Trigonometry CheatSheet

1 How to use this document

This document is not meant to be a list of formulas to be learned by heart. The first few formulas are very basic (they descend from the definition and/or Pythagoras' theorem) and you might want to memorize them, but you should be able to retrieve everything else from these two/three basic formulas, and you should be able to do it quickly (during an exam you don't want to invest ten minutes in recalculating a formula that you might or might not need!!!). These formulas are arranged in a logical order, starting from the most basic, so that each formula can be retrieved using formulas that come first only. Every formula is accompanied by a short explanation about how you can retrieve it (there might be more than one method).

2 Formulas that come from Pythagoras' theorem and/or the definition

$$\cos^2 x + sin^2 x = 1 \tag{1}$$

It's an immediate consequence of the definition and Pythagoras' theorem

$$\tan^2 x + 1 = \sec^2 x \tag{2}$$

It follows from the definition of tangent, secant and the previous formula

$$\cot^2 x + 1 = \csc^2 x \tag{3}$$

Same as before

$$\cos(-x) = \cos x \tag{4}$$

Cosine is even. It follows from the definition.

$$\sin(-x) = -\sin x \tag{5}$$
$$\tan(-x) = -\tan x \tag{6}$$
$$\cot(-x) = -\cot x \tag{7}$$

Sine, tangent and cotangent are all odd. It follows from the definition. The period of sine, cosine secant and cosecant is 2π, and the period of tangent and cotangent is π:

$$\sin(x + 2k\pi) = \sin x \tag{8}$$
$$\cos(x + 2k\pi) = \cos x \tag{9}$$
$$\tan(x + k\pi) = \tan x \tag{10}$$

3 Addition of angles

You might want to take the first two formulas as black boxes and memorize them. However, if you know about the complex numbers you can retrieve them both very quickly.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{11}$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{12}$$

These two formulas can be derived using the property of exponentials

$$e^{i(x+y)} = e^{ix} e^{iy}$$

Plug the Euler's identity $e^{ix} = \cos x + i \sin x$ into the previous equation and compare the left hand side to the right hand side (remember that $i^2 = -1$ and that the complex numbers are a two dimensional vector space over the reals). If this argument doesn't make any sense to you, you can safely ignore it. Combining these formulas with (4) and (5) we easily derive the following:

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \tag{13}$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \tag{14}$$

Using the definition of tangent and equations (11,12) we can derive the addition formula:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \tag{15}$$

4 Formulas for the double and half angle

Using equations (11,12) with $x = y$ we immediately have:

$$\sin 2x = 2 \sin x \cos x \tag{16}$$
$$\cos 2x = \cos^2 x - \sin^2 x \tag{17}$$

By plugging (1) into (17) we have the following two formulas for the squares of sine and cosine:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \tag{18}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2} \tag{19}$$

By substituting x with $\frac{x}{2}$ and taking the square root we have formulas for the half angle:

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}} \tag{20}$$
$$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}} \tag{21}$$

Choose the sign wisely!!! In the same way, but using (15) we have:

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \tag{22}$$

We can easily retrieve formulas for any multiple of an angle just by applying (11,12) recursively:

$$\begin{cases} \sin nx = \sin(x + (n-1)x) = \sin x \cos((n-1)x) + \cos x \sin((n-1)x) \\ \textit{plug in} \quad \sin((n-1)x) = \sin(x + (n-2)x) = \sin x \cos((n-2)x) + \cos x \sin((n-2)x) \\ \textit{and also} \quad \cos((n-1)x) = \cos(x + (n-2)x) = \cos x \cos((n-2)x) - \sin x \sin((n-2)x) \\ \textit{and repeat the process} \ldots \end{cases}$$

until we obtain a formula of this kind:

$$\sin nx = P(\cos x, \sin x)$$

where $P(\cdot, \cdot)$ is a homogeneous polynomial in two variables of degree n.

5 Linear combinations

A linear combination of sine and cosine can always be expressed as the sine of an addition of two angles:

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \theta) \tag{23}$$

where $\theta = \arctan \frac{b}{a}$ for $a > 0$ and $\theta = \pi + \arctan \frac{b}{a}$ for $a < 0$. To retrieve this formula multiply and divide the left hand side by $\sqrt{a^2 + b^2}$ and use equation (11).

6 Prosthaphaeresis

The product of sine and cosine can always be expressed as an addition:

$$\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2} \tag{24}$$

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2} \tag{25}$$

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \tag{26}$$

$$\cos x \sin y = \frac{\sin(x+y) - \sin(x-y)}{2} \tag{27}$$

These formulas can be retrieved as linear combinations of equations (11,12,13,14). For instance the first one is equation (14) plus (12) divided by two.

7 Rational parametric equations

Every trigonometric function of x can be written as a rational function[1] in the variable

$$t = \tan \frac{x}{2}$$

$$t = \tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \sqrt{\frac{1 + \cos x}{1 - \cos x}}$$

[1] A rational function is the quotient of two polynomials

In the last step we have applied (20,21). Take the square and solve for $\cos x$:

$$\cos x = \frac{1-t^2}{1+t^2} \tag{28}$$

From this equation we derive the parametric representation of sine by applying (1):

$$\sin x = \frac{2t}{1+t^2} \tag{29}$$

Putting them together we have the formula for tangent:

$$\tan x = \frac{2t}{1-t^2} \tag{30}$$

Analogous formulas for cotanget, secant and cosecant are obvious to retrieve:

$$\sec x = \frac{1+t^2}{1-t^2} \tag{31}$$

$$\csc x = \frac{1+t^2}{2t} \tag{32}$$

$$\cot x = \frac{1-t^2}{2t} \tag{33}$$

This set of formulas is of the utmost importance in the calculation of integrals.

8 Inverse functions

All the trigonometric functions that we have seen so far are periodic and for this reason they are never injective[2]. In order to invert these functions it's necessary to restrict the domain to a suitable interval[3]. The choice of such an interval is not unique (remember that these functions are periodic!), for this reason we have conventions:

$$\arcsin x, \quad \text{Domain=}[-1,1], \text{ Range=}\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \tag{34}$$

$$\arccos x, \quad \text{Domain=}[-1,1], \text{ Range=}[0,\pi] \tag{35}$$

$$\arctan x,, \quad \text{Domain=}(-\infty,+\infty), \text{ Range=}\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \tag{36}$$

9 Limits and asymptotes

The following limits are fundamental and should be learned by heart:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{37}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \tag{38}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2} \tag{39}$$

[2] A function $f(x)$ is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. It's called *one to one* by certain authors.

[3] the biggest subset of the real line where the function is injective.

These limits can be proved using the squeeze theorem. **These limits are used to calculate the derivatives of sine and cosine and for this reason IT'S WRONG to solve them with de l'Hôpital!**. The formula of de l'Hôpital requires a preliminary knowledge of the derivatives of sine and cosine; even if it provides the right answer the calculation of (37,38,39) with de l'Hôpital generates a vicious circle and as a matter of fact is **WRONG!!!**.

Tangent and cotangent have vertical asymptotes at odd multiples of $\frac{\pi}{2}$:

$$\lim_{x \to \left(\frac{(2k+1)\pi}{2}\right)^{\mp}} \tan x = \pm\infty \tag{40}$$

$$\lim_{x \to \left(\frac{(2k+1)\pi}{2}\right)^{\pm}} \cot x = \pm\infty \tag{41}$$

For this same reason their inverse functions have horizontal asymptotes:

$$\lim_{x \to \pm\infty} \arctan x = \pm\frac{\pi}{2} \tag{42}$$

$$\lim_{x \to \pm\infty} \operatorname{arccot} x = \mp\frac{\pi}{2} \tag{43}$$

10 Derivatives

Using the definition of derivative together with formulas (11,37,38) we can easily retrieve:

$$\frac{d}{dx} \sin x = \cos x \tag{44}$$

Using formulas (12,37,38) we calculate the derivative of cosine:

$$\frac{d}{dx} \cos x = -\sin x \tag{45}$$

Using the formula for the derivative of the quotient and the previous two formulas we obtain:

$$\frac{d}{dx} \tan x = \sec^2 x \tag{46}$$

$$\frac{d}{dx} \cot x = -\csc^2 x \tag{47}$$

$$\frac{d}{dx} \sec x = \tan x \sec x \tag{48}$$

$$\frac{d}{dx} \csc x = -\cot x \csc x \tag{49}$$

To calculate derivatives of inverse trigonometric functions we use the formula for the derivative of the inverse:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

or in a sloppy bu suggestive notation:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

where $y = f(x)$ and $x = f^{-1}(y)$.

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \tag{50}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \tag{51}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \tag{52}$$
$$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2} \tag{53}$$

11 Obvious Primitives

Using the derivatives that we have calculated in the previous section we can quickly fill out a list of primitives:

$$\int \cos x \; dx = \sin x + c \tag{54}$$
$$\int \sin x \; dx = -\cos x + c \tag{55}$$
$$\int \sec^2 x \; dx = \tan x + c \tag{56}$$
$$\int \csc^2 x \; dx = -\cot x + c \tag{57}$$
$$\int \tan x \sec x \; dx = \sec x + c \tag{58}$$
$$\int \cot x \csc x \; dx = -\csc x + c \tag{59}$$
$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \arcsin x + c_1 = -\arccos x + c_2 \tag{60}$$
$$\int \frac{1}{1+x^2} \; dx = \arctan x + c_1 = -\operatorname{arccot} x + c_2 \tag{61}$$