n-kratno nihalo

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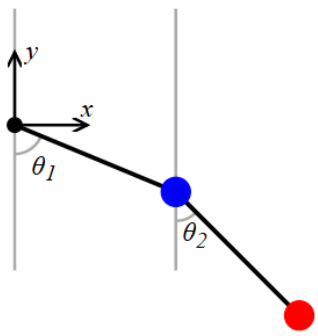
1 Dvojno nihalo

Oglejmo si primer na sliki:



Slika 1: Dvojno nihalo [1]

Imamo dve žogici z masama m_1 in m_2 na palčkah dolžine l_1 in l_2 z zanemarljivo maso.

Recimo, da je prva žogica na (x_1, y_1) in druga žogica na (x_2, y_2). Te koordinate dobimo kot:

$$\begin{aligned}
x_1 &= l_1 \sin(\theta_1), \\
x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2), \\
y_1 &= -l_1 \cos(\theta_1), \\
y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2).
\end{aligned}$$

Potencialna energija

Potencialna energija je definirana kot $V = mgh$, kjer je višina žogice dana z $h = y$.
Za posamezni masi velja:

$$h_1 = y_1, \quad h_2 = y_2.$$

Torej celotna potencialna energija sistema je:

$$\begin{aligned}
V &= V_1 + V_2 \\
&= m_1 g(-l_1 \cos\theta_1) + m_2 g(-l_1 \cos\theta_1 - l_2 \cos\theta_2).
\end{aligned}$$

Kinetična energija

Kinetična energija vsake žogice je podana z izrazom

$$T = \frac{1}{2} m v^2.$$

Hitrost žogic dobimo s pomočjo izraza $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

Odvodi koordinat so:

$$\begin{aligned}
\dot{x}_1 &= l_1\dot{\theta}_1 \cos(\theta_1)\\
\dot{x}_2 &= l_1\dot{\theta}_1 \cos(\theta_1) + l_2\dot{\theta}_2 \cos(\theta_2)\\
\dot{y}_1 &= l_1\dot{\theta}_1 \sin(\theta_1)\\
\dot{y}_2 &= l_1\dot{\theta}_1 \sin(\theta_1) + l_2\dot{\theta}_2 \sin(\theta_2)
\end{aligned}$$

Skupna kinetična energija sistema je torej:

$$T = T_1 + T_2 = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

Lotimo se sedaj zapisa sistema Euler-Lagrangeevih enačb [2] za $\mathcal{L} = T - V$:

$$\begin{aligned}
\mathcal{L} &= \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g(l_1\cos\theta_1) + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)\\
&= \frac{m_1}{2}l_1^2\dot{\theta_1}^2 + \frac{m_2}{2}(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2\cos\theta_1\cos\theta_2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2\sin\theta_1\sin\theta_2)\\
&\quad + m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)\\
&= \frac{1}{2}m_1 l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta_2}^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + (m_1 + m_2)l_1 g\cos\theta_1\\
&\quad + m_2 l_2 g\cos\theta_2.
\end{aligned}$$

Želimo dobiti enačbe oblike:

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i}\Big) = \frac{\partial\mathcal{L}}{\partial\theta_i}, \quad i = 1, 2.$$

Najprej izpeljimo za θ_1:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} &= m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1 - \theta_2),\\
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\Big) &= m_1 l_1^2\ddot{\theta}_1 + m_2 l_1 l_2[\ddot{\theta}_2\cos(\theta_1 - \theta_2) - \dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)]\\
\frac{\partial\mathcal{L}}{\partial\theta_1} &= -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)l_1 g\sin\theta_1.
\end{aligned}$$

Torej za $i = 1$ dobimo:

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\Big) - \frac{\partial\mathcal{L}}{\partial\theta_1} &= (m_1 + m_2)(l_1^2\ddot{\theta}_1) + m_2 l_1 l_2[\ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta_2}^2\sin(\theta_1 - \theta_2)]\\
&\quad + (m_1 + m_2)l_1 g\sin\theta_1 = 0 \quad / : l_1.
\end{aligned}$$

Podobno naredimo za θ_2 in dobimo:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} &= m_2 l_2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1 - \theta_2)\\
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\Big) &= m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2[\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)]\\
\frac{\partial\mathcal{L}}{\partial\theta_2} &= -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 l_2 g\sin\theta_2
\end{aligned}$$

Euler-Lagrangeeva enačba za $i = 2$ se torej glasi:

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\Big) - \frac{\partial\mathcal{L}}{\partial\theta_2} = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2(\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta_1}^2\sin(\theta_1 - \theta_2)) + m_2 l_2 g\sin\theta_2 = 0 \quad / : l_2$$

Dobimo sistem diferencialnih enačb:

$$\begin{aligned}
\theta_1:&\quad (m_1+m_2)[l_1\ddot{\theta_1}+g\sin\theta_1]+m_2l_2[\ddot{\theta_2}\cos(\theta_1-\theta_2)+\dot{\theta_2}^2\sin(\theta_1-\theta_2)]=0\\
\theta_2:&\quad m_2[l_2\ddot{\theta_2}+g\sin\theta_2]+m_2l_1[\ddot{\theta_1}\cos(\theta_1-\theta_2)-\dot{\theta_1}^2\sin(\theta_1-\theta_2)]=0
\end{aligned}$$

2 Trojno nihalo

Kaj pa bi se zgodilo, če vzamemo trojno nihalo? Predpostavimo, da na drugo žogico pripnemo preko vrvice l_3 še eno žogico z maso m_3.

Ta žogica je na položaju

$$\begin{aligned}
x_3 &= l_1\sin\theta_1+l_2\sin\theta_2+l_3\sin\theta_3\\
y_3 &= -l_1\cos\theta_1-l_2\cos\theta_2-l_3\cos\theta_3\\
\dot{x_3} &= l_1\dot{\theta_1}\cos\theta_2+l_2\dot{\theta_2}\cos\theta_2+l_3\dot{\theta_3}\cos\theta_3\\
\dot{y_3} &= l_1\dot{\theta_1}\sin\theta_2+l_2\dot{\theta_2}\sin\theta_2+l_3\dot{\theta_3}\sin\theta_3
\end{aligned}$$

Za ta sistem imamo:

$$\begin{aligned}
V_3 =& m_1gy_1+m_2gy_2+m_3gy_3\\
=& -(m_1+m_2+m_3)l_1g\cos\theta_1-\cos\theta_2gl_2(m_2+m_3)-\cos\theta_3l_3m_3g\\
T_3 =& \frac{1}{2}m_1(\dot{x_1}^2+\dot{y_1}^2)+\frac{1}{2}m_2(\dot{x_2}^2+\dot{y_2}^2)+\frac{1}{2}m_3(\dot{x_3}^2+\dot{y_3}^2)\\
=& \frac{1}{2}m_1l_1^2\dot{\theta_1}^2+\frac{1}{2}m_2[l_1^2\dot{\theta_1}^2+l_2\dot{\theta_2}^2+2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2)]+\frac{1}{2}m_3[l_1^2\dot{\theta_1}^2+l_2\dot{\theta_2}^2\\
&+l_3^2\dot{\theta_3}^2+2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2)+2l_1l_3\dot{\theta_1}\dot{\theta_3}\cos(\theta_1-\theta_3)+2l_2l_3\dot{\theta_2}\dot{\theta_3}\cos(\theta_2-\theta_3)]\\
\mathcal{L} =& T_3-V_3
\end{aligned}$$

Podobno kot za dvojno nihalo tudi tu dobimo sistem enačb za θ_i, $i=1,2,3$. Če jih malo preuredimo, dobimo:

$$\begin{aligned}
\theta_1:&\quad (m_1+m_2+m_3)[l_1\ddot{\theta_1}+g\sin\theta_1]+(m_2+m_3)l_2[\ddot{\theta_2}\cos(\theta_1-\theta_2)+\dot{\theta_2}^2\sin(\theta_1-\theta_2)]\\
&+m_3l_3[\ddot{\theta_3}\cos(\theta_1-\theta_3)+\dot{\theta_3}^2\sin(\theta_1-\theta_3)]=0\\
\theta_2:&\quad (m_2+m_3)[l_2\ddot{\theta_2}+g\sin\theta_2]+(m_2+m_3)l_1[\ddot{\theta_1}\cos(\theta_1-\theta_2)-\dot{\theta_1}^2\sin(\theta_1-\theta_2)]\\
&+m_3l_3[\ddot{\theta_3}\cos(\theta_2-\theta_3)+\dot{\theta_3}^2\sin(\theta_2-\theta_3)]=0\\
\theta_3:&\quad m_3[l_3\ddot{\theta_3}-g\sin\theta_3]+m_3l_1[\ddot{\theta_1}\cos(\theta_1-\theta_3)-\dot{\theta_1}^2\sin(\theta_1-\theta_3)]\\
&+m_3l_2[\ddot{\theta_2}\cos(\theta_2-\theta_3)-\dot{\theta_2}^2\sin(\theta_2-\theta_3)]=0
\end{aligned}$$

Zaenkrat sva dobili idejo za reukrzijo, ki izgleda nekako tako?:

$$\begin{aligned}
\text{Za } \theta_1:&\quad (\sum_{i=1}^{n}m_i)[l_1\ddot{\theta_1}+g\sin\theta_1]+\sum_{i=2}^{n}(\sum_{j=i}^{n}m_j)l_i(\ddot{\theta_i}\cos(\theta_1-\theta_i)+\dot{\theta_i}^2\sin(\theta_1-\theta_i))=0\\
\text{Za } \theta_n:&\quad m_n[l_n\ddot{\theta_n}+g\sin\theta_n]+m_n\sum_{i=1}^{n-1}l_i[\ddot{\theta_i}\cos(\theta_i-\theta_n)-\dot{\theta_i}^2\sin(\theta_i-\theta_n)]=0
\end{aligned}$$

Literatura

[1] Scipython. *The double pendulum*. Ogled: 24. 10. 2025. URL: https://scipython.com/blog/the-double-pendulum/.

[2] Wikipedia. *Lagrangian mechanics*. Ogled: 24. 10. 2025. URL: https://en.wikipedia.org/wiki/Lagrangian_mechanics.