1.723 – Computational Methods for Flow in Porous Media

Homework #3

Due on March 5, 2015

Problem 1 (2 points) Write the linear pressure equation

$$c_t \frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k}{\mu}\frac{\partial p}{\partial x}\right) = 0 \tag{1}$$

in dimensionless form:

$$\frac{\partial p_D}{\partial t_D} + \frac{\partial}{\partial x_D}\left(-\frac{k_D}{\mu_D}\frac{\partial p_D}{\partial x_D}\right) = 0. \tag{2}$$

In the equation above, all the variables are normalized with respect to a characteristic value, that is, $p_D = p/P_c$, $k_D = k/k_c$, and so on. For example, $\xi = x/L$ is the dimensionless distance. You are asked to:

1. Derive the expression for the characteristic time T_c.
2. Derive the expression for the characteristic Darcy velocity u_c.
3. Determine the numeric values (with proper units) of the quantities above for the following system: an aquifer that is 1 km long, with permeability around 10^{-13} m^2, total compressibility 10^{-8} Pa^{-1}, fluid viscosity 1 cP $= 10^{-3}$ Pa·s and characteristic pressure drop $\Delta P_c = 10^5$ Pa.

Problem 2 (8 points) Consider the following initial and boundary value problem in a homogeneous medium ($\lambda = k/\mu = 1$), which we now write directly in dimensionless form:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(-\lambda\frac{\partial p}{\partial x}\right) = 0 \quad x \in (0,1), \tag{3}$$

$$p(x, t=0) = 0 \quad x \in [0,1], \tag{4}$$

$$p(x=0,t) = 1, \quad -\lambda\frac{\partial p}{\partial x}\bigg|_{x=1} = 0, \quad t > 0. \tag{5}$$

1. Be resourceful and find the analytical solution to the problem above in a book (e.g. [1, 2]). Alternatively, you can derive the analytical solution by using the method of separation of variables. Plot the solution as curves of p vs. x at different dimensionless times t, using MATLAB. The solution is an infinite sum (Fourier

series), so you must exercise some caution in evaluating the function. What is the steady-state solution? At what time would you say the solution reaches steady state?

2. Discretize the problem like we did in class (a finite volume method in space and a trapezoidal rule in time with general θ), using $N = 4$. Do it by looping over the elements and, for each element, finding its contribution to the matrix of the system and the right-hand side vector. You must do this for $i = 1$, $i = 2, \ldots, N-1$, and $i = N$, separately—because the first and last elements must incorporate the boundary conditions. Write, clearly, the resulting system of equations: the 4×4 matrix, the 4×1 vector of unknowns, and the 4×1 right-hand side vector.

3. Code the numerical discretization of the problem in MATLAB for a generic grid spacing $\delta x = 1/N$, time step δt and trapezoidal rule θ. You will have a time loop and, for each time step, you will solve the linear system. I strongly suggest that you use a `sparse` matrix in MATLAB by building the system of equations with the command `spdiags`, and that you solve the system with the default method, using the command `x=A\b`.

4. Solve the problem using your code, for $\delta x = 0.01$, $\delta t = 10^{-5}$ and two time-stepping schemes: (1) a Forward Euler explicit scheme ($\theta = 0$); and (2) a Backward Euler implicit scheme ($\theta = 1$). Compare your numerical solution with the analytical solution from Part 1 (that is, overlay the curves from the numerical and analytical solutions at the same dimensionless times). Comment on the results (stability, accuracy, CPU time, etc.)

5. Repeat Part 4 but, now, with $\delta t = 10^{-4}$. What happens? Why?

References

[1] H. S. Carslaw and J. C. Jaeger. *Conduction of Heat in Solids*. Oxford University Press, second edition, 1959.

[2] J. Crank. *Mathematics of Diffusion*. Oxford University Press, second edition, 1975. (First edition, 1956).