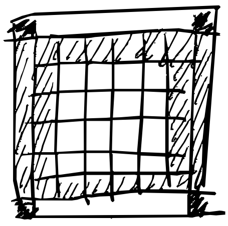
FINITE DISCRETIZATION FOR SOLVING PDEs

$-\nabla u = f$, B.C:

$u(0,y) = u(1,y) = 0 = g_D$

$\frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y}$ @ $y = 0, 1 = 0 = g_N$



$\leftarrow \frac{\partial u(x,1)}{\partial y} = 0$

$N-2$

$\leftarrow \frac{\partial u(x,0)}{\partial y} = 0$

$u(0,y) = 0 \quad u(1,y) = 0$

$Au = f$

Row major ordering:

for $x_i, y_j \rightarrow idx = i + j(N-2)$

$xpos = (i+1)h$ * Generate over entire grid (not just active pts)

$ypos = (j+1)h$

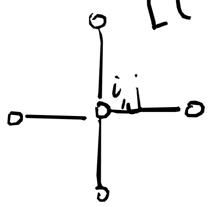
$$-\nabla^2 u = -(u_{xx} + u_{yy})$$

$$\approx -\left[\frac{u_x^{i+1,j} - u_x^{i,j}}{h} + u_{yy}\right]$$

$$= -\left[\left(\frac{u_{i+1,j} - u_{i,j}}{h} - \frac{u_{i,j} - u_{i-1,j}}{h}\right)\frac{1}{h} + u_{yy}\right]$$

$$= -\left[\left(\frac{u_{i+1,j} - 2u_{ij} + u_{i-1j}}{h^2}\right) + \left(\frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h^2}\right)\right]$$

$$= -\left[\underbrace{\left(\frac{1}{h^2}\right)}_{C_1} u_{i+1j} + \left(\frac{1}{h^2}\right)u_{i-1j} + \left(\frac{1}{h^2}\right)u_{ij+1} + \left(\frac{1}{h^2}\right)u_{ij-1} + \underbrace{\left(-\frac{2}{h^2} - \frac{2}{h^2}\right)}_{-4C_1} u_{ij}\right] = b_{ij}$$



$$= -\left[C_1 u_{i+1j} + C_1 u_{i-1j} + C_1 u_{ij+1} + C_1 u_{ij-1} - 4C_1 u_{ij}\right] = b_{ij}$$

Dirichlet Boundary:

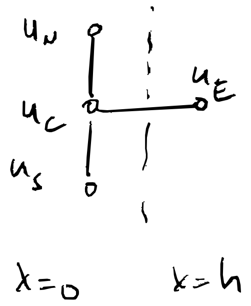
@ dx @ $i = 0, N-1$

$(@\ x=0 \rightarrow u_{i-1,j} = g_D)$

$$-\left[c_1 u_{i+1,j} + \underline{c_1 (g_D)} + c_1 u_{i,j+1} + c_1 u_{i,j-1} - 4c_1 u_{ij}\right]$$

$$= b_{ij} \Rightarrow -[\cdots] = b_{ij} + c_1 g_D$$

consider @ $x=0$, $i=0$:



What matters is that u_c @ $x=0$ is fixed to some value, so the stencil can be ignored for boundary values & instead focus on

$c_1 u_c = b_{ij}$ ← this is the form that the matrix provides

$u_{c,x=0} = g_D$ ← this is the condition that must be ensured

setting $c_1 = 0$ to enforce BC results in $\hat{A}$ being no longer diag. dom.

Instead set:

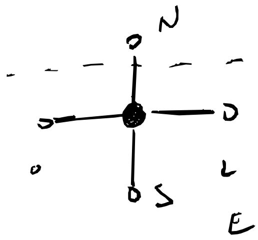
$c_1 = 1$, $b_{0j} = g_D \longrightarrow (1) u_c = g_D$

↑ adjust $\hat{A}$ ↑ adjust source $\vec{b}$ ↑ achieve boundary requirement

Neumann BC:

idx @ $j = 0, N-1$

need because part of interest is center pt



$$\text{CD deriv} = \frac{u_N - u_S}{2h} = g_N$$

$u_S = u_N - 2g_N h$ $u_N = u_S + 2g_N h$

BOT, RIGHT   TOP, LEFT

$$-c_1[u_N + u_S + u_E + u_W - 4u_c]$$

$$= -c_1[(u_S + 2g_N h) + u_S + u_E + u_W - 4u_c]$$

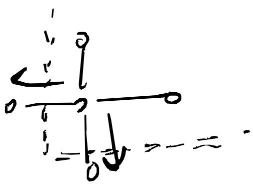
$$= f_j$$

$$-c_1[2u_S + u_E + u_W - 4u_c] = f + 2g_N h$$

$+c_1$ to adjust $\hat{A}$ $+2g_N h$ to adjust $\vec{b}$

for corner (not needed for assignment)

Neumann says:



$$\frac{\partial u}{\partial n} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = g$$

$$u_S = u_N + 2g_N h$$

$$-c_1[u_N + (u_N + 2g_N h) + 2u_E - 4u_c]$$
$$= f + 2g_N h$$

$$= -c_1[2u_N + 2u_E - 4u_c] = f + 4g_N h$$