多元函数积分

曲线积分

$$L:\begin{cases}x=\varphi(t),\\ y=\psi(t),\end{cases}\quad t\in[\alpha,\ \beta],$$

一型曲线积分

$$\int_L f(x,y)ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

二型曲线积分

$$\begin{aligned}&\int_L P(x,y)dx+Q(x,y)dy\\ =&\int_\alpha^\beta[P(\varphi(t),\psi(t))\varphi^{'}(t)+Q(\varphi(t),\psi(t))\psi^{'}(t)]dt\end{aligned}$$

二重积分

直角坐标系

$$\iint\limits_D f(x,y)d\sigma=\int_a^b dx\int_c^d f(x,y)dy=\int_c^d dy\int_a^b f(x,y)dx$$

格林公式

$$\iint\limits_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})d\sigma=\oint_L Pdx+Qdy$$

$$S_D=\frac{1}{2}\oint_L x\,dy-y\,dx$$

曲线积分的路线无关性

对于单连通区域，若函数 $P(x,y),Q(x,y)$ 在 D 内连续，则一下四个条件等价：

1. $$\oint_L Pdx+Qdy=0$$
2. $$\int_L Pdx+Qdy \text{ 与路线无关，仅与 } L \text{ 的起点和终点有关}$$
3. $$\text{在 } D \text{ 内存在 } u(x,y) \text{ 使得}$$
   $$du=Pdx+Qdy$$
4. $$\text{在 } D \text{ 内处处成立}$$
   $$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

全微分的原函数

$$\begin{aligned}&u(x,y)\\&=\int_{x_0}^{x}P(s,y_0)ds+\int_{y_0}^{y}Q(x,t)dt\\&=\int_{x_0}^{x}P(s,y)ds+\int_{y_0}^{y}Q(x_0,t)dt\end{aligned}$$

变量变换

对于 $\iint\limits_D f(x,y)dA$ ，遵照以下步骤进行变量替换

1. 选择变换函数
   定义新的变量 $u=g(x,y),v=h(x,y)$
   解出 $x=x(u,v),y=y(u,v)$
2. 计算变换的雅可比行列式 $|J|$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

3. 确定新变量 (u,v) 对应的新区域 D'
4. 替换积分
   变换为 $\iint\limits_{D'} f(x(u,v),y(u,v))\,|J|\,dudv$

极坐标变换

$$T:\begin{cases}x = arcos\,\theta, \\ y = brsin\,\theta,\end{cases} \quad 0 \le r < +\infty, 0 \le \theta \le 2\pi$$

$$\begin{aligned}&\iint_D f(x,y)dxdy \\ =&\iint_{D'} f(arcos\,\theta, brsin\,\theta)abrdrd\theta \\ =&\int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(arcos\,\theta, brsin\,\theta)abrdr \\ =&\int_{r_1}^{r_2} abrdr \int_{\theta_1(r)}^{\theta_2(r)} f(arcos\,\theta, brsin\,\theta)d\theta\end{aligned}$$

三重积分

基本计算

$$\iiint_E f(x,y,z)\,dV = \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz\right)dy\right)dx$$

坐标变换

雅可比行列式 $|J|$:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

柱坐标变换

$$T:\begin{cases}x = rcos\,\theta, & 0 \le r < +\infty, \\ y = rsin\,\theta, & 0 \le \theta \le 2\pi, \\ z = z, & -\infty < z < +\infty.\end{cases}$$

$$J(r,\theta,z) = r$$

球坐标变换

$$T:\begin{cases}x = rsin\,\varphi cos\,\theta, & 0 \le r < +\infty, \\ y = rsin\,\varphi sin\,\theta, & 0 \le \varphi \le \pi, \\ z = rcos\,\varphi, & 0 \le \theta \le 2\pi.\end{cases}$$

$$J(r,\theta,z) = r^2 sin\,\varphi$$

曲面积分

第一型曲面积分

一般计算

$$\text{对于曲面 } S: z = z(x,y),\ (x,y) \in D,$$

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1+z_x^2+z_y^2}dxdy$$

参量形式曲面

$$S: \begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} \quad (u,v) \in D,$$

$$\downarrow$$

$$\begin{cases} E = x_u^2 + y_u^2 + z_u^2, \\ G = x_u x_v + y_u y_v + z_u z_v, \\ F = x_v^2 + y_v^2 + z_v^2, \end{cases}$$

$$\downarrow$$

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}dudv.$$

$$\text{要求 } \frac{\partial(x,y)}{\partial(u,v)},\ \frac{\partial(x,z)}{\partial(u,v)},\ \frac{\partial(y,z)}{\partial(u,v)} \text{ 之中至少有一个不等于零}$$

第二型曲面积分

一般计算

$$\text{对于曲面 } S: z = z(x,y),\ (x,y) \in D,$$

$$\iint\limits_S f(x,y,z)dxdy = \iint\limits_D f(x,y,z(x,y))dxdy$$

参量形式曲面

$$S:\begin{cases} x=x(u,v), \\ y=y(u,v), \quad (u,v)\in D, \\ z=z(u,v), \end{cases}$$

$$\downarrow$$

$$\begin{cases} \iint\limits_S Pdydz = \pm\iint\limits_D P(x(u,v),y(u,v),z(u,v))\dfrac{\partial(y,z)}{\partial(u,v)}dudv, \\ \iint\limits_S Qdzdx = \pm\iint\limits_D Q(x(u,v),y(u,v),z(u,v))\dfrac{\partial(z,x}{\partial(u,v)}dudv, \\ \iint\limits_S Rdxdy = \pm\iint\limits_D R(x(u,v),y(u,v),z(u,v))\dfrac{\partial(x,y)}{\partial(u,v)}dudv. \end{cases}$$

正负号由法向量$(\dfrac{\partial(x,y)}{\partial(u,v)}, \dfrac{\partial(x,z)}{\partial(u,v)}, \dfrac{\partial(y,z)}{\partial(u,v)})$ 对应 S 内外侧决定

要求 $\dfrac{\partial(x,y)}{\partial(u,v)}, \dfrac{\partial(x,z)}{\partial(u,v)}, \dfrac{\partial(y,z)}{\partial(u,v)}$ 之中至少有一个不等于零

高斯公式

空间区域 V 由双侧封闭曲面 S 围成， P,Q,R 在 V上连续

$$\iiint\limits_V (\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dxdydz$$

$$=\oiint\limits_S Pdydz+Qdzdx+Rdxdy.$$

*常用于化简封闭曲面二重积分

$$\downarrow$$

令高斯公式中的$P=x,Q=y,R=z$

得到封闭空间区域体积公式：

$$\Delta V=\frac{1}{3}\oiint\limits_S xdydz+ydxdz+zdxdy$$

斯托克斯公式

$$
\iint_S (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dzdx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy
$$

$$
= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}
$$

$$
= \oint_L Pdx + Qdy + Rdz
$$

S 的侧以及 L 的方向由右手定则决定