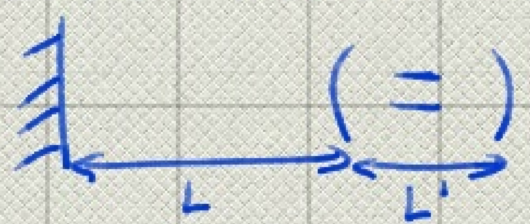
Analytical study of quantum feedback enhanced Rabi oscillations



Previously: PRA 92, 053801 (2015)

$H/\hbar = -\gamma(\sigma^- a^\dagger + \sigma^+ a) - \int dk \left(G(k,t) a^\dagger d_k + G^*(k,t) d_k^\dagger a\right)$ $(L >> L')$

Now comparable sizes $\Rightarrow$ discreet spectrum $\Rightarrow$ multiple delays

$H/\hbar = -\gamma(\sigma^- a^\dagger + \sigma^+ a) - \sum_k \left(G_k(t) a^\dagger d_k + G_k^*(t) d_k^\dagger a\right)$ (interaction picture)

atomic energy is assumed to be on resonance w/ the cavity : ω_0

$\gamma = 50\,\mu eV/\hbar$ $\quad G_k(t) = \underbrace{G_0 \sin(q_k L)} e^{i(\omega_0 - \omega_k)t} = G_0 (-1)^k e^{-i\omega_k' t}$ $\quad \omega_k' \Rightarrow$ centered around ω_0

$k \in \mathbb{Z}$ $\quad q_k = \frac{2k+1}{2L}\pi \Rightarrow G_0 \sin\left(\frac{2k+1}{2L}\pi L\right) = G_0 (-1)^k$ $\quad$ mode function: $u(z) = \sqrt{\frac{2}{L_B}} \sin(q_k(x+L))$

Single-excitation limit

$\omega_l' = \omega_k - \omega_0 \doteq \frac{2l}{2L}\pi c$ (or centering)

$|\psi\rangle = c_e |e, 0, \{0\}\rangle + c_g |g, 1, \{0\}\rangle + \sum_k c_{g,k} |g, 0, \{k\}\rangle$

$\frac{2(k-l)+1}{2L}\pi c = \omega_0$

Schrödinger equation: $\quad i\hbar \partial_t |\psi\rangle = H|\psi\rangle$

$l = -\infty$

$\partial_t c_e = i\gamma c_g$

$\partial_t c_g = i\gamma c_e + i\sum_k G_k(t) c_{g,k}(t)$

$\partial_t c_{gk} = i G_k^*(t) c_g \Rightarrow$ formal integral: $c_{gk}(t) = i\int^t G_k^*(t') c_g(t') dt'$

$$\partial_t c_g = i\gamma c_e - |G_0|^2 \sum_k (-1)^{2k} \int_0^t e^{-i\omega_k'(t-t')} c_g(t')\, dt' = i\gamma c_e - |G_0|^2 \int_0^t e^{-i\frac{\pi}{\tau}(t-t')} c_g(t') \underbrace{\sum_{k=-\infty}^{\infty} e^{-i\frac{2k\pi}{\tau}(t-t')}}_{\tau \sum_{k=-\infty}^{\infty} \delta(t'-t-2\tau)}$$

$\frac{2L}{c} = \tau$

$\omega_k' = \frac{2k+1}{2L} c\pi = \frac{(2k+1)\pi}{\tau}$

$\hookrightarrow$ only $k > 0$ count (causality)

$$\partial_t c_g = i\gamma c_e - |G_0|^2 \sum_{k=-\infty}^{\infty} e^{-i\frac{\pi}{\tau}k\tau} \tau\, \Theta(t - k\tau) c_g(t - k\tau) =$$
$$= i\gamma c_e - |G_0|^2 \tau \sum_{k=0}^{\infty} (-1)^k c_g(t - k\tau)\, \Theta(t - k\tau)$$

PROBLEM:
Forgot to account for
$\int_0^t \delta(t-t') c_g(t') = \frac{1}{2} c_g(t)$!

$c_e(0) = 1 \qquad c_g(0) = c_{gk}(0) = 0 \qquad \tilde{c}_e(s) = \int_0^\infty e^{-st} c_e(t)\, dt$

(I) $\quad s\tilde{c}_e(s) - 1 = i\gamma \tilde{c}_g(s)$

$$\int_0^\infty e^{-st} c_g(t - k\tau)\Theta(t - k\tau)\, dt = \int_{-k\tau}^{\infty} e^{-s(t'+k\tau)} c_g(t')\Theta(t')\, dt' = e^{-sk\tau}\int_0^\infty e^{-st'} c_g(t')\, dt' = e^{-sk\tau}\tilde{c}_g(s)$$

(II) $\quad s\tilde{c}_g(s) = i\gamma\tilde{c}_e(s) - \underbrace{|G_0|^2\tau}_{2\kappa} \tilde{c}_g(s) \sum_{k=0} e^{-sk\tau}(-1)^k$

(I) $\to$ (II)

$$\left(s^2 + \gamma^2 + 2\kappa s \sum_{k=0}^{\infty} (-1)^k e^{-sk\tau}\right)\tilde{c}_g(s) = i\gamma$$

- $t \le \tau$: $\quad \tilde{c}_g(s) = \frac{i\gamma}{s^2 + \gamma^2 + 2\kappa s} = \frac{i\gamma}{(s+\kappa)^2 + \gamma^2 - (\kappa)^2} \Rightarrow c_g(t) = i\gamma \frac{\sin\left[\sqrt{1 - (\frac{\kappa}{\gamma})^2}\,\gamma t\right]}{\sqrt{1 - (\frac{\kappa}{\gamma})^2}} e^{-\kappa t}$

- $\tau \le t \le 2\tau$:

$$\tilde{c}_g(s) = \frac{i\gamma}{s^2 + \gamma^2 + 2\kappa s(1 - e^{-s\tau})} = i\gamma \underbrace{\left[1 - \frac{2s\kappa e^{-s\tau}}{(s+\kappa)^2 + \gamma^2 - \kappa^2}\right]^{-1}}_{\frac{1}{(1-a)} = \sum_{n=0}^{\infty} a^n \text{ geometric series}} \left[(s+\kappa)^2 + \gamma^2 - \kappa^2\right]^{-1} = i\gamma \sum_m \left(\frac{2s\kappa e^{-s\tau}}{(s+\kappa)^2 + \gamma^2 - \kappa^2}\right)^m \left[(s+\kappa)^2 + \gamma^2 - \kappa^2\right]^{-1}$$

- $n\tau \le t \le (n+1)\tau$

$$\tilde{c}_g(s) = \frac{i\gamma}{s^2 + \gamma^2 + 2\kappa s \sum_k^n (-1)^k e^{-ks\tau}} = i\gamma \left[1 - \frac{2s\kappa \sum_{k=1}^{n} e^{-ks\tau}(-1)^k}{(s+\kappa)^2 + \gamma^2 - \kappa^2}\right]^{-1} \left[(s+\kappa)^2 + \gamma^2 - \kappa^2\right]^{-1} = i\gamma \sum_m \left[\frac{2s\kappa \sum_{k=1}^{n} e^{-ks\tau}(-1)^k}{(s+\kappa)^2 + \gamma^2 - \kappa^2}\right]^m \left[(s+\kappa)^2 + \gamma^2 - \kappa^2\right]^{-1}$$

$$\sum_{k=0}^{n} (-e^{-s\tau})^k = \frac{1 - (-e^{-s\tau})^{n+1}}{1 + e^{-s\tau}} \xrightarrow{n \to \infty} \frac{1}{1 + e^{-s\tau}}$$

$n = 0 \to \frac{1 + e^{-s\tau}}{1 + e^{-s\tau}} = 1 \qquad n = 1 \Rightarrow \frac{1 - e^{-2s\tau}}{1 + e^{-s\tau}} = 1 - e^{-s\tau} \qquad n = 2 \Rightarrow \frac{1 + e^{-3s\tau}}{1 + e^{-s\tau}} =$

(i) $s\,\tilde{c}_e(s) - 1 = i\gamma\,\tilde{c}_g(s)$ $\qquad\qquad$ $\tilde{c}_g(s)\sum_{k=0} e^{-sk\tau}(-1)^k$

(ii) $s\,\tilde{c}_g(s) = i\gamma\,\tilde{c}_e(s) - 2\kappa\,\tilde{c}_g(s)\,\underbrace{\frac{1-(-e^{-s\tau})^{n+1}}{1+e^{-s\tau}}}_{\sum_{k=0}^{n}(-e^{-s\tau})^k}$ $\qquad\qquad$ $n\tau < t \leq (n+1)\tau$

$$\tilde{c}_g(s) = \left[s^2+\gamma^2+\kappa s\,\frac{1-(-e^{-s\tau})^{n+1}}{1+e^{-s\tau}}\right]^{-1} i\gamma$$

$n\to\infty \Rightarrow$ $\tilde{c}_g^{(\infty)}(s) = \left[s^2+\gamma^2+\frac{2\kappa s}{1+e^{-s\tau}}\right]^{-1} i\gamma = \left[\frac{s^2+\gamma^2+2\kappa s+(s^2+\gamma^2)e^{-s\tau}}{1+e^{-s\tau}}\right]^{-1} i\gamma = (s^2+\gamma^2+2\kappa s)^{-1}\left[1-\frac{2\kappa s e^{-s\tau}}{(s^2+\gamma^2+\kappa s 2)(1+e^{-s\tau})}\right]^{-1} i\gamma =$

$$= \frac{i\gamma}{s^2+\gamma^2+\kappa s 2}\sum_{m=0}^{\infty}\left[\frac{2\kappa s e^{-s\tau}}{(s^2+\gamma^2+\kappa s 2)(1+e^{-s\tau})}\right]^m$$

$n=0 \Rightarrow$ $\tilde{c}_g^{(0)}(s) = \frac{i\gamma}{s^2+\gamma^2+\kappa s 2}$

$n=1 \Rightarrow$ $\tilde{c}_g^{(1)}(s) = \left[s^2+\gamma^2+2\kappa s-2\kappa s e^{-s\tau}\right]^{-1} i\gamma = \frac{i\gamma}{s^2+\gamma^2+\kappa s 2}\sum_{m=0}^{\infty}\left[\frac{2\kappa s e^{-s\tau}}{(s^2+\gamma^2+2\kappa s)}\right]^m$

$n=2 \Rightarrow$ $\tilde{c}_g^{(2)}(s) = \left[s^2+\gamma^2+2\kappa s-2\kappa s(e^{-s\tau}-e^{-2s\tau})\right]^{-1} i\gamma = \frac{i\gamma}{s^2+\gamma^2+2\kappa s}\sum_{m=0}^{\infty}\left(\frac{2\kappa s e^{-s\tau}}{s^2+\gamma^2+\kappa s 2}\right)^m (1-e^{-s\tau})^m$

Objective: pure oscillation in the long-time limit

$n=0$ $\qquad$ $s^2+\gamma^2+2\kappa s \overset{!}{=} 0$ $\qquad$ $s = \frac{1}{2}\left(-2\kappa \pm \sqrt{4\kappa^2-4\gamma^2}\right) = -\kappa \pm i\gamma\sqrt{1-\frac{\kappa^2}{\gamma^2}}$

oscillation only if $\gamma > \kappa$, it is still damped

$n=1$ $\qquad$ $s^2+\gamma^2+2\kappa s(1-e^{-s\tau}) \overset{!}{=} 0$

pure oscillation only if $\mathrm{Re}(s)=0$

frequency can be set to the pure Rabi freq w/o damping $\Rightarrow s = \pm i\gamma$

$-\gamma^2+\gamma^2+i2\kappa\gamma(1-e^{\pm i\gamma\tau}) \overset{!}{=} 0 \quad \Rightarrow e^{\pm i\gamma\tau} = 1 \Rightarrow \gamma\tau = \mathbb{Z}\frac{\pi}{2}$

$n=\infty$ $\qquad$ $s^2+\gamma^2+\frac{2\kappa s}{1+e^{-s\tau}} \overset{!}{=} 0$ $\qquad$ $1+e^{-s\tau} \neq 0$

same special case: $s = \pm i\gamma \Rightarrow$ not possible

if the feedback loop is not long enough to consider continuous modes, in the long time limit one cannot have steady-state Rabi oscillation by freq. γ

$(-\omega^2+\gamma^2)(1+\cos\omega\tau - i\sin\omega\tau) + 2\kappa i\omega \overset{!}{=} 0$

$(\gamma^2-\omega^2)(1+\cos\omega\tau) \overset{!}{=} 0$ $\qquad$ $(\gamma^2-\omega^2)\sin\omega\tau - 2\kappa\omega \overset{!}{=} 0$

$\cos\omega\tau \overset{!}{=} -1$ $\qquad$ no pure oscillations are possible

$n=2$ $\qquad$ $s^2+\gamma^2+2\kappa s-2\kappa s e^{-s\tau}+2\kappa s e^{-2s\tau} \overset{!}{=} 0$

$s \overset{?}{=} i\gamma$

$-\gamma^2+\gamma^2+2i\gamma\kappa(1-e^{\pm i\gamma\tau}+e^{\pm i2\gamma\tau}) \overset{!}{=} 0$

$(1-\cos\gamma\tau+\cos^2\gamma\tau-\sin^2\gamma\tau) \overset{!}{=} 0 \overset{!}{=} 2\cos^2\gamma\tau - \cos\gamma\tau$ $\qquad$ $\cos\gamma\tau = 0$ ↯

$\pm\sin\gamma\tau \mp 2\sin\gamma\tau\cos\gamma\tau \overset{!}{=} 0$ $\qquad$ $\sin\gamma\tau = 0 \Rightarrow$ doesn't solve the other

$1-2\cos\gamma\tau \overset{!}{=} 0$

$\cos\gamma\tau = \frac{1}{2} = \cos(-\gamma\tau)$ $\qquad$ $\gamma\tau = \pm\frac{\pi}{3} + \mathbb{Z}\pi 2$

$n=3$ $\quad s^2+\gamma^2+2\kappa s-2\kappa s e^{-s\tau}+2\kappa s e^{-2s\tau}-2\kappa s e^{-3s\tau} \stackrel{!}{=} 0$

$s \stackrel{?}{=} i\gamma$

$1-\cos\gamma\tau+\cos^2\gamma\tau-\sin^2\gamma\tau-\cos^3\gamma\tau+3\sin^2\gamma\tau\cos\gamma\tau \stackrel{!}{=} 0$

$-\cos\gamma\tau+2\cos^2\gamma\tau-\cos^3\gamma\tau\cdot 4+3\cos\gamma\tau \stackrel{!}{=} 0$

$-1-\cos\gamma\tau+2\cos^2\gamma\tau \stackrel{!}{=} 0$ $\qquad \cos\gamma\tau=0$ ✓ $\Rightarrow \sin\gamma\tau=\pm 1$

$\cos\gamma\tau=\frac{1}{4}\left(1\pm\sqrt{1+8}\right)=\frac{1}{4}(1\pm 3)\begin{cases}\to 1\\ \to -\frac{1}{2}\end{cases}$

$\sin\gamma\tau-2\sin\gamma\tau\cos\gamma\tau-4\sin^3\gamma\tau+3\sin\gamma\tau \stackrel{!}{=} 0$

$\sin\gamma\tau\left(2-\cos\gamma\tau-2\sin^2\gamma\tau\right) \stackrel{!}{=} 0$ $\qquad \sin\gamma\tau=0$ ✓ $\Rightarrow \cos\gamma\tau=\pm 1$

$\cos\gamma\tau\left(2\cos\gamma\tau-1\right) \stackrel{!}{=} 0$

$e^{\pm i\gamma\tau}=1$ is always a good combination if n is odd

Time-dependent solution for $n=1$

$$\tilde{c}_g^{(1)}(s)=\frac{i\gamma}{s^2+\gamma^2+\kappa s 2}\sum_{m=0}^{\infty}\left[\frac{2\kappa s e^{-s\tau}}{(s^2+\gamma^2+2\kappa s)}\right]^m$$

special case : $\gamma=\kappa$

$$\tilde{c}_g^{(1)}(s)=\frac{i\kappa}{(s+\kappa)^2}\sum_{m=0}^{\infty}(2\kappa)^m e^{-sm\tau}\left(\frac{s}{(s+\kappa)^2}\right)^m=$$

$$=\frac{i\kappa}{(s+\kappa)^2}\sum_{m=0}^{\infty}(2\kappa)^m e^{-sm\tau}\left(\frac{1}{(s+\kappa)}-\frac{\kappa}{(s+\kappa)^2}\right)^m=$$

$$=\frac{i\kappa}{(s+\kappa)^2}\sum_{m=0}^{\infty}(2\kappa)^m e^{-sm\tau}\frac{1}{(s+\kappa)^m}\underbrace{\left[1-\frac{\kappa}{s+\kappa}\right]^m}=$$

binomial series: $\sum_{\ell=0}^{m}\binom{m}{\ell}\left(-\frac{\kappa}{s+\kappa}\right)^\ell$

$$=i\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}2^m\kappa^{m+\ell+1}(-1)^\ell\frac{e^{-s\tau m}}{(s+\kappa)^{2+m+\ell}}\frac{m!}{\ell!(m-\ell)!}=$$

$$=i\sum_{m=0}^{\infty}2^m m!\,e^{-sm\tau}\sum_{\ell=0}^{m}\frac{(-1)^\ell}{\ell!(m-\ell)!}\frac{\kappa^{m+\ell+1}}{(m+\ell+1)!}\frac{(m+\ell+1)!}{(s+\kappa)^{(m+\ell+1)+1}}$$

$$\frac{n!}{(s-a)^{n+1}}\xrightarrow{\text{ILT}}t^n e^{at}$$

$e^{-sm\tau}\to m\tau$ shift

$$c_g^{(1)}(t)=i\sum_{m=0}^{\infty}2^m m!\sum_{\ell=0}^{m}\frac{(-1)^\ell}{\ell!(m-\ell)!}\frac{[\kappa(t-m\tau)]^{m+\ell+1}}{(m+\ell+1)!}e^{-\kappa(t-m\tau)}\Theta(t-m\tau)=$$

$$=i\sum_{m=0}^{\infty}2^m m!\,e^{-\kappa(t-m\tau)}\Theta(t-m\tau)\sum_{\ell=0}^{m}\frac{(-1)^\ell}{\ell!(m-\ell)!}\frac{[\kappa(t-m\tau)]^{m+\ell+1}}{(m+\ell+1)!}$$

$$\tilde{c}_g^{(2)}(s)=\frac{i\gamma}{s^2+\gamma^2+2\kappa s}\sum_{m=0}^{\infty}\left(\frac{2\kappa s e^{-s\tau}}{s^2+\gamma^2+\kappa s 2}\right)^m\left(1-e^{-s\tau}\right)^m$$

special case : $\gamma=\kappa$

$$\tilde{c}_g^{(2)}(s)=\frac{i\kappa}{(s+\kappa)^2}\sum_{m=0}^{\infty}(2\kappa)^m e^{-sm\tau}\frac{1}{(s+\kappa)^m}\underbrace{\left(1-\frac{\kappa}{s+\kappa}\right)^m}_{\sum_{\ell=0}^{m}\binom{m}{\ell}\left(-\frac{\kappa}{s+\kappa}\right)^\ell}\underbrace{\left(1-e^{-s\tau}\right)^m}_{\sum_{\ell'=0}^{m}\binom{m}{\ell'}\left(-e^{-s\tau}\right)^{\ell'}}=$$

$$=i\sum_{m=0}^{\infty}m!\,m!\,2^m\sum_{\ell=0}^{m}\frac{(-1)^\ell}{(s+\kappa)^{2+m+\ell}}\frac{\kappa^{m+\ell+1}}{\ell!(m-\ell)!}\sum_{\ell'=0}^{m}\frac{(-1)^{\ell'}}{\ell'!(m-\ell')!}e^{-s(m+\ell')\tau}=$$

$$=i\sum_{m=0}^{\infty}(m!)^2 2^m\sum_{\ell'=0}^{m}\frac{1}{\ell'!(m-\ell')!}e^{-s(m+\ell')\tau}\sum_{\ell=0}^{m}\frac{(-1)^{\ell+\ell'}}{\ell!(m-\ell)!}\frac{\kappa^{m+\ell+1}}{(m+\ell+1)!}\frac{(m+\ell+1)!}{(s+\kappa)^{m+\ell+2}}$$

$$c_g^{(2)}(t)=i\sum_{m=0}^{\infty}(m!)^2 2^m\sum_{\ell'=0}^{m}\frac{(-1)^{\ell'}}{\ell'!(m-\ell')!}\sum_{\ell=0}^{m}\frac{(-1)^\ell}{\ell!(m-\ell)!}\frac{[\kappa(t-(m+\ell')\tau)]^{m+\ell+1}}{(m+\ell+1)!}e^{-\kappa(t-(m+\ell')\tau)}\Theta(t-(m+\ell')\tau)=$$

$$=i\sum_{m=0}^{\infty}(m!)^2 2^m\sum_{\ell'=0}^{m}\frac{(-1)^{\ell'}}{\ell'!(m-\ell')!}e^{-\kappa(t-(m+\ell')\tau)}\Theta(t-(m+\ell')\tau)\sum_{\ell=0}^{m}\frac{(-1)^\ell}{\ell!(m-\ell)!}\frac{[\kappa(t-(m+\ell')\tau)]^{m+\ell+1}}{(m+\ell+1)!}$$

$$\tilde{c}_g^{(\infty)}(s) = \frac{i\gamma}{s^2+\gamma^2+\kappa s 2} \sum_{m=0}^{\infty} \left[\frac{2\kappa s e^{-s\tau}}{(s^2+\gamma^2+\kappa s 2)(1+e^{-s\tau})}\right]^m$$

special case $\gamma = \kappa$

$$\tilde{c}_g^{(\infty)}(s) = \frac{i\kappa}{(s+\kappa)^2} \sum_{m=0}^{\infty} 2^m \kappa^m e^{-sm\tau} \underbrace{\left(\frac{s}{(s+\kappa)^2}\right)^m}_{\frac{1}{(s+\kappa)^m}\left(1-\frac{\kappa}{s+\kappa}\right)^m = \frac{1}{(s+\kappa)^m}\sum_{\ell=0}^{m}\binom{m}{\ell}\left(-\frac{\kappa}{s+\kappa}\right)^\ell} \underbrace{\frac{1}{(1+e^{-s\tau})^m}}_{\sum_{\ell'=0}^{\infty}\binom{\ell'+m-1}{\ell'}(-e^{-s\tau})^{\ell'}} =$$

$$= i\sum_{m=0}^{\infty} m!\, 2^m \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell!(m-\ell)!} \frac{\kappa^{m+\ell+1}}{(m+\ell+1)!} \frac{(m+\ell+1)!}{(s+\kappa)^{m+\ell+2}} \sum_{\ell'=0}^{\infty} \frac{(\ell'+m-1)!}{\ell'!(m-1)!}(-1)^{\ell'} e^{-s(m+\ell')\tau} =$$

$$= i\sum_{m=0}^{\infty} m 2^m \sum_{\ell'=0}^{\infty} \frac{(\ell'+m-1)!}{\ell'!}(-1)^{\ell'} e^{-s(m+\ell')\tau} \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell!(m-\ell)!} \frac{\kappa^{m+\ell+1}}{(m+\ell+1)!} \frac{(m+\ell+1)!}{(s+\kappa)^{m+\ell+2}}$$

$$c_g^{(\infty)}(t) = i\sum_{m=0}^{\infty} m 2^m \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!}(\ell'+m-1)! \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell!(m-\ell)!} \frac{[\kappa(t-(m+\ell')\tau)]^{m+\ell+1}}{(m+\ell+1)!} e^{-\kappa(t-(m+\ell')\tau)}\, \Theta(t-(m+\ell')\tau) =$$

$$= i\sum_{m=0}^{\infty} m 2^m \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!}(\ell'+m-1)!\, e^{-\kappa(t-(m+\ell')\tau)}\, \Theta(t-(m+\ell')\tau) \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell!(m-\ell)!} \frac{[\kappa(t-(m+\ell')\tau)]^{m+\ell+1}}{(m+\ell+1)!}$$

Continuous case:

$$\partial_t c_e = i\gamma c_g(t)$$
$$\partial_t c_g = i\gamma c_e(t) + i\int^{\infty} dk\, G(k,t)\, c_{g,k}(t)$$
$$\partial_t c_{g,k} = i\, G^*(k,t)\, c_g(t) \Rightarrow c_{g,k}(t) = i\int_0^t G^*(k,t)\, c_g(t)\, dt'$$

$$G(k,t) = \underbrace{G_0}_{[\,]=\frac{\sqrt{m}}{s}} \sin(kL)\, e^{i(\omega_0-\omega_k)t} \qquad \omega_k = c_0 k \qquad \tau = \frac{2L}{c_0}$$

$$\partial_t c_g = i\gamma c_e(t) - |G_0|^2 \int_0^{\infty} \sin^2(kL) \int_0^t e^{i(\omega_0-\omega_k)(t-t')} c_g(t')\, dt'\, dk =$$
$$= i\gamma c_e(t) - |G_0|^2 \int_0^t c_g(t') \int_0^{\infty} e^{i(k_0-k)(t-t')c_0} \sin^2(kL)\, dk\, dt' = \qquad \omega_k = c_0 k$$
$$= i\gamma c_e(t) - |G_0|^2 \int_0^t c_g(t') \underbrace{\int_0^{\infty} e^{i(\omega_0-\omega_k)(t-t')} \sin^2\left(\omega_k \tfrac{\tau}{2}\right) \frac{d\omega_k}{c_0}}\, dt'$$

$$\frac{1}{2i}\int^{\infty} e^{i(\omega_0-\omega_k)(t-t')}\left[e^{i\omega_k\tau} + e^{-i\omega_k\tau} - 2\right]\frac{d\omega_k}{c_0} =$$
$$= -\frac{1}{4}\int_0^{\infty}\left[e^{i(\omega_0-\omega_k)(t-t'-\tau)} e^{i\omega_0\tau} + e^{i(\omega_0-\omega_k)(t-t'+\tau)} e^{-i\omega_0\tau} - 2e^{i(\omega_0-\omega_k)(t-t')}\right]\frac{d\omega_k}{c_0} =$$

$\omega_k' = \omega_k - \omega_0$

$$= -\frac{1}{4}\int_{-\omega_0 \to -\infty}^{\infty}\left[e^{-i\omega_k'(t-t'-\tau)} e^{i\omega_0\tau} + e^{-i\omega_k'(t-t'+\tau)} e^{-i\omega_0\tau} - 2e^{-i\omega_k'(t-t')}\right]\frac{d\omega_k'}{c_0} =$$
$$= -\frac{\pi}{2c_0}\left\{\delta(t'-t+\tau)\, e^{i\omega_0\tau} + \delta(t'-t-\tau)\, e^{-i\omega_0\tau} - 2\delta(t'-t)\right]$$

$$\partial_t c_g = i\gamma c_e(t) - \frac{|G_0|^2\pi}{2c_0}\left\{2\int_0^t c_g(t')\,\delta(t-t')\,dt' - \int_0^t c_g(t')\,\delta(t'-t+\tau)\, e^{i\omega_0\tau} dt' - \int_0^t \cancel{c_g(t')\,\delta(t'-t-\tau)\, e^{-i\omega_0\tau} dt'}\right\} =$$
$$= i\gamma c_e(t) - \underbrace{\frac{|G_0|^2\pi}{2c_0}}_{2\kappa}\left\{c_g(t) - c_g(t-\tau)\, e^{i\omega_0\tau}\,\Theta(t-\tau)\right\}$$

For discrete modes: $G(k,t) = G_q(t) = \frac{G_0}{\sqrt{2}}\sin(k_q L)\, e^{i(\omega_0-\omega_q)t} =$
$$= \frac{G_0}{\sqrt{2}}\sin((k_{q'}+k_0)L)\, e^{-i\omega_{q'}t} \qquad \omega_0 = c_0 k_0$$

$$k_q = \frac{2q+1}{2L}\pi \qquad\qquad k_{q'} = k_q - k_0$$

$$\partial_t c_e = i\gamma c_g(t) \qquad \partial_t c_g = i\gamma c_e(t) + i\sum_{q0}^{\infty} G_q(t)\, c_{g,q}(t) \qquad \underbrace{\partial_t c_{g,q} = i\, G_q^*(t)\, c_g(t)}$$
$$c_{g,q}(t) = i\int_0^t G_q^*(t')\, c_g(t')\, dt'$$

$$\partial_t c_g = i\gamma c_e(t) - \frac{|G_0|^2}{2L}\sum_{q'=-\infty}^{\infty} \sin^2((k_{q'}+k_0)L)\int_0^t e^{-i\omega_{q'}(t-t')} c_g(t')\, dt' =$$
$$= i\gamma c_e(t) - \frac{|G_0|^2}{2L}\int_0^t c_g(t')\sum_{q'=-\infty}^{\infty}\sin^2((k_{q'}+k_0)L)\, e^{-i\omega_{q'}(t-t')}\, dt'$$

$$\sin^2((k_{q'}+k_0)L) = -\frac{1}{4}\left[e^{2i(k_{q'}+k_0)L} + e^{-2i(k_{q'}+k_0)L} - 2\right] =$$

$\tau = \frac{2L}{c_0}$ $\quad$ small $\ll \frac{\pi}{L}$

$$= -\frac{1}{4}\left[e^{i(\omega_{q'}+\omega_0)\tau} + e^{-i(\omega_{q'}+\omega_0)\tau} - 2\right]$$

$k_{q'}$ centered around $k_0 + \frac{\Delta}{c_0}$ $\quad k = q'\frac{\pi}{L} + \frac{\Delta}{c_0} = q'\frac{\pi 2}{\tau c_0} + \frac{\Delta}{c_0}$ $\quad q' \in \mathbb{Z}$ $\quad \omega_{q'} = 2q'\pi\frac{1}{\tau} + \Delta$

$$\sin^2[(k_{q'}+k_0)L] = \sin^2\left[(\omega_{q'}+\omega_0)\frac{\tau}{2}\right] = -\frac{1}{4}\left[e^{2iq'\pi} e^{i(\Delta+\omega_0)\tau} + e^{-2iq'\pi} e^{-i(\Delta+\omega_0)\tau} - 2\right] =$$
$$= \frac{1}{2}\left[1 - \cos(\Delta+\omega_0)\tau\right] = \sin^2\left(\frac{\Delta+\omega_0}{2}\tau\right)$$

$$\sum_{q'=-\infty}^{\infty} e^{-iq'2\pi\frac{t-t'}{\tau}} = \tau\sum_{q'=-\infty}^{\infty}\delta(t-t'-q'\tau)$$

$$\partial_t c_g = i\gamma c_e(t) - \frac{|G_0|^2}{2L}\tau\sin^2\left(\frac{\Delta+\omega_0}{2}\tau\right)\sum_{q'=-\infty}^{\infty}\int_0^t e^{-i\Delta(t-t')}\,\delta(t-t'-q'\tau)\, c_g(t')\, dt' =$$

$\hookrightarrow$ not in the integration range for $q'<0$

$$= i\gamma c_e(t) - \underbrace{\frac{|G_0|^2}{c_0}\sin^2\left(\frac{\Delta+\omega_0}{2}\tau\right)}_{4\kappa'}\left(\sum_{q'=0}^{\infty} c_g(t-q'\tau)\, e^{i\Delta q'\tau}\,\Theta(t-q'\tau) - \frac{1}{2}c_g(t)\right)$$

From now I will assume that $\Delta = 0$ for the sake of simplicity.

$$\partial_t c_e = i\gamma c_g(t)$$

$$\partial_t c_g = i\gamma c_e(t) - \underbrace{\frac{|G_0|^2}{c_0} \sin^2\frac{\omega_0 \tau}{2}}_{4\kappa'} \left[\sum_{q'=0}^{\infty} c_g(t-q'\tau)\,\theta(t-q'\tau) - \frac{1}{2}c_g(t)\right]$$

$$s\tilde{c}_e(s) - 1 = i\gamma \tilde{c}_g(s) \qquad \tilde{c}_e(s) = \frac{1}{s}\left(i\gamma \tilde{c}_g(s) + 1\right)$$

$$s\tilde{c}_g(s) = i\gamma \tilde{c}_e(s) - 4\kappa' \tilde{c}_g(s)\left[\sum_{q'=0}^{\infty} e^{-q's\tau} - \frac{1}{2}\right]$$

$$\left[s^2 + \gamma^2 + 4\kappa' s\left(\sum_{q'=0}^{\infty} e^{-q's\tau} - \frac{1}{2}\right)\right]\tilde{c}_g(s) = i\gamma$$

$t \leq \tau$: $\quad \tilde{c}_g^{(0)}(s) = \frac{i\gamma}{s^2+\gamma^2+2\kappa' s} = \frac{i\gamma}{(s+\kappa')^2+\gamma^2-\kappa'^2}$ $\qquad c_g(t) = i\gamma \frac{\sin\left[\sqrt{1-\left(\frac{\kappa'}{\gamma}\right)^2}\,\gamma t\right]}{\sqrt{1-\left(\frac{\kappa'}{\gamma}\right)^2}} e^{-\kappa' t}$

$n=0$ $\quad$ poles: $\quad s^2 + 2\kappa' s + \gamma^2 = 0$ $\qquad \gamma > \kappa'$

$$s_{1,2} = -\kappa' \pm \sqrt{\kappa'^2 - \gamma^2} = -\kappa' \pm i\gamma\sqrt{1-\frac{\kappa'^2}{\gamma^2}}$$

$\tau < t \leq 2\tau$ $\quad \tilde{c}_g^{(1)}(s) = \frac{i\gamma}{s^2+\gamma^2+2\kappa' s + 4\kappa' s e^{-s\tau}} = \frac{i\gamma}{s^2+\gamma^2+2\kappa' s}\left(1 + \frac{4\kappa' s e^{-s\tau}}{s^2+\gamma^2+2\kappa' s}\right)^{-1} = \frac{i\gamma}{s^2+\gamma^2+2\kappa' s}\sum_{m=0}^{\infty}\left(-\frac{4\kappa' s e^{-s\tau}}{s^2+\gamma^2+2\kappa' s}\right)^m$

Poles: $\quad s^2 + \gamma^2 + 2\kappa' s + 4\kappa' s e^{-s\tau} = 0$

Can I have $s = \pm i\gamma$?

$$-\gamma^2 + \gamma^2 \pm 2i\kappa'\gamma \pm 4i\kappa'\gamma e^{\mp i\gamma\tau} = 0$$

$$-\frac{1}{2} = e^{\mp i\gamma\tau}$$

$$\tau = \pm\frac{2\pi}{3\gamma} + 2n\pi$$

$n\tau < t \leq (n+1)\tau$ $\quad \tilde{c}_g^{(n)}(s) = \frac{i\gamma}{s^2+\gamma^2-2\kappa' s + 4\kappa' s\sum_{q'=0}^{n} e^{-sq'\tau}} = \frac{i\gamma}{s^2+\gamma^2-2\kappa' s}\left(1 + \frac{4\kappa' s \sum_{q'=0}^{n} e^{-sq'\tau}}{s^2+\gamma^2-2\kappa' s}\right)^{-1} =$

$$= \frac{i\gamma}{s^2+\gamma^2-2\kappa' s}\sum_{m=0}^{\infty}\left(-\frac{4\kappa' s\sum_{q'=0}^{n} e^{-sq'\tau}}{s^2+\gamma^2-2\kappa' s}\right)^m$$

Poles: $\quad s^2+\gamma^2-2\kappa' s + 4\kappa' s\sum_{q'=0}^{n} e^{-sq'\tau} = 0$

Can I have $s = \pm i\gamma$? $\qquad$ it can't be just 1 frequency

$$-\gamma^2 + \gamma^2 \mp 2i\kappa'\gamma \pm 4i\kappa'\gamma\sum_{q'=0}^{n} e^{\mp i\gamma q'\tau} = 0$$

$$\sum_{q'=0}^{n} e^{-i\gamma q'\tau} = \frac{1}{2} \Rightarrow \text{for } n \to \infty \text{ this series}$$

can't be convergent

$$\mathrm{Re}(s) \overset{!}{>} 0 \qquad \lightning \qquad |e^{-i\gamma q'\tau}| \not< 1$$

diverges

$\gamma \overset{!}{=} \kappa'$ $\quad \tilde{c}_g^{(\infty)}(s) = \frac{i\kappa}{(s-\kappa')^2}\sum_{m=0}^{\infty}(-1)^m 2^{2m}\kappa'^m \underbrace{\left(\frac{s}{(s-\kappa')^2}\right)^m}_{\left(\frac{1}{s-\kappa'}+\frac{\kappa'}{(s-\kappa')^2}\right)^m = \frac{1}{(s-\kappa')^m}\left(1+\frac{\kappa'}{s-\kappa'}\right)^m} \frac{1}{(1-e^{-s\tau})^m}$

$$\left(1+\frac{\kappa'}{s-\kappa'}\right)^m = \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\kappa'^\ell}{(s-\kappa')^\ell} \qquad \frac{1}{(1-e^{-s\tau})^m} = \sum_{\ell'=0}^{\infty}\binom{\ell'+m-1}{\ell'}e^{-s\ell'\tau}$$

$$\tilde{c}_g^{(\infty)}(s) = i\sum_{m=0}^{\infty}(-1)^m 2^{2m}\sum_{\ell=0}^{m}\frac{m!}{\ell!(m-\ell)!}\kappa'^{m+\ell+1}\frac{1}{(s-\kappa')^{m+\ell+2}}\sum_{\ell'=0}^{\infty}\frac{(\ell'+m-1)!}{\ell'!(m-1)!}e^{-s\ell'\tau} =$$

$$= i\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{\infty}(-1)^m 2^{2m}\frac{m(\ell'+m-1)!}{\ell!\,\ell'!(m-\ell)!}\frac{\kappa'^{m+\ell+1}}{(m+\ell+1)!}\frac{(m+\ell+1)!}{(s-\kappa')^{m+\ell+2}}e^{-s\ell'\tau}$$

$$c_g^{(\infty)}(t) = i\left[\sum_{m=1}^{\infty}\sum_{\ell=0}^{m}\sum_{\ell'=0}^{\infty}(-4)^m \underbrace{\frac{m(\ell'+m-1)!}{\ell!\,\ell'!(m-\ell)!}}_{\binom{m}{\ell}\binom{\ell'+m-1}{\ell'}}\frac{[\kappa'(t-\ell'\tau)]^{m+\ell+1}}{(m+\ell+1)!}e^{\kappa'(t-\ell'\tau)}\,\theta(t-\ell'\tau) + e^{\kappa' t}\right]$$

$$\binom{m}{\ell} = \prod_{i=1}^{\ell}\frac{m+1-i}{i} = \frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdots\frac{m-\ell+2}{\ell-1}\,\frac{m-\ell+1}{\ell}$$

$$\binom{\ell'+m-1}{\ell'} = \prod_{j=1}^{\ell'}\frac{\ell'+m-j}{j}$$

$$\frac{1}{(1-x)^m} = \sum_{k=0}^{\infty} \binom{m-1+k}{k} x^k \qquad m=0 \Rightarrow 1, \quad m=1 \Rightarrow \sum_{k=0}^{\infty} \binom{k}{k} x^k \Rightarrow \text{geometric series}$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \binom{k+1}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)!}{k!\,1!} x^k = \sum_{k=0}^{\infty} (k+1)\, x^k = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots =$$

$$= \frac{1}{1-2x+x^2} = \frac{1}{1-x(2-x)} = \sum_{\ell=0}^{\infty} x^\ell (2-x)^\ell = 1 + x(2-x) + x^2(2-x)^2 + \ldots = 1 + 2x - x^2 + 4x^2 - 4x^3 + x^4 + 8x^3$$

$$\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \binom{k+2}{k} x^k = \sum_{k=0}^{\infty} \frac{(k+2)!}{k!\,2!} x^k = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^k = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$$

$$= \frac{1}{1-3x+3x^2-x^3} = \frac{1}{1-x(3-3x+x^2)} = \sum_{\ell=0}^{\infty} x^\ell (3-3x+x^2)^\ell = 1 + 3x - 3x^2 + x^3 + 9x^2 - 18x^3$$