1.723 - Computational Methods for Flow in Porous Media
Final Examination

Due on Friday, May 15 2015

Problem 1 (10 points) Consider the equations of two-dimensional, horizontal miscible flow in a homogeneous porous medium. In non-dimensional form, and assuming an incompressible system, the concentration transport is modeled by

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(\boldsymbol{u} c - \frac{1}{\mathrm{Pe}} \nabla c \right) = 0, \tag{1}$$

where the Darcy velocity $\boldsymbol{u} = (u_x, u_y)$ is given in terms of the pressure, p, and the concentration-dependent mobility, $\lambda(c)$, as

$$\boldsymbol{u} = -\lambda(c) \nabla p. \tag{2}$$

For an incompressible system, conservation of mass for the mixture reduces to the constraint

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3}$$

The mobility is assumed to depend on the concentration of the mixture as

$$\lambda(c) = e^{Rc}, \tag{4}$$

where R is the natural logarithm of the viscosity ratio, $M = \mu_2/\mu_1$. μ_2 is the dynamic viscosity of the more viscous fluid, and μ_1 is the dynamic viscosity of the less viscous fluid. Writing the pressure equation in stream function-vorticity form, our model problem is

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(\boldsymbol{u} c - \frac{1}{\mathrm{Pe}} \nabla c \right) = 0, \tag{5}$$

$$\nabla^2 \Psi = -\omega, \tag{6}$$

with the stream function Ψ is defined as

$$u_x = \frac{\partial \Psi}{\partial y}, \quad u_y = -\frac{\partial \Psi}{\partial x}, \tag{7}$$

and the magnitude of the vorticity vector expressed as

$$\omega = R \left(\frac{\partial c}{\partial x} u_y - \frac{\partial c}{\partial y} u_x \right). \tag{8}$$

The problem domain is a square of length $L = 1$. Initially, the porous medium is filled with the two perfectly miscible fluids, oil (more viscous) and solvent (less viscous), as shown schematically in Figure 1. For example, the concentration field shown in Figure 1 was generated with the following code

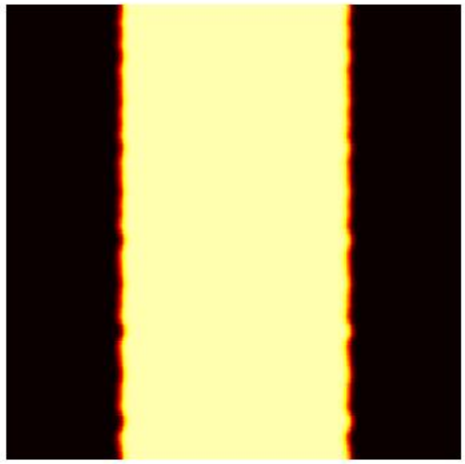

Figure 1. Schematic of the initial concentration field. A strip of less viscous solvent is surrounded by the more viscous oil on both sides.

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L = 1; N = 256;
x = 0:L/N:L; x =  x(1:N); y = x;
[xx,yy] = meshgrid(x,y);
rnd = (-1/N)+(2/N)*rand(1,N);
yr = repmat((y+rnd)',1,N); % Perturbed y coordinate
w = [20 21 30 31 40]; % Fourier modes in the perturbed front
p = 0*yy;
for iw = 1:length(w)
    p = p + 0.004*(rand-0.5)*cos(w(iw)*pi*yr+rand*iw);
end
c = 0.5*(erfc(100*(p+xx-0.75*L)) - erfc(100*(p+xx-0.25*L)));
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The initial, constant velocity field is given by $\boldsymbol{u}_0 = (1, 0)$. The flow is assumed to be periodic in x and y. You are asked to discretize the governing equations using the pseudospectral method, solve for the velocity field, and advance the concentrations explicitly in time. Choose a 256×256 or finer grid, a final simulation time of $t = 4$, a Péclet number $\mathrm{Pe} = 2500$, and three different values of the log-viscosity ratio, $R = 1, 2, 2.8$, corresponding to three simulations. You may choose to decompose the velocity field as $\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}$ where $\boldsymbol{u}_0$ is a non-periodic part corresponding to the background flow imposed by the initial velocity field, and $\tilde{\boldsymbol{u}}$ is a periodic remainder part. This allows for the stream function to be decomposed as

$$\Psi = \Psi_0 + \tilde{\Psi} \tag{9}$$

where $\tilde{\Psi}$ is the periodic part of the stream function with periodic boundary conditions, therefore amenable to solution by a spectral method.

1. Write down Eq. (6) for the periodic stream function $\tilde{\Psi}$. Write down the corresponding equation in the Fourier space in terms of the discrete Fourier coefficients $\hat{\Psi}(k_x, k_y)$ and $\hat{\omega}(k_x, k_y)$, where $k_x = (2\pi/L)[0 : N/2, -N/2+1 : -1]$ and $k_y = (2\pi/L)[0 : N/2, -N/2+1 : -1]$ are wavenumbers in x and y directions. What is the analogous expression in the Fourier space of the Laplacian in Eq. (6)? What is the meaning of $\hat{\Psi}(k_x = 0, k_y = 0)$?

2. Write down an explicit equation to determine $\hat{\Psi}$ at current time step using $\hat{\omega}$ at previous time step. How is $\hat{\omega}$ related to the mobility of the mixture $\lambda(c)$, concentration c, and velocity $\boldsymbol{u}$? Do you see any issue in computing $\hat{\Psi}(k_x = 0, k_y = 0)$?
3. Implement a function in Matlab to solve for the velocity field $\boldsymbol{u}$. This function should take c and $\boldsymbol{u}$ of previous time step as its input arguments and compute $\hat{\Psi}(k_x, k_y)$, $u_x(x, y)$, and $u_y(x, y)$ at the current time step.
4. Implement a function in Matlab to solve for the concentration field c. Use the fourth-order Runge-Kutta (RK4) scheme for time integration:

$$c^n = c^{n-1} + \Delta t \sum_{j=1}^{4} b_j Y_j \tag{10}$$

$$Y_1 = f(c^{n-1}) \tag{11}$$

$$Y_i = f(c^{n-1} + \Delta t \sum_{j=1}^{4} a_{ij} Y_j) \tag{12}$$

where the Butcher's tableu of RK4 method is given as

$$\begin{array}{c|c} e_i & a_{ij} \\ \hline & b_j \end{array} = \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \tag{13}$$

Implement adaptive time stepping where Δt evolves in time while honoring the CFL conditions of the scheme.
5. Analyze and comment on the evolution in time of the following quantities,
   (a) Concentration field
   (b) Variance of the concentration field, which gives an idea of the degree of mixing
   (c) Mean scalar dissipation rate, which gives an idea of the rate of mixing. The scalar dissipation rate, ϵ_c, is defined as $\epsilon_c = \frac{1}{\mathrm{Pe}} \nabla c \cdot \nabla c$.
   (d) (*Bonus*) Average number of fingers in the domain.
6. What are the mechanisms that you identify by visual inspection of the concentration field by which the two fluids interact and mix?
7. What is the dependence of the degree of mixing and rate of mixing on R?
8. What are the accuracy and stability characteristics of this numerical method? Do you think solving for the stream function at current time step using vorticity at previous time step is stable at higher R? Explain. What will you modify in the numerical method to improve its accuracy and stability?

Hints:

1. When solving the Laplacian using the pseudospectral method (FFT), you can remove the singularity at the zero frequency $(k_x, k_y) = (0, 0)$ by setting these frequencies to some non-zero value. Think of the effect of this modification.
2. You can approximate the average number of fingers in the domain by Fourier decomposition of the concentration field in y direction and finding the wavenumber of the most dominant Fourier mode.