Université Libre de Bruxelles

Synthèse

Fluid mechanics and transport processes
MECA-H-300

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Appel à contribution

Synthèse Open Source

Ce document est grandement inspiré de l'excellent cours donné par Allessandro Parente à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi le `README` contient de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

Merci !

Table des matières

Chapitre 1

Introduction

Before beginning the summary, I want to tell you that my English level isn't perfect. Please collaborate and correct the grammatically wrong sentences.

1.1 Reminder

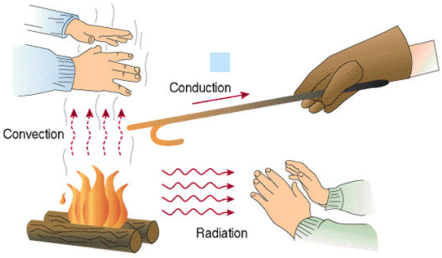
The governing equations in transport processes are the followings :

- Mass conservation :
$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \tag{1.1}$$
- Navier-Stokes :
$$\rho\left(\frac{Dv}{Dt} + v\nabla v\right) = -\nabla p + \mu\nabla^2 v \tag{1.2}$$
- Energy equation :
$$\frac{DT}{Dt} = \nabla(\alpha\nabla T) + \frac{\dot{Q}v}{\rho c} \tag{1.3}$$
- Species conservation
$$\frac{\partial \rho_A}{\partial t} + \nabla(\rho_A v_A) = r_A \tag{1.4}$$

Let's specify that there are many applications using these equations like in the aerospace and automotive industry, in safety and fire prevention or in buildings design.

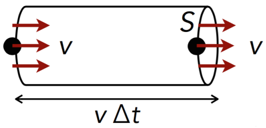
1.2 Convection and diffusion

1.2.1 Definitions



Here is a picture illustrating the principles of **convection**, **conduction** and **radiation**. Imagine that you have a fire and you put your hands above. You will feel a flow of heat transmitted by convection. If someone comes with a stick, there will be conduction in the material transmitting the energy from particles to particles. Finally, if the hands are next to the fire, there is no flow but you feel the heat. The energy is transmitted by radiation.

1.2.2 Convection



Convection is a transfer always associated to **bulk (ensemble) fluid motion**. We consider a fluid with **uniform** velocity and a cylinder of section S and length $v\Delta t$. In that time interval, the fluid in the cylinder will have crossed the section S. We are now able to express the convective flux of momentum (quantité de mouvement), energy and mass knowing that the flux of a physical quantity is given by

$$\text{flux}_A = \frac{A}{S\Delta t} \tag{1.5}$$

Mass

We know that the mass is given by density × volume and that the volume of the cylinder is $Sv\Delta t$. Using the new expression of the mass and (1.5), we can find the **flux of mass**

$$M_c = \rho v S\Delta t \qquad \textit{and} \qquad J_{M_c} = \frac{M_c}{S\Delta t} = \rho v \tag{1.6}$$

Momentum

Similarly to the calculus of the mass

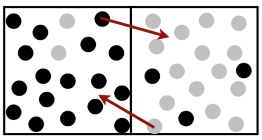
$$Q_c = mv = \rho v^2 S\Delta t \qquad \textit{and} \qquad J_{Q_c} = \rho v^2 \tag{1.7}$$

Energy

The energy in the system is given by the specific heat energy of each particles [1]. If T_0 is the reference temperature,

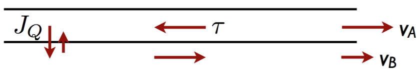
$$E_c = \rho c_v v(T - T_0)S\Delta t \qquad \textit{and} \qquad J_{E_c} = \rho c_v v(T - T_0) \tag{1.8}$$

1.2.3 Diffusion



Diffusion is a transfer associated to the **particles random walk** and is due to the presence of a gradient of physical quantity (temperature for example). The picture on the left illustrates that, for an infinite time, the process will reequilibrate the gradient, difference between the two boxes.

1.3 Diffusive flux



If we consider two parallel fluid layers to bulk velocity $v_A > v_b$, the gradient of velocity will vanish [2] ($v_A = v_b$). This effect is due to the initial velocity gradient that cause the diffusion of faster particles towards the slower ones, transferring a momentum flux J_Q. In fact, the origin of the flux in the transversal direction to the flow is due to the **friction** between the two layers, parallel to the flow direction.

1. See *Chimie générale* for the expression
2. Disparaître

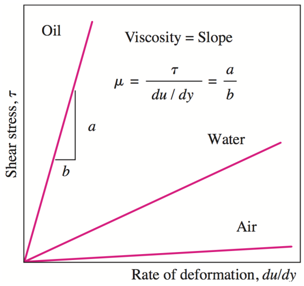
1.3.1 Shear stress

To characterize the friction between to layers, we introduce the **shear stress**[3] that is proportional to the gradient of velocity

$$\tau = \tau\left(\frac{dv}{dy}\right) \tag{1.9}$$

According to the Newton's law, for Newtonian fluids like gases and liquids, the equation becomes

$$\tau = \mu\left(\frac{dv}{dy}\right) \tag{1.10}$$

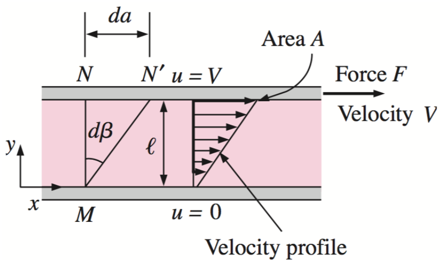


— μ is the **dynamic viscosity**, the **intrinsic resistance** of the fluid to motion $[kg/m.s]$
— dv/dy the gradient of velocity $[s^{-1}]$
— τ a force per unit surface $[N/m^2] = [Pa]$

If we see the shear stress to the **rate of deformation**[4] on a graph, we can see that the **viscosity** is the slope[5]. We can also see that viscosity of air is miner than water that's miner than oil.

1.3.2 Planar Couette flow

Let's consider a fluid layer between two very large plates separated by a distance l. A constant parallel force F (drag force[6]) is applied to the upper plate and after the initial transients, it moves continuously to a constant velocity V. What are the consequences on the fluid ?



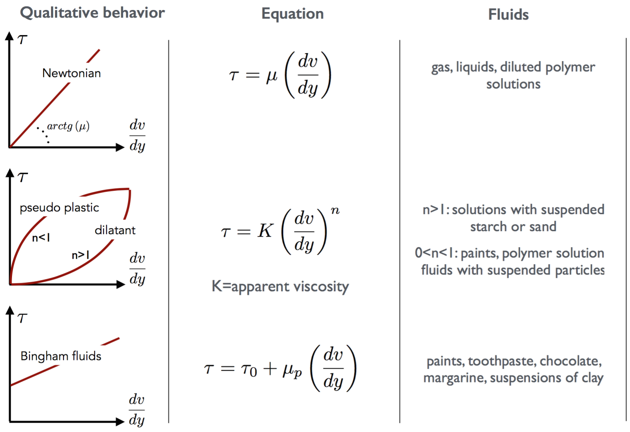
- **Empirical observations :**
  No slip conditions[7] at walls $\rightarrow u(0) = 0$ and $u(l) = V$.

- **The flow is organized in parallel layers :**
  We suppose that the flow is laminar (no turbulence), so the velocity profile is a sole function of y and is **linear** : $u(y) = V\frac{y}{l}$.

- **The drag force is imposed :**
  $\tau = \frac{F}{A} = \mu\frac{dv}{dy} = c$

3. Contrainte de cisaillement
4. Gradient of velocity
5. La pente
6. Force de traînée
7. Condition de non-glissement

1.3.3 Fluid rheology



Rheology is the study of the flow that establish the relation between the shear stress and the velocity gradient. Let's have a look to different fluids.
As first observation, we can see that Newtonian fluids respect a linear relation for shear stress to the rate of deformation. It's not the case for others like second and third line of the table[8]. For Bingham fluids, there is a critical shear stress to reach before the behaviour becomes like Newtonian fluids

Dynamic viscosity of liquids and gases

When we observe the table of dynamic viscosity for the water, the dynamic viscosity decreases with temperature increase for liquid water. It's not the case for gases for which dynamic viscosity increases with temperature. See section 1.5.3 below.

1.4 Constitutive relations

It's an important section because it shows that transport processes have a uniform approach. Let's introduce 3 new variables : **kinematic viscosity** ν, **thermal diffusivity** α, **internal energy** u, **mass diffusivity** D. Let's specify that $\nu, \alpha, D = [L^2T^{-1}]$

- **Momentum diffusion flux** (Newton's law)

$$J_{Q_d} = -\mu \frac{dv}{dy} \quad \xrightarrow{\nu=\frac{\mu}{\rho}} \quad J_{Q_d} = -\nu \frac{d(\rho v)}{dy} \tag{1.11}$$

- **Energy diffusive flux** (Fourier's law)

$$J_{E_d} = -k \frac{dT}{dy} \quad \xrightarrow{\alpha=\frac{k}{\rho c_v} \ and \ u=c_v(T-T_0)} \quad J_{E_d} = -\alpha \frac{d(\rho u)}{dy} \tag{1.12}$$

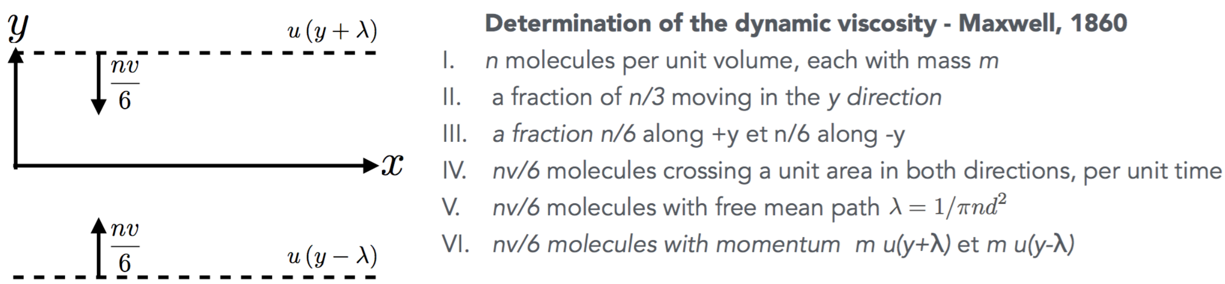
- **Mass diffusion flux** (Fick's law)

$$J_{M_d} = -D \frac{dc}{dy} \qquad (c = \text{concentration}) \tag{1.13}$$

8. Starch = amidon

1.5 Gas viscosity

1.5.1 Another expression of viscosity



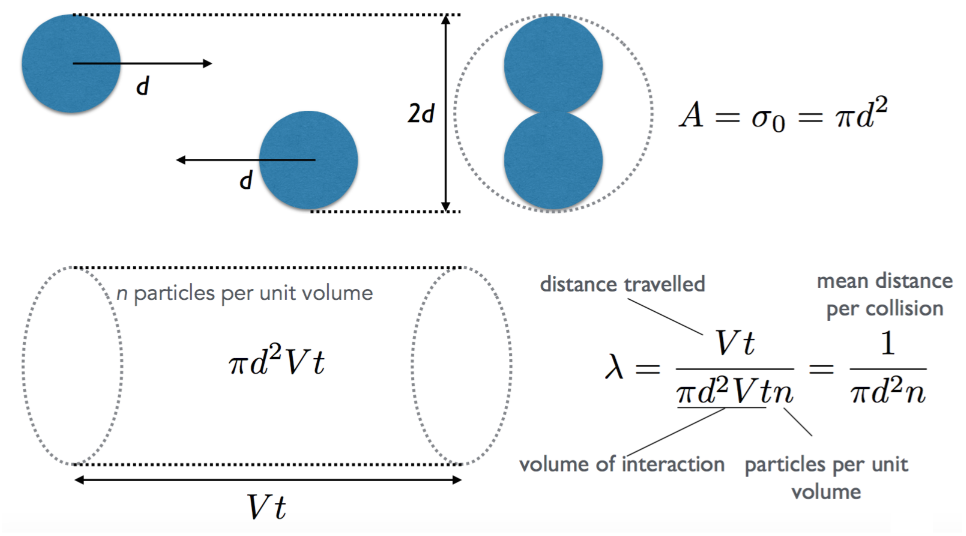
Let's consider a gas moving in the x direction with a velocity $u = u(y)$. The **kinetic theory** gives us the random velocity in y using $v = \sqrt{\frac{3k_bT}{m}}$. The development of Maxwell, above, allows us to calculate the momentum flux crossing the x axis. $\frac{nv}{6}$ gives us the flux of particles for the y direction.

$$J_{Q_d} = \left(\frac{nv}{6}\right) mu(y-\lambda) - \left(\frac{nv}{6}\right) mu(y+\lambda) \tag{1.14}$$

Here we use $u(y-\lambda)$ and $u(y+\lambda)$ because the last particles that can cross an axis on y are those who are situated in a distance of maximum λ from there.
The Taylor expansion of the flux $u(y+\lambda) = u(\lambda) + \lambda\frac{du}{dy} + \ldots$ and the constitutive equation (1.11) gives us the final expression of viscosity

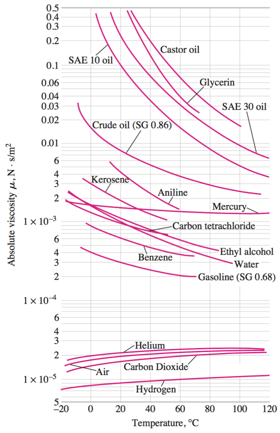
$$\mu = \frac{1}{3}nm\lambda v = \frac{1}{3}\rho\lambda v = \frac{1}{\pi\sqrt{3}}\frac{\sqrt{mk_bT}}{d^2} \tag{1.15}$$

1.5.2 Free mean path



The slide above summarises the way to calculate the free mean path that is the average distance travelled by a particle between 2 collisions. First of all, we have to define a section of interaction. It's given by a circle of radius d. We also have to define a volume within the particles move. It's the cylinder of section A and length Vt. The calculus of λ is given by the distance travelled (without collision) divided by the volume where you can have interactions and by the particles per unit volume.

1.5.3 Viscosity variation with temperature



We have said that gases viscosity increased with temperature. It's due to the kinetic approach that gave us the equation (1.15) where $\mu \propto \sqrt{T}$.
For gases, the intermolecular forces are negligible. The increase in temperature makes particles move randomly at higher velocities conducting to more collisions per unit volume per unit time and so more resistance to flow.
For liquids, the increase in temperature decreases the viscosity because particles have higher energies that help them to oppose to the large cohesive intermolecular forces more strongly. They can thus move more freely.

1.6 Viscosity measurements

I HAVE TO ASK TO THE TEACHER IF WE HAVE TO DO THESE 3 NEXT SECTIONS BECAUSE I DON'T HAVE COMMENTS

1.7 Thermal conductivity

1.8 Mass diffusivity

1.9 Potential flow and boundary layer

1.9.1 Main aspects

- The behaviour of a moving fluid element strongly depends on the **interactions** between the **fluid elements** and the **walls** for both internal and external flows.

- Far from the walls, the shear stress and the dissipative forces are negligible. The fluid behaves **ideally**, is **incompressible** and **non-dissipative**. It's called a **potential** flow.

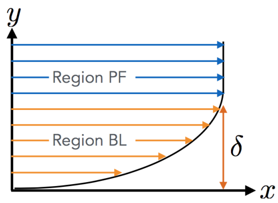
- Potential flow are fully described by **Newtonian mechanics**, using the **conservation laws** for mass and mechanical energy, without any **dissipation** of mechanical **energy into heat**.

- Close to the walls, the fluid velocity is **suddenly modified**, thus introducing **dissipative phenomena** that cannot be described using the potential flow theory.
- The fluid layer thickness [9] affected by the wall is named **boundary layer** [10] (as introduced by Prandtl, 1904).

- Within the boundary layer, the velocity gradients in the flow normal direction as well as the shear stress in the direction parallel to the flow become extremely important.

9. Epaisseur
10. Couche limite

1.9.2 Representation of the boundary layer



We have to delimit two regions. The one (PF) within we have the **potential flow** that conserve his kinetic and potential energy (no dissipation) and the other (BL) within we have the **boundary layer** of thickness δ, a velocity gradient with shear stress and dissipation of mechanical energy into heat. In the BL region appear two forces, a **drag** force that is due to friction, parallel to the flow and **lift** force that is a resistance perpendicular to the flow.
In order to characterize the flow, we introduce the **Reynolds number**

$$Re = \frac{\text{Convection}}{\text{Diffusion}} = \frac{J_{Q_c}}{J_{Q_d}} = \frac{\rho v^2}{\mu \frac{v}{L}} = \frac{vL}{\nu} \tag{1.16}$$

that compares the convective and diffusive effect of the fluid momentum and the **Peclet number**

$$P_{e_t} = \frac{J_{E_c}}{J_{E_d}} = \frac{\rho c_v v(T - T_0)}{k\frac{T}{L}} = \frac{vL}{\alpha} \qquad \textit{and} \qquad P_{e_m} = \frac{J_{M_c}}{J_{M_d}} = \frac{vL}{D} \tag{1.17}$$

that compares the convective and diffusive effect of the fluid energy and mass. Let's specify that if a uniform fluid flows towards a plate, we will have an **inviscid** flow region away from the plate and a **viscous** flow region next to the plate.

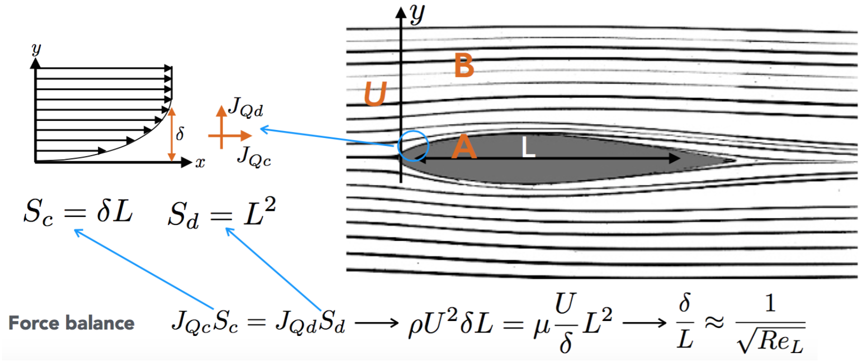
1.9.3 Flow classification and friction force

The classification is based on the Reynolds number. We have two types of fluid :

- **Re ≫ 1 → turbulent flow**, convection > diffusion.
  In that case, the fluid resistance (drag force) is independent of the viscosity and proportional to the momentum flux
  $$F \approx \rho v^2 S \tag{1.18}$$
- **Re ≪ 1 → laminar flow**, convection < diffusion.
  In that case, the fluid resistance is function of the viscosity and proportional to the diffusive momentum flux
  $$F \approx \mu L v \tag{1.19}$$

1.9.4 Determination of the boundary layer thickness

The separation between the two regimes above is not possible. Consider a wind flow around a building. The Reynolds number will be high meaning the flow is turbulent. Indeed, the **approaching flow** can be classified as a potential flow, whereas **close to the walls** the dissipative effects will take place with the conversion of mechanical energy into heat.
So, far from the walls the dominant transport mechanism will be convection and close to the walls viscous transport. At a distance equal to the boundary layer thickness, **convection and diffusion will be comparable**. This allows us determining its **order of magnitude**.



We can qualitatively find δ as the distance from the wall at which the convective momentum flux (**parallel** to the flow) equals the diffusive one (**perpendicular** to the flow). For that, let's take a section in which convection dominates (S_c) and another in which diffusion dominates (S_d). When we equalize the two flux expressed respectively for convection and diffusion, we arrive to the magnitude of δ compared to L.

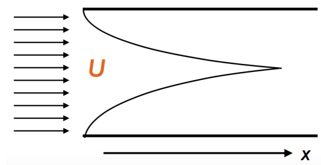
1.9.5 Features of the boundary layer and practical consequences

The relative dimension of the boundary layer is inversely proportional to $Re^{0.5}$ and the absolute dimension is proportional to $L^{0.5}$.

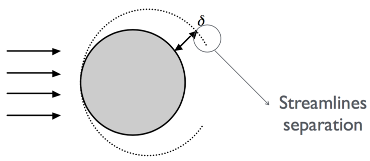
Consequences for internal flows

The boundary layer thickness increases with $x^{0.5}$ and far from the inlet [11], the boundary layer will extend to the full diameter pipe.

$$\frac{\delta}{x} \approx \frac{1}{\sqrt{Re_x}} \qquad \textit{and} \qquad \delta \approx \sqrt{\frac{xv}{U}} \tag{1.20}$$



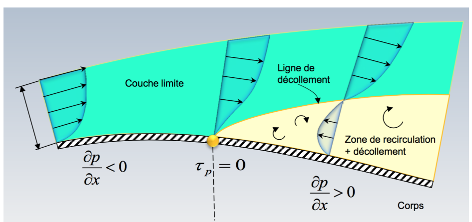
Consequences for external flows



The boundary layer cannot infinitely grow. It will be thinner in front the body and thicker behind it. Behind the body, the flow streamlines detach. If the boundary layer thickness is known, the force exerted by the flow can be directly evaluated, being the velocity gradient of the order $\frac{U}{\delta}$, which gives a shear stress of $\tau = \mu\frac{U}{\delta}$ on the walls. The drag force will be then higher in front of the body (where the boundary layer is thinner) and lower behind it (where the boundary layer is thicker). From symmetry considerations, the drag force only acts in the flow direction.
More commonly, a perpendicular component also exists called **lift** but here it doesn't appear.

$$F = \int_S \mu \frac{dv}{dy}\, dS \tag{1.21}$$

Here's an illustration of the flow separation. There is a **recirculation** region which provides a force opposed to the flow. In order to maximise the body speed, it is necessary to reduce the drag force due to the flow separation.



11. La prise - l'entrée

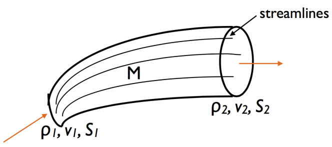
Chapitre 2

Macroscopic and microscopic balances

2.1 Conservation principles

The basic conservation principles in fluid mechanics are the conservation of **mass, energy and momentum.** These conservations are the basis of **continuity, Bernoulli and Navier-Stokes equations** and can be written in **integral** (finite volume of a moving fluid) or **local** (balances in differential form) forms.

2.1.1 Macroscopic mass balance and continuity equation



In a steady [1] flow, the mass balance simply states that the mass entering in a control volume is equal to the one leaving it. But we have to consider the variation of mass for transient [2] problems. We define for that the **mass flow rate** (débit) that gives the mass per time and is easy to measure.

$$\dot{m} = \rho v S \qquad [\dot{m}] = \frac{[M]}{[T]} \tag{2.1}$$

General balance

Let's write what we explained, the mass variation is given by the mass entering at S_1 minus mass leaving at S_2

$$\frac{dM}{dt} = \left(\rho \int_S v dS\right)_1 - \left(\rho \int_S v dS\right)_2 \tag{2.2}$$

To simplify, we introduce the **average velocity** $\bar{v} = \frac{1}{S}\int_S v dS$ giving the final expression

$$\frac{dM}{dt} = (\rho \bar{v} S)_1 - (\rho \bar{v} S)_2 \tag{2.3}$$

Steady balance

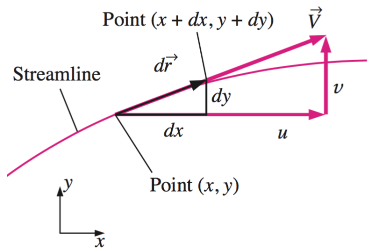
When the fluid is steady, the velocity and the surface are constant in both sections conducting to a constant mass

$$\dot{m} = \rho \bar{v} S = c \tag{2.4}$$

1. Continu, constant.
2. Transitoire.

Streamlines

Let's first define streamline. It's a curve that is everywhere tangent to the instantaneous local velocity vector and so an indicator of the fluid direction. Mathematically, if our velocity vector is



$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \tag{2.5}$$

we can take an infinitesimal arc length along a streamline

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \tag{2.6}$$

Due to the 2 similar triangles, we have the relation (2.7) that gives the 2D equation of a streamline.

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \Rightarrow \qquad \left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \tag{2.7}$$

2.1.2 Macroscopic momentum balance

Unlike the mass, momentum can be created and destroyed due to applied forces. The difference between the momentum entering in the control volume at S_1 and leaving at S_2 is equal to the sum of the forces applied to the fluid element.
The momentum flow through a section per unit time is given by

$$\dot{Q} = \dot{m}v = \rho\bar{v}^2 S \tag{2.8}$$

and the forces acting on the fluid element (positive if exerted by the fluid) are :

- **pressure** that is positive at the entry (the fluid pushes to enter) and negative at the exit (the fluid is pushed).
- **forces on the walls** (normal and tangential). Negative because it's the reaction of the walls.
- **Gravity and other volume forces**. Positive because the forces are due to the presence of a fluid volume.

It allows us to write the equation, considering that the surface is a vector of module equal to the scalar surface and with the same direction of the normal to the surface.

$$\sum F = p_1 S_1 - p_2 S_2 - F_w + F_g \tag{2.9}$$

Steady momentum balance

It's simply the use of equations (2.8) and (2.9)

$$\dot{Q}_2 - \dot{Q}_1 = \sum F \qquad \Leftrightarrow \qquad \dot{m}_2 v_2 - \dot{m}_1 v_1 = p_1 S_1 - p_2 S_2 - F_w + F_g \tag{2.10}$$

We can use equation (2.4) and have a compact final relation

$$\Delta_1^2(\rho\bar{v}S + p) = -F_w + F_g \tag{2.11}$$

General momentum balance (transient)

To consider the case within the velocity is not constant into the 2 sections, we have to remind the mechanical relation for the resultant and do an infinitesimal decomposition in volume V

$$\frac{dR}{dt} = \sum F \qquad \Leftrightarrow \qquad \begin{aligned} \frac{d}{dt}\int_V \rho v dV &= (\rho\bar{v}^2 + p)_1 S_1 - (\rho\bar{v}^2 + p)_2 S_2 - F_w + F_g \\ &= \Delta_1^2(\rho\bar{v}^2 + p)S - F_w + F_g \end{aligned} \tag{2.12}$$

You can find an example of exercise on slide 8 and 9 where we study the forces on an elbow [3]. Keep in mind that we often **consider a steady-state flow** to simplify $d/dt = 0$.

2.1.3 Local form of the conservation principles

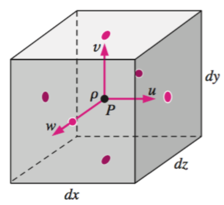
Macroscopic balances

They are used to have a general view of a volume, when we are interested in the **overall features** [4].

Microscopic balances

They consist of differential and local equations and are valid on every point of the fluid to give the velocity, density or pressure.

2.2 Continuity equation



The continuity equation characterizes the net rate of change of mass in a control volume. We start from the fact that this rate is equal to the net rate at which mass flows through the volume.

The variation of mass in function of time in an infinitesimal volume is given by

$$\partial M = \frac{\partial \rho}{\partial t}\partial x \partial y \partial z \tag{2.13}$$

If we do the difference between the enter and the exit of the flow in the volume following axis x, we have $(\rho v S)$

$$\rho u \partial y \partial z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\partial x\right)\partial y \partial z = -\frac{\partial(\rho u)}{\partial x}\partial x \partial y \partial z \tag{2.14}$$

and using the same way for the other axis

$$-\frac{\partial(\rho v)}{\partial y}\partial x \partial y \partial z \qquad \mathit{and} \qquad -\frac{\partial(\rho w)}{\partial z}\partial x \partial y \partial z \tag{2.15}$$

We take care of the contributions in all directions and simplify the infinitesimal volume in each term

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \Rightarrow \qquad \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \tag{2.16}$$

3. Coude
4. Caractéristiques générales

Where the final v is global and not specific to an axis. Let's remind that the material derivative is defined as $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v\nabla\rho$, we have

$$\frac{D\rho}{Dt} + \rho\nabla v = 0 \tag{2.17}$$

2.3 Momentum equation

2.3.1 Cauchy equation

To solve this rapidly we use the Lagrangian approach. We suppose that the control volume is moving with the fluid and contains a fixed mass. The momentum rate of change is equal to the net force acting on the control volume (volumetric and superficial)

$$\frac{D}{Dt}\int_{V_m(t)} \rho v \, dV = \int_{V_m(t)} \rho g \, dV + \int_{S_m(t)} f \, dS \tag{2.18}$$

f is a function of the position r on the surface defined by the unit vector n perpendicular to the surface and pointing outward. f can be expressed as the scalar product between the tensor T and normal n

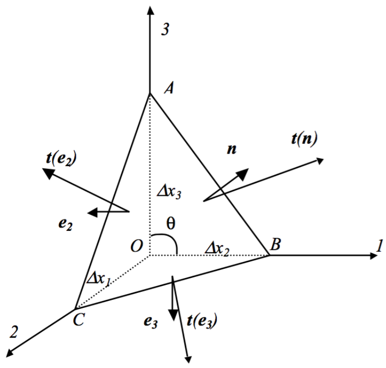
$$f(n, r) = n\,T(r) \tag{2.19}$$

Let's consider a tetrahedron on the axis x_1, x_2, x_3 and the forces exerted by the surrounding fluid. The force balance can be expressed

$$\begin{aligned} f(n)S_{ABC} &= f(e_1)S_{AOB} + f(e_2)S_{AOC} + f(e_3)S_{BOC} \\ &= f(e_1)S_{ABC}ne_1 + f(e_2)S_{ABC}ne_2 + f(e_3)S_{ABC}ne_3 \end{aligned} \tag{2.20}$$

if we do a factorisation and define the tensor T as

$$T = e_1 f(e_1) + e_2 f(e_2) + e_3 f(e_3) \tag{2.21}$$



we find the equation (2.19). So we can write the equation (2.18) with this expression and use the **divergence theorem**

$$\int_S nT \, dS = \int_V \nabla T \, dV \tag{2.22}$$

we obtain the equation

$$\int_V \left[\frac{\partial(\rho v)}{\partial t} + \nabla(\rho v v) - \nabla T - \rho g\right] dV = 0 \tag{2.23}$$

Considering that it must be valid for any and every point of the volume, we get the **Cauchy equation** [5]

$$\rho\frac{Dv}{Dt} = \nabla T + \rho g \tag{2.24}$$

5. We can get ρ out of the material derivative because we consider ρ as constant

2.3.2 Constitutive equations

Static conditions

Imagine that we have a static fluid. The only force f will be due to the pressure p

$$f = -pn \Leftrightarrow T = -pI = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{2.25}$$

In that case, the use of equation (2.24) gives us the information

$$\rho g = \nabla p \tag{2.26}$$

Dynamic conditions

Let's now take care of the viscosity effects on a Newtonian fluid. In that case we have to add to the pressure the forces function of the velocity gradient

$$T = -pI + f(\nabla v) \tag{2.27}$$

where (∇v) has a symmetric (pure deformations) and an antisymmetric (pure rotation) part given by

$$S = \frac{1}{2}(\nabla v + \nabla v^T) \qquad and \qquad A = \frac{1}{2}(\nabla v - \nabla v^T) \tag{2.28}$$

Final form of the stress tensor

Given the upper discussion, the final expression is

$$T = \left[-p + \lambda(\nabla v)\right] I + 2\mu S \tag{2.29}$$

where S is the symmetric part of the gradient. Let's specify that for **incompressible flows**, we have

$$T = -pI + 2\mu S \tag{2.30}$$

Navier-Stokes equation

Theorem : Navier-Stokes equation
For incompressible, Newtonian and isothermal flows

$$\rho \frac{Dv}{Dt} = \rho \left[\frac{\partial v}{\partial t} + v \nabla v\right] = -\nabla p + \mu \nabla^2 v + \rho g \tag{2.31}$$

Where we used the Cauchy equation (2.24) and equation (2.30) for T.

2.4 Bernoulli equation

2.4.1 Without friction effects and energy transfer

To access the Bernoulli equation, let's take the Navier-Stokes equation (2.31) and disregard the viscous term (with μ). The scalar multiplication of that equation provides the **kinetic energy balance**

$$\rho v \frac{Dv}{Dt} = \rho \frac{D}{Dt}\frac{v^2}{2} = -v\nabla(p + \rho\Phi) \qquad with \qquad \Phi = g \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \tag{2.32}$$

For incompressible and steady fluids, $\frac{d\rho}{dt} = 0 = \frac{dp}{dt}$, so

$$\frac{D}{Dt}\left(\rho\frac{v^2}{2} + p + \rho g z\right) = 0 \tag{2.33}$$

leading us to the Bernoulli equation after integration.

Theorem : Bernoulli equation
The sum of the kinetic, potential and flow energies of a fluid particle is **constant** along a streamline during **steady** flow when the **compressibility** and **frictional effects** are negligible.

$$\rho\frac{v^2}{2} + p + \rho g z = c = p_{tot} \tag{2.34}$$

- $\frac{v^2}{2}$ is the **dynamic pressure** : pressure rise when the fluid in motion is brought to a stop isentropically.
- p is the static pressure : thermodynamic pressure of the fluid.
- $\rho g z$ is the **hydrostatic pressure** : for the elevation effects.

Finally, we can say that the sum of these 3 types of pressure is the **total pressure** and that it's constant along a streamline.

Limitations of the Bernoulli equation

To use Bernoulli equation we have to make some hypothesis :

- **Steady flow** : cannot be used for transients (start-up and shut-down) flow or with changes in flow conditions.
- **Frictionless flow** : cannot be used when there are components disturbing the flow.
- **No shaft work** : cannot use in presence of pump, turbine, etc (destroy the streamline structure).
- **Incompressible flow** : cannot be used when the density changes. Liquids and gases (Mach numbers < 0.3) are ok.
- **No heat transfer** : the increase in temperature affects the density.
- **Along a streamline** : the constant c is the same for all streamlines if the flow is irrotational.

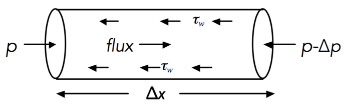
Many applications exist like the Pitot tube, the cavitation or the discharge of a fluid from a tank.

2.4.2 With friction effects and energy transfer

We have to generalize the Bernoulli equation. To take care of the friction forces that dissipate the mechanical energy into heat, we introduce a **pressure drop** ΔP_{tot} for **incompressible fluids**. Friction is an **irreversible process** and these forces are always positive (for potential flow = 0).
When pumps and turbines are used, we introduce an **energy transfer by work** w_p and w_t. The contributions for pumps are positive whereas they are negative for turbines

$$\rho\frac{v_1^2}{2} + p_1 + \rho g z_1 + w_p - w_t - \Delta P_{tot} = c \tag{2.35}$$

2.5 Distributed pressure drops



We want to determine the pressure drops for internal flows. Let's begin with the energy balance

$$\rho\frac{v_2^1}{2} + p_1 + \rho g z_1 + w_p = \rho\frac{v_2^2}{2} + p_2 + \rho g z_2 + w_t + \Delta P_{tot} \tag{2.36}$$

After simplifying the relation, we find that

$$\Delta p = \Delta P_{tot} \tag{2.37}$$

Let's now do the force balance consisting on the pressure force at the entry, the exit and the shear stress within the volume

$$\sum F = \pi R^2 p - \pi R^2(p - \Delta p) - (2\pi R \Delta x)\tau_w = 0 \tag{2.38}$$

After simplifying the terms and isolating the ratio, we can use the relation (2.37)

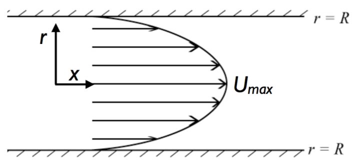
$$\frac{\Delta p}{\Delta x} = \frac{2\tau_w}{R} = \frac{4\tau_w}{D} \qquad \Rightarrow \qquad \Delta P_{tot} = \frac{2\tau_w}{R} = \frac{4\tau_w}{D}L \tag{2.39}$$

Let's finally introduce the **Moody coefficient** λ and the **Fanning coefficient f** to obtain the final expression

$$\lambda = 4f = 4\frac{\tau_w}{\frac{\rho v^2}{2}} \qquad \Rightarrow \qquad \Delta P_{tot} = \lambda\frac{L}{D}\frac{\rho v^2}{2} \tag{2.40}$$

2.6 Newtonian flow in a pipe

2.6.1 Case of a laminar regime (Re < 2100)



Let's take the Navier-Stokes equation (2.31) and suppose that there is no gravity effect on the fluid and that the flow is **steady**.

$$\cancel{\rho\frac{Dv}{Dt}} = -\nabla p + \mu\nabla^2 v + \cancel{\rho g} \qquad \Rightarrow \qquad 0 = -\underbrace{\frac{\partial p}{\partial x}}_{f(x)} + \frac{\mu}{r}\frac{\partial}{\partial r}\underbrace{\left(r\frac{\partial u}{\partial r}\right)}_{f(r)} \tag{2.41}$$

The pressure drop per unit length $\frac{\Delta p}{L}$ must be constant because of the steady regime and the sections are equivalent

$$\frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = c = -\frac{\Delta p}{\mu L} \tag{2.42}$$

The non-slip conditions on the walls ($r = R$) require that $U = 0$ and the maximum velocity at $r = 0$ requires that $\frac{du}{dr} = 0$. The integration part per part gives

$$d\left(r\frac{\partial u}{\partial r}\right) = -\frac{\Delta p}{\mu L}r\,dr \qquad \Rightarrow \frac{\partial u}{\partial r} = -\frac{\Delta p}{\mu L}\frac{r}{2} + \frac{C_1}{r} \qquad \Rightarrow u(r) = -\frac{\Delta p}{\mu L}\frac{r^2}{4} + C_1 \ln r + C_2 \tag{2.43}$$

The upper conditions give the value of constants $C_1 = 0$ and $C_2 = \frac{\Delta p}{\mu L}\frac{R^2}{4}$.

It enables us to calculate the :

- velocity profile

$$U = \frac{\Delta p}{16\mu L}D^2\left(1 - \left(\frac{r}{R}\right)^2\right) \tag{2.44}$$

- Maximum velocity

$$U_{max} = \frac{\Delta p}{16\mu L} D^2 \tag{2.45}$$

- Volumetric flow rate

$$\dot{V} = \int_0^r 2\pi r u(r)\, dr = \frac{\pi}{128\mu} \frac{\Delta p D^4}{L} \tag{2.46}$$

- Mean speed (vitesse moyenne)

$$\frac{\dot{V}}{S} = \frac{\pi}{128\mu} \frac{\Delta p D^4}{L} \frac{4}{\pi D^2} = \frac{U_m ax}{2} \tag{2.47}$$

- Wall shear stress

$$\tau_{max} = -\mu|\frac{du}{dr}|_{r=R} = \frac{4\mu\bar{U}}{R} \tag{2.48}$$

- Friction coefficient

$$f = \frac{\lambda}{4} = \frac{4\mu\bar{U}}{\frac{1}{2}\rho\bar{U}^2 R} = \frac{8\mu\bar{U}}{\frac{1}{2}\rho\bar{U}^2 D} = \frac{16}{Re} \tag{2.49}$$

2.6.2 Case of a turbulent regime (Re > 3500)

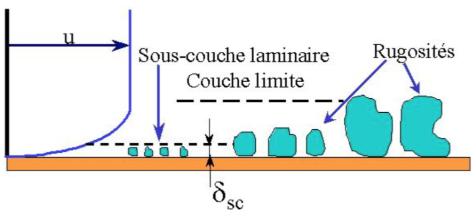
We have to consider two cases in that case. When the walls are smooth [6], the Moody coefficient is given by

$$\lambda = \frac{0.316}{Re_D^{0.25}} \tag{2.50}$$

When the walls are rough [7], we have the **Nikuradse equation** and the **Colebrook equation**

$$\frac{1}{\sqrt{\lambda}} = 1.14 - 0.868 \ln \bar{\Lambda} \qquad and \qquad \frac{1}{\sqrt{\lambda}} = 1.14 - 0.868 \ln \left(\bar{\Lambda} + \frac{9.34}{Re\sqrt{\lambda}}\right) \tag{2.51}$$

$$\bar{\Lambda} = \frac{\Lambda}{D}$$



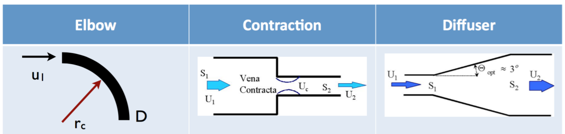
I REALLY HAVEN'T UNDERSTOOD! PLEASE COMPLETE WITH YOUR NOTES.

2.7 Concentrated pressure drops

They are caused by inlets and outlets, curves, change in the cross section, offtakes [8], ...
They are modelled similarly to the distributed pressure drops with the **lack-of-knowledge coefficient** K that differs by geometries and configurations

$$\Delta P_{tot,c} = \sum_j K_j \frac{\rho U_j^2}{2} \tag{2.52}$$

6. Lisse
7. Rugueux
8. Prélèvements



Elbow r_c/D	K	Contraction D_2/D_1	K	Diffuser D_1/D_2	K
1	0.35	0	0.5	0	1
2	0.19	0.2	0.49	0.2	0.87
4	0.16	0.4	0.42	0.4	0.7
6	0.21	0.6	0.27	0.6	0.41
8	0.28	0.8	0.2	0.8	0.15
10	0.32	0.9	0.1	0.9	0.1

Here you can find some values for different situations.

We can now establish the total pressure drops expression

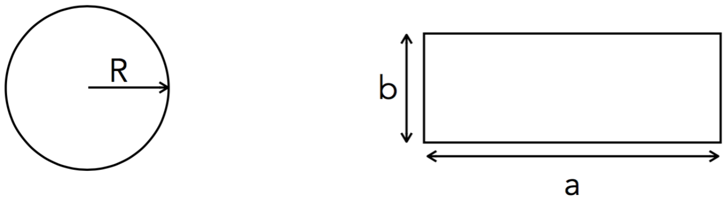
$$\Delta P_{tot} = \Delta P_{tot,d} + \Delta P_{tot,c} = \sum_i \lambda_i \frac{L_i}{D_i} \frac{\rho U_i^2}{2} + \sum_j K_j \frac{\rho U_j^2}{2} \tag{2.53}$$

Hydraulic diameter

To evaluate the Reynolds number we introduce the hydraulic diameter defined by

$$D_h = 4\frac{S}{P} \tag{2.54}$$

Where S is the surface, P the perimeter and the value 4 is introduced to retrieve the normal diameter for a circular section. Let's calculate the diameter.



$$D_h = 4\frac{\pi R^2}{2\pi R} = D \qquad \textit{and} \qquad D_h = 4\frac{ab}{2a+2b} \approx 2b \tag{2.55}$$

Finally, we can find the Reynolds number by

$$Re_h = \frac{D_h U}{\nu} \tag{2.56}$$

Chapitre 3

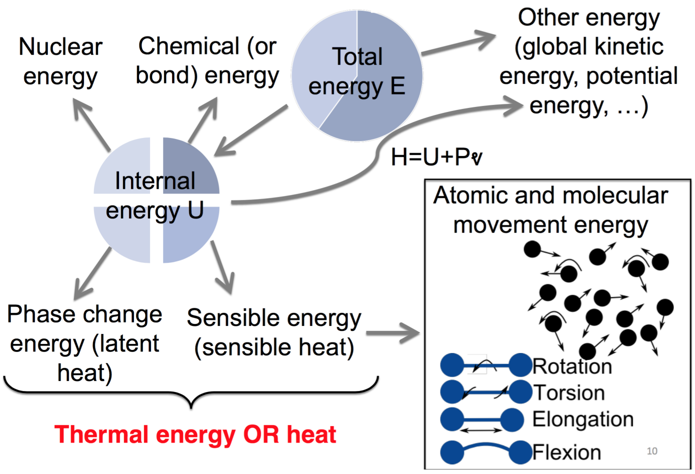
Heat transfer

3.1 Introduction

We retrieve heat transfer in many phenomena like the combustion of coal that we use for the barbecue, the steel production or in power plants. But thermodynamics don't tell us how fast combustion will take place, chemistry tells us it will go too fast and each reaction needs its own reaction rate.
Heat transfer is important in the duration of many processes. It's important to design and understand that principle for engineers.

3.1.1 Definitions

Energy



In a system you have the total energy and is divided in two : internal energy and the rest. The first is the energy you find in the system if you attach yourself to the system. Thermal energy or heat includes the **latent heat** that is the phase change energy and the **sensible heat** that is associated to the atomic move inside the material.

Temperature

It's an interesting principle because it's not simple to define it. You can have thermodynamic definition of temperature : linked to the entropy (system is changing) and microscopic theory linked to the move of particles.

The important for us is that temperature is an intensive quantity correlated to the internal energy of the system. It's a universal feature of matters so it's independent of the material. Temperature is linked to the system energy by the specific heat

$$dU = mc_v dT \qquad and \qquad dH = mc_p dT \tag{3.1}$$

Should remember that gas of low pressure and low velocity can be considered as incompressible. In that case

$$c_v = c_p = c \tag{3.2}$$

Heat flux

With 2 bodies at different temperatures in contact, a quantity $Q(J)$ of heat is transferred through a surface of area A. Heat transfer rate and heat transfer flux density are defined as

$$\dot{Q} = \frac{dQ}{dt} \qquad \Rightarrow \qquad \dot{q} = \frac{\dot{Q}}{A} \tag{3.3}$$

3.1.2 Heat balance

The idea is to see what heat induct. Heat is just a part of energy in the system, only the total energy is conserved. What we will do is make balances on a control volume but we will divide the energy in two parts (heat and the rest). We will take into account the heat flux entrance and exit on a control volume surface, the generation or the consumption and the accumulation :

$$\text{accumulation} + \text{out fluxes} + \text{sinks} = \text{sources} + \text{in fluxes} \tag{3.4}$$

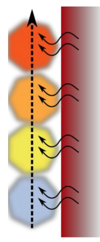
Sources and sinks can be due to the transformation of heat energy into a non thermal energy (sinks) or vice versa (source).
We can divide the situation in 2. The one where we accumulate and the other where we don't. The first case corresponds to stationary systems. The difference between **stationary** and **equilibrium** is that the second is a particular stationary state where we have no flux whereas they are present in a stationary flux.

3.1.3 Heat transfer mechanisms

Conduction

Universal heat transfer by contact. It appears in matter when molecules enter in contact with each others and transfer heat. The entropic transport of heat is quite slow. Universal because it can happen everywhere we have matter and is the main heat transfer. Solid materials are characterized as good heat conductor or not, so it's a material property.

Convection



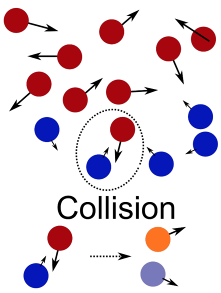
Convection is linked to the movement of the particles but they carry they own heat. During this movement conduction occurs : convection = conduction + advection (illustrated on the picture). It's the major transfer in presence of flows. Convection is complicated, it depends on how moves the fluid, its direction, ... (dependent).
We have to do the difference between forced and natural convection. For forced convection, the flow exists even if there is no heat transfer whereas for natural convection the flow is due to the heat transfer.

Radiation

It's the electromagnetic waves heat transfer between surfaces far from each other. No need to have matter, it's important to have a large temperature difference. It's more complex mathematically but we often neglect it.

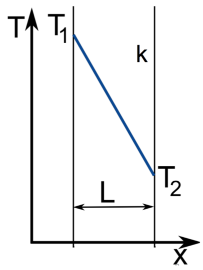
3.2 Conduction

3.2.1 Microscopic approach



Temperature is an indicator of the particles movement. Hot bodies are composed of particles faster than cold bodies and when they are placed in contact, collisions occur, leading the hotter particle to transfer kinetic energy and so heat to the colder one.
The higher the temperature gradient is, the faster the transfer will be. This transfer process continues until the equalization of temperature and the quantity transferred depends on the temperature, the chemical compositions and the densities.

3.2.2 Phenomenological approach - the first Fourier law



Let's take an infinite wall of thickness L. We fix two temperature and we look at the temperature profile of the system. The amount of heat transfer will be proportional to the thermal conductibility $k\,[W/m.K]$ and the heat flux density is

$$\dot{q} = k\frac{T_1 - T_2}{L} \tag{3.5}$$

We can consider the thickness equal to $L = -dx$ giving

$$\dot{q} = -k\frac{dT}{dx} \tag{3.6}$$

3.2.3 Thermal conductibility and diffusivity

Thermal diffusivity

k is mainly dependent of the materials properties but includes the thermal capacity. So we can express a value called **thermal diffusivity** defined as (unity : m^2/s)

$$\alpha = \frac{k}{\rho C_p} \tag{3.7}$$

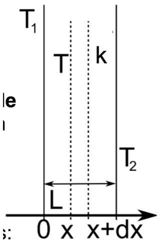
Thermal conductivity of materials

- **Gas :** particles are far from each other, so there are few collision and so few transfer by conduction.
- **Liquid :** many random shocks giving better conduction skills.
- **Solid :** the rigidity makes from them bad conductors but the crystals can transfer heat fast. They can so be good conductors or insulators [1].

k has some dependence with temperature that is neglect for small changes. **We have to beware of phase changes !**

1. Isolant

3.3 1D stationary conductive heat balance



Let's apply a heat balance on the walls of thickness dx using equation (3.4). There is no source and no accumulation term.

$$\text{Flux In} = \text{Flux out} \qquad \Leftrightarrow \qquad A\dot{q}_x = A\dot{q}_{x+dx} \tag{3.8}$$

With a first order Taylor expansion we have the expression of $\dot{q}_{x+dx}$ and replaced in (3.8)

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}_x}{dx}dx \qquad \Rightarrow \qquad \frac{d\dot{q}_x}{dx} = 0 \tag{3.9}$$

We apply the Fourier Law ($k = cst$)

$$\frac{d\dot{q}_x}{dx} = \frac{d}{dx}\left(-k\frac{dT}{dx}\right) = \frac{d^2T}{dx^2} = 0 \qquad \Rightarrow \qquad T = C_1x + C_2 \tag{3.10}$$

3.3.1 Heat resistance

The boundary conditions $T = T_1$ for $x = 0$ and $T = T_2$ for $x = L$ gives the constants, so

$$T = \frac{T_2 - T_1}{L}x + T_1 \tag{3.11}$$

If we reapply the Fourier Law (deriving the equation above)

$$\dot{q} = k\frac{T_1 - T_2}{L} \tag{3.12}$$

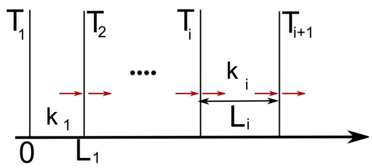
We define now the **thermal resistance** of the wall $R = \frac{L}{kA}$ and obtain the heat transfer rate

$$\dot{Q} = \frac{T_1 - T_2}{R} \tag{3.13}$$

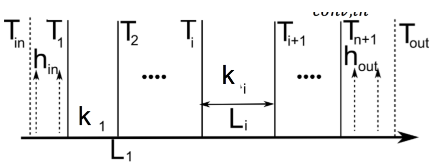
3.3.2 Series of walls

By assuming the temperature continuity and the flux conservation at the boundary of each wall, we have

$$\dot{Q} = \frac{T_1 - T_{n+1}}{R} \qquad with \qquad R = \sum R_i = \sum \frac{L_i}{k_iA} \tag{3.14}$$



3.3.3 Adding two extra resistance - the Biot number



Close to the walls the convective effect appears and is described by

$$\dot{q} = h_{in}(T_{in} - T_1) \quad and \quad \dot{q} = h_{out}(T_{n+1} - T_{out}) \tag{3.15}$$

and the heat transfer rate can be found like

$$\dot{Q} = \frac{T_{in} - T_{out}}{R} \qquad and \qquad R = \frac{1}{h_{in}A} + \sum \frac{L_i}{k_iA} + \frac{1}{h_{out}} \tag{3.16}$$

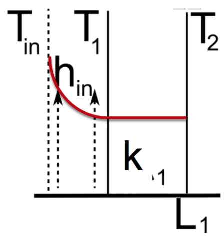
But is it important to take care of the convective effect ? Let's compare the convective and conductive effect in the inner side

$$\left.\begin{aligned} T_{in} - T_1 &= \frac{\dot{Q}}{h_{in}A} \\ T_1 - T_2 &= \frac{L_1\dot{Q}}{k_1 A} \end{aligned}\right\} \Rightarrow \frac{T_1 - T_2}{T_{in} - T_1} = \frac{L_1 h_{in}}{k_1} = Bi \tag{3.17}$$

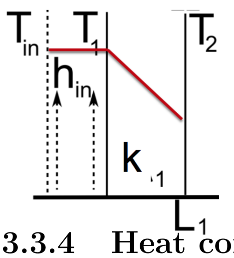
That is the **Biot number** that compares the convection in the fluid around the solid to the conduction in the solid.

Bi « 1

It means that $k \gg h_{in}$ so that the convection is limiting and that the temperature difference is in the fluid. The variation of T in the solid is negligible.



Bi » 1



In that case, the convection is dominating because $h_{in} \gg k$. The temperature difference is in the solid while the variation is in negligible in the fluid.

3.3.4 Heat conduction in tubes

It's the same principle but in cylindrical coordinates. $\dot{Q}$ is conserved for all radius and is expressed like

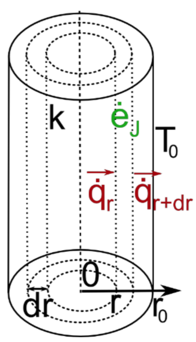
$$\dot{Q} = \dot{q} 2\pi r H \tag{3.18}$$

The flow then writes

$$\dot{Q} = \frac{T_1 - T_2}{R} \qquad R = \frac{1}{2\pi r_1 H h_{in}} + \sum \frac{\ln(r_{i+1}/r_i)}{k_i 2\pi H} + \frac{1}{2\pi r_{n+1} H h_{out}} \tag{3.19}$$

3.4 1D cylindrical problem with source term

Let's consider an infinite cylindrical electric wire [2] of radius r_0 in an environment of temperature T_0. An electric current generates heat **uniformly** due to **Joule effect** at a volumetric rate $\dot{e}_J$ $[W/m^3]$. Let's do the heat balance along the radius



$$\cancel{Accumulation} = (In - Out) Fluxes + (Sources - \cancel{Sink}) \tag{3.20}$$

There are no accumulation because the system is stationary and nothing consume heat. Replacing by mathematical expression

$$\dot{Q}_r = \dot{Q}_{r+dr} - \dot{e}_J 2\pi r H dr \tag{3.21}$$

2. Fil

applying the first order Taylor expansion

$$\dot{Q}_{r+dr} = \dot{Q}_r + \frac{d\dot{Q}_r}{dr}dr \qquad \Rightarrow \qquad \frac{d\dot{Q}_r}{dr} = \dot{e}_J 2\pi r H \tag{3.22}$$

Combining with the Fourier law and equation (3.18)

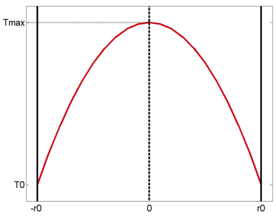
$$\frac{d}{dr}\left(-k2\pi rH\frac{dT}{dr}\right) = \dot{e}_J 2\pi r H \qquad \Leftrightarrow \qquad \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\dot{e}_J \tag{3.23}$$

The integration gives the result

$$T = \frac{-\dot{e}_J}{4k}r^2 + C_1\ln(r) + C_2 \tag{3.24}$$

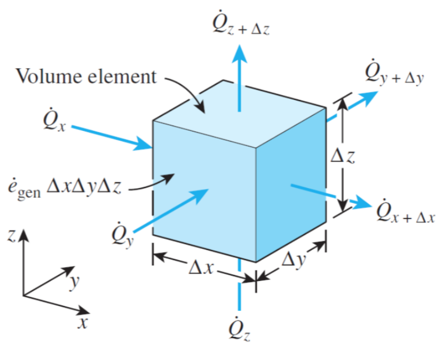
and after applying the boundary conditions $r = 0 \to T\ finite$ and $r = r_0 \to T = T_0$

$$T = T_0 + \frac{\dot{e}_J r_0^2}{4k}\left(1 - \frac{r^2}{r_0^2}\right) \tag{3.25}$$



$T_{max} = T_0 + \frac{\dot{e}_J r_0^2}{4k}$ is obtained in the middle of the cylinder, so there is a moving heat. It's logical to have an equation depending on r_0 because if the radius increases, it will be more difficult for heat to reach the external environment.

3.5 Generalization : unstationary 3D heat balance with heat generation - The second Fourier law



Always the same principle

$$Accumulation = (In - Out)Fluxes + (Sources - Sink) \tag{3.26}$$

replaced by mathematical expressions

$$\rho c_p\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{gen} = 0 \tag{3.27}$$

Using the definition of thermal diffusivity $\alpha = \frac{k}{\rho c_p}$, we obtain the **second Fourier law**

$$\frac{\partial T}{\partial t} + \nabla(\alpha\nabla T) + \dot{e}_{gen} = 0 \tag{3.28}$$

There is the 5 classical boundary conditions to use with the Fourier law :

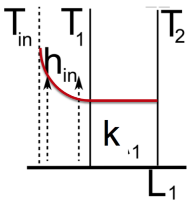
1. Fixed temperature : $T = T_0$
2. Fixed flux density : $\dot{q} = -k\frac{dT}{dx} = \dot{q}_0$
3. Continuity of the flux : $\dot{q}_1 = -k_1\frac{dT}{dx}|_1 = \dot{q}_2 = -k_2\frac{dT}{dx}|_2$
4. Convection at the boundary : $\dot{q} = -k\frac{dT}{dx} = h(T - T_b)$
5. If we look at an insulated material : $\frac{dT}{dx} = 0$

Chapitre 4

Transient heat conduction

Let's imagine that we want to cool a bottle of beer. What time will it take ? We will consider 3 different cases neglecting sometimes convection sometimes conduction.

4.1 Lumped systems analysis



Let's study the case where convection is the limiting factor for heat (dispersion of heat within the solid is negligible compared to heat arrival to the surface). Let's remind the expression of the Biot number (go to sous-section 3.3.3). In this case, the $Bi \ll 1$

$$Bi = \frac{L_1 h_{in}}{k_1} \tag{4.1}$$

4.1.1 Calculation of lumped systems

Let's consider an object of volume V and surface A initially to temperature T_i that we put into an infinite temperature T_∞ environment. We want to know what happens. So my accumulation, is equal to the amount that comes with the convection

$$V\rho c_p \frac{\partial T}{\partial t} = Ah(T_\infty - T) \tag{4.2}$$

Integrating and applying the initial condition $t = 0 \to T = T_i$

$$\underbrace{\frac{T - T_\infty}{T_i - T_\infty}}_{\theta} = \exp\left(-\frac{Ah}{V\rho c_p}t\right) = \exp\left(-\underbrace{\frac{Vh}{Ak}}_{Bi}\underbrace{\frac{A^2k}{V^2\rho c_p}t}_{Fo}\right) \tag{4.3}$$

It can always be expressed like a function of θ. The expression in the exp must be dimensionless so we will make it appear. V/A = length, so the definition of Biot number appears. The other is including t and is called the **Fourier number**.

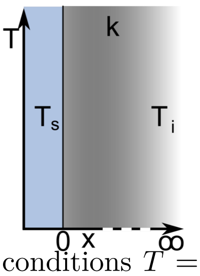
4.1.2 Fourrier number

$$\theta = \exp(-Bi\,Fo) \qquad Fo = \frac{A^2k}{V^2\rho c_p}t = \frac{A^2\alpha}{V^2}t \tag{4.4}$$

If we look to the units, we see that it's the dimensionless time of the process. It compares the time since the external temperature change and the characteristic time of heat conduction in the object.

4.2 Semi-infinite solid

4.2.1 Error function



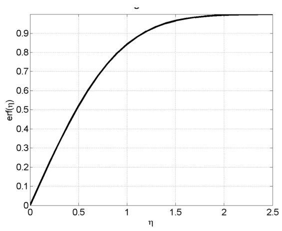
This is the case within the conduction only appears on a small fraction of the total solid and an infinite surface is in contact with a fluid. The $Bi \gg 1$. We respect equation without source term

$$\frac{\partial T}{\partial t} + \alpha \frac{\partial^2 T}{\partial x^2} \tag{4.5}$$

We don't do the calculus in the course but after integrating and applying the conditions $T = T_S$ for $x = 0\ \forall t \leq 0$, $T = T_i$ for $x \to +\infty$ and $T = T_i$ for $t = 0\ \forall x > 0$, we obtain the result

$$\frac{T - T_S}{T_i - T_S} = erf(\eta) \qquad with \qquad \eta = \frac{x}{\sqrt{4\alpha t}} \qquad and \qquad erf(\eta) = \frac{2}{\pi}\int_0^{\eta} \exp(-u^2)\,du \tag{4.6}$$

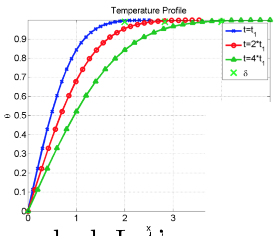
We see that η combines position and time. You progress in the object proportionally to the square root of time. A fixed value of η means a fixed value of temperature. The flux density is expressed



$$\dot{q} = -k\frac{\partial T}{\partial x}|_{x=0} = k\frac{T_S - T_i}{\sqrt{\pi\alpha t}} \tag{4.7}$$

I'm assuming an infinite body, is it realistic ? where does my heat go ?

4.2.2 Penetration length



When we look to the graph, we can see that if the process time is 4 time higher, the x goes 2 times further. We define the penetration length

$$\delta = 4\sqrt{\alpha t} \tag{4.8}$$

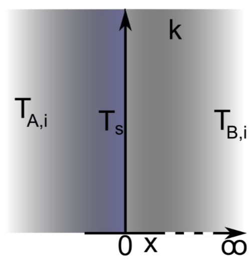
that defines the range of the heat process.

But when is this applicable ? As long as the center of the object is not reached. Let's suppose a characteristic length L. We must respect the condition $\delta = 4\sqrt{\alpha t} < L$. After manipulations (Bi still $\gg 1$)

$$\frac{\alpha t}{L^2} = Fo < \frac{1}{16} \tag{4.9}$$

4.2.3 Two semi-infinite body in contact

What happens when two solids enter in contact ? If the time is short enough ($Fo < 0.16$), we can consider the 2 as semi-infinite bodies. So (for heat coming from B)



$$\dot{q}_{s,A} = \dot{q}_{s,B} \Leftrightarrow k_A\frac{T_S - T_{Ai}}{\sqrt{\pi\alpha_A t}} = k_B\frac{T_{Bi} - T_S}{\sqrt{\pi\alpha_B t}} \tag{4.10}$$

After isolating T_S and expressing α as (3.7)

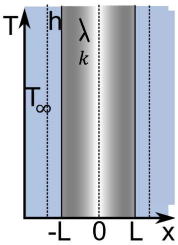
$$T_S = \frac{\sqrt{K_A C_{pA}\rho_A}T_{Ai} + \sqrt{K_B C_{pB}\rho_B}T_{Bi}}{\sqrt{K_A C_{pA}\rho_A} + \sqrt{K_B C_{pB}\rho_B}} \tag{4.11}$$

This is an average temperature independent of time.
We cannot apply that theory in the case of our bottle of beer because we want to cool the whole bottle and not a part.

4.3 Finite body

For that you will still use the second Fourier law but we have to do a more complex design. We can for example have an infinite planar body of thickness $2L$, an infinite cylinder of radius r_0 or a sphere of radius r_0. These can be solved and we will find particular situations with heat flux boundary conditions.

4.3.1 The plane wall case



To describe that, we have to use the general equation again

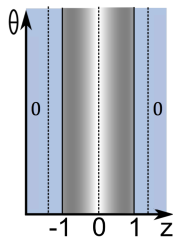
$$\frac{\partial T}{\partial t} + \alpha \frac{\partial^2 T}{\partial x^2} \tag{4.12}$$

To that, we have to apply the boundary conditions :

- no variation of temperature in the middle of the body : $\frac{\partial T}{\partial x} = 0$ for $x = 0$ $\forall t \geq 0$
- convection on the walls : $-k\frac{\partial T}{\partial x} = h(T_L - T_\infty)$ for $|x| = L$ $\forall t \geq 0$
- initial temperature within the solid : $T = T_i$ for $t = 0$ $\forall -L < x < L$

We transcript all that in a non-dimensional form by replacing L by 1, T by θ, T_i by 1, t by Fo, x by X and the second condition becomes $-\frac{\partial \theta}{\partial X} = B_i\theta_1$ for $X = 1$ $\forall Fo \geq 0$. Where

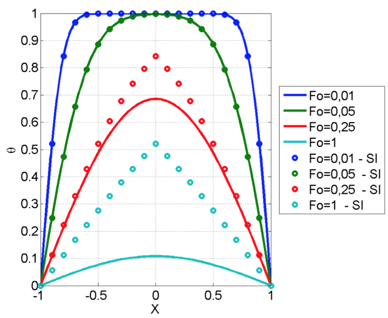
$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad Fo = \frac{\alpha t}{L^2} \qquad X = \frac{x}{L} \qquad Bi = \frac{hL}{k} \tag{4.13}$$



The solution of that (we don't calculate) is

$$\theta = \sum_{i=1}^{\infty} \frac{2Bi\cos(\beta_i X)\exp(-\beta_i^2 Fo)}{\beta_i^2(\beta_i^2 + Bi^2 + Bi)} \tag{4.14}$$

where the β_i are the solutions of $\beta\tan(\beta) = Bi$; these solutions are tabulated.

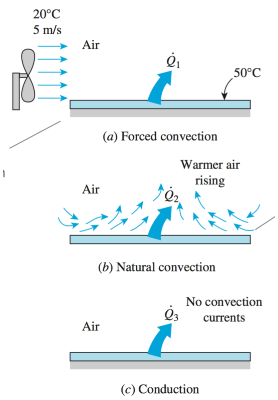


If we compare this case to the semi-infinite solid, we can see that the last theory is more precise when Fo number increases (time increase or length decrease). It's similarly tabulate for the cylinder and the sphere.

Chapitre 5

Forced and natural convection

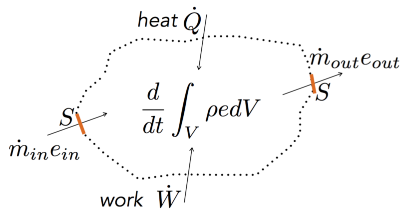
5.1 Conduction and convection



Heat transfer through a solid is always conduction, the molecules position are relatively fixed. Heat transfer in a liquid or gas is convection if there is a **bulk** fluid motion and is conduction when there isn't. Conduction in a fluid is the limiting case of convection where the fluid is **quiescent** [1].
Convection is complicated due to the fact that it involves fluid motion as well as conduction. Fluid motion **enhances** [2] heat motion, it brings the cooler and warmer part of fluid into contact, increasing the rate of heat transfer.
Natural convection is caused by a density gradient.

5.2 Macroscopic energy balance

Let's consider a control volume and let's apply an energy balance, using $e = u + \frac{v^2}{2} + gz$ where the first term is the internal energy, the second the kinetic energy and the last the potential energy



$$\frac{d}{dt}\int_V \rho e\, dV = (\rho veS)_{in} - (\rho veS)_{out} + \dot{W} + \dot{Q} \tag{5.1}$$

Where the work can be decomposed in a **boundary** and a **useful** work and the heat is given by

$$\dot{W} = \dot{W}_f + \dot{W}_p = pvS + \dot{m}w_p = \dot{m}\left(\frac{p}{\rho} + w_p\right) \qquad \textit{and} \qquad \dot{Q} = \dot{m}q \tag{5.2}$$

The transient form is given by

$$\frac{d}{dt}\int_V \rho e\, dV = -\Delta(\frac{1}{2}\rho v^3 S) - g\Delta(\rho vzS) - \Delta(pvS) - \Delta(\dot{m}u) + \dot{m}w_p + \dot{m}q \tag{5.3}$$

The steady form

$$0 = \Delta\left(\frac{1}{2}v^2 + gz + u + \frac{p}{\rho}\right) - q - w_p \tag{5.4}$$

1. Au repos
2. Améliore

5.2.1 Relation with the generalized Bernoulli equation

Let's take the steady form of the energy balance for a finite transformation (5.4) and let's convert it for an infinitesimal transformation

$$0 = -\left(\frac{1}{2}dv^2 + gdz + d\left(\frac{p}{\rho}\right)\right) + (\delta q - du) + dw_p \tag{5.5}$$

Let's remind (see *Chimie-Physique*) that the **Gibbs equation** (corresponding to elemental working change) and the **second principle** of thermodynamics are given by the expressions

$$du = Tds - pdV \qquad and \qquad Tds - \delta q = dh_f \geq 0 \tag{5.6}$$

Combining the two equations around Tds

$$\delta q - du = -dh_f + pdV \tag{5.7}$$

and replacing in (5.5), we obtain the **generalized Bernoulli equation**

$$\frac{1}{2}dv^2 + gdz + \frac{dp}{\rho} = w_p - dh_f \qquad \underset{\rho=cst}{\longrightarrow} \qquad \frac{1}{2}\Delta v^2 + g\Delta z + \frac{\Delta p}{\rho} = w_p - h_f \tag{5.8}$$

5.2.2 Simplification in case of heat exchange

When we analyse heat exchange processes, we assume that we can also consider the variation of enthalpy. So we can rewrite the expression and make appear the heat flux in function of the difference of enthalpy in the streams. So we have

$$u + \frac{p}{\rho} = u + pv = h \qquad and \qquad \underbrace{\Delta h = \int_{T_0}^{T} c_p\, dT}_{Gas} \qquad \underbrace{\Delta h = c\Delta T}_{Liquid} \tag{5.9}$$

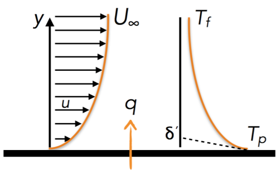
We assume that thermal power is much higher than the power for pumps and compressors and that the enthalpy change is much higher than the kinetic and potential energy variation. Applying all that to equation (5.4), we have

$$\dot{Q} = \dot{m}\Delta h = \Delta\dot{H} \tag{5.10}$$

5.3 The heat transfer coefficient

5.3.1 The Nusselt number

The rate of convection $\dot{Q}$ is proportional to the temperature difference.



$$\dot{Q} = hS(T_p - T_f) \tag{5.11}$$

h here is not the enthalpy but the **heat transfer coefficient**. Let's give a meaning. Imagine that you have a fluid flow and it is approaching a solid. The heat transfer from solid surface to the fluid layer is by pure conduction, since the fluid layer obeys the no-slip conditions. Now if you have T_p miner than T_f we have that type of schema. The heat is then convected. The heat flux to the first layer is given by (where δ' is the layer thickness)

$$\dot{q}_p = -k_f \frac{\partial T}{\partial y}|_p = k_f \frac{(T_p - T_f)}{\delta'} \tag{5.12}$$

Combining that with the **Newton law**, we find an expression for h

$$\dot{q}_p = h(T_p - T_f) \qquad \Rightarrow \qquad h = \frac{k_f}{\delta'} \tag{5.13}$$

We can make that dimensionless with the **Nusselt number** that gives the heat transfer enhancement thanks to convection ($Nu = 1$ represents pure conduction)

$$Nu = \frac{hL}{k_f} \tag{5.14}$$

5.3.2 Comparison with the Biot number

If we look to the two definitions

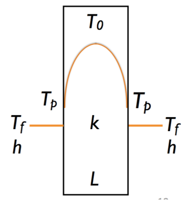
$$Nu = \frac{hL}{k_f} \qquad and \qquad Bi = \frac{hL}{k_s} \tag{5.15}$$

They seem to be the same. Be careful because it's not the case ! Indeed the **Nusselt number** can be written as

$$Nu = \frac{\frac{\partial(T_p - T)}{\partial y}|_p}{\frac{T_p - T_f}{L}} = \frac{\text{Temperature gradient at the surface}}{\text{Reference external temperature gradient}} \tag{5.16}$$

The **Nusselt number** so characterizes the **fluid** while the **Biot number** characterizes the **solid**

$$Bi = \frac{\frac{(T_0 - T_p)}{L}|_p}{\frac{T_p - T_f}{L}} = \frac{\text{Internal temperature gradient}}{\text{Reference external temperature gradient}} \tag{5.17}$$



5.3.3 Forced convection in pipes and channels

We know that the heat flux is transversal to the flow (analogy with the momentum flux). Graetz proposed an analyse and here is the result

$$Nu = \frac{hD_h}{k_f} \tag{5.18}$$

Where D_h is the hydraulic diameter and k_f the fluid conductivity. The value for the case of a constant wall temperature or a constant heat flux are given in slide 11 chapter 5.
Let's specify that for turbulent regime the Nusselt number is given by

$$Nu = 0.023 Re^{\frac{4}{5}} Pr^n \tag{5.19}$$

where Pr is the **Prandtl number** (yes it's correctly written) and $n = 0.4$ or $n = 0.3$ respectively for a heated or cooled fluid. We remind that the Reynolds number $Re = \frac{vD}{\nu}$ compares the momentum convection to the momentum diffusion. The Prandtl number

$$Pr = \frac{c_p \mu}{k} = \frac{c_p \mu \rho}{k \rho} = \frac{\nu}{\alpha} \tag{5.20}$$

compares the momentum diffusion to the energy diffusion (whether momentum diffuse better or not than energy).
That definition of Nusselt number is limited to the range

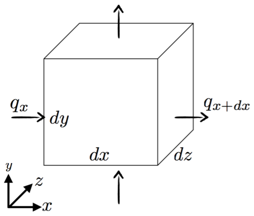
$$Re > 10^4 \qquad and \qquad 0.7 < Pr < 600 \tag{5.21}$$

Bonus : $RePr = \frac{vD}{\alpha}$ is the **Peclet number** (Pe_t)

5.3.4 Laminar convection for the flow around a submerged object

The professor said for slide 13 that all the formula are in the formulaire and that we just have to know that there is a specific Nusselt number for each case (sphere, cylinder, flat plate).

5.4 Local form of the energy equation



We are going to do a simple balance where the flux in and out are associated to mass flow rates and conduction.

$$\text{internal energy variation} = \text{Heat in} - \text{Heat out} + \text{Heat source} \tag{5.22}$$

The internal energy variation is given by

$$\frac{\partial(\rho cT\, dx\, dy\, dz)}{\partial t} = (\rho\, dx\, dy\, dz)\, c\frac{\partial T}{\partial t} \tag{5.23}$$

the difference Heat in - Heat out due to mass transfer by

$$-\rho c_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z})\, dx\, dy\, dz \tag{5.24}$$

by conduction

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dx\, dy\, dz \tag{5.25}$$

and the source term

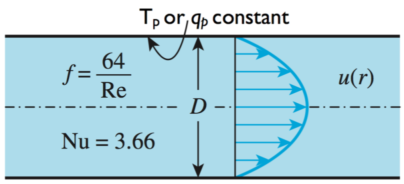
$$\dot{Q}_v\, dx\, dy\, dz \tag{5.26}$$

We put all this together and get the equation and using the **Fourier law** for q

$$\rho c\frac{DT}{Dt} + \nabla(-k\nabla T) = \dot{Q}_v \qquad or \qquad \frac{DT}{Dt} + \nabla(\alpha\nabla T) = \frac{\dot{Q}_v}{\rho c} \tag{5.27}$$

5.5 Heat transfer in pipe flows

We consider a circular pipe of radius R, with constant temperature walls or constant heat flux at the walls. The flow inside is fully-developed, so the velocity is function of the radius and the friction is known. The system is steady and the flow laminar.



5.5.1 Analysis of Graetz

The velocity profile is described by Poiseuille like

$$\frac{u}{U_{max}} = 1 - \left(\frac{r}{R}\right)^2 \tag{5.28}$$

Using the energy equation (5.27), we can express it in polar coordinates

$$u\frac{\partial T}{\partial x} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \cancel{\frac{\partial^2 T}{\partial x^2}}\right] \tag{5.29}$$

where we can neglect the last term because the diffusion in mainly in the radial direction. The boundary conditions due to the symmetry and the imposed flux are

$$r = 0 \Rightarrow \frac{\partial T}{\partial r} = 0 \qquad and \qquad r = R \Rightarrow \frac{\partial T}{\partial r} = -\frac{q}{k} \tag{5.30}$$

Before solving we have to do a consideration. The heat flux q and h are constant everywhere on the wall. So the velocity only change in the vertical direction. We express that, noting the mean temperature variation $< T >$, like

$$\dot{m}c\frac{d < T >}{dx} = qS \Rightarrow \frac{d < T >}{dx} = \frac{qS}{\dot{m}c} = C_0 \tag{5.31}$$

Giving the final equation for (5.29)

$$\alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = u(r)C_0 \tag{5.32}$$

If we solve that we get the temperature profile (T_c for temperature at centreline)

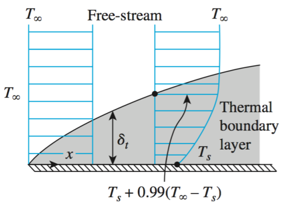
$$T - T_c = \frac{C_0 U_{max} R^2}{4\alpha}\left[\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right] \tag{5.33}$$

5.5.2 Nusselt number for constant temperature and constant heat flux

See slide 20 for the formula. Just to know that the Nusselt numbers are exact values and not an approximations! We can find the temperature of the wall by considering that. With the heat transfer coefficient we can do what? We are able to apply the Newton's law!
The Nusselt number changes for other geometries and other friction factors.

5.6 The thermal boundary layer

5.6.1 Definition



How to estimate the boundary layer thickness in a turbulent flow? What we can see here is that if we have a high Reynolds number ($Re \gg 1$), we will develop a velocity boundary layer and a high temperature difference will develop a thermal boundary layer. Similarly to the non-sleep conditions, the fluid will be in a temperature on the surface, giving a temperature profile ranged from $T_s to T_\infty$.
In this thermal boundary layer temperature varies in the normal direction to the plate. The boundary layer thickness δ_T is the distance at which the difference $T - T_s = 0.99(T_\infty - T_s)$ and increases in the flow direction.
Convection strongly depends on the development of the velocity boundary layer relative to the thermal one.

5.6.2 The Peclet number

We previously defined the **Peclet number** as

$$Pe_t = \frac{J_{E_c}}{J_{E_d}} = \frac{vL}{\alpha} = RePr \tag{5.34}$$

For high-velocity flows over a flat plate, the Peclet number is much higher than 1. It means that :

- Far from the wall : Convection $\gg$ Diffusion ($Re \gg$)
- At the wall : Convection $\ll$ Diffusion ($Re \ll$)
- At the boundary layer height : Convection = Diffusion ($Re = 1$)

5.6.3 Governing equations

For an incompressible fluid in 2 dimensions we will use :

- Continuity equation
- Navier-Stokes along x
- Navier-Stokes along y
- Equation de la chaleur
- Boundary conditions

$$\begin{cases} y = 0 \Rightarrow u = v = 0 \Rightarrow T = T_0 \\ y \to \infty \Rightarrow u = U, v = 0 \Rightarrow T = T_\infty \qquad p = p \end{cases} \tag{5.35}$$

5.6.4 Scaling of the different terms

- Heat flux

$$q = -k \left(\frac{dT}{dy}\right)_{y=0} \approx -k\frac{\Delta T}{\delta_T} \tag{5.36}$$

- Nusselt number

$$Nu = \frac{hL}{k} = \frac{L}{\delta_T} \tag{5.37}$$

- Heat transfer coefficient

5.6.5 Momentum vs thermal boundary layer length

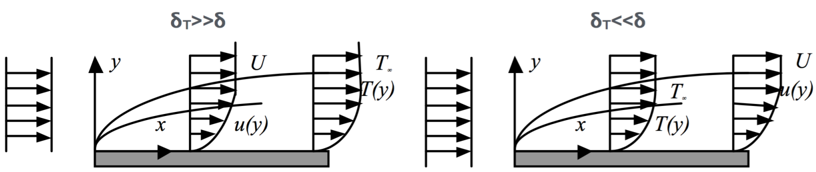
Re » 1

In this case, we have the 2 boundary layer, momentum and thermal. We already know that the thickness of momentum boundary layer decreases relatively to the Reynolds number using

$$\delta \approx LRe^{-\frac{1}{2}} \tag{5.38}$$

We have 2 limiting cases (see definition of Prandtl number in subsection 5.3.3) :

- If $Pr \ll 1$, $\delta_T \gg \delta$ so heat diffuses very quickly compared to momentum (liquid metals)
- If $Pr \gg 1$, $\delta_T \ll \delta$ so heat diffuses very slowly compared to momentum (oil)



Re » 1 and Pr « 1

Let's study that case where the two boundary layer exist. We will assume that $\delta_T \gg \delta$ and verify a posteriori. The heat transfer takes place outside the momentum boundary layer so $u = U$ (fluid upcoming speed) and the velocity v is obtained by **continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad v \approx U\frac{\delta_T}{L} \tag{5.39}$$

The convective heat transfer terms are of the same order of magnitude

$$u\frac{\partial T}{\partial x} \approx v\frac{\partial T}{\partial y} \approx \frac{U\Delta T}{L} \tag{5.40}$$

Only the transversal diffusion term is relevant ($\delta \ll L$)

$$\alpha\frac{\partial^2 T}{\partial x^2} \ll \alpha\frac{\partial^2 T}{\partial y^2} \approx \frac{\alpha\Delta T}{\delta_T^2} \tag{5.41}$$

We know that at the boundary layer border convection = conduction. It allows us to calculate δ_T^2 equalizing the 2 previous equations

$$\delta_T^2 = \frac{\alpha L}{U} \Rightarrow Nu = \frac{L}{\delta_T} = Pe^{\frac{1}{2}} = Re^{\frac{1}{2}} Pr^{\frac{1}{2}} \qquad \Rightarrow \delta_T = LRe^{-\frac{1}{2}} Pr^{-\frac{1}{2}} = \delta Pr^{-\frac{1}{2}} \tag{5.42}$$

Re » 1 and Pr » 1

Let's study that case where the 2 boundary layers exist. We will assume that $\delta_T \ll \delta$ and verify a posteriori. The heat transfer takes place **inside** the boundary layer so we assume a linear velocity profile for $u \Rightarrow u = U\frac{\delta_T}{\delta}$ in the boundary layer and the velocity v is obtained by **continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad v \approx U\frac{\delta_T}{L} \tag{5.43}$$

The convective heat transfer terms are of the same order of magnitude

$$u\frac{\partial T}{\partial x} \approx v\frac{\partial T}{\partial y} \approx \frac{U\Delta T\delta_T}{\delta L} \tag{5.44}$$

Only the transversal diffusion term is relevant ($\delta \ll L$)

$$\alpha\frac{\partial^2 T}{\partial x^2} \ll \alpha\frac{\partial^2 T}{\partial y^2} \approx \frac{\alpha\Delta T}{\delta_T^2} \tag{5.45}$$

We know that at the boundary layer border convection = conduction. It allows us to calculate δ_T^2 equalizing the 2 previous equations

$$\frac{\delta_T^3}{\delta} = \frac{\delta_T^3}{LRe^{-\frac{1}{2}}} = \frac{\alpha L}{U} \Rightarrow Nu = \frac{L}{\delta_T} = Re^{\frac{1}{6}} Pe^{\frac{1}{3}} = Re^{\frac{1}{2}} Pr^{\frac{1}{3}} \qquad \Rightarrow \delta_T = \delta Pr^{-\frac{1}{3}} \tag{5.46}$$

Re « 1

Let's study that case where **only the thermal boundary layer exists**. Let's specify that if $Pe \ll 1, Nu \approx 1$ so convection can be neglected. If $Pe \gg 1$ we have to do a dimensional analysis. There is no momentum boundary layer so u varies linearly with $y \Rightarrow u = U\frac{\delta_T}{L}$ and the velocity v is obtained by **continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad v \approx U\frac{\delta_T}{L} \tag{5.47}$$

The convective heat transfer terms are of the same order of magnitude

$$u\frac{\partial T}{\partial x} \approx v\frac{\partial T}{\partial y} \approx \frac{U\Delta T\delta_T}{L^2} \tag{5.48}$$

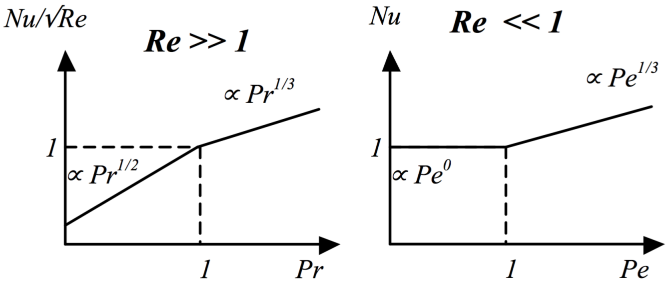
Only the transversal diffusion term is relevant ($\delta \ll L$)

$$\alpha\frac{\partial^2 T}{\partial x^2} \ll \alpha\frac{\partial^2 T}{\partial y^2} \approx \frac{\alpha\Delta T}{\delta_T^2} \tag{5.49}$$

We know that at the boundary layer border convection = conduction. It allows us to calculate δ_T^2 equalizing the 2 previous equations

$$\delta_T^3 = \frac{\alpha L^2}{U} \Rightarrow Nu = \frac{L}{\delta_T} = Pe^{\frac{1}{3}} = Re^{\frac{1}{3}}Pr^{\frac{1}{3}} \tag{5.50}$$

The 2 graphs below summarizes the 2 cases for Re number .



5.7 The Colburn-Chilton analogy

In forced convection, it is important to know the friction coefficient f (shear stress) and Nu number (heat transfer rates). It's so useful to have a relation linking the two. These relations are known as **Colburn-Chilton** analogy.

5.7.1 Relation between f and Nu

Let's begin with the expression of the friction coefficient. We will approximate the shear stress like $|\tau_{xy}|_{y=0} \approx \mu\frac{U}{\delta}$ giving

$$f = \frac{|\tau_{xy}|_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\mu|\frac{\partial u}{\partial y}|_{y=0}}{\frac{1}{2}\rho U^2} \approx \frac{\mu\frac{U}{\delta}}{\frac{1}{2}\rho U^2} \qquad \Rightarrow \qquad \delta = \frac{2\nu}{fU} \tag{5.51}$$

We will make appear the Re number ($Re = \frac{UR}{\nu}$) by dividing the previous equation by the radius

$$\frac{\delta}{R} = \frac{2}{fRe} \tag{5.52}$$

We will now express the Nusselt number (see subsection 5.3.1) differently, using the radius as distance L

$$Nu = \frac{hR}{k} = \frac{R}{\delta_T} = \frac{R}{\delta}\frac{\delta}{\delta_T} \qquad \Rightarrow \qquad Nu = \frac{fRe}{2}\frac{\delta}{\delta_T} \tag{5.53}$$

Colburn concluded his analysis saying that $\delta_T = \delta$ but we've seen it's not the case. Chilton added the dimensional analysis for $Pr \gg 1$ ($Pr > 0.5$ in application) and $Pr \ll 1$

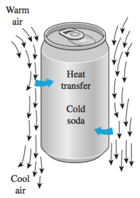
$$Pr \gg 1 : \frac{\delta}{\delta_T} = Pr^{\frac{1}{2}} \qquad and \qquad Pr \ll 1 : \frac{\delta}{\delta_T} = Pr^{\frac{1}{3}} \tag{5.54}$$

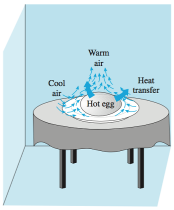
5.7.2 Applications

See slides 35 and 36 for lecture. All the mathematical expression are on the formula sheet.

5.8 Natural convection

5.8.1 Definition





It's probably something that we experience ourself everyday. In forced convection we have to put an external force (ex : impose a pump). This type of convection is not caused by a pomp in a pipe, there is no velocity gradient. Here it is function of the density gradient (a temperature difference or a non-homogeneous concentration field are origins of density changes).
Imagine that you have a coca can. The warmer air coming to the can will be cooled and directed below (due to air density change). This will create a natural convection process. It's the opposite for the egg.
This is the most complex process you can find. We cannot describe the one without the other (natural and forced).

5.8.2 The Navier-Stokes equations

We can adapt the Navier-Stokes equations to the case of a natural convection. For a fluid at temperature T, concentration c and ρ a function of (T, c)

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho g + \nabla \tau \tag{5.55}$$

we define the dynamic pressure P, in respect with a reference state. ρ_0 is a reference density calculated at a reference temperature and concentration

$$\nabla P = \nabla p - \rho_0 g \qquad with \qquad \rho_0 = \rho(T_0, c_0) \tag{5.56}$$

The resulting equation

$$\rho \frac{Dv}{Dt} = -\nabla P + (\rho - \rho_0) g + \nabla \tau \tag{5.57}$$

stresses[3] the role of the term responsible for the natural convection (density gradient).

The Boussinesq hypothesis

We will first assume that the density gradient is smaller than the reference density, leading to $\frac{\Delta\rho}{\rho_0} \ll 1$. We will secondly assume that the density varies linearly with temperature and solute concentration

$$\rho = \rho_0 - \rho\beta_T(T - T_0) - \rho\beta_c(c - c_0) \tag{5.58}$$

where β_T is the **temperature expansion coefficient** and β_c the **concentration expansion coefficient** given by

$$\beta_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}\right)_{p,c} \qquad and \qquad \beta_c = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{p,T} \tag{5.59}$$

The minus sign here is necessary because the density decreases with temperature and solute concentration.

3. Insister sur

Pure fluid

For a **pure ideal gas** there is no notion of concentration so

$$\beta_T = \frac{RT}{pM}\frac{pM}{RT^2} = \frac{1}{T} \qquad and \qquad \beta_c = 0 \tag{5.60}$$

The density changes so only in function of the temperature

$$\frac{\rho_0}{\rho} = 1 + \beta_T(T - T_0) \tag{5.61}$$

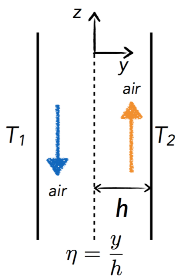
Knowing that the continuity equation gives us the information $\nabla v = 0$, the Navier-Stokes equations become

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v\nabla v = -\frac{1}{\rho_0}\nabla P - (T - T_0)\beta_T g + \nu\nabla^2 v \tag{5.62}$$

In addition to that, we have the energy conservation equation

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v\nabla T = \alpha\nabla^2 T \tag{5.63}$$

5.8.3 Channel flow



This is one of the few problems with an exact solution. Let's consider a channel of length $L = 2h$, infinite height, $v_z = v(y)$ and $T_z = T(y)$. There are 2 walls, one cold T_1 and the other hot $T_2 = T_1 + \Delta T$. The fluid in contact with the colder one will move down and the other up. Finally, there is also conduction in y direction respecting a linear T profile.

Let's now begin with the Fourier law and let's apply the boundary conditions

$$\frac{\partial^2 T}{\partial t^2} = 0 \qquad and \qquad \begin{cases} y = -h & T = T_1 \\ y = h & T = T_2 \end{cases} \tag{5.64}$$

The result is

$$\tilde{T}(\eta) = \frac{T - T_m}{\Delta T} = \frac{1}{2}\eta \qquad where \qquad T_m = \frac{T_1 + T_2}{2} \tag{5.65}$$

The Navier-Stokes equations in z direction

Let see what happens to Navier-Stokes equations. Here is the full version

$$\cancel{\left(u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial z}\right)} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - (T - T_0)\beta_T g + \nu\left(\frac{\cancel{\partial^2 v}}{\cancel{\partial z^2}} + \frac{\partial^2 v}{\partial y^2}\right) \tag{5.66}$$

We don't have variation of v in the directions y and z. We only have one diffusion term in y direction. Let's introduce the non dimensional velocity and length

$$\tilde{v} = \frac{v}{\frac{\nu}{h}} \qquad and \qquad \eta = \frac{y}{h} \tag{5.67}$$

Let's now do the modification to make appear these expressions

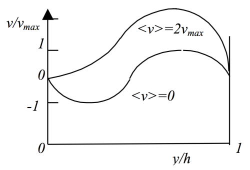
$$\frac{\nu^2}{h^3}\frac{d^2\tilde{v}}{d\eta^2} = \frac{1}{\rho(T_m)}\frac{dP}{dz} - \frac{1}{2}\eta\Delta T\beta_T g \qquad \Leftrightarrow \qquad \frac{d^2\tilde{v}}{d\eta^2} = \frac{h^3}{\nu^2\rho(T_m)}\frac{dP}{dz} - \frac{\eta}{2}\underbrace{\frac{h^3\Delta T\beta_T g}{\nu^2}}_{Gr} \tag{5.68}$$

where Gr is the **Grashof number**. Using the non- sleep condition at the wall ($\tilde{v} = 0$), we have the profile

$$\tilde{v}(\tilde{\eta}) = \frac{1}{12} Gr(\eta - \eta^3) - \frac{h^3}{2\nu^2 \rho(T_m)} \frac{dP}{dz}(1 - \eta^2) \tag{5.69}$$

valid only in the case of a **constant pressure gradient**. The first term is the **Buoyancy term**, a cubic velocity profile with zero mean velocity and the second is the **Pressure term**, the quadratic Poiseuille velocity profile. Extremely complex !

The mean velocity



We don't know the pressure gradient so we introduce the mean velocity to determine it

$$< \tilde{v} >= \frac{1}{2} \int_{-1}^{1} \tilde{v}(\eta)\, d\eta = -\frac{h^3}{3\nu^2 \rho(T_m)} \frac{dP}{dz} \tag{5.70}$$

If $< \tilde{v} >= 0$ the pressure definition contains an average buoyancy term, so $P = c$. If $< \tilde{v} >\neq 0$, it's the general case where the net flux in the z direction exists. The velocity profile is represented on the left.

5.8.4 Dimensional analysis

Let's see a little bit in terms of scaling how we behave in natural convection. A classic situation is the case of a body of temperature $T_0 = T_\infty + \Delta T$ and air around it with temperature T_∞. The buoyancy term can be equilibrated by viscous forces or inertial forces, depending on the Reynolds number. We remind the Navier-Stokes equations 5.62 and consider

- $Re \ll 1$: Buoyancy term equilibrated by viscous forces (channel flow problem)

$$\nu \frac{U_{cv}}{L^2} = g\beta_T \Delta T \qquad \Rightarrow \qquad U_{cv} = \frac{L^2 g \beta_T \Delta T}{\nu} \tag{5.71}$$

- $Re \gg 1$: Buoyancy term equilibrated by inertial forces

$$\frac{U_c^2}{L} = g\beta_T \Delta T \qquad \Rightarrow \qquad U_c = \sqrt{L g \beta_T \Delta T} \tag{5.72}$$

Non-dimensional forms of the Navier-Stokes equations for Re « 1

The non-dimensional terms are

$$U_{cv} = \frac{L^2 g \beta_T \Delta T}{\nu} \qquad \tilde{v} = \frac{v}{U_{cv}} \qquad \tilde{T} = \frac{T - T_\infty}{\Delta T} \qquad \tilde{P} = \frac{P}{\frac{\mu U_{cv}}{L}} \qquad \tilde{r} = \frac{r}{L} \tag{5.73}$$

Transforming the Navier-Stokes equations in (non-dimensional)

$$Gr\, \tilde{v} \cdot \tilde{\nabla} \tilde{v} = -\tilde{\nabla} \tilde{P} - \tilde{T} 1_g + \tilde{\nabla}^2 \tilde{v} \qquad where \qquad \tilde{\nabla} = L\nabla \qquad and \qquad 1_g = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{5.74}$$

The $Gr = \frac{L^3 \beta_T g \Delta T}{\nu^2}$. If $Gr \ll 1$ the inertial term can be omitted but it happens seldom. $Re \ll 1$ is equivalent to $Gr \ll 1$.

Non-dimensional forms of the Navier-Stokes equations for Re » 1

The non-dimensional terms are

$$U_c = \sqrt{Lg\beta_T \Delta T} \qquad \tilde{v} = \frac{v}{U_{cv}} \qquad \tilde{T} = \frac{T - T_\infty}{\Delta T} \qquad \tilde{P} = \frac{P}{\rho U_c^2} \qquad \tilde{r} = \frac{r}{L} \tag{5.75}$$

Transforming the Navier-Stokes equations in (non-dimensional)

$$\tilde{v}\tilde{\nabla}\tilde{v} = -\tilde{\nabla}\tilde{P} - \tilde{T}1_g + \frac{1}{Gr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{v} \qquad where \qquad \tilde{\nabla}\tilde{v} = 0 \qquad and \qquad \tilde{v}\tilde{\nabla}\tilde{T} = \frac{1}{PrGr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{T} \tag{5.76}$$

In this case, $Gr = \frac{L^3\beta_T g\Delta T}{\nu^2} = \left(\frac{U_c L}{\nu}\right)^2 = Re^2$. If $Gr \gg 1$ the viscous term can be omitted and the condition $Re \gg 1$ is equivalent to $Gr \gg 1$.

5.9 The natural convection thermal boundary layer

In natural convection, Re is replaced by Gr, so a momentum boundary layer will form when $Gr \gg 1$. At the edge of the boundary layer, convection and conduction are of the same order of magnitude. Using (5.76), we can approximate

$$\frac{1}{Gr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{v} \approx \frac{1}{Gr^{\frac{1}{2}}}\frac{1}{\tilde{\delta}^2} \approx \frac{1}{Gr^{\frac{1}{2}}}\frac{L^2}{\delta^2} \approx 1 \qquad \Rightarrow \qquad \frac{\delta}{L} \approx \frac{1}{Gr^{\frac{1}{4}}} \tag{5.77}$$

Result equivalent to the Blasius solution for momentum The velocity profile in natural convection is not approximately linear as in forced convection, due to the presence of a buoyancy term.

A thermal boundary layer of thickness δ_T is established when $PrGr^{0.5} \gg 1$ (equivalent to $Pe \gg 1$)

$$\tilde{v}\tilde{\nabla}\tilde{T} = \frac{1}{PrGr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{T} \tag{5.78}$$

- If $Pr \ll 1$, $\delta_T \gg \delta$, the fluid is outside the momentum boundary layer. Using

  $$\tilde{v} \approx 1 \qquad \tilde{v}\tilde{\nabla}\tilde{T} \approx 1 \tag{5.79}$$

  We approximate

  $$\frac{1}{PrGr^{\frac{1}{2}}}\frac{1}{\tilde{\delta}_T^2} \approx 1 \qquad \Rightarrow \qquad \tilde{\delta_T} = \frac{\delta_T}{L} = Gr^{-\frac{1}{4}}Pr^{-\frac{1}{2}} \tag{5.80}$$

  Result equivalent to the solution for forced convection.
- If $Pr \gg 1$, $\delta_T \ll \delta$, the fluid is inside the momentum boundary layer (velocity profile not known). For a given Gr, δ_T will decrease with Pr and v will decrease at the thermal boundary layer border

  $$\frac{\delta_T}{\delta} = Pr^{-a} \qquad \tilde{v} = Pr^{-b} \tag{5.81}$$

  At the border of thermal boundary layer, the viscous term equilibrates the driving force

  $$\frac{1}{Gr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{v} \approx 1 \Rightarrow \frac{1}{Gr^{\frac{1}{2}}}\frac{\tilde{v}}{\tilde{\delta}_T^2} \approx Pr^{2a-b} \approx 1 \qquad \Rightarrow \qquad 2a - b = 0 \tag{5.82}$$

  In addition to that

  $$\tilde{v}\tilde{\nabla}\tilde{T} \approx \tilde{v} \approx Pr^{-b} = \frac{1}{PrGr^{\frac{1}{2}}}\tilde{\nabla}^2\tilde{T} = \frac{1}{PrGr^{\frac{1}{2}}}\frac{1}{\tilde{\delta}_T^2} \approx Pr^{-1+2a} \qquad \Rightarrow \qquad 2a + b = 1 \tag{5.83}$$

  The combination of the two equations gives the values of $a = 0.25$ and $b = 0.5$.

Relation between the momentum and thermal boundary layer thickness

For $Gr \gg 1$

$$\frac{\delta_T}{L} \approx \frac{\delta}{L} Pr^{-\frac{1}{4}} = Gr^{-\frac{1}{4}} Pr^{-\frac{1}{4}} = Ra^{-\frac{1}{4}} \tag{5.84}$$

where Ra is the **Rayleigh number**

$$Ra = GrPr = \frac{L^3 g \beta_T \Delta T}{\nu^2} \frac{\nu}{\alpha} = \frac{L^3 g \beta_T \Delta T}{\nu \alpha} \tag{5.85}$$

Finally, the Nusselt number is given by

$$Nu = \frac{L}{\delta_T} = C \cdot Gr^{\frac{1}{4}} Pr^{\frac{1}{4}} = Ra^{\frac{1}{4}} \tag{5.86}$$