CS321

Introduction to Numerical Methods

Lecture 1

Number Representations and Errors

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Number in Different Bases

Humans use number base $\mathbf{10}$ for reasons of the number of fingers

Computers use number base $\mathbf{2}$ for reasons of on-off switch

A number based on $\mathbf{8}$ can be used to facilitate conversion between the base $\mathbf{2}$ and the base $\mathbf{10}$

Base 10 Expansion:

$$\begin{aligned} 76321 &= 1 + 20 + 300 + 6000 + 70000 \\ &= 1\times 10^{0} + 2\times 10^{1} + 3\times 10^{2} \\ &\quad + 6\times 10^{3} + 7\times 10^{4} \end{aligned}$$

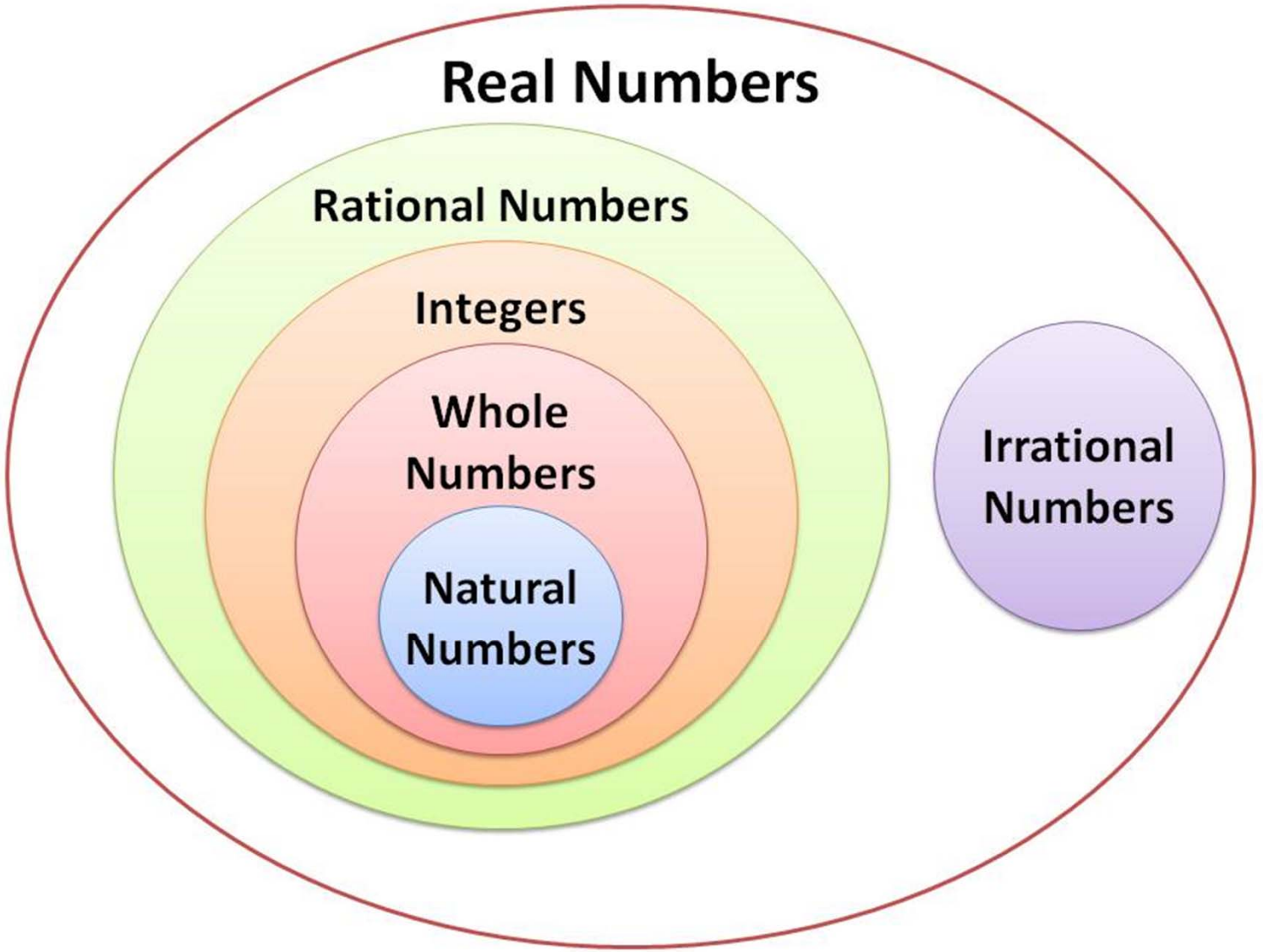
The general formula for a base $\mathbf{10}$ integer is

$$\begin{aligned} a_n a_{n-1} \ldots a_2 a_1 a_0 &= a_0 \times 10^{0} + a_1 \times 10^{1} \\ &+ a_2 \times 10^{2} + \ldots + a_{n-1} \times 10^{n-1} + a_n \times 10^{n} \end{aligned}$$

Binary Numbers

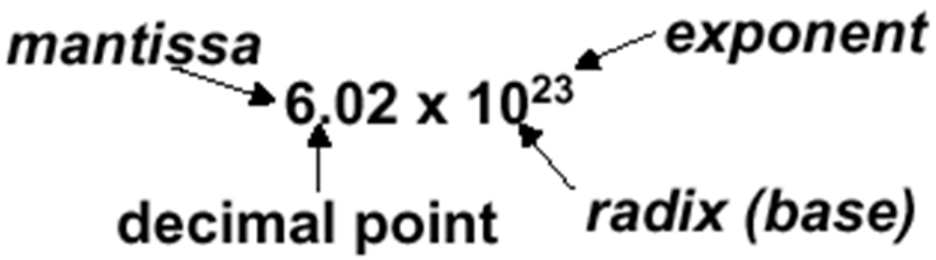
Base 2 number	Base 10 number
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10

Number System

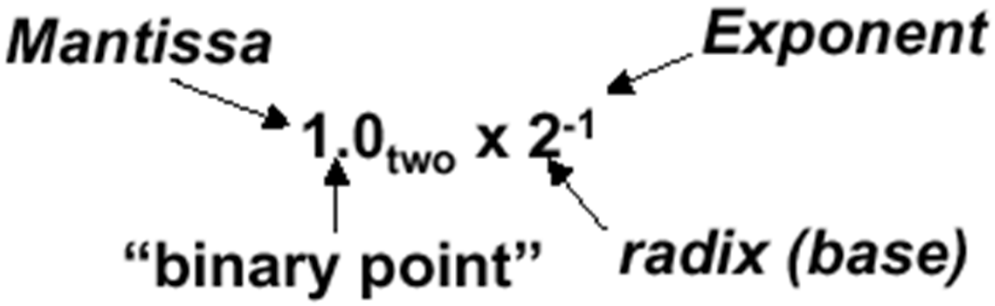


Definitions

There are some definitions for representing a real number in base 10



and in base 2



Nonintegers and Fractions

An example of a fractional number

$$0.6321 = \frac{6}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{1}{10000}$$
$$= 6\times10^{-1} + 3\times10^{-2} + 2\times10^{-3}$$
$$+ 1\times10^{-4}$$

The general formula for a decimal fraction

$$0.b_1b_2b_3\ldots = b_1\times10^{-1} + b_2\times10^{-2} + b_3\times10^{-3} + \ldots$$

Decimal fractional numbers can be repeating or nonrepeating (rational or irrational numbers)

$$(a_n a_{n-1}\ldots a_1 a_0 . b_1 b_2 b_3\ldots)_{10} =$$
$$\sum_{k=0}^{n} a_k \times 10^k + \sum_{k=1}^{\infty} b_k \times 10^{-k}$$

Number Conversion

Base β Numbers

Convert an integer number from base **8** system to base **10** system

$$(7632)_8 = 2\times 8^0 + 3\times 8^1 + 6\times 8^2 + 7\times 8^3$$
$$= 2 + 8(3 + 8(6 + (8(7))))$$
$$= (3994)_{10}$$

Convert a fractional number between **0** and **1**

$$(0.763)_8 = 7\times 8^{-1} + 6\times 8^{-2} + 3\times 8^{-3}$$
$$= 8^{-3}(3 + 8(6 + 8(7)))$$
$$= \frac{499}{512}$$
$$= (0.9746)_{10}$$

Convert integer and fractional parts separately

General formula

$$(a_n a_{n-1}\ldots a_1 a_0 . b_1 b_2 b_3 \ldots)_\beta =$$
$$\sum_{k=0}^{n} a_k \times \beta^k + \sum_{k=1}^{\infty} b_k \times \beta^{-k}$$

Conversion between Systems

From system α to system β with $\alpha < \beta$

- express $(N)_\alpha$ in nested form using powers of α
- replace each digit by corresponding base β number
- carry out indicated arithmetic in base β

Two examples were given in the previous page

Note that one needs to convert the integer and the fractional parts separately

The method is in theory can be used to convert a number between any two bases. But the third step is not easy for humans if the base is not 10

Division Algorithm

Use remainder-quotient-split method to convert an integer, best if $\alpha > \beta$

$$(N)_\alpha = (c_m c_{m-1} \ldots c_1 c_0)_\beta$$
$$= c_0 + \beta(c_1 + \beta(c_2 + \ldots + \beta(c_m)\ldots))$$

1. Divide the representation by β;

2. Take the remainder as the next digit in the base β representation;

3. Repeat the procedure on the quotient.

This procedure is easier to carry out by hand

Note that the first digit obtained is the one closest to the radix point

Repeated Division

To convert a base **10** integer number to a base β**<10** number, we can use repeated division algorithm

$2576 \ / 8 = 322$ **remainder** $0 \rightarrow 0$

$322 \ / 8 = 40$ **remainder** $2 \rightarrow 2$

$40 / 8 = 5$ **remainder** $0 \rightarrow 0$

$5 / 8 = 0$ **remainder** $5 \rightarrow 5$

So we have

$$(2576\ .)_{10} = (5020\ .)_8$$

It can be further converted to a binary number using the binary-octal table (in a later slide)

$$(2576\ .)_{10} = (5020\ .)_8 = (101\ 000\ 010\ 000\ .)_2$$

Note that the first digit you obtain is the first digit next to the radix point

Multiplication Algorithm

Use integer-fraction-split process for converting a fractional part

$$x = \sum_{k=1}^{\infty} c_k \times \beta^{-k} = (0.c_1 c_2 c_3 \ldots)_\beta$$

1.) multiply the (fractional) number by β;

2.) take the integer part as the first (next) digit;

3.) repeat the process on the remaining *fractional* part.

Again, the first digit obtained is the one closest to the radix point

Terminating fractional number may become non-terminating in different base systems, and vice versa

Integer-Fraction-Split

To convert a fractional base $\mathbf{10}$ number to a base $\boldsymbol{\beta}<\mathbf{10}$ number, we can use the integer-fractional-split algorithm

$$0.372 \times 2 = 0.744 \rightarrow 0$$
$$0.744 \times 2 = 1.488 \rightarrow 1$$
$$0.488 \times 2 = 0.976 \rightarrow 0$$
$$0.976 \times 2 = 1.952 \rightarrow 1$$
$$0.952 \times 2 = 1.904 \rightarrow 1$$
$$0.904 \times 2 = 1.808 \rightarrow 1$$

So we have

$$(0.372)_{10} = (0.010111 \ldots)_2$$

Use Intermediate Base

We can convert a base **10** number to a base **8** (octal) number, then to base **2** (binary) number, and vice versa

Binary-octal table

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

Groups of three binary digits can be translated to an octal number using the binary-octal table

$$(110\ 101\ 001.011\ 100\ \ 111)_2$$
$$= (651.347)_8\ = (?)_{10}$$

$$(57.321)_{10} = (?)_8 = (?)_2$$

Base 16 Numbers

Certain computers with word length being multiples of 4 use hexadecimal system, need A, B, C, D, E, and F to represent 10, 11, 12, 13, 14, and 15.

Binary-hexadecimal table

0000	0001	0010	0011	0100	0101
0	1	2	3	4	5

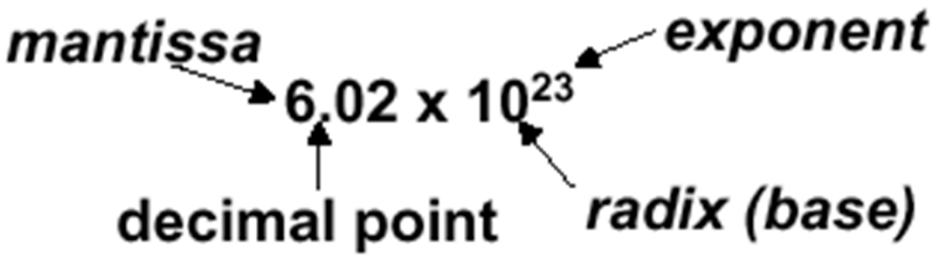
0110	0111	1000	1001	1010	1011
6	7	8	9	A	B

1100	1101	1110	1111	
C	D	E	F	

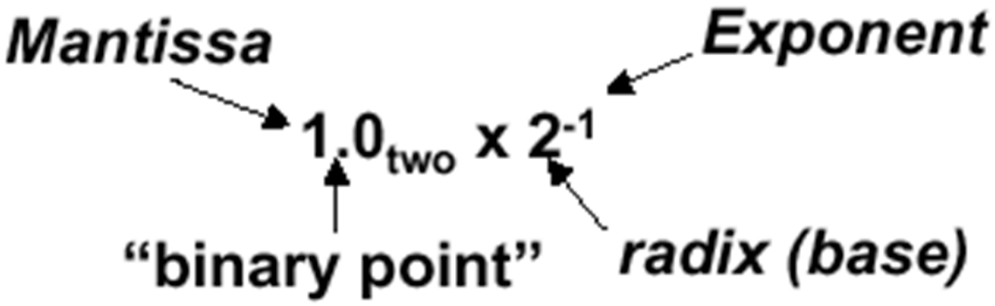
$(010\ 111\ 100.110\ 100\ 101\ 111)_2$
$= (1011\ 1100.1101\ 0010\ 1111)_2$
$= (BC.D2F)_{16}$

Definitions

There are some definitions for representing a real number in base 10



and in base 2



Normalized Scientific Notation

Write a number in the form of

$$728.359 = 0.728359 \times 10^3$$

A normalized floating-point number has the form

$$x = \pm 0.d_1 d_2 d_3 \ldots \times 10^n$$

where $d_1 \neq 0, n$ is an integer

In a simple notation

$$x = \pm r \times 10^n \qquad \left(\frac{1}{10} \leq r < 1 \right)$$

r is called normalized mantissa and n is the exponent. Also for binary representation

$$x = \pm q \times 2^m \qquad \left(\frac{1}{2} \leq q < 1 \right)$$

Machine Numbers

The real numbers that can be represented exactly in a computer are called the *machine numbers* for this computer

Most real numbers are not machine numbers

If a computer has word length of the form $\mathbf{0.d_1d_2d_3d_4}$, then $\mathbf{0.1011}$ is a machine number, but $\mathbf{0.10101}$ is not

Machine numbers are machine dependent. The use of normalized floating-point numbers creates a phenomenon of *hole at zero*, a bunch of numbers close to $\mathbf{0}$ are not representable

This is mainly caused by the under-flow problem, i.e., small numbers close to zero will be treated as zero in a computer.

(**0.0100** cannot be stored in the above mentioned computer, why?)

A 32-bit Machine

Single precision floating-point numbers with 32-bit word length

How to store a floating-point number

$$\pm q \times 2^m$$

with 32 bits?

Sign of q needs 1 bit

Integer $|m|$ needs 8 bits

Number q needs 23 bits

Single precision IEEE standard floating-point format

$$(-1)^s \times 2^{c-127} \times (1.f)_2$$

32-bit Representation

The exponent number c is actually stored, so $m = c - 127$, this is an *excess-127 code* (make sure only "positive numbers" are stored, 0< *c* <255)

With normalized representation, the first bit is always 1 and needs not be stored. The mantissa actually contains 24 binary digits with a *hidden bit*

With a mantissa of 23 bits, a machine can have about six significant decimal digits of accuracy, since $2^{-23} \approx 1.2 \times 10^{-7}$

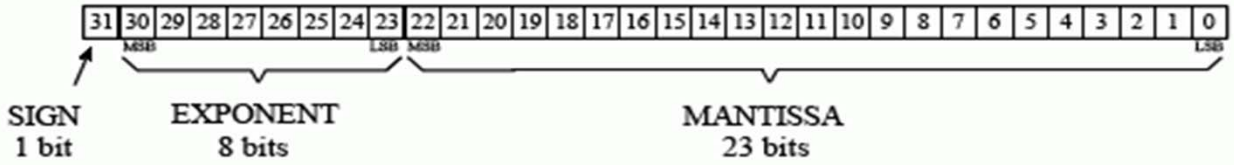
The smallest positive number ϵ such that $1 + \varepsilon \neq 1$ is called the *machine epsilon* or *unit roundoff error*

For single precision, $\varepsilon = 1.2 \times 10^{-7}$

For double precision, $\varepsilon = 2.2 \times 10^{-16}$

32-bit Representation

IEEE 32-bit single precision



Example 1

0 00000111 11000000000000000000000

\+ 7 0.75

$$+1.75 \times 2^{(7-127)} = +1.316554 \times 10^{-36}$$

Example 2

1 10000001 01100000000000000000000

\- 129 0.375

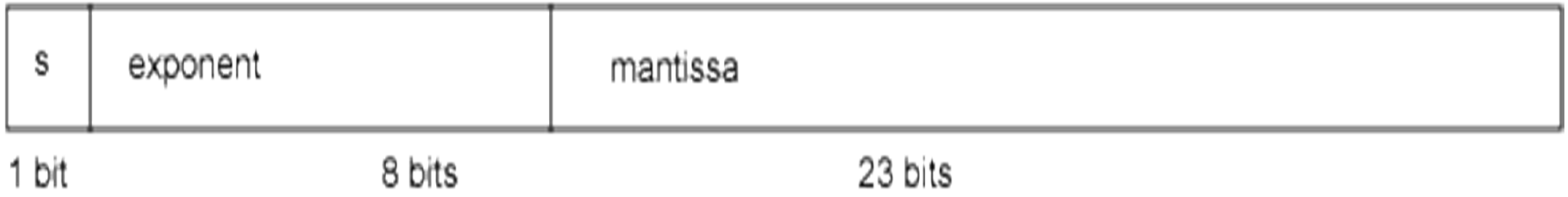
$$-1.375 \times 2^{(129-127)} = -5.500000$$

FIGURE 4-2
Single precision floating point storage format. The 32 bits are broken into three separate parts, the sign bit, the exponent and the mantissa. Equations 4-1 and 4-2 shows how the represented number is found from these three parts. MSB and LSB refer to "most significant bit" and "least significant bit," respectively.

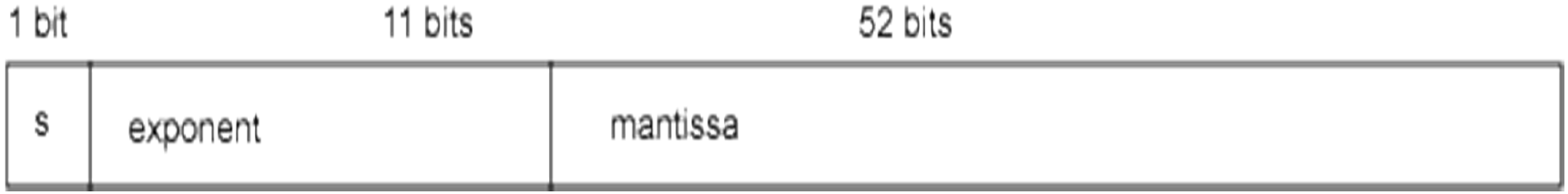
64-bit Representation

IEEE 32-bit single precision and 64-bit double precision

IEEE Floating Point Representation



IEEE Double Precision Floating Point Representation



More About the Exponent

The exponent *c* is stored, with

$$0 < c < 255 = (11\ 111\ 111)_2$$

The actual range of exponent is

$$-126 \leq c - 127 \leq 127$$

$c = 0$ is reserved for the special number ± 0, and $c = 255$ is for $\pm\infty$

This strategy avoids the need to handle sign for the exponent

The largest number can be stored $(2 - 2^{-23})2^{127} \approx 2^{128} \approx 3.4 \times 10^{38}$

The smallest number can be stored $2^{-126} \approx 1.2 \times 10^{-38}$

Representation Procedure

How to represent a real number x?

1.) if *x* is zero, stored it by a full word of zero bits, (a possible sign bit)

2.) for nonzero *x*, first consider sign bit and then consider $|x|$

3.) convert both integer and fractional parts of $|x|$ from decimal to octal, then to binary

4.) one-plus normalize $(|x|)_2$ by shifting the binary point

5.) find the 24 bit 1-plus normalized mantissa

6.) find the exponent of 2 by setting it equal to $c\text{-}127$ and determine c

7.) denote the 32-bit representation as 8 hexadecimal digits

Summary and Examples

A 32-bit single-precision pattern

$$b_1 b_2 b_3 \ldots b_9 b_{10} b_{11} \ldots b_{32}$$

can be interpreted as the real number

$$(-1)^{b_1} \times 2^{(b_2 b_3 \ldots b_9)_2} \times 2^{-127} \times (1.b_{10} b_{11} \ldots b_{32})_2$$

Examples

Find the 32-bit representation of **-52.234375**.

Integer part

$$(52.)_{10} = (64.)_8 = (110\ 100.)_2$$

Fractional part

$$(.234375)_{10} = (.17)_8 = (.001\ 111)_2$$

Examples – cont.

$$(52.234375)_{10} = (110\,100.001111)_2$$
$$= (1.101\,000\,011110)_2 \times 2^5$$

The exponent is $(5)_{10}$, we need to write it as $c - 127 = 5$ so $c = 132$

The stored exponent is

$$(132)_{10} = (204)_8 = (10\,000\,100)_2$$

The representation of **-52.234375** is

$$(1\ 10\ 000\ 100\ 101\ 000\ 011\ 110\ 000\ 000\ 000\ 00)_2$$
$$= (1100\ 0010\ 0101\ 0000\ 1111\ 0000\ 0000\ 0000)_2$$
$$= (C250F000)_{16}$$

What number has the representation
$(45DE4000)_{16}$?

Errors in Representing Numbers

Non-machine numbers are represented by a nearest machine number in computer

Additional digits will be truncated

Correct rounding and **roundoff error**

For a 32-bit single-precision machine with 23 bits for mantissa, the relative error in correct rounding is $\frac{1}{2}\times 2^{-23}$

The unit roundoff error is

$$\varepsilon = 2^{-23}$$

For a computer with a 4 binary digit word length, the number 0.11101011 will be stored as 0.1110 for rounding off, and 0.1111 for correct rounding

Arithmetic Operations

Most computers use double-length arithmetic operations. Numbers are extended to double length, arithmetic operations are performed, and the result is rounded to a single length number

$$0.1231 + 0.3456 * 10^{-3}$$

$$\begin{array}{r} 0.12310000 \\ +\ 0.00034560 \\ \hline 0.12344560 \end{array}$$

On a machine with 4 digit word length, this number needs to be rounded to

$$0.1234$$

resulting in an error 0.00004560

Notation $\mathbf{fl}(x)$

Use $\mathbf{fl}(x)$ to denote the floating-point machine number corresponding to a real number x

$$\mathbf{fl}(x) = x(1 + \delta),$$

For the 32-bit machine, we have

$$\left(|\delta| \leq \frac{1}{2}\varepsilon \right)$$

It is easy to see that if $\epsilon < 2^{-23}$, then

$$\mathbf{fl}(1 + \epsilon) = 1$$

Note that the numbers are smaller than ε. It has more than 23 zeros.

Inverse-Error Analysis

More generally known as **backward error analysis**

Denote $\odot$ as one of the basic arithmetic operations, then

$$\mathbf{fl}(x \odot y) = (x \odot y)(1 + \delta), \qquad \left(|\delta| \le 2^{-24}\right)$$

Two interpretations

$$\mathbf{fl}(z) = (x + y)(1 + \delta) \qquad \text{perturbation of sum}$$
$$\mathbf{fl}(z) = x(1 + \delta) + y(1 + \delta) \qquad \text{sum of perturbations}$$

Direct-error analysis and reverse-error analysis

Forward-error analysis and backward-error analysis

Loss of Significance

Subtraction of two nearly equal numbers may result in loss of significant digits on a finite precision machine

Cure: reprogram or use higher precision arithmetic, may be costly

Use a 4-word length computer:

$$0.11111111 - 0.11101010 = 0.00010101 = 0.1010 \times 10^{-3}$$
$$0.1111 \quad - 0.1110 \quad = 0.00010000 = 0.1000 \times 10^{-3}$$

Loss of Precision Theorem

Let x and y be normalized floating-point machine numbers with $x > y > 0$. If $2^{-p} \leq 1 - y/x \leq 2^{-q}$ for some positive integers p and q, then at most p and at least q significant digits are lost in the subtraction $x - y$

Loss of Significance

$$x = r \times 2^n, \quad y = s \times 2^m, \quad \text{where} \quad \frac{1}{2} \le r, s \le 1$$

Since *y*<*x*, the computer has to shift *y* before carrying out the subtraction

$$y = (s2^{m-n}) \times 2^n$$

so

$$x - y = (r - s2^{m-n}) \times 2^n$$

Look at the mantissa

$$r - s2^{m-n} = r(1 - \frac{s2^m}{r2^n}) = r(1 - \frac{y}{x}) < 2^{-q}$$

$$(> r2^{-p})$$

Hence, to normalize the representation of *x*-*y*, a shift of at least *q* bits to the left is necessary. Thus at least *q* (spurious) zeros are supplied on the right-hand side of the mantissa

An Example from Book

$x = 37.593621$ and $y = 37.584216$

$$1 - \frac{y}{x} = 0.0002501754,$$

which is between $2^{-12} = 0.000244$ and $2^{-11} = 0.000488$. At least 11 and at most 12 binary digits are lost when computing $x - y$

Exactly how many digits lost depends on a computer

$x = 0.37593612 \times 10^2$ and $y = 0.37584216 \times 10^2$

Suppose a machine has 5 decimal digits of accuracy

We have $\tilde{x} = 0.37594 \times 10^2$ and $\tilde{y} = 0.37584 \times 10^2$,
The machine computes

$$\tilde{x} - \tilde{y} = 0.00010 = 0.10000 \times 10^{-3}$$

An Example Cont.

Exact computation

$$x - y = 0.00009396 = 0.9396 \times 10^{-4}$$

The relative error is

$$\frac{|(x-y)-(\tilde{x}-\tilde{y})|}{|x-y|} = \frac{0.604\times10^{-5}}{0.9396\times10^{-4}} = 0.06428$$

This relative error is considered to be large since the machine has 5 decimal digits of accuracy

Note that the double length computing operations are carried out after the numbers are stored
If we use a machine with at least 8 decimal digits of accuracy, we can have the exact value as $0.93960000 \times 10^{-4}$

Avoiding Loss of Significance

Analyze possible loss of significance, reschedule computation to avoid subtraction of two nearly equal numbers, modify algorithm

Example. Evaluating

$$f(x) = \sqrt{x^2 + 1} - 1 \qquad at \qquad x \approx 0$$

Using 5-decimal-digit arithmetic for $x = 10^{-3}$, we have $f(x) = 0$

Rationalizing the numerator, we have

$$f(x) = \frac{x^2}{\sqrt{x^2 + 1} + 1}$$

computing $\sqrt{(10^{-3})^2 + 1} + 1 = 0.2 \times 10^{1}$, and $f(x) = 0.5 \times 10^{-6}$.

More Examples

Evaluating $f(x) = x - \sin x$ at $x \approx 0$

Using Taylor series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Then

$$f(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots$$

Compute $x = 0.1$ with four-decimal-digit arithmetic, $\sin(0.1) = 0.9983 \times 10^{-1}$. So $x - \sin x = 0.17 \times 10^{-3}$. But $x^3/3! = 0.1667 \times 10^{-3}$. This strategy is not good for large x. For $x = \pi$, $x - \sin \pi = \pi$, but $x^3/3! = 5.1677$.

Range reduction for periodic functions

$$\sin(12532.14) \approx \sin(3.47 + 1994 \times 2\pi) = \sin(3.47)$$