

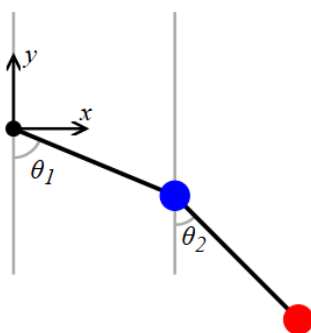
Animacija dvojnega nihala

Nena Šefman Hodnik, Nina Švigelj

1 Osnovni primeri

1.1 Dvojno nihalo

Oglejmo si primer na sliki:



Slika 1: Dvojno nihalo [1]

Imamo dve žogici z masama m_1 in m_2 na palčkah dolžine l_1 in l_2 z zanemarljivo maso.

Recimo, da je prva žogica na (x_1, y_1) in druga žogica na (x_2, y_2) . Te koordinate dobimo kot:

$$\begin{aligned}x_1 &= l_1 \sin(\theta_1), \\x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2), \\y_1 &= -l_1 \cos(\theta_1), \\y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2).\end{aligned}$$

Potencialna energija

Potencialna energija je definirana kot $V = mgh$, kjer je višina žogice dana z $h = y$.

Za posamezni masi velja:

$$h_1 = y_1, \quad h_2 = y_2.$$

Torej celotna potencialna energija sistema je:

$$\begin{aligned}V &= V_1 + V_2 \\&= m_1 g(-l_1 \cos \theta_1) + m_2 g(-l_1 \cos \theta_1 - l_2 \cos \theta_2).\end{aligned}$$

Kinetična energija

Kinetična energija vsake žogice je podana z izrazom

$$T = \frac{1}{2}mv^2.$$

Hitrost žogic dobimo s pomočjo izraza $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

Odvodi koordinat so:

$$\begin{aligned}\dot{x}_1 &= l_1 \dot{\theta}_1 \cos(\theta_1) \\ \dot{x}_2 &= l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2) \\ \dot{y}_1 &= l_1 \dot{\theta}_1 \sin(\theta_1) \\ \dot{y}_2 &= l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2)\end{aligned}$$

Skupna kinetična energija sistema je torej:

$$T = T_1 + T_2 = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

Lotimo se sedaj zapisa sistema Euler-Lagrangeevih enačb [2] za $\mathcal{L} = T - V$:

$$\begin{aligned}\mathcal{L} &= \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g(l_1 \cos \theta_1) + m_2 g(l_1 \cos \theta_1 + l_2 \cos \theta_2) \\ &= \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 \cos \theta_1 \cos \theta_2 + 2l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 \sin \theta_1 \sin \theta_2) \\ &\quad + m_1 g l_1 \cos \theta_1 + m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) + (m_1 + m_2) l_1 g \cos \theta_1 \\ &\quad + m_2 l_2 g \cos \theta_2.\end{aligned}$$

Želimo dobiti enačbe oblike:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \right) = \frac{\partial \mathcal{L}}{\partial \theta_i}, \quad i = 1, 2.$$

Najprej izpeljimo za θ_1 :

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2), \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) &= m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2)] \\ \frac{\partial \mathcal{L}}{\partial \theta_1} &= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) l_1 g \sin \theta_1.\end{aligned}$$

Torej za $i = 1$ dobimo:

$$\begin{aligned}\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) - \frac{\partial \mathcal{L}}{\partial \theta_1} &= (m_1 + m_2) (l_1^2 \ddot{\theta}_1) + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)] \\ &\quad + (m_1 + m_2) l_1 g \sin \theta_1 = 0 \quad / : l_1.\end{aligned}$$

Podobno naredimo za θ_2 in dobimo:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \right) &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2)] \\ \frac{\partial \mathcal{L}}{\partial \theta_2} &= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_2 g \sin \theta_2\end{aligned}$$

Euler-Lagrangeeva enačba za $i = 2$ se torej glasi:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 (\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)) + m_2 l_2 g \sin \theta_2 = 0 \quad / : l_2$$

Dobimo sistem diferencialnih enačb:

$$\begin{aligned}\theta_1 : & (m_1 + m_2)[l_1\ddot{\theta}_1 + g \sin \theta_1] + m_2 l_2[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)] = 0 \\ \theta_2 : & m_2[l_2\ddot{\theta}_2 + g \sin \theta_2] + m_2 l_1[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)] = 0\end{aligned}$$

1.2 Trojno nihalo

Kaj pa bi se zgodilo, če vzamemo trojno nihalo? Predpostavimo, da na drugo žogico pripnemo preko vrvice l_3 še eno žogico z maso m_3 .

Ta žogica je na položaju

$$\begin{aligned}x_3 &= l_1 \sin \theta_1 + l_2 \sin \theta_2 + l_3 \sin \theta_3 \\ y_3 &= -l_1 \cos \theta_1 - l_2 \cos \theta_2 - l_3 \cos \theta_3 \\ \dot{x}_3 &= l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2 + l_3 \dot{\theta}_3 \cos \theta_3 \\ \dot{y}_3 &= l_1 \dot{\theta}_1 \sin \theta_1 + l_2 \dot{\theta}_2 \sin \theta_2 + l_3 \dot{\theta}_3 \sin \theta_3\end{aligned}$$

Za ta sistem imamo:

$$\begin{aligned}V_3 &= m_1 g y_1 + m_2 g y_2 + m_3 g y_3 \\ &= -(m_1 + m_2 + m_3)l_1 g \cos \theta_1 - \cos \theta_2 g l_2(m_2 + m_3) - \cos \theta_3 l_3 m_3 g \\ T_3 &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2) \\ &= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] + \frac{1}{2}m_3[l_1^2 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 \\ &\quad + l_3 \dot{\theta}_3^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + 2l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \cos(\theta_1 - \theta_3) + 2l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3)] \\ \mathcal{L} &= T_3 - V_3\end{aligned}$$

Podobno kot za dvojno nihalo tudi tu dobimo sistem enačb za θ_i , $i = 1, 2, 3$. Če jih malo preuredimo, dobimo:

$$\begin{aligned}\theta_1 : & (m_1 + m_2 + m_3)[l_1\ddot{\theta}_1 + g \sin \theta_1] + (m_2 + m_3)l_2[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)] \\ & + m_3 l_3[\ddot{\theta}_3 \cos(\theta_1 - \theta_3) + \dot{\theta}_3^2 \sin(\theta_1 - \theta_3)] = 0 \\ \theta_2 : & (m_2 + m_3)[l_2\ddot{\theta}_2 + g \sin \theta_2] + (m_2 + m_3)l_1[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)] \\ & + m_3 l_3[\ddot{\theta}_3 \cos(\theta_2 - \theta_3) + \dot{\theta}_3^2 \sin(\theta_2 - \theta_3)] = 0 \\ \theta_3 : & m_3[l_3\ddot{\theta}_3 - g \sin \theta_3] + m_3 l_1[\ddot{\theta}_1 \cos(\theta_1 - \theta_3) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_3)] \\ & + m_3 l_2[\ddot{\theta}_2 \cos(\theta_2 - \theta_3) - \dot{\theta}_2^2 \sin(\theta_2 - \theta_3)] = 0\end{aligned}$$

2 Postopek animacije

Za sistem dvojnega nihala sva v pythonu napisali funkcijo `resen_sistem_n`, ki glede na

- izbrane mase m_1, m_2, \dots, m_n
- dolžine vrvic l_1, l_2, \dots, l_n
- začetne $\theta_1, \theta_2, \dots, \theta_n$

- dt ... korak v času
- $tmax$... časovna dolžina videa

za vsak $t = k \cdot dt$, $k = 0, 1, \dots, \frac{tmax}{dt}$ reši sistem diferencialnih enačb in vrne matriko z elementi $[\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \dots, \theta_n, \dot{\theta}_n]$.

Literatura

- [1] Scipython. *The double pendulum*. Ogled: 24. 10. 2025. URL: <https://scipython.com/blog/the-double-pendulum/>.
- [2] Wikipedia. *Lagrangian mechanics*. Ogled: 24. 10. 2025. URL: https://en.wikipedia.org/wiki/Lagrangian_mechanics.