Simulations of quantum walks on regular graphs

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Quantum walks on regular graphs

- Quantum algorithms for solving search problems.
- Generalization of classical random graph walks.
- Vastly different behaviour.
- Random choice = quantum coin toss.

1 dimension

State registers

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  - $\left|x\right\rangle = \sum_{i=-N}^{N} x_i \left|i\right\rangle \in \mathbb{C}^{2N+1}$
- Coin state:
  - Heads = $\left|0\right\rangle$, Tails = $\left|1\right\rangle$
  - $\left|s\right\rangle = s_0 \left|0\right\rangle + s_1 \left|1\right\rangle \in \mathbb{C}^2$
- Composite state:
  - $\left|x\right\rangle \otimes \left|s\right\rangle$

Evolution

1. Flipping the coin.
2. Stepping according to the result.

- Quantum operators = unitary matrices.

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Fourier coin (QFT)

$$F_N = \left[\frac{1}{\sqrt{N}}e^{\frac{2\pi i}{N}xy}\right]_{x,y}$$

Step

$$|i\rangle\,|0\rangle \to |i-1\rangle\,|0\rangle$$
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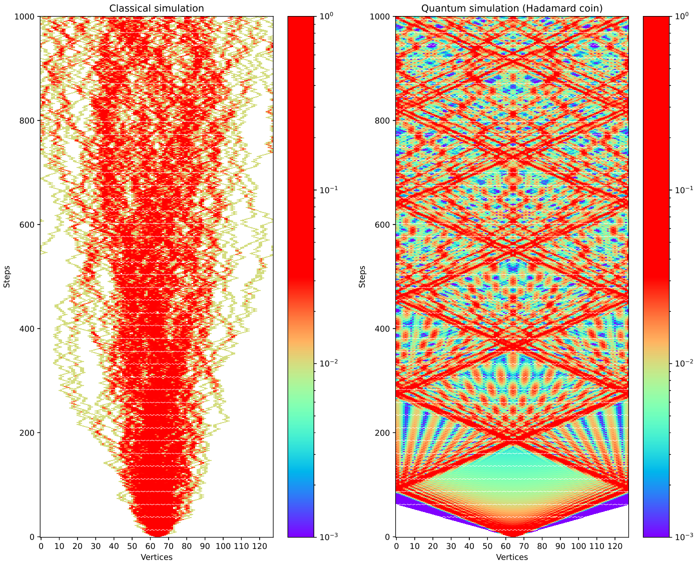
Shift operator

$$\mathsf{S} = \mathsf{L} \otimes |0\rangle\langle 0| + \mathsf{R} \otimes |1\rangle\langle 1|$$

Walk in 1 dimension



Generalization to undirected d-regular graphs

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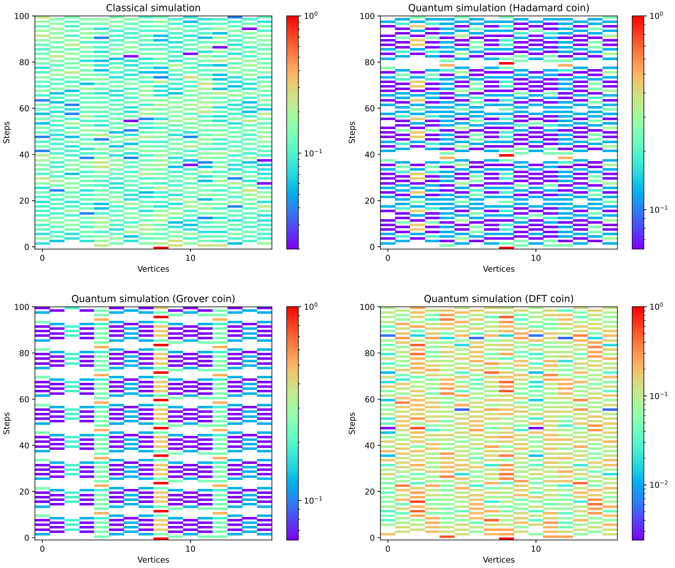
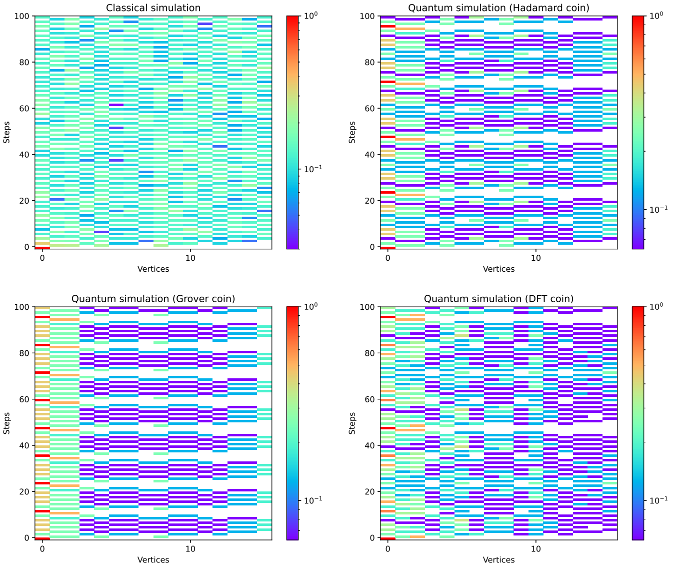
- (Coin toss operator)

Walk in 2 dimension (Grid)



Walk on a Hypercube



Software

- github.com/nemkin/quantum-walk (open source, MIT license)