Simulations of quantum walks on regular graphs

Katalin Friedl   Viktória Nemkin

MŰEGYETEM 1782

Quantum Information National Laboratory HUNGARY

Quantum walks on regular graphs

- Quantum algorithms for solving search problems.
- Generalization of classical random graph walks.
- Vastly different behaviour.
- Random choice = quantum coin toss.

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $\left|-N\right\rangle, \left|-N+1\right\rangle, \ldots, \left|-1\right\rangle, \left|0\right\rangle, \left|1\right\rangle, \ldots, \left|N-1\right\rangle, \left|N\right\rangle$

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $\left|-N\right\rangle, \left|-N+1\right\rangle, \ldots, \left|-1\right\rangle, \left|0\right\rangle, \left|1\right\rangle, \ldots, \left|N-1\right\rangle, \left|N\right\rangle$
  - $\left|x\right\rangle = \sum_{i=-N}^{N} x_i \left|i\right\rangle \in \mathbb{C}^{2N+1}$

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $\left|-N\right\rangle, \left|-N+1\right\rangle, \ldots, \left|-1\right\rangle, \left|0\right\rangle, \left|1\right\rangle, \ldots, \left|N-1\right\rangle, \left|N\right\rangle$
  - $\left|x\right\rangle = \sum_{i=-N}^{N} x_i \left|i\right\rangle \in \mathbb{C}^{2N+1}$
- Coin state:

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $|-N\rangle, |-N+1\rangle, \dots, |-1\rangle, |0\rangle, |1\rangle, \dots, |N-1\rangle, |N\rangle$
  - $|x\rangle = \sum_{i=-N}^{N} x_i |i\rangle \in \mathbb{C}^{2N+1}$
- Coin state:
  - Heads $= |0\rangle$, Tails $= |1\rangle$

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $\left|-N\right\rangle, \left|-N+1\right\rangle, \ldots, \left|-1\right\rangle, \left|0\right\rangle, \left|1\right\rangle, \ldots, \left|N-1\right\rangle, \left|N\right\rangle$
  - $\left|x\right\rangle = \sum_{i=-N}^{N} x_i \left|i\right\rangle \in \mathbb{C}^{2N+1}$
- Coin state:
  - Heads $= \left|0\right\rangle$, Tails $= \left|1\right\rangle$
  - $\left|s\right\rangle = s_0 \left|0\right\rangle + s_1 \left|1\right\rangle \in \mathbb{C}^2$

1 dimension

Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $|-N\rangle, |-N+1\rangle, \ldots, |-1\rangle, |0\rangle, |1\rangle, \ldots, |N-1\rangle, |N\rangle$
  - $|x\rangle = \sum_{i=-N}^{N} x_i |i\rangle \in \mathbb{C}^{2N+1}$
- Coin state:
  - Heads $= |0\rangle$, Tails $= |1\rangle$
  - $|s\rangle = s_0 |0\rangle + s_1 |1\rangle \in \mathbb{C}^2$
- Composite state:

1 dimension

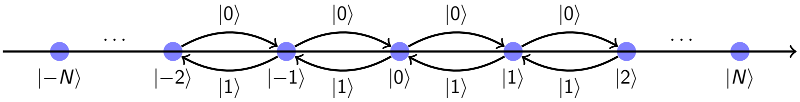
Current state

- Quantum register $\rightarrow$ a unit vector in Hilbert space.
- Position state:
  - Base states: $\left|-N\right\rangle, \left|-N+1\right\rangle, \ldots, \left|-1\right\rangle, \left|0\right\rangle, \left|1\right\rangle, \ldots, \left|N-1\right\rangle, \left|N\right\rangle$
  - $\left|x\right\rangle = \sum_{i=-N}^{N} x_i \left|i\right\rangle \in \mathbb{C}^{2N+1}$
- Coin state:
  - Heads $= \left|0\right\rangle$, Tails $= \left|1\right\rangle$
  - $\left|s\right\rangle = s_0 \left|0\right\rangle + s_1 \left|1\right\rangle \in \mathbb{C}^2$
- Composite state:
  - $\left|x\right\rangle \otimes \left|s\right\rangle$

Evolution

Quantum operators = unitary matrices.

1. Flipping the coin.
2. Stepping according to the result.



Quantum coins

Quantum coins

Hadamard coin

$$\mathrm{H}^{\otimes n} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]^{\otimes n}$$

Quantum coins

Hadamard coin

$$\mathrm{H}^{\otimes n} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]^{\otimes n}$$

Grover coin (diffusion)

$$\begin{aligned}|D\rangle &= \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i\rangle\\ \mathrm{G} &= 2\,|D\rangle\langle D| - \mathrm{I}\end{aligned}$$

Quantum coins

Hadamard coin

$$\mathrm{H}^{\otimes n} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]^{\otimes n}$$

Grover coin (diffusion)

$$\left|D\right\rangle = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}\left|i\right\rangle$$
$$\mathrm{G} = 2\left|D\right\rangle\left\langle D\right| - \mathrm{I}$$

Fourier coin (QFT)

$$\mathrm{F}_N = \left[\frac{1}{\sqrt{N}}e^{\frac{2\pi i}{N}xy}\right]_{x,y}$$

Step

$$|i\rangle\,|0\rangle \rightarrow |i-1\rangle\,|0\rangle$$
$$|i\rangle\,|1\rangle \rightarrow |i+1\rangle\,|1\rangle$$

Step

$$|i\rangle |0\rangle \to |i-1\rangle |0\rangle$$
$$|i\rangle |1\rangle \to |i+1\rangle |1\rangle$$

Left shift operator

$$\mathsf{L} = |N\rangle \langle -N| + \sum_{i=-(N-1)}^{N} |i-1\rangle \langle i|$$

Step

$$|i\rangle|0\rangle \to |i-1\rangle|0\rangle$$
$$|i\rangle|1\rangle \to |i+1\rangle|1\rangle$$

Left shift operator

$$\mathrm{L} = |N\rangle\langle -N| + \sum_{i=-(N-1)}^{N} |i-1\rangle\langle i|$$

Right shift operator

$$\mathrm{R} = |-N\rangle\langle N| + \sum_{i=-N}^{N-1} |i+1\rangle\langle i|$$

Step

$$|i\rangle|0\rangle \rightarrow |i-1\rangle|0\rangle$$
$$|i\rangle|1\rangle \rightarrow |i+1\rangle|1\rangle$$

Left shift operator

$$\mathrm{L} = |N\rangle\langle -N| + \sum_{i=-(N-1)}^{N} |i-1\rangle\langle i|$$

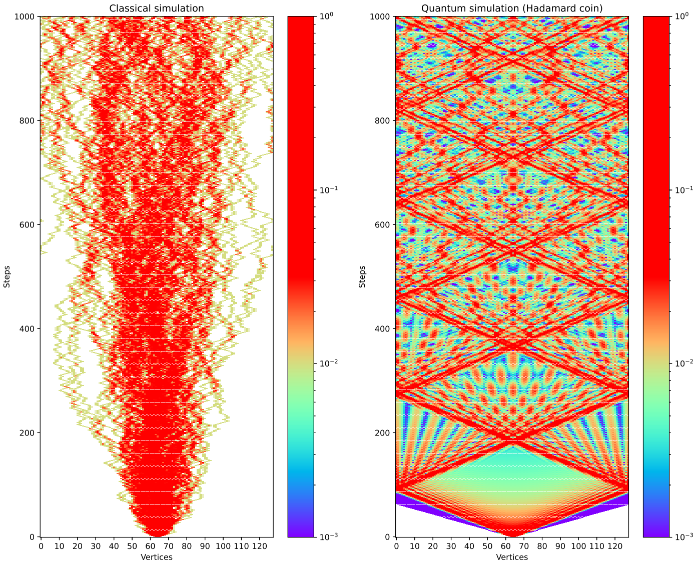
Right shift operator

$$\mathrm{R} = |-N\rangle\langle N| + \sum_{i=-N}^{N-1} |i+1\rangle\langle i|$$

Shift operator

$$\mathrm{S} = \mathrm{L}\otimes|0\rangle\langle 0| + \mathrm{R}\otimes|1\rangle\langle 1|$$

Walk in 1 dimension (Circle)



Generalization to undirected d-regular graphs

Generalization to undirected d-regular graphs

- Position state: $|x\rangle = \sum_{v=0}^{n-1} x_v |v\rangle \in \mathbb{C}^n$

Generalization to undirected d-regular graphs

- Position state: $|x\rangle = \sum_{v=0}^{n-1} x_v |v\rangle \in \mathbb{C}^n$
- Coin state: $|s\rangle = \sum_{i=0}^{d-1} s_i |i\rangle \in \mathbb{C}^d$

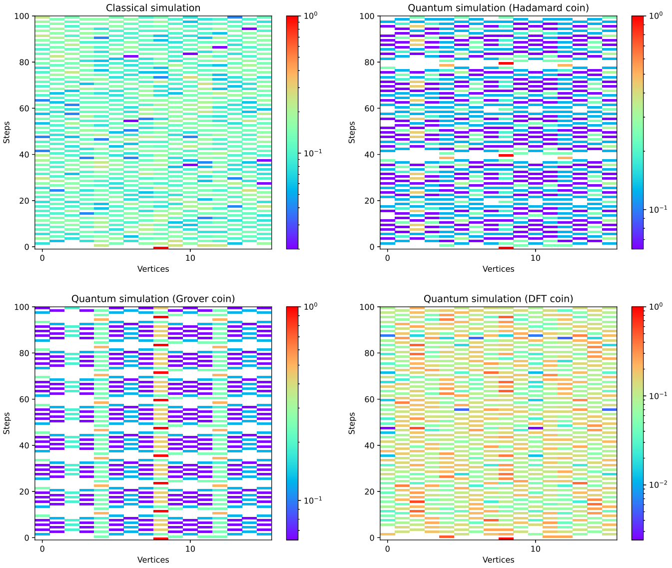
Generalization to undirected d-regular graphs

- Position state: $|x\rangle = \sum_{v=0}^{n-1} x_v |v\rangle \in \mathbb{C}^n$
- Coin state: $|s\rangle = \sum_{i=0}^{d-1} s_i |i\rangle \in \mathbb{C}^d$
- Step operator: $\mathrm{S} = \mathrm{S}_0 \otimes |0\rangle\langle 0| + \mathrm{S}_1 \otimes |1\rangle\langle 1| + \cdots + \mathrm{S}_{d-1} \otimes |d-1\rangle\langle d-1|$
    - $\sum_{i=0}^{d-1} S_i$ is the adjacency matrix of the graph.
    - The evolution operator is unitary $\rightarrow$ the S_i must be permutation matrices.

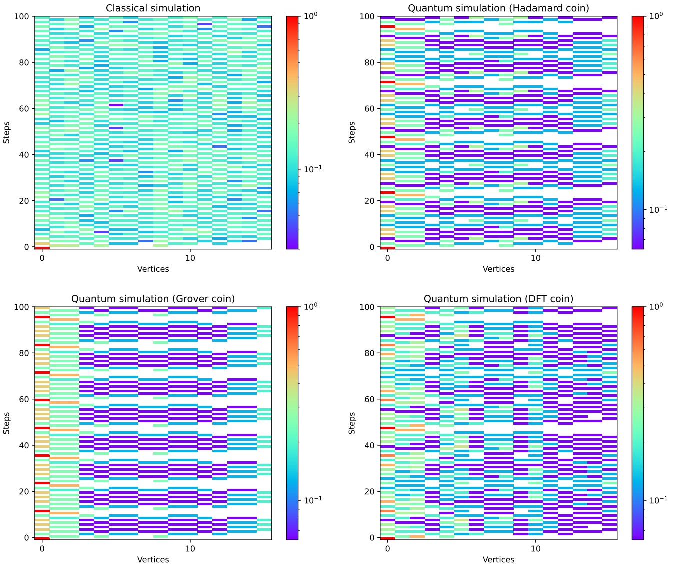
Generalization to undirected d-regular graphs

- Position state: $|x\rangle = \sum_{v=0}^{n-1} x_v |v\rangle \in \mathbb{C}^n$
- Coin state: $|s\rangle = \sum_{i=0}^{d-1} s_i |i\rangle \in \mathbb{C}^d$
- Step operator: $\mathrm{S} = \mathrm{S}_0 \otimes |0\rangle\langle 0| + \mathrm{S}_1 \otimes |1\rangle\langle 1| + \cdots + \mathrm{S}_{d-1} \otimes |d-1\rangle\langle d-1|$
  - $\sum_{i=0}^{d-1} S_i$ is the adjacency matrix of the graph.
  - The evolution operator is unitary $\rightarrow$ the S_i must be permutation matrices.
- (Coin toss operator)

Walk in 2 dimensions (Torus)



Walk in higher dimensions (Hypercube)



Current limitations and future plans

- Current state: coined walks, no errors.
- Simulator in Python, report probability distribution, and various properties (eigenvalues and eigenvectors, graphs on hitting and mixing times).
- Future plans:
  - Szegedy walks.
  - Simulating noise.
  - Moving to C++, memory optimization techniques.
  - The Latex document is too long $\rightarrow$ website.

Software

- `github.com/nemkin/quantum-walk` (open source, MIT license)