$\cancel{B\hat{\beta}|X}$

ДЗ 2

V1 $f(x; \lambda) = \frac{\lambda}{2} e^{-\lambda(|x|-1)}, \quad \lambda > 0$

а) $L(\lambda, x) = \prod_{i=1}^{n} f_{x_i}(X_i, \lambda) = \prod_{i=1}^{n} \frac{\lambda}{2} e^{-\lambda(|X_i|-1)}$

$\ln L(\lambda, x) = \sum_{i=1}^{n} \ln\left[\frac{\lambda}{2} e^{-\lambda(|X_i|-1)}\right] = \sum\left[\ln\frac{\lambda}{2} - \lambda(|X_i|-1)\right] =$

$= n\ln\frac{\lambda}{2} - \lambda\sum(|X_i|-1)$

$S(\theta, x) = n\frac{1}{\frac{\lambda}{2}}\frac{1}{2} - \sum(|X_i|-1) = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum(|X_i|-1)}$

б) $\mathbb{E}\hat{\lambda} = \mathbb{E}\frac{n}{\sum(|X_i|-1)} > \frac{n}{\mathbb{E}\sum(|X_i|-1)} = \frac{n}{n\frac{1}{\lambda}} = \lambda$

т.к. $\mathbb{E}(|X_i|-1) = \frac{1}{\lambda}$ по распредел. оценка смещённая

в) $\frac{1}{n}\sum_{i=1}^{n}(|X_i|-1) \xrightarrow{P} \mathbb{E}[|X|-1] = \frac{1}{\lambda}$ (ЗБЧ)

$\hat{\lambda} = \frac{1}{\frac{1}{n}\sum(|X_i|-1)} \xrightarrow{P} \frac{1}{\frac{1}{\lambda}} = \lambda \Rightarrow \hat{\lambda}$ состоятельная

г) $\cancel{\mathbb{E}(|X_i|-1) = \frac{1}{\lambda}, \ \mathbb{E}|X| = \frac{1}{\lambda}+1 \xrightarrow{\text{метод моментов}} \overline{|X|} = \frac{1}{\hat{\lambda}}+1}$

$\cancel{\hat{\lambda} = \frac{1}{\overline{|X|}-1}}$

$\cancel{\mathbb{E}X}$ $\quad \mathbb{E}X = 0 \quad , \quad \mathbb{E}(X^2) = \int\limits_{-\infty}^{+\infty} x^2 \frac{\lambda}{2} e^{-\lambda(|x|-1)}\, dx =$

$$= 2\frac{\lambda}{2}\int\limits_0^{\infty} x^2 e^{-\lambda(|x|-1)} = \lambda\int\limits_0^{\infty} x^2 e^{-\lambda(x-1)}\, dx = \lambda e^{+\lambda}\int\limits_0^{\infty} x^2 e^{-\lambda x}\, dx =$$

$$= \frac{2}{\lambda^3}\cdot\lambda e^{+\lambda} = \frac{2e^{+\lambda}}{\lambda^2}\ ?$$ Скорее всего в условии ошибка, ведь заданная плотность $f(x,\lambda)$ не соотв. условию нормировки

При условии $f(x;\lambda) = \frac{\lambda}{2} e^{-\lambda(|x|\cancel{-1})}$:

$$\mathbb{E}(X^2) = \int\limits_{-\infty}^{+\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|}\, dx = \frac{\lambda}{2}\int\limits_0^{+\infty} x^2 e^{-\lambda x}\, dx = \frac{\lambda}{2}\frac{2}{\lambda^3} = \frac{1}{\lambda^2}$$

$$\overline{X^2} = \frac{1}{\hat{\lambda}^2} \Rightarrow \hat{\lambda}^2 = \frac{1}{\overline{X^2}},\ \lambda > 0 \Rightarrow \hat{\lambda} = \frac{1}{\sqrt{\overline{X^2}}}$$

e) $I_1(p) = \mathbb{E}S_1^2(\lambda, x) = \mathbb{E}\left(\frac{1}{\lambda} - (|X|-1)\right)^2 = \left\{ Y = |X| - 1 \right\} =$

$$= \mathbb{E}Y = \frac{1}{\lambda} \quad \mathbb{D}Y = \frac{1}{\lambda^2} \quad \mathbb{E}\left(\frac{1}{\lambda} - Y\right) = 0$$

$$I_1(p) = \mathbb{E}\left(\frac{1}{\lambda} + 1 - |X|\right)^2 = \mathbb{E}\left(\left(\frac{1}{\lambda}+1\right)^2 - 2\left(\frac{1}{\lambda}+1\right)|X| + |X|^2\right) =$$

$$= \cancel{\left(\frac{1}{\lambda}+1\right)^2 - 2\left(\frac{1}{\lambda}+1\right)} = \mathbb{E}\left(\frac{1}{\lambda^2} - 2\frac{1}{\lambda}(|X|-1) + (|X|-1)^2\right) =$$

$$= \frac{1}{\lambda^2} - \frac{2}{\lambda}\cdot\frac{1}{\lambda} + \mathbb{E}(|X|-1)^2 = -\frac{1}{\lambda^2} + \mathbb{E}|X|^2 - 2\mathbb{E}|X| + 1 =$$

$$= -\frac{1}{\lambda^2} + \frac{2e}{\lambda^2} - \left(2\mathbb{E}|X| - 2\right) - 1 = -\frac{1}{\lambda^2} + \frac{2e}{\lambda^2} - \frac{2}{\lambda} - 1$$

$\mathbb{E}X^2 = \frac{2e^{\lambda}}{\lambda^2}$ если строго по условию

№2 $f_X(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta} \mathbb{1}_{x>0}$

a) $L(\theta, x) = \prod f_X(x_i, \theta)$ $\qquad \ln L_1(\theta, x) = -\ln\Gamma(\alpha) - \alpha\ln\theta + (\alpha-1)\ln x - \frac{x}{\theta}$

$I_n(\theta) = \mathbb{E}(s^2(\theta, x))$; $-\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\ln L_1(\theta, x)\right)$

$\frac{\partial}{\partial\theta}\ln L_1(\theta, x) = -\frac{\alpha}{\theta} + \frac{x}{\theta^2}$ $\qquad \mathbb{E}(X) = \alpha\theta$

$\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\ln L_1(\theta, x)\right) = \frac{\alpha}{\theta^2} - \frac{2}{\theta^3}\mathbb{E}(X) = \frac{\alpha}{\theta^2} - \frac{2}{\theta^3}\alpha\theta = -\frac{\alpha}{\theta^2}$

$\Rightarrow I_1(\theta) = \frac{\alpha}{\theta^2}$ $\qquad I_n(\theta) = \frac{n\alpha}{\theta^2}$

b) $\frac{\partial}{\partial\theta}\ln L(\theta) = -\frac{n\alpha}{\theta} + \frac{1}{\theta^2}\sum X_i = 0$

$\hat\theta = \frac{1}{n\alpha}\sum X_i = \frac{\overline{X}}{\alpha}$

$\mathbb{E}\hat\theta = \mathbb{E}\frac{\overline{X}}{\alpha} = \frac{1}{\alpha}\alpha\theta = \theta$ $\Rightarrow$ несмещённая

$D(\hat\theta) = D\left(\frac{\overline{X}}{\alpha}\right) = \frac{1}{\alpha^2}D\overline{X} = \frac{1}{\alpha^2}\cdot\frac{DX}{n} = \frac{1}{\alpha^2}\frac{\alpha\theta^2}{n} = \frac{\theta^2}{\alpha n}$

по теореме Рао-Крамера: $D(\hat\theta) \geq \frac{1}{I_n(\theta)} = \frac{\theta^2}{\alpha n}$

получается, что $D(\hat\theta)$ – min из возможных $D(\hat\theta)$ $\Rightarrow$ оценка эффективная

c) $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) = \mathcal{N}\left(0, \frac{\theta^2}{\alpha}\right)$

№3 $T \sim \exp(\lambda)$, $F(t) = 1 - e^{-\lambda t}$

a) $p_1 = \mathbb{P}(T \leq 1) = 1 - e^{-\lambda}$

$p_2 = \mathbb{P}(1 < T \leq 2) = e^{-\lambda} - e^{-2\lambda}$

$p_3 = \mathbb{P}(T > 2) = e^{-2\lambda}$

$L(\lambda, n) = \prod p_i^{n_i} = p_1^{n_1} p_2^{n_2} p_3^{n_3}$, $n_3 = n - n_1 - n_2$

$\ln L(\lambda, n) = n_1 \ln(1 - e^{-\lambda}) + n_2 \ln(e^{-\lambda} - e^{-2\lambda}) + n_3 \ln(e^{-2\lambda}) =$

$= (n_1 + n_2) \ln(1 - e^{-\lambda}) - \lambda n_2 - 2\lambda n_3 + c$

$\frac{\partial}{\partial \lambda} L(\lambda, n) = (n_1 + n_2) \frac{e^{-\lambda}}{1 - e^{-\lambda}} - n_2 - 2n_3 = 0$

$\frac{n_1 + n_2}{e^{\hat{\lambda}} - 1} = n_2 + 2n_3$

$\hat{\lambda} = \ln\left(\frac{n_1 + 2n_2 + 2n_3}{n_2 + 2n_3}\right) = \ln\left(\frac{2n - n_1}{2n - 2n_1 - n_2}\right)$

b) $\hat{\lambda} = \ln\left(\frac{2 - \frac{n_1}{n}}{2 - 2\frac{n_1}{n} - \frac{n_2}{n}}\right)$ (без ЗБЧ)

$n_1 \sim Bin(n, p_1)$

$n_2 \sim Bin(n, p_2)$

$\mathbb{E}\left(\frac{n_1}{n}\right) = p_1$ $\quad \mathbb{D}\left(\frac{n_1}{n}\right) = \frac{p_1(1-p_1)}{n}$

$\mathbb{P}\left(\left|\frac{n_1}{n} - p_1\right| \geq \varepsilon\right) \leq \frac{p_1(1-p_1)}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$

для $\frac{n_2}{n}$ аналогично

$\frac{2 - \frac{n_1}{n}}{2 - 2\frac{n_1}{n} - \frac{n_2}{n}} \xrightarrow{\mathbb{P}} \frac{2 - p_1}{2 - 2p_1 - p_2} = \frac{2 - (1 - e^{-\lambda})}{2 - 2(1 - e^{-\lambda}) - (e^{-\lambda} - e^{-2\lambda})} =$

$= \frac{1 + e^{-\lambda}}{e^{-\lambda} + e^{-2\lambda}} = e^{\lambda}$ $\Rightarrow \hat{\lambda} = \ln\left(\frac{2 - \frac{n_1}{n}}{2 - 2\frac{n_1}{n} - \frac{n_2}{n}}\right) \xrightarrow{\mathbb{P}} \ln(e^{\lambda}) =$

$= \lambda$

оценка состоятельна

№4 $\xi \in U[0, \theta] \quad X_i = e^{\xi}$

a) $F_X(x) = P(X_i < x) = P(e^{\xi_i} \leq x) = P(\xi_i \leq \ln x) =$

$= F_{\xi}(\ln x) = \left\{ F_{\xi}(t) = \frac{t}{\theta} \quad t \in (0, \theta] \right\} = \frac{\ln x}{\theta}$

$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{\ln x}{\theta} & 1 \leq x \leq e^{\theta} \\ 1 & x > e^{\theta} \end{cases} \qquad f(x) = F_X'(x) = \begin{cases} \frac{1}{\theta x} & 1 \leq x \leq e^{\theta} \\ 0 & , \text{иначе} \end{cases}$

$L(\theta, x) = \prod_{i=1}^{n} f_X(X_i) = \prod_{i=1}^{n} \frac{1}{\theta X_i} \mathbb{1}(1 \leq X_i \leq e^{\theta}) =$

$= \frac{1}{\theta^n \cdot \prod X_i} \mathbb{1}(\text{все } 1 \leq X_i \leq e^{\theta})$

$\mathbb{1}(\text{все } 1 \leq X_i \leq e^{\theta}) \Leftrightarrow \theta \geq \ln(X_{(n)})$

$L(\theta, x) \to \max$ при $\theta \to \min$, но $\geq \ln(X_{(n)})$

тогда $\hat{\theta} = \ln(X_{(n)}) = \ln(e^{\xi_{(n)}}) = \xi_{(n)}$

b) $F_{\hat{\theta}_n}(t) = P(\hat{\theta} < t) = P(\xi_{(n)} \leq t) = (P(\xi_1 \leq t))^n = \left(\frac{t}{\theta}\right)^n$

$\forall \varepsilon > 0 \quad \hookrightarrow P(|\hat{\theta}_n - \theta| > \varepsilon) = P(\hat{\theta}_n < \theta - \varepsilon) + \underbrace{P(\hat{\theta} > \theta)}_{=0} =$

$= P(\hat{\theta} < \theta - \varepsilon) = F_{\hat{\theta}}(\theta - \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n = \left(1 - \frac{\varepsilon}{\theta}\right)^n$

$\lim_{n \to \infty} \left(1 - \frac{\varepsilon}{\theta}\right)^n = 0 \Rightarrow P(|\hat{\theta} - \theta| > \varepsilon) \xrightarrow[n \to \infty]{} 0$

$\therefore$ Оценка состоятельна