Task 3 Analysis

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December 5, 2024

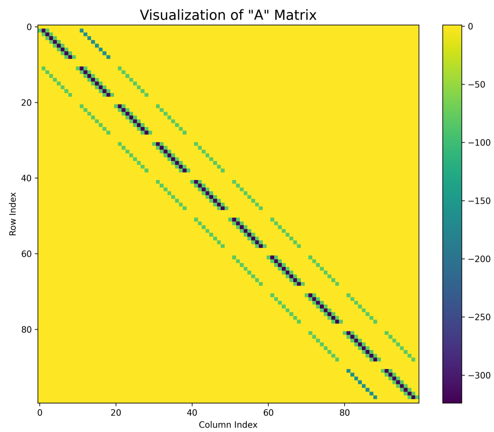

Figure 1: Sample Visualization of Diagonally Dominant Generated Matrix by Solver Generated by 'matrix.py' Using a Resolution of 10 ($10^2 \times 10^2$ matrix)

Parameter Name	*Value*
Resolution	128
Number of Iterations	10, 100, 1000, 10000, 50000
Values for the Dirichlet Boundary Condition on left and right side	0,0
Values for the Neumann Boundary Condition	0
(x,y) center coordinates for the Gaussian Source	(0.5,0.5)
Deviation of the Gaussian Source	0.1

Table 1: Parameters used for Solving the Poisson Equation ($-\Delta\vec{u} = \vec{b}, \vec{b} =$ source function) via Finite Discretization with the Jacobi Matrix Method

1 Discussion of Converged Solution Plot

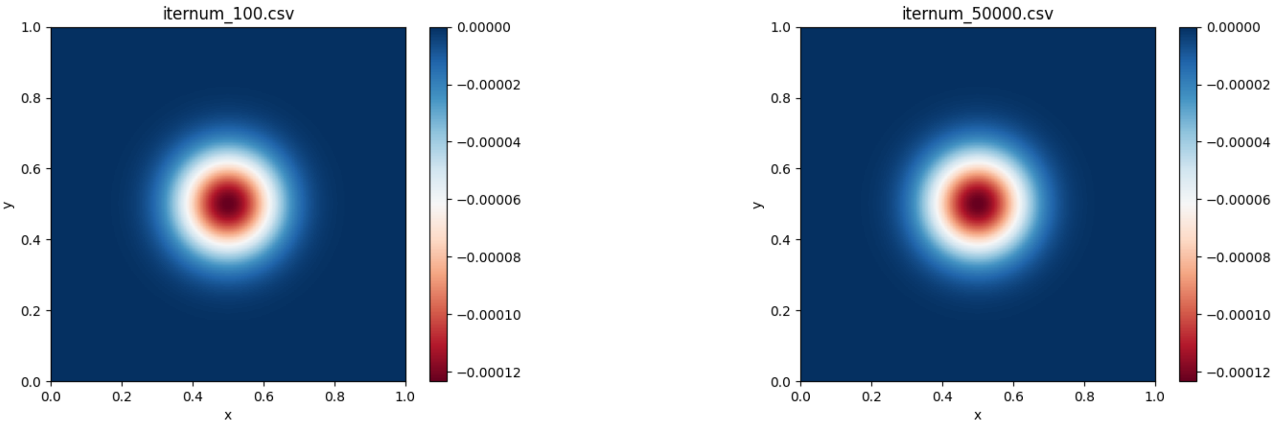

Figure 2: Converged Solution Heatmap after 100 and 50000 Iterations Generated by 'plotty.py'

A couple of key characteristics can be observed from Figure 2 that show signs of correct behavior. The circular behavior of the figure above can be immediately noticed and is to be expected because the form of the Poisson equation is reminiscent of the simple 2D formula of a circle: $x^2 + y^2 = r^2$. Other immediate physical characteristics is that the solution is centered at $(0.5, 0.5)$, which is the center that was set for the Gaussian source function used for this task. Examining the edges of the heatmap reveal that the boundary conditions have also been fulfilled. Along $x = 0, 1$, the values are the Dirichlet values assigned. Looking along the edges at $y = 0, 1$ show that the values at the point and around it are a constant value, which fulfills the Neumann boundary condition as well. The peak of solution vector also grows negative, and is to be expected as the negative of the laplacian operator was used in tandem with the positive values of the Gaussian source function, and therefore is expected to behave inversely with the said function.

2 Discussion of Residual Norm Plots

$$\text{residual} = |A\vec{u} - \vec{b}| \tag{1}$$

Figure 3: Formula for Calculation of Residuals

$$u_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{i \neq j} a_{ij} u_j^{(k)}) \tag{2}$$

Figure 4: Element Based Formula for Jacobi Matrix Method (https://en.wikipedia.org/wiki/Jacobi_method)

In order to examine the error in the converged solution by number of iterations, the 2-norm and infinite norm (inf-norm) produced by the residuals were examined. The residual values were obtained by obtained using the equation provided in Figure 3. In this case, the A matrix consisted of constants formed a 5-point stencil that approximated the laplacian operator acting on $\vec{u}$, which is the solution vector of interest. 2 norms were used on the residual vector to examine behavior of the overall error (2-norm) and its maximum possible error (inf-norm) of the finite discretization method over increasing N iterations (refer to Table 1 for specific parameters used). Because varying iteration number was examined for this task, convergence depends soley on the diagonalization by the Jacobi method. The robustness of the method can easily be seen from the small residual value from 10 iterations, which quickly becomes near zero with increasing iterations. Examining the plots for the residuals show good convergence from 100 to 1000 iterations for both overall and maximum error. While not immediately evident, the residual values beyond 100 iterations and beyond only differ by a negligible magnitude of 10^{-15}, which implies that iterations larger than 100 may be unnecessary for residual minimization. This is especially evident in Figure 2, when comparing the produced nearly identical heatmaps of the solution between 100 and 50000 iterations.

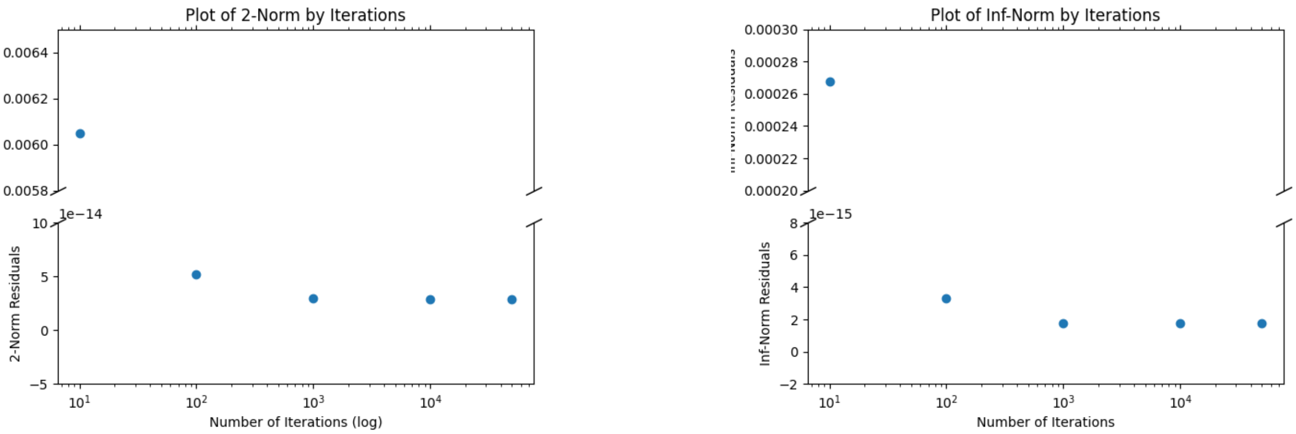

Figure 5: Plots of the Residual Norms by Number of Iterations (Left: 2-norm, Right: inf-norm) Generated by 'plotty.py'