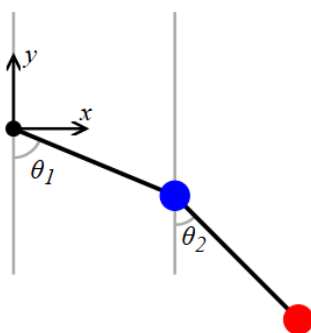# Animacija dvojnega nihala

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1 Osnovni primeri

1.1 Dvojno nihalo

Oglejmo si primer na sliki:

Slika 1: Dvojno nihalo [1]

Imamo dve žogici z masama m_1 in m_2 na palčkah dolžine l_1 in l_2 z zanemarljivo maso.

Recimo, da je prva žogica na (x_1, y_1) in druga žogica na (x_2, y_2). Te koordinate dobimo kot:

$$x_1 = l_1 \sin(\theta_1),$$
$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2),$$
$$y_1 = -l_1 \cos(\theta_1),$$
$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2).$$

Potencialna energija

Potencialna energija je definirana kot $V = mgh$, kjer je višina žogice dana z $h = y$.
Za posamezni masi velja:

$$h_1 = y_1, \quad h_2 = y_2.$$

Torej celotna potencialna energija sistema je:

$$V = V_1 + V_2$$
$$= m_1 g(-l_1 \cos\theta_1) + m_2 g(-l_1 \cos\theta_1 - l_2 \cos\theta_2).$$

Kinetična energija

Kinetična energija vsake žogice je podana z izrazom

$$T = \frac{1}{2}mv^2.$$

Hitrost žogic dobimo s pomočjo izraza $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

Odvodi koordinat so:

$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos(\theta_1)$$
$$\dot{x}_2 = l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2)$$
$$\dot{y}_1 = l_1 \dot{\theta}_1 \sin(\theta_1)$$
$$\dot{y}_2 = l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2)$$

Skupna kinetična energija sistema je torej:

$$T = T_1 + T_2 = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

Lotimo se sedaj zapisa sistema Euler-Lagrangeevih enačb [2] za $\mathcal{L} = T - V$:

$$\begin{aligned}
\mathcal{L} &= \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g(l_1 \cos\theta_1) + m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2) \\
&= \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 \sin\theta_1 \sin\theta_2) \\
&\quad + m_1 g l_1 \cos\theta_1 + m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2) \\
&= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) + (m_1 + m_2) l_1 g \cos\theta_1 \\
&\quad + m_2 l_2 g \cos\theta_2.
\end{aligned}$$

Želimo dobiti enačbe oblike:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) = \frac{\partial \mathcal{L}}{\partial \theta_i}, \quad i = 1, 2.$$

Najprej izpeljimo za θ_1:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2),$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) l_1 g \sin\theta_1.$$

Torej za $i = 1$ dobimo:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} &= (m_1 + m_2)(l_1^2 \ddot{\theta}_1) + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)] \\
&\quad + (m_1 + m_2) l_1 g \sin\theta_1 = 0 \quad / : l_1.
\end{aligned}$$

Podobno naredimo za θ_2 in dobimo:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_2 g \sin\theta_2$$

Euler-Lagrangeeva enačba za $i = 2$ se torej glasi:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 (\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)) + m_2 l_2 g \sin\theta_2 = 0 \quad / : l_2$$

Dobimo sistem diferencialnih enačb:

$$\theta_1: \quad (m_1 + m_2)[l_1\ddot{\theta}_1 + g\sin\theta_1] + m_2l_2[\ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2)] = 0$$

$$\theta_2: \quad m_2[l_2\ddot{\theta}_2 + g\sin\theta_2] + m_2l_1[\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_2)] = 0$$

1.2 Trojno nihalo

Kaj pa bi se zgodilo, če vzamemo trojno nihalo? Predpostavimo, da na drugo žogico pripnemo preko vrvice l_3 še eno žogico z maso m_3.

Ta žogica je na položaju

$$x_3 = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3$$
$$y_3 = -l_1\cos\theta_1 - l_2\cos\theta_2 - l_3\cos\theta_3$$
$$\dot{x}_3 = l_1\dot{\theta}_1\cos\theta_2 + l_2\dot{\theta}_2\cos\theta_2 + l_3\dot{\theta}_3\cos\theta_3$$
$$\dot{y}_3 = l_1\dot{\theta}_1\sin\theta_2 + l_2\dot{\theta}_2\sin\theta_2 + l_3\dot{\theta}_3\sin\theta_3$$

Za ta sistem imamo:

$$
\begin{aligned}
V_3 &= m_1gy_1 + m_2gy_2 + m_3gy_3 \\
&= -(m_1 + m_2 + m_3)l_1g\cos\theta_1 - \cos\theta_2gl_2(m_2 + m_3) - \cos\theta_3l_3m_3g \\
T_3 &= \frac{1}{2}m_1(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}) + \frac{1}{2}m_2(\dot{x}_2^{\,2} + \dot{y}_2^{\,2}) + \frac{1}{2}m_3(\dot{x}_3^{\,2} + \dot{y}_3^{\,2}) \\
&= \frac{1}{2}m_1l_1^2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^{\,2} + l_2\dot{\theta}_2^{\,2} + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)] + \frac{1}{2}m_3[l_1^2\dot{\theta}_1^{\,2} + l_2\dot{\theta}_2^{\,2} \\
&\quad + l_3^2\dot{\theta}_3^{\,2} + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + 2l_1l_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1 - \theta_3) + 2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2 - \theta_3)] \\
\mathcal{L} &= T_3 - V_3
\end{aligned}
$$

Podobno kot za dvojno nihalo tudi tu dobimo sistem enačb za θ_i, $i = 1, 2, 3$. Če jih malo preuredimo, dobimo:

$$
\begin{aligned}
\theta_1: \quad & (m_1 + m_2 + m_3)[l_1\ddot{\theta}_1 + g\sin\theta_1] + (m_2 + m_3)l_2[\ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2)] \\
& + m_3l_3[\ddot{\theta}_3\cos(\theta_1 - \theta_3) + \dot{\theta}_3^{\,2}\sin(\theta_1 - \theta_3)] = 0 \\
\theta_2: \quad & (m_2 + m_3)[l_2\ddot{\theta}_2 + g\sin\theta_2] + (m_2 + m_3)l_1[\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_2)] \\
& + m_3l_3[\ddot{\theta}_3\cos(\theta_2 - \theta_3) + \dot{\theta}_3^{\,2}\sin(\theta_2 - \theta_3)] = 0 \\
\theta_3: \quad & m_3[l_3\ddot{\theta}_3 - g\sin\theta_3] + m_3l_1[\ddot{\theta}_1\cos(\theta_1 - \theta_3) - \dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_3)] \\
& + m_3l_2[\ddot{\theta}_2\cos(\theta_2 - \theta_3) - \dot{\theta}_2^{\,2}\sin(\theta_2 - \theta_3)] = 0
\end{aligned}
$$

2 Postopek animacije

Za sistem dvojnega nihala sva v pythonu napisali funkcijo `resen_sistem_n`, ki glede na

- izbrane mase $m_1, m_2, \ldots, m_n$
- dolžine vrvic $l_1, l_2, \ldots, l_n$
- začetne $\theta_1, \theta_2, \ldots, \theta_n$

- dt ... korak v času

- $tmax$ ... časovna dolžina videa

za vsak $t = k \cdot dt$, $k = 0, 1, \ldots, \frac{tmax}{dt}$ reši sistem diferencialnih enačb in vrne matriko z elementi $[\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \ldots, \theta_n, \dot{\theta}_n]$.

Literatura

[1]   Scipython. *The double pendulum*. Ogled: 24. 10. 2025. URL: https://scipython.com/blog/the-double-pendulum/.

[2]   Wikipedia. *Lagrangian mechanics*. Ogled: 24. 10. 2025. URL: https://en.wikipedia.org/wiki/Lagrangian_mechanics.