16.930 Advanced Topics in Numerical Methods for PDEs
Massachusetts Institute of Technology – Spring 2015

Project 34.C – DWR for DG discretization of the convection-diffusion equation

Handed Out: April 17, 2015 *Due (on-line): May 11 at 12pm*

In this project, you are to apply the Dual Weighted Residual (DWR) error estimation method to the 1D discontinuous Galerkin (DG) discretization using BR2 that you developed in Project 1.

We will again consider the following three test cases:

Poisson: $\nu = 1$, $b = c = 0$, $f = x^2$ and $u(0) = 0$ and $u(1) = 1$.

Reaction-diffusion: $\nu = 10^{-4}$, $b = 1$, $c = 0$, $f = 0$, $u(0) = 0$ and $u(1) = 1$.

Convection-diffusion: $\nu = 10^{-2}$, $b = 0$, $c = 1$, $f = 0$, $u(0) = 0$ and $u(1) = 1$.

The output will be the viscous flux at the $x = 1$ boundary, specifically:

$$J = \nu u_x|_{x=1} \tag{1}$$

Q1: For each test case, derive the corresponding adjoint equation and boundary conditions on the adjoint $\psi(x)$. Then, solve that equation to determine $\psi(x)$.

In Project 1, we wrote the weak form of the BR2 discretization and discretization as: Find $u_{hp} \in \mathcal{V}_{hp}$ and $\bar{F}_{hp} = (\bar{F}_{hp}(0), \bar{F}_{hp}(1)) \in \mathbb{R}^2$ such that

$$R(u_{hp}, \bar{F}_{hp}, w_{hp}, \lambda_{hp}) = 0 \qquad \forall\, w_{hp} \in \mathcal{V}_{hp},\ \forall\, \lambda_{hp} \in \mathbb{R}^2 \tag{2}$$

where

$$\begin{aligned} R(v, \bar{F}, w, \lambda) &= \langle w, \bar{F} \rangle + \langle \lambda, v - g \rangle - (\partial_x w, F) + (w, bv - f) + \sum_{\Gamma_i} \hat{F}_I \, [\![w]\!] && (3) \\ &- \sum_{\Gamma_i} \{\nu \partial_x v\} \, [\![w]\!] - \sum_{\Gamma_i} [\![v]\!] \, \{\nu \partial_x w\} && (4) \\ &- \sum_{\Gamma_i} \eta_f \, \{\nu r_f([\![v]\!])\} \, [\![w]\!] && (5) \end{aligned}$$

Please refer to the FEM1D notes provided with Project 1 for definitions of F, $\hat{F}_I$, r_f, etc. Alternatively, this can be written as: Find $u_{hp} \in \mathcal{V}_{hp}$ and $\bar{F}_{hp} = (\bar{F}_{hp}(0), \bar{F}_{hp}(1)) \in \mathbb{R}^2$ such that

$$a(u_{hp}, \bar{F}_{hp}, w_{hp}, \lambda_{hp}) = l(w_{hp}, \lambda_{hp}) \qquad \forall\, w_{hp} \in \mathcal{V}_{hp},\ \forall\, \lambda_{hp} \in \mathbb{R}^2 \tag{6}$$

where

$$
\begin{aligned}
a(v,\bar{F},w,\lambda) &= \langle w,\bar{F}\rangle + \langle \lambda, v\rangle - (\partial_x w, F) + (w, bv) + \sum_{\Gamma_i} \hat{F}_I \,[\![w]\!] && (7)\\
&- \sum_{\Gamma_i} \{\nu \partial_x v\} \,[\![w]\!] - \sum_{\Gamma_i} [\![v]\!] \,\{\nu \partial_x w\} && (8)\\
&- \sum_{\Gamma_i} \eta_f \,\{\nu r_f([\![v]\!])\} \,[\![w]\!] && (9)\\
l(w_{hp},\lambda_{hp}) &= \langle \lambda, g\rangle + (w,f) && (10)
\end{aligned}
$$

The adjoint problem to this weak form is then given by: Find $\psi_{hp} \in \mathcal{V}_{hp}$ and $\Phi_{hp} = (\Phi_{hp}(0), \Phi_{hp}(1)) \in \mathbb{R}^2$ such that

$$
a(w_{hp},\lambda_{hp},\psi_{hp},\Phi_{hp}) = l^o(w_{hp},\lambda_{hp}) \qquad \forall\, w_{hp} \in \mathcal{V}_{hp},\ \forall\, \lambda_{hp} \in \mathbb{R}^2 \tag{11}
$$

Q2: In the above adjoint problem, ψ_{hp} is an approximation to the adjoint ψ. Provide an interpretation of Φ_{hp}. In particular, what is the exact value of Φ? Further, determine $l^o(w_{hp},\lambda_{hp})$ such that the BR2 discretization is adjoint consistent. Note adjoint consistency is defined as,

$$
a(w_{hp},\lambda_{hp},\psi,\Phi) = l^o(w_{hp},\lambda_{hp}) \qquad \forall\, w_{hp} \in \mathcal{V}_{hp},\ \forall\, \lambda_{hp} \in \mathbb{R}^2 \tag{12}
$$

where ψ and Φ are the exact (infinite dimensional) solutions to the adjoint problem. In other words, an adjoint consistent FEM discretization is one for which the exact adjoint is a solution to the FEM weak form.

Your Tasks

- Implement in your 1D DG FEM code the solution to the above adjoint problems.
- Implement the DWR method to estimate the output error. Specifically, you must calculate,

$$
J - J_{hp} \approx \mathcal{E}_{h\hat{p}} \equiv -R(u_{hp}, \bar{F}_{hp}, \psi_{h\hat{p}}, \Phi_{h\hat{p}}) \tag{13}
$$

  Important: when you evaluate $\mathcal{E}_{h\hat{p}}$, it is important that the lifting operator $r_f([\![u_{hp}]\!])$ in the residual evaluation is calculated using the $\mathcal{V}_{hp}$ space. In other words, even though $\psi_{h\hat{p}}$ is in $\hat{p}$, the weight functions for the lifting operator should remain in p.

- Implement the local sampling procedure to estimate the local rates of convergence, r_κ in each element, where

$$
\eta_\kappa\left(\frac{h_\kappa}{h_{\kappa 0}}\right) = \eta_{\kappa 0}\left(\frac{h_\kappa}{h_{\kappa 0}}\right)^{r_\kappa} \tag{14}
$$

Project deliverables

- You should submit your source directory (in an archive).
- In a short report (submitted as a single PDF), provide the following:

1) The answers to the questions above.

2) Solve the primal and adjoint problems (for each test case) on a series of uniformally refined meshes for $p = 1$, 2, and 3. Investigate the $L^2(\Omega)$ convergence of the error in the adjoint $\psi - \psi_{hp}$. Provide plots of the error versus h and determine the rate of convergence. Further, demonstrate that your implementation produces the same output when evaluated as $l^o(u_{hp}, \bar{F}_{hp})$ and $l(\psi_{hp}, \Phi_{hp})$. Finally, plot the convergence of the actual error in the output, $J - J_{hp}$ (note: J is readily available from the analytic solution of the primal problems) and determine the rate of convergence.

3) For the same meshes as above, calculate $\mathcal{E}_{h\hat{p}}$ for $p = 1$ and 2 (i.e. $\hat{p} = 2$ and 3). Compare $J_{hp} + \mathcal{E}_{h\hat{p}}$ to $J_{h\hat{p}}$.

4) For $p = 1$, create plots of the distribution of the localized error indicator, η_κ on the meshes used above. Describe how the error indicator is distributed and its dependence on the mesh size.

5) Again for $p = 1$ on the same meshes, use the local sampling procedure to estimate the rates of convergence r_κ in each element, creating plots of the distribution of r_κ, and describing how it is distributed and its dependence on the mesh size.

6) EXTRA: If you are really looking for something to do, you could then use the rates to optimize the mesh sizes and then remesh the one-dimensional domain according to those mesh sizes. If you want to do this, I can give you a few hints. Come and talk to me.