1.723 - Computational Methods for Flow in Porous Media
Homework #8

Due on Thursday, May 7 2015

Problem 1 (4 points) Differentiation of $u(x) = e^{ikx}$ multiplies it by $g_\infty(k) = ik$. Determine the analogous functions $g_2(k)$ and $g_4(k)$ corresponding to the standard (explicit) centered finite difference methods of second and fourth order accuracy. Plot $\mathrm{Im}(g_2(k))$, $\mathrm{Im}(g_4(k))$, and $\mathrm{Im}(g_\infty(k))$ vs. k. Where in the plot do we see the order of accuracy of a finite difference formula?

Problem 2 (6 points) Consider the linear advection-diffusion equation

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} - \frac{1}{\mathrm{Pe}}\frac{\partial^2 c}{\partial x^2} = 0 \qquad x \in [0, 2\pi], \quad t \in [0, T], \quad c \in [0, 1] \tag{1}$$

with a velocity u that reflects an oscillatory behavior on x

$$u = \begin{cases} 2\sin(x/2), & \theta \leq \pi, \\ -2\sin(x/2), & \theta > \pi, \end{cases}$$

due to rotation of a particle with the phase angle θ, where $\theta = \omega t$, $\theta \in [0, 2\pi]$, on a circle with center $(\pi, 0)$. Choose $T = 9$, $\omega = 1$, $\mathrm{Pe} = 10000$. Use periodic boundary conditions for the concentration, and the initial condition

$$c(x, t = 0) = \exp(-200((x - 0.7)^2)),$$

which is almost periodic on the domain $[0, 2\pi]$.

(a) Discretize equation (1) in space using a fourth-order standard (explicit) finite difference and a first-order explicit time integration scheme. Use grid resolutions of $N = 128, 256$ and a time step of $dt = 4E - 05$. Plot the concentrations at times when the peak of the concentration plume crosses $x = \pi$ ($t = 1.73, 4.55, 8.01$). Report the run time of the time loop in your code using Matlab's `tic` and `toc` commands. What differences do you observe in the concentration profile between $N = 128$ and $N = 256$ simulations, and why?

(b) Repeat (a) using a fourth-order compact (implicit) finite difference scheme for spatial discretization. The fourth-order compact scheme for the first derivative is

$$\alpha c'_{i-1} + c'_i + \alpha c'_{i+1} = a\left(\frac{c_{i+1} - c_{i-1}}{2h}\right) + b\left(\frac{c_{i+2} - c_{i-2}}{4h}\right),$$

with the coefficients $\alpha = 1/4$, $a = 3/2$ and $b = 0$. The fourth-order compact scheme for the second derivative is given by

$$\alpha c''_{i-1} + c''_i + \alpha c''_{i+1} = a\left(\frac{c_{i+1} - 2c_i + c_{i-1}}{h^2}\right) + b\left(\frac{c_{i+2} - 2c_i + c_{i-2}}{4h^2}\right),$$

with the coefficients $\alpha = 1/10$, $a = 6/5$ and $b = 0$.

(c) Repeat (a) with a spectral method for the spatial derivatives. Looking at the concentration profiles, how will you rank the three methods in terms of accuracy and run time, and why?