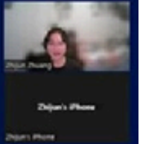
Localization and Mapping

Introduction to Bayes Filter

Zhijun Zhuang
University of Pennsylvania

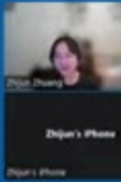


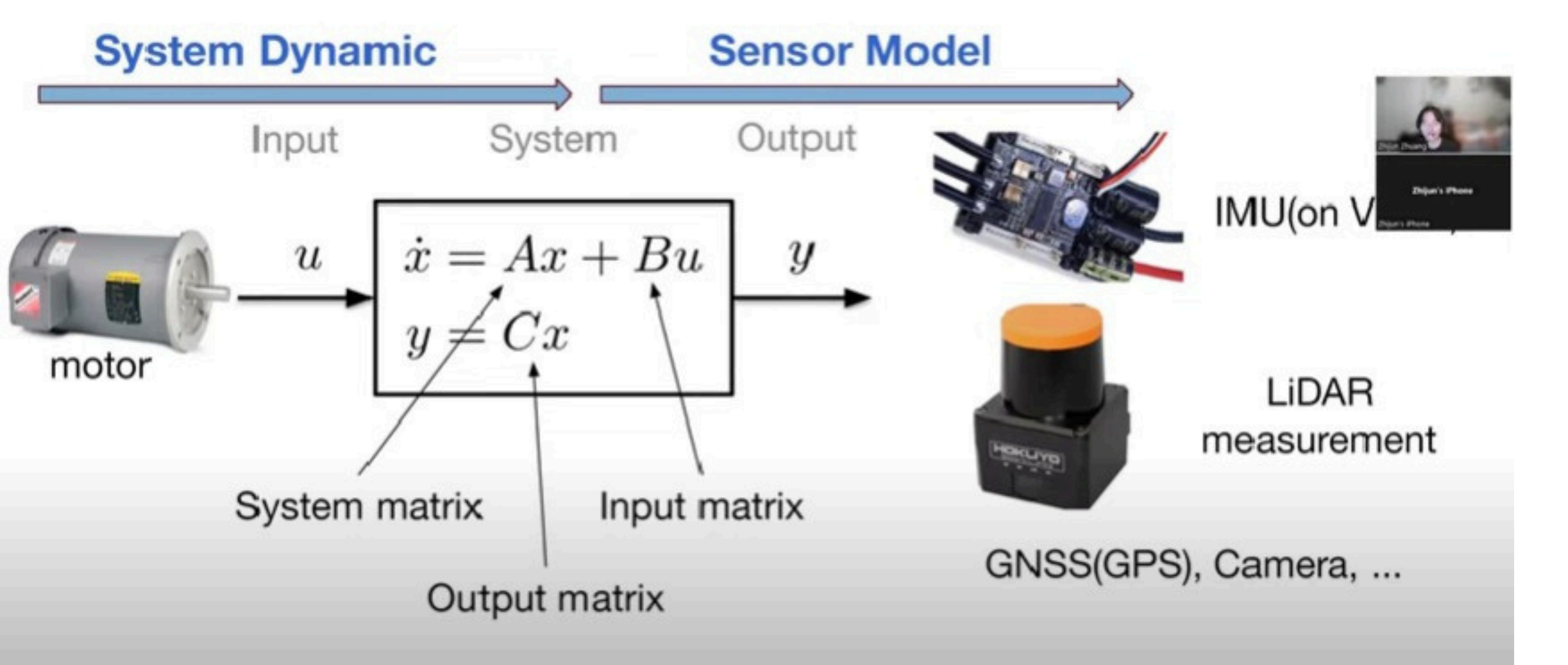
Penn Engineering

Lesson Plan: Localization and Mapping

1. Introduction to State Estimation
2. Recap of Probability and Bayes Rule
3. Recursive Bayes Filter
4. Variants of Bayes Filter: KF, Particle filter
5. Running Particle Filter in ROS2

State Estimation




System Dynamic
Sensor Model
Input
System
Output
u
$\dot{x} = Ax + Bu$
$y = Cx$
y
motor
System matrix
Input matrix
Output matrix
IMU(on V
LiDAR
measurement
GNSS(GPS), Camera, ...

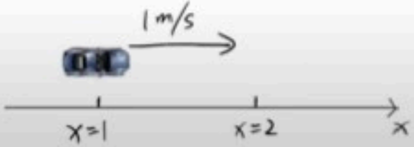
Dead reckoning

Definition:

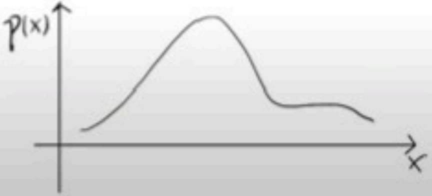
Calculate the current position of the car by using a previously position, and incorporate estimates of s
heading (or direction) , and elapsed time.

Does it work?

However, there is always uncertainty in the measurement and system model, which will create cumulative errors. Recall the parameters you tuned for the car. Can you get an accurate estimate for the speed and position when echo /odom?

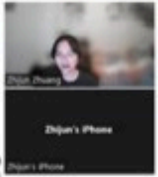


Always have noise!



Frame the state estimation as a question to approximate a distribution of the position.

IMU(on VESC)



Input System Output

motor

u

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

y

LiDAR measurement

GNSS(GPS), Camera, ...

Control Input

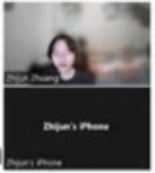
Prediction

State Estimation: Approximate a distribution

Observation

Correction

Bayes Rule: Update estimation w/ evidence



Conditional Probability

$P(B|A)$ - The chance of event B when event A has already happened: probability of B given

Bayes Rule

$$P(AB) = P(A) \times P(B|A) \quad (1)$$
$$P(AB) = P(B) \times P(A|B) \quad (2)$$

$$\Rightarrow P(B|A) = \frac{P(AB)}{P(A)} \Rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

$A : evidence(observation)$

$B : hypothesis(state)$

Law of Total Probability: Decompose the problem

Discrete Case

Suppose $B_1, \ldots, B_k$ are mutually exclusive and exhaustive events in a sample space.
Then for any event A in that sample space:

$$P(A) = P(A \cap B_1) + \ldots + P(A \cap B_k) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A \mid B_i) P(B_i)$$

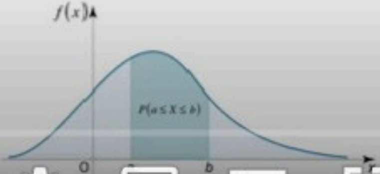
Continuous Case

$$\mathrm{P}(x \leq X \leq x + \delta) = F(x + \delta) - F(x) \approx f_X(x)\delta$$

Cumulative Distribution Function

Probability Density Function

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x) dF(x) = \int_{-\infty}^{\infty} P(A \mid X = x) f_X(x) dx$$



20:16 / 56:42

More evidences -> More conditions

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \Rightarrow P(B|A, C_1, \ldots C_n) = \frac{P(A|B, C_1, \ldots C_n)P(B|C_1, \ldots C}{P(A|C_1, \ldots C_n)}$$

When we combine historical information: Recursive Bayes filter
When we combine multiple sensor measurements: Sensor fusion

Control Input → **Prediction**

State Estimation: Approximate a distribution

Observation ← **Correction**

$$\mathrm{Bel}\left(x_t\right) = P\left(x_t \mid o_t, u_t, o_{t-1}, u_{t-1}, \ldots\right)$$

Belief or posterior

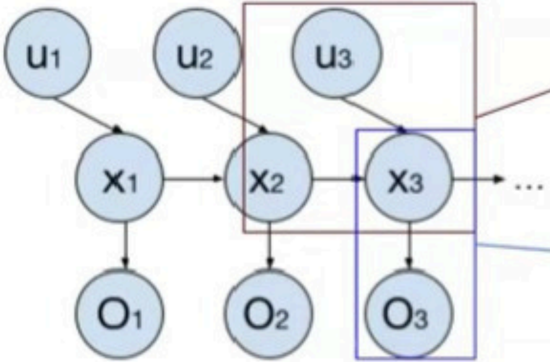
Robot state

Observation

Control Input

History Observation

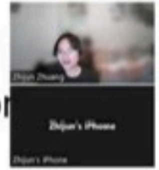
Hidden Markov Model



$p\left(x_t \mid u_t, x_{t-1}\right)$ **Likelihood** of next state, given current control and previous state

$p\left(o_t \mid x_t\right)$ **Likelihood** of current observatio

current state



$$\mathrm{Bel}\left(x_t\right) = P\left(x_t \mid o_t, u_t, o_{t-1}, u_{t-1}, \ldots\right)$$

Goal: Take belief of time t-1, advance the estimation of x to time t.

Bayes Filter

Step1: Prediction with the control input (like in dead reckoning)

$$\overline{bel}(x_t) = P(x_t \mid o_{1:t-1}, u_{1:t})$$

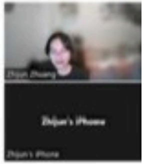
Law of Total Probability

$$= \int P(x_t \mid \boxed{x_{t-1}}, o_{1:t-1}, u_{1:t})\, P(\boxed{x_{t-1}} \mid o_{1:t-1}, u_{1:t})\, d(x_{t-1})$$

Break the problem by conditioning on x_{t-1}

Bayes Filter

Step1: Prediction with the control input (like in dead reckoning)



$$\overline{\mathrm{bel}}(x_t) = P(x_t \mid o_{1:t-1}, u_{1:t})$$

Law of Total Probability

$$= \int P(x_t \mid x_{t-1}, o_{1:t-1}, u_{1:t}) \boxed{P(x_{t-1} \mid o_{1:t-1}, u_{1:t})}\, d(x_{t-1})$$

Recursive term

$$= \int P(x_t \mid x_{t-1}, o_{1:t-1}, u_{1:t}) \boxed{\mathrm{bel}(x_{t-1})}\, d(x_{t-1})$$

Markov Property

$$= \int \boxed{P(x_t \mid x_{t-1}, u_t)} \boxed{\mathrm{bel}(x_{t-1})}\, d(x_{t-1})$$

Action model **Posterior of x_{t-1}**

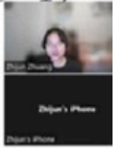
Bayes Filter

Step1: Prediction with the control input (like in dead reckoning)

$$\overline{\text{bel}}(x_t) = P(x_t | o_{1:t-1}, u_{1:t}) = \int P(x_t \mid x_{t-1}, u_t)\,\text{bel}(x_{t-1})\,d(x_{t-1})$$

Prior

Action model



Step2: Correction with the observation

$$\text{bel}(x_t) = P(x_t \mid o_{1:t-1}, o_t, u_{1:t})$$

Posterior

Bayes Rule

$$= \frac{P(o_t \mid o_{1:t-1}, x_t, u_{1:t})\,\boxed{P(x_t \mid o_{1:t-1}, u_{1:t})}}{P(o_t \mid o_{1:t-1}, u_{1:t})}$$

Prior

Markov Property

Sensor model

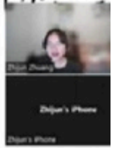
$$= \frac{\boxed{P(o_t \mid x_t)}\,\boxed{P(x_t \mid o_{1:t-1}, u_{1:t})}}{\boxed{P(o_t \mid o_{1:t-1}, u_{1:t})}}$$

Normalization term

Practical Issue of Bayes Filter

Step1: Prediction with the control input

$$\overline{\mathrm{bel}}\,(x_t) = P(x_t|o_{1:t-1}, u_{1:t}) = \int \boxed{P\,(x_t \mid x_{t-1}, u_t)}\,\boxed{\mathrm{bel}\,(x_{t-1})}\, d\,(x$$



Multiplication and even integration of two probability distributions!

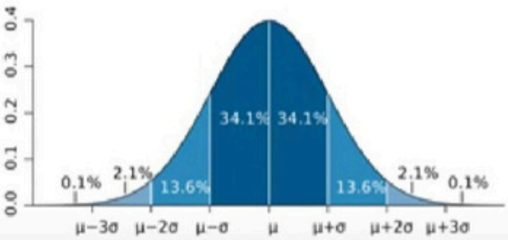
Step2: Correction with the observation

$$\mathrm{bel}\,(x_t) = \frac{P\,(o_t \mid x_t)\,\overline{\mathrm{bel}}\,(x_t)}{P\,(o_t \mid o_{1:t-1}, u_{1:t})} = \boxed{\eta P\,(o_t \mid x_t)}\,\boxed{\overline{\mathrm{bel}}\,(x_t)}$$

Variants of Bayes Filter

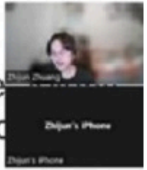
- **Assume the state and noise follow a simple distribution (Gaussian)**

  Kalman Filter (KF)
  Extended Kalman Filter (EKF)
  Unscented Kalman Filter (UKF)



Conditional distribution: If two sets of variables are Gaussian, then the conditional distribution of one set conc on the other set is again Gaussian.

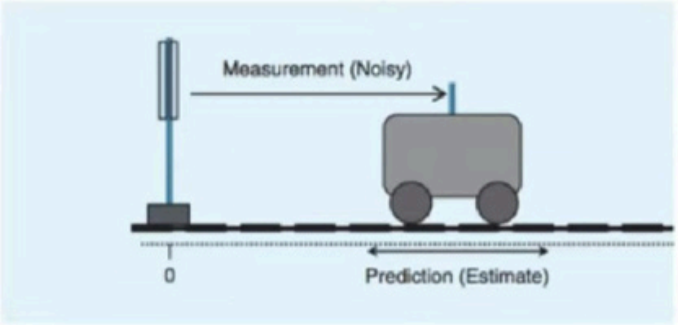
Self-conjugate: given the Gaussian likelihood function, choosing the Gaussian prior will result in Gaussian posterior.

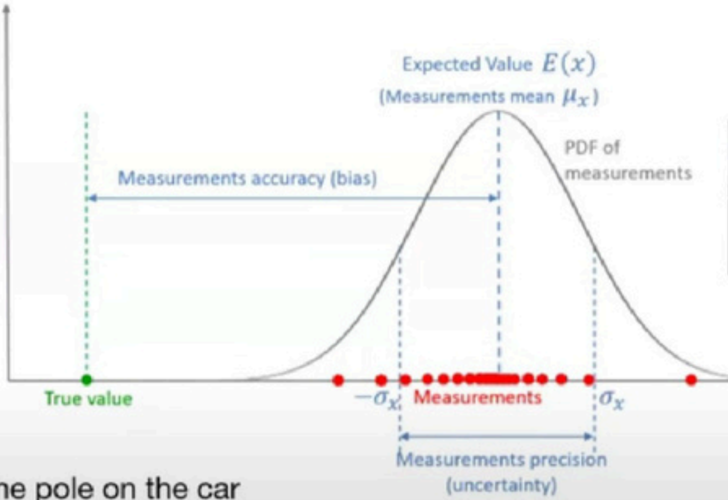


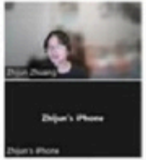
- **Use sampling-based method to estimate the complicated distribution**

  Particle Filter (PF), coming next!

KF example



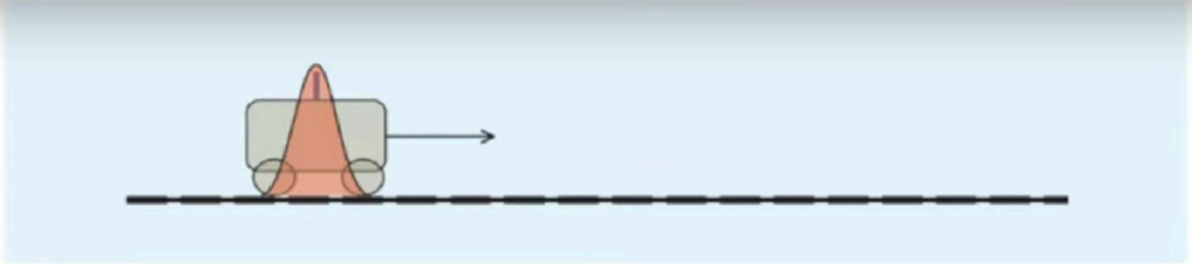




State: Position of the car

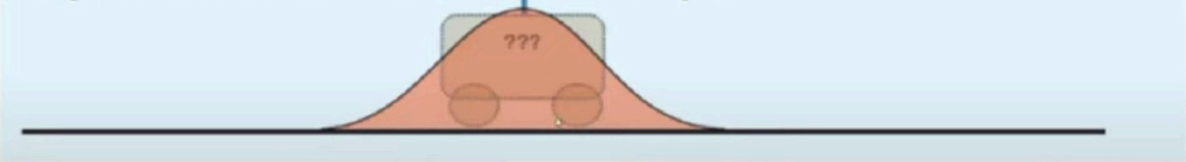
Observation: Measurement from the sensor to the pole on the car

Assumption: Both the state and the observation follow the Gaussian distribution



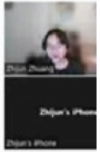
[Fig1] The initial knowledge of the system at time T=0. The arrow pointing to the right represents the known initial velocity of the car.

Step1: Prediction with the control input

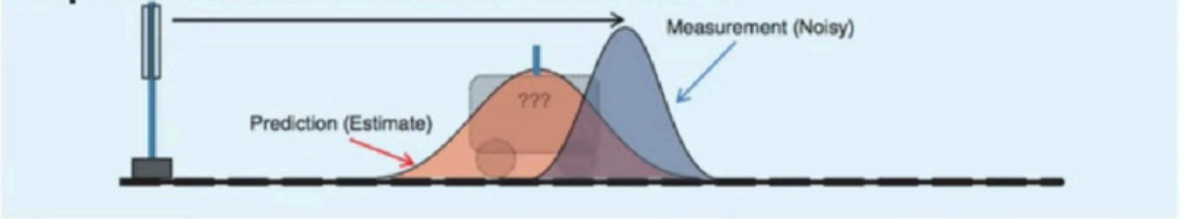


[Fig2] The prediction of the location of the car at time T=1 and the level of uncertainty in that prediction.

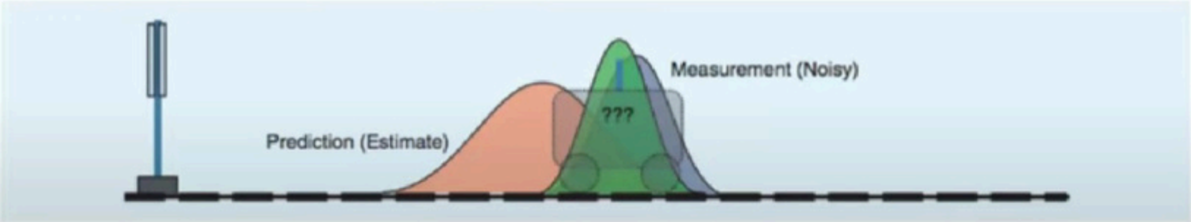
The confidence in the knowledge of the position of the car has decreased, since we introduce the uncertainty in the action model.



Step2: Correction with the observation

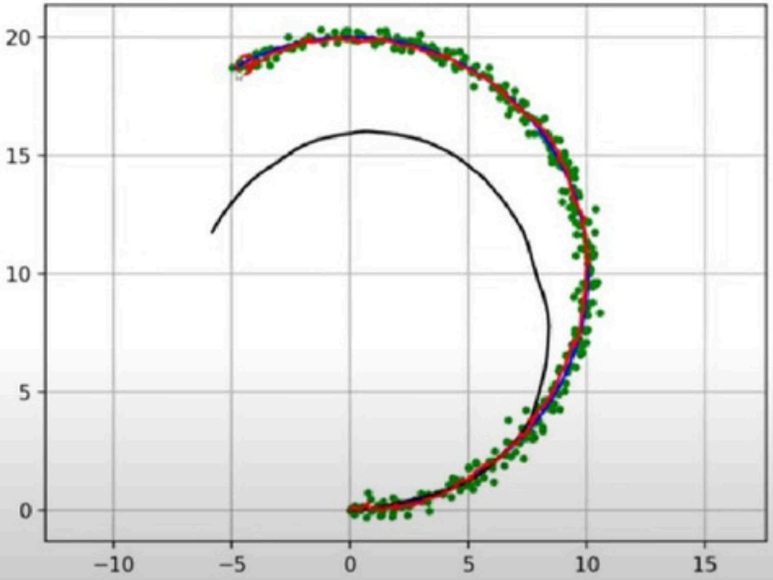


[Fig3] Shows the measurement of the location of the car at time T=1 and the level of uncertainty in that noisy measurement represented by the blue Gaussian.



[Fig4] The combined knowledge of this system is provided by multiplying these two PDFs(orange and blue) together. This new PDF provides the best estimate of the location of the car, by fusing the data from the prediction and the observation(measurement).

EKF example



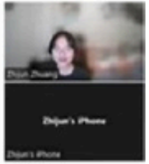
Blue line: True trajectory.

Black line: Dead reckoning trajectory,

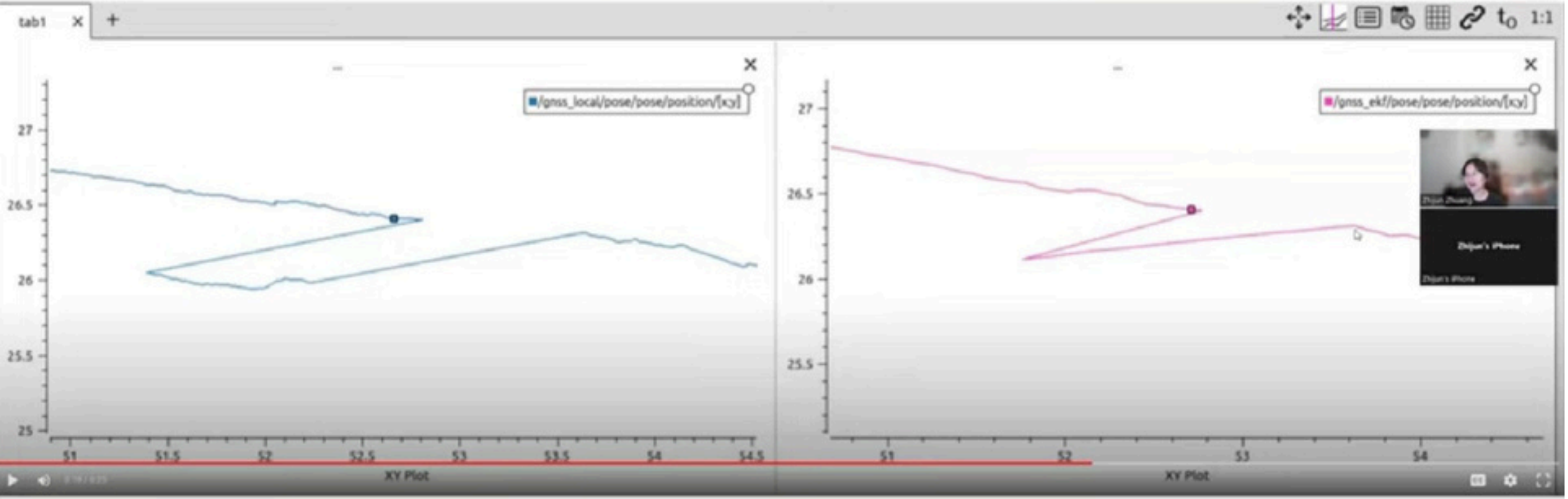
Green points: observation(GNSS).

Red line: estimated trajectory with EKF.

Red ellipse: estimated covariance ellipse with EKF.



Practice Problem of KF: non-gaussian noise



Positioning observation(GNSS)

EKF result