

UNIVERSITÉ LIBRE DE BRUXELLES

SYNTHÈSE

Fluid mechanics and transport processes
MECA-H-300

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Appel à contribution

Synthèse OpenSource



Ce document est grandement inspiré de l'excellent cours donné par Allessandro Parente à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

<https://github.com/nenglebert/Syntheses>

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer L^AT_EX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi le README contient de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

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Chapitre 1

Introduction

Before beginning the summary, I want to tell you that my English level isn't perfect. Please collaborate and correct the grammatically wrong sentences.

1.1 Reminder

The governing equations in transport processes are the followings :

- Mass conservation :

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \quad (1.1)$$

- Navier-Stokes :

$$\rho \left(\frac{Dv}{Dt} + v \nabla v \right) = -\nabla p + \mu \nabla^2 v \quad (1.2)$$

- Energy equation :

$$\frac{DT}{Dt} = \nabla(\alpha \nabla T) + \frac{\dot{Q}_v}{\rho c} \quad (1.3)$$

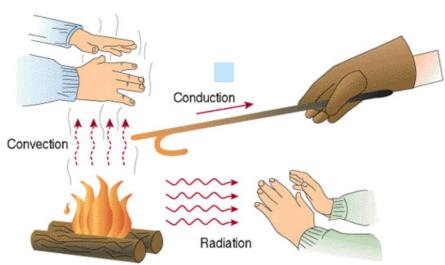
- Species conservation

$$\frac{\partial \rho_A}{\partial t} + \nabla(\rho_A v_A) = r_A \quad (1.4)$$

Let's precise that there are many applications using these equations like in the aerospace and automotive industry, in safety and fire prevention or in buildings design.

1.2 Convection and diffusion

1.2.1 Definitions

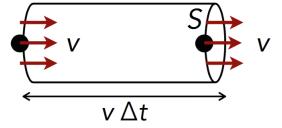


Here is a picture illustrating the principles of **convection**, **conduction** and **radiation**. Imagine that you have a fire and you put your hands above. You will feel a flow of heat transmitted by convection. If someone comes with a stick, there will be conduction in the material transmitting the energy from particles to particles. Finally, if the hands are next to the fire, there is no flow but you feel the heat. The energy is transmitted by radiation.

1.2.2 Convection

Convection is a transfer always associated to **bulk (ensemble) fluid motion**. We consider a fluid with **uniform** velocity and a cylinder of section S and lenght Δt . In that time interval, the fluid in the cylinder will have crossed the section S . We are now able to express the convective flux of momentum (quantité de mouvement), energy and mass knowing that the flux of a physical quantity is given by

$$flux_A = \frac{A}{S\Delta t} \quad (1.5)$$



Mass

We know that the mass is given by density \times volume and that the volume of the cylindre is $Sv\Delta t$. Using the new expression of the mass and (1.5), we can find the **flux of mass**

$$M_c = \rho v S \Delta t \quad and \quad J_{M_c} = \frac{M_c}{S \Delta t} = \rho v \quad (1.6)$$

Momentum

Similarly to the calcul of the mass

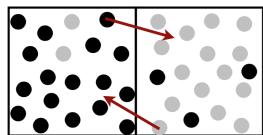
$$Q_c = mv = \rho v^2 S \Delta t \quad and \quad J_{Q_c} = \rho v^2 \quad (1.7)$$

Energy

The energy in the system is given by the specific heat energy of each particles¹. If T_0 is the reference temperature,

$$E_c = \rho c_v v (T - T_0) S \Delta t \quad and \quad J_{E_c} = \rho c_v v (T - T_0) \quad (1.8)$$

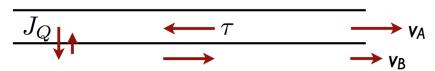
1.2.3 Diffusion



Diffusion is a transfer associated to the **particles random walk** and is due to the presence of a gradient of physical quantity (temperature for example). The picture on the left illustrates that, for an infinite time, the process will reequilibrate the gradient, difference between the two boxes.

1.3 Diffusive flux

If we consider two parallel fluid layers to bulk velocity $v_A > v_B$, the gradient of velocity will vanish² ($v_A = v_B$). This effect is due to the initial velocity gradient that cause the diffusion of faster particles towards the slower ones, transferring a momentum flux J_Q . In fact, the origin of the flux in the transversal direction to the flow is due to the **friction** between the two layers, parallel to the flow direction.



1. See *Chimie générale* for the expression

2. Disparaître

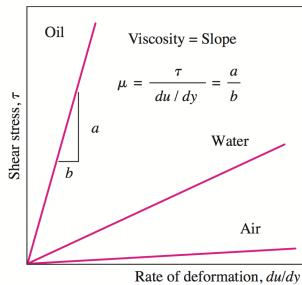
1.3.1 Shear stress

To characterize the friction between two layers, we introduce the **shear stress**³ that is proportional to the gradient of velocity

$$\tau = \tau \left(\frac{dv}{dy} \right) \quad (1.9)$$

According to the Newton's law, for newtonian fluids like gases and liquids, the equation becomes

$$\tau = \mu \left(\frac{dv}{dy} \right) \quad (1.10)$$

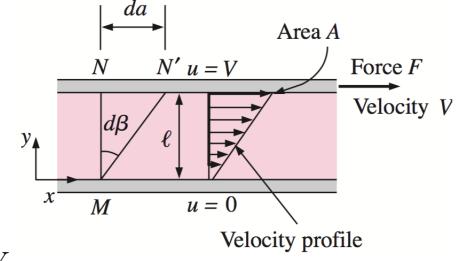


- μ is the **dynamic viscosity**, the **intrinsic resistance** of the fluid to motion [$kg/m.s$]
- dv/dy the gradient of velocity [s^{-1}]
- τ a force per unit surface [N/m^2] = [Pa]

If we see the shear stress to the **rate of deformation**⁴ on a graph, we can see that the **viscosity** is the slope⁵. We can also see that viscosity of air is lower than water that's lower than oil.

1.3.2 Planar Couette flow

Let's consider a fluid layer between two very large plates separated by a distance l . A constant parallel force F (drag force⁶) is applied to the upper plate and after the initial transients, it moves continuously to a constant velocity V . What are the consequences on the fluid?



- **Empirical observations :**

No slip conditions⁷ at walls $\rightarrow u(0) = 0$ and $u(l) = V$.

- **The flow is organized in parallel layers :**

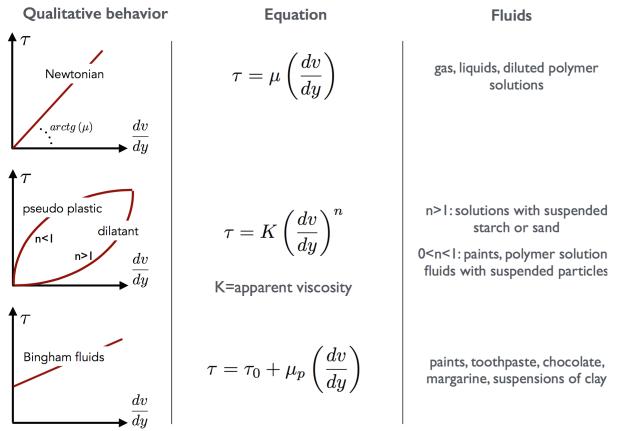
We suppose that the flow is laminar (no turbulence), so the velocity profile is a sole function of y and is **linear** : $u(y) = V \frac{y}{l}$.

- **The drag force is imposed :**

$$\tau = \frac{F}{A} = \mu \frac{dv}{dy} = c$$

3. Contrainte de cisaillement
 4. Gradient of velocity
 5. La pente
 6. Force de traînée
 7. Condition de non-glissement

1.3.3 Fluid rheology



Rheology is the study of the flow that establish the relation between the shear stress and the velocity gradient. Let's have a look to different fluids.

As first observation, we can see that newtonian fluids respect a linear relation for shear stress to the rate of deformation. It's not the case for others like second and third line of the table⁸. For bingham fluids, there is a critical shear stress to reach before the behavior becomes like newtonian fluids

Dynamic viscosity of liquids and gases

When we observe the table of dynamic viscosity for the water, the dynamic viscosity decreases with temperature increase for liquid water. It's not the case for the gases for which dynamic viscosity increases with temperature. See section 1.5.3 below.

1.4 Constitutive relations

It's an important section because it shows that transport processes have a uniform approach. Let's introduce 3 new variables : **kinematic viscosity** ν , **thermal diffusivity** α **internal energy** u **mass diffusivity** D . Let's precise that $\nu, \alpha, D = [L^2 T^{-1}]$

- **Momentum diffusion flux** (Newton's law)

$$J_{Q_d} = -\mu \frac{dv}{dy} \xrightarrow{\nu = \frac{\mu}{\rho}} J_{Q_d} = -\nu \frac{d(\rho v)}{dy} \quad (1.11)$$

- **Energy diffusive flux** (Fourrier's law)

$$J_{E_d} = -k \frac{dT}{dy} \xrightarrow{\alpha = \frac{k}{\rho c_v} \text{ and } u = c_v(T - T_0)} J_{E_d} = -\alpha \frac{d(\rho u)}{dy} \quad (1.12)$$

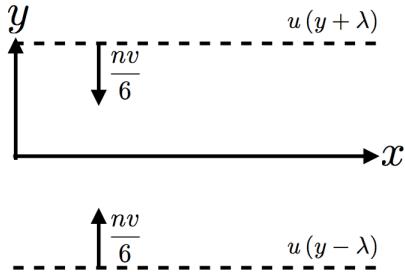
- **Mass diffusion flux** (Fick's law)

$$J_{M_d} = -D \frac{dc}{dy} \quad (c = \text{concentration}) \quad (1.13)$$

8. Starch = amidon

1.5 Gas viscosity

1.5.1 Another expression of viscosity



Determination of the dynamic viscosity - Maxwell, 1860

- I. n molecules per unit volume, each with mass m
- II. a fraction of $n/3$ moving in the y direction
- III. a fraction $n/6$ along $+y$ et $n/6$ along $-y$
- IV. $nv/6$ molecules crossing a unit area in both directions, per unit time
- V. $nv/6$ molecules with free mean path $\lambda = 1/\pi n d^2$
- VI. $nv/6$ molecules with momentum $m u(y+\lambda)$ et $m u(y-\lambda)$

Let's consider a gas moving in the x direction with a velocity $u = u(y)$. The **kinetic theory** gives us the random velocity in y using $v = \sqrt{\frac{3k_b T}{m}}$. The development of Maxwell, above, allows us to calculate the momentum flux crossing the x axis. $\frac{nv}{6}$ gives us the flux of particles for the y direction.

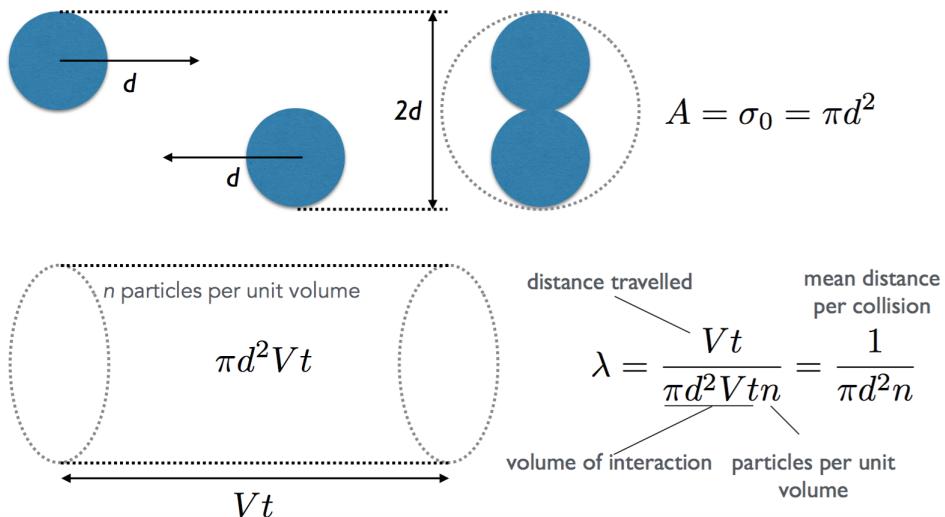
$$J_{Q_d} = \left(\frac{nv}{6} \right) mu(y - \lambda) - \left(\frac{nv}{6} \right) mu(y + \lambda) \quad (1.14)$$

Here we use $u(y - \lambda)$ and $u(y + \lambda)$ because the last particles that can cross an axis on y are those who are situated in a distance of maximum λ from there.

The Taylor expansion of the flux $u(y + \lambda) = u(\lambda) + \lambda \frac{du}{dy} + \dots$ and the constitutive equation (1.11) gives us the final expression of viscosity

$$\mu = \frac{1}{3} nm \lambda v = \frac{1}{3} \rho \lambda v = \frac{1}{\pi \sqrt{3}} \frac{\sqrt{mk_b T}}{d^2} \quad (1.15)$$

1.5.2 Free mean path



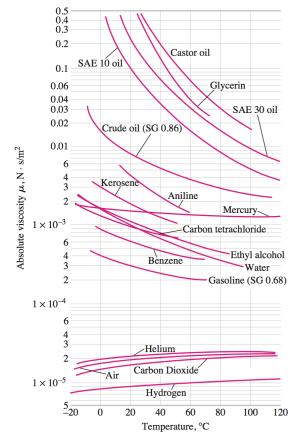
The slide above summarise the way to calculate the free mean path that is the average distance travelled by a particle between 2 collisions. First of all, we have to define a section of interaction. It's given by a circle of radius d . We also have to define a volume within the particles move. It's the cylinder of section A and length Vt . The calculus of λ is given by the distance travelled (without collision) divided by the volume where you can have interactions and by the particles per unit volume.

1.5.3 Viscosity variation with temperature

We have said that gases viscosity increased with temperature. It's due to the kinetic approach that gives us the equation (1.15) where $\mu \propto \sqrt{T}$.

For gases, the intermolecular forces are negligible. The increase of temperature makes particles move randomly at higher velocities conducting to more collisions per unit volume per unit time and so more resistance to flow.

For liquids, the increase of temperature decreases the viscosity because particles have higher energies that help them to oppose to the large cohesive intermolecular forces more strongly. They can thus move more freely.



1.6 Viscosity measurements

I HAVE TO ASK TO THE TEACHER IF WE HAVE TO DO THESE 3 NEXT SECTIONS BECAUSE I DON'T HAVE COMMENTS

1.7 Thermal conductivity

1.8 Mass diffusivity

1.9 Potential flow and boundary layer

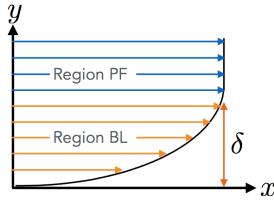
1.9.1 Main aspects

- The behavior of a moving fluid element strongly depends on the **interactions** between the **fluid elements** and the **walls** for both internal and external flows.
- Far from the walls, the shear stress and the dissipative forces are negligible. The fluid behaves **ideally**, is **incompressible** and **non-dissipative**. It's called a **potential flow**.
- Potential flow are fully described by **Newtonian mechanics**, using the **conservation laws** for mass and mechanical energy, without any **dissipation** of mechanical **energy into heat**.
- Close to the walls, the fluid velocity is **suddenly modified**, thus introducing **dissipative phenomena** that cannot be described using the potential flow theory.
- The fluid layer thickness⁹ affected by the wall is named **boundary layer**¹⁰ (as introduced by Prandtl, 1904).
- Within the boundary layer, the velocity gradients in the flow normal direction as well as the shear stress in the direction parallel to the flow become extremely important.

9. Epaisseur

10. Couche limite

1.9.2 Representation of the boundary layer



We have to delimit two regions. The one (PF) within we have the **potential flow** that conserve his kinetic and potential energy (no dissipation) and the other (BL) within we have the **boundary layer** of thickness δ , a velocity gradient with shear stress and dissipation of mechanical energy into heat. In the BL region appear two forces, a **drag** force that is due to friction, parallel to the flow and **lift** force that is a resistance perpendicular to the flow.

In order to characterize the flow, we introduce the **Reynolds number**

$$Re = \frac{\text{Convection}}{\text{Diffusion}} = \frac{J_{Q_c}}{J_{Q_d}} = \frac{\rho v^2}{\mu \frac{v}{L}} = \frac{vL}{\nu} \quad (1.16)$$

that compares the convective and diffusive effect of the fluid momentum and the **Peclet number**

$$P_{et} = \frac{J_{E_c}}{J_{E_d}} = \frac{\rho c_v v (T - T_0)}{k \frac{T}{L}} = \frac{vL}{\alpha} \quad \text{and} \quad P_{em} = \frac{J_{M_c}}{J_{M_d}} = \frac{vL}{D} \quad (1.17)$$

that compares the convective and diffusive effect of the fluid energy and mass. Let's precise that if a uniform fluid flows towards a plate, we will have an **inviscid** flow region away from the plate and a **viscous** flow region next to the plate.

1.9.3 Flow classification and friction force

The classification is based on the Reynolds number. We have two types of fluid :

- **Re $\gg 1$** \rightarrow **turbulent flow**, convection > diffusion.

In that case, the fluid resistance (drag force) is independent of the viscosity and is proportional to the momentum flux

$$F \approx \rho v^2 S \quad (1.18)$$

- **Re $\ll 1$** \rightarrow **laminar flow**, convection < diffusion.

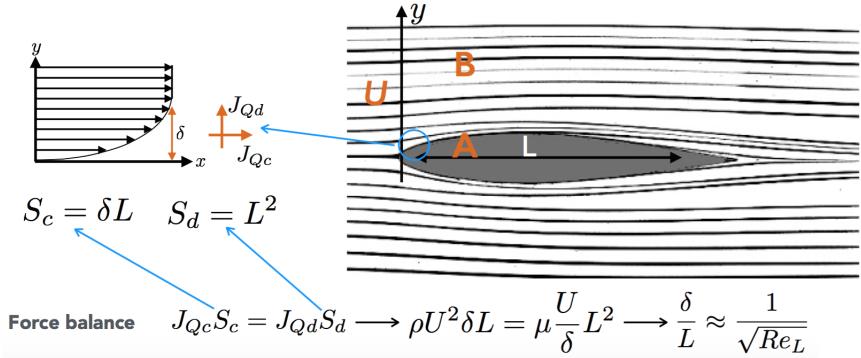
In that case, the fluid resistance is a function of the viscosity and is proportional to the diffusive momentum flux

$$F \approx \mu Lv \quad (1.19)$$

1.9.4 Determination of the boundary layer thickness

The separation between the two regimes above is not possible. Consider a wind flow around a building. The Reynolds number will be high meanig the flow is turbulent. Indeed, the **approaching flow** can be classified as a potential flow, whereas **close to the walls** the dissipative effects will take place with the conversion of mechanical energy into heat.

So, far from the walls the dominant transport mechanism will be convection and close to the walls viscous transport. At a distance equal to the boundary layer thickness, **convection and diffusion will be comparable**. This allows us determining its **order of magnitude**.



We can qualitatively find δ as the distance from the wall at which the convective momentum flux (**parallel** to the flow) equals the diffusive one (**perpendicular** to the flow). For that, let's take a section within convection dominate S_c and another for diffusion S_d . When we equalize the two flux expressed respectively for convection and diffusion, we arrive to the magnitude of δ compared to L .

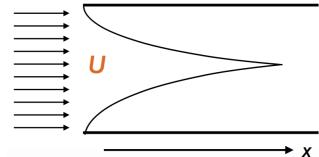
1.9.5 Features of the boundary layer and practical consequences

The relative dimension of the boundary layer is inversely proportional to $Re^{0.5}$ and the absolute dimension is proportional to $L^{0.5}$.

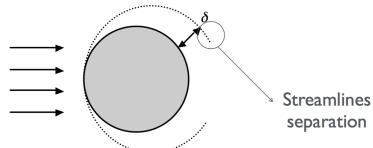
Consequences for internal flows

The boundary layer thickness increases with $x^{0.5}$ and far from the inlet¹¹, the boundary layer will extend to the full diameter pipe.

$$\frac{\delta}{x} \approx \frac{1}{\sqrt{Re_x}} \quad \text{and} \quad \delta \approx \sqrt{\frac{xv}{U}} \quad (1.20)$$



Consequences for external flows



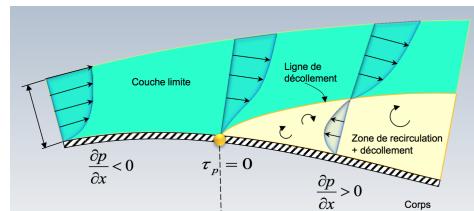
The boundary layer cannot infinitely grow. It will be thinner in front of the body and thicker behind it. Behind the body, the flow streamlines detach. If the boundary layer thickness is known, the force exerted by the flow can be directly evaluated, being the velocity gradient of the order $\frac{U}{\delta}$, which gives a shear stress

of $\tau = \mu \frac{U}{\delta}$ on the walls. The drag force will be then higher in front of the body (where the boundary layer is thinner) and lower behind it (where the boundary layer is thicker). From symmetry considerations, the drag force only acts in the flow direction.

More commonly, a perpendicular component also exists called **lift** but here it doesn't appear.

$$F = \int_S \mu \frac{dv}{dy} dS \quad (1.21)$$

Here's an illustration of the flow separation. There is a **recirculation** region which provides a force opposed to the flow. In order to maximise the body speed, it is necessary to reduce the drag force due to the flow separation.



11. La prise - l'entrée

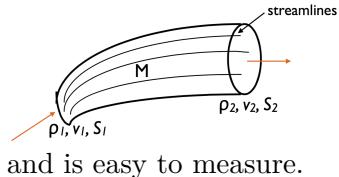
Chapitre 2

Macroscopic and microscopic balances

2.1 Conservation principles

The basic conservation principles in fluid mechanics are the conservation of **mass**, **energy** and **momentum**. These conservations are the basis of **continuity**, **Bernoulli** and **Navier-Stokes equations** and can be written in **integral** (finite volume of a moving fluid) or **local** (balances in differential form) form.

2.1.1 Macroscopic mass balance and continuity equation



and is easy to measure.

In a steady¹ flow, the mass balance simply states that the mass entering is equal to the one leaving a volume. But we have to consider the variation of mass for transient² problems. We define for that the **mass flow rate** (débit) that gives the mass per time

$$\dot{m} = \rho v S \quad [\dot{m}] = \frac{[M]}{[T]} \quad (2.1)$$

General balance

Let's write what we explained, the mass variation is given by the mass entering at S_1 minus mass leaving at S_2

$$\frac{dM}{dt} = \left(\rho \int_S v dS \right)_1 - \left(\rho \int_S v dS \right)_2 \quad (2.2)$$

To simplify, we introduce the **average velocity** $\bar{v} = \frac{1}{S} \int_S v dS$ giving the final expression

$$\frac{dM}{dt} = (\rho \bar{v} S)_1 - (\rho \bar{v} S)_2 \quad (2.3)$$

Steady balance

When the fluid is steady, the velocity and the surface are constant in both 2 sections conducting to a constant mass

$$\dot{m} = \rho \bar{v} S = c \quad (2.4)$$

1. Continu, constant.
2. Transitoire.

Streamlines

Let's first define what's a streamline. It's a curve that is everywhere tangent to the instantaneous local velocity vector and so an indicator of the fluid direction. Mathematically, if our velocity vector is

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \quad (2.5)$$

we can take an infinitesimal arc length along a streamline

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \quad (2.6)$$

Due to the 2 similar triangles, we have the relations (2.7) that gives the 2D equation of a streamline.

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \Rightarrow \quad \left(\frac{dy}{dx} \right) \text{ along a streamline} = \frac{v}{u} \quad (2.7)$$

2.1.2 Macroscopic momentum balance

Unlike the mass, momentum can be created and destroyed due to applied forces. The difference between the momentum entering in the control volume at S_1 and leaving at S_2 is equal to the sum of the forces applied to the fluid element.

The momentum flow through a section per unit time is given by

$$\dot{Q} = \dot{m}v = \rho\bar{v}^2S \quad (2.8)$$

and the forces acting on the fluid element (positive if exerted by the fluid) are :

- **pressure** that is positive at the entry (the fluid push to enter) and negative at the exit (the fluid is pushed).
- **forces on the walls** (normal and tangential). Negative because it's the reaction of the walls.
- **Gravity and other volume forces**. Positive because the forces are due to the presence of a fluid volume.

It allows us to write the equation, considering that the surface is a vector of module equal to the scalar surface and of same direction of the normal to the surface.

$$\sum F = p_1S_1 - p_2S_2 - F_w + F_g \quad (2.9)$$

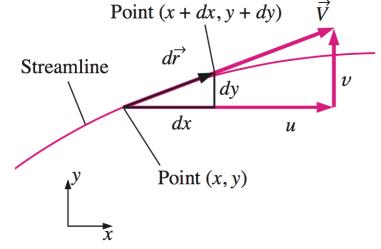
Steady momentum balance

It's simply the use of equations (2.8) and (2.9)

$$\dot{Q}_2 - \dot{Q}_1 = \sum F \quad \Leftrightarrow \quad \dot{m}_2v_2 - \dot{m}_1v_1 = p_1S_1 - p_2S_2 - F_w + F_g \quad (2.10)$$

We can use equation (2.4) and have a compact final relation

$$\Delta_1^2(\rho\bar{v}S + p) = -F_w + F_g \quad (2.11)$$



General momentum balance (transient)

To consider the case within the velocity is not constant into the 2 sections, we have to remind the mechanical relation for the resultant and do an infinitesimal decomposition in volume V

$$\frac{dR}{dt} = \sum F \quad \Leftrightarrow \quad \frac{d}{dt} \int_V \rho v dV = (\rho \bar{v}^2 + p)_1 S_1 - (\rho \bar{v}^2 + p)_2 S_2 - F_w + F_g \quad (2.12)$$

$$= \Delta_1^2(\rho \bar{v}^2 + p)S - F_w + F_g$$

You can find an example of exercice on slide 8 and 9 where we study the forces on an elbow³. Keep in mind that we often **consider a steady-state flow** to simplify $d/dt = 0$.

2.1.3 Local form of the conservation principles

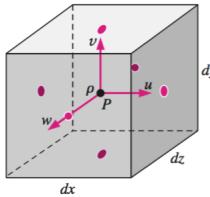
Macroscopic balances

They are used to have a general view an a volume, when we are interested in the **overall features**⁴.

Microscopic balances

They consist on differential and local equations and are valid on every point of the fluid to give the velocity, density or pressure.

2.2 Continuity equation



The continuity equation characterise the net rate of change of mass in a control volume. We start from the fact that this rate is equal to the net rate at which mass flows through the volume.

The variation of mass in function of time in an infinitesimal volume is given by

$$\partial M = \frac{\partial \rho}{\partial t} \partial x \partial y \partial z \quad (2.13)$$

If we do the difference between the enter and the exit of the flow in the volume following axis x, we have ($\rho v S$)

$$\rho u \partial y \partial z - \left(\rho u + \frac{\partial(\rho u)}{\partial x} \partial x \right) \partial y \partial z = - \frac{\partial(\rho u)}{\partial x} \partial x \partial y \partial z \quad (2.14)$$

and using the same way for the other axis

$$- \frac{\partial(\rho v)}{\partial y} \partial x \partial y \partial z \quad \text{and} \quad - \frac{\partial(\rho w)}{\partial z} \partial x \partial y \partial z \quad (2.15)$$

We take care of the contributions in all direction and simplify the infinitesimal volume in each term

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \quad (2.16)$$

3. Coude

4. Caractéristiques générales

Where the final v is global and not specific to an axis. Let's remind that the material derivative is defined as $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + v\nabla\rho$, we have

$$\frac{D\rho}{Dt} + \rho\nabla v = 0 \quad (2.17)$$

2.3 Momentum equation

2.3.1 Cauchy equation

To solve this rapidly we use the Langrangian approach. We suppose that the control volume is moving with the fluid and contain a fixed mass. The momentum rate of change is equal to the net force acting on the control volume (volumetric and superficial)

$$\frac{D}{Dt} \int_{V_m(t)} \rho v dV = \int_{V_m(t)} \rho g dV + \int_{S_m(t)} f dS \quad (2.18)$$

f is a function of the position r on the surface defined by the unit vector n perpendicular to the surface and pointing outward. f can be expressed as the scalar product between the tensor T and normal n

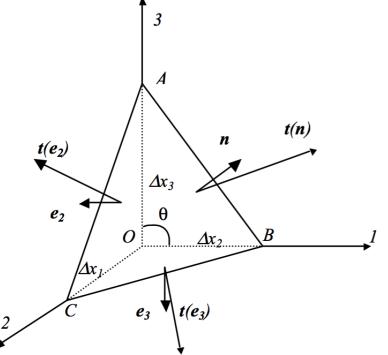
$$f(n, r) = nT(r) \quad (2.19)$$

Let's consider a tetrahedron on the axis x_1, x_2, x_3 and the forces exerted by the surrounding fluid. The force balance can be expressed

$$\begin{aligned} f(n)S_{ABC} &= f(e_1)S_{AOB} + f(e_2)S_{AOC} + f(e_3)S_{BOC} \\ &= f(e_1)S_{ABC}ne_1 + f(e_2)S_{ABC}ne_2 + f(e_3)S_{ABC}ne_3 \end{aligned} \quad (2.20)$$

if we do a factorisation and we define the tensor T as

$$T = e_1 f(e_1) + e_2 f(e_2) + e_3 f(e_3) \quad (2.21)$$



we find the equation (2.19). We can so write the equation (2.18) with this expression and use the **divergence theorem**

$$\int_S nT dS = \int_V \nabla T dV \quad (2.22)$$

we obtain the equation

$$\int_V \left[\frac{\partial(\rho v)}{\partial t} + \nabla(\rho v v) - \nabla T - \rho g \right] dV = 0 \quad (2.23)$$

Considering that it must be valid for any and every point of the volume, we get the **Cauchy equation**⁵

$$\rho \frac{Dv}{Dt} = \nabla T + \rho g \quad (2.24)$$

5. We can get ρ out of the material derivative because we consider a constant ρ

2.3.2 Constitutive equations

Static conditions

Imagine that we have a static fluid. The only force f will be due to the pressure p

$$f = -pn \Leftrightarrow T = -pI = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \quad (2.25)$$

In that case, the use of equation (2.24) give us the information

$$\rho g = \nabla p \quad (2.26)$$

Dynamic conditions

Let's now take care of the viscosity effects on a Newtonian fluid. In that case we have to add to the pressure the forces function of the velocity gradient

$$T = -pI + f(\nabla v) \quad (2.27)$$

where (∇v) has a symmetrical (pure deformations) and an antisymmetrical (pure rotation) part given by

$$S = \frac{1}{2}(\nabla v + \nabla v^T) \quad \text{and} \quad A = \frac{1}{2}(\nabla v - \nabla v^T) \quad (2.28)$$

Final form of the stress tensor

Given the upper discussion, the final expression is

$$T = [-p + \lambda(\nabla v)] I + 2\mu S \quad (2.29)$$

where S is the symmetric part of the gradient. Let's precise that for **incompressible flows**, we have

$$T = -pI + 2\mu S \quad (2.30)$$

Navier-Stokes equation

THÉORÈME : NAVIER-STOKES EQUATION

For incompressible, Newtonian and isothermal flows

$$\rho \frac{Dv}{Dt} = \rho \left[\frac{\partial v}{\partial t} + v \nabla v \right] = -\nabla p + \mu \nabla^2 v + \rho g \quad (2.31)$$

Where we used the Cauchy equation (2.24) and equation (2.30) for T .

2.4 Bernoulli equation

2.4.1 Without friction effects and energy transfer

To access the Bernoulli equation, let's take the Navier-Stokes equation (2.31) and disregard the viscous term (with μ). The scalar multiplication of that equation provides the **kinetic energy balance**

$$\rho v \frac{Dv}{Dt} = \rho \frac{D}{Dt} \frac{v^2}{2} = -v \nabla(p + \rho \Phi) \quad \text{with} \quad \Phi = g \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \quad (2.32)$$

For incompressible and steady fluids, $\frac{d\rho}{dt} = 0 = \frac{dp}{dt}$, so

$$\frac{D}{Dt} \left(\rho \frac{v^2}{2} + p + \rho g z \right) = 0 \quad (2.33)$$

leading us to the Bernouilli equation after integration.

THÉORÈME : BERNOUILLI EQUATION

The sum of the kinetic, potential and flow energies of a fluid particle is **constant** along a streamline during **steady** flow when the **compressibility** and **frictional effects** are negligible.

$$\rho \frac{v^2}{2} + p + \rho g z = c = p_{tot} \quad (2.34)$$

- $\frac{v^2}{2}$ is the **dynamic pressure** : pressure rise when the fluid in motion is brought to a stop isentropically.
- p is the static pressure : thermodynamic pressure of the fluid.
- $\rho g z$ is the **hydrostatic pressure** : for the elevation effects.

Finally, we can say that the sum of these 3 type of pressure is the **total pressure** and that it's constant along a streamline.

Limitations of the Bernouilli equation

To use Bernouilli equation we have to make some hypothesis :

- **Steady flow** : cannot be used for transients (start-up and shut-down) flow or with changes in flow conditions.
- **Frictionless flow** : cannot be used when there are components disturbing the flow.
- **No shaft work** : cannot use in presence of pump, turbine, etc (destroy the streamline structure).
- **Incompressible flow** : cannot be used when the density changes. Liquids and gases (Mach numbers < 0.3) are ok.
- **No heat transfer** : the increase of temperature influe on the density.
- **Along a streamline** : the constant c is the same for all streamlines if the flow is irrotational.

Many applications exists like the the Pitot tube, the cavitation or the discharge of a fluid from a tank.

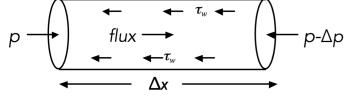
2.4.2 With friction effects and energy transfer

We have to generalize the Bernouilli equation. To take care of the friction forces that dissipate the mechanical energy into heat, we introduce a **pressure drop** ΔP_{tot} for **incompressible fluids**. Friction is an **irreversible process** and this forces are always positive (for potential flow = 0).

When pumps and turbines are used, we introduce an **energy transfer by work** w_p and w_t . The contributions for pumps are positive whereas they are negative for turbines

$$\rho \frac{v_1^2}{2} + p_1 + \rho g z_1 + w_p - w_t - \Delta P_{tot} = c \quad (2.35)$$

2.5 Distributed pressure drops



We want to determine the pressure drops for internal flows. Let's begin with the energy balance

$$\rho \frac{v_2^2}{2} + p_1 + \rho g z_1 + w_p = \rho \frac{v_2^2}{2} + p_2 + \rho g z_2 + w_t + \Delta P_{tot} \quad (2.36)$$

After simplifying the relation, we find that

$$\Delta p = \Delta P_{tot} \quad (2.37)$$

Let's now do the force balance consisting on the pressure force at the entry, the exit and the shear stress whithin the volume

$$\sum F = \pi R^2 p - \pi R^2 (p - \Delta p) - (2\pi R \Delta x) \tau_w = 0 \quad (2.38)$$

After simplifying the terms and isolating the ratio, we can use the relation (2.37)

$$\frac{\Delta p}{\Delta x} = \frac{2\tau_w}{R} = \frac{4\tau_w}{D} \quad \Rightarrow \quad \Delta P_{tot} = \frac{2\tau_w}{R} = \frac{4\tau_w}{D} L \quad (2.39)$$

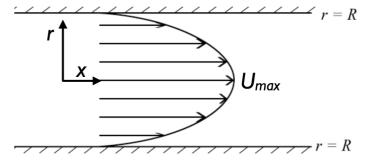
Let's finally introduce the **Moody coefficient** λ and the **Fanning coefficient** f to obtain the final expression

$$\lambda = 4f = 4 \frac{\tau_w}{\rho v^2} \quad \Rightarrow \quad \Delta P_{tot} = \lambda \frac{L}{D} \frac{\rho v^2}{2} \quad (2.40)$$

2.6 Newtonian flow in a pipe

2.6.1 Case of a laminar regime ($Re < 2100$)

Let's take the Nvier-Stokes equation (2.31) and suppose that there is no gravity effect on the fluid and that the flow is **steady**.



$$\frac{\partial u}{\partial t} = -\nabla p + \mu \nabla^2 v + \rho g \quad \Rightarrow \quad 0 = -\underbrace{\frac{\partial p}{\partial x}}_{f(x)} + \underbrace{\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)}_{f(r)} \quad (2.41)$$

The pressure pressure drop per unit lenght $\frac{\Delta p}{L}$ must be constant because of the steady regime and the sections are equivalent

$$\frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = c = -\frac{\Delta p}{\mu L} \quad (2.42)$$

The non-slip conditions on the walls ($r = R$) impose that $U = 0$ and the maximum velocity at $r = 0$ impose that $\frac{du}{dr} = 0$. The integration part per part gives

$$d \left(r \frac{\partial u}{\partial r} \right) = -\frac{\Delta p}{\mu L} r dr \quad \Rightarrow \frac{\partial u}{\partial r} = -\frac{\Delta p}{\mu L} \frac{r}{2} + \frac{C_1}{r} \quad \Rightarrow u(r) = -\frac{\Delta p}{\mu L} \frac{r^2}{4} + C_1 \ln r + C_2 \quad (2.43)$$

The upper conditions give the value of constants $C_1 = 0$ and $C_2 = \frac{\Delta p}{\mu L} \frac{R^2}{4}$.

It enable us to calcul the :

- velocity profile

$$U = \frac{\Delta p}{16\mu L} D^2 \left(1 - \left(\frac{r}{R} \right)^2 \right) \quad (2.44)$$

- Maximum velocity

$$U_{max} = \frac{\Delta p}{16\mu L} D^2 \quad (2.45)$$

- Volumetric flow rate

$$\dot{V} = \int_0^r 2\pi r u(r) dr = \frac{\pi}{128\mu} \frac{\Delta p D^4}{L} \quad (2.46)$$

- Mean speed (vitesse moyenne)

$$\frac{\dot{V}}{S} = \frac{\pi}{128\mu} \frac{\Delta p D^4}{L} \frac{4}{\pi D^2} = \frac{U_{max}}{2} \quad (2.47)$$

- Wall shear stress

$$\tau_{max} = -\mu \left| \frac{du}{dr} \right|_{r=R} = \frac{4\mu \bar{U}}{R} \quad (2.48)$$

- Friction coefficient

$$f = \frac{\lambda}{4} = \frac{4\mu \bar{U}}{\frac{1}{2}\rho \bar{U}^2 R} = \frac{8\mu \bar{U}}{\frac{1}{2}\rho \bar{U}^2 D} = \frac{16}{Re} \quad (2.49)$$

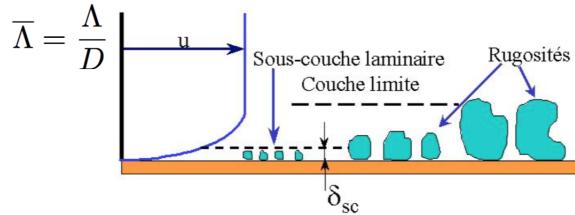
2.6.2 Case of a turbulent regime ($Re > 3500$)

We have to consider two case in that case. When the walls are smooth⁶, the Moody coefficient is given by

$$\lambda = \frac{0.316}{Re_D^{0.25}} \quad (2.50)$$

When the walls are rough⁷, we have the **Nikuradse equation** and the **Colebrook equation**

$$\frac{1}{\sqrt{\lambda}} = 1.14 - 0.868 \ln \bar{\Lambda} \quad \text{and} \quad \frac{1}{\sqrt{\lambda}} = 1.14 - 0.868 \ln \left(\bar{\Lambda} + \frac{9.34}{Re \sqrt{\lambda}} \right) \quad (2.51)$$



I REALLY DON'T HAVE UNDERSTOOD ! PLEASE COMPLETE WITH YOUR NOTES.

2.7 Concentrated pressure drops

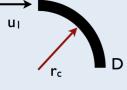
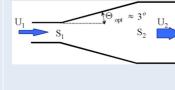
They are caused by inlets and outlets, curves, change in the cross section, offtakes⁸, ... They are modelled similarly to the distributed pressure drops with the **lack-of-knowledge coefficient** K that differs by geometries and configurations

$$\Delta P_{tot,c} = \sum_j K_j \frac{\rho U_j^2}{2} \quad (2.52)$$

6. Lisse

7. Rugueux

8. Prélèvements

Elbow	Contraction	Diffuser																																										
																																												
<table border="1"> <thead> <tr> <th>r/D</th><th>K</th></tr> </thead> <tbody> <tr><td>1</td><td>0.35</td></tr> <tr><td>2</td><td>0.19</td></tr> <tr><td>4</td><td>0.16</td></tr> <tr><td>6</td><td>0.21</td></tr> <tr><td>8</td><td>0.28</td></tr> <tr><td>10</td><td>0.32</td></tr> </tbody> </table>	r/D	K	1	0.35	2	0.19	4	0.16	6	0.21	8	0.28	10	0.32	<table border="1"> <thead> <tr> <th>D2/D1</th><th>K</th></tr> </thead> <tbody> <tr><td>0</td><td>0.5</td></tr> <tr><td>0.2</td><td>0.49</td></tr> <tr><td>0.4</td><td>0.42</td></tr> <tr><td>0.6</td><td>0.27</td></tr> <tr><td>0.8</td><td>0.2</td></tr> <tr><td>0.9</td><td>0.1</td></tr> </tbody> </table>	D2/D1	K	0	0.5	0.2	0.49	0.4	0.42	0.6	0.27	0.8	0.2	0.9	0.1	<table border="1"> <thead> <tr> <th>D2/D1</th><th>K</th></tr> </thead> <tbody> <tr><td>0</td><td>1</td></tr> <tr><td>0.2</td><td>0.87</td></tr> <tr><td>0.4</td><td>0.7</td></tr> <tr><td>0.6</td><td>0.41</td></tr> <tr><td>0.8</td><td>0.15</td></tr> <tr><td>0.9</td><td>0.1</td></tr> </tbody> </table>	D2/D1	K	0	1	0.2	0.87	0.4	0.7	0.6	0.41	0.8	0.15	0.9	0.1
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Here you can find some values for different situations.

We can now establish the total pressure drops expression

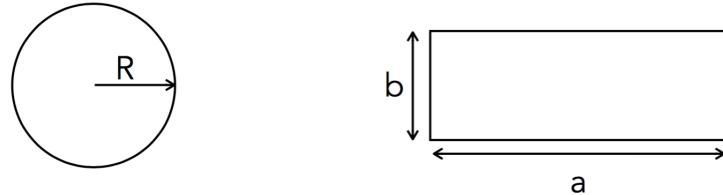
$$\Delta P_{tot} = \Delta P_{tot,d} + \Delta P_{tot,c} = \sum_i \lambda_i \frac{L_i}{D_i} \frac{\rho U_i^2}{2} + \sum_j K_j \frac{\rho U_j^2}{2} \quad (2.53)$$

Hydraulic diameter

To evaluate the Reynolds number we introduce the hydraulic diameter defined by

$$D_h = 4 \frac{S}{P} \quad (2.54)$$

Where S is the surface, P the perimeter and the value 4 is introduced to retrieve the normal diameter for a circular section. Let's calcul this diameter.



$$D_h = 4 \frac{\pi R^2}{2\pi R} = D \quad \text{and} \quad D_h = 4 \frac{ab}{2a+2b} \approx 2b \quad (2.55)$$

Finally, we can find the Reynolds number by

$$Re_h = \frac{D_h U}{\nu} \quad (2.56)$$

Chapitre 3

Heat transfer

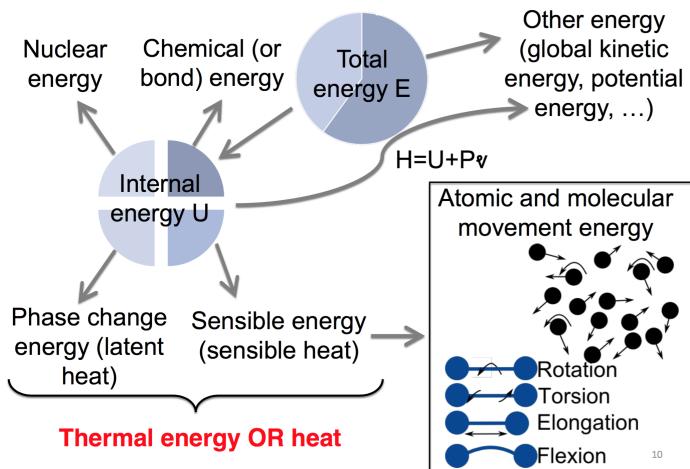
3.1 Introduction

We retrieve heat transfer in many phenomena like the combustion of coal that we use for the barbecue, the steel production or in power plants. But thermodynamics don't tell us how fast combustion will take place, chemistry tells us it will go too fast and each reaction needs its own reaction rate.

Heat transfer is important in the duration of many processes. It's important to design and understand that principle for engineers.

3.1.1 Definitions

Energy



In a system you have the total energy and is divided in two : internal energy and the rest. The first is the energy you find in the system if you attach yourself to the system. Thermal energy or heat includes the **latent heat** that is the phase change energy and the **sensible heat** that is associated to the atomic move inside the material.

Temperature

It's an interesting principle because it's not simple to define it. You can have thermodynamic definition of temperature : linked to the entropie (system is changing) and microscopic theory linked to the move of particles.

The important for us is that temperature is an intensive quantity correlated to the internal energy of the system. It's a universal feature of matters so it's independent of the material. Temperature is linked to the system energy by the specific heat

$$dU = mc_v dT \quad \text{and} \quad dH = mc_p dT \quad (3.1)$$

Should remember that gas of low pressure and low velocity can be considered as incompressible. In that case

$$c_v = c_p = c \quad (3.2)$$

Heat flux

With 2 body at different temperatures in contact, a quantity $Q(J)$ of heat is transferred through a surface of area A . Heat transfer rate and heat transfer flux density are defined as

$$\dot{Q} = \frac{dQ}{dt} \quad \Rightarrow \quad \dot{q} = \frac{\dot{Q}}{A} \quad (3.3)$$

3.1.2 Heat balance

The idea is to see what heat induct. Heat is just a part of energy in the system, only the total energy is conserved. What we will do is make balances on a control volume but we will divide the energy in two parts (heat and the rest). We will take into account the heat flux entrance and exit on a control volume surface, the generation or the consumption and the accumulation :

$$\text{accumulation} + \text{out fluxes} + \text{sinks} = \text{sources} + \text{in fluxes} \quad (3.4)$$

Sources and sinks can be due to the transformation of heat energy into a non thermal energy (sinks) or vice versa (source).

We can divide the situation in 2. The one where we accumulate and the other where we don't. The first case corresponds to stationary systems. The difference between **stationnary** and **equilibrium** is that the second is a particular stationary states where we have no flux whereas they are present in a stationnary flux.

3.1.3 Heat transfer mechanisms

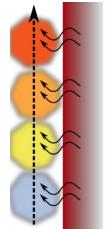
Conduction

Universal heat transfer by contact. It appears in matter when molecules enter in contact with each others and tranfer heat. The entropic transport of heat is quite slow. Universal because it can happen everywhere we have matter and is the main heat transfer. Solid materials are characterized as good heat conductor or not, so it's a material property.

Convection

Convection is linked to the movement of the particles but they carry they own heat. During this movement conduction occurs : convection = conduction + advection (illustrated on the picture). It's the major tranfer in presence of flows. Convection is complicated, it depends on how moves the fluid, its direction, ... (dependent).

We have to do the difference between forced and natural convection. For forced convection, the flow exists even if there is no heat transfer whereas for natural convection the flow is due to the heat transfer.

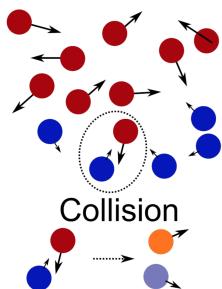


Radiation

It's the electromagnetic waves heat transfer between surfaces far from each other. No need to have matter, it's important to have a large temperature difference. It's more complex mathematically but we often neglect it.

3.2 Conduction

3.2.1 Microscopic approach



Temperature is an indicator of the particles movement. Hot body are composed of particles faster than cold bodies and when they are placed in contact, collisions occur, leading the hotter particle to transfer kinetic energy and so heat to the colder one. More high is the temperature gradient, the more the transfer will be fast. This transfer process dure until the egalization of temperature and the quantity transferred depends on the temperature, the chemical compositions and the densities.

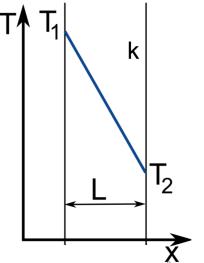
3.2.2 Phenomenological approach - the first Fourier law

Let's take an infinite wall of thickness L . We fix two temperature and we look at the temperature profile of the system. The amount of heat transfer will be proportional to the termal conductibility $k [W/m.K]$ and the heat flux density is

$$\dot{q} = k \frac{T_1 - T_2}{L} \quad (3.5)$$

We can consider the thickness equal to $L = -dx$ giving

$$\dot{q} = -k \frac{dT}{dx} \quad (3.6)$$



3.2.3 Thermal conductivity and diffusivity

Thermal diffusivity

k is mainly dependent of the materials properties but includes the thermal capacity. We can so express a value called **thermal diffusivity** defined as (unity : m^2/s)

$$\alpha = \frac{k}{\rho C_p} \quad (3.7)$$

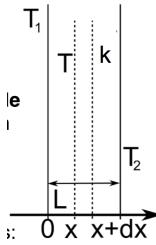
Thermal conductivity of materials

- **Gas** : particles are far from each other, so there are few collision and so few transfer by conduction.
- **Liquid** : many randoms shocks giving better conduction skills.
- **Solid** : the rigidity makes from them bad conductors but the cristals can transfer heat fast. They can so be good conductors or insulators¹.

k has some dependence with temperature that is neglect for small changes. **We have to beware of phase changes !**

1. Isolateurs

3.3 1D stationnary conductive heat balance

 Let's apply a heat balance on the walls of thickness dx using equation (3.4). There is no source and no accumulation term.

$$\text{Flux In} = \text{Flux out} \Leftrightarrow A\dot{q}_x = A\dot{q}_{x+dx} \quad (3.8)$$

With a first order Taylor expansion we have the expression of \dot{q}_{x+dx} and replaced in (3.8)

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}_x}{dx} dx \Rightarrow \frac{d\dot{q}_x}{dx} = 0 \quad (3.9)$$

We apply the Fourier Law ($k = cst$)

$$\frac{d\dot{q}_x}{dx} = \frac{d}{dx} \left(-k \frac{dT}{dx} \right) = \frac{d^2T}{dx^2} = 0 \Rightarrow T = C_1 x + C_2 \quad (3.10)$$

3.3.1 Heat resistance

The boundary conditions $T = T_1$ for $x = 0$ and $T = T_2$ for $x = L$ gives the constants, so

$$T = \frac{T_2 - T_1}{L} x + T_1 \quad (3.11)$$

If we reapply the Fourier Law (derivating the equation above)

$$\dot{q} = k \frac{T_1 - T_2}{L} \quad (3.12)$$

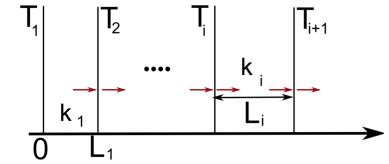
We define now the **thermal resistance** of the wall $R = \frac{L}{kA}$ and obtain the heat transfer rate

$$\dot{Q} = \frac{T_1 - T_2}{R} \quad (3.13)$$

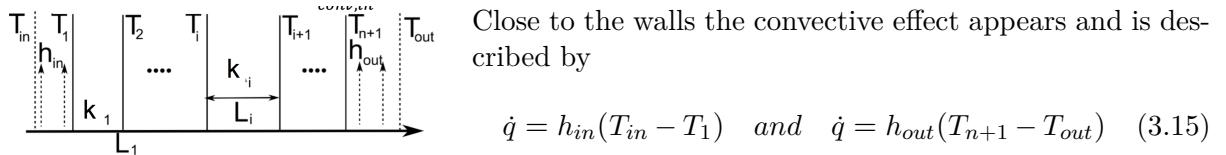
3.3.2 Serie of walls

By assuming the temperature continuity and the flux conservation at the boundary of each wall, we have

$$\dot{Q} = \frac{T_1 - T_{n+1}}{R} \quad \text{with} \quad R = \sum R_i = \sum \frac{L_i}{k_i A} \quad (3.14)$$



3.3.3 Adding two ewtra resistance - the Biot number



and the heat transfer rate can be found like

$$\dot{Q} = \frac{T_{in} - T_{out}}{R} \quad \text{and} \quad R = \frac{1}{h_{in}A} + \sum \frac{L_i}{k_i A} + \frac{1}{h_{out}A} \quad (3.16)$$

But is it important to take care of the convective effect? Let's compare the convective and conductive effect in the inner side

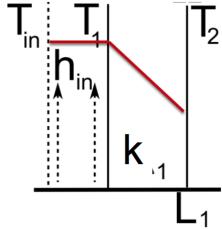
$$\left. \begin{array}{l} T_{in} - T_1 = \frac{\dot{Q}}{h_{in}A} \\ T_1 - T_2 = \frac{L_1 \dot{Q}}{k_1 A} \end{array} \right\} \Rightarrow \frac{T_1 - T_2}{T_{in} - T_1} = \frac{L_1 h_{in}}{k_1} = Bi \quad (3.17)$$

That is the **Biot number** that compares the convection in the fluid around the solid to the conduction in the solid.

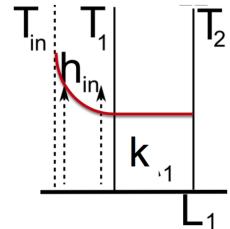
$Bi < 1$

It means that $k \ll h_{in}$ so that the convection is limiting and that the temperature difference is in the fluid. The variation of T in the solid is negligible.

$Bi < 1$



In that case, the convection is dominating because $h_{in} \gg k$. The temperature difference is in the solid while the variation is in negligible in the fluid.



3.4 Heat conduction in tubes

It's the same principle but in cylindrical coordinates. \dot{Q} is conserved for all radius and is expressed like

$$\dot{Q} = \dot{q}2\pi rH \quad (3.18)$$

The flux then writes

$$\dot{Q} = \frac{T_1 - T_2}{R} \quad R = \frac{1}{2\pi r_1 H h_{in}} + \sum \frac{\ln(r_{i+1}/r_i)}{k_i 2\pi H} + \frac{1}{2\pi r_{n+1} H h_{out}} \quad (3.19)$$