Exercise 3: - a Report

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Introduction

The following is a report, discussing the results we obtained for the three tasks of exercise 1, as a part of the lecture 360.242 Numerical Simulation and Scientific Computing I.

1 Task 1: Benchmark Vector Triad

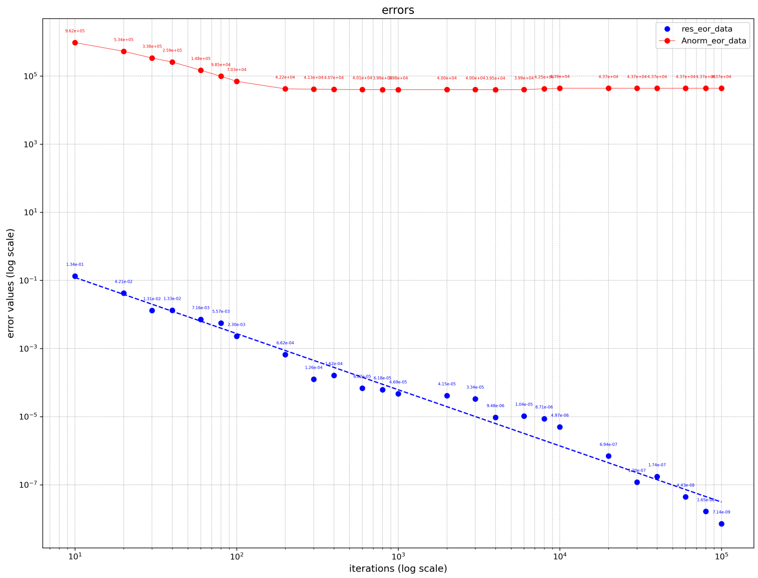

Figure 1: Plot $\frac{\|r_k\|_2}{\|r_0\|_2}$ and $\|e_k\|_A$ as a function of the iteration number for the matrix BCSSTK13 [1]

In the first task, we discuss our findings from two plotted errors as a function of iteration numbers of the conjugate gradients (CG) algorithm. The algorithm for iteratively solving $Ax = b$, where A should be symmetric positive definite. The input vector starts from an initial guess $x_0 = [0, ..., 0]$.

Algorithm 1 CG algorithm

$x_0 = 0$, $r_0 = b$, $p_0 = r_0$
for $k = 1, 2, ...$ **do**
  $\alpha_k = \frac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A p_{k-1}}$
  $x_k = x_{k-1} + \alpha_k p_{k-1}$
  $r_k = r_{k-1} - \alpha_k A p_{k-1}$
  $\beta_k = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}}$
  $p_k = r_k + \beta_k p_{k-1}$
end for
return x_k

Residual: $r_0 = b - A \cdot 0$ is the residual of the initial not converged result. r_k is the residual of the approximation of the k-th iteration. Then $\frac{\|r_k\|_2}{\|r_0\|_2}$ is a metric of the approximation accuracy over k iterations. As shown in figure 1, the CG method reduces the metric almost linearly in the log-log scale. We can conclude that the CG method finds appropriate descent directions and moves x_k closer to the true result $x^* = [1, ..., 1]$ as expected.
A-norm error: The error on A-norm is defined by:

$$\|e_k\|_A = \|x^* - x_k\|_A = \sqrt{e_k^T A e_k}$$

$e_k = x_k - x^*$ is the error between the approximation of the k-th iteration and the true result. $\sqrt{e_k^T A e_k}$ reflects the method's convergence in the Krylov subspace. The behavior of the convergence is shown in figure 1. The trend becomes stable when $k > 200$, which means the result converges since then. However, the error value is rather large because the metric incorporates elements from A.

References

[1] *BCSSTK13 matrix*. `https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/bcsstruc1/bcsstk13.html`.