Antiderivatives

Before: Given $f(x)$, find $f'(x)$.
Now: $f(x)$, find $F(x)$ such that $F'(x) = f(x)$.

Definition: A function F is called an **antiderivative** of f on an interval $\boldsymbol{I}$ if $F'(x) = f(x)$ for all x in $\boldsymbol{I}$.

Ex. Because $(\sin x)' = \cos x$, therefore $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$.

Recall that, as a consequence of the *Mean Value Theorem*, all functions with the same derivative differ from each other by a constant. Therefore, continue the example above, functions of the form $F(x) = \sin x + C$, where C is any constant, is the set of <u>all</u> antiderivatives of $f(x) = \cos x$.

Theorem: If F is an antiderivative of f on an interval $\boldsymbol{I}$, then the most general antiderivative of f on $\boldsymbol{I}$ is

$$F(x) + C$$

where C is an arbitrary constant.

> *Comment*: This theorem tells us that, in order to find all antiderivatives of a given function f, all we need to do is to find ONE function F whose derivative is $F' = f$. The rest of the antiderivatives of f can then be found by adding constants to F. The family of all antiderivatives of a function f, $F(x) + C$, is called the **indefinite integral** of f.

The process of finding antiderivatives of a given function is called **integration**. It is the reverse process of differentiation.

Notation: Integration and Indefinite Integral

The fact that the set of functions $F(x) + C$ represents all antiderivatives of $f(x)$ is denoted by:

$$\int f(x)dx = F(x) + C$$

where the symbol $\int$ is called the **integral sign**, $f(x)$ is the **integrand**, C is the **constant of integration**, and dx denotes the independent variable we are integrating with respect to.

Ex. **Power rule of integration**

Because $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = x^n$, the function $F(x) = \frac{1}{n+1}x^{n+1}$ is an antiderivative of $f(x) = x^n$, for $n \neq -1$. Therefore, the set of all antiderivatives of x^n is:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \text{ for all numbers } n,\ n \neq -1.$$

Math 141 notes: Later you'll learn the "missing" case of the above power rule, the case where $n = -1$. It is

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

Ex. $\int \sqrt[5]{x}dx = \int x^{1/5}dx = \frac{1}{6/5}x^{6/5} + C = \frac{5}{6}x^{6/5} + C$

Ex. $\int\left(x^2 + \frac{1}{x^2} - 2\sqrt{x}\right)dx = \int\left(x^2 + x^{-2} - 2x^{1/2}\right)dx = \frac{1}{3}x^3 + \frac{1}{-1}x^{-1} - 2\frac{1}{3/2}x^{3/2} + C$

$= \frac{1}{3}x^3 - x^{-1} - \frac{4}{3}x^{3/2} + C$

Some other basic integration formulas

sum/difference rule: $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

constant-multiple rule: $\int c\, f(x)dx = c\int f(x)dx$

$\int \cos x\, dx = \sin x + C$

$\int \sin x\, dx = -\cos x + C$

$\int \sec^2 x\, dx = \tan x + C$

$\int \sec x \tan x\, dx = \sec x + C$

$\int \csc^2 x\, dx = -\cot x + C$

$\int \csc x \cot x\, dx = -\csc x + C$

Ex. $\int (3\cos x - 2\csc x \cot x)dx = 3\int \cos x\, dx - 2\int \csc x \cot x\, dx$

$= 3\sin x - 2(-\csc x) + C = 3\sin x + 2\csc x + C$

Initial value problems

If given some extra condition, a particular antiderivative of a given function can then be found. That is, we can find the value of the constant of integration that satisfies the extra condition and thus uniquely determine an antiderivative of the function. A common type of such additional condition is called the **initial condition** or **initial value**.

Ex. Find the antiderivative $F(x)$ of $f(x) = 6x^2 + 5$, such that $F(1) = 2$.

$$F(x) = \int (6x^2 + 5)\,dx = 6\left(\frac{1}{3}x^3\right) + 5x + C = 2x^3 + 5x + C$$

$$F(1) = 2 = 2(1^3) + 5(1) + C$$

$$2 = 7 + C \quad \rightarrow \quad C = -5$$

Therefore, $F(x) = 2x^3 + 5x - 5$.

Ex. Find y if $y'' = 1 - \sin x$, $y(0) = 1$ and $y'(0) = -3$.

$$y' = \int y''\,dx = \int (1 - \sin x)\,dx = x + \cos x + C$$

$$y'(0) = -3 = 0 + \cos(0) + C = 1 + C$$

$$-3 = 1 + C \quad \rightarrow \quad C = -4$$

$$\rightarrow \; y' = x + \cos x - 4$$

$$y = \int y'\,dx = \int (x + \cos x - 4)\,dx = \frac{x^2}{2} + \sin x - 4x + C_1$$

$$y(0) = 1 = 0 + \sin(0) - 0 + C_1 = C_1 \quad \rightarrow \quad C_1 = 1$$

$$\rightarrow \; y = \frac{x^2}{2} + \sin x - 4x + 1$$

Ex. A ball is tossed straight up from the top of a 30-meter building with an initial velocity of 20 m/sec. Find the position function describing its height.

Known: $a(t) = -9.8$ (m/sec^2), $v(0) = 20$ (m/sec), $s(0) = 30$ (m).

$v(t) = \int a(t)\,dt = \int -9.8\,dt = -9.8t + C$

$v(0) = 20 = -9.8(0) + C = C \;\rightarrow\; C = 20$

$v(t) = -9.8t + 20$

$s(t) = \int v(t)\,dt = \int (-9.8t + 20)\,dt = -4.9t^2 + 20t + C_1$

$s(0) = 30 = -4.9(0) + 20(0) + C_1 = C_1$

$C_1 = 30$

Therefore, $s(t) = -4.9t^2 + 20t + 30$ (meters)

The above example can be easily generalized to obtain the formula from the physics class describing the height of a free-falling object that was tossed straight up in the air with an initial velocity of v_0 and an initial height of s_0:

In the metric system: $s(t) = -4.9t^2 + v_0 t + s_0$ (meters)

In the English system: $s(t) = -16t^2 + v_0 t + s_0$ (feet)