1.723 – Computational Methods for Flow in Porous Media
Homework #5
Due on April 2, 2015

Problem 1 (10 points) Consider a two-dimensional horizontal porous medium with a heterogeneous permeability field. You are asked to solve the flow and tracer transport equations in the so-called quarter five-spot configuration. In this flow set-up, all boundaries are no-flow boundaries, and fluid is injected in the lower-left corner and extracted from the upper-right corner (Figure 1).

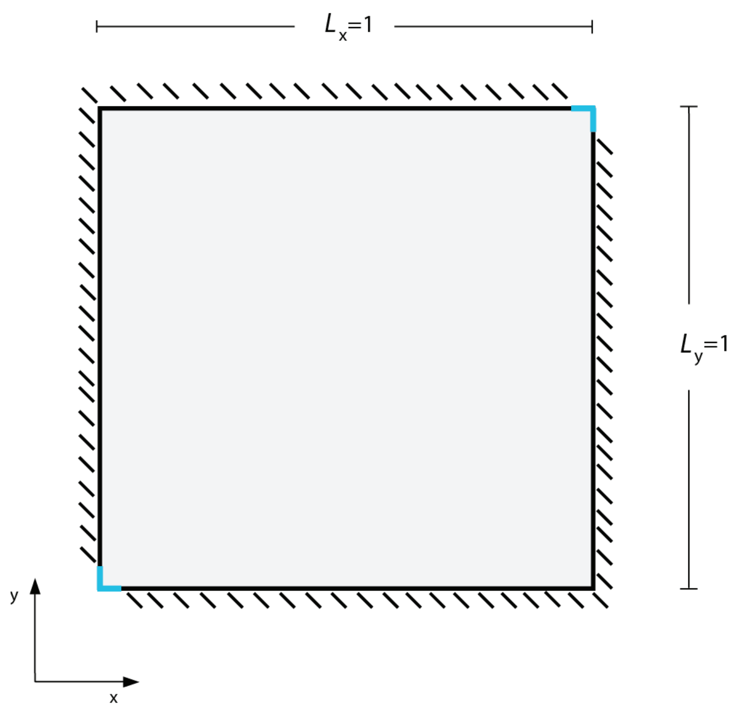

Figure 1: Schematic representation of the quarter five-spot problem set-up.

The governing equations for flow and tracer transport in non dimensional form are, respectively:

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} = -\frac{k(x,y)}{\mu}\nabla p, \tag{1}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(\mathbf{u}c - \frac{1}{\mathrm{Pe}}\nabla c\right) = 0, \tag{2}$$

where p is the fluid pressure, $k(x, y)$ is the spatially-variable permeability field, $\mu = 1$ is the fluid viscosity, $\mathbf{u}$ is the Darcy velocity, c is the tracer concentration, and Pe is the Péclet number. In dimensionless form, the domain is the unit square: $[0, L_x] \times [0, L_y] = [0, 1] \times [0, 1]$. The flow rate (i.e., integrated flux) in and out of the corners of the domain is also set to unity, $Q_{\text{in}} = Q_{\text{out}} = 1$. The pressure at the outflow boundary is set to zero, $p_{\text{out}} = 0$. The initial concentration is zero everywhere, and the injected fluid is assumed to carry a known concentration $c_{\text{in}} = 1$.

1. Discretize the problem using the Finite Volume Method with $N_x = N_y = 3$. You should use the choices explained in class, i.e., a two-point flux approximation for the pressure equation, and a one-point upstream flux approximation with Forward Euler time stepping for the tracer transport equation. You should distribute the inflow integrated flux equally between the left and bottom edges of the lower-left grid cell (i.e., $U_{1/2,1} = U_{1,1/2} = 1/2$), and do the same for the outflow flux (i.e., $U_{N_x,N_y+1/2} = U_{N_x+1/2,N_y} = 1/2$). You can disregard the diffusive component of the flux at the inflow and outflow boundaries. Write, clearly, the resulting system of equations (the 9×9 matrix, the 9×1 vector of unknowns, and the 9×1 right-hand side vector), incorporating the boundary conditions.

2. Generate and display several permeability fields, using the provided function

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   random_perm(var_lnk,corr_lenx,corr_leny,Nx,Ny,Lx,Ly).
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   This function has as inputs the variance of the log-permeability field, $\sigma^2_{\log k}$, the correlation lengths in x and y directions, l_x and l_y, the number of cells in the two dimensions, N_x and N_y, and the size of the domain, L_x and L_y.

3. Code the numerical discretization of the problem in MATLAB for a generic grid spacing $\delta x = \delta y = 1/N$ and time step δt, and for a random permeability field of variance $\sigma^2_{\log k}$ and correlation lengths l_x, l_y.

4. For $\sigma^2_{\log k} = 0.1$, $l_x = l_y = 4\delta x$, plot the pressure distribution along the diagonal with respect to a datum at the center of the domain, $p(x, x) - p(L_x/2, L_y/2)$, and comment on the behavior as a function of the number of grid cells N (take, e.g., $N = 10$, 20, 50, 100, 200).

5. For a sufficiently large N (e.g., $N = 200$) and for different values of the Péclet number Pe (e.g., Pe = 100, 200, 500, 1000, 2000), plot the breakthrough curves at the outlet grid cell, $c_{\text{out}}(t) = c_{N_x,N_y}(t)$ up to simulation time $t = 2$.

6. Repeat steps 4–5 for permeability fields with different properties:

   (a) Variance $\sigma^2_{\log k} = 2$ and correlation lengths $l_x = l_y = 4\delta x$.

   (b) Variance $\sigma^2_{\log k} = 5$ and correlation lengths $l_x = l_y = 4\delta x$.

   Comment on the results.