EXERCISES WEEK 38

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Show that $\mathbb{E}[(y-\tilde{y})^2] = \text{Bias}[\tilde{y}] + \text{var}[\tilde{y}] + \sigma^2$

$$C(X,\beta) = \frac{1}{n}\sum_{i=0}^{n-1}(y_i-\tilde{y}_i)^2 = \mathbb{E}[(y-\tilde{y})^2]$$

Can be written out as:

$$\mathbb{E}[y^2-2y\tilde{y}+\tilde{y}^2] = \mathbb{E}[y^2] + 2\mathbb{E}[y\tilde{y}] + \mathbb{E}[\tilde{y}^2]$$

$\mathbb{E}[y^2]$ can be written as:

$$\mathbb{E}[y^2] = \mathbb{E}[(f+\varepsilon)^2] = \mathbb{E}[f^2+2f\varepsilon+\varepsilon^2] = \mathbb{E}[f^2] + 2\mathbb{E}[f\varepsilon] + \mathbb{E}[\varepsilon^2]$$

Here we assume that $\mathbb{E}[f]=f$, $\mathbb{E}[\varepsilon]=0$, $\mathbb{E}[\varepsilon^2]=\sigma^2$

$$\Rightarrow\ = f^2 + 2f\mathbb{E}[\varepsilon] + \sigma^2 = f^2+\sigma^2$$

$\mathbb{E}[y\tilde{y}]$ can be written as:

$$\mathbb{E}[y\tilde{y}] = \mathbb{E}[(f+\varepsilon)\tilde{y}] = \mathbb{E}[f\tilde{y}+\varepsilon\tilde{y}] = \mathbb{E}[f\tilde{y}] + \mathbb{E}[\varepsilon\tilde{y}] = f\mathbb{E}[\tilde{y}] + \mathbb{E}[\varepsilon]\mathbb{E}[\tilde{y}]$$
$$= f\mathbb{E}[\tilde{y}]$$

If we use that $\text{var}[x] = \mathbb{E}[x^2] - (\mathbb{E}[x])^2$, we can write $\mathbb{E}[\tilde{y}^2]$ as:

$$\mathbb{E}[\tilde{y}^2] = \text{var}[\tilde{y}] + (\mathbb{E}[\tilde{y}])^2$$

We put all the terms together:

$$\mathbb{E}[(y-\tilde{y})^2] = f^2+\sigma^2 - 2f\mathbb{E}[\tilde{y}] + \text{var}[\tilde{y}] + (\mathbb{E}[\tilde{y}])^2$$
$$= (f^2 - 2f\mathbb{E}[\tilde{y}] + (\mathbb{E}[\tilde{y}])^2) + \text{var}[\tilde{y}] + \sigma^2$$
$$= \mathbb{E}(y-\mathbb{E}[\tilde{y}])^2 + \text{var}[\tilde{y}] + \sigma^2$$
$$= \text{Bias}[\tilde{y}] + \text{var}[\tilde{y}] + \sigma^2 \quad \square$$