1.723 - Computational Methods for Flow in Porous Media
Homework #7

Due on Thursday, April 30 2015

Problem 1 (3 points) Consider a one-dimensional finite difference grid with N points in the interval $[0, 1]$.

(a) Use Taylor expansions to develop difference formulas for the first derivative at the points near the boundary, namely i={1,2,$N-1$, N}. Use stencils with four points.

(b) Combine your boundary formulas with the centered, fourth-order formula for interior points,

$$w_j = \frac{-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2}}{12h}. \tag{1}$$

Write a code that generates the differentiation matrix for a grid with N nodes, and approximates the derivative of the function $u(x) = e^{\sin(2\pi x)}$. Study the convergence as N increases, and discuss your results.

Problem 2 (3 points) Consider a one-dimensional finite difference periodic grid with N points in the interval $[-\pi + h, \pi]$: $\{x_1, x_2, \ldots, x_N\} = \{-\pi + h, -\pi + 2h, \ldots, \pi\}$, where the grid spacing is $h = 2\pi/N$. Note that for a periodic grid $x_0 = x_N$. We are interested in approximating the second derivative, $u''(x_j)$, $j = 1, 2, \ldots, N$, of a function $u(x)$ using a centered, fourth-order accurate, finite difference method.

(a) Construct the differentiation matrix $\boldsymbol{D}^{(2)}$ in the formula $\boldsymbol{u}^{(2)} = \boldsymbol{D}^{(2)}\boldsymbol{u}$.

(b) Let $u(x) = e^{\sin^2(x)}$. Study convergence of your finite difference approximation for this function. Plot the error of approximation vs. N for $N = \{2^3, 2^4, \ldots, 2^{12}\}$. What rate of convergence do you observe? Comment.

(c) Repeat (b) for $u(x) = e^{\sin(x)|\sin(x)|}$. What rate of convergence do you observe? Comment.

Problem 3 (4 points) Consider a one-dimensional finite difference grid with $N = 4$ points, $x_j = \{x_1, x_2, x_3, x_4\}$ and non-uniform spacing $\{x_2 - x_1, x_3 - x_2, x_4 - x_3\} = \{h_1, h_2, h_3\}$ in the interval $[-1, 1]$.

(a) Use polynomial interpolation to derive a finite difference approximation for $u''(x_2)$ that is as accurate as possible for smooth functions $u(x)$, based on the four values $u_j = u(x_j)$. Give an expression for the dominant term in the error. Verify your expression for the error by testing your formula with a specific function and various values of h_1, h_2, h_3.

(b) Let $h_1 = h_2 = h_3 = h = 2/(N-1)$ and $u(x) = \sin(x)$. Calculate the global polynomial interpolant $p(x) = \sum_{j=1}^{N} l_j(x)u_j$ using the Lagrange polynomials $l_j(x)$. Plot $u(x)$ and $p(x)$ vs. x on the same plot for `x_in = -1:0.1:1`. Does the global interpolant approximate the function well?

(c) Repeat (b) with $u(x) = 1/(1+16x^2)$. Use two uniformly-spaced grids: $N = 4$ and $N = 20$. Use `P=lagrangepoly(x_j, u_j)` to generate the polynomial coefficients `P` and use the in-built function `polyval(P, x_in)` to generate the polynomial $p(x)$. What is different from your observation in (b), and why?

(d) Repeat (c) for a non-uniform grid $x_j = \{\cos \frac{\pi j}{N-1}\}$, $j = 0, 1, \ldots, N-1$. Between (c) and (d), which interpolant is doing a better job in interpolating the underlying function $u(x)$, and why?