Introduction to

Algorithm Design and Analysis

[08] logn search

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In the last class …

- **Selection - warm up**
  - Max and min
  - Second largest
- **Selection - rank k (median)**
  - Expected linear time
  - Worst-case linear time
- **Adversary argument**
  - Lower bound

The Searching Problem

- **Searching v.s. Selection**
  - Search for “Alice” or “Bob”
    - The key itself matters
  - Select the “rank 2” student
    - The partial order relation matters
- **Expected cost for searching**
  - Brute force case: O(n)
  - Ideal case: O(1)
  - Can we achieve O(logn)?

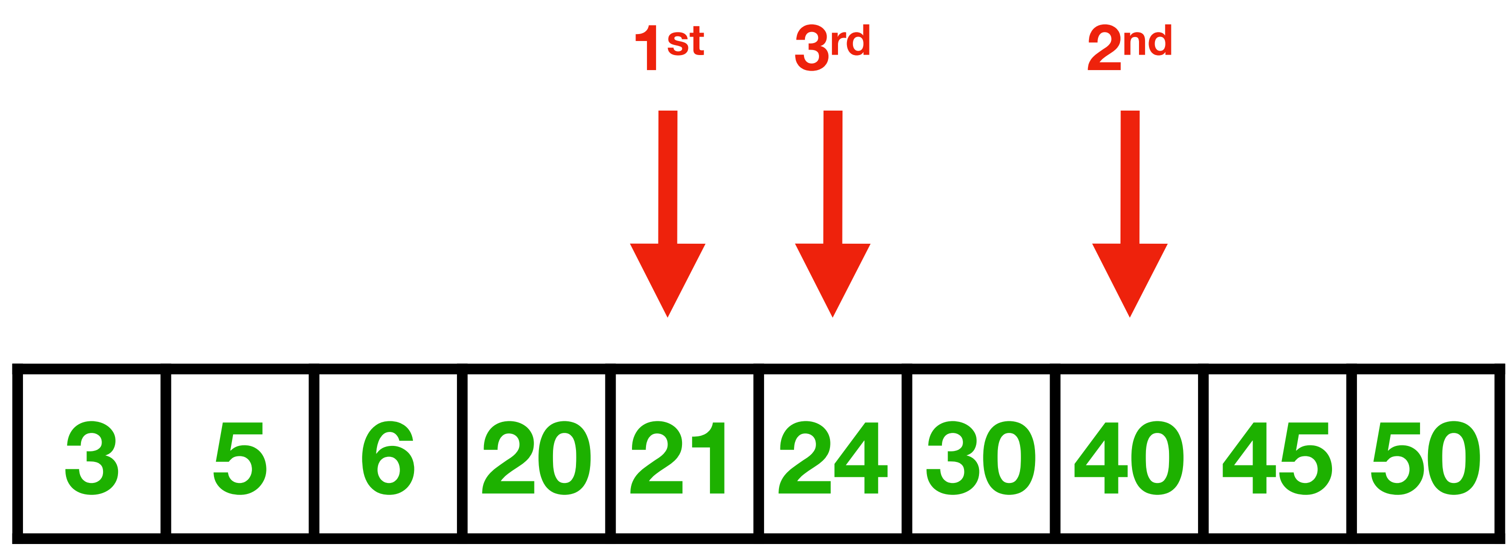
The Searching Problem

- **Essential of searching**
    - How to organize the data to enable efficient search
    - logn search
        - Each search cuts off half of the search space
        - How to organize the data to enable logn search
- **logn search techniques**
    - Warmup
        - Binary search over sorted sequences
    - Balanced Binary Search Tree (BST)
        - Red-black tree

Binary Search by Example

- **Binary search for “24”**
  - Divide the search space
  - Cut off half the space after each search

The sequence is already sorted



Binary Search Generalized

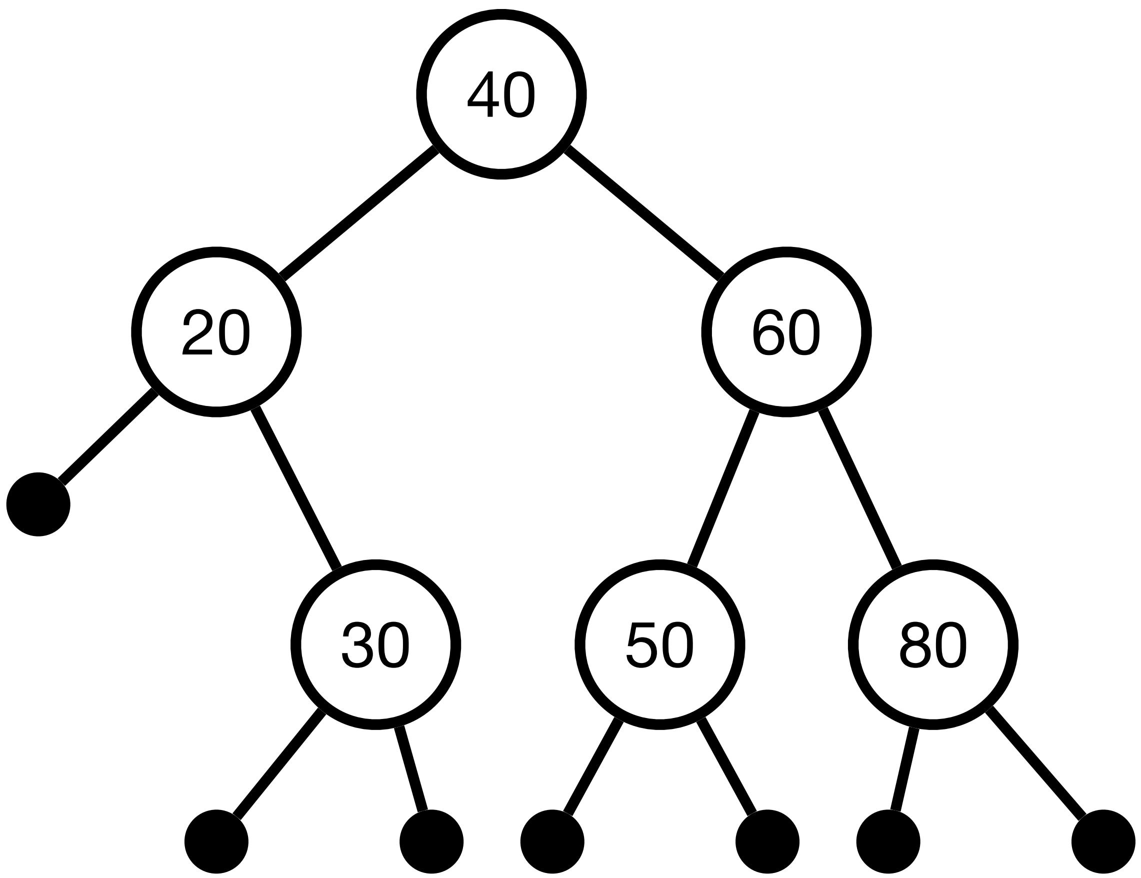
- **Peak-number**
  - Uni-modal array
- **Least number not in the array**
  - Sorted array of natural numbers
- **A[i]=i**
  - Sorted array of integers

Balanced Binary Search Tree

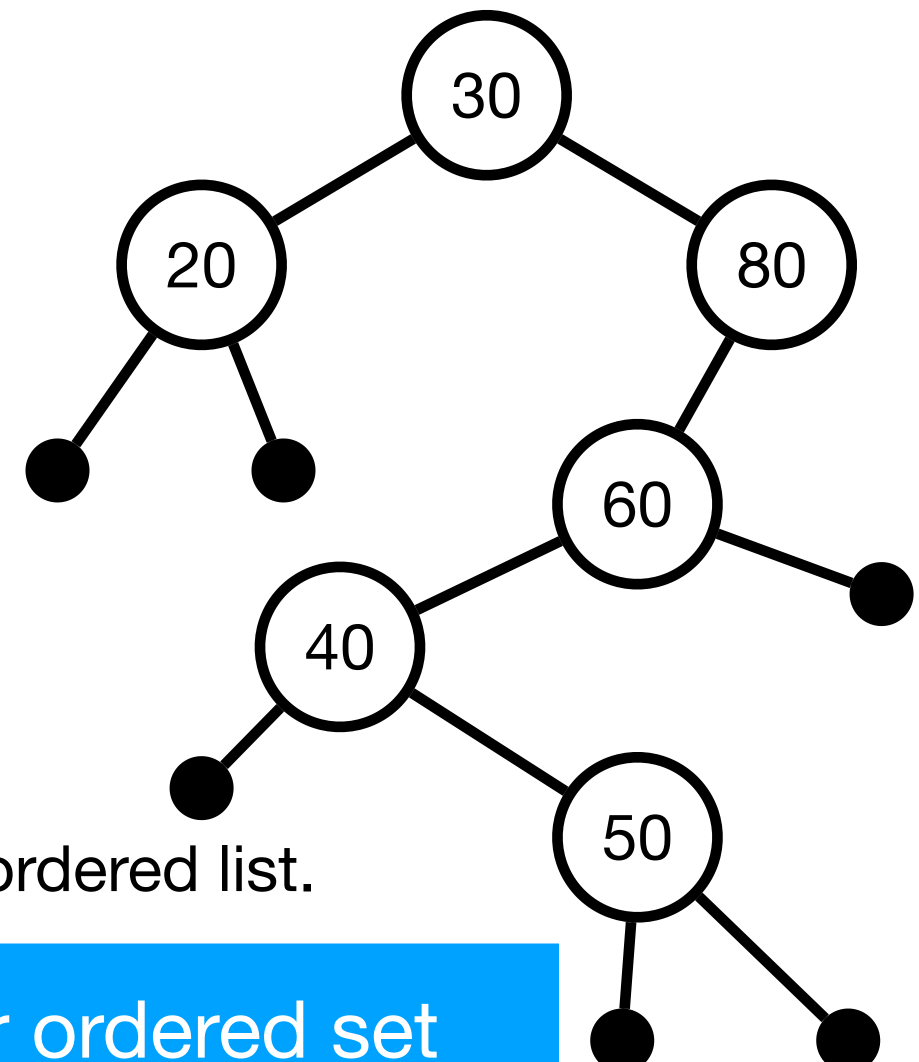
- **Binary search tree (BST)**
  - Definitions and basic operations
- **Definition of Red-Black Tree (RBT)**
  - Black height
- **RBT operations**
  - Insertion into a red-black tree
  - Deletion from a red-black tree

Binary Search Tree Revisited

Good balancing
$\Theta(\log n)$



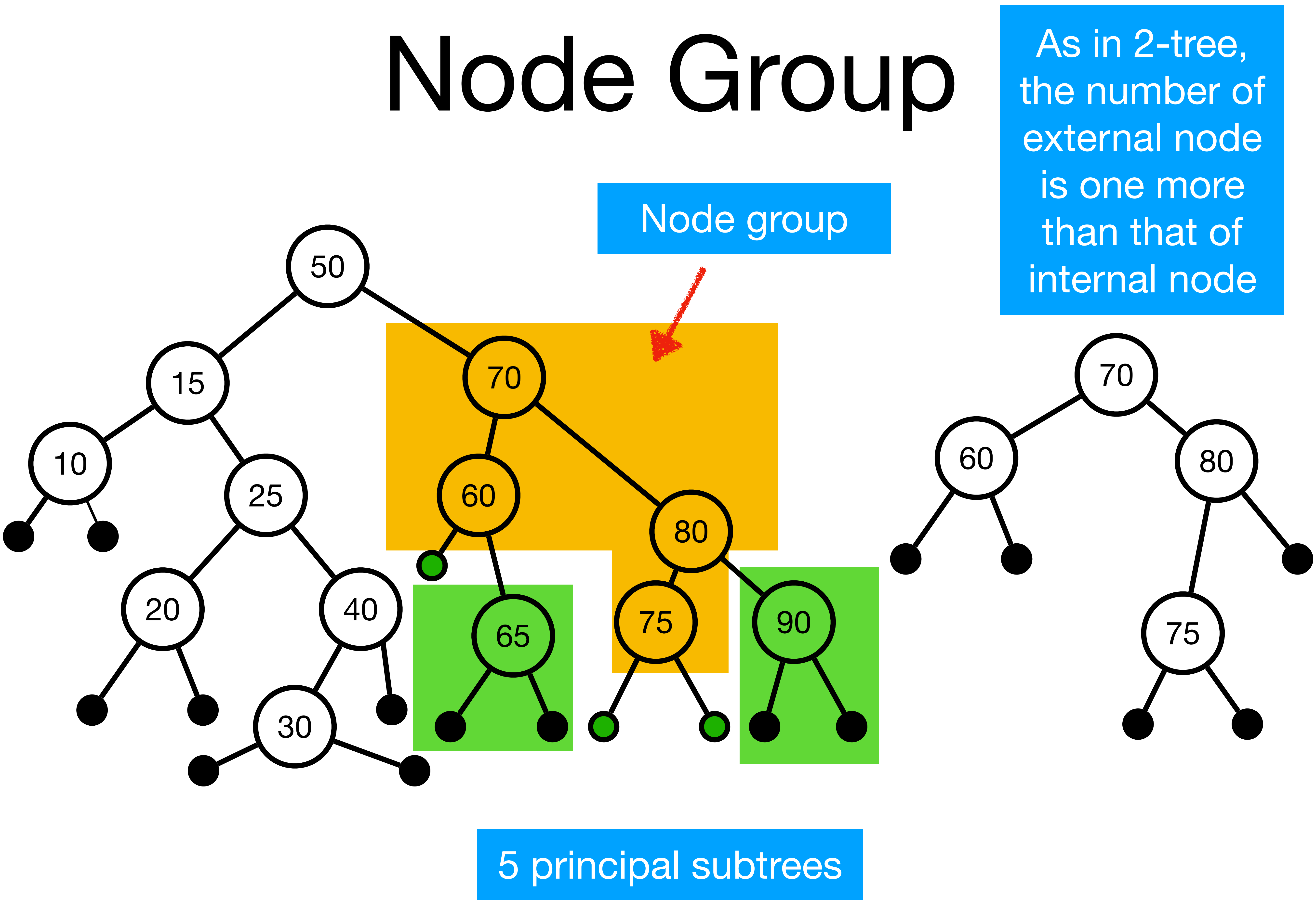
Poor balancing
$\Theta(n)$



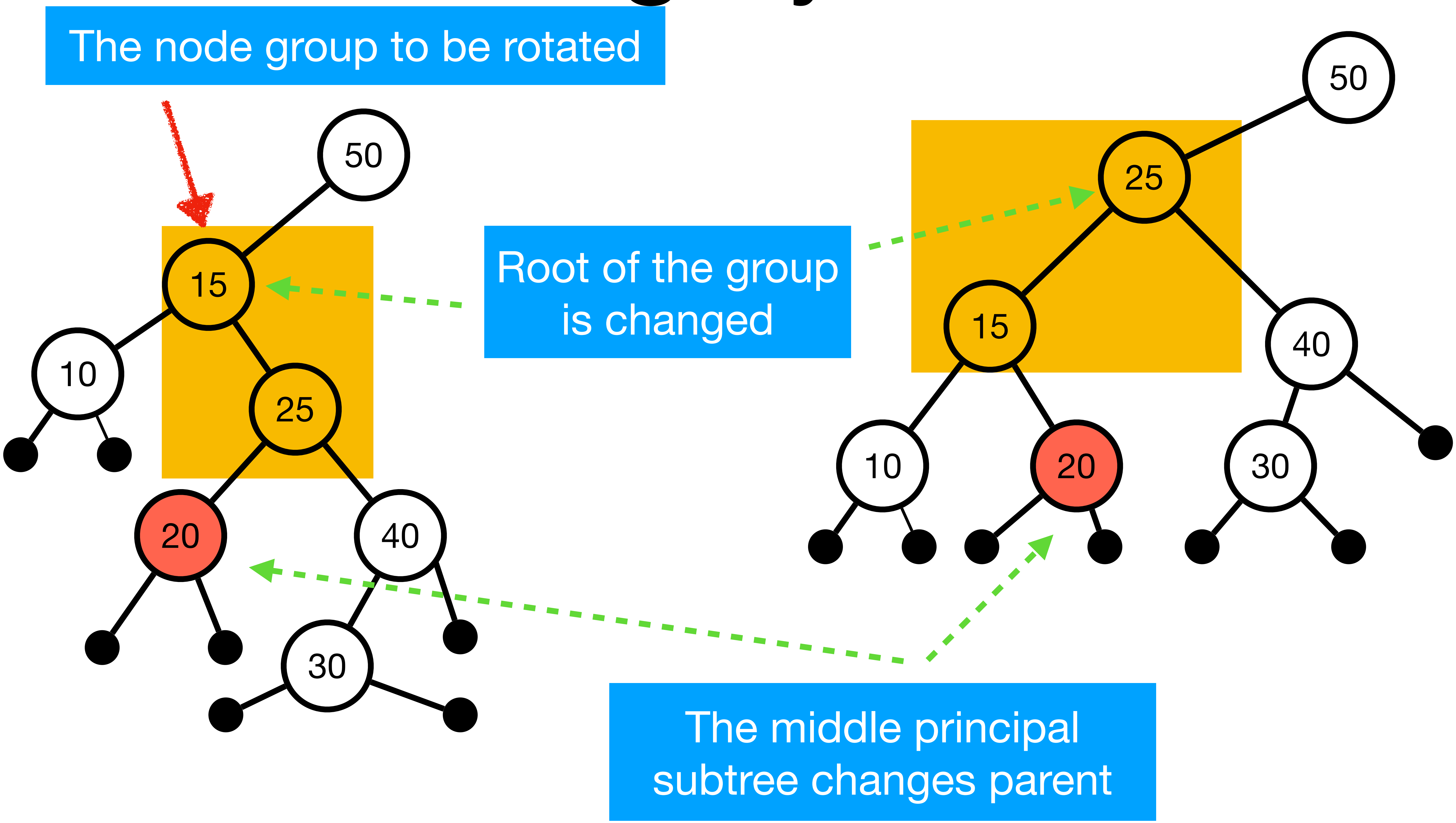
In a properly drawn tree, pushing forward to get the ordered list.

Each node has a key, belonging to a linear ordered set
An inorder traversal produces a sorted list of the keys

Node Group


As in 2-tree, the number of external node is one more than that of internal node
Node group
50
15
70
10
25
60
80
20
40
65
75
90
30
70
60
80
75
5 principal subtrees

Balancing by Rotation



Red-Black Tree: Definition

- **If T is a binary search tree in which each node has a color, red or black, and all external nodes are black, then T is a red-black tree if and only if:**
  - [Color constraint] No red node has a red child
  - [Black height constraint] The black length of all external paths from a given node u is the same (the black height of u)
  - The root is black.
- **Almost-red-black tree (ARB tree)**
  - Root is red, satisfying the other constraints.

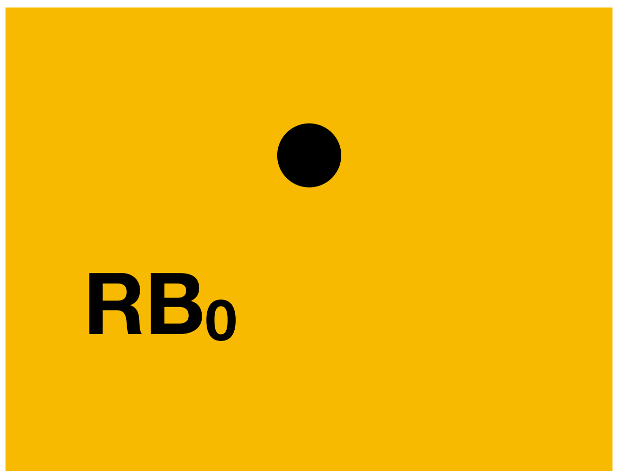
Balancing is under control

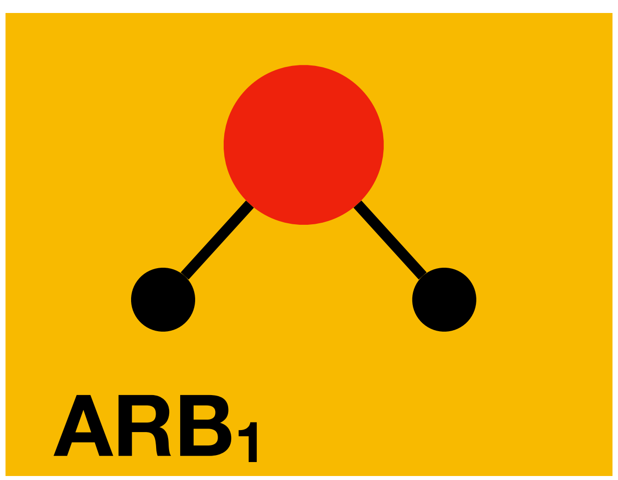
Recursive Definition of RBT

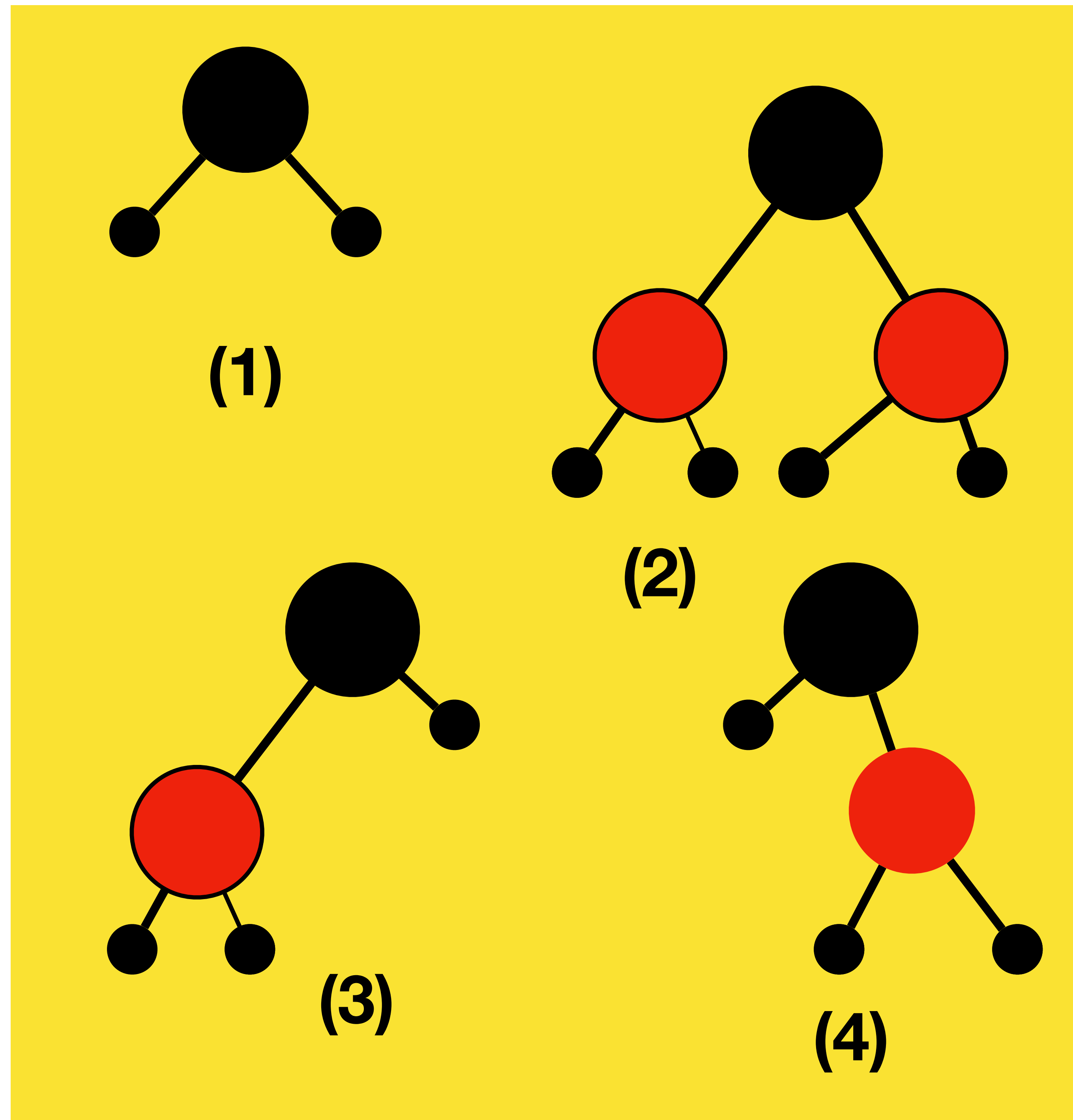
(A red-black tree of black height h is denoted as RB_h)

- **Definition**
    - An external node is an RB_0 tree, and the node is black.
    - A binary tree is an ARB_h ($h \geq 1$) tree if:
        - Its root is red, and
        - Its left and right sub trees are each an RB_{h-1} tree.
    - A binary tree is an RB_h ($h \geq 1$) tree if:
        - Its root is black, and
        - Its left and right sub trees are each either an RB_{h-1} tree or an ARB_h tree.

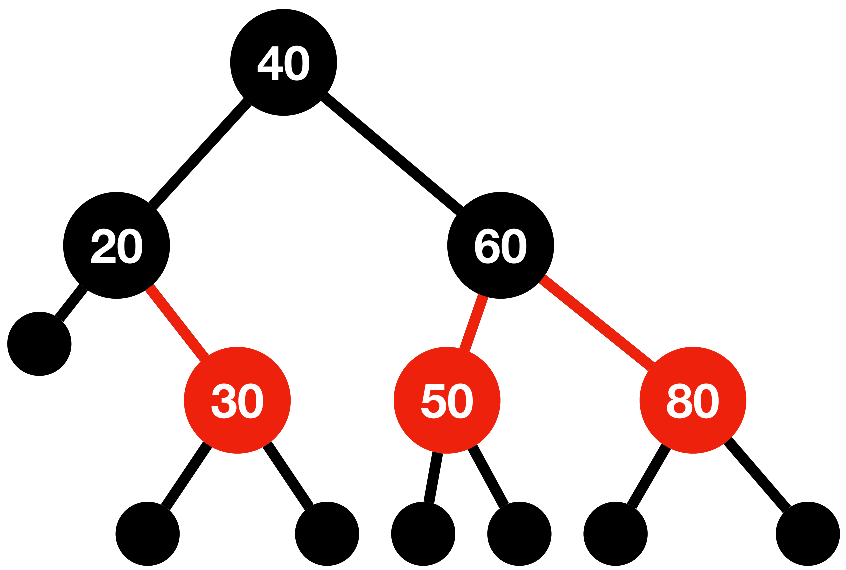
RB_i and ARB_i



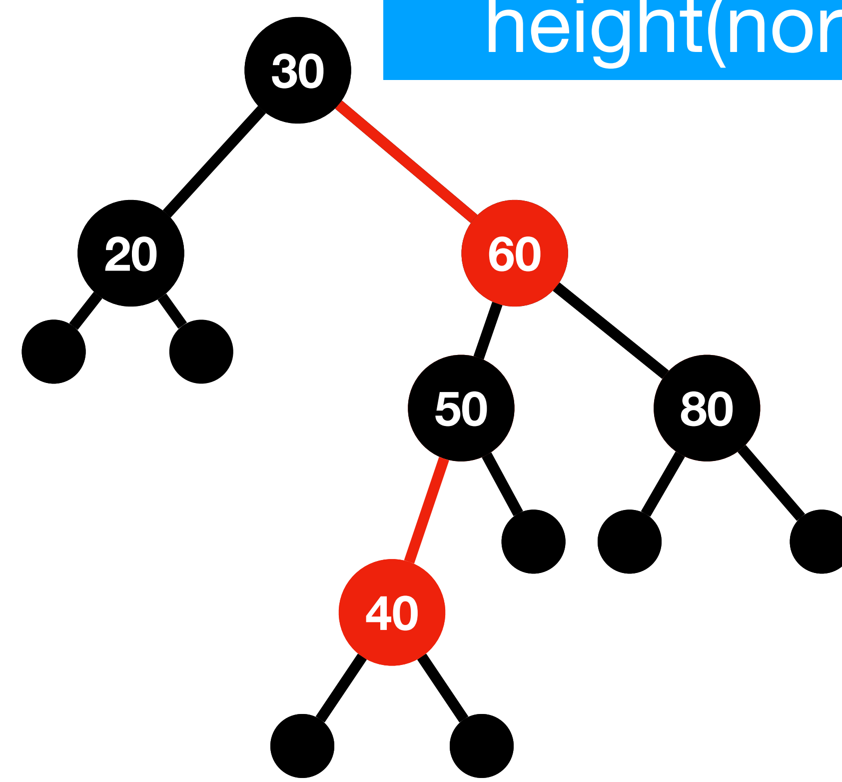




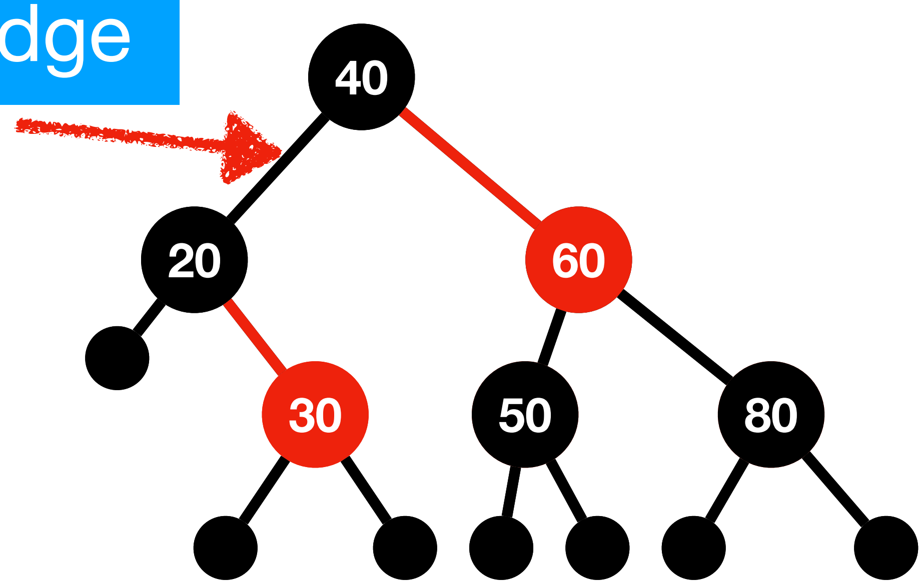
Red-Black Tree with 6 Nodes



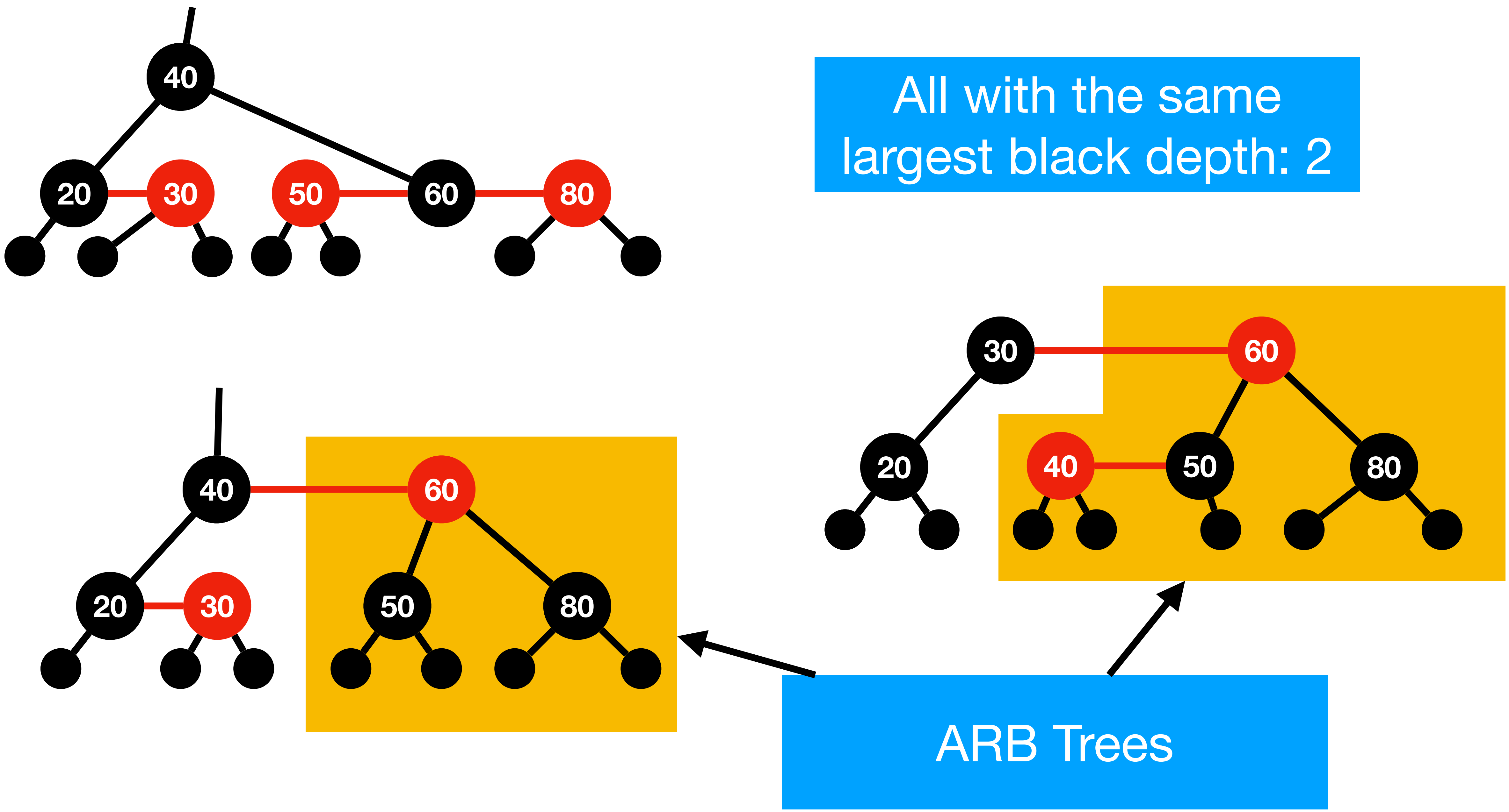
poorest balancing: height(normal) is 4



Black edge



Black-depth Convention



Properties of Red-Black Tree

- **The black height of any RB_h tree or ARB_h tree is well-defined and is h.**
- **Let T be an RB_h tree, then:**
  - T has at least 2^h-1 internal black nodes.
  - T has at most 4^h-1 internal nodes.
  - The depth of any black node is at most twice its black depth.
- **Let A be an ARB_h tree, then:**
  - A has at least 2^h-2 internal black nodes.
  - A has at most $(4^h)/2$-1 internal nodes.
  - The depth of any black node is at most twice its black depth.

Well-defined Black Height

- **That "the black height of any RB_h tree or ARB_h tree is well defined" means the black length of all external paths from the root is the same.**
- **Proof: induction on h**
- **Base case: h=0, that is RB_0 (there is no ARB_0)**
- **In ARB_{h+1}, its two subtrees are both RB_h. Since the root is red, the black length of all external paths from the root is h, that's the same as its two subtrees.**
- **In RB_{h+1}:**
    - Case 1: two subtrees are RB_h's
    - Case 2: two subtrees are ARB_{h+1}'s
    - Case 3: one subtree is an RB_h (black height=h), and the another is an ARB_{h+1} (black height=h+1)

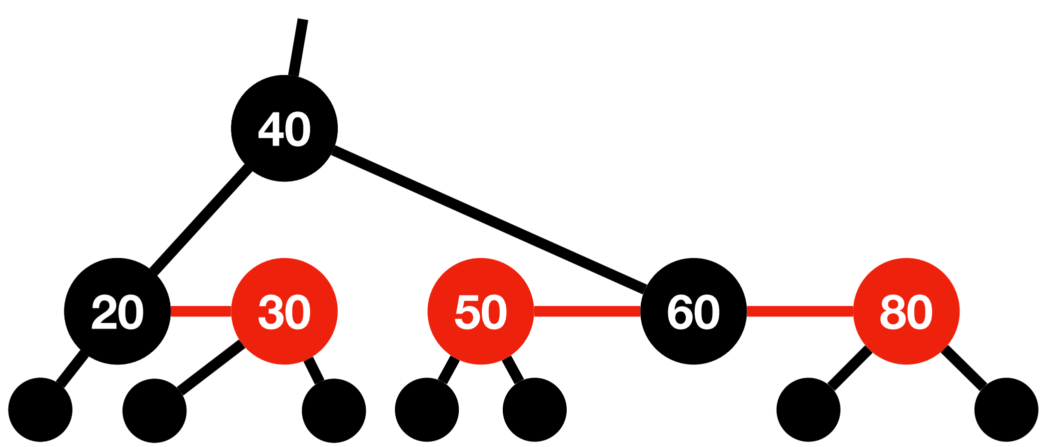
Bound on Depth of Node in RBTree

- **Let T be a red-black tree with n internal nodes. Then no node has black depth greater than log(n+1), which means that the height of T in the usual sense is at most 2log(n+1).**
  - Proof:
  - Let h be the black height of T. The number of internal nodes, n, is at least the number of internal black nodes, which is at least 2^h-1, so $h \leq \log(n+1)$. The node with greatest depth is some external node. All external nodes are with black depth h. So, the depth is at most 2h.

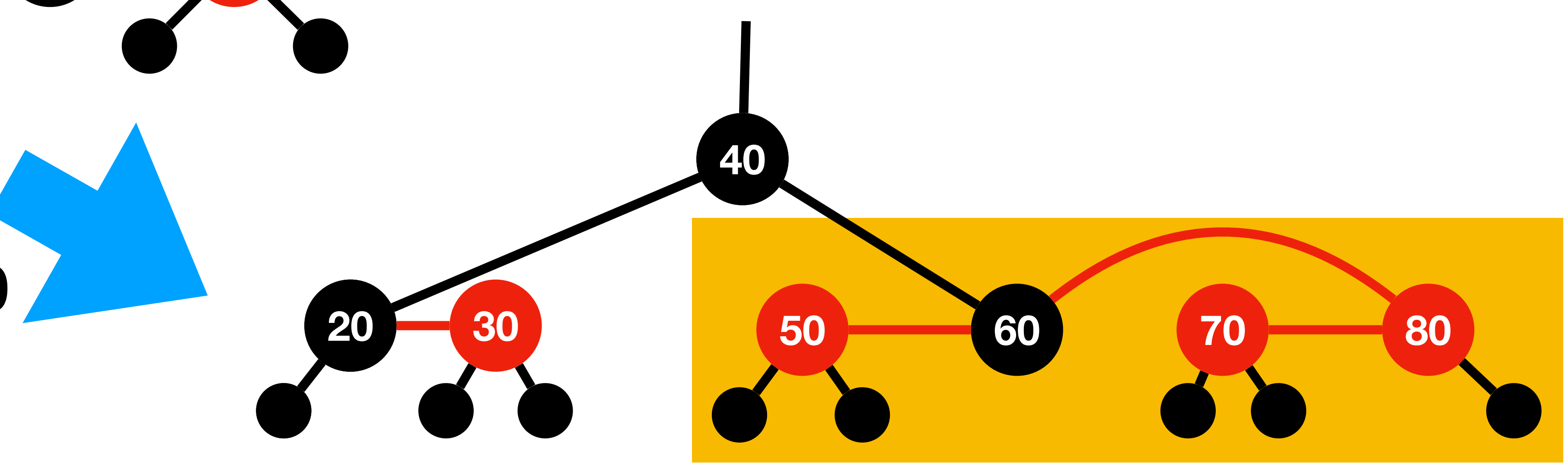
Influences of Insertion to an RBT

- **Black height constraint:**
  - No violation if inserting a red node.
- **Color constraint:**

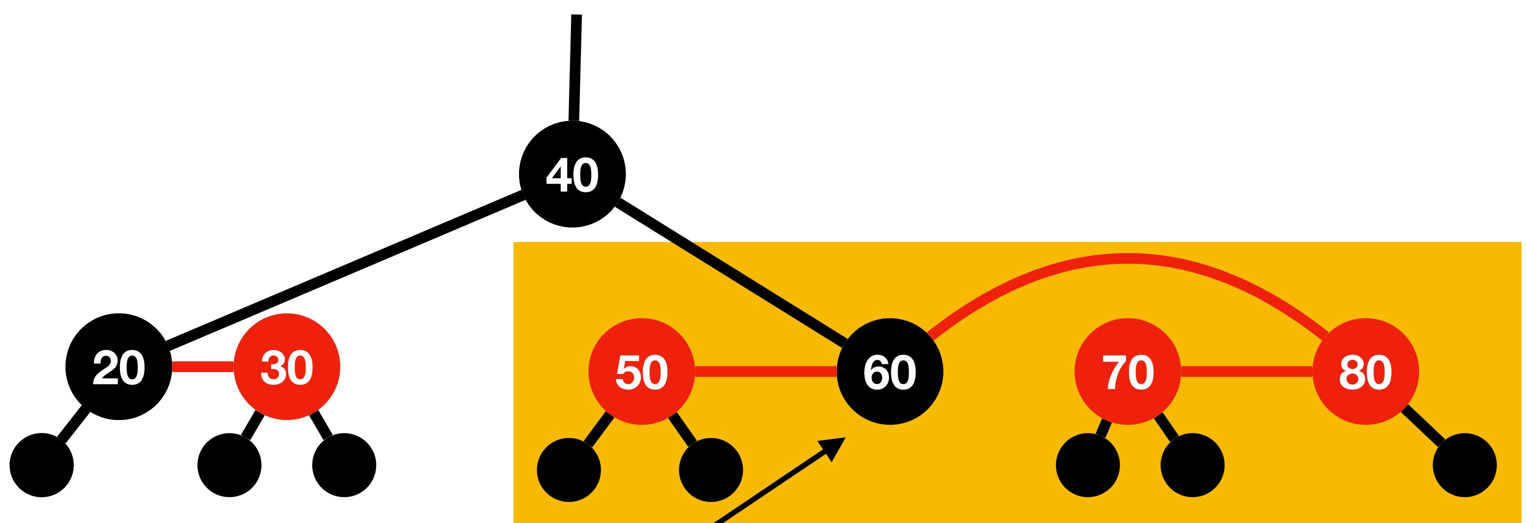
Critical clusters(external nodes excluded), which originated by color violation, with 3 or 4 red nodes



Inserting 70

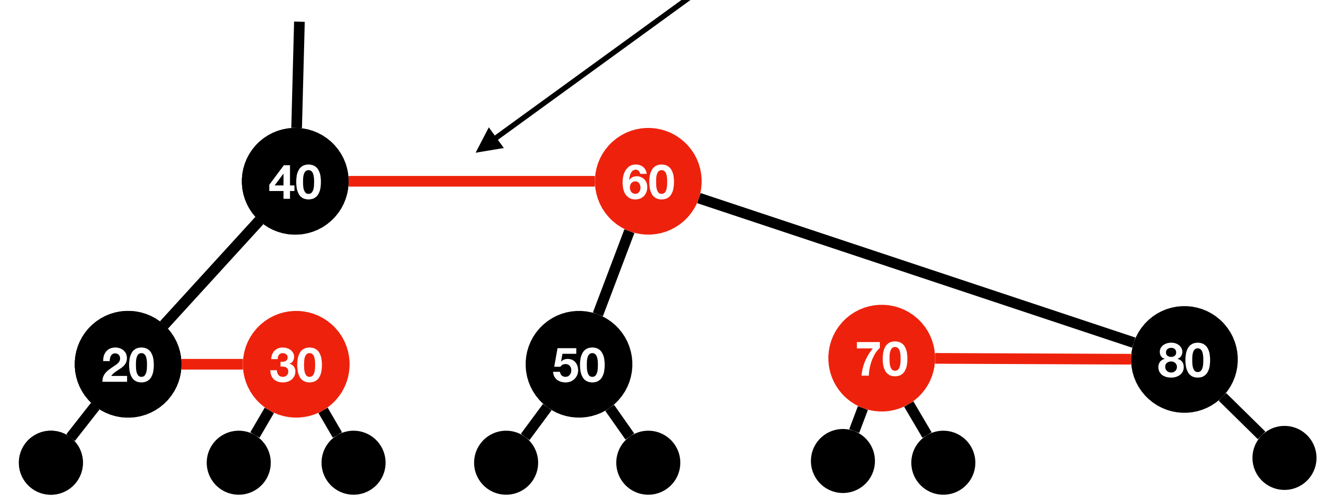


Repairing 4-node Critical Cluster



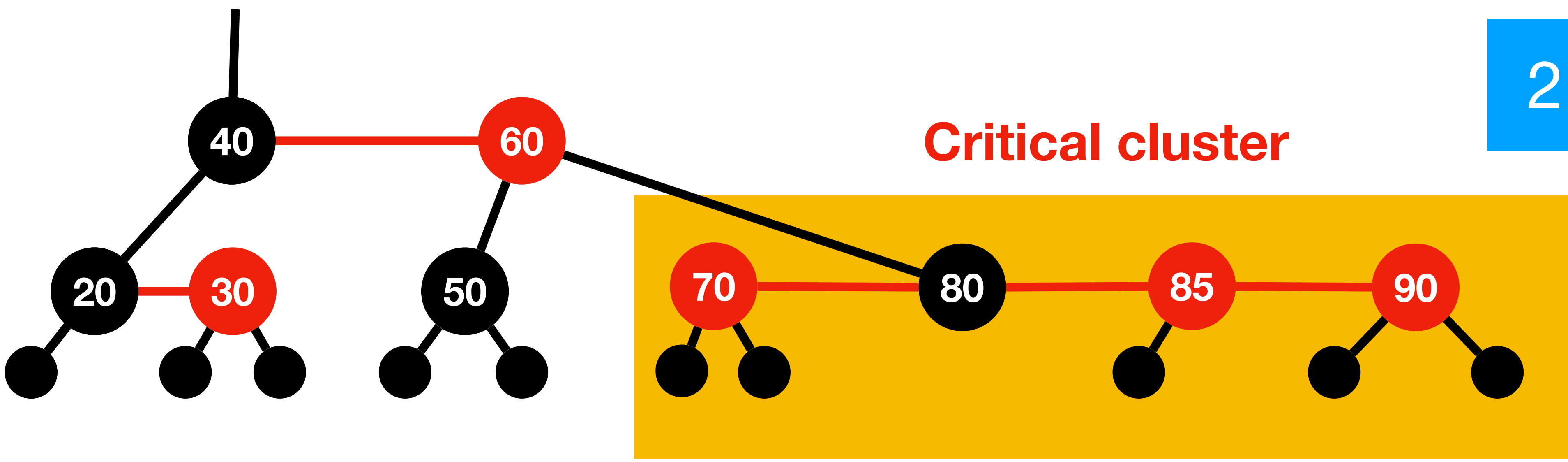
No new critical cluster occurs, inserting finished

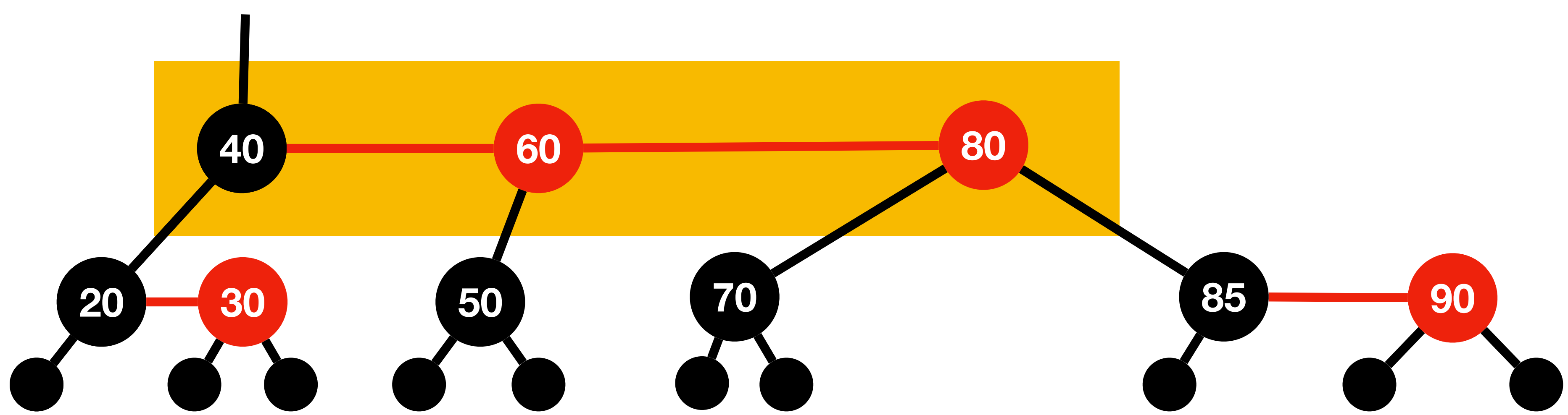
Color flip:
Root of the critical cluster exchanges color with its subtrees



Repairing 4-node Critical Cluster

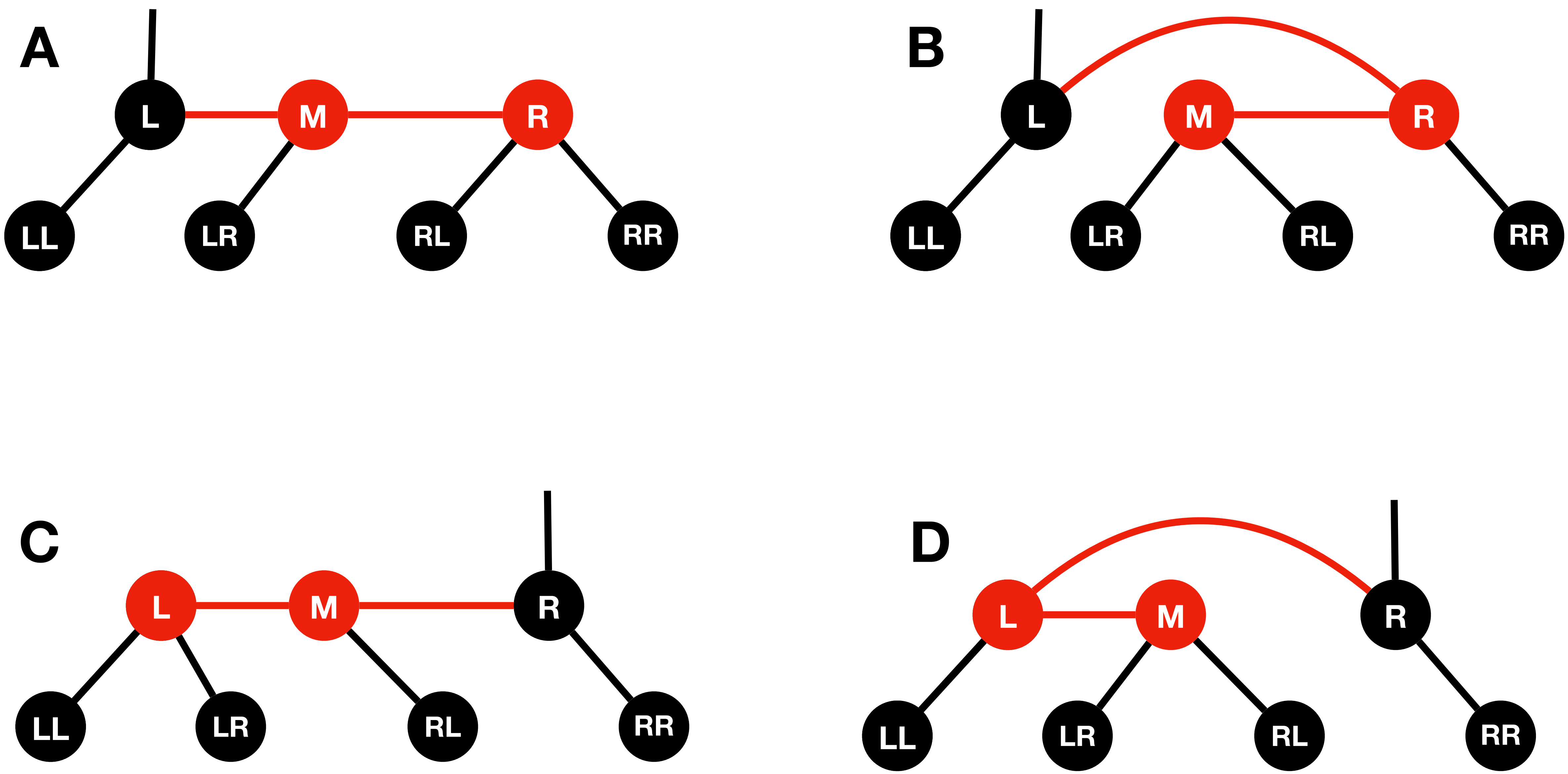
2 more insertions





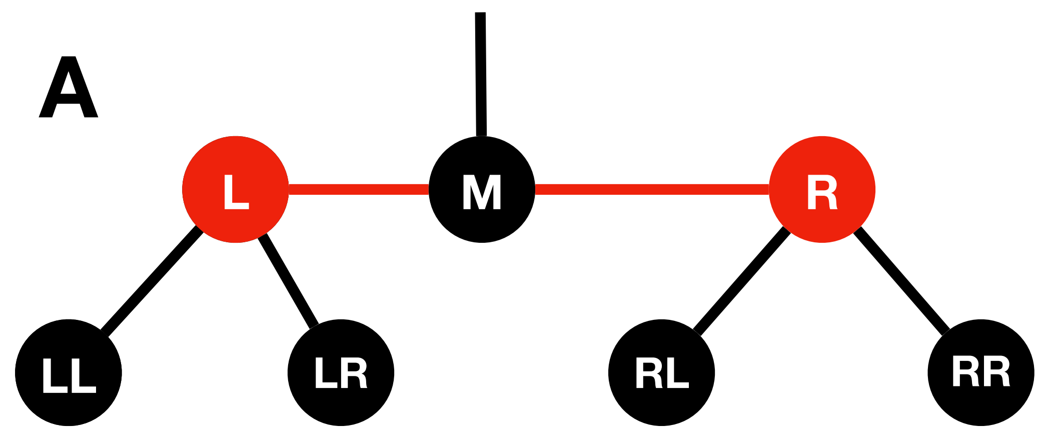
New critical cluster with 3 nodes. Color flip doesn't work, why?

Patterns of 3-node Critical Cluster

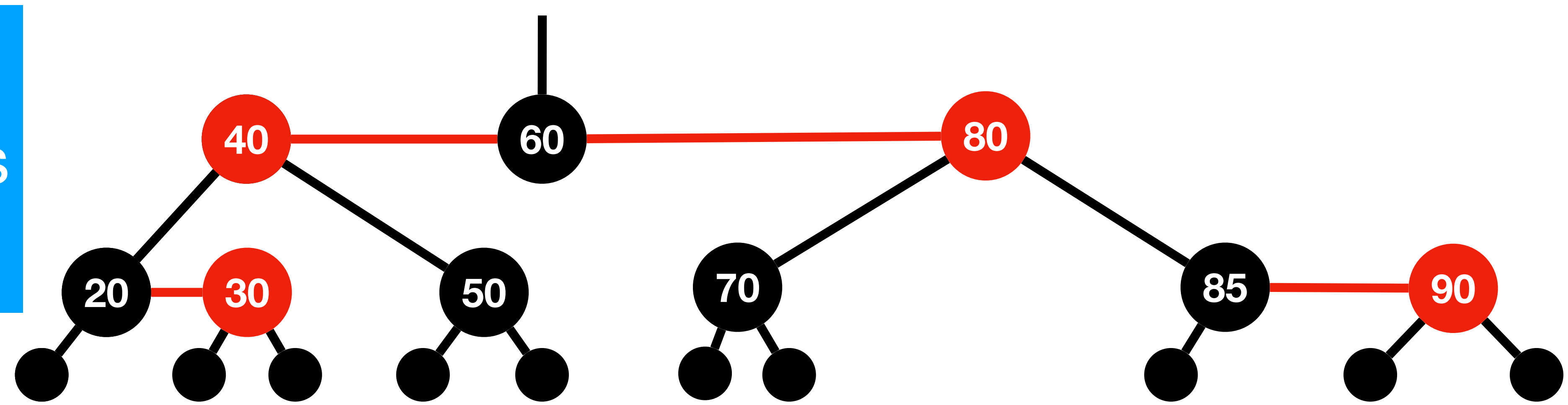


Repairing 3-node Critical Cluster

Root of the critical cluster is changed to M, and the parent ship is adjusted accordingly



The incurred critical cluster is of pattern A



Implementing Insertion: Class

```
class RBTree
    Element root;
    RBTree leftSubtree;
    RBTree rightSubtree;
    int color; /*red, black*/;

    static class InsReturn
        public RBTree newTree;
        public int status /* ok, rbr, brb, rrb, brr */
```

Implementing Insertion: Procedure

```
RBTree rbtInsert(RBtree oldRBtree, Element newNode)
    InsReturn ans = rbtIns(oldREtree, newNode);
    if(ans.newTree.color != black)
        ans.newTree.color = black;
    return ans.newTree;
```

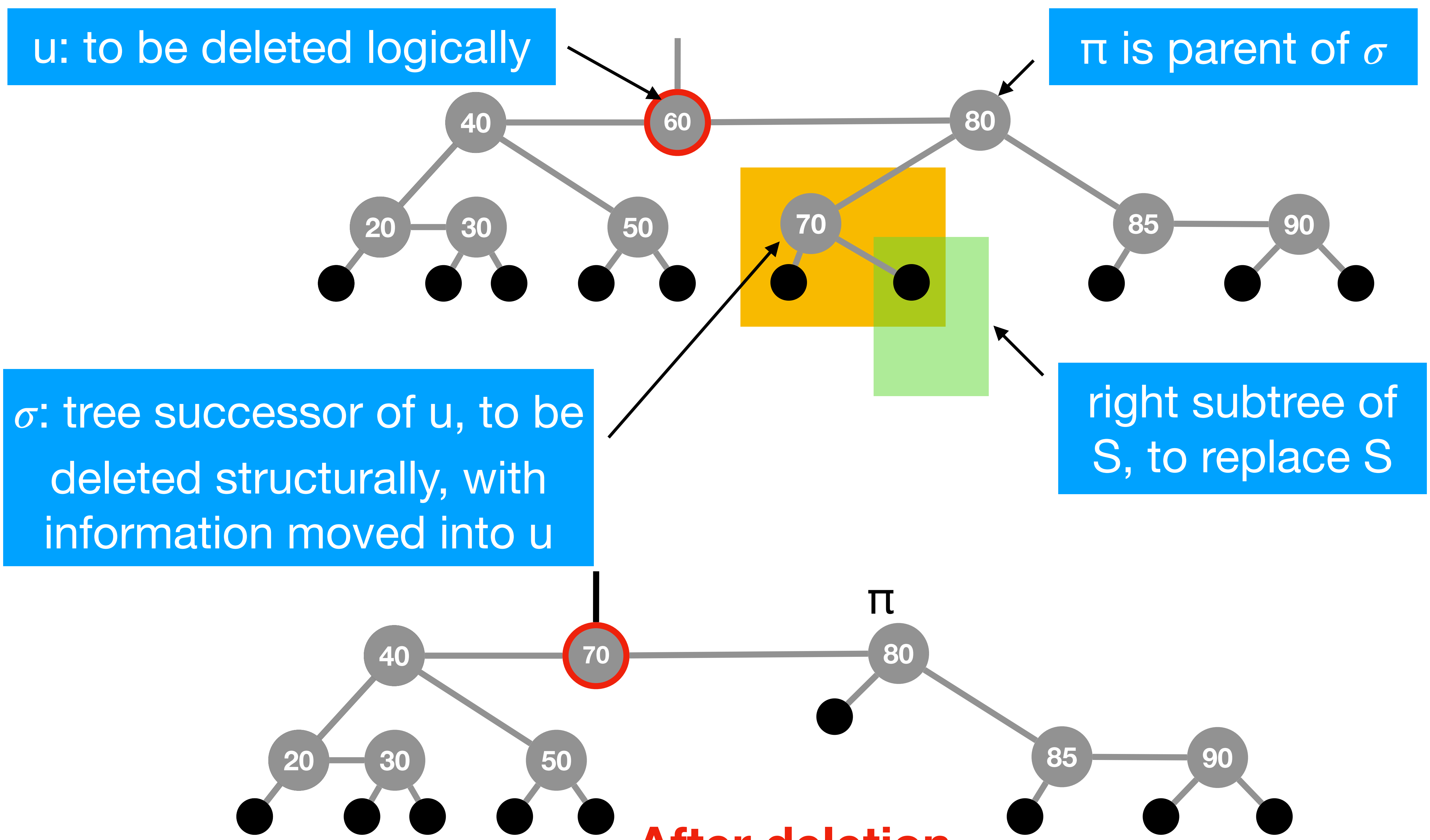
Implementing Insertion: Procedure

```
InsReturn rbtIns(RBtree oldRBtree, Element newNode)
   InsReturn ans, ansLeft, ansRight;
   if (oldRBtree = nil) then <Inserting simply>;
   else
      if (newNode.key < oldRBtree.root.key)
         ansLeft = rbtIns(oldRBtree.leftSubtree, newNode);
         ans = repairLeft(oldRBtree, ansLeft);
      else
         ansRight = rbtIns(oldRBtree.rightSubtree, newNode);
         ans = repairRight(oldRBtree, ansRight);
   return ans
```

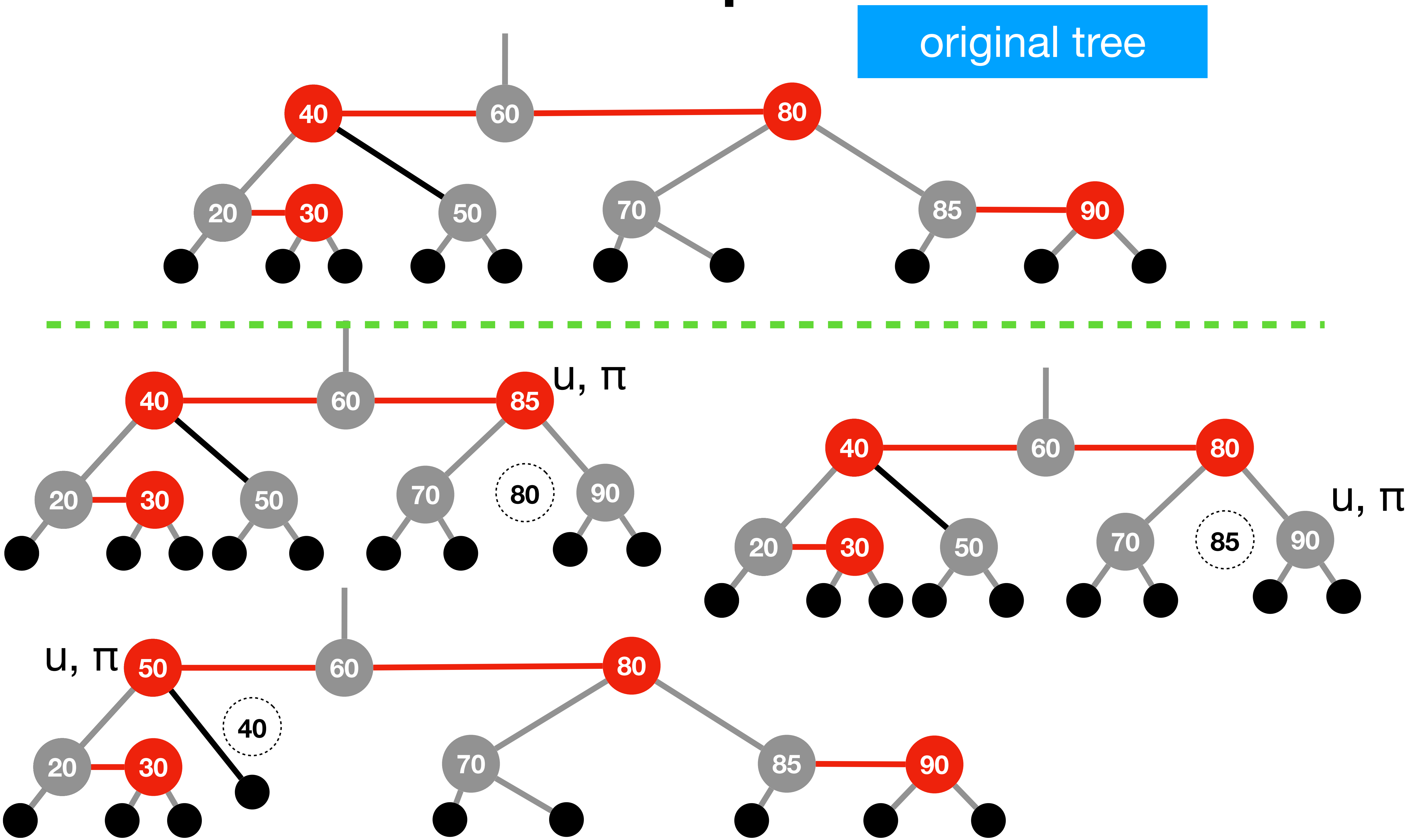
Correctness of Insertion

- **If the parameter oldRBtree of rbtIns is an RB_h tree or an ARB_{h+1} tree (which is true for the recursive calls on rbtIns), then the newTree and status fields returned are one of the following combinations:**
  - Status=ok, and newTree is an RB_h or an ARB_{h+1} tree,
  - Status=rbr, and newTree is an RB_h,
  - Status=brb, and newTree is an ARB_{h+1} tree,
  - Status=rrb, and newTree.color=red, newTree.leftSubtree is an ARB_{h+1} tree and newTree.rightSubtree is an RB_h tree,
  - Status=brr, and newTree.color=red, newTree.rightSubtree is an ARB_{h+1} tree and newTree.leftSubtree is an RB_h tree
- **For those cases with red root, the color will be changed to black, with other constraints satisfied by repairing subroutines.**

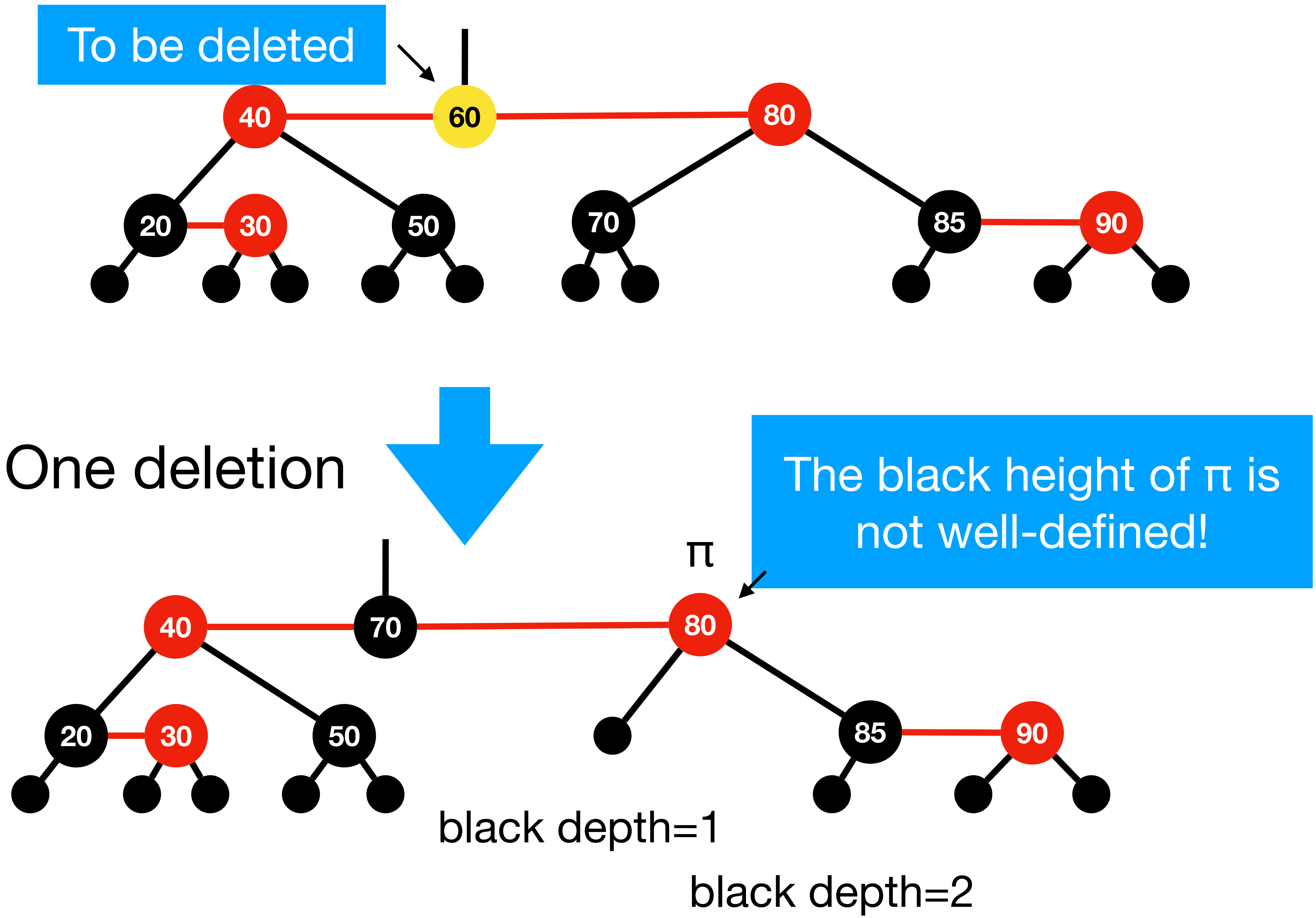
Deletion: Logical and Structural



Deletion from RBT - Examples



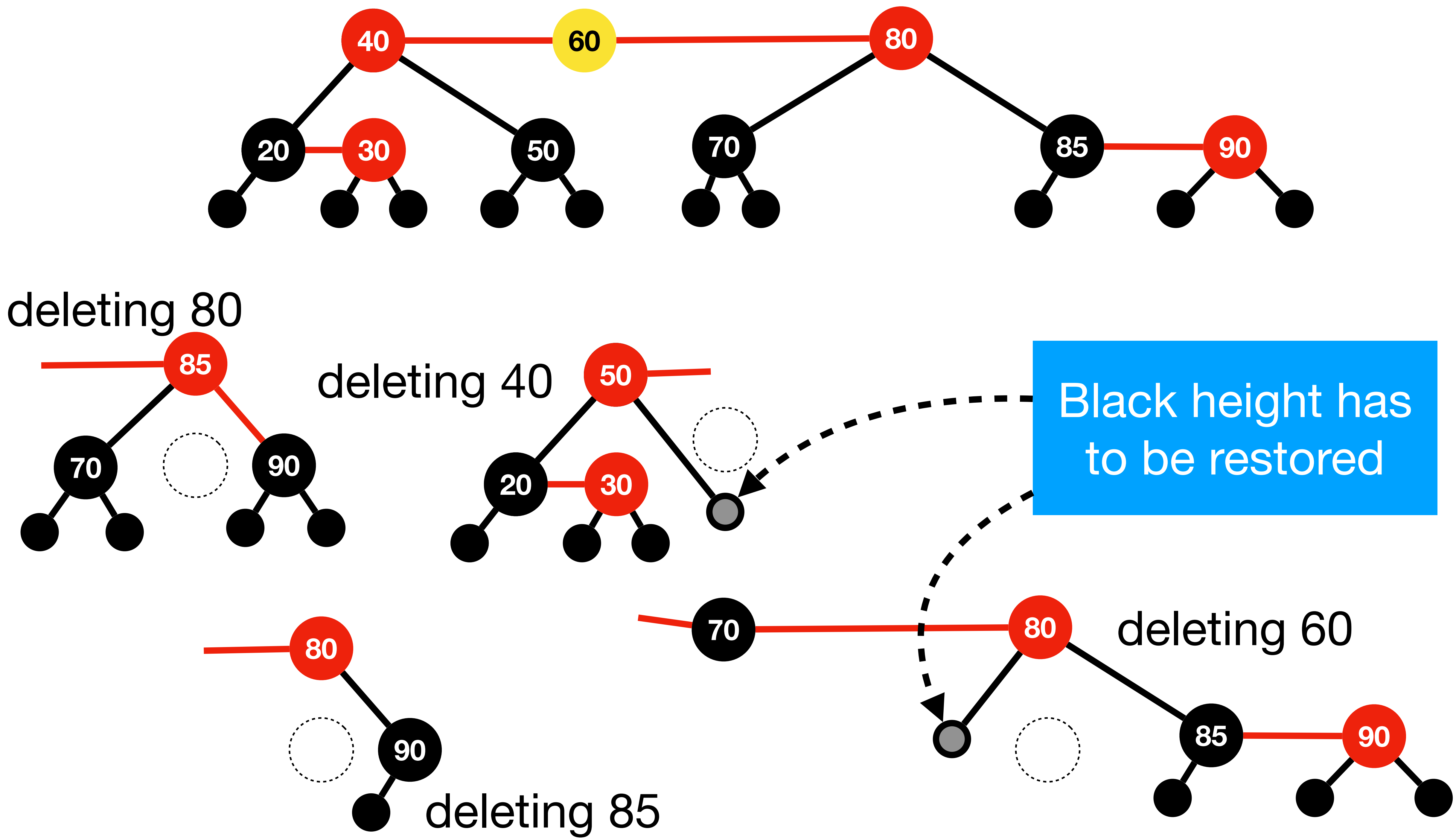
Deletion in RBT



Procedure of Red-Black Deletion

- Do a standard BST search to locate the node to be logically deleted, call it u
- If the right child of u is an external node, identify u as the node to be structurally deleted.
- If the right child of u is an internal node, find the tree successor of u, call it σ, copy the key and information from σ to u. (color of u not changed) Identify σ as the node to be deleted structurally.
- Carry out the structural deletion and repair any imbalance of black height.

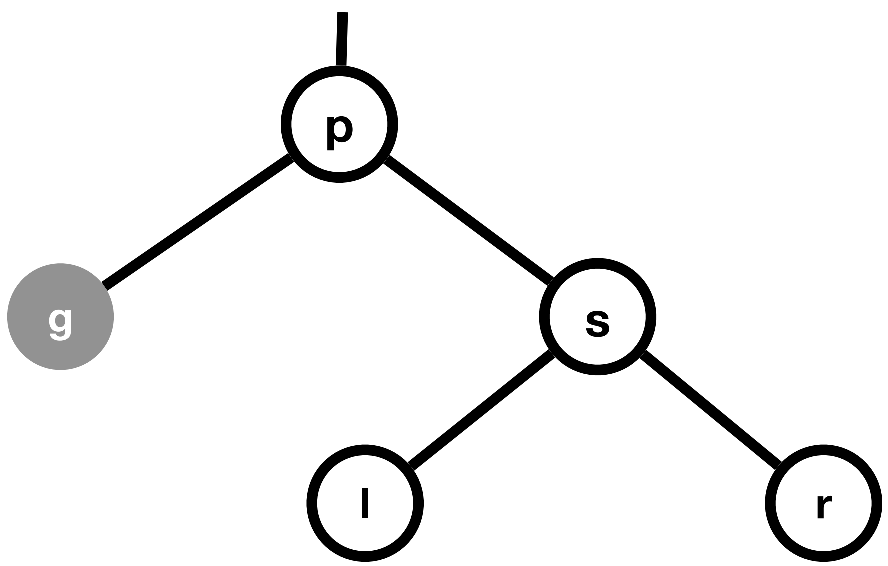
Imbalance of Black Height



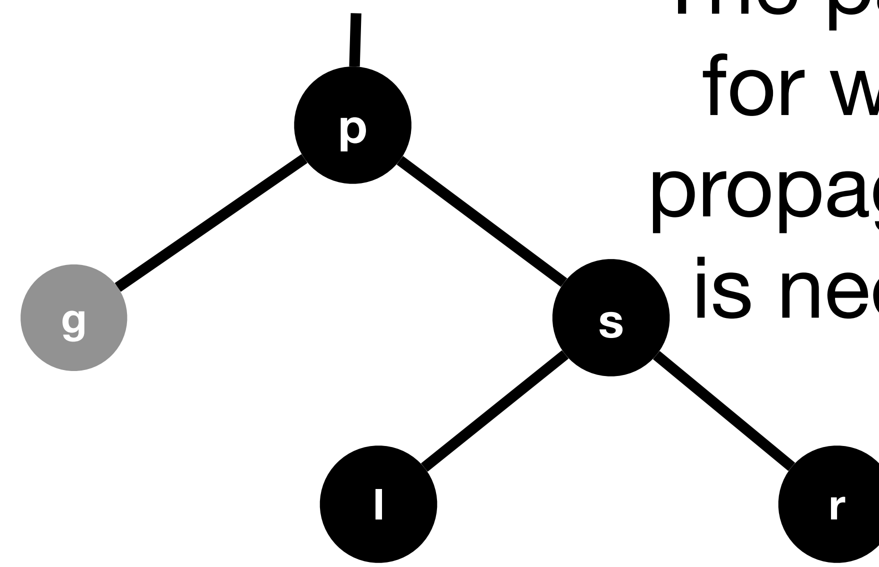
Analysis of Black Imbalance

- **The imbalance occurs when:**
  - A black node is deleted structurally, and
  - Its right subtree is black (external)
- **The result is:**
  - An RB_{h-1} occupies the position of an RB_h as required by its parent, coloring it as a "gray" node.
- **Solution:**
  - Find a red node and turn it black as locally as possible.
  - The gray color might propagate up the tree.

Propagation of Gray Node

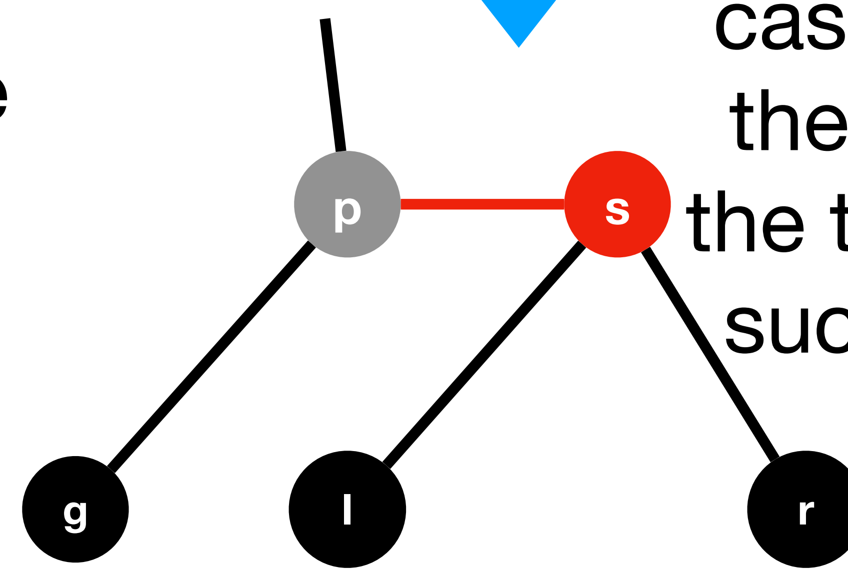


Map of the vicinity of **g**, the gray node



The pattern for which propagation is needed

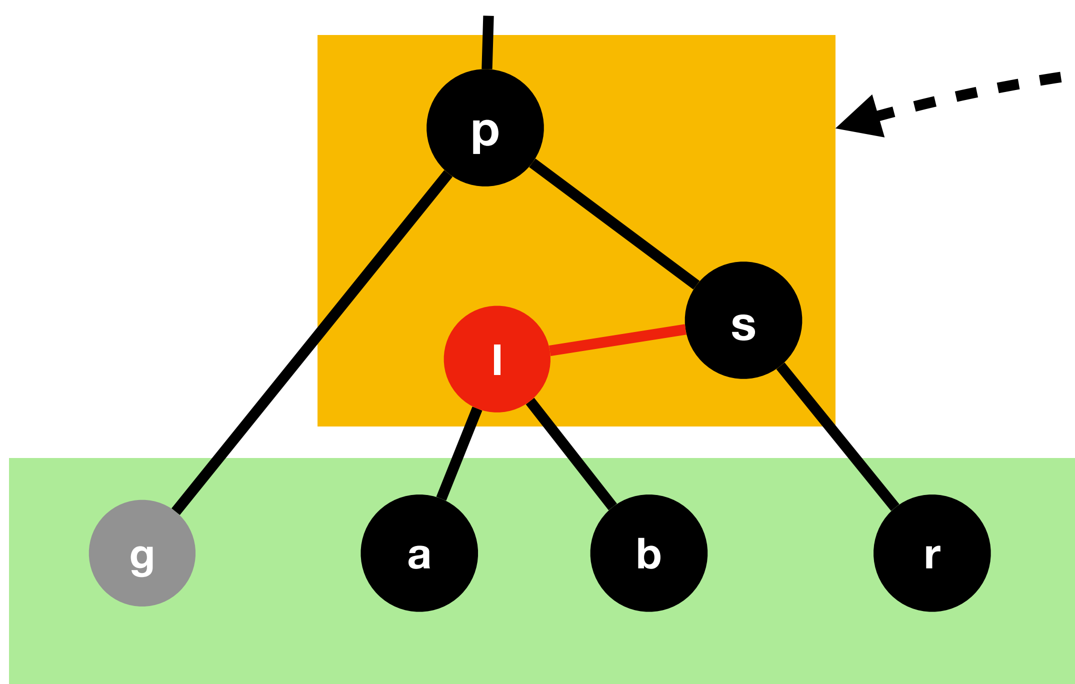
Gray up



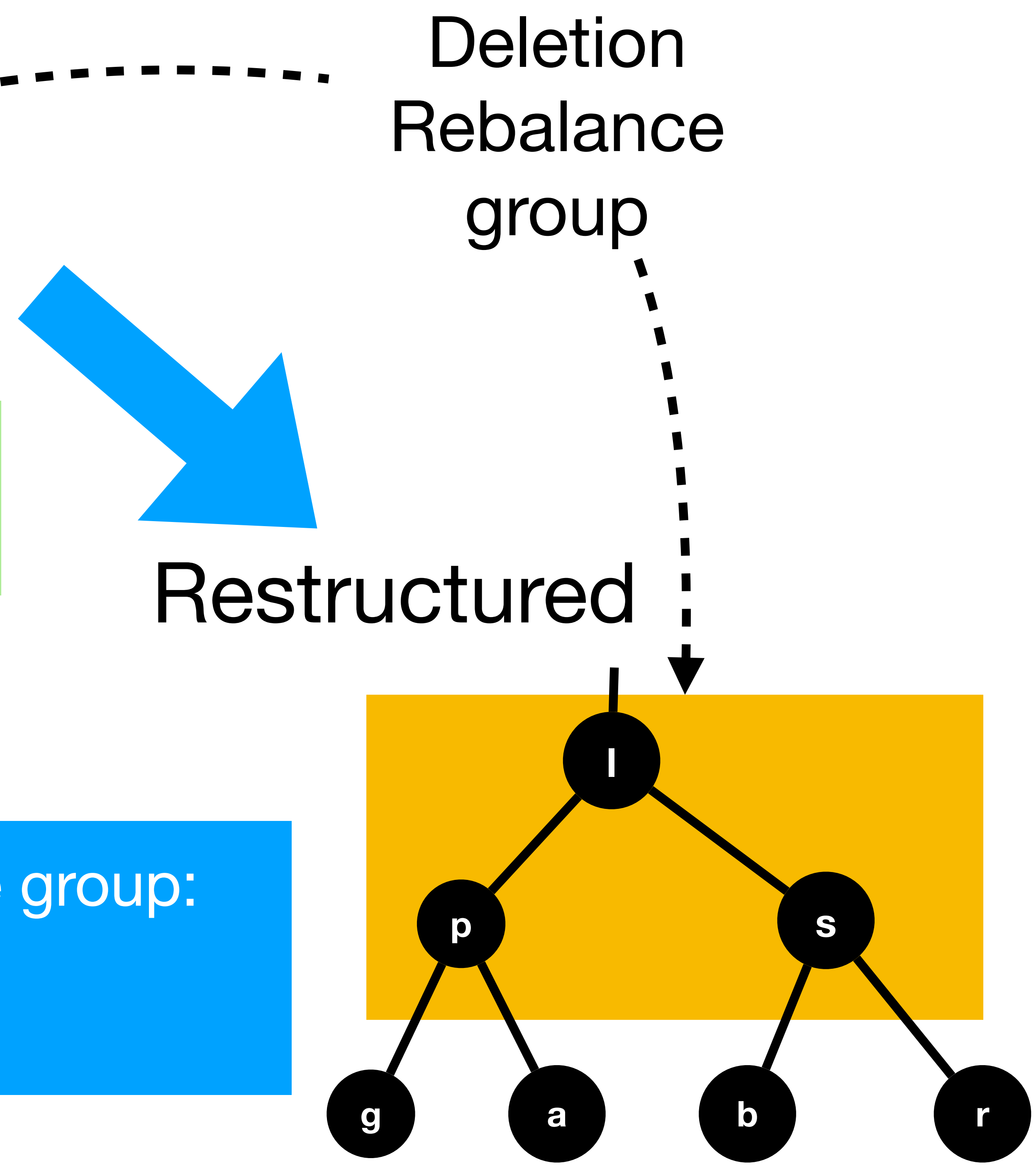
In the worst case, up to the root of the tree, and successful

G-subtree gets well-defined black height, but that is less than that required by its parent

Repairing without Propagation

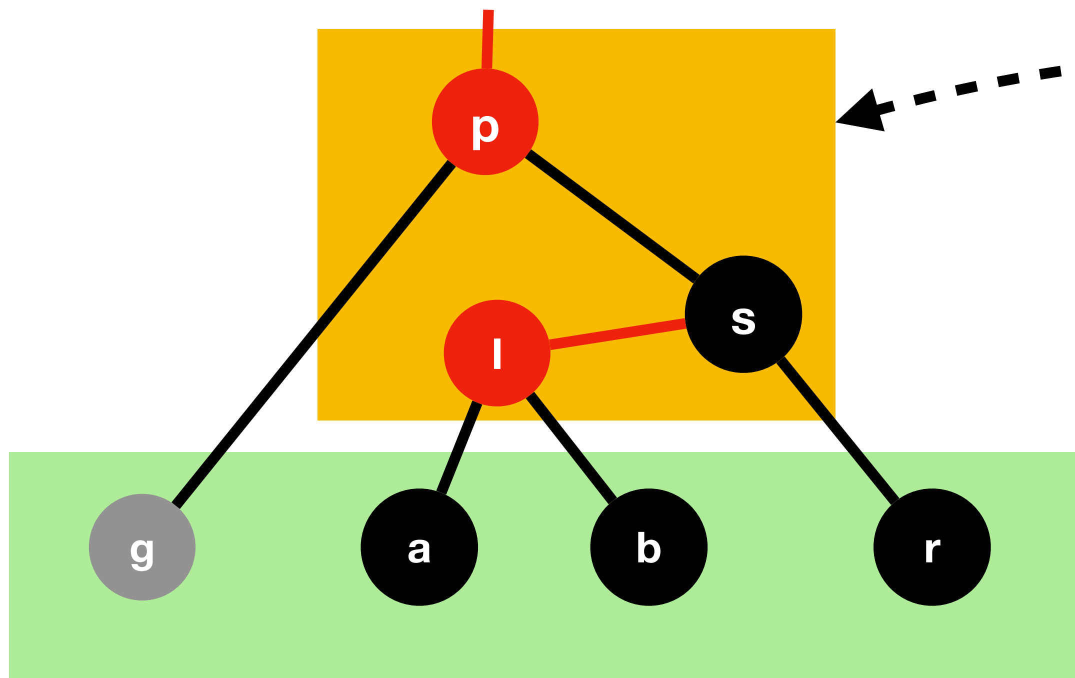


4 principal subtrees, RB_{h-1}

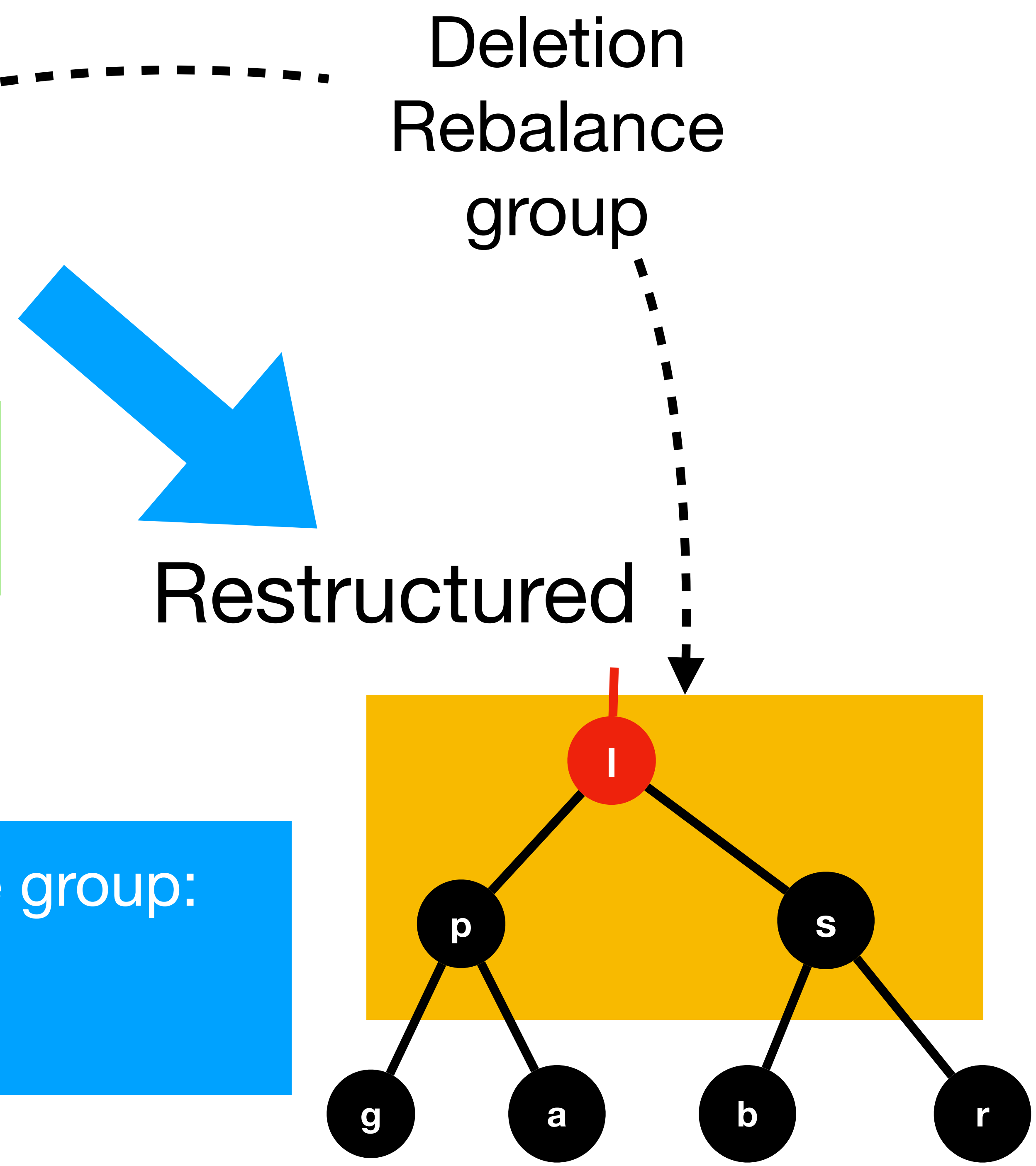


Restructuring the deletion rebalance group:
Red p: form an RB_1 or ARB_2 tree
Black p: form an RB_2 tree

Repairing without Propagation



4 principal subtrees, RB_{h-1}



Restructuring the deletion rebalance group:
Red p: form an RB_1 or ARB_2 tree
Black p: form an RB_2 tree

Complexity of Operations on RBT

- **With reasonable implementation**
  - A new node can be inserted correctly in a red-black tree with n nodes in (logn) time in the worst case.
  - Repairs for deletion do O(1) structural changes, but may do O(logn) color changes.

Thank you!
Q & A