HW0 → EE19MTECH01008

Thursday, 21 November 2019 8:10 PM

Q) Complete the missing steps in the derivation of the sum-product form of the marginal distribution of a probabilistic graphical model defined on an undirected graph?

A:) For undirected graphs :-

$$p(x_1 \dots x_n) = \frac{1}{Z} \prod_{C \in \mathcal{C}} \psi_C(x_C)$$

ψ_C = Compatibility function $\mathcal{X}_1 \times \mathcal{X}_2 \dots \mathcal{X}_K \rightarrow \mathbb{R}_+$

Z = normalization factor

$\mathcal{C}$ = set of maximal clique of G (graph)

For a tree,

$$p(x_1 \dots x_n) = \frac{1}{Z} \prod_{u \in V} \psi_u(x_u) \prod_{s,t \in E} \psi_{st}(x_s, x_t)$$

$$\mu_s(x_s) = \sum_{V \setminus x_s} p(x_1 \dots x_n) \Rightarrow \text{marginal where } s \text{ is root}$$

<u>To Show</u> :- (Sum product Algorithm)

$$\mu_s(x_s) = K \psi_s(x_s) \prod_{t \in N(s)} M_{ts}(x_s)$$

$$M_{ts}(x_s) = \sum_{x_{V_t}} \psi_{st}(x_s, x_t) \, \phi(x_{V_t}; T_t)$$

Now, notice $V = \{s\} \cup \{\bigcup_{t \in N(s)} V_t\}$

and $E = \{\bigcup_{t \in N(s)} (s,t)\} \cup \{\bigcup_{t \in N(s)} E_t\}$

$$p(x_1 \dots x_n) = \frac{1}{Z} \prod_{u \in V} \psi_u(x_u) \prod_{(s,t) \in E} \psi_{st}(x_s, x_t)$$

$$= \frac{1}{Z} \psi_s(x_s) \prod_{u \in \cup V_t} \psi_u(x_u) \cdot \prod_{(s,t) \in \cup E_t} \psi_{st}(x_s, x_t).$$

$$= \frac{1}{Z} \psi_s(x_s) \prod_{\substack{u \in V_t \\ t \in N(s)}} \psi_u(x_u) \cdot \prod_{\substack{(s,t) \in VE_r \\ t \in N(s)}} \psi_{sr}(x_s, x_t) \cdot \prod_{\substack{(s,t) \\ t \in N(s)}} \psi_{sr}(x_s, x_t)$$

Note,

$$\mu_s(x_s) = \sum_{x \setminus x_s} p(x_1 \ldots x_n)$$

$$= \sum_{x \setminus x_s} \frac{1}{Z} \psi_s(x_s) \prod_{\substack{u \in V_t \\ t \in N(s)}} \psi_u(x_u) \prod_{\substack{(s,t) \in VE_r \\ t \in N(s)}} \psi_{sr}(x_s, x_t) \prod_{\substack{(s,t) \\ \in N(s)}} \psi_{sr}(x_s, x_t)$$

$$\left(\text{But, } p(x_{V_t}; T_t) \propto \prod_{u \in V_t} \psi_u(x_u) \prod_{(v,w) \in E_t} \psi_{vw}(x_v, x_w) \right)$$

$$\mu_s(x_s) = k_s \psi_s(x_s) \sum_{x \setminus x_s} \left(\prod_{t \in N(s)} p(x_{V_t}; T_t) \right) \left(\prod_{t \in N(s)} \psi_{sr}(x_s, x_t) \right)$$

$$= k_s \psi_s(x_s) \sum_{x_1} \ldots \sum_{x_{s-1}} \ldots \sum_{x_n} \prod_{t \in N(s)} \psi_{sr}(x_s, x_t) \, p(x_{V_t}; T_t)$$

$$= k_s \psi_s(x_s) \sum_{V_1} \sum_{V_2} \ldots \sum_{V_{t \in N(s)}} \sum_{t \in N(s)} \psi_{sr}(x_s, x_t) \, p(x_{V_t}; T_t)$$

Note:- Since they are disconnected parts of the tree each summation terms are independent of another i.e. V_i independent of V_j $\forall i, j, i \neq j$.

$$\left\{ \begin{array}{l} \text{Ex: } i, j, k \rightarrow \text{indep (assume)} \\ \sum_i \sum_j \sum_k i\, j\, k \left(\sum_i i\right) \left(\sum_j j\right) \left(\sum_k k\right) = \prod_{x \in \{i, j, k\}} \sum_x x \end{array} \right\}$$

$$\Rightarrow \mu_s(x_s) = k_s \psi_s(x_s) \prod_{t \in N(s)} \left(\sum \psi_{sr}(x_s, x_t) \, p(x_{V_t}, T_t) \right)$$

$$\Rightarrow \boxed{\mu_s(x_s) = k_s \psi_s(x_s) \prod_{t \in N(s)} M_{ts}(x_s)}$$

$$\Rightarrow \boxed{\begin{aligned} &\mu_s(x_s) = k\,\psi_s(x_s) \prod_{t \in N(s)} M_{ts}(x_s) \\ &\text{where } \; M_{ts}(x_s) = \sum_{x_{v_t}} \psi_{st}(x_s, x_t)\, p(x_{v_t}; T_t) \end{aligned}}$$