepsilon^{*}\tilde{a} = -\sqrt{2\kappa}\,\tilde{a}^{+}_{in}(-\nu)$$

$$d_{-}(\nu)\,\tilde{a} - \varepsilon \tilde{a}^{+}(-\nu) = -\sqrt{2\kappa}\,\tilde{a}_{in}$$

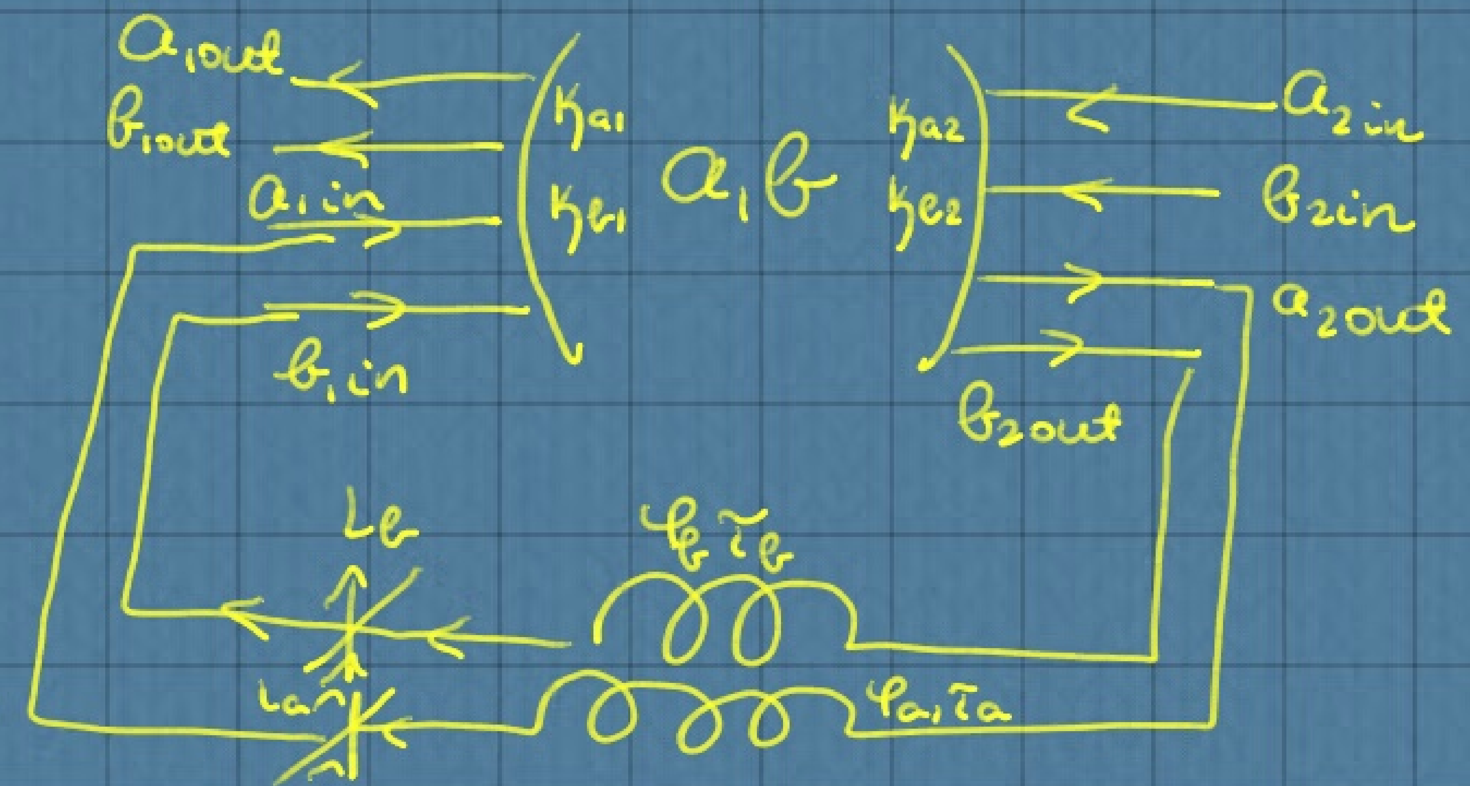
$$d_{+}(\nu)\,\tilde{a}^{+}(-\nu) - \varepsilon^{*}\tilde{a} = -\sqrt{2\kappa}\,\tilde{a}^{+}_{in}(-\nu)$$

$$\left(d_{-}(\nu) d_{+}(\nu) - |\varepsilon|^{2}\right)\tilde{a} = -\sqrt{2\kappa}\left(d_{+}(\nu)\tilde{a}_{in} + \varepsilon \tilde{a}^{+}_{in}(-\nu)\right)$$

$$H = \Delta_1 a^{+}a + \Delta_2 b^{+}b + \frac{1}{2} i\left[\varepsilon a^{+}b^{+} - \varepsilon^{*} a b\right]$$

$$\dot{a} = -i\Delta_a a + \varepsilon b^{+} - \kappa_a a - \sqrt{2\kappa_a}\, a_{in} - e^{i\varphi_a} g_a\, a(t-\tau_a)$$

$$\dot{b} = -i\Delta_b b + \varepsilon a^{+} - \kappa_b b - \sqrt{2\kappa_b}\, b_{in} - e^{i\varphi_b} g_b\, b(t-\tau_b)$$



$$\underbrace{\left[-i(\nu-\Delta_a) + \kappa_a + g_a e^{i(\varphi_a+\nu\tau_a)}\right]}_{d_{-}^{a}(\nu)}\tilde{a} - \varepsilon \tilde{b}^{+}(-\nu) = -\sqrt{2\kappa_a}\,\tilde{a}_{in}$$

$$\underbrace{\left[-i(\nu-\Delta_b) + \kappa_b + g_b e^{i(\varphi_b+\nu\tau_b)}\right]}_{d_{-}^{b}(\nu)}\tilde{b} - \varepsilon \tilde{a}^{+}(-\nu) = -\sqrt{2\kappa_b}\,\tilde{b}_{in}$$

$$\kappa_{a1} + \kappa_{a2} = \kappa_a \qquad \kappa_{b1} + \kappa_{b2} = \kappa_b$$

- $$\left.\begin{array}{l} d_{-}^{a}\tilde{a} - \varepsilon \tilde{b}^{+}(-\nu) = -\sqrt{2\kappa_a}\,\tilde{a}_{in} \\ d_{+}^{b}\tilde{b}^{+}(-\nu) - \varepsilon^{*}\tilde{a} = -\sqrt{2\kappa_b}\,\tilde{b}^{+}_{in}(-\nu) \end{array}\right\} \tilde{a} = \frac{-\sqrt{2\kappa_a}\, d_{+}^{b}\tilde{a}_{in} - \sqrt{2\kappa_b}\,\varepsilon \tilde{b}^{+}_{in}(-\nu)}{d_{+}^{b} d_{-}^{a} - |\varepsilon|^{2}}$$

$$\left.\begin{array}{l} d_{-}^{b}\tilde{b}(\nu) - \varepsilon \tilde{a}^{+}(-\nu) = -\sqrt{2\kappa_b}\,\tilde{b}_{in}(\nu) \\ d_{+}^{a}\tilde{a}^{+}(-\nu) - \varepsilon^{*}\tilde{b}(\nu) = -\sqrt{2\kappa_a}\,\tilde{a}^{+}_{in}(-\nu) \end{array}\right\} \tilde{b} = \frac{-\sqrt{2\kappa_b}\, d_{+}^{a}\tilde{b}_{in}(\nu) - \sqrt{2\kappa_a}\,\varepsilon \tilde{a}^{+}_{in}}{d_{+}^{a} d_{-}^{b} - |\varepsilon|^{2}}$$

- $$a_{in}(t) = \frac{1}{2\kappa_a}\left\{\sqrt{2\kappa_{a1}}\, a'_{1in}(t) + \sqrt{2\kappa_{a2}}\, a_{2in}(t)\right\} \qquad a'_{1in} = \sqrt{1-L_a}\, e^{i\varphi_a} a_{2in}(t-\tau) + \sqrt{L_a}\,\xi_a(t)$$

$$\tilde{a}_{in} = \frac{1}{2\kappa_a}\left\{\left[\sqrt{2\kappa_{a1}(1-L_a)}\, e^{i(\varphi_a+\nu\tau_a)} + \sqrt{2\kappa_{a2}}\right]\tilde{a}_{2in} + \sqrt{L_a}\,\tilde{\xi}_a\right\} \qquad \tilde{a}'_{1in} = \sqrt{1-L_a}\, e^{i(\varphi_a+\nu\tau_a)}\tilde{a}_{2in}(\nu) + \sqrt{L_a}\,\tilde{\xi}_a(\nu)$$

$$\tilde{b}_{in} = \frac{1}{2\kappa_b}\left\{\left[\sqrt{2\kappa_{b1}(1-L_b)}\, e^{i(\varphi_b+\nu\tau_b)} + \sqrt{2\kappa_{b2}}\right]\tilde{b}_{2in} + \sqrt{L_b}\,\tilde{\xi}_b\right\}$$

- $$f_{a2} = 2\kappa_{a2} + k_a e^{i(\varphi_a+\nu\tau_a)} \qquad f_{ij} = 2\kappa_{ij} + k_i e^{i(\varphi_i+\nu\tau_i)}$$

$$\tilde{a}_{in} = \frac{1}{2\kappa_a}\left\{\frac{1}{\sqrt{2\kappa_{a2}}} f_{a2}\tilde{a}_{2in} + \sqrt{L_a}\,\tilde{\xi}_a\right\} \qquad \tilde{b}_{in} = \frac{1}{2\kappa_b}\left\{\frac{1}{\sqrt{2\kappa_{b2}}} f_{b2}\tilde{b}_{2in} + \sqrt{L_b}\,\tilde{\xi}_b\right\}$$

$$\tilde{a}(\nu) = \frac{1}{d_{+}^{b}(\nu) d_{-}^{a}(\nu) - |\varepsilon|^{2}}\left\{-\frac{d_{+}^{b}}{\sqrt{2\kappa_a}}\left[\frac{1}{\sqrt{2\kappa_{a2}}} f_{a2}\tilde{a}_{2in} + \sqrt{L_a}\,\tilde{\xi}_a\right] - \frac{\varepsilon}{\sqrt{2\kappa_b}}\left[\frac{1}{\sqrt{2\kappa_{b2}}} f^{*}_{b2}\tilde{b}^{+}_{2in}(-\nu) + \sqrt{L_b}\,\tilde{\xi}^{+}_{b}(-\nu)\right]\right\}$$

$$\tilde{a}^{+}(-\nu) = \frac{1}{d_{-}^{b}(\nu) d_{+}^{a}(\nu) - |\varepsilon|^{2}}\left\{-\frac{d_{-}^{b}}{\sqrt{2\kappa_a}}\left[\frac{1}{\sqrt{2\kappa_{a2}}} f^{*}_{a2}(-\nu)\tilde{a}^{+}_{2,in}(-\nu) + \sqrt{L_a}\,\tilde{\xi}^{+}_{a}(-\nu)\right] - \frac{\varepsilon^{*}}{\sqrt{2\kappa_b}}\left[\frac{1}{\sqrt{2\kappa_{b2}}} f_{b2}(\nu)\tilde{b}_{2in}(\nu) + \sqrt{L_b}\,\tilde{\xi}_b(\nu)\right]\right\}$$

$$\tilde{b}(\nu) = \frac{1}{d_{+}^{a} d_{-}^{b} - |\varepsilon|^{2}}\left\{-\frac{d_{+}^{a}}{\sqrt{2\kappa_b}}\left[\frac{1}{\sqrt{2\kappa_{b2}}} f_{b2}\tilde{b}_{2in} + \sqrt{L_b}\,\tilde{\xi}_b\right] - \frac{\varepsilon}{\sqrt{2\kappa_a}}\left[\frac{1}{\sqrt{2\kappa_{a2}}} f^{*}_{a2}(-\nu)\tilde{a}^{+}_{2in}(-\nu) + \sqrt{L_a}\,\tilde{\xi}^{+}_{a}(-\nu)\right]\right\}$$

$$\tilde{b}^{+}(\nu) = \frac{1}{d_{-}^{a} d_{+}^{b} - |\varepsilon|^{2}}\left\{-\frac{d_{-}^{a}}{\sqrt{2\kappa_b}}\left[\frac{1}{\sqrt{2\kappa_{b2}}} f^{*}_{b2}(-\nu)\tilde{b}^{+}_{2in}(-\nu) + \sqrt{L_b}\,\tilde{\xi}^{+}_{b}(-\nu)\right] - \frac{\varepsilon^{*}}{\sqrt{2\kappa_a}}\left[\frac{1}{\sqrt{2\kappa_{a2}}} f_{a2}\tilde{a}_{2in} + \sqrt{L_a}\,\tilde{\xi}_a\right]\right\}$$

$$\langle \tilde{a}(\nu), \tilde{a}(\nu')\rangle = \emptyset = \langle \tilde{b}(\nu), \tilde{b}(\nu')\rangle \qquad d_{+}^{i}(-\nu) = d_{-}^{i*}(\nu)$$

$$\langle \tilde{a}^{+}(-\nu), \tilde{a}(\nu)\rangle = \frac{1}{\left(d_{-}^{b}(\nu) d_{+}^{a}(\nu) - |\varepsilon|^{2}\right)\left(d_{+}^{b}(\nu') d_{-}^{a}(\nu') - |\varepsilon|^{2}\right)} \frac{|\varepsilon|^{2}}{2\kappa_b}\left[\frac{1}{2\kappa_{b2}} f_{b2}(\nu) f^{*}_{b2}(-\nu') + L_b\right]\delta(\nu+\nu') =$$

$$= \frac{1}{|d_{-}^{b}(\nu) d_{+}^{a}(\nu) - |\varepsilon|^{2}|^{2}} \frac{|\varepsilon|^{2}}{2\kappa_b}\left[\frac{|f_{b2}(\nu)|^{2}}{2\kappa_{b2}} + L_b\right]\delta(\nu+\nu')$$

$$\langle \tilde{a}^{+}(-\nu'), \tilde{a}(\nu)\rangle = \frac{1}{|d_{+}^{b}(\nu) d_{-}^{a}(\nu) - |\varepsilon|^{2}|^{2}} \frac{|\varepsilon|^{2}}{2\kappa_b}\left[\frac{|f_{b2}(-\nu)|^{2}}{2\kappa_{b2}} + L_b\right]\delta(\nu+\nu')$$

$$\langle \tilde{X}^{a}_{\theta}(\nu), \tilde{X}^{a}_{\theta}(\nu')\rangle = \frac{|\varepsilon|^{2}}{2\kappa_b}\left\{L_b\left[\frac{1}{|d_{-}^{b}(\nu) d_{+}^{a}(\nu) - |\varepsilon|^{2}|^{2}} + \frac{1}{|d_{+}^{b}(\nu) d_{-}^{a}(\nu) - |\varepsilon|^{2}|^{2}}\right] + \frac{1}{2\kappa_{b2}}\left[\frac{|f_b(\nu)|^{2}}{|d_{-}^{b} d_{+}^{a} - |\varepsilon|^{2}|^{2}} + \frac{|f_b(-\nu)|^{2}}{|d_{+}^{b} d_{-}^{a} - |\varepsilon|^{2}|^{2}}\right]\right\}$$

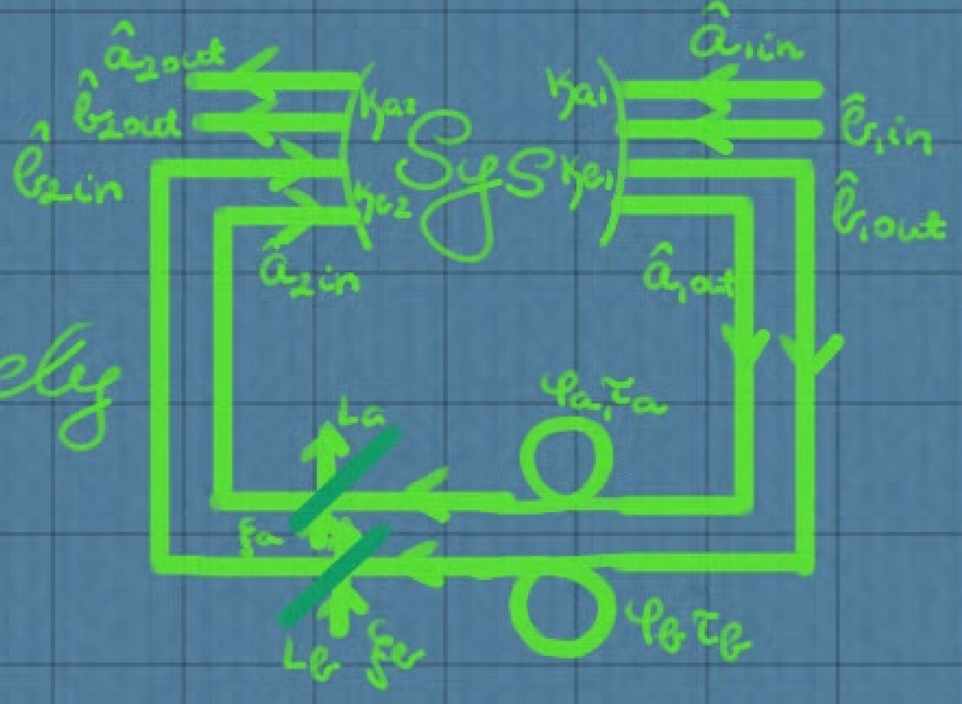
$$\langle \tilde{X}^{b}_{\theta}(\nu), \tilde{X}^{b}_{\theta}(\nu')\rangle = \frac{|\varepsilon|^{2}}{2\kappa_a}\left\{L_a\left[\frac{1}{|d_{-}^{a} d_{+}^{b} - |\varepsilon|^{2}|^{2}} + \frac{1}{|d_{+}^{a} d_{-}^{b} - |\varepsilon|^{2}|^{2}}\right] + \frac{1}{2\kappa_{a2}}\left[\frac{|f_a(\nu)|^{2}}{|d_{-}^{a} d_{+}^{b} - |\varepsilon|^{2}|^{2}} + \frac{|f_a(-\nu)|^{2}}{|d_{+}^{a} d_{-}^{b} - |\varepsilon|^{2}|^{2}}\right]\right\}$$

$$\hat{a}_{1in}(t) = \sqrt{L_a}\,\hat{\xi}_a(t) + \sqrt{1-L_a}\, e^{i\varphi_a}\hat{a}_{2out}(t-\tau_a)$$

Non-degenerate conversion model (NDPA, optomech)

$$\boxed{\hat{H} = \hbar\Delta_a \hat{a}^+\hat{a} + \hbar\Delta_b \hat{b}^+\hat{b} + i\hbar|\varepsilon|\left[\hat{a}^+\hat{b}^+ e^{i\vartheta} - \hat{a}\hat{b}e^{-i\vartheta}\right] + i\hbar|g|\left[\hat{a}^+\hat{b}e^{i\chi} - \hat{a}\hat{b}^+e^{-i\chi}\right]}$$

- internal dynamics with the Hamiltonian above
- two-channels of input & output in each mode
- time-delayed feedback in each mode separately



$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{a}(t)\right] - (\kappa_{a1} + \kappa_{a2})\hat{a}(t) - \sqrt{2\kappa_{a1}}\,\hat{a}_{1in}(t) - \sqrt{2\kappa_{a2}}\,\hat{a}_{2in}(t)$$

$$\frac{d\hat{b}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{b}(t)\right] - (\kappa_{b1} + \kappa_{b2})\hat{b}(t) - \sqrt{2\kappa_{b1}}\,\hat{b}_{1in}(t) - \sqrt{2\kappa_{b2}}\,\hat{b}_{2in}(t)$$

$$\hat{a}_{2in}(t) = \sqrt{L_a}\,\hat{\xi}_a + \sqrt{1-L_a}\,e^{i\varphi_a}\left(\hat{a}_{1in}(t-\tau_a) + \sqrt{2\kappa_{a1}}\,\hat{a}(t-\tau_a)\right)$$

$$\kappa_{a1} + \kappa_{a2} = \kappa_a$$

$$\tilde{a}_{2in}(\nu) = \sqrt{L_a}\,\tilde{\xi}_a + \sqrt{1-L_a}\,e^{i(\varphi_a+\nu\tau_a)}\left(\tilde{a}_{1in} + \sqrt{2\kappa_{a1}}\,\tilde{a}\right)$$

$$k_a = 2\sqrt{\kappa_{a1}\kappa_{a2}(1-L_a)}$$

$$\hat{a}_{in}(t) = \frac{1}{\sqrt{2\kappa_a}}\left\{\sqrt{2\kappa_{a1}}\,\hat{a}_{1in}(t) + \sqrt{2\kappa_{a2}}\left[\sqrt{L_a}\,\hat{\xi}_a(t) + \sqrt{1-L_a}\,e^{i\varphi_a}\,\hat{a}_{1in}(t-\tau_a)\right]\right\}$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{a}(t)\right] - \kappa_a\hat{a}(t) - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - k_a e^{i\varphi_a}\hat{a}(t-\tau_a)$$

$$\frac{d\hat{b}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{b}(t)\right] - \kappa_b\hat{b}(t) - \sqrt{2\kappa_b}\,\hat{b}_{in}(t) - k_b e^{i\varphi_b}\hat{b}(t-\tau_b)$$

$$\frac{d\hat{a}}{dt} = -i\Delta_a\hat{a} + |\varepsilon|\hat{b}^+e^{i\vartheta} + |g|\hat{b}e^{i\chi} - \kappa_a\hat{a}(t) - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - k_a e^{i\varphi_a}\hat{a}(t-\tau_a) =$$

$$= -(\kappa_a + i\Delta_a)\,\hat{a}(t) + |g|\hat{b}(t)e^{i\chi} + |\varepsilon|\hat{b}^+(t)e^{i\vartheta} - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - k_a e^{i\varphi_a}\,\hat{a}(t-\tau_a)$$

$$-i\nu\tilde{a} = -(\kappa_a + i\Delta_a)\tilde{a} + |g|\tilde{b}e^{i\chi} + |\varepsilon|\tilde{b}^+(-\nu)e^{i\vartheta} - \sqrt{2\kappa_a}\,\tilde{a}_{in} - k_a e^{i(\varphi_a+\nu\tau_a)}\tilde{a}$$

$$0 = \left[\kappa_a - i(\nu - \Delta_a) + k_a e^{i(\varphi_a+\nu\tau_a)}\right]\tilde{a} - |g|\tilde{b}e^{i\chi} - |\varepsilon|\tilde{b}^+e^{i\vartheta} + \sqrt{2\kappa_a}\,\tilde{a}_{in}$$

$$d_-^a\,\tilde{a}(\nu) - g\tilde{b}(\nu) - \varepsilon\tilde{b}^+(-\nu) = -\sqrt{2\kappa_a}\,\tilde{a}_{in}(\nu) \qquad d_+^a\tilde{a}^+(-\nu) - g^*\tilde{b}^+(-\nu) - \varepsilon^*\tilde{b}(\nu) = -\sqrt{2\kappa_a}\,\tilde{a}^+_{in}(-\nu)$$

$$\frac{d\hat{b}}{dt} = -i\Delta_b\hat{b} + |\varepsilon|\hat{a}^+e^{i\vartheta} - |g|\hat{a}e^{-i\chi} - \kappa_b\hat{b}(t) - \sqrt{2\kappa_b}\,\hat{b}_{in}(t) - k_b e^{i\varphi_b}\hat{b}(t-\tau_b)$$

$$d_-^b\tilde{b}(\nu) + g^*\tilde{a}(\nu) - \varepsilon\tilde{a}^+(-\nu) = -\sqrt{2\kappa_b}\,\tilde{b}_{in}(\nu) \qquad d_+^b\tilde{b}^+(-\nu) + g\tilde{a}^+(-\nu) - \varepsilon^*\tilde{a}(\nu) = -\sqrt{2\kappa_b}\,\tilde{b}^+_{in}(-\nu)$$

$$\begin{pmatrix} d_-^a & 0 & -g & -\varepsilon \\ 0 & d_+^a & -\varepsilon^* & -g^* \\ g^* & -\varepsilon & d_-^b & 0 \\ -\varepsilon^* & g & 0 & d_+^b \end{pmatrix}\begin{pmatrix} \tilde{a}(\nu) \\ \tilde{a}^+(-\nu) \\ \tilde{b}(\nu) \\ \tilde{b}^+(-\nu) \end{pmatrix} = \begin{pmatrix} -\sqrt{2\kappa_a}\,\tilde{a}_{in}(\nu) \\ -\sqrt{2\kappa_a}\,\tilde{a}^+_{in}(-\nu) \\ -\sqrt{2\kappa_b}\,\tilde{b}_{in}(\nu) \\ -\sqrt{2\kappa_b}\,\tilde{b}^+_{in}(-\nu) \end{pmatrix}$$

$$d_-^a = \kappa_a - i(\nu - \Delta_a) + k_a e^{i(\varphi_a+\nu\tau_a)}$$
$$d_+^a = \kappa_a - i(\nu + \Delta_a) + k_a e^{-i(\varphi_a-\nu\tau_a)}$$
$$d_-^b = \kappa_b - i(\nu - \Delta_b) + k_b e^{i(\varphi_b+\nu\tau_b)}$$
$$d_+^b = \kappa_b - i(\nu + \Delta_b) + k_b e^{-i(\varphi_b-\nu\tau_b)}$$

$$\Rightarrow \tilde{a}(\nu)\ \text{nominator:} -2\Big\{\sqrt{\kappa_a}\left[g\varepsilon(d_+^b - d_-^b)a^+_{in}(-\nu) + d_-^b d_+^a d_+^b a_{in}(\nu)\right] + |g|^2\left[\sqrt{\kappa_a}\,d_-^b a_{in}(\nu) + \sqrt{\kappa_b}\left(\varepsilon b^+_{in}(-\nu) + g\,b_{in}(\nu)\right)\right] -$$
$$- |\varepsilon|^2\left[\sqrt{\kappa_a}\,d_+^b a_{in}(\nu) + \sqrt{\kappa_b}\left(\varepsilon b^+_{in}(-\nu) + g\,b_{in}(\nu)\right)\right] + \sqrt{\kappa_b}\,d_+^a\left[\varepsilon d_-^b b^+_{in}(-\nu) + g\,d_+^b b_{in}(\nu)\right]\Big\}$$

$$a^+_{in}(-\nu) \Rightarrow g\varepsilon(d_+^b - d_-^b) = g\varepsilon\left[-2i\Delta_b - k_b e^{i\nu\tau_b}2i\sin\varphi_b\right] = -2ig\varepsilon\big(\underbrace{\Delta_b + k_b\sin\varphi_b\,e^{i\nu\tau_b}}_{q_b}\big) = -2ig\varepsilon q_b$$

$$a_{in}(\nu) \Rightarrow \sqrt{\kappa_a}\left[d_+^a d_-^b d_+^b + d_-^b|g|^2 - d_+^b|\varepsilon|^2\right] = \sqrt{\kappa_a}\left[(p_b^2 + q_b^2)d_+^a + p_b(|g|^2 - |\varepsilon|^2) + iq_b(|g|^2 + |\varepsilon|^2)\right]$$

$$d_-^b|g|^2 - d_+^b|\varepsilon|^2 = \big(\underbrace{\kappa_b - i\nu + k_b e^{i\nu\tau_b}\cos\varphi_b}_{p_b}\big)(|g|^2 - |\varepsilon|^2) + \big(\underbrace{i\Delta_b + ik_b e^{i\nu\tau_b}\sin\varphi_b}_{iq_b}\big)(|g|^2 + |\varepsilon|^2)$$

$$d_-^b d_+^b = p_b^2 + q_b^2$$

$$b_{in}(\nu) \Rightarrow \sqrt{\kappa_b}\left[g|g|^2 - g|\varepsilon|^2 + g\,d_+^a d_+^b\right] = \sqrt{\kappa_b}\,g\left[|g|^2 - |\varepsilon|^2 + d_+^a d_+^b\right]$$

$$b^+_{in}(-\nu) \Rightarrow \sqrt{\kappa_b}\left[\varepsilon|g|^2 - \varepsilon|\varepsilon|^2 + \varepsilon d_-^b d_+^a\right] = \sqrt{\kappa_b}\,\varepsilon\left[|g|^2 - |\varepsilon|^2 + d_+^a d_-^b\right]$$

$$\frac{d\hat{a}}{dt} = -i\Delta_a \hat{a} + |\varepsilon| \hat{b}^{+} e^{i\theta} + |g| \hat{b} e^{i\chi} - \kappa_a \hat{a}(t) - \sqrt{2\kappa_a}\, \hat{a}_{in}(t) - k_a e^{i\varphi_a} \hat{a}(t-\tau_a)$$
$$\frac{d\hat{b}}{dt} = -i\Delta_b \hat{b} + |\varepsilon| \hat{a}^{+} e^{i\theta} - |g| \hat{a} e^{-i\chi} - \kappa_b \hat{b}(t) - \sqrt{2\kappa_b}\, \hat{b}_{in}(t) - k_b e^{i\varphi_b} \hat{b}(t-\tau_b)$$

Classical nonlinear model

- $\Delta_a = \Delta_b = 0$

- $\frac{|g|}{\kappa_a} = \tilde{g}_a, \quad \frac{k_a}{\kappa_a} = \tilde{k}_a, \quad \frac{k_b}{\kappa_b} = \tilde{k}_b, \quad \frac{\kappa_a}{\kappa_b} = \tilde{r}, \quad \frac{\kappa_p}{\kappa} = \tilde{\kappa}_p$
- $\alpha = \langle \hat{a} \rangle, \beta = \langle \hat{b} \rangle, \rho \sim$ pump, $t = \kappa t$

$$\frac{d\alpha}{d\tilde{t}} = -\alpha + \rho\beta^* + \tilde{g}_a \beta e^{i\chi} - \tilde{k}_a e^{i\varphi_a} \alpha(\tilde{t} - \tilde{\tau}_a)$$
$$\frac{d\beta}{d\tilde{t}} = -\frac{\beta}{\tilde{r}} + \tilde{r}\rho\alpha^* - \tilde{g}_a e^{-i\chi} \tilde{r} \alpha - \tilde{k}_a \tilde{r} e^{i\varphi_b} \beta(\tilde{t} - \tilde{\tau}_b)$$
$$\frac{d\rho}{d\tilde{t}} = -\tilde{\kappa}_p \rho - \tilde{\kappa}_p \alpha\beta + \tilde{\kappa}_p \eta e^{i\theta} = \tilde{\kappa}_p\left(-\rho - (\alpha+\beta)^2 + \eta e^{i\theta}\right)$$

equations of motion in real variables:

$$\frac{d\alpha_1}{d\tilde{t}} = -\alpha_1 + \rho_1\beta_1 + \rho_2\beta_2 + \tilde{g}_a \cos\chi\, \beta_1 - \tilde{g}_a \sin\chi\, \beta_2 - \tilde{k}_a \cos\varphi_a\, \alpha_1(\tilde{t} - \tilde{\tau}_a) + \tilde{k}_a \sin\varphi_a\, \alpha_2(\tilde{t} - \tilde{\tau}_a)$$
$$\frac{d\alpha_2}{d\tilde{t}} = -\alpha_2 + \rho_2\beta_1 - \rho_1\beta_2 + \tilde{g}_a \cos\chi\, \beta_2 + \tilde{g}_a \sin\chi\, \beta_1 - \tilde{k}_a \cos\varphi_a\, \alpha_2(\tilde{t} - \tilde{\tau}_a) - \tilde{k}_a \sin\varphi_a\, \alpha_1(\tilde{t} - \tilde{\tau}_a)$$
$$\frac{d\beta_1}{d\tilde{t}} = -\frac{\beta_1}{\tilde{r}} + \tilde{r}\rho_1\alpha_1 + \tilde{r}\rho_2\alpha_2 - \tilde{g}_a \tilde{r} \cos\chi\, \alpha_1 - \tilde{g}_a \tilde{r} \sin\chi\, \alpha_2 - \tilde{k}_a \tilde{r} \cos\varphi_b\, \beta_1(\tilde{t} - \tilde{\tau}_b) + \tilde{k}_a \tilde{r} \sin\varphi_b\, \beta_2(\tilde{t} - \tilde{\tau}_b)$$
$$\frac{d\beta_2}{d\tilde{t}} = -\frac{\beta_2}{\tilde{r}} + \tilde{r}\rho_2\alpha_1 - \tilde{r}\rho_1\alpha_2 - \tilde{g}_a \tilde{r} \cos\chi\, \alpha_2 + \tilde{g}_a \tilde{r} \sin\chi\, \alpha_1 - \tilde{k}_a \tilde{r} \cos\varphi_b\, \beta_2(\tilde{t} - \tilde{\tau}_b) - \tilde{k}_a \tilde{r} \sin\varphi_b\, \beta_1(\tilde{t} - \tilde{\tau}_b)$$
$$\frac{d\rho_1}{d\tilde{t}} = -\tilde{\kappa}_p \rho_1 - \tilde{\kappa}_p \alpha_1\beta_1 + \tilde{\kappa}_p \alpha_2\beta_2 + \tilde{\kappa}_p \eta \cos\theta$$
$$\frac{d\rho_2}{d\tilde{t}} = -\tilde{\kappa}_p \rho_2 - \tilde{\kappa}_p \alpha_1\beta_2 - \tilde{\kappa}_p \alpha_2\beta_1 + \tilde{\kappa}_p \eta \sin\theta$$

Final equations

$par = (\overset{1}{\tilde{g}_a}, \overset{2}{\chi}, \overset{3}{\tilde{k}_a}, \overset{4}{\varphi_a}, \overset{5}{\varphi_b}, \overset{6}{\tilde{r}}, \overset{7}{\tilde{\kappa}_p}, \overset{8}{\eta}, \overset{9}{\theta}, \overset{10}{\tau_a}, \overset{11}{\tau_b})$ $\quad xx = (\overset{1}{\alpha_1}, \overset{2}{\alpha_2}, \overset{3}{\beta_1}, \overset{4}{\beta_2}, \overset{5}{\rho_1}, \overset{6}{\rho_2})$

- $-xx(1,1) + [par(1)\cos(par(2)) + xx(5,1)]\, xx(3,1) - [par(1)\sin(par(2)) - xx(6,1)]\, xx(4,1) -$
  $- par(3)[\cos(par(4))\, xx(1,2) - \sin(par(4))\, xx(2,2)]$
- $-xx(2,1) + [par(1)\sin(par(2)) + xx(6,1)]\, xx(3,1) + [par(1)\cos(par(2)) - xx(5,1)]\, xx(4,1) -$
  $- par(3)[\sin(par(4))\, xx(1,2) + \cos(par(4))\, xx(2,2)]$
- $-\frac{1}{par(6)} xx(3,1) - par(6)\{[par(1)\cos(par(2)) - xx(5,1)]\, xx(1,1) + [par(1)\sin(par(2)) - xx(6,1)]\, xx(2,1) +$
  $+ par(3)[\cos(par(5))\, xx(3,3) - \sin(par(5))\, xx(4,3)]\}$
- $-\frac{1}{par(6)} xx(4,1) + par(6)\{[par(1)\sin(par(2)) + xx(6,1)]\, xx(1,1) - [par(1)\cos(par(2)) + xx(5,1)]\, xx(2,1) -$
  $- par(3)[\sin(par(5))\, xx(3,3) + \cos(par(5))\, xx(4,3)]\}$
- $-par(7)\{xx(5,1) + xx(1,1)\, xx(3,1) - xx(2,1)\, xx(4,1) - par(8)\cos(par(9))\}$
- $-par(7)\{xx(6,1) + xx(1,1)\, xx(4,1) + xx(2,1)\, xx(3,1) - par(8)\sin(par(9))\}$

BIFTOOL

Steady-state solutions:

$$\frac{d\alpha}{d\tilde{t}} = 0 = -(1 + \tilde{k}_a e^{i\varphi_a})\alpha + \tilde{g}_a \beta e^{-i\chi} + \rho\beta^*$$
$$\frac{d\beta}{d\tilde{t}} = 0 = -\tilde{g}_a e^{-i\chi} \tilde{r} \alpha + \tilde{r}\rho\alpha^* - \frac{1 + \tilde{k}_a \tilde{r}^2 e^{i\varphi_b}}{\tilde{r}} \beta$$
$$\frac{d\rho}{d\tilde{t}} = 0 = -\tilde{\kappa}_p(\rho + \alpha\beta - \eta e^{i\theta})$$

Special case $\kappa_a = \kappa_b = \kappa_p$

$$0 = -(1 + \tilde{k} e^{i\varphi_a})\alpha + \tilde{g}\beta e^{-i\chi} + \rho\beta^*$$
$$0 = -\tilde{g} e^{-i\chi} \alpha + \rho\alpha^* - (1 + \tilde{k} e^{i\varphi_b})\beta$$
$$0 = -\rho - \alpha\beta + \eta e^{i\theta}$$

Special case: $\kappa_a = \kappa_b = \kappa_p$

(1) $0 = -(1+\tilde{\Delta}e^{i\varphi_a})\alpha + \tilde{g}\beta e^{i\chi} + p\beta^*$ $\Rightarrow$ $-p\beta^* + (\eta e^{i\theta}-\alpha\beta)\beta^* = -K_a\alpha + \tilde{g}\beta e^{i\chi} + \overbrace{(\eta e^{i\theta}-\alpha\beta)}^{\Gamma}\beta^* = 0$

(2) $0 = -\tilde{g}e^{-i\chi}\alpha + p\alpha^* - (1+\tilde{\Delta}e^{i\varphi_b})\beta$ $\Rightarrow$ $-p\alpha^* + (\eta e^{i\theta}-\alpha\beta)\alpha^* = -K_b\beta - \tilde{g}e^{-i\chi}\alpha + (\eta e^{i\theta}-\alpha\beta)\alpha^* = 0$

(3) $0 = -p - \alpha\beta + \eta e^{i\theta}$

$$(1)\quad \alpha = \frac{1}{K_a}\left[\tilde{g}e^{i\chi}\beta + \Gamma\beta^*\right]$$

$$-K_b\beta K_a - \tilde{g}^2\beta - \Gamma\tilde{g}e^{-i\chi}\beta^* + \Gamma\tilde{g}e^{-i\chi}\beta^* + |\Gamma|^2\beta = 0$$

$$\left[|\Gamma|^2 - K_bK_a - \tilde{g}^2\right]\beta = 0$$

- Trivial case: $\beta = 0 \overset{(1)}{\Rightarrow} \alpha = 0 \overset{(3)}{\Rightarrow} p = \eta e^{i\theta}$ undepleted pump approx.
- $|\Gamma|^2 = K_bK_a + \tilde{g}^2 \Longrightarrow K_bK_a \in \mathbb{R}$

$$|\Gamma| = \sqrt{K_bK_a + \tilde{g}^2}$$

$$\Gamma = \pm\sqrt{K_bK_a + \tilde{g}^2}$$

$$\eta e^{i\theta} - \alpha\beta = \pm\sqrt{K_bK_a+\tilde{g}^2} \Rightarrow \alpha\beta = \eta e^{i\theta} \mp \sqrt{K_bK_a+\tilde{g}^2} = \eta e^{i\theta} \mp \xi$$

$$\hookrightarrow (3)\quad p = -\eta e^{i\theta} \pm \xi + \eta e^{i\theta} = \pm\xi$$

$$(1)\quad -K_a\frac{\eta e^{i\theta}\mp\xi}{\beta} + \tilde{g}\beta e^{i\chi} \pm \xi\beta^* = 0$$

$$-K_a(\eta e^{i\theta}\mp\xi) + \tilde{g}|\beta|^2e^{i(\chi+2\varphi_b)} \pm \xi|\beta|^2 = 0$$

$$\pm(K_a+|\beta|^2)\xi - K_a\eta e^{i\theta} + \tilde{g}|\beta|^2e^{i(\chi+2\varphi_b)} = 0$$

$$\pm(K_a+|\beta|^2)\xi - K_a\eta\cos\theta + \tilde{g}|\beta|^2\cos(\chi+2\varphi_b) = 0$$

$$-K_a\eta\sin\theta + \tilde{g}|\beta|^2\sin(\chi+2\varphi_b) = 0$$

Special case: $\theta = \chi = 0$ $\quad \varphi_b = 0,\ \varphi_a = 0, \pi$

$$\beta^2 = K_a\frac{\eta\mp\xi}{\tilde{g}\pm\xi}$$

$$\frac{|\beta|^2}{K_a}\sin(\chi+2\varphi_b) = \frac{|\alpha|^2}{K_b}\sin(\chi-2\varphi_a)$$

$$(2)\quad -K_b\frac{\eta e^{i\theta}\mp\xi}{\alpha} - \tilde{g}e^{-i\chi}\alpha \pm \xi\alpha^* = 0$$

$$-K_b(\eta e^{i\theta}\mp\xi) - \tilde{g}|\alpha|^2e^{-i(\chi-2\varphi_a)} \pm \xi|\alpha|^2 = 0$$

$$\pm(K_b+|\alpha|^2)\xi - K_b\eta e^{i\theta} - \tilde{g}|\alpha|^2e^{-i(\chi-2\varphi_a)} = 0$$

$$\pm(K_b+|\alpha|^2)\xi - K_b\eta\cos\theta - \tilde{g}|\alpha|^2\cos(\chi-2\varphi_a) = 0$$

$$-K_b\eta\sin\theta + \tilde{g}|\alpha|^2\sin(\chi-2\varphi_a) = 0$$

Special case: $\theta = \chi = 0$ $\quad \varphi_a = 0,\ \varphi_b = 0, \pi$

$$\alpha^2 = K_b\frac{\eta\mp\xi}{-\tilde{g}\pm\xi}$$

As $\xi \geq \tilde{g} \Rightarrow \alpha = \sqrt{K_b\frac{\eta-\xi}{\xi-\tilde{g}}}$ & $\beta = \sqrt{K_a\frac{\eta-\xi}{\tilde{g}+\xi}}$ $\Rightarrow$ threshold $\eta \geq \xi = \sqrt{\tilde{g}^2 + (1\pm\tilde{\Delta})(1\pm\tilde{\Delta})}$

$$p = \xi$$

The equations of motion with detuning

- $\frac{|g|}{\kappa_a} = \tilde{g}_a$, $\frac{k_a}{\kappa_a} = \tilde{k}_a$, $\frac{k_b}{\kappa_b} = \tilde{k}_b$, $\frac{\kappa_a}{\kappa_b} = \tilde{r}$, $\frac{\kappa_p}{\kappa} = \tilde{\kappa}_p$
- $\alpha = \langle \hat{a} \rangle$, $\beta = \langle \hat{b} \rangle$, $\tilde{\rho}$ pump, $t = \kappa t$

$$\frac{d\alpha}{d\tilde{t}} = -(1+i\tilde{\delta}_a)\alpha + \rho\beta^* + \tilde{g}_a \beta e^{i\chi} - \tilde{k}_a e^{i\varphi_a} \alpha(\tilde{t}-\tilde{\tau}_a)$$

$$\frac{d\beta}{d\tilde{t}} = -(1+i\tilde{\delta}_b)\frac{\beta}{\tilde{r}} + \tilde{r}\rho\alpha^* - \tilde{g}_a e^{-i\chi}\tilde{r}\alpha - \frac{\tilde{k}_b}{\tilde{r}} e^{i\varphi_b}\beta(\tilde{t}-\tilde{\tau}_b)$$

$$\frac{d\rho}{d\tilde{t}} = -\tilde{\kappa}_p\rho - \tilde{\kappa}_p\alpha\beta + \tilde{\kappa}_p\eta e^{i\theta} = \tilde{\kappa}_p\left(-\rho - (\alpha+\beta)^2 + \eta e^{i\theta}\right)$$

These equations cover 2 cases:

- NDPA $\tilde{\delta}_i$: respective detuning of signal & idler from resonance
- optomechanics: δ_a = detuning $\qquad \delta_b = \Omega_m$

  $\tilde{\delta}_a = \frac{\Delta_a}{\kappa_a} \qquad \tilde{\delta}_b = \frac{\Delta_b}{\kappa_a} = \frac{\Omega_m}{\kappa_a} \qquad \tilde{\delta}_b \gg 1$ resolved sideband

  $\tilde{\delta}_b \ll 1$

$$d_-^a = \kappa_a - i(\nu-\Delta_a) + k_a e^{i(\varphi_a+\nu\tau_a)} \qquad d_-^b = \kappa_b - i(\nu-\Delta_b) + k_b e^{i(\varphi_b+\nu\tau_b)}$$
$$d_+^a = \kappa_a - i(\nu+\Delta_a) + k_a e^{-i(\varphi_a-\nu\tau_a)} \qquad d_+^b = \kappa_b - i(\nu+\Delta_b) + k_b e^{i(\varphi_b-\nu\tau_b)}$$

$\tilde a(\nu)$ nominator: $-\sqrt2\{\sqrt{\kappa_a}[g\varepsilon(d_+^b-d_-^b)a^+_{in}(-\nu)+d_-^b d_+^a d_+^b a_{in}(\nu)] + |g|^2[\sqrt{\kappa_a} d_-^b a_{in}(\nu)+\sqrt{\kappa_b}(\varepsilon b^+_{in}(-\nu)+g b_{in}(\nu))] -$
$-|\varepsilon|^2[\sqrt{\kappa_a} d_+^b a_{in}(\nu)+\sqrt{\kappa_b}(\varepsilon b^+_{in}(-\nu)+g b_{in}(\nu))] + \sqrt{\kappa_b} d_+^a[\varepsilon d_-^b b^+_{in}(-\nu)+g d_+^b b_{in}(\nu)]\}$ $\circledast$

$a^+_{in}(-\nu) \Rightarrow g\varepsilon(d_+^b-d_-^b) = g\varepsilon[-2i\Delta_b - k_b e^{i\nu\tau_b} 2i\sin\varphi_b] = -2ig\varepsilon(\underbrace{\Delta_b + k_b\sin\varphi_b e^{i\nu\tau_b}}_{q_b}) = -2ig\varepsilon q_b$

$a_{in}(\nu) \Rightarrow \sqrt{\kappa_a}[d_+^a d_-^b d_+^b + d_-^b|g|^2 - d_+^b|\varepsilon|^2] = \sqrt{\kappa_a}[(p_b^2+q_b^2)d_+^a + p_b(|g|^2-|\varepsilon|^2) + iq_b(|g|^2+|\varepsilon|^2)]$

$d_-^b|g|^2 - d_+^b|\varepsilon|^2 = (\underbrace{\kappa_b - i\nu + k_b e^{i\nu\tau_b}\cos\varphi_b}_{p_b})(|g|^2-|\varepsilon|^2) + (\underbrace{i\Delta_b + ik_b e^{i\nu\tau_b}\sin\varphi_b}_{iq_b})(|g|^2+|\varepsilon|^2)$

$d_-^b d_+^b = p_b^2+q_b^2$ $\quad$ q_b & p_b doesn't change when $*(-\nu)$!

$b_{in}(\nu) \Rightarrow \sqrt{\kappa_b}[g|g|^2 - g|\varepsilon|^2 + g d_+^a d_+^b] = \sqrt{\kappa_b}\, g[|g|^2-|\varepsilon|^2+d_+^a d_+^b] = \sqrt{\kappa_b}\, g\Lambda_+$

$b^+_{in}(-\nu) \Rightarrow \sqrt{\kappa_b}[\varepsilon|g|^2 - \varepsilon|\varepsilon|^2 + \varepsilon d_-^b d_+^a] = \sqrt{\kappa_b}\,\varepsilon[|g|^2-|\varepsilon|^2+d_+^a d_-^b] = \sqrt{\kappa_b}\,\varepsilon\Lambda_-$

$a_{in}(\nu) \Rightarrow \sqrt{\kappa_a}[d_-^b\Lambda_+ - (d_+^b-d_-^b)|\varepsilon|^2] = \sqrt{\kappa_a}[d_+^b\Lambda_- - (d_+^b-d_-^b)|g|^2] =$
$= \sqrt{\kappa_a}[d_-^b\Lambda_+ + 2iq_b|\varepsilon|^2] = \sqrt{\kappa_a}[d_+^b\Lambda_- + 2iq_b|g|^2]$

$\circledast$ $-\sqrt2\{\sqrt{\kappa_a}[(d_-^b\Lambda_+ + 2iq_b|\varepsilon|^2)a_{in}(\nu) - 2ig\varepsilon q_b a^+_{in}(-\nu)] + \sqrt{\kappa_b}[g\Lambda_+ b_{in}(\nu) + \varepsilon\Lambda_- b^+_{in}(-\nu)]\}$

$\tilde a(\nu)$ denominator: $-|\varepsilon|^2(2|g|^2 + d_-^a d_+^b + d_-^b d_+^a) + (|g|^2 + d_-^a d_-^b)(|g|^2 + d_+^a d_+^b) + |\varepsilon|^4 =$
$= |\varepsilon|^4 + |g|^2(d_-^a d_-^b + d_+^a d_+^b - 2|\varepsilon|^2) - d_-^a d_+^b|\varepsilon|^2 - d_+^a d_-^b|\varepsilon|^2 + d_-^a d_+^a d_-^b d_+^b + |g|^4 =$
$= (|g|^2-|\varepsilon|^2)^2 + |g|^2(d_-^a d_-^b + d_+^a d_+^b) - |\varepsilon|^2(d_-^a d_+^b + d_+^a d_-^b) + d_-^a d_+^a d_-^b d_+^b = \Omega^2$

$d_-^\sigma(\nu) = d_+^{\sigma *}(-\nu)$

$\tilde a^+(-\nu)$ nominator: $\sqrt2\{\varepsilon^*[a^+_{in}(-\nu) d_-^b\varepsilon\sqrt{\kappa_a} + g^*(a_{in}(\nu)\sqrt{\kappa_a}(d_+^b-d_-^b) + \sqrt{\kappa_b}(b^+_{in}(-\nu)\varepsilon - b_{in}(\nu) g))] - b_{in}(\nu) d_-^a d_+^b\sqrt{\kappa_b} -$
$-[|g|^2 + d_-^a d_-^b][a^+_{in}(-\nu) d_+^b\sqrt{\kappa_a} + b^+_{in}(-\nu)\sqrt{\kappa_b}\, g^*] + b_{in}\varepsilon^*\sqrt{\kappa_b}|\varepsilon|^2\}$ $\circledast$

$a_{in}(\nu) \Rightarrow \sqrt{\kappa_a}\, g^*\varepsilon^*(d_+^b - d_-^b) = -2i\sqrt{\kappa_a}\, g^*\varepsilon^* q_b$

$a^+_{in}(-\nu) \Rightarrow \sqrt{\kappa_a}[|\varepsilon|^2 d_-^b - |g|^2 d_+^b - d_-^a d_-^b d_+^b] = \sqrt{\kappa_a}[d_-^b\Lambda_-^\times - 2i|g|^2 q_b] = \sqrt{\kappa_a}[d_+^b\Lambda_+^\times - 2i|\varepsilon|^2 q_b]$

$b_{in}(\nu) \Rightarrow \sqrt{\kappa_b}[|g|^2\varepsilon^* - d_-^a d_+^b\varepsilon^* + \varepsilon^*|\varepsilon|^2] = -\sqrt{\kappa_b}\,\varepsilon^*[|g|^2 - |\varepsilon|^2 + d_-^a d_+^b] = -\sqrt{\kappa_b}\,\varepsilon^*\Lambda_-^\times$ $\quad$ $\Lambda_\sigma^\times(\nu) = \Lambda_\sigma^*(-\nu)$

$b^+_{in}(-\nu) \Rightarrow \sqrt{\kappa_b}[|\varepsilon|^2 g^* - g^*(|g|^2 + d_-^a d_-^b)] = -\sqrt{\kappa_b}\, g^*[|g|^2 - |\varepsilon|^2 + d_-^a d_-^b] = -\sqrt{\kappa_b}\, g^*\Lambda_+^\times$

$\circledast$ $-\sqrt2\{\sqrt{\kappa_a}[(d_+^b\Lambda_+^\times - 2i|\varepsilon|^2 q_b)a^+_{in}(-\nu) + 2ig^*\varepsilon^* q_b a_{in}(\nu)] + \sqrt{\kappa_b}[g^*\Lambda_+^\times b^+_{in}(-\nu) + \varepsilon^*\Lambda_-^\times b_{in}(\nu)]\}$

$\tilde a^+(\nu)$ denominator: same! Ω^2

$\tilde b(\nu)$ nominator: $\sqrt2\{g^*[\sqrt{\kappa_a}(a_{in} d_+^a d_+^b - a^+_{in}\varepsilon g) - a_{in}\sqrt{\kappa_a}|\varepsilon|^2 - \sqrt{\kappa_b}(b^+_{in}\varepsilon(d_-^a - d_+^a) + b_{in} d_-^a g)] +$
$+[|\varepsilon|^2 - d_-^a d_+^b][a^+_{in}\varepsilon\sqrt{\kappa_a} + b_{in} d_+^a\sqrt{\kappa_b}] + a_{in} g^*\sqrt{\kappa_a}|g|^2\}$ $\circledast$

$a_{in}(\nu) \Rightarrow \sqrt{\kappa_a}\, g^*[d_+^a d_+^b - |\varepsilon|^2 + |g|^2] = \sqrt{\kappa_a}\, g^*\Lambda_+$

$a^+_{in}(-\nu) \Rightarrow \sqrt{\kappa_a}\,\varepsilon[-|g|^2 + |\varepsilon|^2 - d_-^a d_+^b] = -\sqrt{\kappa_a}\,\varepsilon\Lambda_-^\times$ $\qquad$ denom: Ω^2

$b_{in}(\nu) \Rightarrow \sqrt{\kappa_b}[-|g|^2 d_-^a + d_+^a|\varepsilon|^2 - d_-^a d_+^a d_+^b] = -\sqrt{\kappa_b}[d_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a]$

$b^+_{in}(-\nu) \Rightarrow -\sqrt{\kappa_b}\,\varepsilon g^*(d_-^a - d_+^a) = -2i\sqrt{\kappa_b}\,\varepsilon g^* q_a$

$\circledast$ $\sqrt2\{\sqrt{\kappa_a}[g^*\Lambda_+ a_{in}(\nu) - \varepsilon\Lambda_-^\times a^+_{in}(-\nu)] - \sqrt{\kappa_b}[(d_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a) b_{in}(\nu) + 2i\varepsilon g^* q_a b^+_{in}(-\nu)]\}$

$\tilde b^+(-\nu)$ nominator: $\sqrt2\{-\varepsilon^*[\sqrt{\kappa_a}(a^+_{in}\varepsilon g + a_{in} d_-^b d_+^a) + a_{in} g\sqrt{\kappa_a}\, g^* - \sqrt{\kappa_b}(b^+_{in} d_-^a\varepsilon + b_{in} g(d_-^a - d_+^a))] +$
$+[|g|^2 + d_-^a d_-^b][a^+_{in} g\sqrt{\kappa_a} - b^+_{in} d_+^a\sqrt{\kappa_b}] + a_{in}\varepsilon^*\sqrt{\kappa_a}|\varepsilon|^2\}$ $\circledast$

$a_{in}(\nu) \Rightarrow -\varepsilon^*\sqrt{\kappa_a}[d_+^a d_-^b + |g|^2 - |\varepsilon|^2] = -\varepsilon^*\sqrt{\kappa_a}\Lambda_-$

$a^+_{in}(-\nu) \Rightarrow g\sqrt{\kappa_a}[-|\varepsilon|^2 + |g|^2 + d_-^a d_-^b] = g\sqrt{\kappa_a}\Lambda_+^\times$

$b_{in}(\nu) \Rightarrow -\varepsilon^* g(d_-^a - d_+^a)\sqrt{\kappa_b} = -2i\sqrt{\kappa_b}\,\varepsilon^* g q_a$

$b^+_{in}(-\nu) \Rightarrow -\sqrt{\kappa_b}[-|\varepsilon|^2 d_-^a + |g|^2 d_+^a + d_+^a d_-^a d_-^b] = -\sqrt{\kappa_b}[d_+^a\Lambda_+^\times - 2i|\varepsilon|^2 q_a] = -\sqrt{\kappa_b}[d_-^a\Lambda_- - 2i|g|^2 q_a]$

$\circledast$ $\sqrt2\{\sqrt{\kappa_a}[g\Lambda_+^\times a^+_{in}(-\nu) - \varepsilon^*\Lambda_- a_{in}(\nu)] - \sqrt{\kappa_b}[(d_+^a\Lambda_+^\times - 2i|\varepsilon|^2 q_a) b^+_{in}(-\nu) + 2i\varepsilon^* g q_a b_{in}(\nu)]\}$

$$\tilde{a}(\nu) = \frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[(d_-^b \Lambda_+ + 2iq_b|\varepsilon|^2)a_{in}(\nu) + 2ig\varepsilon q_b\, a_{in}^+(-\nu)\right] - \sqrt{\kappa_b}\left[g\Lambda_+\, b_{in}(\nu) + \varepsilon\Lambda_-\, b_{in}^+(-\nu)\right]\right\}$$

$$\tilde{b}(\nu) = \frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[g^*\Lambda_+ a_{in}(\nu) - \varepsilon\Lambda_-^\times a_{in}^+(-\nu)\right] - \sqrt{\kappa_b}\left[(d_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a)b_{in}(\nu) + 2i\varepsilon g^* q_a b_{in}^+(-\nu)\right]\right\}$$

$$\Omega = (|g|^2-|\varepsilon|^2)^2 + |g|^2(d_-^a d_-^b + d_+^a d_+^b) - |\varepsilon|^2(d_-^a d_+^b + d_+^a d_-^b) + d_-^a d_+^a d_-^b d_+^b \qquad \Omega^*(-\nu) = \Omega(\nu) \Rightarrow \Omega^*(\nu) = \Omega(-\nu)$$

$$d_-^\sigma = \kappa_\sigma - i(\nu - \Delta_\sigma) + k_\sigma e^{i(\varphi_\sigma + \nu\tau_\sigma)} \qquad d_+^\sigma = \kappa_\sigma - i(\nu + \Delta_\sigma) + k_\sigma e^{-i(\varphi_\sigma - \nu\tau_\sigma)} \qquad d_+^\sigma(\nu) = d_-^{\sigma *}(-\nu)$$

$$q_\sigma = \frac{i}{2}(d_+^\sigma - d_-^\sigma) = \Delta_\sigma + k_\sigma \sin\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_- = |g|^2 - |\varepsilon|^2 + d_+^a d_-^b \qquad \Lambda_-^\times(\nu) = \Lambda_-^*(-\nu)$$

$$p_\sigma = \frac{1}{2}(d_+^\sigma + d_-^\sigma) = \kappa_\sigma - i\nu + k_\sigma \cos\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_+ = |g|^2 - |\varepsilon|^2 + d_+^a d_+^b \qquad \Lambda_+^\times(\nu) = \Lambda_+^*(-\nu)$$

Special case: degenerate par. amp, no coupling g

$d_\mp^b = d_\mp^a = d_\mp$ $\qquad \square_a = \square_b$

$$\Omega^2 = |\varepsilon|^4 - 2d_-d_+|\varepsilon|^2 + (d_-d_+)^2 = (d_-d_+ - |\varepsilon|^2)^2 = m^2$$

$$\Lambda_\mp = -|\varepsilon|^2 + d_+ d_\mp$$

$$\tilde{a}(\nu) = \frac{\sqrt{2\kappa}}{m^2}\left\{-(d_-d_+^2 - |\varepsilon|^2 d_- + d_-|\varepsilon|^2 - d_+|\varepsilon|^2)a_{in}(\nu) - \varepsilon(d_+d_- - |\varepsilon|^2)a_{in}^+(-\nu)\right\} =$$

$$= \frac{\sqrt{2\kappa}}{m^2}\left\{-d_+(d_-d_+ - |\varepsilon|^2)a_{in}(\nu) - \varepsilon(d_-d_+ - |\varepsilon|^2)a_{in}^+(-\nu)\right\} = -\frac{\sqrt{2\kappa}}{m}\left[d_+a_{in}(\nu) + \varepsilon a_{in}^+(-\nu)\right] \checkmark$$

$$\tilde{b}(\nu) \overset{?}{=} \tilde{a}(\nu) = \frac{\sqrt{2\kappa}}{m^2}\left\{-\varepsilon(d_+d_- - |\varepsilon|^2)a_{in}^+(-\nu) - (d_-d_+^2 - d_-|\varepsilon|^2 + d_-|\varepsilon|^2 - d_+|\varepsilon|^2)a_{in}(\nu)\right\} =$$

$$= -\frac{\sqrt{2\kappa}}{m}\left(\varepsilon a_{in}^+(-\nu) + d_+a_{in}(\nu)\right) \checkmark$$

$$\tilde{a}(\nu)=\frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[(\alpha_-^b\Lambda_+ + 2iq_b|\varepsilon|^2)a_{in}(\nu)+2ig\varepsilon q_b a_{in}^+(-\nu)\right]-\sqrt{\kappa_b}\left[g\Lambda_+ b_{in}(\nu)+\varepsilon\Lambda_- b_{in}^+(-\nu)\right]\right\}$$

$$\tilde{b}(\nu)=\frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[g^*\Lambda_+ a_{in}(\nu)-\varepsilon\Lambda_-^\times a_{in}^+(-\nu)\right]-\sqrt{\kappa_b}\left[(\alpha_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a)b_{in}(\nu)+2i\varepsilon g^* q_a b_{in}^+(-\nu)\right]\right\}$$

$$\Omega=(|g|^2-|\varepsilon|^2)^2+|g|^2(\alpha_-^a\alpha_-^b+\alpha_+^a\alpha_+^b)-|\varepsilon|^2(\alpha_-^a\alpha_+^b+\alpha_+^a\alpha_-^b)+\alpha_-^a\alpha_+^a\alpha_-^b\alpha_+^b \qquad \Omega^*(-\nu)=\Omega(\nu)\Rightarrow\Omega^*(\nu)=\Omega(-\nu)$$

$$\alpha_-^\sigma=\kappa_\sigma-i(\nu-\Delta_\sigma)+k_\sigma e^{i(\varphi_\sigma+\nu\tau_\sigma)} \qquad \alpha_+^\sigma=\kappa_\sigma-i(\nu+\Delta_\sigma)+k_\sigma e^{-i(\varphi_\sigma-\nu\tau_\sigma)} \qquad \alpha_+^\sigma(\nu)=\alpha_-^{\sigma *}(-\nu)$$

$$q_\sigma=\frac{i}{2}(\alpha_+^\sigma-\alpha_-^\sigma)=\Delta_\sigma+k_\sigma\sin\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_-=|g|^2-|\varepsilon|^2+\alpha_+^a\alpha_-^b \qquad \Lambda_-^\times(\nu)=\Lambda_-^*(-\nu)$$

$$p_\sigma=\frac{1}{2}(\alpha_+^\sigma+\alpha_-^\sigma)=\kappa_\sigma-i\nu+k_\sigma\cos\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_+=|g|^2-|\varepsilon|^2+\alpha_+^a\alpha_+^b \qquad \Lambda_+^\times(\nu)=\Lambda_+^*(-\nu)$$

$$\hat{a}_{in}(t)=\frac{1}{\sqrt{2\kappa_a}}\left\{\sqrt{2\kappa_{a1}}\,\hat{a}_{1,in}(t)+\sqrt{2\kappa_{a2}}\left[\sqrt{L_a}\hat{f}_a(t)+\sqrt{1-L_a}e^{i\varphi_a}\hat{a}_{1in}(t-\tau_a)\right]\right\} \qquad k_a=2\sqrt{\kappa_{a1}\kappa_{a2}(1-L_a)}$$

$$\tilde{a}_{in}(\nu)=\frac{1}{\sqrt{2\kappa_a}}\left\{\sqrt{2\kappa_{a1}}\,\tilde{a}_{1in}(\nu)+\sqrt{2\kappa_{a2}}\left[\sqrt{L_a}\tilde{f}_a(\nu)+\sqrt{1-L_a}e^{i(\varphi_a+\nu\tau_a)}\tilde{a}_{1in}(\nu)\right]\right\}=$$

$$=\frac{1}{\sqrt{2\kappa_a}}\left\{\frac{1}{\sqrt{2\kappa_{a1}}}\left[2\kappa_{a1}+k_a e^{i(\varphi_a+\nu\tau_a)}\right]a_{1in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a(\nu)\right\}=\frac{1}{\sqrt{2\kappa_a}}\left\{\frac{1}{\sqrt{2\kappa_{a1}}}f_a a_{1in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a(\nu)\right\}$$

$$\tilde{b}_{in}(\nu)=\frac{1}{\sqrt{2\kappa_b}}\left\{\frac{1}{\sqrt{2\kappa_{b1}}}\left[2\kappa_{b1}+k_b e^{i(\varphi_b+\nu\tau_b)}\right]b_{1in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right\}=\frac{1}{\sqrt{2\kappa_b}}\left\{\frac{1}{\sqrt{2\kappa_{b1}}}f_b b_{1in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right\}$$

$$\boxed{\tilde{a}(\nu)}$$

$$\sqrt{2\kappa_a}\left[-(\alpha_-^b\Lambda_+ + 2iq_b|\varepsilon|^2)a_{in}(\nu)+2ig\varepsilon q_b a_{in}^+(-\nu)\right]=-(\alpha_-^b\Lambda_+ + 2iq_b|\varepsilon|^2)\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a a_{1in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a(\nu)\right)+$$

$$+2ig\varepsilon q_b\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)a_{1in}^+(-\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a^+(-\nu)\right)$$

$$-\sqrt{2\kappa_b}\left[g\Lambda_+ b_{in}(\nu)+\varepsilon\Lambda_- b_{in}^+(-\nu)\right]=-g\Lambda_+\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b b_{1in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right)-\varepsilon\Lambda_-\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)b_{1in}^+(-\nu)+\sqrt{2\kappa_{b2}}\tilde{f}_b^+(-\nu)\right)$$

$$\boxed{\tilde{b}(\nu)}$$

$$\sqrt{2\kappa_a}\left(g^*\Lambda_+ a_{in}(\nu)-\varepsilon\Lambda_-^\times a_{in}^+(-\nu)\right)=g^*\Lambda_+\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a a_{1in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}(\nu)\right)-\varepsilon\Lambda_-^\times\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)a_{1in}^+(-\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}^+(-\nu)\right)$$

$$-\sqrt{2\kappa_b}\left[(\alpha_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a)b_{in}(\nu)+2i\varepsilon g^* q_a b_{in}^+(-\nu)\right]=-(\alpha_-^a\Lambda_+ + 2i|\varepsilon|^2 q_a)\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b b_{1in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right)-$$

$$-2i\varepsilon g^* q_a\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)b_{1in}^+(-\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b^+(-\nu)\right)$$

Intracavity quadratures

$$\tilde{X}_{\theta'}^a(\nu)=\frac{1}{2}\left(e^{-i\frac{\theta'}{2}}\tilde{a}(\nu)+e^{i\frac{\theta'}{2}}\tilde{a}^+(-\nu)\right) \qquad \tilde{X}_{\theta'}^b(\nu)=\frac{1}{2}\left(e^{-i\frac{\theta'}{2}}\tilde{b}(\nu)+e^{i\frac{\theta'}{2}}\tilde{b}^+(-\nu)\right)$$

$$\langle:\tilde{X}_{\theta'}(\nu),\tilde{X}_{\theta'}(\nu'):\rangle=\frac{1}{4}\left\{e^{-i\theta'}\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle+e^{i\theta'}\langle\tilde{a}^+(-\nu),\tilde{a}^+(-\nu')\rangle+\langle\tilde{a}^+(-\nu),\tilde{a}(\nu')\rangle+\langle\tilde{a}^+(-\nu'),\tilde{a}(\nu)\rangle\right\}$$

$$\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle=\frac{1}{\Omega^2(\nu)\Omega^2(\nu')}\left\{-\left[\alpha_-^b(\nu)\Lambda_+(\nu)+2iq_b(\nu)|\varepsilon|^2\right]2ig\varepsilon q_b(\nu')\left[\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right]\delta(\nu+\nu')+\right.$$

$$\left.+g\varepsilon\Lambda_+(\nu)\Lambda_-(\nu')\left[\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\right]\delta(\nu+\nu')\right\}=$$

$$=\frac{1}{|\Omega(\nu)^2|^2}\left\{-\left[\alpha_-^b\Lambda_+ + 2iq_b|\varepsilon|^2\right]2ig\varepsilon q_b^*\left[\frac{1}{2\kappa_{a1}}|f_a|^2+2\kappa_{a2}L_a\right]+g\varepsilon\Lambda_+\Lambda_-^{*\times}\left[\frac{1}{2\kappa_{b1}}|f_b|^2+2\kappa_{b2}L_b\right]\right\}\delta(\nu+\nu')=$$

$$=\frac{g\varepsilon}{|\Omega^2|^2}\left\{-2iq_b^*\left[\alpha_-^b\Lambda_+ + 2iq_b|\varepsilon|^2\right]\left[\frac{|f_a|^2}{2\kappa_{a1}}+2\kappa_{a2}L_a\right]+\Lambda_+\Lambda_-^{\times *}\left[\frac{|f_b|^2}{2\kappa_{b2}}+2\kappa_{b2}L_b\right]\right\}\delta(\nu+\nu')=(*)$$

$$-2i\alpha_-^b\Lambda_+ q_b^*=-\alpha_-^b(\alpha_+^{*b}-\alpha_-^{*b})(|g|^2-|\varepsilon|^2+\alpha_+^a\alpha_+^b)=(|g|^2-|\varepsilon|^2)\alpha_-^b(\alpha_+^{*b}-\alpha_-^{*b})-\alpha_+^a\alpha_-^b\alpha_+^b(\alpha_+^{*b}-\alpha_-^{*b})$$

$$4|q_b|^2|\varepsilon|^2=|\varepsilon|^2(|\alpha_+^b|^2+|\alpha_-^b|^2-\alpha_+^b\alpha_-^{*b}-\alpha_+^{*b}\alpha_-^b)$$

$$\Lambda_+\Lambda_-^{\times *}=(|g|^2-|\varepsilon|^2+\alpha_+^a\alpha_+^b)(|g|^2-|\varepsilon|^2+\alpha_-^{*a}\alpha_+^{*b})=(|g|^2-|\varepsilon|^2)^2+(|g|^2-|\varepsilon|^2)(\alpha_+^a\alpha_+^b+\alpha_-^{*a}\alpha_+^{*b})+|\alpha_+^b|^2\alpha_+^a\alpha_-^{*a}$$

$$(*)=\frac{g\varepsilon}{|\Omega^2|^2}\left\{\left[|g|^2(\alpha_-^{*b}-\alpha_+^{*b})\alpha_-^b+|\varepsilon|^2\alpha_+^b(\alpha_+^{*b}-\alpha_-^{*b})-\alpha_+^a\alpha_-^b\alpha_+^b(\alpha_+^{*b}-\alpha_-^{*b})\right]\left[\frac{|f_a|^2}{2\kappa_{a1}}+2\kappa_{a2}L_a\right]+\right.$$

$$\left.+\left[(|g|^2-|\varepsilon|^2)^2+(|g|^2-|\varepsilon|^2)(\alpha_+^a\alpha_+^b+\alpha_-^{*a}\alpha_+^{*b})+|\alpha_+^b|^2\alpha_+^a\alpha_-^{*a}\right]\underbrace{\left[\frac{|f_b|^2}{2\kappa_{b1}}+2\kappa_{b2}L_b\right]}_{A_b}\right\}\delta(\nu+\nu')=$$

$$=\frac{g\varepsilon}{|\Omega^2|^2}\left\{-2i\overbrace{\left(|g|^2\alpha_-^b-|\varepsilon|^2\alpha_+^b+\alpha_+^a\alpha_-^b\alpha_+^b\right)}^{\Gamma_{ba}}q_b^*A_a+\Lambda_+\Lambda_-^{\times *}A_b\right\}=\langle\tilde{a}^+(-\nu),\tilde{a}^+(-\nu')\rangle^*$$

$$\langle\tilde{a}^+(-\nu),\tilde{a}(\nu')\rangle=\left\{4|g|^2|\varepsilon|^2q_b(\nu)q_b(\nu')\left[\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right]+|\varepsilon|^2\Lambda_-^\times(\nu)\Lambda_-(\nu')\left[\frac{1}{2\kappa_{b1}}f_b^*(-\nu')f_b(\nu)+2\kappa_{b2}L_b\right]\right\}\delta(\nu+\nu')=$$

$$=\left\{4|g|^2|\varepsilon|^2|q_b|^2\left[\frac{1}{2\kappa_{a1}}|f_a|^2+2\kappa_{a2}L_a\right]+|\varepsilon|^2|\Lambda_-^\times|^2\left[\frac{1}{2\kappa_{b1}}|f_b|^2+2\kappa_{b2}L_b\right]\right\}\delta(\nu+\nu')=$$

$$=\frac{|\varepsilon|^2}{|\Omega^2|^2}\left\{4|g|^2|q_b|^2A_a+|\Lambda_-^\times|^2A_b\right\}\delta(\nu+\nu')$$

$$\langle\tilde{a}^+(-\nu'),\tilde{a}(\nu)\rangle=\frac{|\varepsilon|^2}{|\Omega^2|^2}\left\{4|g|^2|q_b|^2A_a(-\nu)+|\Lambda_-^\times|^2A_b(-\nu)\right\}\delta(\nu+\nu') \qquad \theta_a=\arg(\varepsilon)+\arg(g)$$

$$\langle:\tilde{X}_{\theta'}(\nu),\tilde{X}_{\theta'}(\nu'):\rangle=\frac{1}{|\Omega^2|^2}\left\{\mathrm{Re}\left[e^{i(\theta_a-\theta')}\left(2i\Gamma_{ba}q_b^*A_a+\Lambda_+\Lambda_-^{\times *}A_b\right)\right]|g||\varepsilon|2+|\varepsilon|^2\left[4|g|^2|q_b|^2(A_a(\nu)+A_a(-\nu))+\right.\right.$$

$$\left.\left.+|\Lambda_-|^2A_b(\nu)+|\Lambda_-^\times|^2A_b(-\nu)\right]\right\}\delta(\nu+\nu')=$$

$$=\frac{|\varepsilon|}{|\Omega^2|^2}\left\{2|g|\mathrm{Re}\left[e^{i(\theta_a-\theta')}\left(2i\Gamma_{ba}q_b^*A_a+\Lambda_+\Lambda_-^{\times *}A_b\right)\right]+|\varepsilon|\left[4|g|^2|q_b|^2(A_a(\nu)+A_a(-\nu))+|\Lambda_-|^2A_b(\nu)+|\Lambda_-^\times|^2A_b(-\nu)\right]\right\}\delta(\nu+\nu')$$

$$\tilde{a}(\nu)=\frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[(d_-^b\Lambda_+ + 2iq_b|\mathcal{E}|^2)a_{in}(\nu)+2ig\mathcal{E}q_b a_{in}^+(-\nu)\right]-\sqrt{\kappa_b}\left[g\Lambda_+ b_{in}(\nu)+\mathcal{E}\Lambda_- b_{in}^+(-\nu)\right]\right\}$$

$$\tilde{b}(\nu)=\frac{\sqrt{2}}{\Omega^2}\left\{\sqrt{\kappa_a}\left[g^*\Lambda_+ a_{in}(\nu)-\mathcal{E}\Lambda_-^\times a_{in}^+(-\nu)\right]-\sqrt{\kappa_b}\left[(d_-^a\Lambda_+ + 2i|\mathcal{E}|^2 q_a)b_{in}(\nu)+2i\mathcal{E}g^* q_a b_{in}^+(-\nu)\right]\right\}$$

$$\Omega = (|g|^2-|\mathcal{E}|^2)^2+|g|^2(d_-^a d_-^b+d_+^a d_+^b)-|\mathcal{E}|^2(d_-^a d_+^b+d_+^a d_-^b)+d_-^a d_+^a d_-^b d_+^b \qquad \Omega^*(-\nu)=\Omega(\nu)\Rightarrow\Omega^*(\nu)=\Omega(-\nu)$$

$$d_-^\sigma=\kappa_\sigma-i(\nu-\Delta_\sigma)+k_\sigma e^{i(\varphi_\sigma+\nu\tau_\sigma)} \qquad d_+^\sigma=\kappa_\sigma-i(\nu+\Delta_\sigma)+k_\sigma e^{-i(\varphi_\sigma-\nu\tau_\sigma)} \qquad d_+^\sigma(\nu)=d_-^{\sigma *}(-\nu)$$

$$q_\sigma=\frac{i}{2}(d_+^\sigma-d_-^\sigma)=\Delta_\sigma+k_\sigma\sin\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_-=|g|^2-|\mathcal{E}|^2+d_+^a d_-^b \qquad \Lambda_-^\times(\nu)=\Lambda_-^*(-\nu)$$

$$p_\sigma=\frac{1}{2}(d_+^\sigma+d_-^\sigma)=\kappa_\sigma-i\nu+k_\sigma\cos\varphi_\sigma e^{i\nu\tau_\sigma} \qquad \Lambda_+=|g|^2-|\mathcal{E}|^2+d_+^a d_+^b \qquad \Lambda_+^\times(\nu)-\Lambda_+^*(-\nu)$$

$\boxed{\tilde{b}(\nu)}$

$$\sqrt{2\kappa_a}\left(g^*\Lambda_+ a_{in}(\nu)-\mathcal{E}\Lambda_-^\times a_{in}^+(-\nu)\right)=g^*\Lambda_+\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a a_{in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a(\nu)\right)-\mathcal{E}\Lambda_-^\times\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)a_{1in}^+(-\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a^+(-\nu)\right)$$

$$-\sqrt{2\kappa_b}\left[(d_-^a\Lambda_+ + 2i|\mathcal{E}|^2 q_a)b_{in}(\nu)+2i\mathcal{E}g^* q_a b_{in}^+(-\nu)\right]=-(d_-^a\Lambda_+ + 2i|\mathcal{E}|^2 q_a)\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b b_{in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right)-$$

$$-2i\mathcal{E}g^* q_a\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)b_{1in}^+(-\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b^+(-\nu)\right)$$

$$\langle:\tilde{X}_{\theta'}^b(\nu),\tilde{X}_{\theta'}^b(\nu'):\rangle=\frac{1}{4}\left\{e^{-i\theta'}\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle+e^{i\theta'}\langle\tilde{b}^+(-\nu),\tilde{b}^+(-\nu')\rangle+\langle\tilde{b}^+(-\nu),\tilde{b}(\nu')\rangle+\langle\tilde{b}^+(-\nu'),\tilde{b}(\nu)\rangle\right\} \quad \ast$$

$$\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle=\frac{1}{|\Omega|^4}\left\{g^*\mathcal{E}\Lambda_+(\nu)\Lambda_-^\times(\nu')\left[\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right]+(d_-^a\Lambda_+ + 2i|\mathcal{E}|^2 q_a(\nu))2i\mathcal{E}g^* q_a(\nu')\left[\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\right]\right\}$$

$$\delta(\nu+\nu')=\frac{1}{|\Omega|^4}\left\{g^*\mathcal{E}\Lambda_+\Lambda_-^*\left[\frac{1}{2\kappa_{a1}}|f_a|^2+2\kappa_{a2}L_a\right]+(d_-^a\Lambda_+ + 2i|\mathcal{E}|^2 q_a)2i\mathcal{E}g^* q_a^*\left[\frac{1}{2\kappa_{b1}}|f_b|^2+2\kappa_{b2}L_b\right]\right\}\delta(\nu+\nu')=\ast$$

$$+d_-^a\Lambda_+ 2iq_a^*=+d_-^a(d_+^{a*}-d_-^{a*})(|g|^2-|\mathcal{E}|^2+d_+^a d_+^b)=(|g|^2-|\mathcal{E}|^2)(d_+^{a*}-d_-^{a*})d_-^a+d_-^a d_+^a d_+^b(d_+^{a*}-d_-^{a*})$$

$$-4|\mathcal{E}|^2|q_a|^2=-|\mathcal{E}|^2\left(|d_+^a|^2+|d_-^a|^2-d_+^{a*}d_-^a-d_+^a d_-^{a*}\right)$$

$$\ast=\frac{g^*\mathcal{E}}{|\Omega^2|^2}\left\{\Lambda_+\Lambda_-^* A_a+(|g|^2 d_-^a-|\mathcal{E}|^2 d_+^a+d_-^a d_+^a d_+^b)2iq_a^* A_b\right\}= \qquad A_a(\nu)=\frac{1}{2\kappa_{a1}}|f_a(\nu)|^2+2\kappa_{a2}L_a$$

$$=\frac{g^*\mathcal{E}}{|\Omega^2|^2}\left\{\Lambda_+\Lambda_-^* A_a+\Gamma_{ab}2iq_a^* A_b\right\}=\langle\tilde{b}^+(-\nu),\tilde{b}^+(-\nu')\rangle^*$$

$$\langle\tilde{b}^+(-\nu),\tilde{b}(\nu')\rangle=\frac{1}{\Omega^*(\nu)\Omega^*(\nu')}\left\{4|\mathcal{E}|^2|g|^2 q_a(\nu)q_a(\nu')\left(\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_2\right)+|\mathcal{E}|^2\Lambda_-(\nu)\Lambda_-^\times(\nu')\left(\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right)\right\}$$

$$\delta(\nu+\nu')=\frac{|\mathcal{E}|^2}{|\Omega^2|^2}\left\{|g|^2|q_a(\nu)|^2 A_b+|\Lambda_-|^2 A_a\right\}\delta(\nu+\nu')$$

$$\langle\tilde{b}^+(-\nu'),\tilde{b}(\nu)\rangle=\frac{|\mathcal{E}|^2}{|\Omega^2|^2}\left\{4|g|^2|q_a(\nu)|^2 A_b(-\nu)+|\Lambda_-^\times|^2 A_a(-\nu)\right\}\delta(\nu+\nu') \qquad \theta_b=\arg(\mathcal{E})+\arg(g^*)=\arg(\mathcal{E})-\arg(g)$$

$$\ast\ \langle:\tilde{X}_{\theta'}^b(\nu),\tilde{X}_{\theta'}^b(\nu'):\rangle=\frac{|\mathcal{E}|}{4|\Omega^2|^2}\left\{2\,\mathrm{Re}\left[e^{i(\theta_b-\theta')}(\Lambda_+\Lambda_-^* A_a-2iq_a^*\Gamma_{ab}A_b)\right]|g|+|\mathcal{E}|\left[4|g|^2|q_a|^2(A_b(\nu)+A_b(-\nu))+|\Lambda_-|^2 A_a(\nu)+|\Lambda_-^*|^2 A_a(-\nu)\right]\right\}$$

$$\delta(\nu+\nu')$$

$\boxed{\tilde{a}(\nu)}$

$$\sqrt{2\kappa_a}\left[-(d_-^b\Lambda_+ + 2iq_b|\mathcal{E}|^2)a_{in}(\nu)+2ig\mathcal{E}q_b a_{in}^+(-\nu)\right]=-(d_-^b\Lambda_+ + 2iq_b|\mathcal{E}|^2)\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a a_{1in}(\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a(\nu)\right)+$$

$$+2ig\mathcal{E}q_b\left(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)a_{1in}^+(-\nu)+\sqrt{2\kappa_{a2}L_a}\tilde{f}_a^+(-\nu)\right)$$

$$-\sqrt{2\kappa_b}\left[g\Lambda_+ b_{in}(\nu)+\mathcal{E}\Lambda_- b_{in}^+(-\nu)\right]=-g\Lambda_+\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b b_{1in}(\nu)+\sqrt{2\kappa_{b2}L_b}\tilde{f}_b(\nu)\right)-\mathcal{E}\Lambda_-\left(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)b_{1in}^+(-\nu)+\sqrt{2\kappa_{b2}}\tilde{f}_b^+(-\nu)\right)$$

$$\tilde{c}(\nu)=(\tilde{a}(\nu)+\tilde{b}(\nu))\frac{1}{\sqrt{2}} \qquad \tilde{c}^+(\nu)=(\tilde{a}^+(-\nu)+\tilde{b}^+(-\nu))\frac{1}{\sqrt{2}} \qquad \tilde{Y}_{\theta'}(\nu)=\frac{1}{2}\left(\tilde{c}(\nu)e^{-i\theta'/2}+\tilde{c}^+(-\nu)e^{i\frac{\theta'}{2}}\right)$$

$$\langle:\tilde{Y}_{\theta'}(\nu),\tilde{Y}_{\theta'}(\nu'):\rangle=\frac{1}{4}\left\{e^{-i\theta'}\langle\tilde{c}(\nu),\tilde{c}(\nu')\rangle+e^{i\theta'}\langle\tilde{c}^+(-\nu),\tilde{c}^+(-\nu')\rangle+\langle\tilde{c}^+(-\nu),\tilde{c}(\nu')\rangle+\langle\tilde{c}^+(-\nu'),\tilde{c}(\nu)\rangle\right\}$$

$$\langle\tilde{c}(\nu),\tilde{c}(\nu')\rangle=\frac{1}{2}\left\{\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle+\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle+\langle\tilde{a}(\nu),\tilde{b}(\nu')\rangle+\langle\tilde{b}(\nu),\tilde{a}(\nu')\rangle\right\}$$

$$\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle=\frac{g\mathcal{E}}{|\Omega^2|^2}\left\{-2iq_b^*\Gamma_{ba}A_a+\Lambda_+\Lambda_-^{\times *}A_b\right\} \qquad A_b(\nu)=\frac{|f_b(\nu)|^2}{2\kappa_{b1}}+2\kappa_{b2}L_b \qquad \Gamma_{ba}=|g|^2 d_-^b-|\mathcal{E}|^2 d_+^b+d_+^a d_-^b d_+^b$$

$$\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle=\frac{g^*\mathcal{E}}{|\Omega^2|^2}\left\{2iq_a^*\Gamma_{ab}A_b-\Lambda_+\Lambda^* A_a\right\}$$

$$\langle\tilde{a}(\nu),\tilde{b}(\nu')\rangle=+\left[(d_-^b(\nu)\Lambda_+(\nu)+2iq_b(\nu)|\mathcal{E}|^2)\mathcal{E}\Lambda_-^\times(\nu')\left(\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right)+\right.$$

$$\left.+g\Lambda_+(\nu)2i\mathcal{E}g^* q_a(\nu')\left(\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\right)\right]\delta(\nu+\nu')= \quad \ast$$

$$d_-^b\Lambda_+\Lambda_-^* + 2iq_b|\mathcal{E}|^2\Lambda_-^*=\left[d_-^b(|g|^2-|\mathcal{E}|^2+d_+^a d_+^b)-(d_+^b-d_-^b)|\mathcal{E}|^2\right]\Lambda_-^*=$$

$$=\left[d_-^b|g|^2-d_+^b|\mathcal{E}|^2+d_+^a d_-^b d_+^b\right]\Lambda_-^*=\Gamma_{ba}\Lambda_-^*$$

$$\ast=\mathcal{E}\left\{\Lambda_-^*\Gamma_{ba}A_a+2iq_a^*|g|^2\Lambda_+A_b\right\}\delta(\nu+\nu')$$

$$\langle\tilde{b}(\nu),\tilde{a}(\nu')\rangle=\left[g^*\Lambda_+(\nu)2ig\mathcal{E}q_b(\nu')\left(\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\right)+(d_-^a(\nu)\Lambda_+(\nu)+2i|\mathcal{E}|^2 q_a(\nu))\mathcal{E}\Lambda_-(\nu')\left(\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\right)\right]$$

$$\delta(\nu+\nu')=\mathcal{E}\left\{\Lambda_-^{\times *}\Gamma_{ab}A_b+2iq_b^*|g|^2\Lambda_+A_a\right\}\delta(\nu+\nu')$$

$$\tilde{a}(\nu)=\frac{1}{\Omega^2}\Big\{-(d_-^b\Lambda_+ + 2iq_b|\mathcal{E}|^2)\Big(\frac{1}{\sqrt{2\kappa_{a1}}}f_a\tilde{a}_{in}(\nu)+\sqrt{2\kappa_{a2}L_a}\,\tilde{\xi}_a(\nu)\Big)+2ig\mathcal{E}q_b\Big(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)\tilde{a}^+_{in}(-\nu)+\sqrt{2\kappa_{a2}L_a}\,\tilde{\xi}^+_a(-\nu)\Big)-$$
$$-g\Lambda_+\Big(\frac{1}{\sqrt{2\kappa_{b1}}}f_b\tilde{b}_{in}(\nu)+\sqrt{2\kappa_{b2}L_b}\,\tilde{\xi}_b(\nu)\Big)-\mathcal{E}\Lambda_-\Big(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)\tilde{b}^+_{in}(-\nu)+\sqrt{2\kappa_{b2}L_b}\,\tilde{\xi}^+_b(-\nu)\Big)\Big\}$$
$$\tilde{b}(\nu)=\frac{1}{\Omega^2}\Big\{g^*\Lambda_+\Big(\frac{1}{\sqrt{2\kappa_{a1}}}f_a\tilde{a}_{in}(\nu)+\sqrt{2\kappa_{a2}L_a}\,\tilde{\xi}_a(\nu)\Big)-\mathcal{E}\Lambda_-^{\times}\Big(\frac{1}{\sqrt{2\kappa_{a1}}}f_a^*(-\nu)\tilde{a}^+_{in}(-\nu)+\sqrt{2\kappa_{a2}L_a}\,\tilde{\xi}^+_a(-\nu)\Big)-$$
$$-(d_-^a\Lambda_+ + 2iq_a|\mathcal{E}|^2)\Big(\frac{1}{\sqrt{2\kappa_{b1}}}f_b\tilde{b}_{in}(\nu)+\sqrt{2\kappa_{b2}L_b}\,\tilde{\xi}_b(\nu)\Big)-2iq_a\mathcal{E}g^*\Big(\frac{1}{\sqrt{2\kappa_{b1}}}f_b^*(-\nu)\tilde{b}^+_{in}(-\nu)+\sqrt{2\kappa_{b2}L_b}\,\tilde{\xi}^+_b(-\nu)\Big)\Big\}$$
$$q_\sigma=\frac{i}{2}(d_+^\sigma-d_-^\sigma)=\Delta_\sigma+k_\sigma\sin\varphi_\sigma e^{i\nu\tau_\sigma}\qquad \Lambda_-=|g|^2-|\mathcal{E}|^2+d_+^a d_-^b\qquad \Lambda_-^{\times}(\nu)=\Lambda_-^*(-\nu)$$
$$\Gamma_{ba}=|g|^2d_-^b-|\mathcal{E}|^2d_+^b+d_+^ad_-^bd_+^b\qquad \Lambda_+=|g|^2-|\mathcal{E}|^2+d_+^ad_+^b\qquad \Lambda_+^{\times}(\nu)-\Lambda_+^*(-\nu)$$

$$\langle\tilde{c}(\nu),\tilde{c}(\nu')\rangle=\frac{1}{2}\{\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle+\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle+\langle\tilde{a}(\nu),\tilde{b}(\nu')\rangle+\langle\tilde{b}(\nu),\tilde{a}(\nu')\rangle\}=\circledast$$
$$\langle\tilde{a}(\nu),\tilde{a}(\nu')\rangle=\frac{g\mathcal{E}}{|\Omega^2|^2}\{-2iq_b^*\Gamma_{ba}A_a+\Lambda_+\Lambda_-^{\times *}A_b\}\delta(\nu+\nu')\quad A_b(\nu)=\frac{|f_b(\nu)|^2}{2\kappa_{b1}}+2\kappa_{b2}L_b\qquad \Gamma_{ba}=|g|^2d_-^b-|\mathcal{E}|^2d_+^b+d_+^ad_-^bd_+^b$$
$$\langle\tilde{b}(\nu),\tilde{b}(\nu')\rangle=\frac{g^*\mathcal{E}}{|\Omega^2|^2}\{2iq_a^*\Gamma_{ab}A_b-\Lambda_+\Lambda_-^*A_a\}\delta(\nu+\nu')$$
$$\langle\tilde{a}(\nu),\tilde{b}(\nu')\rangle=\frac{\mathcal{E}}{|\Omega^2|^2}\{\Lambda_-^*\Gamma_{ba}A_a+2iq_a^*|g|^2\Lambda_+A_b\}\delta(\nu+\nu')$$
$$\langle\tilde{b}(\nu),\tilde{a}(\nu')\rangle=\frac{\mathcal{E}}{|\Omega^2|^2}\{\Lambda_-^{\times *}\Gamma_{ab}A_b+2iq_b^*|g|^2\Lambda_+A_a\}\delta(\nu+\nu')$$
$$\circledast=\frac{\mathcal{E}}{2|\Omega^2|^2}\{[-2iq_b^*g\Gamma_{ba}-g^*\Lambda_+\Lambda_-^*+\Lambda_-^*\Gamma_{ba}+2iq_b^*|g|^2\Lambda_+]A_a+[g\Lambda_+\Lambda_-^{\times *}+2iq_a^*g^*\Gamma_{ab}+2iq_a^*|g|^2\Lambda_++\Lambda_-^{\times *}\Gamma_{ab}]A_b\}\delta(\nu+\nu')=$$
$$=\frac{\mathcal{E}}{2|\Omega^2|^2}\{[2iq_b^*g(g^*\Lambda_+-\Gamma_{ba})-\Lambda_-^*(g^*\Lambda_+-\Gamma_{ba})]A_a+[2iq_a^*g^*(g\Lambda_++\Gamma_{ab})+\Lambda_-^{\times *}(g\Lambda_++\Gamma_{ab})]A_b\}\delta(\nu+\nu')=$$
$$=\frac{\mathcal{E}}{2|\Omega^2|^2}\{(2iq_b^*g-\Lambda_-^*)(g^*\Lambda_+-\Gamma_{ba})A_a+(2iq_a^*g^*+\Lambda_-^{\times *})(g\Lambda_++\Gamma_{ab})A_b\}\delta(\nu+\nu')=\langle\tilde{c}^+(-\nu),\tilde{c}^+(-\nu')\rangle^*$$
$$\langle\tilde{c}^+(-\nu),\tilde{c}(\nu')\rangle=\frac{1}{2}\{\langle\tilde{a}^+(-\nu),\tilde{a}(\nu')\rangle+\langle\tilde{b}^+(-\nu),\tilde{b}(\nu')\rangle+\langle\tilde{a}^+(-\nu),\tilde{b}(\nu')\rangle+\langle\tilde{b}^+(-\nu),\tilde{a}(\nu')\rangle\}=\circledast$$
$$\langle\tilde{a}^+(-\nu),\tilde{a}(\nu')\rangle=\frac{|\mathcal{E}|^2}{|\Omega^2|^2}\{4|g|^2|q_b|^2A_a+|\Lambda_-^{\times}|^2A_b\}\delta(\nu+\nu')$$
$$\langle\tilde{b}^+(-\nu),\tilde{b}(\nu')\rangle=\frac{|\mathcal{E}|^2}{|\Omega^2|^2}\{4|g|^2|q_a|^2A_b+|\Lambda_-|^2A_a\}\delta(\nu+\nu')$$
$$\langle\tilde{a}^+(-\nu),\tilde{b}(\nu')\rangle=\frac{1}{|\Omega^2|^2}\Big\{2ig^*\mathcal{E}^*q_b(\nu)\mathcal{E}\Lambda_-^{\times}(\nu')\Big(\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\Big)+\mathcal{E}^*\Lambda_-^{\times}(\nu)2iq_a(\nu')\mathcal{E}g^*\Big(\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\Big)\Big\}\delta(\nu+\nu')=$$
$$=2i\frac{|\mathcal{E}|^2g^*}{|\Omega^2|^2}\{q_b\Lambda_-^*A_a+q_a^*\Lambda_-^{\times}A_b\}\delta(\nu+\nu')$$
$$\langle\tilde{b}^+(-\nu),\tilde{a}(\nu')\rangle=\frac{1}{|\Omega^2|^2}\Big\{-\mathcal{E}^*\Lambda_-(\nu)2ig\mathcal{E}q_b(\nu')\Big(\frac{1}{2\kappa_{a1}}f_a(\nu)f_a^*(-\nu')+2\kappa_{a2}L_a\Big)-2iq_a(\nu)\mathcal{E}^*g\mathcal{E}\Lambda_-(\nu')\Big(\frac{1}{2\kappa_{b1}}f_b(\nu)f_b^*(-\nu')+2\kappa_{b2}L_b\Big)\Big\}\delta(\nu+\nu')=$$
$$=2i\frac{|\mathcal{E}|^2g}{|\Omega^2|^2}\{-q_b^*\Lambda_-A_a-q_a\Lambda_-^{\times *}A_b\}\delta(\nu+\nu')=-2i\frac{|\mathcal{E}|^2g}{|\Omega^2|^2}[q_b^*\Lambda_-A_a-q_a\Lambda_-^{\times *}A_b]\delta(\nu+\nu')$$
$$\circledast=\frac{|\mathcal{E}|^2}{2|\Omega^2|^2}\{[4|g|^2|q_b|^2+|\Lambda_-|^2+2ig^*q_b\Lambda_-^*-2igq_b^*\Lambda_-]A_a+[|\Lambda_-^{\times}|^2+4|g|^2|q_a|^2+2ig^*q_a^*\Lambda_-^{\times}-2igq_a\Lambda_-^{\times *}]A_b\}\delta(\nu+\nu')=$$
$$=\frac{|\mathcal{E}|^2}{2|\Omega^2|^2}\{|2ig^*q_b+\Lambda_-|^2A_a(\nu)+|-2igq_a+\Lambda_-^{\times}|^2A_b(\nu)\}\delta(\nu+\nu')$$
$$\langle\tilde{c}^+(-\nu'),\tilde{c}(\nu)\rangle=\frac{|\mathcal{E}|^2}{2|\Omega^2|^2}\{|-2igq_b+\Lambda_-^{\times}|^2A_a(-\nu)+|2ig^*q_a+\Lambda_-|^2A_b(\nu)\}\delta(\nu+\nu')$$
$$\langle:\tilde{Y}_{\theta'}(\nu),\tilde{Y}_{\theta'}(\nu'):\rangle=\frac{|\mathcal{E}|}{8|\Omega^2|^2}\Big\{2\mathrm{Re}\big[e^{i(\theta_\mathcal{E}-\theta)}(-\Xi_{ab}^{\times *}\Sigma_{ba}A_a+\Xi_{ba}^{*}\Sigma_{ab}A_b)\big]+$$
$$+|\mathcal{E}|\big[|\Xi_{ba}^{\times}|^2A_a(\nu)+|\Xi_{ba}|^2A_a(-\nu)+|\Xi_{ab}|^2A_b(\nu)+|\Xi_{ab}^{\times}|^2A_b(-\nu)\big]\Big\}\delta(\nu+\nu')$$
$$\Xi_{ba}(\nu)=-2igq_b(\nu)+\Lambda_-^{\times}(\nu)=g(d_+^b(\nu)-d_-^b(\nu))+|g|^2-|\mathcal{E}|^2+d_-^ad_+^b$$
$$\Xi_{ab}^{\times}(\nu)=2ig^*q_a(\nu)+\Lambda_-(\nu)=g^*(d_+^a(\nu)-d_-^a(\nu))+|g|^2-|\mathcal{E}|^2+d_+^ad_-^b$$
$$\Sigma_{ba}(\nu)=g^*\Lambda_+-\Gamma_{ba}=g^*(|g|^2-|\mathcal{E}|^2+d_+^ad_+^b)-|g|^2d_-^b+|\mathcal{E}|^2d_+^b-d_+^ad_-^bd_+^b=(g^*-d_-^b)(|g|^2-|\mathcal{E}|^2+d_+^ad_+^b)+|\mathcal{E}|^2(d_+^b-d_-^b)$$
$$\Sigma_{ab}(\nu)=g\Lambda_++\Gamma_{ab}=g(|g|^2-|\mathcal{E}|^2+d_+^ad_+^b)+|g|^2d_-^a-|\mathcal{E}|^2d_+^a+d_+^bd_-^ad_+^a=(g+d_-^a)(|g|^2-|\mathcal{E}|^2+d_+^ad_+^b)-|\mathcal{E}|^2(d_+^a-d_-^a)$$

)

$$\hat{H} = \hbar\Delta_a \hat{a}^\dagger \hat{a} + \hbar\Delta_b \hat{b}^\dagger \hat{b} + i\hbar|\mathcal{E}|\left[\hat{a}^\dagger \hat{b}^\dagger e^{i\theta} - \hat{a}\hat{b}e^{-i\theta}\right] + i\hbar|g|\left[\hat{a}^\dagger \hat{b} e^{i\alpha} - \hat{a}\hat{b}^\dagger e^{-i\alpha}\right] + \hbar\left[\Omega(t)\hat{a}^\dagger + \Omega^*(t)\hat{a}\right]$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{a}(t)\right] - \kappa_a \hat{a}(t) - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - k_a e^{i\varphi_a}\hat{a}(t-\tau_a)$$

$$\frac{d\hat{b}}{dt} = \frac{i}{\hbar}\left[\hat{H}(t), \hat{b}(t)\right] - \kappa_b \hat{b}(t) - \sqrt{2\kappa_b}\,\hat{b}_{in}(t) - k_b e^{i\varphi_b}\hat{b}(t-\tau_b)$$

$$k_a e^{i\varphi_a} (t-\tau_a) = \int_0^t dt'\, k_a e^{i\varphi_a}\delta(t-\tau_a-t')\,\hat{a}(t') = \int_0^t dt'\, f_a(t-t')\,\hat{a}(t')$$

$$k_b e^{i\varphi_b}\ \ t-\tau_b) = \int_0^t dt'\, k_b e^{i\varphi_b}\delta(t-\tau_b-t')\,\hat{b}(t') = \int_0^t dt'\, f_b(t-t')\,\hat{b}(t')$$

$$\frac{d\hat{a}}{dt} = -(\kappa_a + i\Delta_a)\hat{a}(t) + \mathcal{E}\hat{b}^\dagger(t) + g\,\hat{b}(t) - \int_0^t dt'\, f_a(t-t')\,\hat{a}(t') - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - i\,\Omega(t)$$

$$\frac{d\hat{b}}{dt} = -(\kappa_b\ \ i\Delta_b)\hat{b}(t) + \mathcal{E}\hat{a}^\dagger(t) - g^*\hat{a}(t) - \int_0^t dt'\, f_b(t-t')\,\hat{b}(t') - \sqrt{2\kappa_b}\,\hat{b}_{in}(t)$$

Scope the model

- NDPA: - $\hat{a}$ & $\hat{b}$ different optical modes
  - $g=0$ ⇒ no particle conversion, $\Omega(t)=0$ no driving
  - Δ_a & Δ_b: respective detuning from resonance
  - observed quantity: quadrature variance ($\hat{a}$&$\hat{b}$) for entanglement
- Optomech: - $\hat{a}$: optical, $\hat{b}$: mechanical mode
  - Δ_a: carrier freq. detuning from resonance, $\Delta_b = \Omega_m$
  - $k_b = 0$
  - erved quantity: phonon number for cooling

- Atom-cavity sys: - $\hat{a}$: optical, $\hat{b} = \hat{\sigma}_-$ lowering op
  - Δ_a: carrier detuning from resonance, $\Delta_b = \omega_a$
  - low driving strength $\Rightarrow \langle\sigma_z\rangle \approx -1$ ⇒ low excitation number
  - $k_b = 0$
  - observed quantity: photon number

Simon's notation:

$$\left[\hat{a}_{in}(t), \hat{a}^\dagger_{in}(t')\right] = \frac{1}{2\kappa_a}\Big\{\Big[\Big(\sqrt{2\kappa_{a1}}\,\hat{a}_{1,in}(t) + \sqrt{2\kappa_{a2}(1-L)}\,e^{i\varphi_a}\hat{a}_{1,in}(t-\tau_a)\Big), \Big(\sqrt{2\kappa_{a1}}\,\hat{a}^\dagger_{1,in}(t') + \sqrt{2\kappa_{a2}(1-L)}\,e^{-i\varphi_{a1}}\hat{a}^\dagger_{in}(t'-\tau_a)\Big)\Big]$$

$$+ 2\kappa_{a2}L\left[\hat{\xi}(t), \hat{\xi}^\dagger(t')\right]\Big\} =$$

$$= \frac{1}{2\kappa_a}\Big\{\left(2\kappa_{a1} + 2\kappa_{a2}(1-L) + 2\kappa_{a2}L\right)\delta(t-t') + k_a\left(e^{i\varphi_a}\delta(t-\tau_a-t') + e^{-i\varphi_a}\delta(t+\tau_a-t')\right)\Big\} =$$

$$= \delta(t-t') + \frac{k_a}{2\kappa_a}\left(e^{i\varphi_a}\delta(t-\tau_a-t') + e^{-i\varphi_a}\delta(t+\tau_a-t')\right) = f(t)/2\kappa_a$$

in frequency space:

$$\left[\tilde{a}_{in}(\nu), \tilde{a}^\dagger_{in}(\nu')\right] = \frac{1}{2\kappa_a}\Big\{\left(\sqrt{2\kappa_{a1}} + \sqrt{2\kappa_{a2}(1-L)}\,e^{i(\varphi_a+\nu\tau_a)}\right)\left(\sqrt{2\kappa_{a1}} + \sqrt{2\kappa_{a2}(1-L)}\,e^{-i(\varphi_a-\nu'\tau_a)}\right) + 2\kappa_{a2}L\Big\}\delta(\nu+\nu') =$$

$$= \frac{1}{2\kappa_a}\Big\{2\kappa_{a1} + 2\kappa_{a2}(1-L) + 2\kappa_{a2}L + 2k_a\cos(\varphi_a+\nu\tau_a)\Big\}\delta(\nu+\nu') =$$

$$= \left(1 + \frac{k_a}{\kappa_a}\cos(\varphi_a+\nu\tau_a)\right)\delta(\nu+\nu') = 2\pi\,\frac{\mathcal{F}'(\nu)}{2\kappa_a}$$

f_a & f_b is consitent with $\left[\hat{a}_{in}(t), \hat{a}^\dagger_{in}(t')\right]$ & $\left[\hat{b}_{in}(t), \hat{b}^\dagger_{in}(t')\right]$

$$\frac{d\hat{a}}{dt} = -(\kappa_a + i\Delta_a)\hat{a}(t) + \mathcal{E}\hat{b}^{+}(t) + g\,\hat{b}(t) - \int_0^t dt'\, f_a(t-t')\,\hat{a}(t') - \sqrt{2\kappa_a}\,\hat{a}_{in}(t) - i\,\Omega(t)$$
$$\frac{d\hat{b}}{dt} = -(\kappa_b \;\; i\Delta_b)\hat{b}(t) + \mathcal{E}\hat{a}^{+}(t) - g^{*}\hat{a}(t) - \int_0^t dt'\, f_b(t-t')\,\hat{b}(t') - \sqrt{2\kappa_b}\,\hat{b}_{in}(t)$$
$$k_a e^{i\theta_a}\,(t-\tau_a) = \int_0^t dt'\, k_a e^{i\theta_a}\,\delta(t-\tau_a-t')\,\hat{a}(t') = \int_0^t dt'\, f_a(t-t')\,\hat{a}(t')$$
$$k_b e^{i\theta_b}\;\; t-\tau_b) = \int_0^t dt'\, k_b e^{i\theta_b}\,\delta(t-\tau_b-t')\,\hat{b}(t') = \int_0^t dt'\, f_b(t-t')\,\hat{b}(t')$$

$$\frac{d}{dt}\underline{\hat{a}}(t) = -i\,\underline{\underline{\Delta}}\,\underline{\hat{a}}(t) - \int_0^t dt'\,\underline{\underline{F}}(t-t')\,\underline{\hat{a}}(t') - i\left(\underline{\Omega}(t) + \underline{\hat{B}}(t)\right)$$

where

- $\underline{\hat{a}}(t) = (\hat{a}(t), \hat{a}^{+}(t), \hat{b}(t), \hat{b}^{+}(t))$
- $\underline{\underline{\Delta}} = \begin{pmatrix} -(\kappa_a + i\Delta_a) & 0 & g & \mathcal{E} \\ 0 & -(\kappa_a - i\Delta_a) & \mathcal{E}^{*} & g^{*} \\ -g^{*} & \mathcal{E} & -(\kappa_b + i\Delta_b) & 0 \\ \mathcal{E}^{*} & -g & 0 & -(\kappa_b - i\Delta_b) \end{pmatrix}$
- $\underline{\underline{F}}(t) = \mathrm{diag}\left(k_a e^{i\theta_a}\delta(t-\tau_a),\, k_a e^{-i\theta_a}\delta(t-\tau_a),\, k_b e^{i\theta_b}\delta(t-\tau_b),\, k_b e^{-i\theta_b}\delta(t-\tau_b)\right)$
- $\underline{\Omega}(t) = (\Omega(t), \Omega^{*}(t), 0, 0)$
- $\underline{\hat{B}}(t) = -i\left(\sqrt{2\kappa_a}\,\hat{a}_{in}(t), \sqrt{2\kappa_a}\,\hat{a}^{+}_{in}(t), \sqrt{2\kappa_b}\,\hat{b}_{in}(t), \sqrt{2\kappa_b}\,\hat{b}^{+}_{in}(t)\right)$

Solution:

$$s\,\underline{\tilde{a}}(s) - \underline{\hat{a}}(0) = -i\,\underline{\underline{\Delta}}\,\underline{\tilde{a}}(s) - \underline{\underline{\tilde{F}}}(s)\,\underline{\tilde{a}}(s) - i\left(\underline{\tilde{\Omega}}(s) + \underline{\tilde{B}}(s)\right)$$
$$\underline{\tilde{a}}(s) = \left(s\mathbb{1} + i\,\underline{\underline{\Delta}} + \underline{\underline{\tilde{F}}}(s)\right)^{-1}\left(\underline{\hat{a}}(0) - i\left(\underline{\tilde{\Omega}}(s) + \underline{\tilde{B}}(s)\right)\right) = \underline{\underline{\tilde{g}}}(s)\left(\underline{\hat{a}}(0) - i\left(\underline{\tilde{\Omega}}(s) + \underline{\tilde{B}}(s)\right)\right)$$
$$\underline{\hat{a}}(t) = \underline{\underline{G}}(t)\,\hat{a} - i(\underline{\underline{G}} * \underline{\Omega})(t) - i(\underline{\underline{G}} * \underline{\hat{B}})(t) = \underline{\underline{G}}(t)\,\hat{a} - i(\underline{\underline{G}} * \underline{\Omega})(t) - i\,\underline{\hat{C}}(t)$$

$\underline{\underline{G}}(t)$: Green-function that satisfies:

$$\frac{d}{dt}\underline{\underline{G}}(t) = -i\,\underline{\underline{\Delta}}\,\underline{\underline{G}}(t) - \int_0^t dt'\,\underline{\underline{F}}(t-t')\,\underline{\underline{G}}(t')$$

According to arXiv 1602.03971 the time-local master equation is:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i\sum_j\left[\xi_j(t)\,\hat{a}_j^{+} + \xi_j^{*}(t)\,\hat{a}_j, \hat{\rho}_S(t)\right] + \sum_{jk}\left\{\gamma_{jk}(t)\left[\hat{a}_k\hat{\rho}_S(t), \hat{a}_j^{+}\right] + \gamma_{jk}^{*}(t)\left[\hat{a}_j, \hat{\rho}_S(t)\hat{a}_k^{+}\right]\right\} +$$
$$+ \sum_{jk}\lambda_{jk}(t)\left[\left[\hat{a}_k, \hat{\rho}_S(t)\right]\hat{a}_j^{+}\right]$$

where

- $\underline{\underline{\gamma}}(t) = -\left(\frac{d}{dt}\underline{\underline{G}}(t)\right)\underline{\underline{G}}^{-1}(t)$
- $\underline{\xi}(t) = \left[\underline{\underline{\gamma}}(t) + \frac{d}{dt}\mathbb{1}\right](\underline{\underline{G}} * \underline{\Omega})(t)$
- $\underline{\underline{\lambda}}(t) = \frac{d}{dt}\underline{\underline{W}}(t) + \underline{\underline{\gamma}}(t)\,\underline{\underline{W}}(t) + \underline{\underline{W}}(t)\,\underline{\underline{\gamma}}^{+}(t)$
  - $\hookrightarrow \underline{\underline{W}}(t) = \int_0^t dt_1 \int_0^t dt_2\, \underline{\underline{G}}(t-t_1)\,\underline{\underline{V}}(t_{12})\,\underline{\underline{G}}^{+}(t-t_2)$
    - $\hookrightarrow V_{jk}(t_{12}) = \langle B_k^{+}(t_2)\,B_j(t_1)\rangle$

$$\underline{\underline{V}}(t_{12}) = -\begin{pmatrix} 2\kappa_a\langle\hat{a}^{+}_{in}(t_2)\hat{a}_{in}(t_1)\rangle & 2\kappa_a\langle\hat{a}_{in}(t_2)\hat{a}_{in}(t_1)\rangle & 0 & 0 \\ 2\kappa_a\langle\hat{a}^{+}_{in}(t_2), \hat{a}^{+}_{in}(t_1)\rangle & 2\kappa_a\langle\hat{a}_{in}(t_2)\hat{a}^{+}_{in}(t_1)\rangle & 0 & 0 \\ 0 & 0 & 2\kappa_b\langle\hat{b}^{+}_{in}(t_2)\hat{b}_{in}(t_1)\rangle & 2\kappa_b\langle\hat{b}_{in}(t_2)\hat{b}_{in}(t_1)\rangle \\ 0 & 0 & 2\kappa_b\langle\hat{b}^{+}_{in}(t_2)\hat{b}^{+}_{in}(t_1)\rangle & 2\kappa_b\langle\hat{b}_{in}(t_2)\hat{b}^{+}_{in}(t_1)\rangle \end{pmatrix}$$