1.723 – Computational Methods for Flow in Porous Media

Midterm

Due on April 9, 2015

We are interested in the Boussinesq approximation of density-driven convective mixing (see Figure 1) [1, 2]. The governing equations, in dimensionless form, read:

$$\nabla \cdot (-\nabla p) = -\nabla \cdot (c \nabla z), \tag{1}$$

$$\partial_t c + \nabla \cdot \left(\boldsymbol{u} c - \frac{1}{\mathrm{Ra}} \nabla c \right) = 0, \tag{2}$$

where $\boldsymbol{u} = -(\nabla p - c \nabla z)$. The domain is a rectangle of dimensions $\alpha \times 1$, where $\alpha = L/H$ is the aspect ratio. The boundary conditions are no-flow everywhere, and the concentration gradient in the direction of the unit normal is also zero everywhere except at the top boundary ($0 < x < \alpha, z = 0$), where $c = 1$.

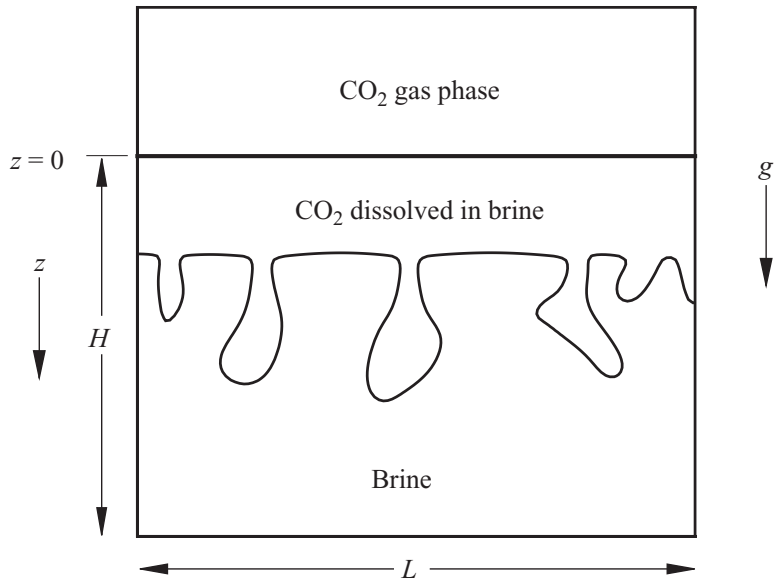

Figure 1: Problem set-up (taken from [1]).

Problem 1 (10 points) Implement a finite volume solution to these equations (with the appropriate boundary conditions) in MATLAB. Use an IMPES (IMplicit Pressure, Explicit Concentration) solution strategy. The code should be sufficiently general to allow for an arbitrary aspect ratio α and user-defined discretization δx, δz and δt.

Note: The pressure is determined only up to an arbitrary constant. In addition, in the Boussinesq approximation, the matrix of the system is constant. To take advantage of this fact, and to remove the singularity in the system of equations, it proves useful to factorize the matrix (i.e., express it as a matrix $A = LDL^T$, where L is a lower triangular matrix, and D is a diagonal matrix) outside of the time loop; then use back-substitution for solving the system of equations at every time step. The factorization step can be implemented with the following lines of code, outside of the time loop:

```
% LDLt Factorization
[Lmat,Dmat,perm] = ldl(Amat,'vector');
```

The solution of the linear system by back-substitution can be done as follows, at each time step:

```
% Solve system
clear pvec;
pvec(perm,:) = Lmat'\(Dmat\(Lmat\(bvec(perm,:))));
```

In addition, when solving the transport equation, one should introduce a small random perturbation to the prescribed concentration at the top boundary, which changes at every time step. That is, inside the time loop, you can have a statement like the following:

```
% Fluctuating boundary condition
c0vec = 1 + 2*epsrnd*(rand(1,Nx)-0.5);
```

Problem 2 (10 points) Compute the solution (pressure, velocity and concentration) on a unit square ($\alpha = 1$) for different values of the Rayleigh number, Ra = 1000, 2000 and 4000. Use the finest spatial discretization you can afford (grids of 128×128 or even 256×256 gridblocks should be affordable) and tune the time step so that the scheme is stable.

1. Plot the concentration $c(x, z)$ field at several well-chosen times to illustrate its evolution, and comment on the results.
2. Plot the horizontally-averaged concentration $\bar{c}(z)$ at those times, and comment on the behavior before and after the fingers hit the bottom boundary.
3. For each case, evaluate the total mass of CO_2 in the system as a function of time. Can you distinguish separate regimes (diffusive, advective, saturated, ...)? Do you think it is possible to predict the transition times between regimes?

References

[1] A. Riaz, M. Hesse, H. A. Tchelepi, and F. M. Orr, Jr. Onset of convection in a gravitationally unstable, diffusive boundary layer in porous media. *J. Fluid Mech.*, 548:87–111, 2006.

[2] J. J. Hidalgo, J. Fe, L. Cueto-Felgueroso, and R. Juanes. Scaling of convective mixing in porous media. *Phys. Rev. Lett.*, 109:264503, doi:10.1103/PhysRevLett.109.264503, 2012.