1.723 – Computational Methods for Flow in Porous Media

Homework #4

Due on March 17, 2015

Problem 1 (2 points) Perform an analysis of the discretization error of the finite volume method for the one-dimensional pressure equation, with Crank-Nicolson time stepping ($\theta = 0.5$). What is the order of approximation in space and time?

Problem 2 (2 points) Perform a Von Neumann stability analysis of the finite volume method for the one-dimensional pressure equation, with Backward Euler time stepping ($\theta = 1$). Is it conditionally or unconditionally stable?

Problem 3 (2 points) Perform an analysis of the discretization error of the finite volume method for the one-dimensional linear advection equation, with a one-point upstream flux approximation, and with Backward Euler time stepping ($\theta = 1$). Do you expect that the solution will be more or less diffusive than for the Forward Euler method? Why?

Problem 4 (2 points) Perform a Von Neumann stability analysis of the finite volume method for the one-dimensional linear advection equation, with a centered two-point flux approximation, and with Forward Euler time stepping ($\theta = 0$). Is it conditionally or unconditionally stable?

Problem 5 (2 points) Write the linear tracer transport equation

$$\phi \frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(uc - D\frac{\partial c}{\partial x}\right) = 0 \tag{1}$$

in dimensionless form:

$$\phi_D \frac{\partial c_D}{\partial t_D} + \frac{\partial}{\partial x_D}\left(u_D c_D - \frac{1}{\mathrm{Pe}} D_D \frac{\partial c_D}{\partial x_D}\right) = 0. \tag{2}$$

In the equation above, all the variables are normalized with respect to a characteristic value, that is, $c_D = c/c_c$, $\phi_D = \phi/\phi_c$, $u_D = u/u_c$, and so on. For example, $x_D = x/L$ is the dimensionless distance. You are asked to:

1. Derive the expression for the characteristic *advective* time T_c.
2. Derive the expression for the key dimensionless group in the equation, the Péclet number Pe. What is the physical meaning of this variable?

Problem 4 (10 points) Consider the following problem (in dimensionless form) in a homogeneous medium ($\phi_D = 1$), and with uniform flow ($u_D = D_D = 1$) where for convenience we have dropped the 'D' subscript:

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(c - \frac{1}{\mathrm{Pe}}\frac{\partial c}{\partial x}\right) = 0, \quad x \in (0,1), \tag{3}$$

$$c(x, t=0) = 0, \quad x \in [0,1], \tag{4}$$

$$F\Big|_{x=0} = c - \frac{1}{\mathrm{Pe}}\frac{\partial c}{\partial x}\Big|_{x=0} = 1, \quad t > 0, \tag{5}$$

$$F\Big|_{x=1} = c\Big|_{x=1}, \quad t > 0 \ \text{(natural outflow condition)}. \tag{6}$$

1. For an infinite medium ($-\infty < x < \infty$) with initial condition
$$c = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \tag{7}$$
the problem has the following analytical solution:
$$c(x,t) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x-t}{2\sqrt{t/\mathrm{Pe}}}\right). \tag{8}$$
Plot the solution as curves of c vs. x at different dimensionless times t, using MATLAB. Create plots for several values of the Péclet number: Pe = 1, 10, and 100. How long does it take for the tracer to reach the right boundary? What is the effect of the Péclet number on the sharpness of the front?

2. Discretize the problem like we did in class (a finite volume method in space and a trapezoidal rule in time with general θ), using $N = 4$. Do it by looping over the elements and, for each element, finding its contribution to the matrix of the system and the right-hand side vector. You must do this for $i = 1$, $i = 2, \ldots, N-1$, and $i = N$, separately—because the first and last elements must incorporate the boundary conditions. Write, clearly, the resulting system of equations: the 4×4 matrix, the 4×1 vector of unknowns, and the 4×1 right-hand side vector.

3. Code the numerical discretization of the problem in MATLAB for a generic grid spacing $\delta x = 1/N$, time step δt and trapezoidal rule θ. You will have a time loop and, for each time step, you will solve the linear system. I strongly suggest that you use a `sparse` matrix in MATLAB by building the system of equations with the command `spdiags`, and that you solve the system with the default method, using the command `x=A\b`.

4. Solve the problem using your code, for Pe = 100, $\delta x = 0.01$, $\delta t = 10^{-3}$ and two time-stepping schemes: (1) a Forward Euler explicit scheme ($\theta = 0$); and (2) a Backward Euler implicit scheme ($\theta = 1$). Compare your numerical solution with the analytical solution from Part 1 (that is, overlay the curves from the numerical and analytical solutions at the same dimensionless times). Comment on the results (stability, accuracy, CPU time, and agreement with the analytical solution).

5. Repeat Part 4 but, now, with $\delta t = 10^{-2}$. What happens? Why?