Explicit time-evolution in first order:

$$|\psi(t_{k+1})\rangle = U(t_{k+1}, t_k)|\psi(t_k)\rangle$$

$$U(t_{k+1}, t_k) = \exp\left[-\frac{i}{\hbar}\int_{t_k}^{t_{k+1}} (H_s + H_{fb}(t))\,dt\right]$$

first order

$$U = \exp(M_s + M_B) = \mathbb{1} + M_B + M_s + \frac{1}{2}M_B^2 \Big| + \frac{1}{2}(M_s M_B + M_B M_s) + \frac{1}{6}M_B^3 + \frac{1}{2}M_s^2 + \frac{1}{6}(M_B^2 M_s + M_B M_s M_B + M_s M_B^2) + \frac{1}{24}M_B^4 + \mathcal{O}(\Delta t^{5/2})$$

$$M_s = -\frac{i}{\hbar}H_s\Delta t \qquad M_B = -\frac{i}{\hbar}\int_{t_k}^{t_{k+1}} H_{fb}(t)\,dt$$

$$H_{fb}(t) = -i\hbar\left\{\left[\sqrt{\gamma_R}\, b(t-\tau)e^{-i\varphi} + \sqrt{\gamma_L}\, b(t)\right]c^+ - \left[\sqrt{\gamma_R}\, b^+(t-\tau)e^{i\varphi} + \sqrt{\gamma_L}\, b^+(t)\right]c\right\}$$

$$H_s = \hbar\Delta_c c^+c + \hbar\Delta_e\sigma_+\sigma_- + \hbar g(c^+\sigma_- + c\sigma_+) + \frac{\hbar\Omega_c}{2}(c + c^+) + \frac{\hbar\Omega_e}{2}(\sigma_- + \sigma_+)$$

$$M_B = -\Big\{\Big[\sqrt{\gamma_R}e^{-i\varphi}\underbrace{\int_{t_k}^{t_{k+1}} b(t-\tau)\,dt}_{\tau = e\Delta t} + \sqrt{\gamma_L}\underbrace{\int_{t_k}^{t_{k+1}} b(t)\,dt}_{\Delta B(t_k)}\Big]c^+ - \Big[\sqrt{\gamma_R}e^{i\varphi}\underbrace{\int_{t_k}^{t_{k+1}} b^+(t-\tau)\,dt}_{\Delta B^+(t_{k-e})} + \sqrt{\gamma_L}\underbrace{\int_{t_k}^{t_{k+1}} b^+(t)\,dt}_{\Delta B^+(t_k)}\Big]c\Big\}$$

$$\int_{t_{k-e}}^{t_{k-e+1}} b(t)\,dt = \Delta B(t_{k-e})$$

$$[\Delta B(t_k), \Delta B^+(t_j)] = \int_{t_k}^{t_{k+1}}\int_{t_j}^{t_{j+1}} \underbrace{[b(t), b^+(t')]}^{\delta(t-t')}\,dt'\,dt = \int_{t_k}^{t_{k+1}} dt\,\delta_{jk} = \Delta t\,\delta_{jk}$$

$$\begin{aligned}M_B^2 = &\left[\gamma_R e^{-2i\varphi}\Delta B(t_{k-e})\Delta B(t_{k-e}) + \sqrt{\gamma_R\gamma_L}\left(\Delta B(t_{k-e})\Delta B(t_k) + \Delta B(t_k)\Delta B(t_{k-e})\right)e^{-i\varphi} + \gamma_L\Delta B(t_k)\Delta B(t_k)\right]c^+c^+ -\\ &-\left[\gamma_R\Delta B(t_{k-e})\Delta B^+(t_{k-e}) + \sqrt{\gamma_L\gamma_R}\left(\Delta B(t_{k-e})\Delta B^+(t_k)e^{-i\varphi} + \Delta B(t_k)\Delta B^+(t_{k-e})e^{i\varphi}\right) + \gamma_L\Delta B(t_k)\Delta B^+(t_k)\right]c^+c -\\ &-\left[\gamma_R\Delta B^+(t_{k-e})\Delta B(t_{k-e}) + \sqrt{\gamma_L\gamma_R}\left(e^{i\varphi}\Delta B^+(t_{k-e})\Delta B(t_k) + e^{-i\varphi}\Delta B^+(t_k)\Delta B(t_{k-e})\right) + \gamma_L\Delta B^+(t_k)\Delta B(t_k)\right]cc^+ +\\ &+\left[\gamma_R e^{2i\varphi}\Delta B^+(t_{k-e})\Delta B^+(t_{k-e}) + \sqrt{\gamma_L\gamma_R}\left(\Delta B^+(t_{k-e})\Delta B^+(t_k) + \Delta B^+(t_k)\Delta B^+(t_{k-e})\right)e^{i\varphi} + \gamma_L\Delta B^+(t_k)\Delta B^+(t_k)\right]cc\end{aligned}$$

$$|i_p\rangle = \frac{(\Delta B^+(t_p))^{i_p}}{\sqrt{i_p!\,\Delta t^{i_p}}}|vac\rangle \qquad \Delta B(t_k)|i_k\rangle = \frac{\Delta B(t_k)[\Delta B^+(t_k)]^{i_k}}{\sqrt{i_k!\,\Delta t^{i_k}}}|vac\rangle = \frac{i_k\Delta t\,(\Delta B^+(t_k))^{i_k-1}}{\sqrt{i_k!\,\Delta t^{i_k}}}|vac\rangle = \sqrt{i_k\Delta t}\,|i_k-1\rangle$$

$$[\Delta B(t_k), (\Delta B^+(t_k))^{i_k}] = [\Delta B(t_k), \Delta B^+(t_k)](\Delta B^+(t_k))^{i_k-1} + \Delta B^+(t_k)[\Delta B(t_k), (\Delta B^+(t_k))^{i_k-1}] = \Delta t(\Delta B^+(t_k))^{i_k-1} + \Delta B^+(t_k)\Delta t(\Delta B^+(t_k))^{i_k-2} + \ldots + (\Delta B^+(t_k))^{i_k-1}\Delta t = i_k\Delta t(\Delta B^+(t_k))^{i_k-1}$$

$$\Delta B^+(t_k)|i_k\rangle = \frac{\Delta B^+(t_k)[\Delta B^+(t_k)]^{i_k}}{\sqrt{i_k!\,\Delta t^{i_k}}}|vac\rangle = \sqrt{(i_k+1)\Delta t}\,|i_k+1\rangle$$

$\Phi = |i_c, i_e\rangle \bigotimes_{p=-\infty}^{\infty} |i_p\rangle$ $\quad p$ shows the different timebins for the environment

Matrix elements of the evolution operator:

$$\langle j_c, j_e, \{j_p\}|U(t_{k+1}, t_k)|i_c, i_e, \{i_p\}\rangle = \delta_{j_{-\infty}i_{-\infty}}\cdots\delta_{j_{k-e-1}i_{k-e-1}}\delta_{j_{k-e+1}i_{k-e+1}}\cdots\delta_{j_{k-1}i_{k-1}}\delta_{j_{k+1}i_{k+1}}\cdots\delta_{j_\infty i_\infty}\cdot$$

$$\begin{aligned}\cdot\Big\{&\delta_{j_ci_c}\delta_{j_ei_e}\delta_{j_ki_k}\delta_{j_{k-e}i_{k-e}} +\\ &+(-\sqrt{\Delta t})\,\delta_{j_ei_e}\Big[\delta_{j_ki_k}\sqrt{\gamma_R}e^{-i\varphi}\delta_{j_{k-e}i_{k-e}-1}\delta_{j_ci_c+1}\sqrt{i_{k-e}(i_c+1)} + \delta_{j_{k-e}i_{k-e}}\sqrt{\gamma_L}\delta_{j_ki_k-1}\delta_{j_ci_c+1}\sqrt{i_k(i_c+1)} -\\ &\qquad -\delta_{j_ki_k}\sqrt{\gamma_R}e^{i\varphi}\delta_{j_{k-e}i_{k-e}+1}\delta_{j_ci_c-1}\sqrt{(i_{k-e}+1)i_c} + \delta_{j_{k-e}i_{k-e}}\sqrt{\gamma_L}\delta_{j_ki_k+1}\delta_{j_ci_c-1}\sqrt{(i_k+1)i_c}\Big] +\\ &-i\,\delta_{j_ki_k}\delta_{j_{k-e}i_{k-e}}\Big[\Delta_c i_c\delta_{j_ci_c}\delta_{j_ei_e} + \Delta_e\delta_{j_ci_c}\delta_{j_ee}\delta_{i_ee} + g\left(\sqrt{i_c+1}\,\delta_{j_ci_c+1}\delta_{j_eg}\delta_{i_ee} + \sqrt{i_c}\,\delta_{j_ci_c-1}\delta_{j_ee}\delta_{i_eg}\right) + \frac{\Omega_c}{2}\left(\sqrt{i_c}\,\delta_{j_ci_c-1} + \sqrt{i_c+1}\,\delta_{j_ci_c+1}\right)\delta_{j_ei_e} +\\ &\qquad + \frac{\Omega_e}{2}\left(\delta_{j_eg}\delta_{i_ee} + \delta_{j_ee}\delta_{i_eg}\right)\delta_{j_ci_c}\Big]\Delta t +\\ &+\delta_{j_ei_e}\frac{\Delta t}{2}\Big[\left(\gamma_R e^{-2i\varphi}\delta_{j_{k-e}i_{k-e}-2}\sqrt{i_{k-e}(i_{k-e}-1)}\,\delta_{j_ki_k} + \sqrt{\gamma_L\gamma_R}\,e^{-i\varphi}2\,\delta_{j_{k-e}i_{k-e}-1}\delta_{j_ki_k-1}\sqrt{i_ki_{k-e}} + \gamma_L\delta_{j_ki_k-2}\sqrt{i_k(i_k-1)}\right)\delta_{j_ci_c+2}\sqrt{(i_c+1)(i_c+2)} -\\ &\qquad -\left(\gamma_R\delta_{j_{k-e}i_{k-e}}\delta_{j_ki_k}(i_{k-e}+1) + \sqrt{\gamma_L\gamma_R}\left\{\delta_{j_{k-e}i_{k-e}-1}\delta_{j_ki_k+1}\sqrt{i_{k-e}(i_k+1)}e^{-i\varphi} + \delta_{j_ki_k-1}\delta_{j_{k-e}i_{k-e}+1}\sqrt{i_k(i_{k-e}+1)}e^{i\varphi}\right\} + \gamma_L\delta_{j_{k-e}i_{k-e}}\delta_{j_ki_k}(i_k+1)\right)\delta_{j_ci_c}i_c -\\ &\qquad -\left(\gamma_R\delta_{j_{k-e}i_{k-e}}\delta_{j_ki_k}i_{k-e} + \sqrt{\gamma_L\gamma_R}\left\{e^{i\varphi}\delta_{j_{k-e}i_{k-e}+1}\delta_{j_ki_k-1}\sqrt{i_k(i_{k-e}+1)} + e^{-i\varphi}\delta_{j_ki_k+1}\delta_{j_{k-e}i_{k-e}-1}\sqrt{i_{k-e}(i_k+1)}\right\} + \gamma_L\delta_{j_{k-e}i_{k-e}}\delta_{j_ki_k}i_k\right)\delta_{j_ci_c}(i_c+1) +\\ &\qquad +\left(\gamma_R e^{2i\varphi}\delta_{j_{k-e}i_{k-e}+2}\delta_{j_ki_k}\sqrt{(i_{k-e}+1)(i_{k-e}+2)} + \sqrt{\gamma_L\gamma_R}\,2e^{i\varphi}\delta_{j_{k-e}i_{k-e}+1}\delta_{j_ki_k+1}\sqrt{(i_k+1)(i_{k-e}+1)} + \gamma_L\delta_{j_ki_k+2}\sqrt{(i_k+1)(i_k+2)}\right)\delta_{j_ci_c-2}\sqrt{i_c(i_c-1)}\Big]\Big\}\end{aligned}$$

Evolving the wave function in MPS representation:

$$U^{i_c, i_e, \{i_p\}}_{j_c, j_e, \{j_p\}}\,\psi_{i_c, i_e, \{i_p\}}(0) = \psi_{\underbrace{j_c, j_e}_{A_c\otimes A_e}, \{j_p\}}$$

diagonalization → diagonalized Hamiltonian gives only 1 index for the system

$$\psi(0) = A_\infty \otimes \ldots \otimes A_1 \otimes A_0 \otimes \overbrace{A_s} \otimes A_{-1} \otimes \ldots \otimes A_{-e+1} \otimes \underset{\tau}{A_{-e}} \otimes A_{e-1} \otimes \ldots \otimes A_{-\infty}$$

tensor product of vectors

Basis for the system Hamiltonian:

$$\hbar\Delta_c c^+c + \hbar\Delta_e \sigma_+\sigma_- + \hbar g\,(c^+\sigma_- + c\,\sigma_+) + \frac{\hbar\Omega_c}{2}(c + c^+) + \frac{\hbar\Omega_e}{2}(\sigma_- + \sigma_+)$$

i_s i_c i_e

0 $|0\rangle|g\rangle \rightarrow \frac{\hbar\Omega_c}{2}|1\rangle|g\rangle + \frac{\hbar\Omega_e}{2}|0\rangle|e\rangle$

1 $|0\rangle|e\rangle \rightarrow \hbar\Delta_e|0\rangle|e\rangle + \hbar g|1\rangle|g\rangle + \frac{\hbar\Omega_c}{2}|1\rangle|e\rangle + \frac{\hbar\Omega_e}{2}|0\rangle|g\rangle$

2 $|1\rangle|g\rangle \rightarrow \hbar\Delta_c|1\rangle|g\rangle + \hbar g|0\rangle|e\rangle + \frac{\hbar\Omega_c}{2}(|2\rangle|g\rangle + |0\rangle|g\rangle) + \frac{\hbar\Omega_e}{2}|1\rangle|e\rangle$

3 $|1\rangle|e\rangle \rightarrow \hbar(\Delta_c + \Delta_e)|1\rangle|e\rangle + \hbar g|2\rangle|g\rangle + \frac{\hbar\Omega_c}{2}(|2\rangle|e\rangle + |0\rangle|e\rangle) + \frac{\hbar\Omega_e}{2}|1\rangle|g\rangle$

4 $|2\rangle|g\rangle \rightarrow \hbar\Delta_c \cdot 2|2\rangle|g\rangle + \hbar g\sqrt{2}|1\rangle|e\rangle + \frac{\hbar\Omega_c}{2}(\sqrt{2}|1\rangle|g\rangle + \sqrt{3}|3\rangle|g\rangle) + \frac{\hbar\Omega_e}{2}|2\rangle|e\rangle$

5 $|2\rangle|e\rangle \rightarrow \hbar(2\Delta_c + \Delta_e)|2\rangle|e\rangle + \hbar g\sqrt{3}|3\rangle|g\rangle + \frac{\hbar\Omega_c}{2}(\sqrt{2}|1\rangle|e\rangle + \sqrt{3}|3\rangle|e\rangle) + \frac{\hbar\Omega_e}{2}|2\rangle|g\rangle$

6 $|3\rangle|g\rangle \rightarrow \hbar 3\Delta_c|3\rangle|g\rangle + \hbar g\sqrt{3}|2\rangle|e\rangle + \frac{\hbar\Omega_c}{2}(\sqrt{3}|2\rangle|g\rangle + \sqrt{4}|4\rangle|g\rangle) + \frac{\hbar\Omega_e}{2}|3\rangle|e\rangle$

7 $|3\rangle|e\rangle \rightarrow \hbar(3\Delta_c + \Delta_e)|3\rangle|e\rangle + \hbar g\sqrt{4}|4\rangle|g\rangle + \frac{\hbar\Omega_c}{2}(\sqrt{3}|2\rangle|e\rangle + \sqrt{4}|4\rangle|e\rangle) + \frac{\hbar\Omega_e}{2}|3\rangle|g\rangle$

8 $|4\rangle|g\rangle \rightarrow \hbar 4\Delta_c|4\rangle|g\rangle + \hbar g\sqrt{4}|3\rangle|e\rangle + \frac{\hbar\Omega_c}{2}(\sqrt{4}|3\rangle|g\rangle + \sqrt{5}|5\rangle|g\rangle) + \frac{\hbar\Omega_e}{2}|4\rangle|e\rangle$

$i_c = \mathrm{int}\left(\frac{i_s}{2}\right)$ $i_e = \mathrm{mod}_2(i_s)$

$i_c = \mathrm{int}\left(\frac{i_s}{2}\right)$ $i_e = \mathrm{mod}_2(i_s)$

$$\langle j_s, \{j_p\} | U(t_{k+1}, t_k) | i_s, \{i_p\} \rangle = \delta_{j_{-\infty} i_{-\infty}} \cdots \delta_{j_{k-e-1} i_{k-e-1}} \delta_{j_{k-e+1} i_{k-e+1}} \cdots \delta_{j_{k-1} i_{k-1}} \delta_{j_{k+1} i_{k+1}} \cdots \delta_{j_\infty i_\infty} \cdot$$

$$\cdot \Big\{ \delta_{j_s i_s} \delta_{j_k i_k} \delta_{j_{k-e} i_{k-e}} +$$

$$+ (-\sqrt{\Delta t}) \Big[\delta_{j_k i_k} \sqrt{\gamma_R} e^{-i\varphi} \delta_{j_{k-e} i_{k-e}-1} \delta_{j_s i_s+2} \sqrt{i_{k-e}(i_c+1)} + \delta_{j_{k-e} i_{k-e}} \sqrt{\gamma_L} \delta_{j_k i_k-1} \delta_{j_s i_s+2} \sqrt{i_k(i_c+1)} -$$

$$- \delta_{j_k i_k} \sqrt{\gamma_R} e^{i\varphi} \delta_{j_{k-e} i_{k-e}+1} \delta_{j_s i_s-2} \sqrt{(i_{k-e}+1)\, i_c} + \delta_{j_{k-e} i_{k-e}} \sqrt{\gamma_L} \delta_{j_k i_k+1} \delta_{j_s i_s-2} \sqrt{(i_k+1)\, i_c} \Big] +$$

$$- i\, \delta_{j_k i_k} \delta_{j_{k-e} i_{k-e}} \Big[\mathrm{int}\left(\frac{i_s}{2}\right) \Delta_c + \mathrm{mod}_2(i_s)\ \Delta_e \Big] \delta_{j_s, i_s} + g \Big[\mathrm{mod}_2(i_s) \sqrt{\mathrm{int}\left(\frac{i_s}{2}\right)+1}\ \delta_{j_s, i_s+1} + \mathrm{mod}_2(i_s+1) \sqrt{\mathrm{int}\left(\frac{i_s}{2}\right)}\ \delta_{j_s, i_s-1} \Big] +$$

$$+ \frac{\Omega_c}{2} \Big[\sqrt{\mathrm{int}\left(\frac{i_s}{2}\right)}\ \delta_{j_s, i_s-2} + \sqrt{\mathrm{int}\left(\frac{i_s}{2}\right)+1}\ \delta_{j_s, i_s+2} \Big] + \frac{\Omega_e}{2} \Big[\mathrm{mod}_2(i_s)\, \delta_{j_s, i_s-1} + \mathrm{mod}_2(i_s+1)\, \delta_{j_s, i_s+1} \Big] \Delta t +$$

$$+ \frac{1}{2}\Delta t \Big[\Big(\gamma_R e^{-2i\varphi} \delta_{j_{k-e} i_{k-e}-2} \sqrt{i_{k-e}(i_{k-e}-1)}\ \delta_{j_k i_k} + \sqrt{\gamma_L \gamma_R}\, e^{-i\varphi}\, 2\, \delta_{j_{k-e} i_{k-e}-1} \delta_{j_k i_k-1} \sqrt{i_k i_{k-e}} + \gamma_L \delta_{j_k i_k-2} \sqrt{i_k(i_k-1)} \Big) \delta_{j_s i_s+4} \sqrt{(i_c+1)(i_c+2)} -$$

$$- \Big(\gamma_R \delta_{j_{k-e} i_{k-e}} \delta_{j_k i_k} (i_{k-e}+1) + \sqrt{\gamma_L \gamma_R} \big\{ \delta_{j_{k-e} i_{k-e}-1} \delta_{j_k i_k+1} \sqrt{i_{k-e}(i_k+1)}\, e^{-i\varphi} + \delta_{j_k i_k-1} \delta_{j_{k-e} i_{k-e}+1} \sqrt{i_k(i_{k-e}+1)}\, e^{i\varphi} \big\} + \gamma_L \delta_{j_{k-e} i_{k-e}} \delta_{j_k i_k} (i_k+1) \Big) \delta_{j_s i_s}\, i_c -$$

$$- \Big(\gamma_R \delta_{j_{k-e} i_{k-e}} \delta_{j_k i_k}\, i_{k-e} + \sqrt{\gamma_L \gamma_R} \big\{ e^{-i\varphi} \delta_{j_{k-e} i_{k-e}+1} \delta_{j_k i_k-1} \sqrt{i_k(i_{k-e}+1)} + e^{i\varphi} \delta_{j_k i_k+1} \delta_{j_{k-e} i_{k-e}-1} \sqrt{i_{k-e}(i_k+1)} \big\} + \gamma_L \delta_{j_{k-e} i_{k-e}} \delta_{j_k i_k}\, i_k \Big) \delta_{j_s i_s} (i_c+1) +$$

$$+ \Big(\gamma_R e^{2i\varphi} \delta_{j_{k-e} i_{k-e}+2} \delta_{j_k i_k} \sqrt{(i_{k-e}+1)(i_{k-e}+2)} + \sqrt{\gamma_L \gamma_R}\, 2 e^{i\varphi} \delta_{j_{k-e} i_{k-e}+1} \delta_{j_k i_k+1} \sqrt{(i_k+1)(i_{k-e}+1)} + \gamma_L \delta_{j_k i_k+2} \sqrt{(i_k+1)(i_k+2)} \Big) \delta_{j_s i_s-4} \sqrt{i_c(i_c-1)} \Big] \Big\}$$

SWAP gate

$|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|0\rangle|2\rangle$, $|0\rangle|3\rangle$, $|0\rangle|4\rangle$

$|1\rangle|0\rangle$, $|1\rangle|1\rangle$, $|1\rangle|2\rangle$, $|1\rangle|3\rangle$, $|1\rangle|4\rangle$

$|2\rangle|0\rangle$, $|2\rangle|1\rangle$, $|2\rangle|2\rangle$, $|2\rangle|3\rangle$, $|2\rangle|4\rangle$

$|3\rangle|0\rangle$, $|3\rangle|1\rangle$, $|3\rangle|2\rangle$, $|3\rangle|3\rangle$, $|3\rangle|4\rangle$

$|4\rangle|0\rangle$, $|4\rangle|1\rangle$, $|4\rangle|2\rangle$, $|4\rangle|3\rangle$, $|4\rangle|4\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \\ |0\rangle|0\rangle & |0\rangle|1\rangle & |1\rangle|0\rangle & |1\rangle|1\rangle \end{array} \\ \begin{array}{c} |0\rangle|0\rangle \\ |0\rangle|1\rangle \\ |1\rangle|0\rangle \\ |1\rangle|1\rangle \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} |0\rangle|0\rangle & |0\rangle|1\rangle & |1\rangle|0\rangle & |1\rangle|1\rangle & |1\rangle|2\rangle & |2\rangle|1\rangle & |2\rangle|2\rangle & |0\rangle|2\rangle & |2\rangle|0\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$|4\rangle|4\rangle$

5 diag & $\frac{5 \cdot 4}{2}$ pair (10)

3 diag & $\frac{3 \cdot 2}{2}$ = 3 pair

The code

1) Creation of the wave function

- The wave function is initially a tensor product of the states in the different time bins. The objective of the MPS representation is to keep these time bins separate, recording the connections between them. In this way the created tensors can be stored separately on the hard drive.
- Depending on the order of precision, the photon limit in each environment bin is given. E.g. for $O(\Delta t^{3/2})$ it is 4. $\Rightarrow$ basis size: 5
- Simplest system: 1 TLS with feedback. System basis size: 2

$$H_{SYS,TLS} = \hbar\omega_{TLS}\sigma_+\sigma_- + \hbar\Omega_{TLS}\left(e^{-i\omega_L t}\sigma_+ + e^{i\omega_L t}\sigma_-\right)$$

- Number of time bin: "$\gamma\tau \gg 1$" + "$\tau = \Delta t \cdot e$"

2.) SWAP: Actually it might just be easier to swap the indices in numpy.

- The time bins to be SWAP-ed have to be merged into a common state vector.
- SWAP matrix should be built as n = max num of photons in a bin:

$$n+1 \text{ diagonal } \& \ \frac{(n+1)n}{2} \text{ pairs} \qquad \left(n+1+2\frac{(n+1)n}{2} = n^2+2n+1 = (n+1)^2\right)$$

- Act on the common state vector with the SWAP
- Decompose state vectors

e.g. $\begin{matrix}|0\rangle\\|1\rangle\\|2\rangle\end{matrix}\left(\ \right) \otimes \begin{matrix}|0\rangle\\|1\rangle\\|2\rangle\end{matrix}\left(\ \right) = \begin{matrix}|00\rangle & |01\rangle & |02\rangle\\|10\rangle & |11\rangle & |12\rangle\\|20\rangle & |21\rangle & |22\rangle\end{matrix}\left(\qquad\right) \Rightarrow \begin{matrix}|00\rangle\\|01\rangle\\|02\rangle\\|10\rangle\\|11\rangle\\|12\rangle\\|20\rangle\\|21\rangle\\|22\rangle\end{matrix}\left|\ \right|$

SWAP

$$\begin{array}{c|ccccccccc}
 & |00\rangle & |01\rangle & |02\rangle & |10\rangle & |11\rangle & |12\rangle & |20\rangle & |21\rangle & |22\rangle \\
\hline
|00\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|01\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
|02\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
|10\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|11\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
|12\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
|20\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
|21\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
|22\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$$\delta_{00} + \delta_{13} + \delta_{26} + \delta_{31} + \delta_{44} + \delta_{57} + \delta_{62} + \delta_{75} + \delta_{88}$$

From this, the general vector state is: MERGED WAVE FUNCTION

$|00\rangle, |01\rangle, \dots, |0n\rangle, |10\rangle, \dots |1,n\rangle, |2,0\rangle \dots \dots |n-1,n\rangle, |n,0\rangle \dots |n,n\rangle$

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \delta_{(n+1)i+j,\,(n+1)j+i}$$

SWAP OPERATOR

$n=2 \Rightarrow \delta_{00}+\delta_{13}+\delta_{26}+\delta_{31}+\delta_{44}+\delta_{57}+\delta_{62}+\delta_{75}+\delta_{88}$ ✓

Check: $n=3$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\|03\rangle$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\|13\rangle$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\|23\rangle$	$\|30\rangle$	$\|31\rangle$	$\|32\rangle$	$\|33\rangle$

$$\delta_{00}+\delta_{14}+\delta_{28}+\delta_{3,12}+\delta_{41}+\delta_{55}+\delta_{69}+\delta_{7,13}+\delta_{8,2}+\delta_{96}+\delta_{10,10}+\delta_{11,14}+\delta_{12,3}+\delta_{13,7}+\delta_{14,11}+\delta_{5,15}$$

$$\delta_{00}+\delta_{41}+\delta_{82}+\delta_{12,3}+\delta_{1,4}+\delta_{55}+\delta_{96}+\delta_{13,7}+\delta_{2,8}+\delta_{6,9}+\delta_{10,10}+\delta_{14,11}+\delta_{3,12}+\delta_{7,13}+\delta_{11,14}+\delta_{15,15}$$

Decomposing: convert back to $(n+1)\otimes(n+1) \longrightarrow$ SVD

$$i_1 \cdot i_2 \cdot i_3 \qquad \left(\begin{matrix} \overset{000}{0} & \overset{001}{0} & \overset{002}{0} \\ \overset{010}{0} & \overset{011}{0} & \overset{012}{0} \end{matrix}\right), \left(\begin{matrix} \overset{100}{0} & \overset{101}{0} & \overset{102}{0} \\ \overset{110}{0} & \overset{111}{1} & \overset{112}{2} \end{matrix}\right), \left(\begin{matrix} \overset{200}{0} & \overset{201}{0} & \overset{202}{0} \\ \overset{210}{0} & \overset{211}{2} & \overset{212}{4} \end{matrix}\right)$$

$$\overset{i_2=0}{0},0,0,\overset{i_2=1}{0},0,0,\overset{i_2=0}{0},0,0,\overset{i_2=1}{0},1,2,\overset{i_2=0}{0},0,0,\overset{i_2=1}{0},2,4$$

$$H^{1}_{SYS,TLS} = \hbar\omega_{TLS}\sigma_+\sigma_- + \hbar\Omega_{TLS}\left(e^{-i\omega_L t}\sigma_+ + e^{i\omega_L t}\sigma_-\right)$$

$$H^{1}_{fb} = \int_B d\omega\, \hbar\omega\, b^+(\omega)b(\omega) + i\hbar\int_B d\omega \left[\left(\kappa_R(\omega)e^{i\omega L/c_0} - \kappa_L(\omega)e^{-i\omega L/c_0}\right)b^+(\omega)\sigma_- - \left(\kappa_R^*(\omega)e^{-i\omega L/c_0} - \kappa_L^*(\omega)e^{i\omega L/c_0}\right)b(\omega)\sigma_+\right]$$

$$H_{SYS,TLS} = \hbar\Delta_T\sigma_+\sigma_- + \hbar\Omega_{TLS}(\sigma_+ + \sigma_-)$$

$$H_{fb} = -i\hbar\left\{\left[\sqrt{\gamma_R}\, b(t-\tau)e^{-i\varphi} + \sqrt{\gamma_L}\, b(t)\right]\sigma_+ - \left[\sqrt{\gamma_R}\, b^+(t-\tau)e^{i\varphi} + \sqrt{\gamma_L}\, b^+(t)\right]\sigma_-\right\}$$

$$\langle j_{TLS}, \{j\} | U(t_{k+1}, t_k) | i_{TLS}, \{i_p\}\rangle = \delta_{j_{-\infty} i_{-\infty}} \cdots \delta_{j_{k-e-1} i_{k-e-1}} \delta_{j_{k-e+1} i_{k-e+1}} \cdots \delta_{j_{k-1} i_{k-1}} \delta_{j_{k+1} i_{k+1}} \cdots \delta_{j_\infty i_\infty} \cdot$$

$$\cdot \Big\{ \delta_{j_T i_T}\delta_{j_k i_k}\delta_{j_{k-e} i_{k-e}} +$$

$$+(-\sqrt{\Delta t})\Big[\delta_{j_k i_k}\sqrt{\gamma_R}e^{-i\varphi}\delta_{j_{k-e} i_{k-e}-1}\sqrt{i_{k-e}}\,\delta_{i_T,g}\delta_{j_T,e} + \delta_{j_{k-e} i_{k-e}}\sqrt{\gamma_L}\,\delta_{j_k i_k-1}\sqrt{i_k}\,\delta_{i_T,g}\delta_{j_T,e} -$$

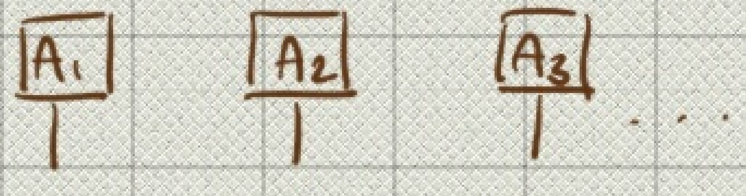
$$-\delta_{j_k i_k}\sqrt{\gamma_R}e^{i\varphi}\delta_{j_{k-e} i_{k-e}+1}\sqrt{i_{k-e}+1}\,\delta_{i_T,e}\delta_{j_T,g} - \delta_{j_{k-e} i_{k-e}}\sqrt{\gamma_L}\,\delta_{j_k i_k+1}\sqrt{i_k+1}\,\delta_{i_T,e}\delta_{j_T,g}\Big] +$$

$$-i\delta_{j_k i_k}\delta_{j_{k-e} i_{k-e}}\left[\Delta_T\delta_{j_T,e}\delta_{i_T,e} + \Omega_{TLS}\left(\delta_{j_T e}\delta_{i_T g} + \delta_{j_T g}\delta_{i_T e}\right)\right]\Delta t +$$

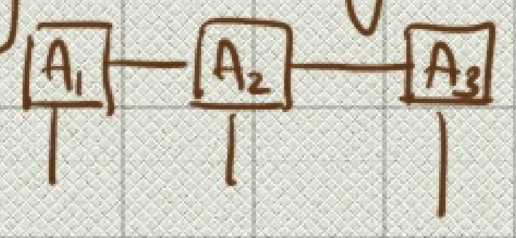
$$-\frac{i}{2}\Big[\Big(\gamma_R\,\delta_{j_{k-e} i_{k-e}}\delta_{j_k i_k}(i_{k-e}+1) + \sqrt{\gamma_L\gamma_R}\left\{\delta_{j_{k-e} i_{k-e}-1}\delta_{j_k i_k+1}\sqrt{i_{k-e}(i_k+1)}\,e^{-i\varphi} + \delta_{j_k i_k-1}\delta_{j_{k-e} i_{k-e}+1}\sqrt{i_k(i_{k-e}+1)}\,e^{i\varphi}\right\} + \gamma_L\delta_{j_{k-e} i_{k-e}}\delta_{j_k i_k}(i_k+1)\Big)\delta_{i_T,e}\delta_{j_T,e} +$$

$$+\Big(\gamma_R\,\delta_{j_{k-e} i_{k-e}}\delta_{j_k i_k}\, i_{k-e} + \sqrt{\gamma_L\gamma_R}\left\{e^{i\varphi}\delta_{j_{k-e} i_{k-e}+1}\delta_{j_k i_k-1}\sqrt{i_k(i_{k-e}+1)} + e^{-i\varphi}\delta_{j_k i_k+1}\delta_{j_{k-e} i_{k-e}-1}\sqrt{i_{k-e}(i_k+1)}\right\} + \gamma_L\,\delta_{j_{k-e} i_{k-e}}\delta_{j_k i_k}\, i_k\Big)\delta_{i_T,g}\delta_{j_T,g}\Big]\Delta t\Big\}$$

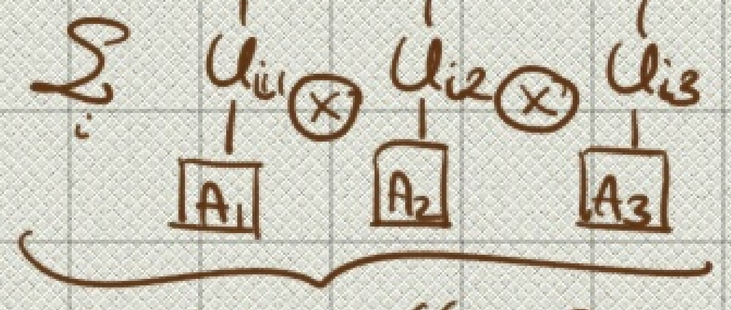
Initially



After the first time step



$$U = \sum_i U_i$$



quantity with 3 indices

formulate as a matrix with S up front

$$M_S = -\frac{i}{\hbar} H_S \Delta t = -i\Delta t \{\Delta_e \sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\} \qquad M_B = -\frac{i}{\hbar}\int_{t_k}^{t_{k+1}} H_{fB}(t)\, dt$$

$$H_{fB}(t) = -i\hbar\{[\sqrt{\gamma_R}\, b(t-\tau)e^{-i\varphi} + \sqrt{\gamma_L}\, b(t)]\sigma_+ - [\sqrt{\gamma_R}\, b^+(t-\tau)e^{i\varphi} + \sqrt{\gamma_L}\, b^+(t)]\sigma_-\}$$

$$H_S = \hbar\Delta_e\sigma_+\sigma_- + \hbar\Omega_e(\sigma_- + \sigma_+)$$

$$M_B = -\Big\{\Big[\sqrt{\gamma_R}e^{-i\varphi}\underbrace{\int_{t_k}^{t_{k+1}} b(t-\tau)dt}_{\tau = \ell\Delta t} + \sqrt{\gamma_L}\underbrace{\int_{t_k}^{t_{k+1}} b(t)dt}_{\Delta B(t_k)}\Big]\sigma_+ - \Big[\sqrt{\gamma_R}e^{i\varphi}\underbrace{\int_{t_k}^{t_{k+1}} b^+(t-\tau)dt}_{\Delta B^+(t_{k-\ell})} + \sqrt{\gamma_L}\underbrace{\int_{t_k}^{t_{k+1}} b^+(t)dt}_{\Delta B^+(t_k)}\Big]\sigma_-\Big\}$$

$$\int_{t_{k-\ell}}^{t_{k-\ell+1}} b(t)dt = \Delta B(t_{k-\ell}) \qquad [\Delta B(t_k), \Delta B^+(t_j)] = \int_{t_k}^{t_{k+1}}\int_{t_j}^{t_{j+1}} \underbrace{[b(t), b^+(t')]}^{\delta(t-t')} dt'dt = \int_{t_k}^{t_{k+1}} dt\, \delta_{jk} = \Delta t\, \delta_{jk}$$

$$M_B = -\{[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B(t_k)]\sigma_+ - [\sqrt{\gamma_R}\Delta B^+(t_{k-\ell})e^{i\varphi} + \sqrt{\gamma_L}\Delta B^+(t_k)]\sigma_-\}$$

$$M_B^2 = [\gamma_R e^{-2i\varphi}\Delta B(t_{k-\ell})\Delta B(t_{k-\ell}) + \sqrt{\gamma_R\gamma_L}(\Delta B(t_{k-\ell})\Delta B(t_k) + \Delta B(t_k)\Delta B(t_{k-\ell}))e^{-i\varphi} + \gamma_L \Delta B(t_k)\Delta B(t_k)]\overset{0}{\widetilde{\sigma_+\sigma_+}} -$$
$$-[\gamma_R \Delta B(t_{k-\ell})\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L\gamma_R}(\Delta B(t_{k-\ell})\Delta B^+(t_k)e^{-i\varphi} + \Delta B(t_k)\Delta B^+(t_{k-\ell})e^{i\varphi}) + \gamma_L \Delta B(t_k)\Delta B^+(t_k)]\sigma_+\sigma_- -$$
$$-[\gamma_R\Delta B^+(t_{k-\ell})\Delta B(t_{k-\ell}) + \sqrt{\gamma_L\gamma_R}(e^{i\varphi}\Delta B^+(t_{k-\ell})\Delta B(t_k) + e^{-i\varphi}\Delta B^+(t_k)\Delta B(t_{k-\ell})) + \gamma_L \Delta B^+(t_k)\Delta B(t_k)]\sigma_-\sigma_+ +$$
$$+[\gamma_R e^{2i\varphi}\Delta B^+(t_{k-\ell})\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L\gamma_R}(\Delta B^+(t_{k-\ell})\Delta B^+(t_k) + \Delta B^+(t_k)\Delta B^+(t_{k-\ell}))e^{i\varphi} + \gamma_L\Delta B^+(t_k)\Delta B^+(t_k)]\underbrace{\sigma_-^2}_{0}$$

$$\langle\sigma_+\sigma_+\sigma_+\rangle = \langle\sigma_+\sigma_+\sigma_-\rangle = \langle\sigma_+\sigma_-\sigma_-\rangle = \langle\sigma_-\sigma_+\sigma_+\rangle = \langle\sigma_-\sigma_-\sigma_+\rangle = \langle\sigma_-\sigma_-\sigma_-\rangle = 0$$

$2^3 = 8$ 2 left: $\langle\sigma_+\sigma_-\sigma_+\rangle$ & $\langle\sigma_-\sigma_+\sigma_-\rangle$

$\delta_{g i_T}\delta_{e j_T}$ $\delta_{g j_T}\delta_{e i_T}$

$$\Delta B(t_k)|i_k\rangle = \frac{\Delta B(t_k)[\Delta B^+(t_k)]^{i_k}}{\sqrt{i_k!\,\Delta t^{i_k}}}|vac\rangle = \frac{i_k\Delta t\,(\Delta B^+(t_k))^{i_k-1}}{\sqrt{i_k!\,\Delta t^{i_k}}}|vac\rangle = \sqrt{i_k\Delta t}\,|i_k - 1\rangle$$

$$M_B^3 = \{\gamma_R^{3/2}e^{-i\varphi}\Delta B(t_{k-\ell})\Delta B^+(t_{k-\ell})\Delta B(t_{k-\ell}) + \gamma_L^{3/2}\Delta B(t_k)\Delta B^+(t_k)\Delta B(t_k) +$$
$$+\gamma_R\sqrt{\gamma_L}[\Delta B(t_{k-\ell})\Delta B^+(t_{k-\ell})\Delta B(t_k) + \Delta B(t_k)\Delta B^+(t_{k-\ell})\Delta B(t_{k-\ell}) + \Delta B(t_{k-\ell})\Delta B^+(t_k)\Delta B(t_{k-\ell})]\cdot e^{-2i\varphi} +$$
$$+\gamma_L\sqrt{\gamma_R}[\Delta B(t_k)\Delta B^+(t_k)\Delta B(t_{k-\ell})e^{-i\varphi} + \Delta B(t_{k-\ell})\Delta B^+(t_k)\Delta B(t_k)e^{-i\varphi} + \Delta B(t_k)\Delta B^+(t_{k-\ell})\Delta B(t_k)e^{i\varphi}]\}\sigma_+ +$$
$$-\{\gamma_R^{3/2}e^{i\varphi}\Delta B^+(t_{k-\ell})\Delta B(t_{k-\ell})\Delta B^+(t_{k-\ell}) + \gamma_L^{3/2}\Delta B^+(t_k)\Delta B(t_k)\Delta B^+(t_k) +$$
$$+\gamma_R\sqrt{\gamma_L}[\Delta B^+(t_{k-\ell})\Delta B(t_{k-\ell})\Delta B^+(t_k) + \Delta B^+(t_k)\Delta B(t_{k-\ell})\Delta B^+(t_{k-\ell}) + \Delta B^+(t_{k-\ell})\Delta B(t_k)\Delta B^+(t_{k-\ell})]e^{2i\varphi} +$$
$$+\gamma_L\sqrt{\gamma_R}[\Delta B^+(t_k)\Delta B(t_k)\Delta B^+(t_{k-\ell})e^{i\varphi} + \Delta B^+(t_{k-\ell})\Delta B(t_k)\Delta B^+(t_k)e^{i\varphi} + \Delta B^+(t_k)\Delta B(t_{k-\ell})\Delta B^+(t_k)e^{-i\varphi}]\}\sigma_-$$

$$M_S M_B = i(\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+))\{[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B(t_k)]\sigma_+ - [\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B^+(t_k)]\sigma_-\} =$$
$$= i\{[\Delta_e\sigma_+ + \Omega_e\sigma_-\sigma_+][\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B(t_k)] - \Omega_e\sigma_+\sigma_-[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B^+(t_k)]\}$$

$$M_B M_S = i\Delta t\{[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B(t_k)]\sigma_+ - [\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B^+(t_k)]\sigma_-\}\{\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\} =$$
$$= i\Delta t\{\Omega_e[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B(t_k)]\sigma_+\sigma_- - [\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_{k-\ell}) + \sqrt{\gamma_L}\Delta B^+(t_k)](\Delta_e\sigma_- + \Omega_e\sigma_-\sigma_+)\}$$

$$\langle j_T, j_{\{k\}}|U(t_k, t_{k+1})|i_T, i_{\{k\}}\rangle = \delta_{j_T i_T}\delta_{j_k i_k} -$$
$$-\{[\sqrt{\gamma_R}e^{-i\varphi}\sqrt{i_{k-\ell}}\,\delta_{j_{k-\ell}\, i_{k-\ell}-1} + \sqrt{\gamma_L}\sqrt{i_k}\,\delta_{j_k\, i_k-1}]\delta_{j_T e}\delta_{i_T g} - [\sqrt{\gamma_R}e^{i\varphi}\sqrt{i_{k-\ell}+1}\,\delta_{j_{k-\ell}\, i_{k-\ell}+1} + \sqrt{\gamma_L}\sqrt{i_k+1}\,\delta_{j_k\, i_k+1}]\delta_{j_T g}\delta_{i_T e}\}\sqrt{\Delta t} -$$
$$-i\{\Delta_e\delta_{j_T e}\delta_{i_T e} + \Omega_e(\delta_{j_T e}\delta_{i_T g} + \delta_{i_T e}\delta_{j_T g})\}\Delta t - [\gamma_R(i_{k-\ell}+1)\delta_{i_{k-\ell} j_{k-\ell}} + \sqrt{\gamma_R\gamma_L}(\sqrt{(i_k+1)i_{k-\ell}}\,e^{-i\varphi}\delta_{j_{k-\ell}\, i_{k-\ell}-1}\delta_{j_k\, i_k+1} + \sqrt{(i_{k-\ell}+1)i_k}\,e^{i\varphi}\delta_{j_{k-\ell}\, i_{k-\ell}+1}\delta_{j_k\, i_k-1}) +$$
$$+\gamma_L(i_k+1)\delta_{j_k i_k}]\tfrac{1}{2}\delta_{i_T e}\delta_{j_T e}\Delta t - \tfrac{1}{2}[\gamma_R i_{k-\ell}\delta_{j_{k-\ell} i_{k-\ell}} + \sqrt{\gamma_L\gamma_R}(e^{i\varphi}\sqrt{(i_{k-\ell}+1)i_k}\,\delta_{j_{k-\ell}\, i_{k-\ell}+1}\delta_{j_k\, i_k-1} + e^{-i\varphi}\sqrt{(i_k+1)i_{k-\ell}}\,\delta_{j_{k-\ell}\, i_{k-\ell}-1}\delta_{j_k\, i_k+1}) + \gamma_L i_k]\delta_{j_T g}\delta_{i_T g}\Delta t +$$
$$+\tfrac{1}{6}\{\gamma_R^{3/2}e^{-i\varphi}i_{k-\ell}^{3/2}\delta_{j_{k-\ell}\, i_{k-\ell}-1} + \gamma_L^{3/2}i_k^{3/2}\delta_{j_k\, i_k-1} + \gamma_R\sqrt{\gamma_L}[\sqrt{i_k}(i_{k-\ell}+1)\delta_{j_k\, i_k-1}\delta_{j_{k-\ell}\, i_{k-\ell}} + \sqrt{i_k}\, i_{k-\ell}\delta_{j_k\, i_k-1}\delta_{j_{k-\ell}\, i_{k-\ell}} + \sqrt{(i_k+1)i_{k-\ell}(i_{k-\ell}-1)}\,e^{-2i\varphi}\delta_{j_{k-\ell}\, i_{k-\ell}-2}\delta_{j_k\, i_k+1}] +$$
$$+\gamma_L\sqrt{\gamma_R}[(i_k+1)\sqrt{i_{k-\ell}}\,\delta_{j_k i_k}\delta_{j_{k-\ell}\, i_{k-\ell}-1}e^{-i\varphi} + i_k\sqrt{i_{k-\ell}}\,\delta_{j_k i_k}\delta_{j_{k-\ell}\, i_{k-\ell}-1}e^{-i\varphi} + \sqrt{(i_{k-\ell}+1)i_k(i_k-1)}\,\delta_{j_k\, i_k-2}\delta_{j_{k-\ell}\, i_{k-\ell}+1}e^{i\varphi}]\}\delta_{j_T e}\delta_{i_T g}\Delta t^{3/2} -$$
$$-\tfrac{1}{6}\{\gamma_R^{3/2}e^{i\varphi}(i_{k-\ell}+1)^{3/2}\delta_{j_{k-\ell}\, i_{k-\ell}+1} + \gamma_L^{3/2}(i_k+1)^{3/2}\delta_{j_k\, i_k+1} + \gamma_R\sqrt{\gamma_L}[i_{k-\ell}\sqrt{i_k+1}\,\delta_{j_{k-\ell} i_{k-\ell}}\delta_{j_k\, i_k+1} + (i_{k-\ell}+1)\sqrt{i_k+1}\,\delta_{j_{k-\ell} i_{k-\ell}}\delta_{j_k\, i_k+1} + \sqrt{(i_{k-\ell}+1)(i_{k-\ell}+2)i_k}\,\delta_{j_{k-\ell}\, i_{k-\ell}+2}\delta_{j_k\, i_k-1}e^{2i\varphi}] +$$
$$+\gamma_L\sqrt{\gamma_R}[\sqrt{i_{k-\ell}+1}\, i_k\,\delta_{j_k i_k}\delta_{j_{k-\ell}\, i_{k-\ell}+1}e^{i\varphi} + \sqrt{i_{k-\ell}+1}(i_k+1)\delta_{j_{k-\ell}\, i_{k-\ell}+1}\delta_{j_k i_k}e^{i\varphi} + \sqrt{(i_k+1)(i_k+2)i_{k-\ell}}\,\delta_{j_k\, i_k+2}\delta_{j_{k-\ell}\, i_{k-\ell}-1}e^{-i\varphi}]\}\delta_{j_T g}\delta_{i_T e}\Delta t^{3/2} +$$
$$+\tfrac{1}{2}i\{[\Delta_e\delta_{j_T e}\delta_{i_T g} + \Omega_e\delta_{j_T g}\delta_{i_T g}][\sqrt{\gamma_R}e^{-i\varphi}\sqrt{i_{k-\ell}}\,\delta_{j_{k-\ell}\, i_{k-\ell}-1} + \sqrt{\gamma_L}\sqrt{i_k}\,\delta_{j_k\, i_k-1}] - \Omega_e\delta_{j_T e}\delta_{i_T e}[\sqrt{\gamma_R}e^{i\varphi}\sqrt{i_{k-\ell}+1}\,\delta_{j_{k-\ell}\, i_{k-\ell}+1} + \sqrt{\gamma_L}\sqrt{i_k+1}\,\delta_{j_k\, i_k+1}]\}\Delta t^{3/2} +$$
$$+\tfrac{i}{2}\{\Omega_e[\sqrt{\gamma_R}e^{-i\varphi}\delta_{j_{k-\ell}}\delta_{i_{k-\ell}-1}\sqrt{i_{k-\ell}} + \sqrt{\gamma_L}\delta_{j_k\, i_k-1}\sqrt{i_k}]\delta_{j_T e}\delta_{i_T e} - [\sqrt{\gamma_R}e^{i\varphi}\delta_{j_{k-\ell}\, i_{k-\ell}+1}\sqrt{i_{k-\ell}+1} + \sqrt{\gamma_L}\delta_{j_k\, i_k+1}\sqrt{i_k+1}](\Delta_e\delta_{j_T g}\delta_{i_T e} + \Omega_e\delta_{j_T g}\delta_{i_T g})\}\Delta t^{3/2}$$

$$M_S = -\frac{i}{\hbar} H_S \Delta t = -i\Delta t\left\{\Delta_e \sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\right\}$$

$$M_B = -\left\{\left[\sqrt{\gamma_R}\, e^{-i\varphi}\underbrace{\int_{t_\ell}^{t_{\ell+1}} b(t-\tau)\,dt}_{\substack{\tau=\ell\Delta t\\ \Delta B(t_{\ell-e})}} + \sqrt{\gamma_L}\underbrace{\int_{t_\ell}^{t_{\ell+1}} b(t)\,dt}_{\Delta B(t_\ell)}\right]\sigma_+ - \left[\sqrt{\gamma_R}\, e^{i\varphi}\underbrace{\int_{t_\ell}^{t_{\ell+1}} b^\dagger(t-\tau)\,dt}_{\Delta B^\dagger(t_{\ell-e})} + \sqrt{\gamma_L}\underbrace{\int_{t_\ell}^{t_{\ell+1}} b^\dagger(t)\,dt}_{\Delta B^\dagger(t_\ell)}\right]\sigma_-\right\}$$

$$M_S^2 = -\Delta t^2\left\{\Delta_e^2\sigma_+\sigma_- + \Omega_e\Delta_e(\sigma_+ + \sigma_-) + \Omega_e^2(\sigma_-\sigma_+ + \sigma_+\sigma_-)\right\} = -\Delta t^2\left\{\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\right\}\left\{\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\right\}$$

$$\langle\sigma_+\sigma_+\sigma_+\sigma_-\rangle = \langle\sigma_+\sigma_+\sigma_+\sigma_+\rangle = \langle\sigma_+\sigma_+\sigma_-\sigma_+\rangle = \langle\sigma_+\sigma_+\sigma_-\sigma_-\rangle = \langle\sigma_+\sigma_-\sigma_-\sigma_-\rangle = \langle\sigma_-\sigma_+\sigma_-\sigma_-\rangle = \langle\sigma_-\sigma_-\sigma_-\sigma_-\rangle =$$
$$= \langle\sigma_+\sigma_-\sigma_-\sigma_+\rangle = \langle\sigma_-\sigma_+\sigma_+\sigma_-\rangle = \langle\sigma_-\sigma_+\sigma_+\sigma_+\rangle = \langle\sigma_+\sigma_-\sigma_+\sigma_+\rangle = \langle\sigma_-\sigma_-\sigma_+\sigma_+\rangle = \langle\sigma_-\sigma_-\sigma_-\sigma_+\rangle = \langle\sigma_-\sigma_-\sigma_+\sigma_-\rangle = 0$$

$2^4 - 14 = 2$ $\quad\langle\sigma_+\sigma_-\sigma_+\sigma_-\rangle$ & $\langle\sigma_-\sigma_+\sigma_-\sigma_+\rangle$ $\quad$ only alternating terms can survive

$$\begin{aligned}
M_B^3 = &\Big\{\gamma_R^{3/2}e^{-i\varphi}\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e}) + \gamma_L^{3/2}\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_\ell) + \\
&+\gamma_R\sqrt{\gamma_L}\big[\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell) + \Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e}) + \Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\cdot e^{-2i\varphi}\big] + \\
&+\gamma_L\sqrt{\gamma_R}\big[\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})e^{-i\varphi} + \Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_\ell)e^{-i\varphi} + \Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)e^{i\varphi}\big]\Big\}\sigma_+ + \\
-&\Big\{\gamma_R^{3/2}e^{i\varphi}\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e}) + \gamma_L^{3/2}\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_\ell) + \\
&+\gamma_R\sqrt{\gamma_L}\big[\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell) + \Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e}) + \Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})e^{2i\varphi}\big] + \\
&+\gamma_L\sqrt{\gamma_R}\big[\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})e^{i\varphi} + \Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_\ell)e^{i\varphi} + \Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)e^{-i\varphi}\big]\Big\}\sigma_-
\end{aligned}$$

$$\begin{aligned}
M_B^4 = &\Big\{\gamma_R^2\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e}) + \gamma_L\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_\ell) + \\
&+\gamma_L^{3/2}\sqrt{\gamma_R}\big[\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})e^{i\varphi} + \Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)e^{-i\varphi} + \Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_\ell)e^{-i\varphi} + \\
&+\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_\ell)e^{i\varphi}\big] + \gamma_L\gamma_R\big[\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_\ell) + \Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell) + \\
&+\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)e^{-2i\varphi} + \Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e}) + \Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e}) + \\
&+\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})e^{2i\varphi}\big] + \sqrt{\gamma_L}\gamma_R^{3/2}\big[e^{-i\varphi}\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell) + \\
&+\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})e^{+i\varphi} + \Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})e^{i\varphi} + \Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})e^{-i\varphi}\big]\Big\}\sigma_+\sigma_- + \\
+&\Big\{\gamma_R^2\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e}) + \gamma_L\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_\ell) + \\
&+\gamma_L^{3/2}\sqrt{\gamma_R}\big[\Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})e^{-i\varphi} + \Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)e^{i\varphi} + \Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_\ell)e^{i\varphi} + \\
&+\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_\ell)e^{-i\varphi}\big] + \gamma_L\gamma_R\big[\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_\ell) + \Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell) + \\
&+\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)e^{+2i\varphi} + \Delta B^\dagger(t_\ell)\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e}) + \Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e}) + \\
&+\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})e^{-2i\varphi}\big] + \sqrt{\gamma_L}\gamma_R^{3/2}\big[e^{+i\varphi}\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell) + \\
&+\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})\Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})e^{-i\varphi} + \Delta B^\dagger(t_\ell)\Delta B(t_{\ell-e})\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})e^{-i\varphi} + \Delta B^\dagger(t_{\ell-e})\Delta B(t_\ell)\Delta B^\dagger(t_{\ell-e})\Delta B(t_{\ell-e})e^{+i\varphi}\big]\Big\}\sigma_-\sigma_+
\end{aligned}$$

$$\begin{aligned}
M_S M_B = &+i\left(\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_- + \sigma_+)\right)\left\{\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right]\sigma_+ - \left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\sigma_-\right\} = \\
= &\,i\left\{\left[\Delta_e\sigma_+ + \Omega_e\sigma_-\sigma_+\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right] - \Omega_e\sigma_+\sigma_-\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
M_S M_B^2 = &\,i\Big\{\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right]\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\left[\Delta_e\sigma_+\sigma_- + \Omega_e\sigma_-\right] + \\
&+\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right]\Omega_e\sigma_+\Big\}\Delta t
\end{aligned}$$

$$\begin{aligned}
M_B M_S M_B = &\,i\Big\{\Delta_e\sigma_-\sigma_+\left[\sqrt{\gamma_R}e^{+i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right] - \\
&-\Omega_e\sigma_+\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B(t_\ell)\right] - \\
&-\Omega_e\sigma_-\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^\dagger(t_{\ell-e}) + \sqrt{\gamma_L}\Delta B^\dagger(t_\ell)\right]\Big\}\Delta t
\end{aligned}$$

$$M_3 M_S = i\Delta t\left\{\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_{2-e})+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\sigma_+-\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\sigma_-\right\}\left\{\Delta_e\sigma_+\sigma_-+\Omega_e(\sigma_-+\sigma_+)\right\}=$$

$$=i\Delta t\left\{\Omega_e\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\sigma_+\sigma_- -\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\left(\Delta_e\sigma_-+\Omega_e\sigma_-\sigma_+\right)\right\}$$

$$M_3=-\left\{\left[\sqrt{\gamma_R}e^{-i\varphi}\underbrace{\int_{t_2}^{t_{2+1}}b(t-\tau)dt}_{\substack{\tau=e\Delta t\\ \Delta B(t_{2-e})}}+\sqrt{\gamma_L}\underbrace{\int_{t_2}^{t_{2+1}}b(t)dt}_{\Delta B(t_2)}\right]\sigma_+-\left[\sqrt{\gamma_R}e^{i\varphi}\underbrace{\int_{t_2}^{t_{2+1}}b^+(t-\tau)dt}_{\Delta B^+(t_{2-e})}+\sqrt{\gamma_L}\underbrace{\int_{t_2}^{t_{2+1}}b^+(t)dt}_{\Delta B^+(t_2)}\right]\sigma_-\right\}$$

$$M_3^2M_S=i\left\{\Omega_e\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\sigma_-+\right.$$
$$\left.+\left(\Delta_e\sigma_+\sigma_-+\Omega_e\sigma_+\right)\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\right\}$$

$j_T e\ i_T e$ $\quad j_T g\ i_T g$ $\quad j_T e\ i_T g$ $\quad j_T g\ i_T e$

σ_{j2} is $\sigma_{j_T i_T}$

$$M_3=\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\sigma_- -\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\sigma_+$$

$$M_S=-i\Delta t\left\{\Delta_e\sigma_+\sigma_-+\Omega_e(\sigma_++\sigma_-)\right\}$$

$$\frac{1}{2}M_3^2=-\frac{1}{2}\left\{\left[\gamma_R\Delta B^+(t_2-e)\Delta B(t_2-e)+\sqrt{\gamma_R\gamma_L}\left(e^{i\varphi}\Delta B^+(t_2-e)\Delta B(t_2)+e^{-i\varphi}\Delta B^+(t_2)\Delta B(t_2-e)\right)+\gamma_L\Delta B^+(t_2)\Delta B(t_2)\right]\sigma_-\sigma_++\right.$$
$$\left.+\left[\gamma_R\Delta B(t_2-e)\Delta B^+(t_2-e)+\sqrt{\gamma_R\gamma_L}\left(e^{-i\varphi}\Delta B(t_2-e)\Delta B^+(t_2)+e^{i\varphi}\Delta B(t_2)\Delta B^+(t_2-e)\right)+\gamma_L\Delta B(t_2)\Delta B^+(t_2)\right]\sigma_+\sigma_-\right\}$$

$$\frac{1}{2}M_3M_S=\frac{i}{2}\Delta t\left\{\Omega_e\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]\sigma_+\sigma_- -\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\left(\Delta_e\sigma_-+\Omega_e\sigma_-\sigma_+\right)\right\}$$

$$\frac{1}{2}M_SM_3=\frac{i}{2}\Delta t\left\{\left[\Delta_e\sigma_++\Omega_e\sigma_-\sigma_+\right]\left[\sqrt{\gamma_R}e^{-i\varphi}\Delta B(t_2-e)+\sqrt{\gamma_L}\,\Delta B(t_2)\right]-\Omega_e\sigma_+\sigma_-\left[\sqrt{\gamma_R}e^{i\varphi}\Delta B^+(t_2-e)+\sqrt{\gamma_L}\,\Delta B^+(t_2)\right]\right\}$$

$$\frac{1}{6}M_3^3=\frac{1}{6}\left\{\gamma_R^{3/2}e^{-i\varphi}\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2-e)+\gamma_L^{3/2}\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2)+\right.$$
$$+\gamma_R\sqrt{\gamma_L}\left[\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2)+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2-e)+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2-e)\cdot e^{-2i\varphi}\right]+$$
$$\left.+\gamma_L\sqrt{\gamma_R}\left[\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2-e)e^{-i\varphi}+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2)e^{-i\varphi}+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2)e^{i\varphi}\right]\right\}\sigma_++$$
$$-\left\{\gamma_R^{3/2}e^{i\varphi}\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2-e)+\gamma_L^{3/2}\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2)+\right.$$
$$+\gamma_R\sqrt{\gamma_L}\left[\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2)+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2-e)+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2-e)e^{2i\varphi}\right]+$$
$$\left.+\gamma_L\sqrt{\gamma_R}\left[\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2-e)e^{i\varphi}+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2)e^{i\varphi}+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2)e^{-i\varphi}\right]\right\}\sigma_-$$

$$\frac{1}{2}M_S^2=-\frac{1}{2}\Delta t^2\left\{\Delta_e^2\sigma_+\sigma_-+\Omega_e\Delta_e(\sigma_++\sigma_-)+\Omega_e^2(\sigma_-\sigma_++\sigma_+\sigma_-)\right\}$$

$$\frac{1}{6}M_3^2M_S=\frac{i\Delta t}{6}\left\{\Omega_e\left[\gamma_R\Delta B^+(t_2-e)\Delta B(t_2-e)+\sqrt{\gamma_R\gamma_L}\left(e^{i\varphi}\Delta B^+(t_2-e)\Delta B(t_2)+e^{-i\varphi}\Delta B^+(t_2)\Delta B(t_2-e)\right)+\gamma_L\Delta B^+(t_2)\Delta B(t_2)\right]\sigma_-+\right.$$
$$\left.+\left[\gamma_R\Delta B(t_2-e)\Delta B^+(t_2-e)+\sqrt{\gamma_R\gamma_L}\left(e^{-i\varphi}\Delta B(t_2-e)\Delta B^+(t_2)+e^{i\varphi}\Delta B(t_2)\Delta B^+(t_2-e)\right)+\gamma_L\Delta B(t_2)\Delta B^+(t_2)\right]\left(\Delta_e\sigma_+\sigma_-+\Omega_e\sigma_+\right)\right\}$$

$$\frac{1}{6}M_3M_SM_3=\frac{i\Delta t}{6}\left\{\Delta_e\sigma_-\sigma_+\left[\gamma_R\Delta B^+(t_2-e)\Delta B(t_2-e)+\overset{\sqrt{\gamma_L\gamma_R}\left(\Delta B^+(t_2-e)\Delta B(t_2)+\Delta B^+(t_2)\Delta B(t_2-e)\right)}{+}\gamma_L\Delta B^+(t_2)\Delta B(t_2)\right]-\Omega_e\sigma_+\left[\gamma_Re^{-2i\varphi}\Delta B(t_2-e)^2+e^{-i\varphi}\sqrt{\gamma_R\gamma_L}\left(\Delta B(t_2-e)\Delta B(t_2)+\Delta B(t_2)\Delta B(t_2-e)\right)+\right.\right.$$
$$\left.\left.+\gamma_L\Delta B(t_2)^2\right]-\Omega_e\sigma_-\left[\gamma_Re^{2i\varphi}\Delta B^{+2}(t_2-e)+\sqrt{\gamma_L\gamma_R}\left(\Delta B^+(t_2)\Delta B^+(t_2-e)+\Delta B^+(t_2-e)\Delta B^+(t_2)\right)e^{i\varphi}+\gamma_L\Delta B^{+2}(t_2)\right]\right\}$$

$$\frac{1}{6}M_SM_3^2=\frac{i}{6}\left\{\left[\gamma_R\Delta B(t_2-e)\Delta B^+(t_2-e)+\sqrt{\gamma_L\gamma_R}\left(\Delta B(t_2)\Delta B^+(t_2-e)e^{i\varphi}+\Delta B(t_2-e)\Delta B^+(t_2)e^{-i\varphi}\right)+\gamma_L\Delta B(t_2)\Delta B^+(t_2)\right]\left[\Delta_e\sigma_+\sigma_-+\Omega_e\sigma_-\right]+\right.$$
$$\left.+\left[\gamma_R\Delta B^+(t_2-e)\Delta B(t_2-e)+\sqrt{\gamma_L\gamma_R}\left(\Delta B^+(t_2)\Delta B(t_2-e)e^{-i\varphi}+\Delta B^+(t_2-e)\Delta B(t_2)e^{i\varphi}\right)+\gamma_L\Delta B^+(t_2)\Delta B(t_2)\right]\Omega_e\sigma_+\right\}\Delta t$$

$$\frac{1}{24}M_3^4=\frac{1}{24}\left\{\gamma_R^2\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2-e)+\gamma_L\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2)+\right.$$
$$+\gamma_L^{3/2}\sqrt{\gamma_R}\left[\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2-e)e^{i\varphi}+\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2)e^{-i\varphi}+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2)e^{-i\varphi}+\right.$$
$$\left.+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2)e^{i\varphi}\right]+\gamma_L\gamma_R\left[\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2)+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2)+\right.$$
$$+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2)e^{-2i\varphi}+\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2-e)+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2-e)+$$
$$\left.+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2-e)e^{2i\varphi}\right]+\sqrt{\gamma_L}\gamma_R^{3/2}\left[e^{-i\varphi}\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2)+\right.$$
$$\left.\left.+\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2-e)e^{i\varphi}+\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2-e)e^{i\varphi}+\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2-e)e^{-i\varphi}\right]\right\}\sigma_-\sigma_++$$

$$+\frac{1}{24}\left\{\gamma_R^2\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2-e)+\gamma_L\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2)+\right.$$
$$+\gamma_L^{3/2}\sqrt{\gamma_R}\left[\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2-e)e^{-i\varphi}+\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2)e^{i\varphi}+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2)e^{i\varphi}+\right.$$
$$\left.+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2)e^{-i\varphi}\right]+\gamma_L\gamma_R\left[\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2)+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2)+\right.$$
$$+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2)e^{2i\varphi}+\Delta B^+(t_2)\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2-e)+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2)\Delta B(t_2-e)+$$
$$\left.+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2-e)e^{-2i\varphi}\right]+\sqrt{\gamma_L}\gamma_R^{3/2}\left[e^{i\varphi}\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2)+\right.$$
$$\left.\left.+\Delta B^+(t_2-e)\Delta B(t_2-e)\Delta B^+(t_2)\Delta B(t_2-e)e^{-i\varphi}+\Delta B^+(t_2)\Delta B(t_2-e)\Delta B^+(t_2-e)\Delta B(t_2-e)e^{-i\varphi}+\Delta B^+(t_2-e)\Delta B(t_2)\Delta B^+(t_2-e)\Delta B(t_2-e)e^{i\varphi}\right]\right\}\sigma_+\sigma_-$$

$U_m^k = \sqrt{\gamma_L}\,\Delta B(t_k) \qquad U_p^k = \sqrt{\gamma_L}\,\Delta B^\dagger(t_k) \qquad U_m^e = \sqrt{\gamma_R}\, e^{-i\varphi}\Delta B(t_{k-e}) \qquad U_p^e = \sqrt{\gamma_R}\, e^{i\varphi}\Delta B^\dagger(t_{k-e})$

$U_{2np1}^k = (2n_k+1)\gamma_L\Delta t \qquad U_{2np1}^e = (2n_{k-e}+1)\gamma_R\Delta t \qquad U_n^k = n_k\gamma_L\Delta t \qquad U_n^e = n_e\gamma_R\Delta t$

$\delta_{j_k i_k}\,\delta_{j_{k-e} i_{k-e}}$ $\qquad \delta_{je}\delta_{ig} \qquad \delta_{jg}\delta_{ie} \qquad \delta_{je}\delta_{ie} \qquad \delta_{jg}\delta_{ig}$

$M_B = [U_p^e + U_p^k]\sigma_- - [U_m^e + U_m^k]\sigma_+$

$M_S = -i\Delta t\{\Delta_e\sigma_+\sigma_- + \Omega_e(\sigma_+ + \sigma_-)\}$

$\frac{1}{2}M_B^2 = -\frac{1}{2}\{[U_n^e\mathbb{1} + U_p^e U_m^k + U_p^k U_m^e + U_n^k\mathbb{1}]\sigma_-\sigma_+ + [(U_n^e+\gamma_R\Delta t)\mathbb{1} + U_p^k U_m^e + U_m^k U_p^e + (U_n^k+\gamma_L\Delta t)\mathbb{1}]\sigma_+\sigma_-\}$

$\frac{1}{2}M_B M_S = \frac{i}{2}\Delta t\{\Omega_e[U_m^e + U_m^k]\sigma_+\sigma_- - [U_p^e + U_p^k](\Delta_e\sigma_- + \Omega_e\sigma_-\sigma_+)\}$

$\frac{1}{2}M_S M_B = \frac{i}{2}\Delta t\{-\Omega_e[U_p^e + U_p^k]\sigma_+\sigma_- + [U_m^e + U_m^k](\Delta_e\sigma_+ + \Omega_e\sigma_-\sigma_+)\}$

$\frac{1}{6}M_B^3 = \frac{1}{6}\{[(U_n^e+\gamma_R\Delta t)U_m^e + (U_n^k+\gamma_L\Delta t)U_m^k + U_{2np1}^e U_m^k + U_{2np1}^k U_m^e + (U_m^e)^2 U_p^k + (U_m^k)^2 U_p^e]\sigma_+ -$
$\quad -[U_n^e U_p^e + U_n^k U_p^k + U_{2np1}^e U_p^k + U_{2np1}^k U_p^e + (U_p^e)^2 U_m^k + (U_p^k)^2 U_m^e]\sigma_-\}$

$\frac{1}{2}M_S^2 = \frac{1}{2}\Delta t^2\{\Delta_e^2\sigma_+\sigma_- + \Omega_e\Delta_e(\sigma_+ + \sigma_-) + \Omega_e^2(\sigma_-\sigma_+ + \sigma_+\sigma_-)\}$

$\frac{1}{6}M_B^2 M_S = \frac{i\Delta t}{6}\{\Omega_e[U_n^e\mathbb{1} + U_m^k U_p^e + U_p^k U_m^e + U_n^k\mathbb{1}]\sigma_- + [(U_n^e+\gamma_R\Delta t)\mathbb{1} + U_m^e U_p^k + U_m^k U_p^e + (U_n^k+\gamma_L\Delta t)\mathbb{1}](\Delta_e\sigma_+\sigma_- + \Omega_e\sigma_+)\}$

$\frac{1}{6}M_S M_B^2 = \frac{i\Delta t}{6}\{\Omega_e[U_n^e\mathbb{1} + U_m^k U_p^e + U_p^k U_m^e + U_n^k\mathbb{1}]\sigma_+ + [(U_n^e+\gamma_R\Delta t)\mathbb{1} + U_m^k U_p^e + U_m^e U_p^k + (U_n^k+\gamma_L\Delta t)\mathbb{1}](\Delta_e\sigma_+\sigma_- + \Omega_e\sigma_-)\}$

$\frac{1}{6}M_B M_S M_B = \frac{i\Delta t}{6}\{[U_n^e\mathbb{1} + U_m^k U_p^e + U_p^k U_m^e + U_n^k\mathbb{1}]\Delta_e\sigma_-\sigma_+ - [U_m^{e\,2} + 2U_m^e U_m^k + U_m^{k\,2}]\Omega_e\sigma_+ - [U_p^{e\,2} + 2U_p^e U_p^k + U_p^{k\,2}]\Omega_e\sigma_-\}$

$\frac{1}{24}M_B^4 = \frac{1}{24}\{[(U_n^e+\gamma_R\Delta t)^2 + (U_n^k+\gamma_L\Delta t)^2 + U_m^k U_p^e U_{2np1}^k + U_{2np1}^k U_m^e U_p^k + U_{2np1}^k(U_n^e+\gamma_R\Delta t) + U_{2np1}^e(U_n^k+\gamma_L\Delta t) +$
$\quad + (U_p^e)^2(U_m^k)^2 + (U_m^e)^2(U_p^k)^2 + U_m^e U_p^k U_{2np1}^e + U_{2np1}^e U_p^e U_m^k]\sigma_+\sigma_- +$
$\quad + [(U_n^e)^2 + (U_n^k)^2 + U_p^k U_m^e U_{2np1}^k + U_{2np1}^k U_p^e U_m^k + U_{2np1}^k U_n^e + U_{2np1}^e U_n^k + (U_p^e)^2(U_m^k)^2 + (U_m^e)^2(U_p^k)^2 +$
$\quad + U_p^e U_m^k U_{2np1}^e + U_{2np1}^e U_m^e U_p^k]\sigma_-\sigma_+\}$

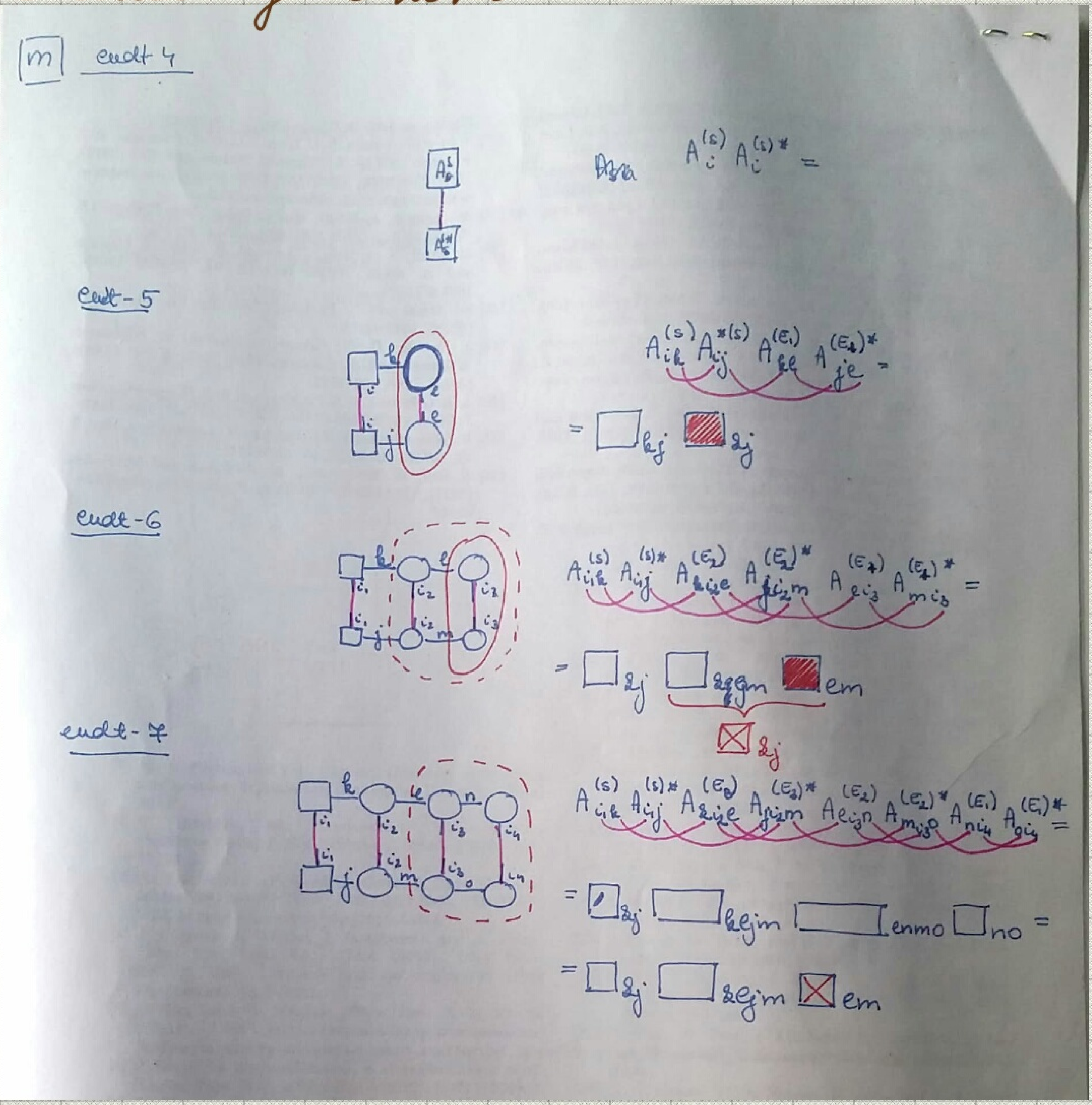
Notes:

$[\Delta B(t_k), \Delta B^\dagger(t_k)] = \Delta t \Rightarrow [\Delta B(t_k), (\Delta B^\dagger(t_k))^{i_k}] = \Delta t(\Delta B^\dagger(t_k))^{i_k-1} + \Delta B^\dagger(t_k)\Delta t(\Delta B^\dagger(t_k))^{i_k-2} + \ldots + (\Delta B^\dagger(t_k))^{i_k-1}\Delta t =$
$= i_k\Delta t(\Delta B^\dagger(t_k))^{i_k-1}$

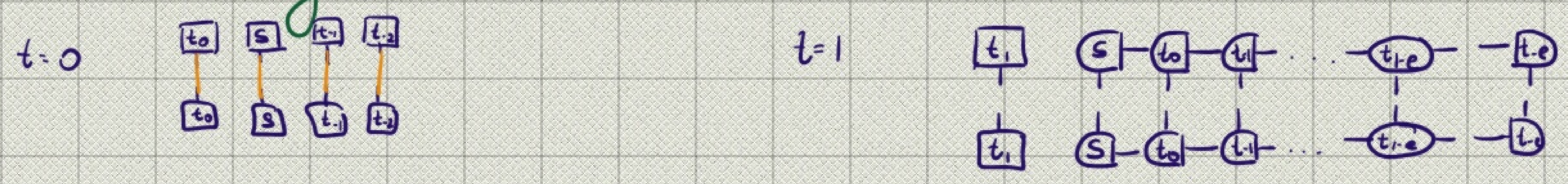
$\Delta B(t_k)|i_k\rangle = \frac{\Delta B(t_k)[\Delta B^\dagger(t_k)]^{i_k}}{\sqrt{i_k!(\Delta t)^{i_k}}}|vac\rangle = \frac{i_k\Delta t(\Delta B^\dagger(t_k))^{i_k-1}}{\sqrt{i_k!(\Delta t)^{i_k}}}|vac\rangle = \sqrt{i_k\Delta t}\,|i_k-1\rangle$

$\Delta B^\dagger(t_k)|i_k\rangle = \frac{[\Delta B^\dagger(t_k)]^{i_k+1}}{\sqrt{i_k!(\Delta t)^{i_k}}}|vac\rangle = \sqrt{(i_k+1)\Delta t}\,\frac{[\Delta B^\dagger(t_k)]^{i_k+1}}{\sqrt{(i_k+1)!(\Delta t)^{i_k+1}}} = \sqrt{(i_k+1)\Delta t}\,|i_k+1\rangle$

Calculating the norm

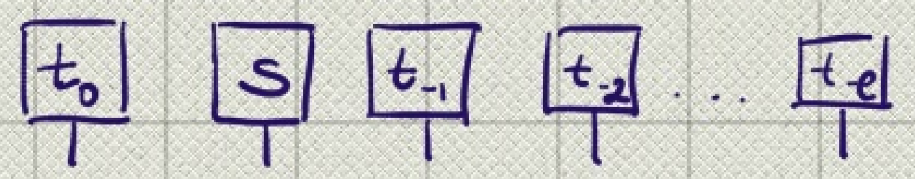


with feedback



Adding feedback

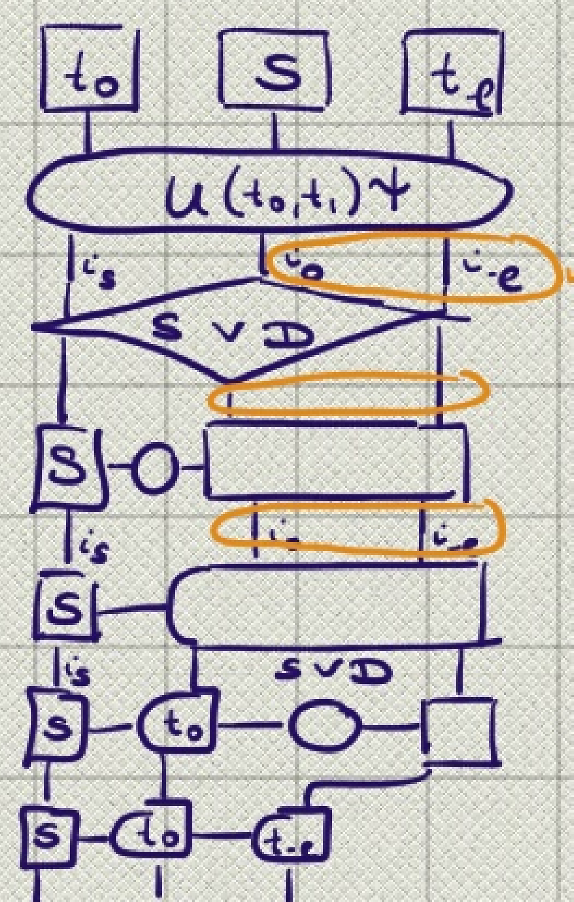
t = 0



e = 3

independent bins

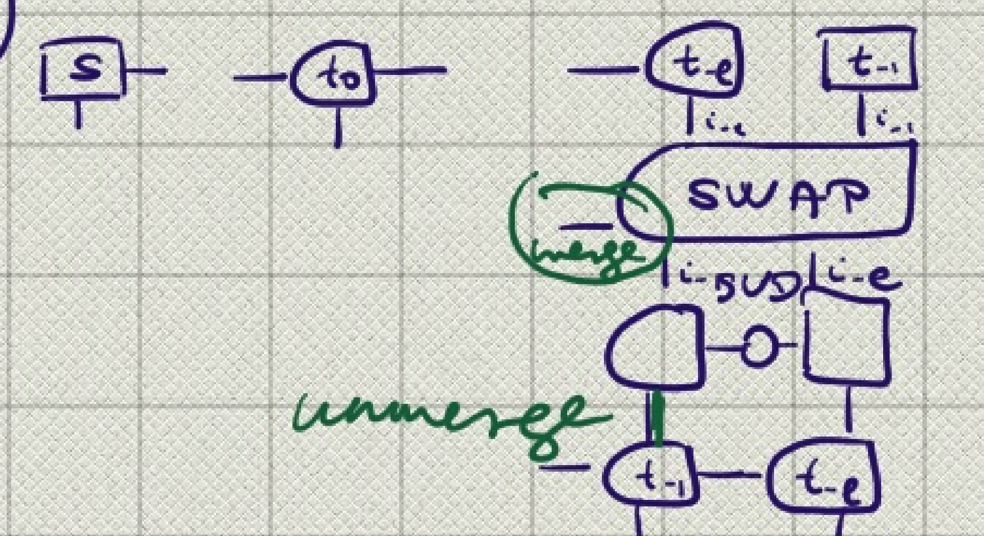
3.)



tensor prod

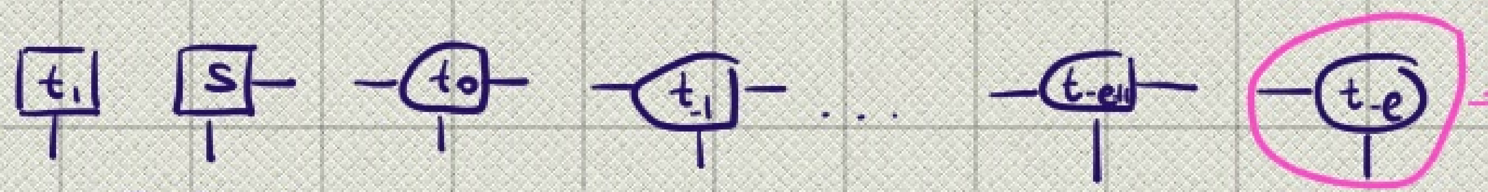
OC & SWAP-step not needed

⇒ 4.)



tensor prod

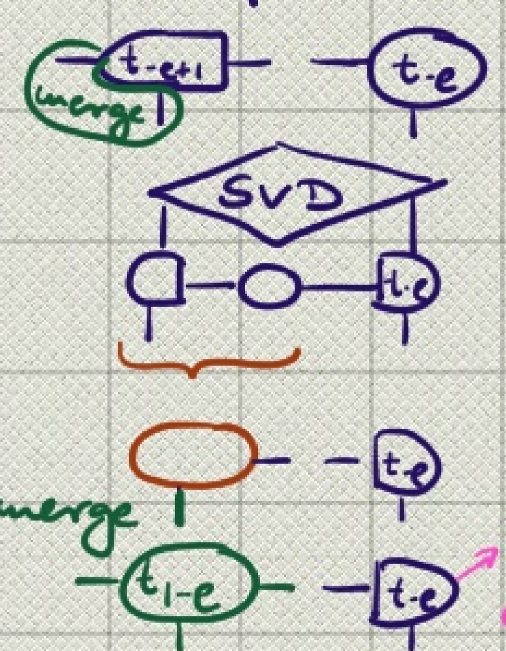
t = 1



→ won't change anymore

OC-step

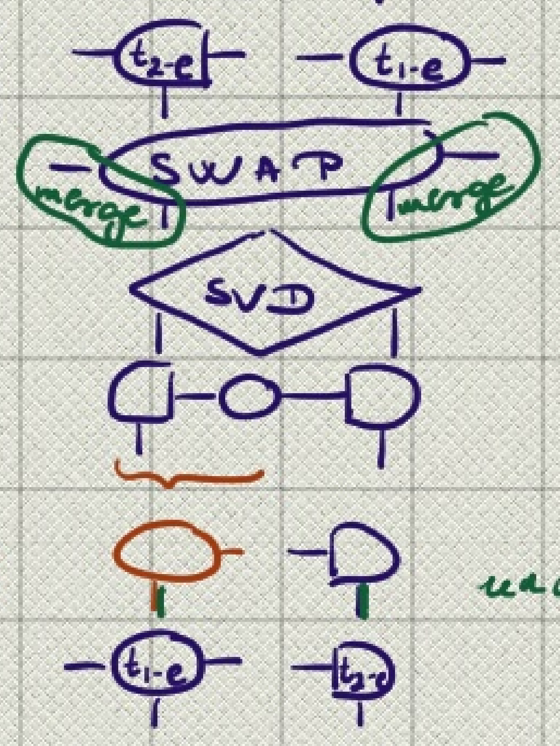
1.)



calculations with this can be stored

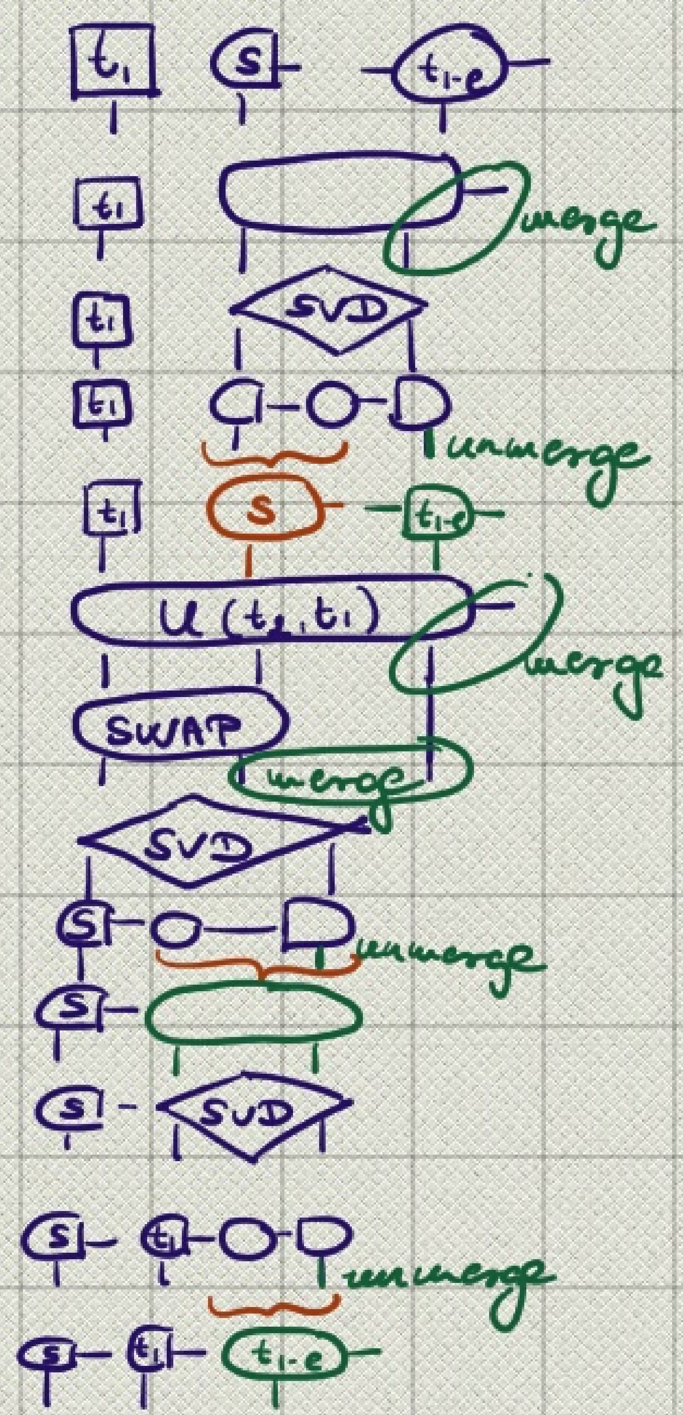
SWAP-step

2.)



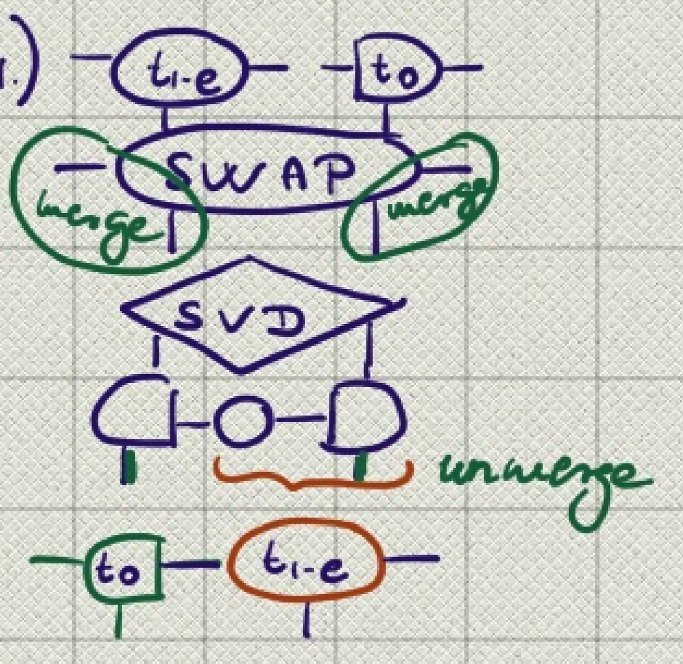
EVOLUTION step

3.)

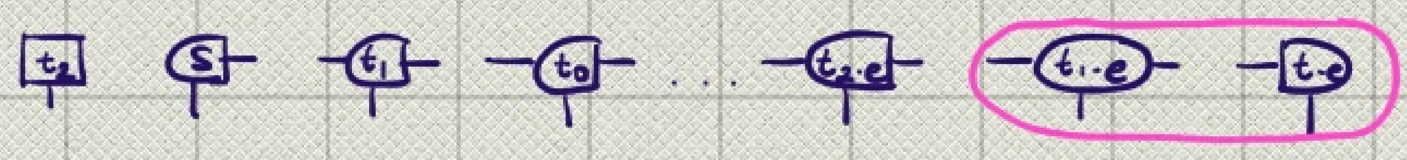


UNSWAP-step

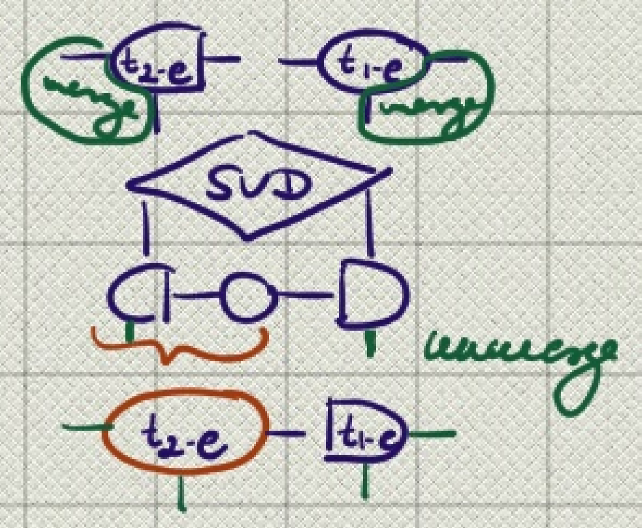
4.)



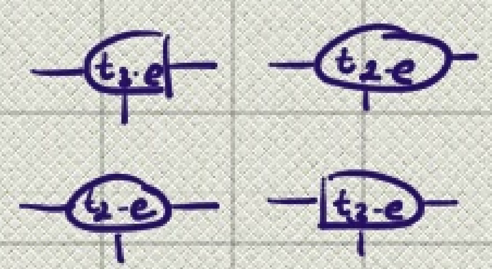
t = 2



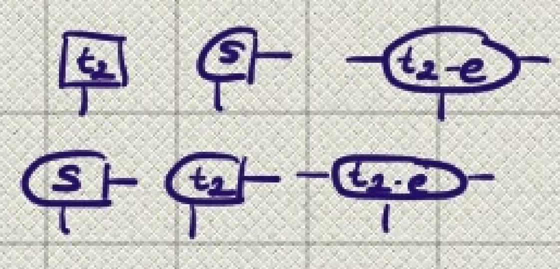
1.)



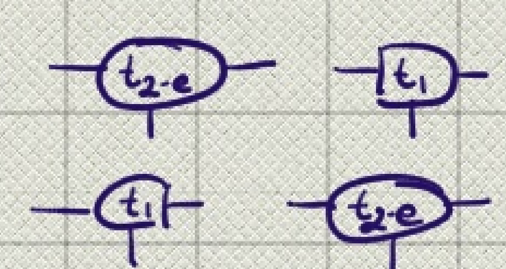
2.)



3.)

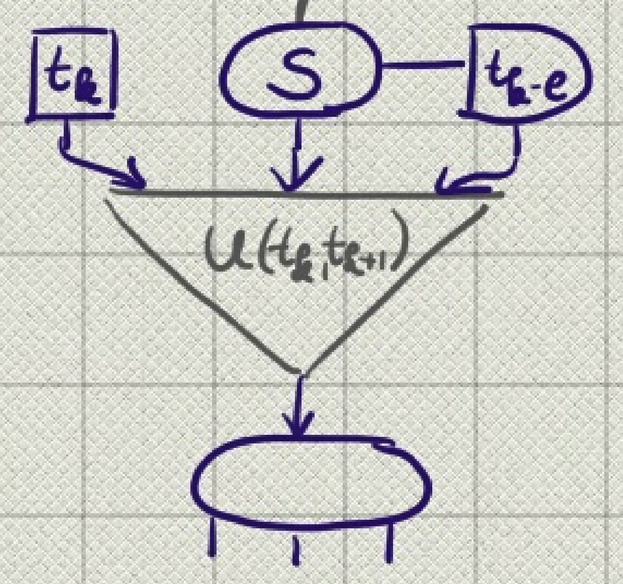


4.)

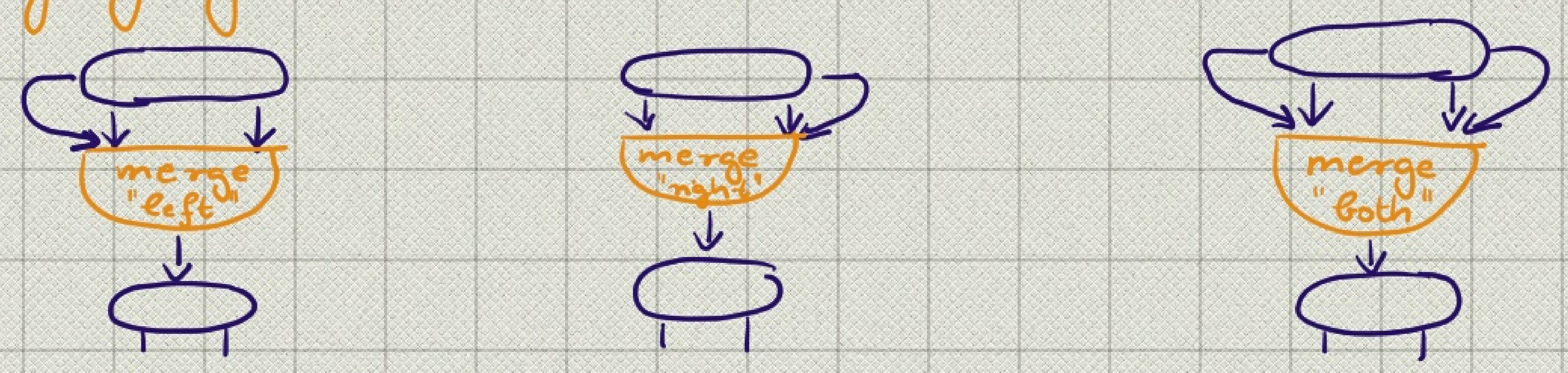


Blocks that I've defined:

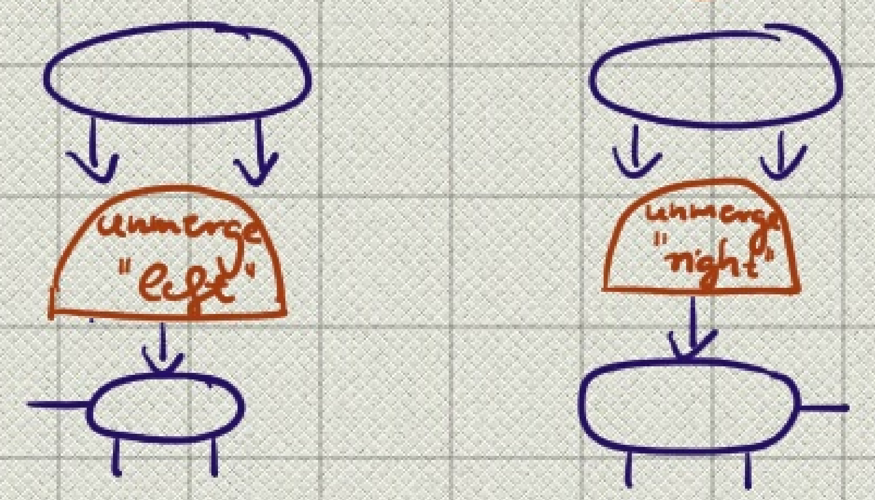
1.) time-evolution step U to 2nd order:



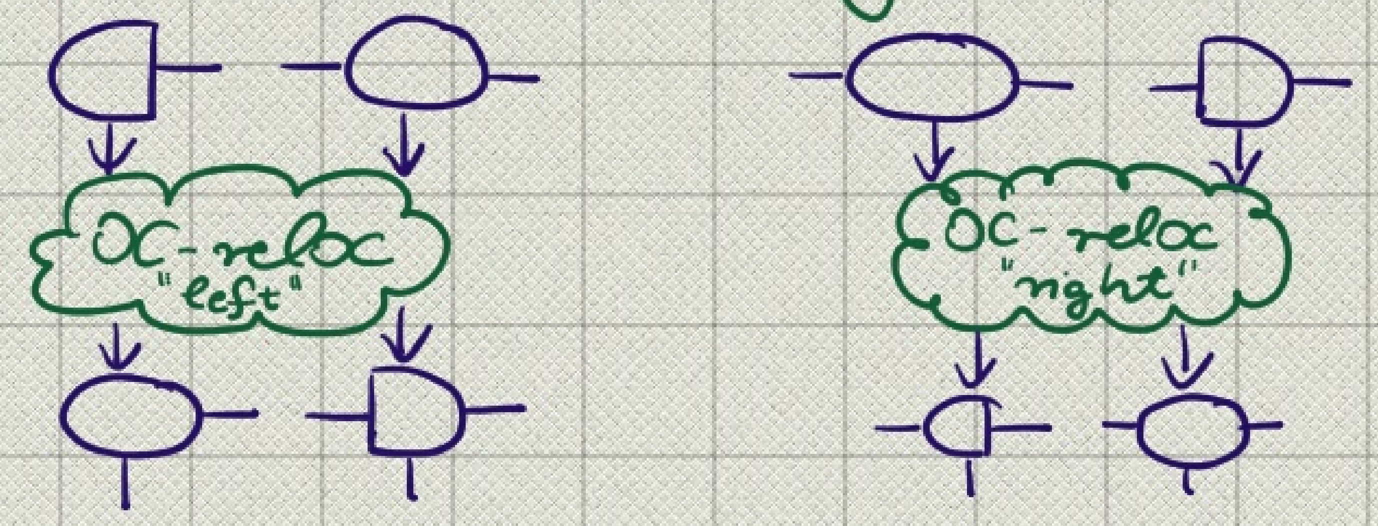
2.) merging of indices



3.) undoing the merge of indices



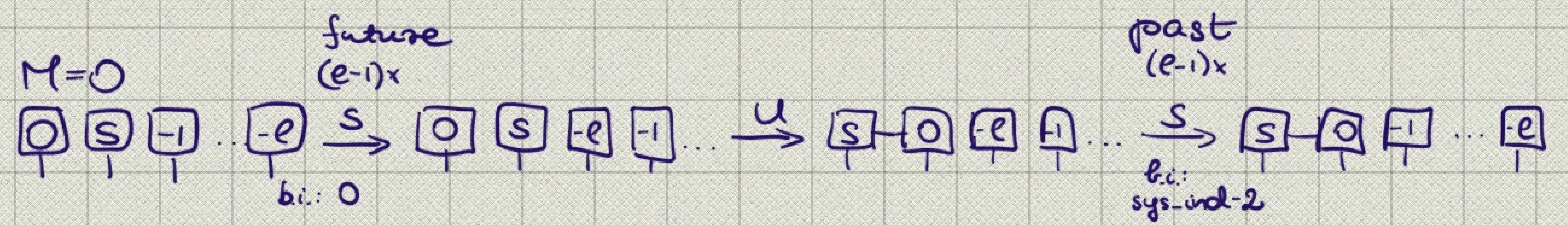
4.) shifting the Orthogonality Centre (OC)



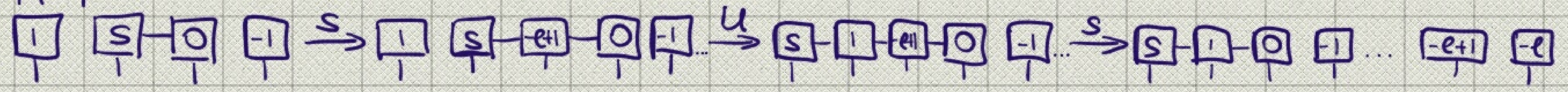

t=0 1.)
t_0 S t_{-e}
$u(t_0, t_1)$
merge "right"
cut "right"
SVD
S
unmerge "right"
merge "left"
cut "left"
SVD
"left"
S t_0 t_{-e}
2.) (l-1) times
t_{-e} t_{-1}
"left"
cut "left"
SVD
"left"
t_{-1} t_{-e}
3.) t_{1-e} t_{-e}
OC - shift "left"
t_{1-e} t_{-e}
t=1 1.) (l-1) times
t_{2-e} t_{1-e}
"both"
cut "both" "OC left"
SVD
"right"
t_{2-e}
"left"
t_{1-e} t_{2-e}
2.) S t_{1-e}
OC-shift "left"
S t_{1-e}
3.) t_1 S t_{1-e}
$u(t_1, t_2)$
"right"
"right"
cut "right"
SVD
S
"right"
"left"
cut "left"
SVD
"left"
t_1
"right"
S t_1 t_{1-e}
4.) t_{1-e} t_0 (l-1) times
"both"
cut "both" "OC right"
"left"
t_0
"right"
t_{1-e}
5.) t_{2-e} t_{1-e}
OC-shift "left"
t_{2-e} t_{1-e}
t=k general
1.) t_{k+1-e} t_{k-e}

SWAP stages:

M=0



M=1



M=2