

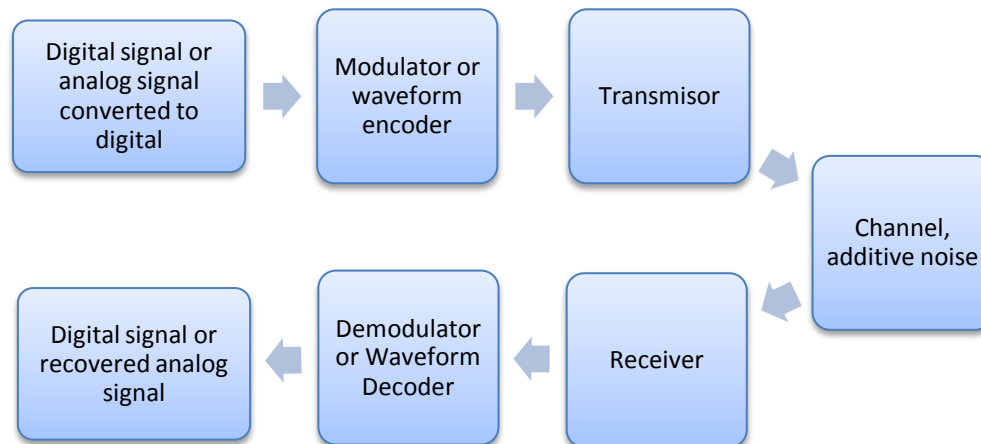
Communication Systems based on Software Defined Radio (SDR)

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Review of Digital Communication Systems

Introduction. baseband digital transmission

A baseband system is represented in the block diagram of the figure



Introducción

The purpose of this chapter is to analyze different problems associated with baseband transmission, namely:

The effect of the **noise** in the channel, and the determination of the properties of the system in function of the noise, the energy and spectrum of the signal.

The effect of channel filtering on signal characteristics, the effect of **intersymbol interference** and methods to avoid it.

The generation of baseband signals with digital characteristics, coming from an analog source. **Digitization** of analog signals.

Digital transmission parameters

The parameters of digital transmission are related to the classic parameters of analog communication:

The concept of signal/noise ratio will be replaced by the estimate of the **error rate**, that is, the amount of errors produced in a transmission of a large number of data.

The bandwidth will be equivalent to the amount of data transmitted in a given time interval, that is, the **transmission rate**. When more data will be sent, the data will be shorter duration, and a greater bandwidth required for that purpose.

PAM Digital signals (Pulse Amplitude Modulation)

The baseband digital transmission signal is modeled as a train of pulses of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kD)$$

Where the amplitude modulation is represented by the coefficient a_k which is the k -th symbol in the sequence, in such a way that the symbols a_k correspond to some of the M values of the discrete alphabet of signals.

In the previous expression, the basic pulse that travels in the channel and that represents the waveform of each symbol sent is $p(t)$, which will be considered as a pulse such that:

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm D, \pm 2D, \pm 3D \dots \end{cases}$$

PAM Digital signals (Pulse Amplitude Modulation)

The pulse has a value normalized to 1 at the instant $t=0$, and passes through zero at the sampling instants $t=\pm D, \pm 2D, \pm 3D, \dots$, of the neighboring data to the analyzed one.

The sampling of the signal occurs synchronously at instants kD , where $k = 0, \pm 1, \pm 2, \dots$ in such a way that for a certain value of $k=K$:

$$x(KD) = \sum_{k=-\infty}^{\infty} a_K p(KD - kD) = a_K$$

Since $KD - kD$ is zero, except for $k=K$, and $p(KD - KD) = p(0) = 1$

The sequence of the transmitted signal, a train of pulses, typically rectangular, has a cadence or symbol rate.

PAM Digital signals (Pulse Amplitude Modulation)

The symbol r will be used to represent the repetition period between pulses. In this way the inverse will be equal to the signaling rate, r :

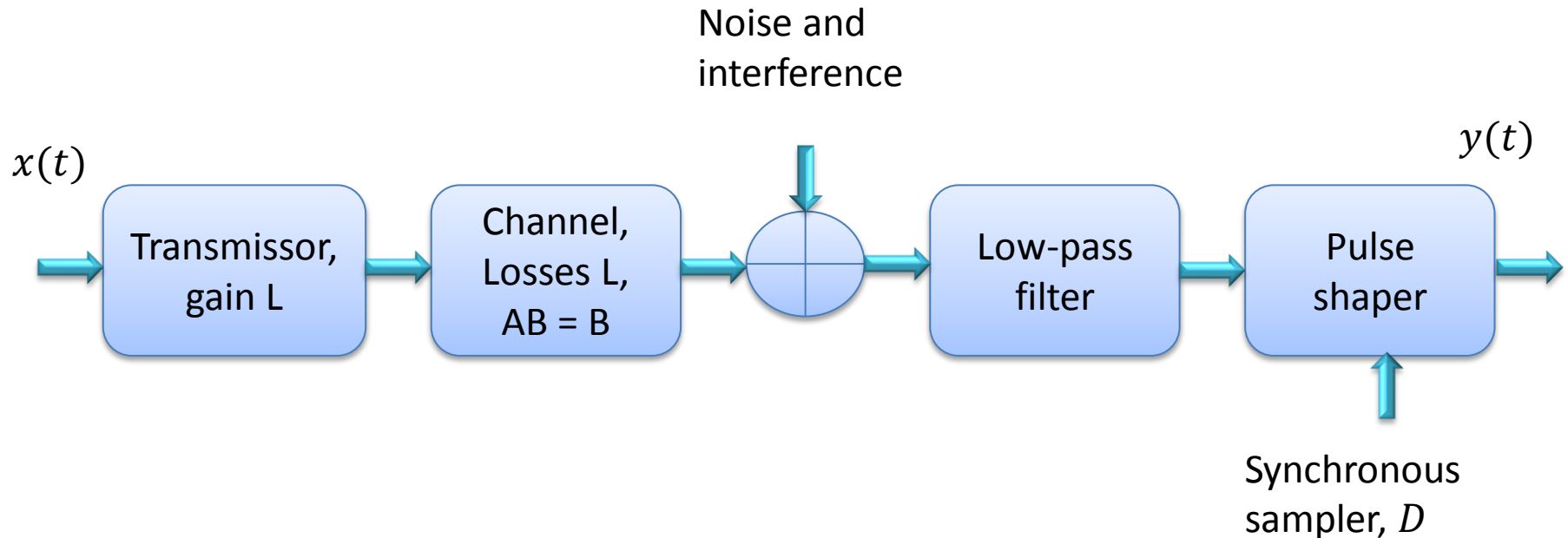
$$r = \frac{1}{D} \text{ (baud)}$$

As mentioned in the introduction, digital transmission is based on sending any of the M available symbols of the discrete alphabet. When in the binary case, $M = 2$, the symbol r_b will be used to represent the repetition period between pulses in the case of binary transmission. In this way the inverse of the time interval $D=T_b$ will be equal to the signaling speed, r_b :

$$r_b = \frac{1}{T_b} \text{ (bits/second, bps)}$$

Limitations of the Transmission

A baseband transmission system is presented in the block diagram shown in the figure:



Limitations of the Transmission

a_K is the ideal sampled value, the amplitude that the symbol originally had when transmitted unaffected by noise and interference

$\sum_{k \neq K} a_k \tilde{p}(t - t_d - kD)$ is the contribution of all the other pulses when, due to imperfections, they do not cross zero at the sampling instants, $\pm D, \pm 2D, \pm 3D, \dots$, that is, it is the **ISI** term

$n(t_K)$ is the value added to the signal that provides the **additive noise** present in the channel

Multilevel Signaling

To increase the rate in bps, the binary information can be grouped in sets of more than one bit. The message may be sent as a sequence of bits that alternately represent a one or a zero. However, the same message could be sent by grouping the bits in pairs, such that an alphabet of $M = 4$ symbols is used for example. If each element of the M of the set is assigned a number according to the natural code, the following is obtained:

Bit grouping	Ampl
1 1	$3A/2$
1 0	$A/2$
0 1	$-A/2$
0 0	$-3A/2$

Baseband signal formats

The properties of each format can be useful in one case or another, according to the needs imposed by the transmission channel.

Some desirable characteristics of the formats are listed below:

- *Auto-synchronization capability.*
- *Spectrum adaptable to the characteristics of the transmission channel.*
- *Transmission bandwidth: It should be as low as possible.*

Power Spectral Density of an amplitude modulated digital signal

Spectral efficiency

The spectral efficiency of a system is the number of bits per second that are transmitted for each Herz of bandwidth;

$$\frac{r_b}{B_T} \text{ bps/Hz}$$

When there are bandwidth limitations and it is desired to increase the bit signaling rate, M-ary signaling is normally used. In this case the typical result is that the efficiency is equal to $\log_2 M$.

For the most popular formats:

Power Spectral Density of an amplitude modulated digital signal

Format	bandwidth for 1 ^{er} zero crossing	Efficiency η_B
Unipolar NRZ	r_b	1
Polar RZ	$2r_b$	1/2
Unipolar RZ	$2r_b$	1/2
Polar NRZ	r_b	1
Manchester	$2r_b$	1/2
M-ary NRZ	r_b/n	n

Part 2

Noise and errors

Noise is one of the main problems to be solved in the design of a communications system. The case for binary and M-ary signals will be analyzed.

Using the concept of superposition, the effect of noise is analyzed as if intersymbol interference (ISI) did not exist. Then the ISI will be considered, canceling the effect of the noise. Contribution of the different effects on the transmitted signal after sampling the received signal. Power of an amplitude modulated digital signal:

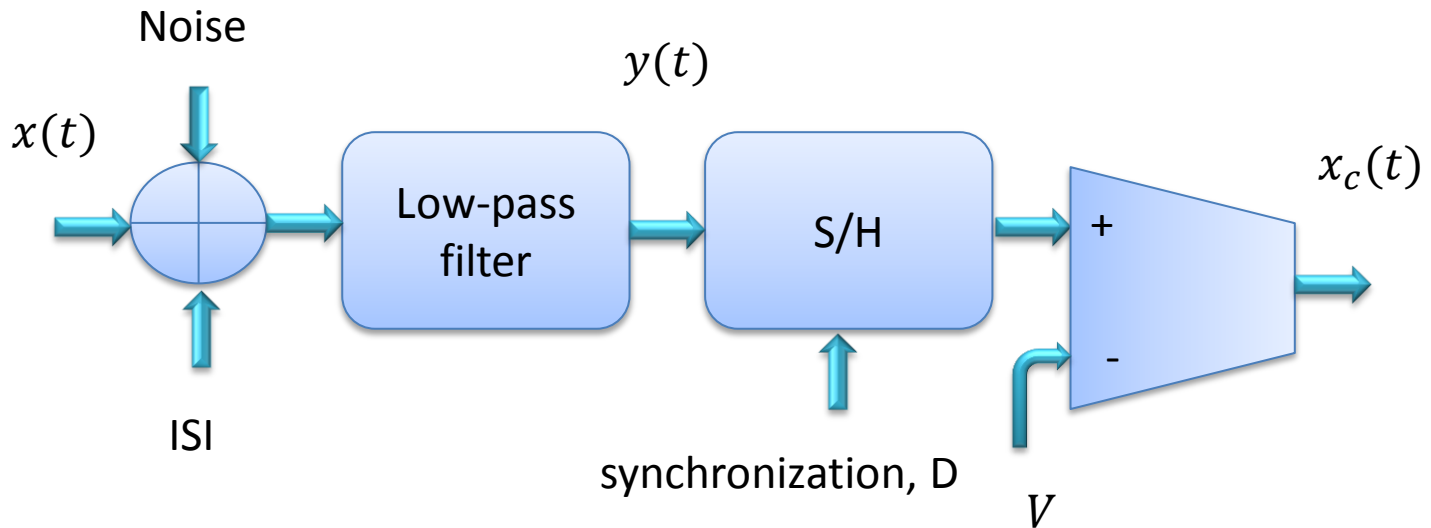
$$y(t_K) = a_K + \sum_{k \neq K} a_k \tilde{p}(t - t_d - kD) + n(t_K)$$

If we consider noise only:

$$y(t_K) = a_K + n(t_K)$$

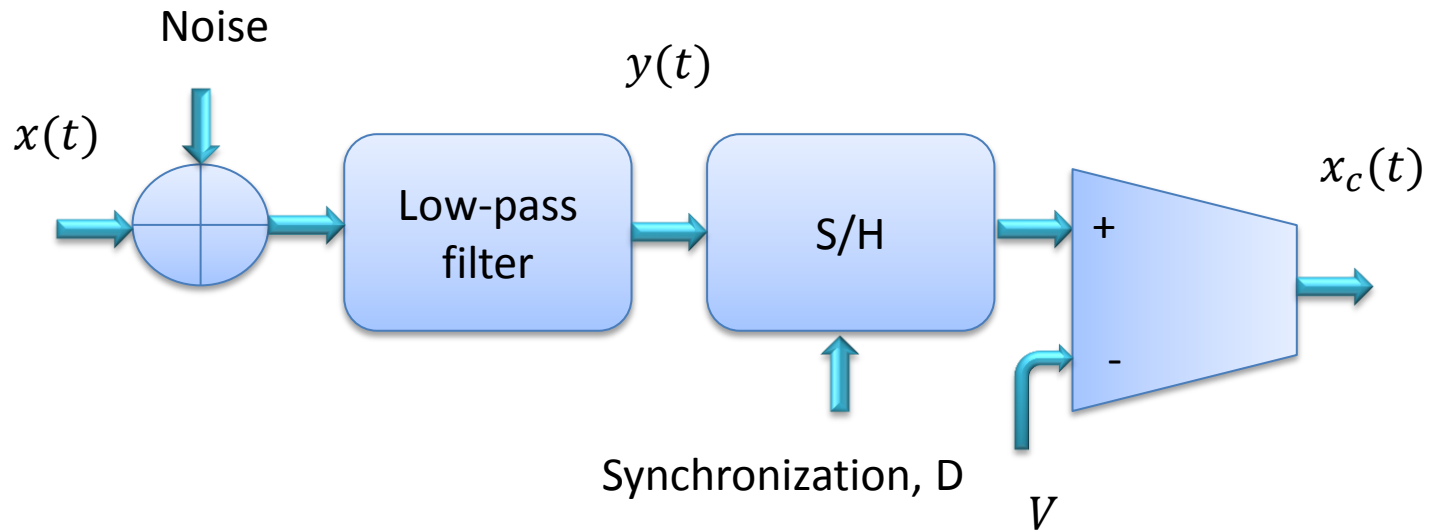
Noise and errors

In the receiver is the sum of the signal of interest, the noise and the possible ISI interference.



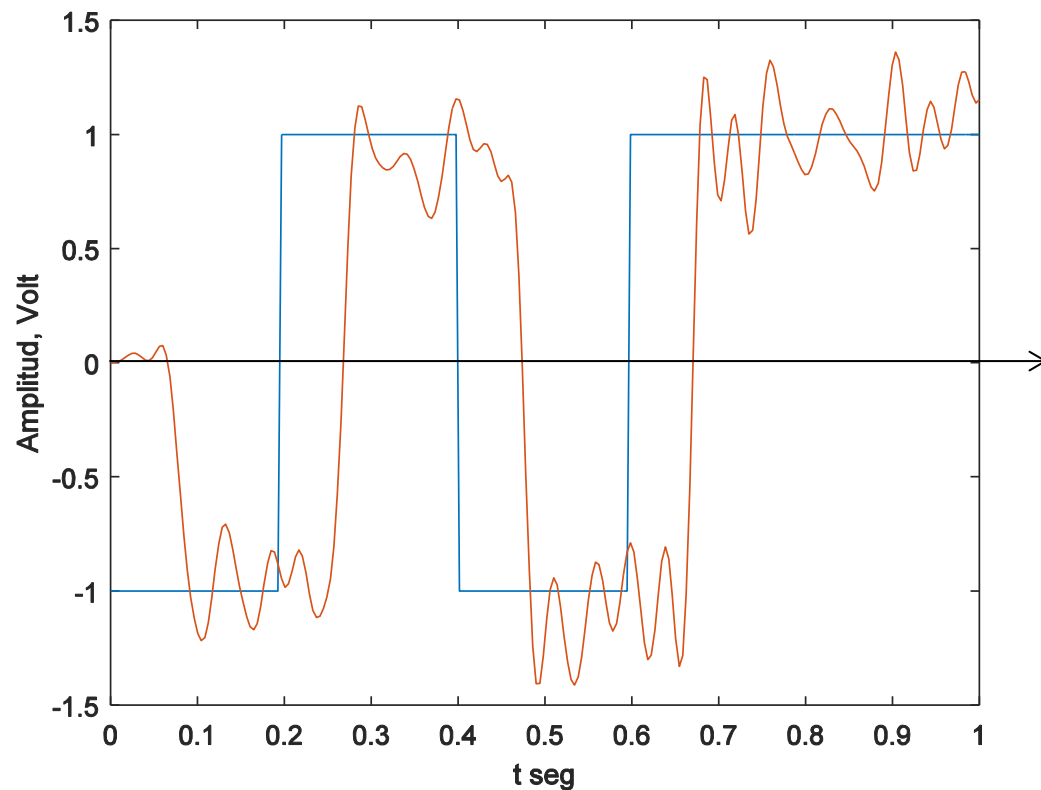
Noise and Errors

Applying the superposition theorem we analyze only the effect of noise:



Noise and Errors

The transmission of a data stream affected by noise and filtering would look like in the figure:



Binary probability of error

The signal and the noise enter the receiver. The first stage consists of a low-pass filter that removes part of the incoming noise without producing intersymbol interference. We will study the effect of noise assuming that the ISI caused is zero. The value detected by the receiver after the sample and hold operation is:

$$y(t_K) = a_K + n(t_K)$$

The received value is calculated by applying what is called “hard decision”, that is, the received value $y(t_K)$ is compared with a certain threshold value V .

Binary probability of error

A decision rule is adopted such that:

$$\begin{cases} \text{Si } y(t_K) > V, \text{ then the data is a '1'} \\ \text{Si } y(t_K) < V, \text{ then the data is a '0'} \end{cases}$$

In this way, the receiver converts the samples of the received signal $y(t_K)$, which corresponds to a sampled noisy signal, into a signal $x_c(t)$, which is a signal similar to the transmitted one $x(t)$, without noise, but with any errors.

The sampled values $y(t_K)$ correspond to a continuous random variable Y , while the values $n(t_K)$ belong to the samples of a random variable n .

Binary probability of error

We will then say that:

H_0 : It is the hypothesis that a zero ('0') has been sent :

$$a_k = 0; \quad Y = n$$

H_1 : It is the hypothesis that a one ('1') has been sent :

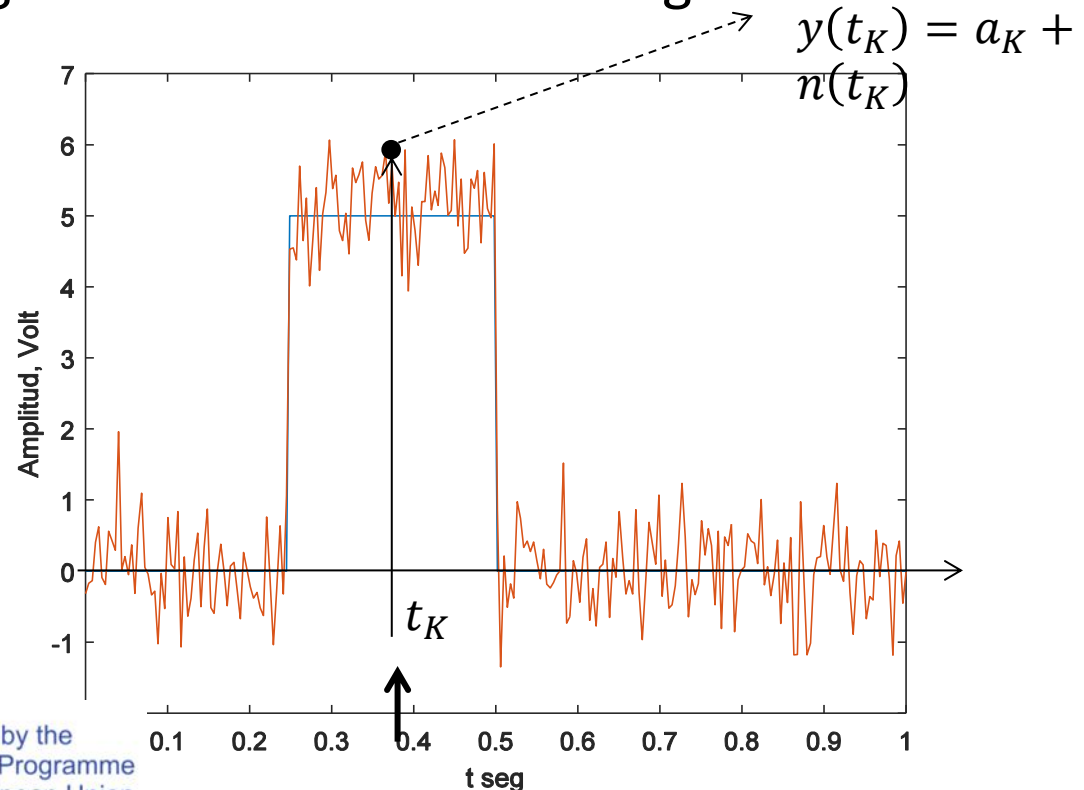
$$a_k = A; \quad Y = A + n.$$

The conditional probability density functions for the random variable Y , for the events H_0 and H_1 are given by:

$$p_Y(y/H_0) = p_n(y) \qquad p_Y(y/H_1) = p_n(y - A)$$

Binary probability of error

Where $p_n(y)$ is the pdf of the noise in the studied channel, in our case the Gaussian channel. This is the pdf function of the noise added to the signal, characterized as a Gaussian pdf function. When the signal is transmitted in a Unipolar format in the presence of this noise, it would be observed, in the absence of filtering, something like what is seen in the figure :



Binary probability of error

P_0 y P_1 are the probabilities of occurrence of zeros and ones respectively.

If the source data is equiprobable:

$$P_e = (P_{e0} + P_{e1})/2$$

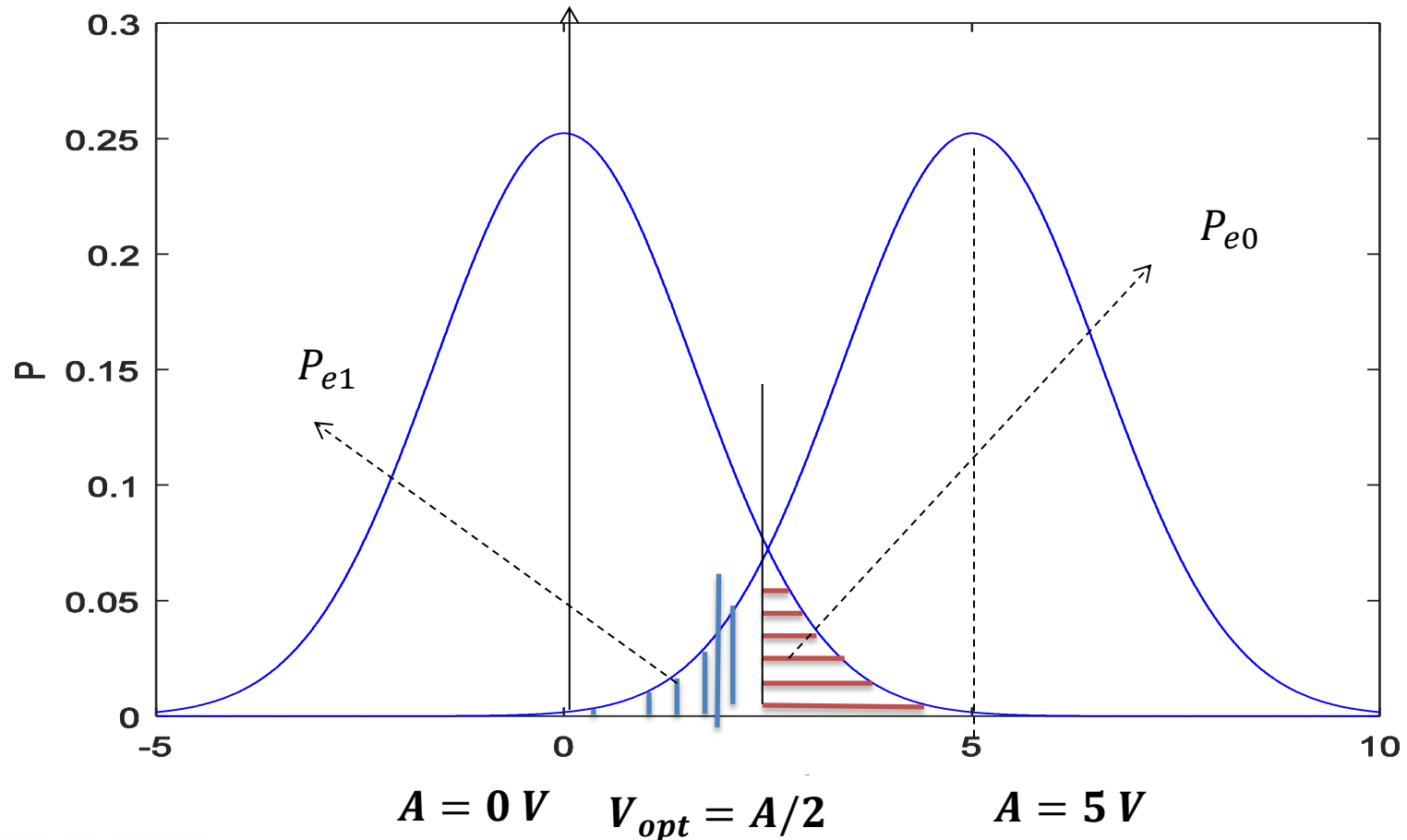
The location of the decision threshold is certainly important. A value of very close to '0' reduces the error for the value '1', but strongly increases it for the value '0', and vice versa.

To calculate the optimal position of the threshold value V , the expression of the probability of error with respect to the threshold must be derived, and set equal to zero:

$$\frac{dP_e}{dV} = 0$$

Binary probability of error

In the case of equiprobable data, the optimal value of the threshold is: $V_{opt} = A/2$



Binary probability of error

$$P_{e0} = \frac{1}{\sqrt{2\pi}} \int_{V/\sigma}^{\infty} e^{-\frac{(\lambda)^2}{2}} d\lambda = Q\left(\frac{V}{\sigma}\right)$$

$$P_{e1} = \int_{-\infty}^V p_Y(y/H_1) dy = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^V e^{-\frac{(y-A)^2}{2\sigma^2}} dy$$

With $\lambda = (y - A)/\sigma$, the upper limit is $(V - A)/\sigma$

$$P_{e1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(V-A)/\sigma} e^{-\frac{(\lambda)^2}{2}} d\lambda = Q\left(\frac{V - A}{\sigma}\right)$$

If the optimal threshold is used in the equiprobable case, then $V = A/2$,

$$P_e = P_{e0} = P_{e1} = Q\left(\frac{A}{2\sigma}\right)$$

Binary probability of error

This is the minimum probability of binary error in the presence of Gaussian white noise when the digits are equally likely.

As can be seen, the term $A/(2\sigma)$ or its corresponding squared value $A^2/(4\sigma^2)$, defines the magnitude of the errors in the system, that is, the error probability for a given receiver. This quantity is a **signal-to-noise ratio** expressed in amplitude or power.

This result is the same for polar signaling when the distance between the symbols remains constant and equal to A ,
($a_k = \pm A/2$).

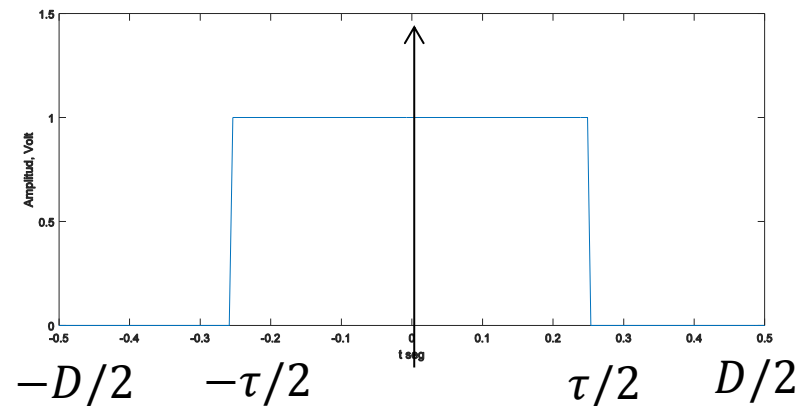
Expression of received power

To relate the probability of error and the signal noise ratio, the calculation of the power associated with the transmitted sequence is first proposed.

Assuming that $T=ND$ is a period large enough for a sequence of values transmitted in time slots D , such that $N \gg 1$, and that rectangular pulses of the form

$$p(t) = \begin{cases} 1 & \text{si } |t| < \tau/2 \\ 0 & \text{si } |t| > \tau/2 \end{cases}$$

$$\tau \leq$$



Expression of received power

If the format NRZ is used, $\tau = T_b = D$

$$S_R = P_0(a_0)^2 \frac{1}{D} \int_{-\frac{D}{2}}^{\frac{D}{2}} p^2(t) dt + P_1(a_1)^2 \frac{1}{D} \int_{-\frac{D}{2}}^{\frac{D}{2}} p^2(t) dt$$
$$S_R = P_0(a_0)^2 + P_1(a_1)^2$$

$$S_R = \frac{1}{2}(A)^2 + \frac{1}{2}(0)^2 = \frac{A^2}{2}, \text{ Unipolar NRZ}$$

$$S_R = \frac{1}{2}\left(\frac{A}{2}\right)^2 + \frac{1}{2}\left(-\frac{A}{2}\right)^2 = \frac{A^2}{4}, \text{ Polar NRZ}$$

Signal-to-Noise ratio

Then

$$A = \sqrt{2S_R}, \text{ Unipolar NRZ}$$

$$A = \sqrt{4S_R}, \text{ Polar NRZ}$$

$$\left(\frac{A}{2\sigma}\right)^2 = \frac{A^2}{4N_R} = \begin{cases} \frac{1}{2} \left(\frac{S}{N}\right)_R & \text{Unipolar NRZ} \\ \left(\frac{S}{N}\right)_R & \text{Polar NRZ} \end{cases}$$

$$N_R = \eta B \geq \frac{1}{2} \eta r_b$$

Adapted filter

The low-pass filter at the input of the receiver is intended to limit the noise entering the system. As explained, excessive filtering could cause ISI, so there will be a trade off between noise filtering and the interference.

If the pulses are limited in time, and the noise present is white and Gaussian, the filter with optimal characteristics in terms of noise and ISI is the matched filter.

The matched filter maximizes the signal-to-noise ratio, which is actually the ratio that exists between the sampled value at the optimum instant and the noise power that enters through the filter.

Filter Adapted to pulse detection

In the analyzed case, the incoming pulse has a known shape $p(t)$:

$$x_R(t) = A_R p(t - t_0)$$

With a known delay t_0 .

Its Fourier transform is:

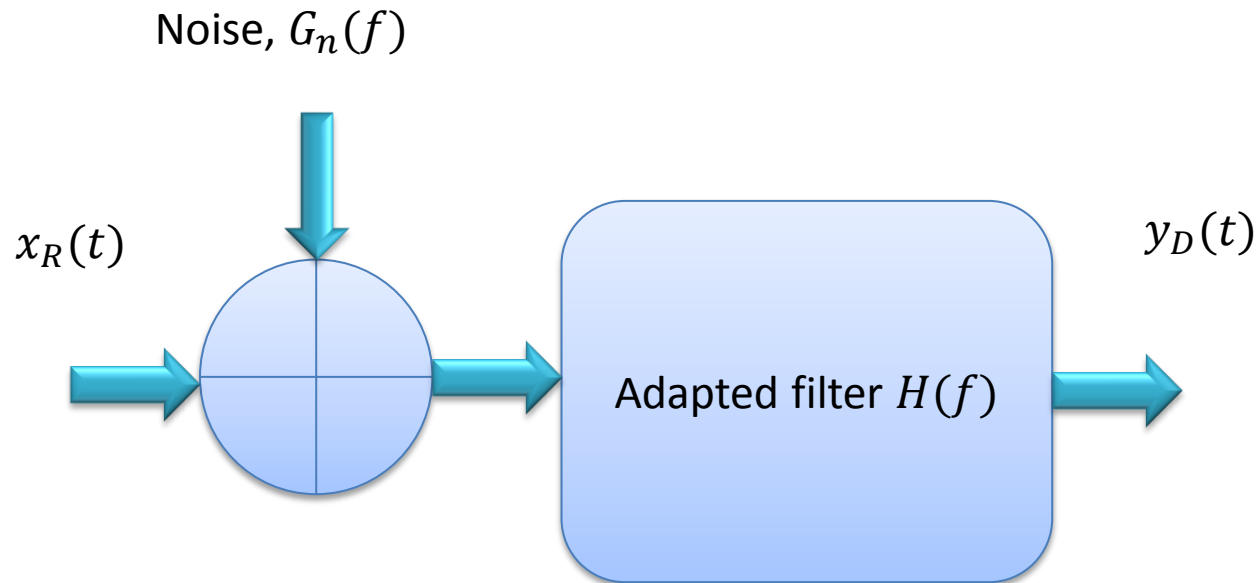
$$X_R(f) = A_R P(f) e^{-j\omega t_0}$$

Signal whose energy is:

$$E_R = A_R^2 \int_{-\infty}^{\infty} |P(f)|^2 df$$

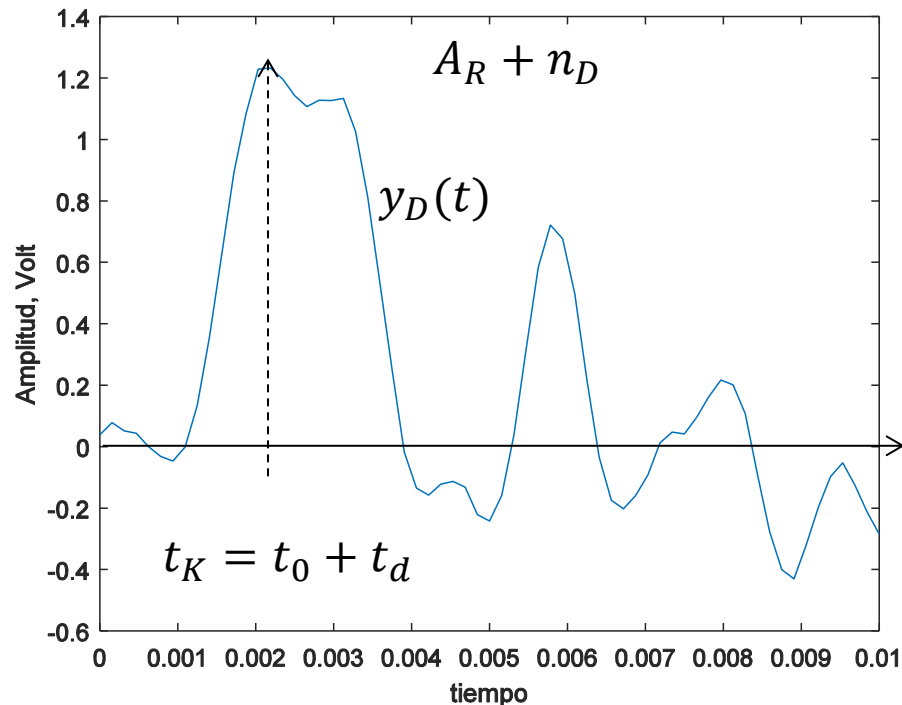
Filter Adapted to pulse detection

The received pulse, which is of arbitrary shape, enters the receiver, where the matched filter is located, in the presence of spectral density noise $G_n(f)$:



Filter Adapted to pulse detection

The received pulse is to be sampled at an optimal time instant to maximize the received amplitude in comparison to the noise amplitude entering the receiver. That is the goal of the matched filter, to maximize A with respect to the incoming noise.



Filter Adapted to pulse detection

If the matched filter is applied, the signal-to-noise ratio assumes its maximum value of the form:

$$\left(\frac{A}{\sigma}\right)_{max}^2 = A_R^2 \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df$$

If the noise is white, the distribution of spectral components is flat:

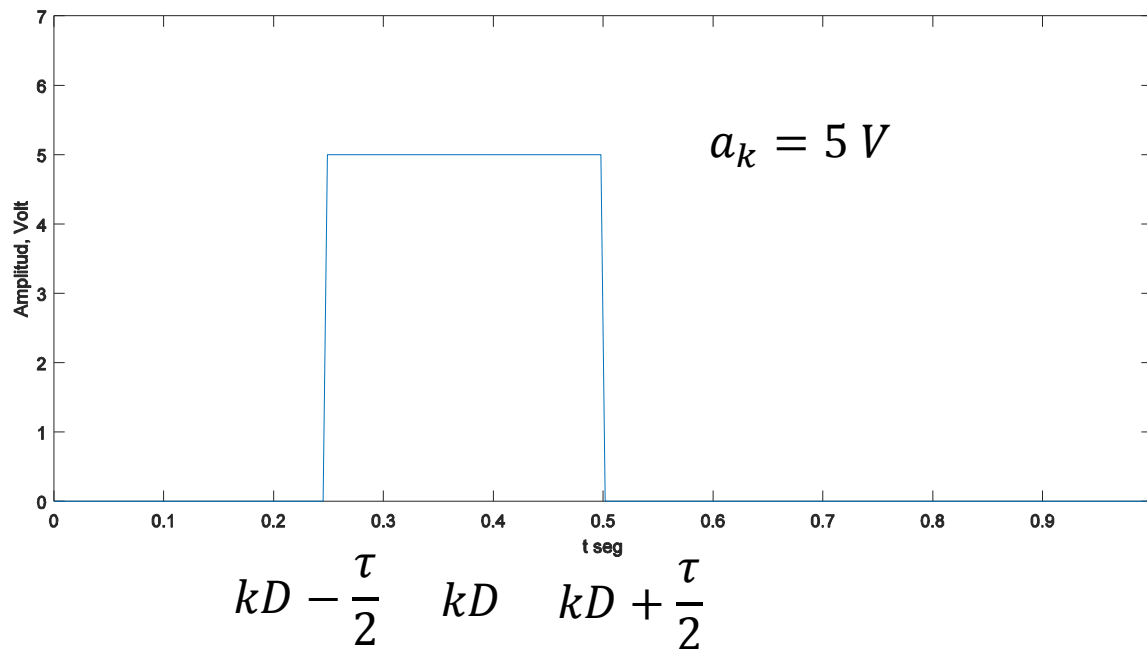
$$\left(\frac{A}{\sigma}\right)_{max}^2 = \frac{2A_R^2}{\eta} \int_{-\infty}^{\infty} |P(f)|^2 df = \frac{2E_R}{\eta}$$

The signal-to-noise ratio $A/(2\sigma)$, argument of the function $Q(k)$ that determines the error rate, depends on the energy of the pulse with respect to the power spectral density of the noise.

Rectangular Pulse Matched Filter

If the input signal is a rectangular pulse of duration τ , limited in time, and centered at $(t - kD)$:

$$x(t) = a_k p(t - kD)$$



Parameters of binary digital transmission

In general the incoming noise power has its minimum if the matched filter is used, and is greater than that amount in the case of any other filter:

$$N_R \geq \frac{\eta}{2\tau_{eq}}$$

Remembering the expression for S_R , and $D = T_b$:

$$S_R = P_0 \frac{1}{T_b} \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_0)^2 p^2(t) dt + P_1 \frac{1}{T_b} \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_1)^2 p^2(t) dt$$
$$S_R = \frac{1}{T_b} \left[P_0 \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_0)^2 p^2(t) dt + P_1 \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_1)^2 p^2(t) dt \right]$$

Parameters of binary digital transmission

$$S_R = \frac{1}{T_b} \left[P_0 \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_0)^2 p^2(t) dt + P_1 \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_1)^2 p^2(t) dt \right]$$

$$S_R = \frac{1}{T_b} [P_0 E_0 + P_1 E_1] = r_b E_b$$

With $E_b = P_0 E_0 + P_1 E_1$

$$E_0 = \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_0)^2 p^2(t) dt; \quad E_1 = \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} (a_1)^2 p^2(t) dt$$

The average energy per bit E_b is a statistical average of the energies of the signals corresponding to data '0' y '1'.

Parameters of binary digital transmission

$$S_R = r_b E_b$$
$$E_b = \frac{S_R}{r_b}$$

It is possible to define:

$$\gamma_b = \frac{E_b}{\eta} = \frac{S_R}{\eta r_b}$$

Which we call the ratio of the energy per bit to the noise power spectral density. Since E_b is energy measured in *Joules*, and η is a power spectral density measured in $\frac{\text{Watts}}{\text{Hz}} = \frac{\text{Joules}}{\text{seg}}$, then γ_b is a dimensionless quantity, a number that conceptually equals signal-to-noise ratio s .

Parameters of binary digital transmission

$$E_b = P_0 E_0 + P_1 E_1 = E \left[a_k^2 \int_{-\infty}^{\infty} p^2(t - kD) dt \right]$$
$$E_b = E \left[a_k^2 \int_{-\infty}^{\infty} p^2(t) dt \right] = \overline{a_k^2} \tau_{eq}$$
$$\overline{a_k^2} = \begin{cases} E[a_k^2] = \frac{1}{2} A^2 + \frac{1}{2} 0^2 = \frac{A^2}{2} & U \\ E[a_k^2] = \frac{1}{2} \left(\frac{A}{2} \right)^2 + \frac{1}{2} \left(-\frac{A}{2} \right)^2 = \frac{A^2}{4} & P \end{cases}$$

Parameters of binary digital transmission

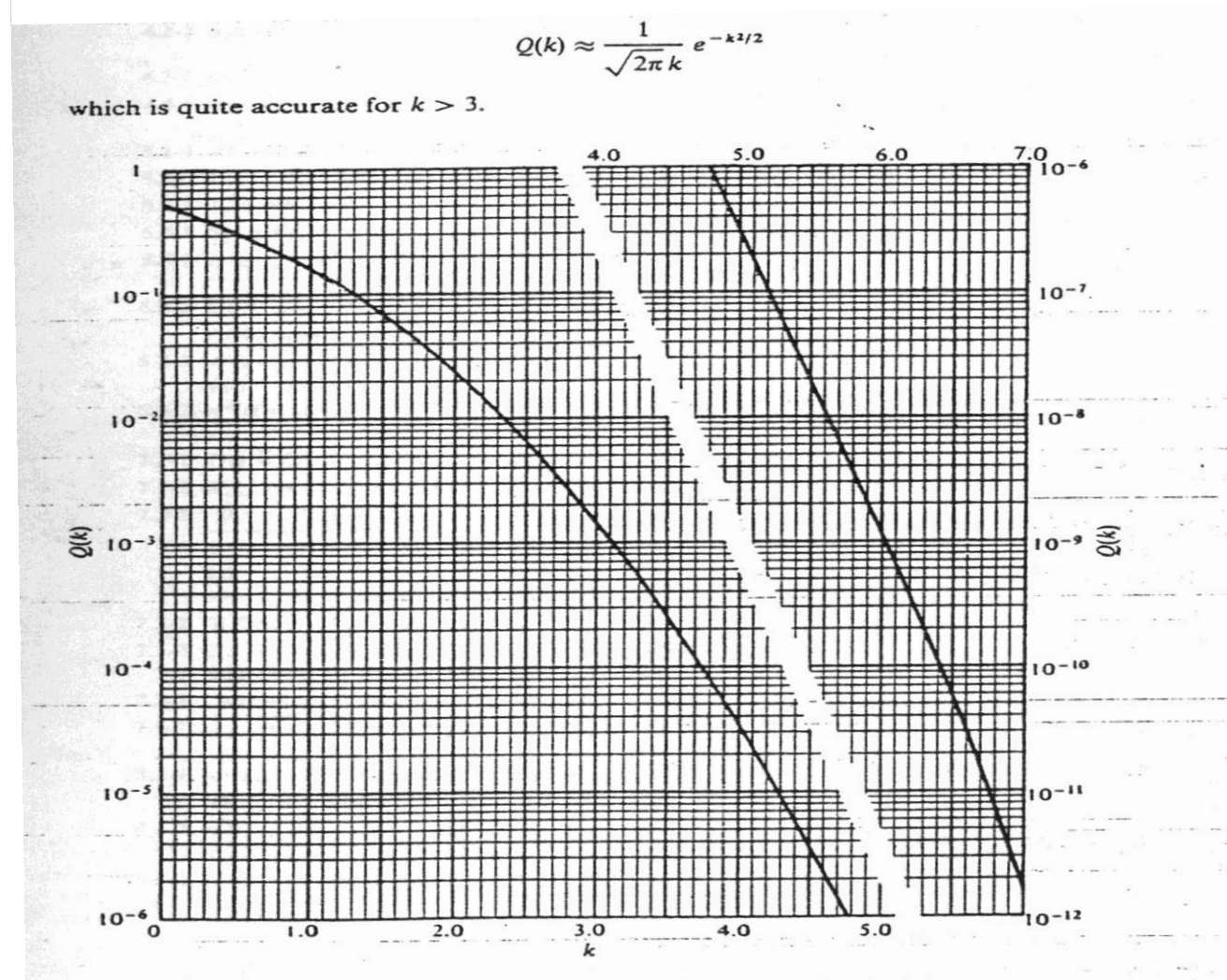
If a matched filter is applied in the receiver, the noise power is $N_R = \frac{\eta}{2\tau_{eq}}$, then:

$$\left(\frac{A}{2\sigma}\right)^2 = \begin{cases} \frac{2\overline{a_k^2}}{4\sigma^2} = \frac{2E_b 2\tau_{eq}}{4\tau_{eq}\eta} = \frac{E_b}{\eta} = \gamma_b & U \\ \frac{4\overline{a_k^2}}{4\sigma^2} = \frac{4E_b 2\tau_{eq}}{4\tau_{eq}\eta} = \frac{2E_b}{\eta} = 2\gamma_b & P \end{cases}$$

The binary probability of error for each format is a function of the factor $A/(2\sigma)$, for each format:

$$P_e = \begin{cases} Q(\sqrt{\gamma_b}) & U \\ Q(\sqrt{2\gamma_b}) & P \end{cases}$$

Binary probability of error



M-ary Probability of Error

Binary signaling provides the best noise isolation because the full dynamic range of voltages is used by only two levels. The purpose of using M-ary or multilevel signaling is to increase signaling speed while maintaining constant bandwidth, but this objective is achieved at the expense of error performance, or to preserve the same error probability, at expense of more power.

The error probability will be calculated in the presence of white and Gaussian noise, with zero mean value and noise power σ^2 .

Polar signaling will be used to simplify some calculations:

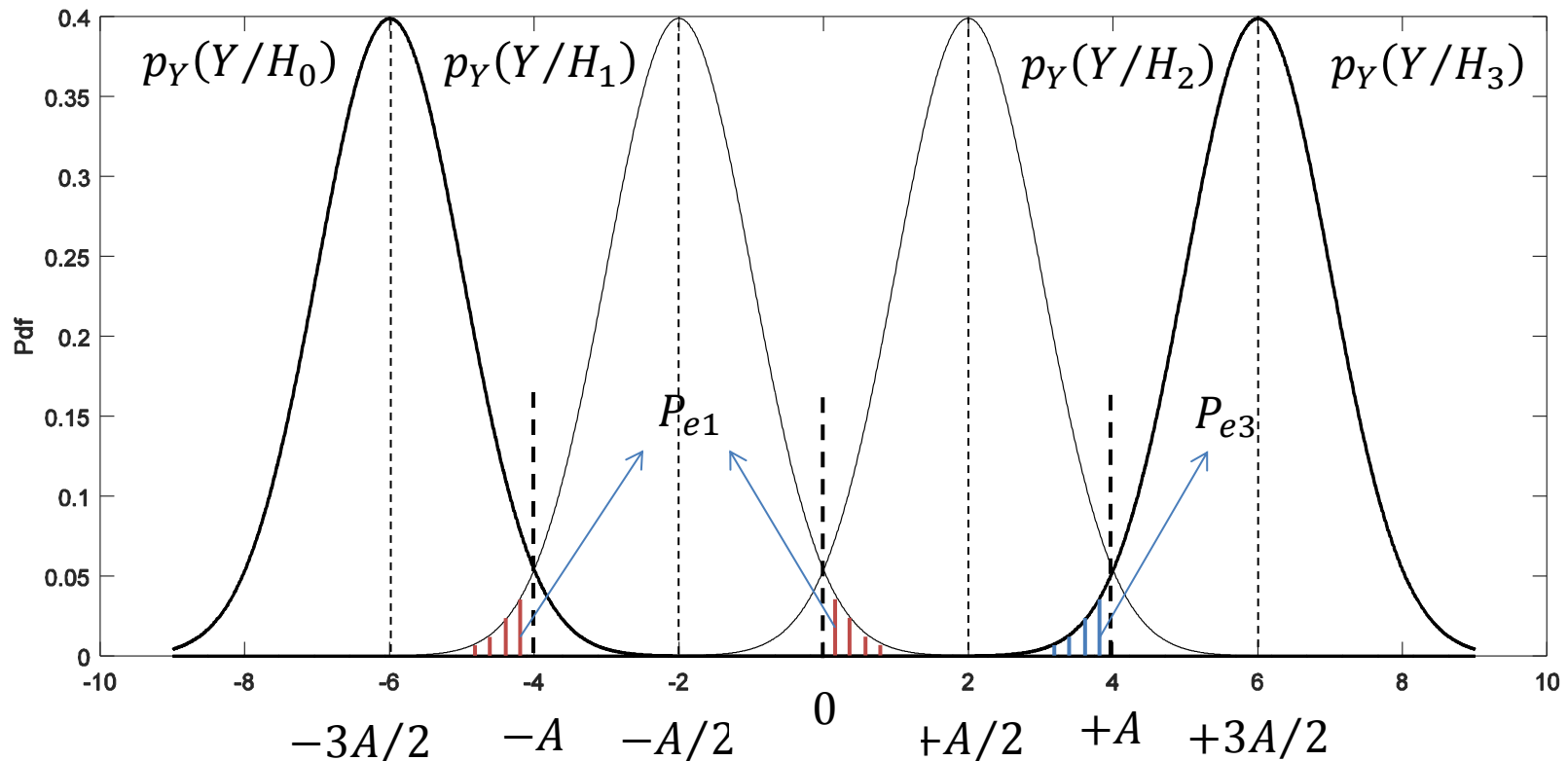
$$a_k = \pm A/2, \pm 3 A/2, \pm \dots \pm (M - 1)A/2$$

M-ary Probability of Error

If all the symbols of the value alphabet are equiprobable:

$$P_e = \frac{1}{M} (P_{e0} + P_{e1} + \cdots P_{eM-1})$$

The probability density function scheme for an M-ary signal of, for example, 4 levels is presented in the figure



M-ary Probability of Error

The decision thresholds are determined assuming that all values are equally likely so that they are located at the midpoint of the mean values of each probability density function. The thresholds are at $-A$, 0 and A .

The symbols do not have the same probability of error. For the left and right extreme symbols:

$$P_{e0} = P_{e3} = Q(A/2\sigma)$$

While for the symbols of the middle zone:

$$P_{e1} = P_{e2} = 2Q(A/2\sigma)$$

M-ary Probability of Error

$$P_e = \frac{1}{4} [2Q(A/2\sigma) + 2 \times 2Q(A/2\sigma)] = \frac{3}{2} Q(A/2\sigma)$$

Extending this result to a signal of M possible values, where $M = 2^n$, with decision thresholds located at $y = 0, \pm A, \pm 2A, \dots, \pm \left[\frac{M-2}{2} \right] A$:

$$\begin{aligned} P_e &= \frac{1}{M} [2Q(A/2\sigma) + 2 \times (M-2)Q(A/2\sigma)] \\ &= 2 \left(1 - \frac{1}{M} \right) Q(A/2\sigma) \end{aligned}$$

The expression matches that of the binary probability if $M = 2$.

M-ary Probability of Error

$$\left(\frac{A}{2\sigma}\right)^2 \leq \frac{6S_R \log_2(M)}{(M^2 - 1)\eta r_b} = \frac{6 \log_2(M)}{(M^2 - 1)} \frac{S_R}{\eta r_b} = \frac{6 \log_2(M)}{(M^2 - 1)} \gamma_b$$

The relation $\left(\frac{A}{2\sigma}\right)^2$ is thus expressed as a function of γ_b . Thus:

$$\lim_{M \rightarrow \infty} \frac{6 \log_2(M)}{(M^2 - 1)} = 0$$

In other words, the quotient $\left(\frac{A}{2\sigma}\right)^2$ degrades strongly as M increases.

When the adapted filter is used, the maximized value of $\left(\frac{A}{2\sigma}\right)^2$ is obtained.

M-ary Probability of Error

The derived expressions for the error probability P_e of an M-ary signaling can be related to the corresponding binary error probability P_{be} .

This calculation of the probability of error depends on the way in which the correspondence between the M-ary symbols and the binary information is established.