8)

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (7-\lambda)\big((6-\lambda)(5-\lambda) - 4\big) + 2\big(\cancel{4\lambda} - 2(5-\lambda)\big)$$

$$\Rightarrow (7-\lambda)(30 - 11\lambda + \lambda^2 - 4) \cancel{+8} - 4(5-\lambda)$$

$$\Rightarrow (7-\lambda)(\lambda^2 - 11\lambda + 26) \cancel{+8} - 20 + 4\lambda$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 26\lambda + 7\lambda^2 - 77\lambda + 182 \cancel{-8} - 20 + 4\lambda$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 99\lambda + 162$$

$$\Rightarrow (\lambda-3)(\lambda-6)(\lambda-9) = 0$$

$$\Rightarrow \lambda = \{3, 6, 9\}$$

Eigen vectors:

$$\underline{A}v - \lambda v = 0$$

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 7v_1 - 2v_2 \\ -2v_1 + 6v_2 - 2v_3 \\ -2v_2 + 5v_3 \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 7v_1 - 2v_2 - \lambda v_1 \\ -2v_1 + 6v_2 - 2v_3 - \lambda v_2 \\ -2v_2 + 5v_3 - \lambda v_3 \end{bmatrix} = 0$$

$$v_1(7-\lambda) - 2v_2 = 0$$
$$-2v_1 + \cancel{6} v_2(6-\lambda) - 2v_3 = 0$$
$$-2v_2 + v_3(5-\lambda) = 0$$

$$\Rightarrow \begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

1) $\lambda = 3$

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$4v_1 - 2v_2 = 0 \Rightarrow v_2 = 2v_1$

$-2v_1 + 3v_2 - 2v_3 = 0 \Rightarrow -2v_1 + 6v_1 - 2v_3 = 0$

$\Rightarrow 4v_1 - 2v_3 = 0$

$\Rightarrow 4v_1 - 2v_3 = 0$

$\Rightarrow v_3 = 2v_1$

$-2v_2 + 2v_3 = 0$

$\Rightarrow v_2 = v_3$ -

First eigenvector $= \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

2) $\lambda = 6$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$\rightarrow v_1 - 2v_2 = 0$

$v_1 = 2v_2$

Second vector $= \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\rightarrow -2v_1 - 2v_3 = 0$

$v_1 = -v_3$

$\rightarrow -2v_2 - v_3 = 0$

$2v_2 = -v_3$

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -\cancel{3}4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$-2v_1 - 2v_2 = 0$

$v_1 = -v_2$

$-2v_2 - 4v_3 = 0$

$4v_3 = -2v_2$

$2v_3 = -v_2$

$-2v_1 - 3v_2 - 2v_3 = 0$

3^{rd} vec $= \cancel{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$ $\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{c_1}{3} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \frac{c_2}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \frac{c_3}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$c_1 + 2c_2 + 2c_3 = 3$ —(a)

$2c_1 + c_2 - 2c_3 = 3$ —(b)

$2c_1 - 2c_2 + c_3 = 3$ —(c)

a+b ⇒ $3c_1 + 3c_2 = \cancel{2}6 \Rightarrow c_2 = 2\cancel{3} - c_1$

a+c ⇒ $3c_1 + 3c_3 = \cancel{2}6 \Rightarrow c_3 = 2\cancel{3} - c_1$

$c_2 = c_3$

a ⇒ $\cancel{c_1 + (\frac{4}{3}(}$

$c_1 + 2(\frac{2}{\cancel{3}} - c_1) + 2(\frac{2}{\cancel{3}} - c_1) = 3$

$c_1 - 2c_1 - 2c_1 + \frac{8}{\cancel{3}} = 3$

$-3c_1 = -5$ ∴

$$\boxed{\begin{aligned} c_1 &= 5/3 \\ c_2 &= 1/3 \\ c_3 &= 1/3 \end{aligned}}$$

c. To solve the above system of equations;

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = u.$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 3\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}^{-1} u$$

$$\Rightarrow S^{-1}u \Rightarrow$$

& where $S = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

= Matrix of eigenvectors.

d. Using $A = S\Lambda S^{-1}$

$A^{10} u = S\Lambda^{10}S^{-1}u$

1.

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = S$$

$$\therefore \underline{\underline{c}} = S^{*} u$$

$$\Rightarrow \begin{bmatrix} 5/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

d.) ~~Sing~~

~~Since~~

$$A^{10} u = S \Lambda^{10} S^{-1} u$$

$$\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 1/3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Lambda^{10} = \frac{1}{3^{10}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix}$$

$$S^{-1} u = S u = \begin{bmatrix} 5/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\cancel{A'}\ \Lambda^{10} S^{-1} u = \frac{1}{3^{10}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix} \begin{bmatrix} 5/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3^{10}} \begin{bmatrix} 5/3 \\ 2^{10}/3 \\ 3^{9} \end{bmatrix}$$

$$\cancel{SA}\ S \Lambda^{10} S^{-1} u = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{3^{10}} \begin{bmatrix} 5/3 \\ 2^{10}/3 \\ 3^{9} \end{bmatrix} = \boxed{10^{4} \times 3^{10} \begin{bmatrix} 1.335 \\ -1.3 \\ 0.63 \end{bmatrix}}$$

5. FTCS :

$$u(x,y,z,t) = G(t)\cdot F(x,y,z)$$

let F be seperable as ~~f.g.h~~ $f(x)\,g(y)\,h(z)$

$$f = e^{ikx}$$
$$g = e^{iky}$$
$$h = e^{ikz}$$

Discretize as:

$$u(a,b,c,n) = \cancel{G^n\cdot F(a\Delta x, b\Delta y, c\Delta z)}$$
$$G^n\cdot F_{abc}, \text{ where } x = a\Delta x,\ y = b\Delta y,\ z = c\Delta z$$

~~FCC~~ FTCS form of heat equation:

$$\frac{G^{n+1}F_{abc} - G^n F_{abc}}{\Delta t} = \alpha G^n\left[\begin{aligned}&\tfrac{1}{\Delta x}(F_{a+1\,bc} - 2F_{abc} + F_{a-1\,bc}) + \\ &\tfrac{1}{\Delta y}(F_{a\,b+1\,c} - 2F_{abc} + F_{ab-1c}) + \\ &\tfrac{1}{\Delta z}(F_{abc+1} - 2F_{abc} + F_{abc-1})\end{aligned}\right]$$

$$\frac{G^{n+1} - G^n}{G^n} = \frac{\alpha\Delta t}{F_{abc}}\left[\begin{aligned}&\tfrac{1}{\Delta x}\Big(e^{ik(a+1)\Delta x}\cdot e^{ikb\Delta y}\cdot e^{ikc\Delta z} \\ &\quad - 2e^{ika\Delta x}\,e^{ikb\Delta y}\cdot e^{ikc\Delta z} \\ &\quad + e^{ik(a-1)\Delta x}\,e^{ikb\Delta y}\cdot e^{ikc\Delta z}\Big) \\ &+ \cdots\end{aligned}\right]$$

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Assuming $\Delta x = \Delta y = \Delta z = \Delta d$

$$\frac{G^{n+1}}{G^n} - 1 = \frac{1}{e^{ik(a+b+c)\Delta d}} \frac{\alpha \Delta t}{\Delta d} \Big(e^{ik(a+1+b+c)\Delta d} - 2e^{ik(a+b+c)\Delta d} + e^{ik((a-1)+b+c)\Delta d} + e^{ik(a+b+1+c)\Delta d} - 2e^{ik(a+b+c)\Delta d} + e^{ik(a+b-1+c)\Delta d} + \cdots \Big)$$

$$\frac{G^{n+1}}{G^n} - 1 = \underbrace{\frac{\alpha \Delta t}{\Delta d}}_{C} \left(3e^{ik\Delta d} - 6 + 3e^{-ik\Delta d} \right)$$

For stability : $-1 \le \frac{G^{n+1}}{G^n} \le 1$

$$\Rightarrow -2 \le \frac{G^{n+1}}{G^n} - 1 \le 0$$

$$\Rightarrow -2 \le C \left(6\cos(k\Delta d) - 6 \right) \le 0$$

$\Rightarrow$ lowest value of $\cos(k\Delta d) = -1$

$\therefore -2 \le C(-12)$

$\therefore \boxed{C \le 1/6}$

highest value of $\cos(k\Delta d) = 1$

$\therefore$ a no limit on the other side.

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Similarly, for 2d:

$$\frac{G^{n+1}}{G^n} - 1 = \frac{\alpha \Delta t}{\Delta d}\left(2e^{ik\Delta d} - 4 + 2e^{-ik\Delta d}\right)$$

$$\Rightarrow \quad -2 \le C\left(4(\cos(k\Delta d) - 4)\right) \le 0$$

$$\Rightarrow \quad \boxed{C \le 1/4}$$

BECS :

BECS form of heat equation:

$$\frac{G^{n+1} F_{abc} - G^n F_{abc}}{\Delta t} = \alpha G^{n+1}\left[\begin{array}{l}\frac{1}{\Delta d}\left(F_{a+1bc} - 2F_{abc} + F_{a-1bc}\right) \\ + \frac{1}{\Delta d}\left(F_{ab+1c} - 2F_{abc} + F_{ab-1c}\right) \\ + \frac{1}{\Delta d}\left(F_{abc+1} - 2F_{abc} + F_{abc-1}\right)\end{array}\right]$$

$$\Rightarrow \quad \frac{G^{n+1} - G^n}{G^{n+1}} = \frac{\alpha \Delta t}{\Delta d}\left[3e^{ik\Delta d} - 6 + 3e^{-ik\Delta d}\right]$$

$$1 - \frac{G^n}{G^{n+1}} = \underbrace{\frac{\alpha \Delta t}{\Delta d}}_{C}\left(6\cos(k\Delta d) - 6\right)$$

$$\frac{G^n}{G^{n+1}} = 1 - c\left(6\cos(k\Delta d) - 6\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - 6c\left(\cos(k\Delta d) - 1\right)}$$

$$-1 \le \frac{G^{n+1}}{G^n} \le 1$$

$$\Rightarrow \quad \frac{G^n}{G^{n+1}} \le -1 \quad \text{or} \quad \frac{G^n}{G^{n+1}} \ge 1$$

a) $\Rightarrow \quad 1 - 6c\left(\cos(k\Delta d) - 1\right) \le -1$

$$6c\left(\cos(k\Delta d) - 1\right) \ge 2$$

b) $\Rightarrow \quad 1 - 6c\left(\cos(k\Delta d) - 1\right) \ge 1$

$$6c\left(\cos(k\Delta d) - 1\right) \le 0$$

Always true, since $\cos(k\Delta d) \le 1$

$\therefore$ $\boxed{\text{unconditionally stable}}$

Similarly for 2d:

$$4c\,(\cos(k\Delta d) - 1) \leq 0$$

and 1d:

$$2c\,(\cos(k\Delta d) - 1) \leq 0$$

Alway unconditionally true.