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Constructions for Regular Polytopes

P. McMullen

University College London, Mathematics Department, London WCIE 6BT, England

AND

E. Schulte*

Mathematics Department, MIT, Cambridge, Massachusetts 02139

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The paper discusses a general method for constructing regular incidence-polytopes $\mathcal{P}$ from certain operations on the generators of a group W, which is generated by involutions. When W is the group of a regular incidence-polytope $\mathcal{L}$, this amounts to constructing the regular skew polytopes (or skew polyhedra) $\mathcal{P}$ associated with $\mathcal{L}$.

1. Introduction

The study of regular polytopes and their generalizations has a long history (cf. Coxeter [6]). In recent years there has been a renewed interest in this subject leading to new insights into the geometrical and combinatorial structure of regular polytopes in the most general sense (McMullen [17], Grünbaum [14, 15], Dress [11, 12]).

A purely combinatorial generalization is given by the concept of regular incidence-polytopes. This concept is patterned after the classical theory of regular polytopes and provides a suitable framework for the study of combinatorially regular structures. For an introduction see Danzer and Schulte [10], but note that closely related notions also occur in McMullen [17, 18], Grünbaum [15], Buekenhout [1], Tits [28, 29], and Dress [13].

In this paper we shall describe a general method for constructing regular incidence-polytopes $\mathcal{P}$ from groups W, which are generated by involutions $\sigma_0, ..., \sigma_{m-1}$. Given such a group W, we derive a new group A from W by taking as generators $\rho_0, ..., \rho_{d-1}$ for A certain suitably chosen products of

* Present address: Mathematics Department, Northeastern University, Boston, MA 02115.

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the generators σ_i of W. This process is called a mixing operation μ, or simply an operation, on (the generators σ_i of) W; note that mixing operations are not (twisting) operations in the sense of [20, 21]. Then in suitable cases the resulting group A will be the group of a regular d-incidence-polytope $\mathscr{P}$.

Particularly interesting is the case where W is the group of a regular m-incidence-polytope $\mathscr{L}$. Then, in many situations $\mathscr{P}$ can be regarded as some kind of d-dimensional regular skew polytope "sitting" inside the m-dimensional $\mathscr{L}$. This relationship can be made more precise by suitably employing a combinatorial analogue of Wythoff's construction (cf. [6]). In particular, if $\mathscr{L}$ is a classical 4-dimensional regular polytope or star-polytope, then a suitable choice of μ gives us Coxeter's regular skew star-polyhedra in euclidean 4-space $\mathbb{E}^4$ (cf. [4, p. 102]). This illustrates that mixing operations provide an approach to a combinatorial theory of regular incidence-polytopes and their associated skew polytopes.

2. Definitions

Following [10], an *incidence-polytope $\mathscr{P}$ of dimension d* (or briefly, a *d-incidence-polytope*) is a partially ordered set, with a strictly monotone rank function $\dim(\cdot)$, such that the properties (I1) to (I4) below hold. The elements of rank i are called the *i-faces* of $\mathscr{P}$:

(I1) There is a unique least face F_{-1} with $\dim(F_{-1}) = -1$ and a unique greatest face F_d with $\dim(F_d) = d$. These are the *improper faces* of $\mathscr{P}$.

(I2) The maximal totally ordered subsets or *flags* of $\mathscr{P}$ contain exactly $d+2$ faces (including F_{-1} and F_d), that is, a face of each dimension.

(I3) $\mathscr{P}$ is *strongly flag-connected* in the following sense: for any two flags Φ and Ψ, there is a sequence $\Phi = \Phi_0, \Phi_1, ..., \Phi_{n-1}, \Phi_n = \Psi$ of *adjacent* flags, meaning that $\mathrm{card}(\Phi_j \backslash \Phi_{j-1}) = 1$, such that $\Phi \cap \Psi$ is contained in Φ_j for all j.

(I4) For any $(i-1)$-face F and any $(i+1)$-face G with $F < G$ there are exactly two i-faces H of $\mathscr{P}$ with $F < H < G$.

If F and G are faces of $\mathscr{P}$ with $F < G$, we call the incidence-polytope $G/F := \{H \mid F \leqslant H \leqslant G\}$ (of dimension $\dim(G) - \dim(F) - 1$) a *section* of $\mathscr{P}$. There is little possibility of confusion if we identify a face F with the section F/F_{-1} (and thus F_d with $\mathscr{P}$). The faces of dimension 0, 1, and $d-1$ are also called the *vertices*, *edges*, and *facets*, respectively. If F is a face, then F_d/F is said to be *co-face* of F, or the *vertex-figure* of F if F is a vertex.

The *dual* $\mathscr{P}^*$ of $\mathscr{P}$ is obtained from $\mathscr{P}$ by replacing " $<$ " by " $>$ " while leaving the set of faces unchanged (see also Section 4). $\mathscr{P}$ is called *self-dual* if $\mathscr{P}^*$ is isomorphic to $\mathscr{P}$.

The *automorphism group* or simply *group* $A(\mathscr{P})$ of $\mathscr{P}$ consists of the order preserving permutations of the faces or *automorphisms* of $\mathscr{P}$. The *extended group* $D(\mathscr{P})$ of a self-dual $\mathscr{P}$ is the group of all automorphisms and *dualities* (order reversing permutations of the faces) of $\mathscr{P}$; it contains $A(\mathscr{P})$ as a subgroup of index 2.

We say that $\mathscr{P}$ is *regular* if $A(\mathscr{P})$ is transitive on its flags. (Note that (I3) and (I4) imply that $A(\mathscr{P})$ is actually simply transitive on the flags of $\mathscr{P}$.) For a regular $\mathscr{P}$, all its sections are also regular and any two comparable sections are isomorphic.

Let $\mathscr{P}$ be regular, $\Phi := \{F_{-1}, F_0, ..., F_d\}$ a fixed *base flag* of $\mathscr{P}$ (occasionally we omit F_{-1} and F_d) and ρ_i the unique automorphism which keeps all but the i-face F_i of Φ fixed $(i=0, ..., d-1)$. Then $\rho_0, ..., \rho_{d-1}$ generate $A(\mathscr{P})$ and satisfy relations

$$(\rho_i\rho_j)^{p_{ij}} = 1 \qquad (i, j = 0, ..., d-1), \tag{1}$$

where $p_{ii} = 1$, $p_{ji} = p_{ij} =: p_{i+1}$ if $j = i+1$, and $p_{ij} = 2$ otherwise; here, p_{i+1} is the number of i-faces of $\mathscr{P}$ in the section F_{i+2}/F_{i-1}. Thus (1) is determined by the *type* $\{p_1, ..., p_{d-1}\}$ of $\mathscr{P}$.

The group of $\mathscr{P}$ has the *intersection property with respect to its generators* ρ_i; that is,

$$\langle \rho_i \mid i \in I\rangle \cap \langle \rho_i \mid i \in J\rangle = \langle \rho_i \mid i \in I \cap J\rangle \qquad \text{for} \quad I, J \subset \{0, ..., d-1\}, \tag{2}$$

where we define $\langle \rho_i \mid i \in \emptyset\rangle := \{1\}$. If $i<j$, the subgroup $\langle \rho_k \mid k = i+1, ..., j-1\rangle$ is the group of the section F_j/F_i. Furthermore, within $\mathscr{P}$ incidence of faces is characterized by

$$\varphi(F_i) \leqslant \psi(F_j) \Leftrightarrow (-1 \leqslant i \leqslant j \leqslant d \quad \text{and} \quad \varphi\langle \rho_k \mid k \neq i\rangle \cap \psi\langle \rho_k \mid k \neq j\rangle \neq \emptyset); \tag{3}$$

here, by the commutation rules for the ρ_k, the condition on the right-hand side is equivalent to "$\psi^{-1}\varphi \in \langle \rho_k \mid k \leqslant j-1\rangle \cdot \langle \rho_k \mid k \geqslant i+1\rangle$." Note that, by (3) with $i=j$, we can identify the i-faces of $\mathscr{P}$ with the left cosets of $\langle \rho_k \mid k \neq i\rangle$ in $A(\mathscr{P})$.

By a *C*-group we mean a group A generated by involutions $\rho_0, ..., \rho_{d-1}$ such that (1) and (2) are satisfied for the ρ_k. (Here, "C" stands for "Coxeter" but clearly C-groups will not be Coxeter groups in general.) In this paper many considerations are based on the fact that the C-groups are in one-to-one correspondence with the groups of regular incidence-polytopes. More precisely, if A is a C-group with generating involutions $\rho_0, ..., \rho_{d-1}$, then A is the group of a regular d-incidence-polytope $\mathscr{P}$ of type $\{p_1, ..., p_{d-1}\}$ and $\rho_0, ..., \rho_{d-1}$ are the distinguished generators defined with respect to a base flag of $\mathscr{P}$, and vice versa (cf. [25]).

For example, if $p_1, ..., p_{d-1} \geqslant 2$ and A is the Coxeter group with the linear diagram $\bullet\underset{p_1}{\text{—}}\bullet\underset{p_2}{\text{—}}\bullet\cdots\bullet\underset{p_{d-1}}{\text{—}}\bullet$ (cf. [6]), that is the group abstractly defined by the relations (1), then A is a C-group and is the group of the *universal incidence-polytope of type* $\{p_1, ..., p_{d-1}\}$; this will simply be denoted by $\{p_1, ..., p_{d-1}\}$, and its group by $[p_1, ..., p_{d-1}]$ (cf. Coxeter [6], Tits [28], and [25]). Here "universal" means that any regular $\mathscr{P}$ of the same type can be derived from $\{p_1, ..., p_{d-1}\}$ by suitable identifications.

Given regular d-incidence-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ such that the vertex-figures of $\mathscr{P}_1$ are isomorphic to the facets of $\mathscr{P}_2$, we denote by $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ the class of all regular $(d+1)$-incidence-polytopes $\mathscr{P}$ with facets isomorphic to $\mathscr{P}_1$ and vertex-figures isomorphic to $\mathscr{P}_2$. If $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle \neq \varnothing$, then any such $\mathscr{P}$ is obtained from the universal member of $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ denoted by $\{\mathscr{P}_1, \mathscr{P}_2\}$ (cf. [26]).

Next we recall the notions of Petrie-polygons and j-chains (cf. [4, 6]).

Following the inductive definition of *Petrie-polygons* in [6], we declare that the Petrie polygon of a 2-incidence-polytope is that incidence-polytope itself. Then, for $d \geqslant 3$, a Petrie polygon of a regular d-incidence-polytope $\mathscr{P}$ is a path within the *1-skeleton* $\mathrm{skel}_1(\mathscr{P})$ of $\mathscr{P}$ (the graph consisting of the vertices and edges of $\mathscr{P}$) such that any $d-1$ consecutive edges, but no d, belong to a Petrie polygon of a facet of $\mathscr{P}$. In most of our applications, $\mathscr{P}$ will be a regular map of type $\{p, q\}$ on a surface, that is, $d=3$ (cf. [8]). In this case, if $\mathscr{P}$ has Petrie polygons of length h, then we have

$$(\rho_0 \rho_1 \rho_2)^h = 1, \tag{4}$$

and $\mathscr{P}$ is said to be of *type* $\{p, q\}_h$. If (1) (with $d=3$) and (4) suffice for a presentation of $A(\mathscr{P})$, then $\mathscr{P}$ is denoted by $\{p, q\}_h$ (cf. [8]).

Let $\mathscr{P}$ be a regular map of type $\{p, q\}$ on a surface S, possibly an infinite map on a non-compact surface. For $j = 1, ..., [q/2]$, a *j-chain* of $\mathscr{P}$ is a path along edges of $\mathscr{P}$ which leaves, at each vertex, exactly j faces on the right (and $p-j$ on the left) according to a local orientation carried on the path. When $j=1$ this is a 2-face of $\mathscr{P}$, when $j=2$ it is the "first hole" (cf. [4]). If r_j is the length of the j-chains of $\mathscr{P}$, then we have the additional relations

$$(\rho_0 \rho_1 (\rho_2 \rho_1)^{j-1})^{r_j} = 1 \qquad \text{for} \quad j = 1, ..., [q/2]; \tag{5}$$

again, if (1) (with $d=3$) and (5) suffice for a presentation of $A(\mathscr{P})$, then $\mathscr{P}$ is denoted by $\{p, q \mid r_2, ..., r_{[q/2]}\}$. An important example is Coxeter's map

$$M_q := \{4, q \mid 4^{[q/2]-1}\} \qquad (q \geqslant 3),$$

where the upper index gives the number of 4's (cf. [4, p. 100; 3]); its group is $C_2 \wr D_q$, the wreath-product of C_2 and the dihedral group D_q (with D_q acting as usual), of order $2^{q+1} \cdot q$.

If only the first l (say) numbers r_j are of interest, we simply write $\mathscr{P} = \{p, q \mid r_2, ..., r_l\}$ if (1) and the corresponding relations (5) with $j \leqslant l$ suffice for a presentation of $A(\mathscr{P})$. Examples with $l = 2$ are Coxeter's finite skew polyhedra $\{4, 6 \mid 3\}$, $\{6, 4 \mid 3\}$, $\{4, 8 \mid 3\}$ and $\{8, 4 \mid 3\}$ in euclidean 4-space (cf. [4]). More generally, we leave a space r_j blank if it is not needed; examples are the maps $\{p, q \mid \cdot, r\}$.

We conclude this section with some remarks on Wythoff's construction (cf. [6]).

Given a regular d-incidence-polytope $\mathscr{P}$, the *barycentric subdivision* $\mathscr{C}(\mathscr{P})$ of $\mathscr{P}$ is the simplicial complex whose faces are the totally ordered subsets of $\mathscr{P}$ not containing the improper faces F_{-1} and F_d. Call $\mathscr{P}$ (faithfully) *realizable* if, for $j = 1, ..., d-1$, all the j-faces of $\mathscr{P}$ are uniquely determined by their $(j-1)$-faces (cf. [19]). If $\mathscr{P}$ is a lattice, then $\mathscr{P}$ is realizable.

Any realizable $\mathscr{P}$ can be recovered from its barycentric subdivision $\mathscr{C}(\mathscr{P})$ by the following process, the geometrical analogue of which is often called *Wythoff's construction.* Pick the vertex F_0 in the base flag of $\mathscr{P}$ (which remains a vertex for $\mathscr{C}(\mathscr{P})$), set $G_0 := F_0$ and define recursively

$$G_j = \{\varphi(G_{j-1}) \mid \varphi \in \langle \rho_0, ..., \rho_{j-1} \rangle\}$$

for $j = 1, ..., d$. This gives the base flag $\Psi := \{G_0, ..., G_d\}$ (with the (-1)-face omitted) of a d-incidence-polytope $\mathscr{P}'$, whose remaining flags are the transforms of Ψ by $A(\mathscr{P})$. Clearly, since $\mathscr{P}$ is realizable, $\mathscr{P}'$ is isomorphic to $\mathscr{P}$.

3. Mixing Operations

In this work the incidence-polytopes are constructed by so-called mixing operations, or simply operations, on groups W generated by involutions $\sigma_0, ..., \sigma_{m-1}$. In many applications W will be the group of some regular m-incidence-polytope and $\sigma_0, ..., \sigma_{m-1}$ the distinguished system of generators.

Given such a group W, we derive a new group A from W by taking as generators $\rho_0, ..., \rho_{d-1}$ for A certain suitably chosen products of the generators σ_i of W; hence A is a subgroup of W. This process is called an *operation* on (the generators of) W and is denoted by

$$\mu\colon (\sigma_0, ..., \sigma_{m-1}) \mapsto (\rho_0, ..., \rho_{d-1}).$$

For example, if W is the group of a regular m-incidence-polytope $\mathscr{L}$, then the operation

$$\mu\colon (\sigma_0, ..., \sigma_{m-1}) \mapsto (\sigma_0\sigma_2\sigma_4 \cdot ..., \sigma_1\sigma_3\sigma_5 \cdot ...,) =: (\rho_0, \rho_1)$$

of taking the two products of alternate generators σ_i yields a dihedral group which is the group of a Petrie polygon of $\mathscr{L}$. In the general situation, operations on W can be chosen in several ways which lead to quite different groups and generators.

We assume (as we may) that among the defining relations for W in terms of the σ_i there are relations of the form

$$(\sigma_i\sigma_j)^{q_{ij}} = 1 \qquad (i, j = 0, ..., m-1), \tag{6}$$

with $q_{ii} = 1$ and $q_{ji} = q_{ij}$ for all i and j, but not necessarily with $q_{ij} = 2$ if $|i-j| \geqslant 2$; here, q_{ij} is supposed to be the exact order of $\sigma_i\sigma_j$. This way W is exhibited as a quotient of the Coxeter group $\hat{W}$ abstractly defined by the relations (6). Now, if the same operation is applied to W and $\hat{W}$ to give new groups $A = \langle \rho_0, ..., \rho_{d-1} \rangle$ and $\hat{A} = \langle \hat{\rho}_0, ..., \hat{\rho}_{d-1} \rangle$, respectively, then $\hat{\rho}_i \mapsto \rho_i$ $(i = 0, ..., d-1)$ defines a homomorphism of $\hat{A}$ onto A. Hence, if A and $\hat{A}$ are the groups of regular d-incidence-polytopes $\mathscr{P}$ and $\hat{\mathscr{P}}$, respectively, then $\mathscr{P}$ is obtained from $\hat{\mathscr{P}}$ by identifying faces. More generally, these considerations extend to situations where W is a homomorphic image of some other group to which the same operation μ applies.

As to the admissible choice of generators ρ_i for A, there is the simple restriction that $(\rho_i\rho_j)^2 = 1$ if $|i-j| \geqslant 2$. This cuts down many possibilities, but still is not sufficient for A to be the group of an incidence-polytope, unless A also satisfies the intersection property (2), that is, A is a C-group. In the sequel, when an operation μ is described, we do not assert in any case that the resulting group A has the intersection property. In fact, this property has to be checked in each case separately.

Occasionally we shall apply our operations to groups W of regular incidence polytopes $\mathscr{L}$. Then we shall see that in some cases we may think of the new incidence polytope $\mathscr{P}$ as some kind of skew polyhedron associated with $\mathscr{L}$.

Many of our operations μ on W can be "extended" to larger groups W', with the σ_i among their generators σ'_i. For example, if $W' = \langle \sigma'_0, ..., \sigma'_{k-1} \rangle$ is the group of a regular incidence-polytope $\mathscr{L}'$ of dimension $k \geqslant 4$, then the Petrie-operation (see Section 4.1.)

$$\mu\colon (\sigma_{k-3}, \sigma_{k-2}, \sigma_{k-1}) \mapsto (\sigma_{k-3} \cdot \sigma_{k-1}, \sigma_{k-2}, \sigma_{k-1}) =: (\rho_{k-3}, \rho_{k-2}, \rho_{k-1})$$

applied to the subgroup $W = \langle \sigma_{k-3}, \sigma_{k-2}, \sigma_{k-1} \rangle$ of W' extends trivially to an operation μ' on W' by setting

$$\mu'\colon (\sigma'_0, ..., \sigma'_{k-1}) \mapsto (\sigma'_0, ..., \sigma'_{k-4}, \sigma'_{k-3} \cdot \sigma'_{k-1}, \sigma'_{k-2}, \sigma'_{k-1}) =: (\rho_0, ..., \rho_{k-1}).$$

This means that in describing an operation it suffices to consider only the "essential" part of it. On the other hand, in some instances, such extensions can have interesting geometric interpretations. An example for this is our facetting operation of Section 4.2.

We remark that the twisting operations discussed in [20, 21] are not (mixing) operations in the above sense. Twisting operations also involve outer automorphisms of W permiting the σ_i, while mixing operations are defined by products of the σ_i only.

4. Operations on Regular Maps

In this section we discuss some (mixing) operations on regular maps, or more exactly, on the groups of regular maps. Note that in [16] the term "operation on maps" was used in a quite different sense.

Throughout, let $\mathscr{L}$ be a (finite or infinite) regular map of type $\{p, q\}$ on a surface and $W := A(\mathscr{L}) = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ its group, with $\sigma_0, \sigma_1, \sigma_2$ the distinguished generators. We shall derive groups $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ of regular maps $\mathscr{P}$ by operations μ on W; then we write $\mathscr{P} = \mathscr{L}^\mu$. A trivial example is the duality operation

$$\delta\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_2, \sigma_1, \sigma_0) =: (\rho_0, \rho_1, \rho_2)$$

leading to the dual $\mathscr{L}^\delta = \mathscr{L}^*$ of $\mathscr{L}$.

4.1. *The Petrie Operation* π

Another well-known example is the Petrie operation defined by

$$\pi\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_2, \sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

Here, the group $A := \langle \rho_0, \rho_1, \rho_2 \rangle$ is the group of a regular map $\mathscr{L}^\pi$ which is sometimes called the *Petrie dual* of $\mathscr{L}$. The reason is that π has as its combinatorial counterpart the process of replacing all the 2-faces of $\mathscr{L}$ by new 2-faces given by the various Petrie polygons of $\mathscr{L}$ (cf. [8, 14]). In particular, if $\mathscr{L}$ is of type $\{p, q\}_h$, then $\mathscr{L}^\pi$ is of type $\{h, q\}_p$. Note that π is involutory, that is, $\pi^{-1} = \pi$ and $(\mathscr{L}^\pi)^\pi = \mathscr{L}$. If $\mathscr{L}^\pi$ is isomorphic to $\mathscr{L}$, then $\mathscr{L}$ is called *self-Petrie*. The hemi-dodecahedron $\{5, 3\}_5 = \{5, 3\}/2$ is an example of a self-Petrie map (cf. [8]). Further examples of self-Petrie maps are the maps $\mathscr{L}^\sigma$ discussed in Section 4.4.

4.2. *The Facetting Operation* φ_k

For $k = 1, ..., [q/2]$ we define the facetting operation φ_k by

$$\varphi_k\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0, \sigma_1(\sigma_2\sigma_1)^{k-1}, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

Then, $\sigma_1\sigma_2 = (\sigma_1\sigma_2)^k$. Hence, if $(k, q) > 1$, then in general the new group

$A := \langle \rho_0, \rho_1, \rho_2 \rangle$ is a proper subgroup of the old, but in the most interesting case where $(k, q) = 1$, the two groups coincide. In the former case, in general A will not be the group of a regular map, as can be seen for $\mathscr{L} = \{3, 6\}_{2,0}$ and $k = 2$.

If $(k, q) = 1$, the (restricted) operation $(\sigma_1, \sigma_2) \mapsto (\rho_1, \rho_2)$ corresponds to passing from a regular q-gon $\{q\}$ to a regular star-q-gon $\{q/k\}$. This means that φ_k corresponds to substituting for the given vertex-figure of $\mathscr{L}$ another with the same vertices and group; to see that the vertices are preserved, note that the neighbouring vertices in $\mathscr{L}$ of the base vertex F_0 in $\mathscr{L}$ are the transforms of $\sigma_0(F_0) = \rho_0(F_0)$ by the elements of $\langle \sigma_1, \sigma_2 \rangle = \langle \rho_1, \rho_2 \rangle$ and thus remain neighbours of F_0 in $\mathscr{P} := \mathscr{L}^{\varphi_k}$. Here, it is worth remarking that instead of checking the intersection proper (2) for A it is easier to employ Wythoff's construction for $\mathscr{P}$ with initial vertex F_0 (on the surface). Then the new map $\mathscr{P}$ is of type $\{p', q\}$ for some p', but is in general on another surface. The 1-skeleton of $\mathscr{P}$ is the same as that of $\mathscr{L}$. Note that any k with $k \leqslant [q/2]$ and $(k, p) = 1$ can actually occur as the "density" of the $\{q/k\}$.

From

$$\rho_0 \rho_1 = \sigma_0 \sigma_1 (\sigma_2 \sigma_1)^{k-1}$$

we see that the number p' of sides in a 2-face of $\mathscr{P}$ equals the length of the k-chains of $\mathscr{L}$. More generally we have

$$\begin{aligned} \rho_0 \rho_1 (\rho_2 \rho_1)^{m-1} &= \sigma_0 (\sigma_1 \sigma_2)^{k-1} \, \sigma_1 (\sigma_2 \sigma_1 (\sigma_2 \sigma_1)^{k-1})^{m-1} \\ &= \sigma_0 (\sigma_1 \sigma_2)^{k-1} \, \sigma_1 (\sigma_2 \sigma_1)^{k(m-1)} \\ &= \sigma_0 \sigma_1 (\sigma_2 \sigma_1)^{mk-1} \end{aligned}$$

which relates the m-chains of $\mathscr{P}$ to the mk-chains of $\mathscr{L}$, with $\pm mk$ considered modulo q.

For example, if $\mathscr{L} = \{3, 5\} = \{3, 5 \mid 5\}$ is the icosahedron and $k = 2$, then $\mathscr{P}$ is the Kepler–Poinsot polyhedron $\{5, \frac{5}{2}\}$ which, as a map, is isomorphic to $\{5, 5 \mid 3\}$ (cf. [4]). See also Coxeter [6, p. 98] for a discussion of metrical facetting of the Platonic solids. If $\mathscr{L}$ is Coxeter's map M_q and $1 \leqslant k \leqslant [q/2]$ with $(k, q) = 1$, then $\mathscr{P}$ is an isomorphic copy of M_q.

In the general situation, if $1 \leqslant k \leqslant [q/2]$ and $(k, q) = 1$, then the operation φ_k can be reversed and $\varphi_k^{-1} = \varphi_{k'}$, where k' is uniquely defined by $1 \leqslant k' \leqslant [q/2]$ and $kk' \equiv \pm 1 \pmod{q}$. Observe also that for any k the operations π and φ_k commute.

We remark that the reciprocal operations $\delta \varphi_k \delta$ (with δ the duality operation) can be thought of as stellating operations, in agreement with the discussion in [6].

4.3. *The Halving Operation* η

This operation applies only to regular maps $\mathscr{L}$ of type $\{4, q\}$, $q \geqslant 3$, turning them into regular maps $\mathscr{P} := \mathscr{L}^{\eta}$ of type $\{q, q\}$. We define η by

$$\eta : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0 \sigma_1 \sigma_0, \sigma_2, \sigma_1) =: (\rho_0, \rho_1, \rho_2).$$

The intersection property is easily checked for $A := \langle \rho_0, \rho_1, \rho_2 \rangle$. The effect of η can be studied best by employing Wythoff's construction.

Consider the barycentric subdivision $\mathscr{C}(\mathscr{L})$ of $\mathscr{L}$, realized topologically on the underlying surface S (see Fig. 1). Let T be the triangular fundamental region for $A(\mathscr{L})$ in $\mathscr{C}(\mathscr{L})$ whose sides are the "reflecting lines" for the generators σ_0, σ_1, and σ_2; the vertices of T are the "centres" of the faces in the base flag $\{F_0, F_1, F_2\}$ of $\mathscr{P}$, say $\hat{F}_0 = F_0, \hat{F}_1$, and $\hat{F}_2$, respectively. Further let T' be the union of T and its reflexion image $\sigma_0(T)$.

Now, as $\mathscr{L}$ has quadrangular 2-faces, T' has a "right" angle at $\hat{F}_2$. Then, A can be regarded as the subgroup of $A(\mathscr{L})$ generated by the "reflexions" in the sides of T'. When we employ Wythoff's construction for $\mathscr{P}$ with initial vertex $\hat{F}_0$, we find that two cases are possible.

First, let the 1-skeleton of $\mathscr{L}$ be bipartite; that is, all edge-circuits of $\mathscr{L}$ have even length. Then $\mathscr{P}$ is a map on the same surface S with half as many vertices as $\mathscr{P}$, namely the vertices in the class of $\hat{F}_0$. Its group A has index 2 in $A(\mathscr{L})$, with T' as its fundamental region. Also, $\mathscr{P}$ is self-dual, since the conjugation by σ_0 is an automorphism of $A = A(\mathscr{P})$ which interchanges ρ_0 and ρ_2 but keeps ρ_1 fixed; this is an outer automorphism, since σ_0 interchanges the two kinds of vertices of $\mathscr{L}$ and thus $\sigma_0 \notin A$. Another way to see

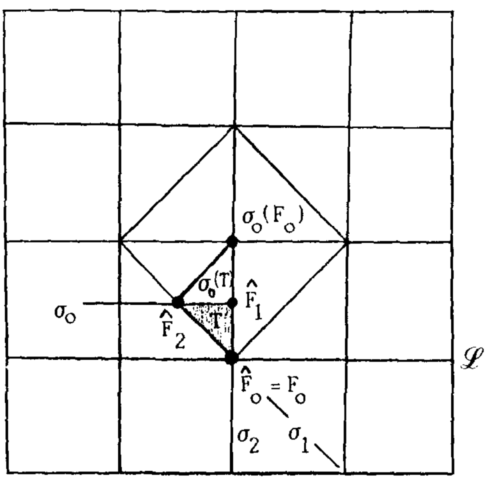


FIGURE 1

the self-duality is to derive a copy of the dual $\mathscr{P}^*$ of $\mathscr{P}$ by Wythoff's construction with initial vertex $\sigma_0(F_0)$; this is clearly isomorphic to $\mathscr{P}$ and is the reflexion image $\sigma_0(\mathscr{P})$ of $\mathscr{P}$. Observe also that the old group $A(\mathscr{L})$ and its generators $\sigma_0, \sigma_1, \sigma_2$ can be restored by a twisting operation from the new group A together with its generators ρ_0, ρ_1, ρ_2 and its outer automorphism σ_0 (cf. [20]).

Second, assume that the 1-skeleton of $\mathscr{L}$ is not bipartite. Unless $q=4$, $\mathscr{P}$ is a map on a different surface from $\mathscr{L}$. In any case, $A=A(\mathscr{P})=A(\mathscr{L})$ and $\mathscr{P}$ has the same number of vertices as $\mathscr{L}$. Also, $\mathscr{P}$ is self-dual, though this time the conjugation by σ_0 is an inner automorphism.

For both possibilities we have

$$\rho_0\rho_1\rho_2\rho_1=\sigma_0\sigma_1\sigma_0\sigma_2\sigma_1\sigma_2=(\sigma_0\sigma_1\sigma_2)^2.$$

This shows that, if $\mathscr{L}$ has Petrie polygons of length h, then the 2-chains of $\mathscr{P}$ have length $\frac{1}{2}h$ or h as h is even or odd, respectively.

By these considerations, if $\mathscr{L}=\{4,4\}_{c,c}=\{4,4\}_{2c}$ and thus $h=2c$, then $\mathscr{P}=\{4,4\,|\,c\}=\{4,4\}_{c,0}$ (cf. [8]). If $\mathscr{L}=\{4,4\}_{2b,0}$, then $\mathscr{P}=\{4,4\}_{b,b}$; if b is odd and $\mathscr{L}=\{4.\,4\}_{b,0}$, then $\mathscr{P}=\mathscr{L}$. For Gordan's map $\mathscr{L}=\{4,5\}_6$ of genus 4 we have $\mathscr{P}=\{5,5\,|\,3\}$ $(\simeq\{5,\frac{5}{2}\})$; see [8].

Note that the halving operation η could be generalized to operations $\eta_{k,l}$ on regular maps of type $\{2n,q\}$, with $n=k+l$, by defining

$$\eta_{k,l}\colon(\sigma_0,\sigma_1,\sigma_2)\mapsto((\sigma_0\sigma_1)^k\,\sigma_0,\sigma_2,(\sigma_1\sigma_0)^{l-1}\,\sigma_1)=:(\rho_0,\rho_1,\rho_2).$$

Then, $\eta_{1,1}=\eta$ if $n=2$. In any case, $(\rho_0\rho_2)^2=1$, but it seems that unless k, l, and q are chosen properly the structure of the new map is fairly complicated (if it exists at all).

Concluding this section let us remark on an interesting connexion between the halving operation η and a particular twisting operation discussed in [21].

Let $\mathscr{K}$ be a regular 4-incidence-polytope of type $\{4,q,4\}$, $q\geqslant 3$, with group $A(\mathscr{K})=\langle\tau_0,\ldots,\tau_3\rangle$. If η is applied to both the facets and the vertex-figures of $\mathscr{K}$, the operation

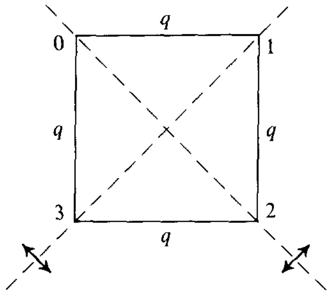
$$\hat{\eta}\colon(\tau_0,\ldots,\tau_3)\mapsto(\tau_0\tau_1\tau_0,\tau_2,\tau_1,\tau_3\tau_2\tau_3)=:(\varphi_0,\ldots,\varphi_3)$$

gives us a new group $U:=\langle\varphi_0,\ldots,\varphi_3\rangle$. The involutions φ_i satisfy the relations

$$(\varphi_0\varphi_2)^2=(\varphi_1\varphi_3)^2=(\varphi_0\varphi_1)^q=(\varphi_1\varphi_2)^q=(\varphi_2\varphi_3)^q=(\varphi_0\varphi_3)^q=1$$

but in general other independent relations too. Also, the conjugation by the elements τ_0 and τ_3 of $A(\mathscr{K})$ is an (inner or outer) automorphism of A

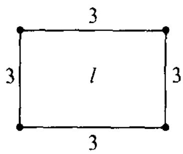
permuting the φ_i. Hence U is a quotient of the infinite Coxeter group with the symmetrical diagram



(7)

(cf. [7]), and the factorization is such that it respects the two indicated reflexions of the diagram; we shall simply say that U belongs to (7).

Now, the discussion of [21] covers, in a sense, the inverse process. Given a group U which belongs to (7), then in suitable cases a regular 4-incidence-polytope $\mathscr{K}$ of type $\{4, q, 4\}$ can be constructed from U by adjoining to U the two outer automorphisms corresponding to the two indicated reflexions of (7). In [21] this was used to construct from the unitary reflexion group $U = [1\ 1\ 2^l]^3$ with diagram



(cf. [5, 27]) an infinite series of regular 4-incidence-polytopes $\mathscr{K}$ of type $\{4, 3, 4\}$, whose group is a semidirect product of U by $C_2 \times C_2$, of order $96l^3$.

4.4. *The Skewing Operation* σ

Skewing, or skew halving, as it perhaps should be called, is remotely related to halving. It applies only to regular maps $\mathscr{L}$ of type $\{p, 4\}$ and is defined by

$$\sigma : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_0\sigma_2, (\sigma_1\sigma_2)^2) =: (\rho_0, \rho_1, \rho_2).$$

The relationship with halving is given by

$$\sigma = \pi\delta\eta\pi\delta;$$

in fact, because of $(\sigma_1\sigma_2)^4 = 1$ we have

$$\begin{aligned}(\sigma_0, \sigma_1, \sigma_2) &\underset{\pi}{\longmapsto} (\sigma_0\sigma_2, \sigma_1, \sigma_2) \underset{\delta}{\longmapsto} (\sigma_2, \sigma_1, \sigma_0\sigma_2)\\ &\underset{\eta}{\longmapsto} (\sigma_2\sigma_1\sigma_2, \sigma_0\sigma_2, \sigma_1)\\ &\underset{\pi}{\longmapsto} ((\sigma_1\sigma_2)^2, \sigma_0\sigma_2, \sigma_1) \underset{\delta}{\longmapsto} (\sigma_1, \sigma_0\sigma_2, (\sigma_1\sigma_2)^2),\end{aligned}$$

as required. This indicates also that σ halves the order of the group just when η does (modulo the double application of $\pi\delta$), that is, if and only if the 1-skeleton of $\mathcal{L}^{\pi\delta}$ is bipartite. If this is the case, then σ_2 is not in the group $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ but operates on A as an outer automorphism. In any case, the new map $\mathcal{P} = \mathcal{L}^{\sigma}$ is self-Petrie (that is, $\mathcal{P}^{\pi} \simeq \mathcal{P}$), since the corresponding groups are conjugate by σ_2.

The type $\{s, t\}$ of $\mathcal{P}$ is determined by the periods of

$$\rho_0\rho_1 = \sigma_1\sigma_0\sigma_2 = \sigma_0(\sigma_0\sigma_1\sigma_2)\,\sigma_0$$

and

$$\rho_1\rho_2 = \sigma_0\sigma_2(\sigma_1\sigma_2)^2 = \sigma_0\sigma_1\sigma_2\sigma_1.$$

Hence, s is the length of the Petrie polygons of $\mathcal{L}$ and t is the length of the 2-chains of $\mathcal{L}$.

As in the case of halving, the skewing operation σ admits a generalization $\sigma_{k,l}$ for maps of type $\{p, 2n\}$, with $k + l = n$. This is defined by

$$\sigma_{k,l} = \pi\delta\eta_{k,l}\pi\delta \colon (\sigma_0, \sigma_1, \sigma_2) \mapsto ((\sigma_1\sigma_2)^{l-1}\,\sigma_1, \sigma_0\sigma_2, (\sigma_1\sigma_2)^n),$$

so that $\sigma_{1,1} = \sigma$ if $n = 2$. If $\sigma_{k,l}$ defines a new map $\mathcal{P}$, of type $\{r, s\}$ (say), then s is the length of the n-chains and r the length of the l-zigzags of $\mathcal{L}$. (Just as $\rho_0\rho_1(\rho_2\rho_1)^{j-1}$ takes the jth exit from a vertex preserving local sense, giving a j-chain, so $\rho_0(\rho_1\rho_2)^j$ takes the jth exit from a vertex reversing local sense at each stage, giving a j-zigzag. This generalizes the Petrie polygon which corresponds to the case $j = 1$.)

4.5. *Infinite Regular Polytopes in* $\mathbb{E}^3$

As an illustration of the various techniques we have discussed, let us consider the relationships between the infinite regular polytopes of dimension 3 in $\mathbb{E}^3$ which were described by Coxeter [4], Grünbaum [14], and Dress [12]. Half of them are derived from the plane regular tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$ and their Petrie duals, by blending (see Section 5) with a line segment or the ordinary apeirogon $\{\infty\}$ in $\mathbb{E}^1$, which is why the families depend (up to similarity) on a parameter.

The remaining 12 are all related, as Fig. 2 indicates. (The notation used is that due to Grünbaum [14], to which we refer the reader for details. We take the opportunity to correct a small error of notation in the lists of [14, 12], which have $\{\infty^{2\pi/3,\, 2\pi/3}, 4^{\pi/3}/1\}$ instead of $\{\infty^{2\pi/3,\, 2\pi/3}, 4^{\alpha^*}/1\}$.)

$$
\begin{tikzcd}
\{\infty^{2\pi/3,\, 2\pi/3}, 4^{\alpha^*}/1\} \arrow[r, leftrightarrow, "\pi"] & \{6, 4^{\alpha^*}/1\} \arrow[r, leftrightarrow, "\delta"] \arrow[d, "\sigma"'] & \{4, 6^{\pi/3}/1\} \arrow[r, leftrightarrow, "\pi"] \arrow[d, "\eta"'] & \{\infty^{\pi/2,\, 2\pi/3}, 6^{\pi/3}/1\} \\
& \{\infty^{\pi/2,\, 2\pi/3}, 4\} & \{6^{\pi/3}/1, 6\} \arrow[loop left, "\delta"] \arrow[r, "\varphi_2"] \arrow[d, leftrightarrow, "\pi"'] & \{\infty^{2\pi/3,\, 2\pi/3}, 3\} \arrow[d, leftrightarrow, "\pi"'] \\
& \{6^{\pi/2}/1, 4\} \arrow[r, leftrightarrow, "\delta"] \arrow[d, phantom, "\sigma\delta"] & \{4^{\pi/3}/1, 6\} \arrow[r, "\varphi_2"] \arrow[d, "\eta"'] & \{\infty^{2\pi/3,\, \pi/2}, 3\} \\
& \{\infty^{2\pi/3,\, \pi/2,}\, 6^{\alpha^{**}}/1\} \arrow[r, leftrightarrow, "\pi"] & \{6, 6^{\alpha^{**}}/1\} \arrow[loop right, "\delta"] &
\end{tikzcd}
$$

FIGURE 2

In an analogous way, the skew polyhedra $\{4, 6\,|\,3\}$ and $\{4, 8\,|\,3\}$ of [4] are the starting points for families in $\mathbb{E}^4$ containing 7 and 26 members, respectively; in the latter case, the 26 fall into 13 pairs of isomorphic but not similar polyhedra.

5. THE BLENDING OPERATION β

We begin our discussion on operations on groups with more than three generators with an operation β we call *blending*. The geometric version of this operation plays a key role in the study of geometrical realizations of incidence-polytopes (cf. [19]).

Though blending could be defined in a more general setting we shall restrict ourselves to the case where the (blending) components are groups of regular incidence-polytopes. Then, when interpreted geometrically, blending essentially amounts to the following procedure: take n regular polytopes in n mutually orthogonal subspaces of euclidean space, or more exactly, their groups, the *components*; extend all the components in the trivial way to isometry groups of the whole space; then define the new group by taking as generators the products of corresponding generators of the components (cf. [19]).

To cover a slightly more general situation we also allow the components to be shifted in the following sense. Let $\mathscr{L}$ be a regular d-incidence-polytope with group $A(\mathscr{L}) = \langle \tau_0, ..., \tau_{d-1} \rangle$, where $\tau_0, ..., \tau_{d-1}$ are the distinguished generators. Further, let k be a non-negative integer. Define $\sigma_i := 1$ if $0 \leqslant i \leqslant k-1$, $\sigma_i := \tau_{i-k}$ if $k \leqslant i \leqslant d-1+k$, and, for purely technical reasons, $\sigma_i := 1$ if $i \geqslant d+k$. Then, $A(\mathscr{L}) = \langle \sigma_i \mid i \geqslant 0 \rangle$, and $\{\sigma_i \mid i \geqslant 0\}$ is the distinguished system of generators for $A(\mathscr{L})$ if $A(\mathscr{L})$ is interpreted as the group of the partially ordered set $\mathscr{L}^k$ obtained from $\mathscr{L}$ by shifting the rank function of $\mathscr{L}$ by k and then adjoining to $\mathscr{L}$ a single i-face for each i with $-1 \leqslant i < k-1$ or $i > d+k$. We shall call $\mathscr{L}^k$ the *k-shift* of $\mathscr{L}$.

Now, let $\mathscr{L}_1, ..., \mathscr{L}_n$ be regular incidence-polytopes and let $k_1, ..., k_n$ be non-negative integers one of which is 0. Assume that $\mathscr{L}_j$ is of dimension d_j and that the group of the k_j-shift $\mathscr{L}^{k_j}$ is given by $A(\mathscr{L}_j^{k_j}) = \langle \sigma_{ji} \mid i \geqslant 0 \rangle = A(\mathscr{L}_j)$, for $j = 1, ..., n$. Define

$$W := A(\mathscr{L}_1) \times \cdots \times A(\mathscr{L}_n),$$

the direct product of these groups. We again write φ for the element $(1, ..., 1, \varphi, 1, ..., 1)$ of W, whose jth component is the element φ of $A(\mathscr{L}_j)$.

As generators of the new group A we take the elements

$$\rho_i := \sigma_{1i} \sigma_{2i} \cdot ... \cdot \sigma_{ni} \qquad (i \geqslant 0)$$

of W. Then, $\rho_i^2 = 1$ for each i, and $\rho_i = 1$ if and only if $i \geqslant d := \max(d_j + k_j \mid j = 1, ..., n)$. The corresponding *blending operation* $\beta = \beta(k_1, ..., k_n)$ is defined by

$$\beta \colon (\sigma_{ji} \mid j = 1, ..., n; i \geqslant 0) \mapsto (\rho_0, ..., \rho_{d-1}).$$

If $A = \langle \rho_0, ..., \rho_{d-1} \rangle$ is the group of a regular d-incidence-polytope $\mathscr{P}$ and $\rho_0, ..., \rho_{d-1}$ the distinguished system of generators, then $\mathscr{P}$ is called a *blend* or, more exactly, the $(k_1, ..., k_n)$-*blend* of $\mathscr{L}_1, ..., \mathscr{L}_n$ and is denoted by

$$\mathscr{P} = \mathscr{L}_1^{k_1} \# \mathscr{L}_2^{k_2} \# \cdots \# \mathscr{L}_n^{k_n}.$$

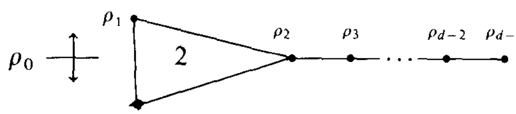
The shifts $\mathscr{L}_1^{k_1}, ..., \mathscr{L}_n^{k_n}$ are the *components* of $\mathscr{P}$. Here, if $k_j = 0$, we suppress the upper index and write $\mathscr{L}_j$ instead of $\mathscr{L}_j^{k_j}$. Observe that, for any regular $\mathscr{L}$, the blend $\mathscr{L} \# \mathscr{L} \# \cdots \# \mathscr{L}$ of n copies of $\mathscr{L}$ is again isomorphic to $\mathscr{L}$. More generally, $\mathscr{L}_1 \# \mathscr{L}_2 \simeq \mathscr{L}_1$ if $\mathscr{L}_1$ collapses onto $\mathscr{L}_2$ (that is, if $\mathscr{L}_2$ is obtained from $\mathscr{L}_1$ by identifications). Let us consider some specific cases.

Let $n = 2$, $k_1 = k_2 = 0$, and V_1 and V_2 be the vertex sets of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Assume that all the faces of $\mathscr{L}_1$ and $\mathscr{L}_2$ are uniquely determined by their vertices, so that $\mathscr{L}_1$ and $\mathscr{L}_2$ can be regarded as subcomplexes of the power sets of V_1 and V_2. Then, if $\mathscr{P} = \mathscr{L}_1 \# \mathscr{L}_2$ exists, it can be realized as a subcomplex of the power set of $V_1 \times V_2$ by suitably applying Wythoff's construction, with initial vertex the pair of vertices in the base flags of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. For example, if $\mathscr{L}_1 = \{p_1\}$ and $\mathscr{L}_2 = \{p_2\}$, then $\mathscr{L}_1 \# \mathscr{L}_2 = \{p\}$, where p is the lowest common multiple of p_1 and p_2.

Let $n=2$, $k_1=0$, $k_2=1$, and let $\mathscr{L}_1$ and $\mathscr{L}_2$ be of the same dimension $d_1=d_2=d$, say. Further assume that the vertex-figures of $\mathscr{L}_1$ are isomorphic to the facets of $\mathscr{L}_2$, say isomorphic to the $(d-1)$-incidence-polytope $\mathscr{K}$. It was proved in [26, Theorem 2] that $\mathscr{P}=\mathscr{L}_1 \# \mathscr{L}_2^1$ is a regular $(d+1)$-incidence-polytope with facets isomorphic to $\mathscr{L}_1$ and vertex-figures isomorphic to $\mathscr{L}_2$ provided $\mathscr{L}_1$ and $\mathscr{L}_2$ have an additional property with respect to $\mathscr{K}$, in [26] called the degenerate amalgamation property. (In particular, $p_1=p_1(\mathscr{L}_1)$ and $p_d=p_{d-1}(\mathscr{L}_2)$ must be even.) In this case $\mathscr{P}$ has the remarkable property of being combinatorially flat, meaning that each vertex is incident with each facet. For example, if $\mathscr{L}_1=\{6,3\}_{c,c}$ (cf. [8]) and $\mathscr{L}_2=\{3,4\}$, then $\mathscr{P}$ is a member of $\langle\{6,3\}_{c,c}, \{3,4\}\rangle$, with only $6c^2$ vertices and 8 facets.

In the general situation, it seems to be difficult to decide whether or not a given regular incidence-polytope $\mathscr{P}$ is a (non-trivial) blend. Here, the following two problems are of particular interest. Given $\mathscr{P}$, what can be said about the possible decompositions of $\mathscr{P}$ as a $(0,0,...,0)$-blend of regular incidence-polytopes of the same or different dimensions? Is it possible to characterize the unblended regular incidence-polytopes? For geometrical blending and finite polytopes, the solutions for these problems have been obtained in [19].

Of particular interest in this context is the regular d-incidence-polytope obtained from twisting the group with diagram



(cf. [21]). Combinatorially, the result is $\alpha_1 \# \alpha_d$ (with α_k the k-simplex). But this has a pure faithful realization in $\mathbb{E}^d$; here "pure" means that it is not a non-trivial geometrical blend. Among other things, this tells us that combinatorial and geometric blending are different concepts: a combinatorial blend need not be a geometrical blend.

6. Operations on Higher Dimensional Incidence-Polytopes

Given a regular m-incidence-polytope $\mathscr{L}$, with group $W:=A(\mathscr{L})=\langle\delta_0, ..., \delta_{m-1}\rangle$, there is a natural way of constructing operations on W. In fact, as generators ρ_j for the new group A we can take elements of the form

$$\rho_j := \prod_{i \in I_j} \sigma_i \qquad (j=0, ..., d-1), \tag{8}$$

where the non-empty subsets $I_0, \ldots, I_{d-1}$ of $\{0, \ldots, m-1\}$ are chosen subject to the following restrictions: for all j, if $i_1, i_2 \in I_j$, then $|i_1 - i_2| \geqslant 2$; if $|j_1 - j_2| \geqslant 2$, $i_1 \in I_{j_1}$, and $i_2 \in I_{j_2}$, then $|i_1 - i_2| \geqslant 2$; for $j = 0, \ldots, d-2$ there exists an i in I_j such that $i-1$ or $i+1$ is in I_{j+1}. Here, the first condition implies $\rho_j^2 = 1$ (but $\rho_j \neq 1$), the second, $\rho_{j_1}\rho_{j_2} = \rho_{j_2}\rho_{j_1}$, and the third, $\rho_j\rho_{j+1} \neq \rho_{j+1}\rho_j$ in all non-trivial cases. Then, the corresponding operation is given by

$$\mu : (\sigma_0, \ldots, \sigma_{m-1}) \mapsto (\rho_0, \ldots, \rho_{d-1}).$$

We shall restrict ourselves to some particularly interesting cases.

Among the various possibilities for choices of the I_j the following operations α of taking suitable products of alternating generators are suggested by the kind of relations derived from Petrie polygons.

Let $-1 := m_{-1} \leqslant m_0 < m_1 < \cdots < m_{d-1} \leqslant m_d := m-1$ with $m_j - m_{j-1}$ odd ($j = 1, \ldots, d-1$). For $j = 0, \ldots, d-1$ define

$$I_j := \{ i \mid m_{j-1} < i \leqslant m_{j+1} \text{ and } i \equiv m_j (\operatorname{mod} 2) \} \tag{9}$$

and ρ_j by (8). Here, possibly I_0 is empty, in which case $\rho_0 = 1$; in this case we simply omit the generator. These subsets I_j have the properties mentioned above. By construction, for $j = 1, \ldots, d-1$, the period p_j of $\rho_{j-1}\rho_j$ is the length h_j or twice this length of the Petrie polygons of a suitable section of $\mathscr{L}$, the value depending on the parity of h_j and the differences $m_{j+1} - m_j$ and $m_{j-1} - m_{j-2}$.

Of particular interest is the case $d = 3$. Though many of the following considerations extend to more general situations, we shall mainly discuss the construction of regular maps from 4-incidence-polytopes $\mathscr{L}$.

6.1. *The Regular Skew Polyhedra of 4-Incidence-Polytopes*

Let $\mathscr{L}$ be a regular 4-incidence-polytope with group $W = A(\mathscr{L}) = \langle \sigma_0, \ldots, \sigma_3 \rangle$, so that $m = 4$.

We begin by discussing the case $m_0 := -1$, $m_1 := 2$, and $m_2 := 3 = m - 1$. The corresponding operation μ is given by

$$\mu : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_1, \sigma_0\sigma_2, \sigma_3) =: (\rho_0, \rho_1, \rho_2). \tag{10}$$

As we shall see below, the new group $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ is in fact the group of a regular map $\mathscr{P} = \mathscr{L}^{\mu}$. If $\mathscr{L}$ is of type $\{p, q, r\}$, and if the Petrie-polygons of its facets have length t, then $\mathscr{P}$ is of type $\{t, r\}$ or $\{t, 2r\}$ as r is even or odd, respectively. Also, the Petrie polygons of $\mathscr{L}$ and $\mathscr{P}$ have the same length. Furthermore, from

$$\rho_0\rho_1(\rho_2\rho_1)^{j-1} = \sigma_1\sigma_0\sigma_2(\sigma_3\sigma_0\sigma_2)^{j-1} = \sigma_1\sigma_2(\sigma_3\sigma_2)^{j-1}\sigma_0^j$$

we deduce that for even j the j-chains of $\mathscr{P}$ and of the vertex-figures $\mathscr{M}$ of $\mathscr{L}$ have the same length.

To illustrate the operation μ (and to prove the existence of $\mathscr{P}$) let us apply again the combinatorial version of Wythoff's construction. Take the barycentric subdivision $\mathscr{C}(\mathscr{L})$ of $\mathscr{L}$ and pick as the initial vertex G_0 the barycentre of the edge in the base flag of $\mathscr{L}$; note that G_0 is invariant under ρ_1 and ρ_2 but not under ρ_0. Then, if Wythoff's construction is applied to A and G_0, we obtain a model of $\mathscr{P}$. The incidence-polytope $\mathscr{L}$ "splits" into several copies of $\mathscr{P}$, each an image of $\mathscr{P}$ under an automorphism of $\mathscr{L}$; these copies are in one-to-one correspondence with the left cosets of A in $A(\mathscr{L})$.

This system of copies of $\mathscr{P}$ can equally well be described as follows; in particular, this will show that we can reasonably think of $\mathscr{P}$ as a "skew (star-) polyhedon" belonging to $\mathscr{L}$. In fact, when applied to a classical regular 4-polytope the operation gives a regular skew star-polyhedron in the sense of Coxeter [4, p. 102]. For example, if $\mathscr{L} = \{4, 3, 3\}$ is the 4-cube, then $\mathscr{P}$ is Coxeter's map $\{6, 6 \mid 3, 4\}$, whose hexagonal 2-faces are the equatorial hexagons of all the bounding cuboctahedra of the truncated 4-cube $t_1\{4, 3, 3\}$.

Now, to get the system consider all the edges of $\mathscr{L}$ and pick as new "vertices" the barycentres of all the edges. Take all the Petrie polygons of all the facets of $\mathscr{L}$ and replace any such Petrie polygon Π by a new polygon, with vertices the barycentres of the edges in Π and with edges connecting the barycentres of consecutive edges in Π. These new polygons give the 2-faces of the maps in the system, and any two such maps are disjoint except for vertices. Now, two cases can occur.

Let r be odd. Then, going around a vertex in one of the maps corresponds to going around the corresponding edge in $\mathscr{L}$ twice, in agreement with the fact that $\mathscr{P}$ is of type $\{t, 2r\}$. This follows also from

$$(\rho_1 \rho_2)^r = \sigma_0^r (\sigma_2 \sigma_3)^r = \sigma_0, \tag{11}$$

which further implies $A = A(\mathscr{L})$. Hence, the system consists of a single copy of $\mathscr{P}$.

Let r be even. Then any new vertex is at the same time a vertex of two maps in the system, one a reflexion image of the other. In fact, going around the vertex corresponds to going around the respective edge in $\mathscr{L}$ just once, in agreement with the fact that $\mathscr{P}$ is of type $\{t, r\}$. Note that the two maps might well coincide, in which case the new vertex becomes a doubly counted vertex of the (single) map. However, this can happen if and only if there is an element ψ in A which fixes the edge in the base flag of $\mathscr{L}$ but is not in $\langle \rho_1, \rho_2 \rangle$, that is, if and only if $A \cap \langle \sigma_0, \sigma_2, \sigma_3 \rangle \neq \langle \sigma_0 \cdot \sigma_2, \sigma_3 \rangle$.

Since r is even, and thus $\langle\sigma_0\sigma_2, \sigma_3\rangle \simeq D_r$ is a subgroup of $\langle\sigma_0, \sigma_2, \sigma_3\rangle \simeq C_2 \times D_r$ of index 2, this is equivalent to $\langle\sigma_0, \sigma_2, \sigma_3\rangle \subset A$ and thus to $A = A(\mathscr{L})$. Hence, if for one new vertex the two maps are the same, then there is only one map in the system and this counts any new vertex twice; further, $A = A(\mathscr{L})$. In all the other cases A is a proper subgroup of $A(\mathscr{L})$, whose index gives the number of maps in the system. For example, if $\mathscr{L} = \{3, 3, 4\}$ is the 4-crosspolytope, then we get 3 copies of toroidal maps $\mathscr{P} = \{4, 4 | 4\} = \{4, 4\}_{4,0}$ (cf. [8]).

These geometrical considerations are supported by the fact that A indeed has the intersection property with respect to the ρ_i. This can be seen as follows. We trivially have

$$\begin{aligned}&\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle\\ &\quad= \langle\sigma_1, \sigma_0\sigma_2\rangle \cap \langle\sigma_0\sigma_2, \sigma_3\rangle \subset \langle\sigma_0, \sigma_1, \sigma_2\rangle \cap \langle\sigma_0, \sigma_2, \sigma_3\rangle\\ &\quad= \langle\sigma_0, \sigma_2\rangle \simeq C_2 \times C_2.\end{aligned}$$

Hence, if $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle \neq \langle\rho_1\rangle = \langle\sigma_0\sigma_2\rangle$, then necessarily

$$\langle\sigma_0, \sigma_1, \sigma_2\rangle = \langle\sigma_1, \sigma_0\sigma_2\rangle, \tag{12}$$

which is the group of the facets $\mathscr{N}$ of $\mathscr{L}$. But this is impossible, since the right side of (12) is the group of a Petrie polygon of $\mathscr{N}$, of order twice the number of vertices of $\mathscr{N}$. This proves $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle = \langle\rho_1\rangle$. The remaining cases are trivial.

Observe that for odd r the old generators of $A = A(\mathscr{L})$ can be recovered from the new generators by the inverse operation

$$\mu^{-1}\colon (\rho_0, \rho_1, \rho_2) \mapsto ((\rho_1\rho_2)^r, \rho_0, (\rho_1\rho_2)^r \rho_1, \rho_2) = (\sigma_0, ..., \sigma_3).$$

This allows us to transform any presentation for $A(\mathscr{L})$ in terms of the σ_i into one for $A(\mathscr{P}) = A(\mathscr{L})$ in terms of the ρ_i. For example, if $\mathscr{L} = \{p, q, r\}$ is universal and thus $A(\mathscr{L})$ the Coxeter group $[p, q, r]$, then the new map $\mathscr{P}$ has the same group but with the presentation

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_1\rho_2)^{2r} = ((\rho_1\rho_2)^r \rho_0)^p = (\rho_0(\rho_1\rho_2)^r \rho_1)^q = 1. \tag{13}$$

Note that also for even r this map is universal among all maps derived by operation μ from any regular 4-incidence-polytope of type $\{p, q, r\}$.

Except for the trivial operations $(\sigma_0, ..., \sigma_3) \mapsto (\sigma_0, \sigma_1, \sigma_2)$ and $(\sigma_0, ..., \sigma_3) \mapsto (\sigma_1, \sigma_2, \sigma_3)$ the above operation μ is the only operation on $\mathscr{L}$ with index sets I_j as in (9). However, there are other possibilities of choosing the I_j, for example the "dual" operation

$$\delta\mu\delta\colon (\sigma_0, ..., \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_3, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

This operation has the effect of first applying a Petrie operation to the vertex-figures of $\mathscr{L}$ (trivially extended to the fourth generator), and then passing from the resulting 4-incidence-polytope to its facet. The result is a regular map $\mathscr{P}$. Again, this can be realized in the barycentric subdivision $\mathscr{C}(\mathscr{L})$ of $\mathscr{L}$ by applying Wythoff's construction to $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ with initial vertex the vertex in the base flag of $\mathscr{L}$. So, in a sense, we can again think of $\mathscr{P}$ as a skew polyhedron related to $\mathscr{L}$. As an example, if $\mathscr{L} = \{4, 3, 4\}$ is the cubical tessellation of euclidean 3-space $\mathbb{E}^3$, then $\mathscr{P} = \{4, 6 \mid 4\}$ is one of the Petrie–Coxeter polyhedra in $\mathbb{E}^3$ (cf. [4]).

Similar considerations apply also to other operations on $\mathscr{L}$, giving further classes of skew polyhedra. In particular, this holds for operations on $\mathscr{L}$ built on the operations of Section 4.

Clearly, there is no need to restrict the discussion to 4-dimensional $\mathscr{L}$ (and 3-dimensional $\mathscr{P}$). For example, if $m = \dim(\mathscr{L}) \geqslant 5$ and $A(\mathscr{L}) = \langle \sigma_0, ..., \sigma_{m-1} \rangle$, then the operation

$$\begin{aligned} \nu: (\sigma_0, ..., \sigma_{m-1}) &\mapsto (\sigma_{m-3}\sigma_{m-5}\sigma_{m-7} \cdot ..., \sigma_{m-2}\sigma_{m-4}\sigma_{m-6} \cdot ..., \sigma_{m-1}) \\ &=: (\rho_0, \rho_1, \rho_2) \end{aligned}$$

is a generalization of μ and gives us a regular map $\mathscr{P} = \mathscr{L}^{\nu}$. Here, if Wythoff's construction is applied to $\mathscr{C}(\mathscr{L})$ and A, we can take as the initial vertex the barycentre of the face $\{F_{m-3}, F_{m-5}, F_{m-7}, ...\}$ of $\mathscr{C}(\mathscr{L})$ (where $\{F_{-1}, F_0, ..., F_m\}$ is the base flag of $\mathscr{L}$), but there are other choices too.

For example, let $\mathscr{L} = \{3^{m-2}, 4\}$ be the m-crosspolytope and hence $\mathscr{L}^* = \{4, 3^{m-2}\}$ the m-cube. Then, applying to $A(\mathscr{L}^*) = \langle \tau_0, ..., \tau_{m-1} \rangle$ the operation

$$\nu^* : (\tau_0, ..., \tau_{m-1}) \mapsto (\tau_0, \tau_1\tau_3 \cdot ..., \tau_2\tau_4 \cdot ...)$$

gives us the dual $\mathscr{P}^*$ of $\mathscr{P} = \mathscr{L}^{\nu}$. Now, realizing the generators $\tau_0, ..., \tau_{m-1}$ by the euclidean reflexions

$$\begin{aligned} &\tau_0 : \xi_1 \leftrightarrow -\xi_1, \\ &\tau_j : \xi_j \leftrightarrow \xi_{j+1} \qquad (j = 1, ..., m-1) \end{aligned}$$

in $\mathbb{E}^m$, we find that the point $x := (1, 1, ..., 1)$ is invariant under $\tau_1, ..., \tau_{m-1}$. Hence, applying Wythoff's construction to $A(\mathscr{L}^*)$ with initial vertex x yields a realization of $\mathscr{P}^*$, which is exactly the realization of Coxeter's map $M_m = \{4, m \mid 4^{[m/2]-1}\}$ described in [4, p. 100]. By the above considerations there is also a realization of $M_m^* = \mathscr{P} = \mathscr{L}^{\nu}$ in $\mathbb{E}^m$ (cf. [19]).

Another possible generalization of μ is

$$\begin{aligned} &\nu_k: (\sigma_0, ..., \sigma_{m-1}) \\ &\mapsto \begin{cases} (... \cdot \sigma_{k-5}\sigma_{k-3}\sigma_{k-1}, ... \cdot \sigma_{k-4}\sigma_{k-2}\sigma_k\sigma_{k+2}\sigma_{k+4} \cdot ..., \sigma_{k+1}\sigma_{k+3}\sigma_{k+5} \cdot ...) \\ \quad =: (\rho_0, \rho_1, \rho_2), \end{cases} \end{aligned}$$

with $1 \leqslant k \leqslant m-2$; then $v_{m-2} = v$. For Wythoff's construction we can take the barycentre of the face $\{..., F_{k-5}, F_{k-3}, F_{k-1}\}$ of $\mathscr{C}(\mathscr{L})$, but again there are other possible choices. For example, if $\mathscr{L} = \{4, 3^{m-3}, 4\}$ is the cubical tessellation in $\mathbb{E}^{m-1}$ $(m \geqslant 4)$, then the resulting map $\mathscr{P} = \mathscr{L}^{v_k}$ is of type $\{2(k+1), 2(m-k)\}$ and is infinite. Again, $\mathscr{P}$ and its dual $\mathscr{P}^*$ admit realizations in $\mathbb{E}^{m-1}$ (or occasionally in $\mathbb{E}^{m-2}$).

6.2. *Some Skew Polyhedra Derived from 4-Incidence-Polytopes*

In this section we discuss some interesting examples of regular maps $\mathscr{P} = \mathscr{L}^{\mu}$, with $\mathscr{L}$ a regular 4-incidence-polytope of type $\{p, q, r\}$ and μ as in (10).

(a) First, let us consider the case where $\mathscr{L}$ is a classical regular convex polytope or star-polytope in euclidean 4-space $\mathbb{E}^4$ and $\{p, q, r\}$ is its combinatorial type (cf. [6, 18]). Here the 4-crosspolytope $\mathscr{L} = \{3, 3, 4\}$ is the only instance with r even. As mentioned in Section 6.1, in this case $\mathscr{P}$ is the toroidal map $\{4, 4\}_{4,0}$ and $\mathscr{L}$ splits into three copies of $\mathscr{P}$. In the remaining cases r is 3 or 5, and there is only one copy of $\mathscr{P}$, with the same group as $\mathscr{L}$. Then $\mathscr{P}$ is of type $\{t, 2r\}$, with t the length of the Petrie polygons of the facets of $\mathscr{L}$.

If $\mathscr{L} = \{3, 3, 3\}$, then $\mathscr{P}$ is of type $\{4, 6\}$ with $A(\mathscr{P}) \simeq S_5$; the map $\{4, 6\}_5$, with group $C_2 \times S_5$, is a two-fold covering of $\mathscr{P}$ (cf. [8]). From the polytopes $\mathscr{L} = \{4, 3, 3\}$, $\{3, 4, 3\}$, $\{3, 3, 5\}$, and $\{5, 3, 3\}$, respectively, we derive maps $\mathscr{P}$ of type $\{6, 6\}$, $\{6, 6\}$, $\{4, 10\}$, and $\{10, 6\}$, with 32, 96, 720, and 1200 vertices. The former map of type $\{6, 6\}$ is Coxeter's $\{6, 6 \mid 3, 4\}$ (cf. [4, p. 101]). As the regular convex 4-polytope with Schläfli-symbol $\{p, q, r\}$ is the universal $\{p, q, r\}$ (cf. [6, 25]), we observe that in all these cases (with r odd) the relations (13) give a presentation for $A(\mathscr{P})$ in terms of ρ_0, ρ_1, and ρ_2. Also, when $\mathscr{L}$ is considered as a geometric polytope, with a geometric barycentric subdivision $\mathscr{C}(\mathscr{L})$, then the model of $\mathscr{P}$ described in Section 6.1 is a realization of $\mathscr{P}$ in the sense of [19]. Observing that the midpoints of the edges in a Petrie polygon of a Platonic solid lie in a plane, we see that $\mathscr{P}$ has planar 2-faces bounding a convex t-gon. When thinking of these t-gons as the 2-faces of $\mathscr{P}$, we obtain a polyhedron P with convex faces in $\mathbb{E}^4$. However, P has many self-intersections, arising from the fact that each edge of a Platonic solid lies in precisely two Petrie polygons. This is why Coxeter [4, p. 102] called these generalized Kepler–Poinsot polyhedra regular skew star polyhedra in $\mathbb{E}^4$. Note that we can suitably project these starpolyhedra into 3-space to get highly symmetric 3-dimensional analogues of the classical Kepler–Poinsot polyhedra. For example, from $\{3, 3, 5\}$ and $\{5, 3, 3\}$ we obtain Kepler–Poinsot-type polyhedra with icosahedral symmetry other than the classical examples (cf. [23]).

The situation is similar for the ten regular star polytopes in $\mathbb{E}^4$. Figure 3 is taken from [6, p. 266] and indicates isomorphism and duality relations between these star polytopes and the polytopes $\{3, 3, 5\}$ and $\{5, 3, 3\}$: isomorphic polytopes are diametrically opposite to each other (and are given by interchanging 5 and $\frac{5}{2}$), while reciprocal polytopes are joined by horizontal lines. (See also Table 2 in [18].)

Considering just isomorphism classes of corresponding regular maps $\mathscr{P}$ we find that there are only four further maps. They are derived from $\{5, \frac{5}{2}, 5\}$, $\{3, 5, \frac{5}{2}\}$, $\{5, 3, \frac{5}{2}\}$, and $\{5, \frac{5}{2}, 3\}$, and are of type $\{6, 10\}$, $\{10, 10\}$, $\{10, 10\}$, and $\{6, 6\}$, respectively. The first three maps have 720 vertices each, the last map has 1200 vertices. These maps are different from each other and from the maps constructed from the regular convex polytopes. This is either trivial from the group order or follows by comparing the length of Petrie polygons or 2-chains. For example, the length of the Petrie polygons of the two maps of type $\{10, 10\}$ is the same as the length of the Petrie polygons of the corresponding star polytope, but this is 20 or 12, respectively (cf. [6, p. 294]). Similar arguments (in fact, the same for the two maps of type $\{10, 10\}$) show that no two of this total of 8 maps can be duals of each other.

Again, as in the case of the regular convex polytopes, we can obtain realizations of $\mathscr{P}$ in $\mathbb{E}^4$ with planar 2-faces. However, this time the geometrical shape of the resulting star polyhedron depends essentially on which representative of an isomorphism class of regular polytopes or star polytopes has been chosen for $\mathscr{L}$. For example, from $\mathscr{L} = \{\frac{5}{2}, 3, 3\}$ we get

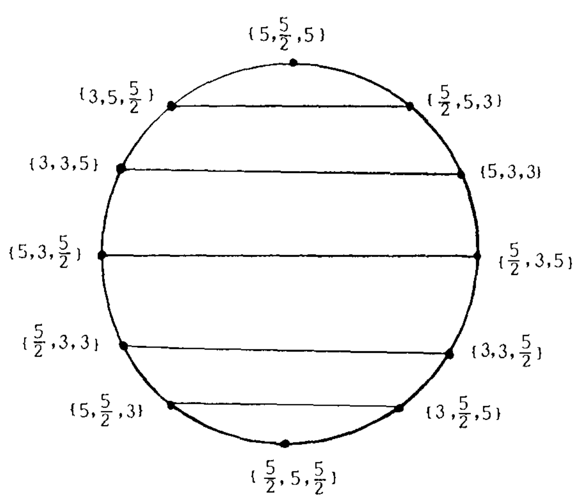


FIGURE 3

a star polyhedron with star-gons $\{\frac{10}{3}\}$ as 2-faces, while the star polyhedron derived from $\mathscr{L} = \{5, 3, 3\}$ has convex 10-gons as 2-faces. Again, these star polyhedra can be suitably projected down into $\mathbb{E}^3$ to give Kepler–Poinsot-type polyhedra with icosahedral symmetry.

Concluding the discussion on the classical regular polytopes in $\mathbb{E}^4$ we point out that it would be very desirable to have a complete classification of "all" the regular skew polyhedra related to these polytopes, both in the combinatorial and geometrical sense. It seems that such an enumeration has to be along the lines of this paper and of [22].

(b) As a further application we shall determine the regular skew polyhedra $\mathscr{P} = \mathscr{L}^{\mu}$ for the duals $\mathscr{L}$ of the regular 4-incidence-polytopes $2^{\mathscr{K}}$, with $\mathscr{K}$ a regular map. These incidence-polytopes $2^{\mathscr{K}}$ were originally found by Danzer [9], but see also [20] for a more detailed discussion. We briefly recall the construction.

Let $\mathscr{K}$ be a finite regular $(m-1)$-incidence-polytope which is a lattice $(m \geqslant 2)$, and let $V = \{0, 1, ..., v-1\}$ (say) denote its vertex set. Write $\hat{V} := \{-1, 1\}^v$ for the vertex set of the v-cube in euclidean space $\mathbb{E}^v$. As j-faces $(j = 0, ..., m)$ of $2^{\mathscr{K}}$ we take, for any $(j-1)$-face F of $\mathscr{K}$ and any $a := (a_0, ..., a_{v-1})$ in $\hat{V}$, the subsets $F(a)$ of $\hat{V}$ defined by

$$F(a) := \{(b_0, ..., b_{v-1}) \mid b_i = a_i \text{ for any } i \text{ which is not a vertex of } F\},$$

or, by an abuse of notation,

$$F(a) = \Big(\mathop{\times}_{i \leqslant F} \{-1, 1\}\Big) \times \Big(\mathop{\times}_{i \nleqslant F} \{a_i\}\Big).$$

Then, if F, F' are faces of $\mathscr{K}$ and a, a' elements of $\hat{V}$, we have $F(a) \subset F'(a')$ if and only if $F \leqslant F'$ and $a_i = a'_i$ for every i with $i \nleqslant F'$. In particular, $2^{\mathscr{K}}$ is a regular m-incidence-polytope which is a lattice and is of type $\{4, p_1, ..., p_{m-2}\}$ if $\mathscr{K}$ is of type $\{p_1, ..., p_{m-2}\}$. The group $A(2^{\mathscr{K}})$ is the wreath-product $C_2 \wr A(\mathscr{K})$, with $A(\mathscr{K})$ acting naturally on V. As an example, for any $k \geqslant 3$ we have $M_k = 2^{\{k\}}$, with group $C_2 \wr D_k$ of order $2^k \cdot 2k$ (cf. [18]).

Now let $\mathscr{K}$ be a regular map, that is, $m = 4$. To describe the generators $\sigma_0, ..., \sigma_3$ of $A(2^{\mathscr{K}})$, let the base flag $\{G_0, G_1, G_2\}$ of $\mathscr{K}$ be chosen such that $G_0 = 0$ and 0, 1 are the vertices of G_1. Representing the σ_i as permutations on the vertex set $\hat{V}$ of $2^{\mathscr{K}}$, we find that σ_0 is given by

$$\sigma_0((a_0, ..., a_{v-1})) = (-a_0, a_1, ..., a_{v-1}) \qquad \text{for} \quad (a_0, ..., a_{v-1}) \in \hat{V}, \tag{14}$$

while σ_1, σ_2, and σ_3, respectively, do not involve a change of signs but just permute the components of $(a_0, ..., a_{v-1})$ according to the effect of the distinguished generators τ_0, τ_1, τ_2 (say) of $A(\mathcal{K})$ on the vertices $0, ..., v-1$ of $\mathcal{K}$.

Now, applying operation μ to the dual $\mathcal{L}$ of $2^{\mathcal{K}}$, we observe that the generators ρ_0, ρ_1, and ρ_2 of the group A of the regular skew polyhedron $\mathcal{P} = \mathcal{L}^{\mu}$ are given by

$$(\rho_0, \rho_1, \rho_2) = (\sigma_2, \sigma_1\sigma_3, \sigma_0).$$

Hence, $\mathcal{P}$ is of type $\{t, 4\}$, where t is the length of the Petrie polygons of $\mathcal{K}$. Suppose that the vertices of $\mathcal{K}$ are labelled in such a way that $0, 1, 2, ..., t-1$ are consecutive vertices in the Petrie polygon of $\mathcal{K}$ defined by the base flag $\{G_0, G_1, G_2\}$. Then ρ_0 and ρ_1 permute $0, 1, ..., t-1$ among themselves; in particular, $\rho_1\rho_0$ corresponds to the cyclic permutation $(0\ 1\ 2 \cdots t-2\ t-1)$. This implies that the "restriction" A' of A to $\{-1, 1\}^t$ (or more precisely, to the first t components of the vectors in $\hat{V}$) is the wreath-product $C_2 \wr D_t$, with D_t acting on $0, 1, ..., t-1$ as usual. But A' is isomorphic to A; in fact, if $\psi \in A$, then $\psi = (\varphi_1\rho_2\varphi_1^{-1}) \cdot ... \cdot (\varphi_n\rho_2\varphi_n^{-1}) \cdot \varphi$ with $\varphi_1, ..., \varphi_n, \varphi \in \langle \rho_0, \rho_1 \rangle$, giving at most $2^t \cdot 2t$ elements ψ.

All these considerations imply that, if $\mathcal{K}$ is a finite regular map which is a lattice and has Petrie polygons of length t, then the regular skew polyhedron $\mathcal{P} = \mathcal{L}^{\mu}$ of the dual $\mathcal{L}$ of $2^{\mathcal{K}}$ is the dual of Coxeter's map $M_t = 2^{\{t\}}$; that is, $((2^{\mathcal{K}})^*)^{\mu} = (2^{\{t\}})^*$ or, equivalently, $(2^{\mathcal{K}})^{\delta\mu} = (2^{\{t\}})^{\delta}$. As an example, if $\mathcal{K} = \{3, 5\}_5$ is the hemi-icosahedron, then $2^{\mathcal{K}} = \{\{4, 3\}, \{3, 5\}_5\}$ (cf. [2, 20]), $\mathcal{L} = \{\{5, 3\}_5, \{3, 4\}\}$, and $\mathcal{P} = M_5^* = \{5, 4 \mid 4\}$ (of genus 5). Similarly, if $\mathcal{K}$ is the toroidal map $\{4, 4\}_{3,0}$, then $2^{\mathcal{K}} = \{\{4, 4\}_{4,0}, \{4, 4\}_{3,0}\}$, $\mathcal{L} = \{\{4, 4\}_{3,0}, \{4, 4\}_{4,0}\}$, and $\mathcal{P} = \{4, 6 \mid 4, 4\}^*$ (of genus 17).

(c) Our last application of the operation μ illustrates for Coxeter's maps M_p the interaction of μ with a specific twisting operation κ in the sense of [20, 22]; see the diagram (24) at the end of this section. Here we shall follow the discussion in [22]. We begin by recalling some facts about twisting operations.

Let $\mathcal{L}$ be a self-dual regular 4-incidence-polytope of type $\{p, q, p\}$ with group $A(\mathcal{L}) = \langle \sigma_0, ..., \sigma_3 \rangle$. There is a unique polarity τ (duality of order 2) which keeps the base flag Φ fixed and so interchanges the i-faces and $(3-i)$-faces of $\mathcal{L}$; then

$$\tau\sigma_i\tau = \sigma_{3-i} \qquad (i = 0, ..., 3), \tag{15}$$

so that τ induces an outer automorphism of $A(\mathcal{L})$ permuting the basic generators. Further, the extended group $D(\mathcal{L})$ of $\mathcal{L}$ is generated by $\sigma_0, ..., \sigma_3$ and τ, and hence is a semi-direct product of $A(\mathcal{L})$ by C_2.

Now a twisting operation κ on $\mathscr{L}$ (or on $A(\mathscr{L})$) is obtained by choosing as generators for the new group $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ the elements

$$\rho_0 := \sigma_0, \qquad \rho_1 := \tau, \qquad \rho_2 := \sigma_2 \tag{16}$$

of $D(\mathscr{L})$; for short,

$$\kappa \colon (\sigma_0, ..., \sigma_3; \tau) \mapsto (\sigma_0, \tau, \sigma_2) =: (\rho_0, \rho_1, \rho_2). \tag{17}$$

Note that this is not a (mixing) operation in the sense of this paper. Then, by (15), we have $A = D(\mathscr{L})$. The most important point is that A is the group of a regular map $\mathscr{N} = \mathscr{L}^{\kappa}$ of type $\{4, 2q\}$, with ρ_0, ρ_1, ρ_2 the distinguished generators of $A = A(\mathscr{N})$. In terms of ρ_0, ρ_1, ρ_2 the standard relations (1) and (15) for $A = D(\mathscr{L})$ take the form

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^{2q} = (\rho_0\rho_2)^2 = 1, \tag{18}$$

$$(\rho_0\rho_1\rho_2\rho_1)^p = 1. \tag{19}$$

In general these relations do not suffice for a presentation. However, if $\mathscr{L} = \{p, q, p\}$ is universal, then (18), (19) gives a presentation for A, and hence $\mathscr{N} = \{4, 2q \mid p\}$. For example, if $\mathscr{L} = \{4, 3, 4\}$ is the cubical tessellation in $\mathbb{E}^3$, then $\mathscr{N}$ is the Petrie–Coxeter polyhedron $\{4, 6 \mid 4\}$ (cf. [4]). In the general situation we shall refer to $\mathscr{N}$ as the *Petrie–Coxeter polyhedron* of $\mathscr{L}$ (cf. [22]).

One of the most interesting properties of these Petrie–Coxeter polyhedra is that $\mathscr{L}$ can be recovered from $\mathscr{N}$; in fact, by (15) and (16), we have the inverse operation

$$\kappa^{-1} : (\rho_0, \rho_1, \rho_2) \mapsto (\rho_0, \rho_1\rho_2\rho_1, \rho_2, \rho_1\rho_0\rho_1; \rho_1) =: (\sigma_0, \sigma_1, \sigma_2, \sigma_3; \tau). \tag{20}$$

As an interesting byproduct, (20) also gives a method for constructing from suitable regular maps $\mathscr{N}$ of type $\{4, 2q\}$ a self-dual regular 4-incidence-polytope $\mathscr{L}$ with Petrie–Coxeter polyhedron $\mathscr{N}$. In [22] this was discussed in some detail for Coxeter's map $\mathscr{N} = M_{2q} = \{4, 2q \mid 4^{q-1}\}$. Here the resulting $\mathscr{L} = \mathscr{L}_q$ is of type $\{4, q, 4\}$, with a group of order $2^{2q} \cdot 2q = \frac{1}{2}|A(M_{2q})|$ and with 2^q facets and 2^q vertex-figures isomorphic to M_q and M_q^*, respectively. Further, $\mathscr{L}_q$ is combinatorially flat, meaning as above that each vertex is incident with each facet.

Consider the group of M_k, with $k \geqslant 3$. Since $M_k = 2^{\{k\}}$ we have a representation of $A(M_k) = C_2 \wr D_k$ on $V = \{-1, 1\}^k$ as in Section (b), with three generators only, denoted again by ρ_0, ρ_1, and ρ_2. The ρ_i can be chosen such that

$$\begin{aligned} \rho_0((a_0, ..., a_{k-1})) &= (-a_0, a_1, ..., a_{k-1}), \\ \rho_1((a_0, ..., a_{k-1})) &= (a_1, a_0, a_{k-1}, ..., a_2), \\ \rho_2((a_0, ..., a_{k-1})) &= (a_0, a_{k-1}, ..., a_1) \end{aligned} \tag{21}$$

for $(a_0, ..., a_{k-1}) \in \hat{V}$. We easily find that $\rho_0\rho_1\rho_2$ has period $2k$, implying that M_k has Petrie polygons of length $2k$. The involution $\alpha_k := (\rho_0\rho_1\rho_2)^k$ is the central reflexion on $\hat{V}$ and generates the centre of $A(M_k)$. We remark that the elliptic contraction $M_k/2$ is a self-Petrie map in the sense of Section 4.1. However, except for $k = 2q$,

$$\alpha_{2q} = (\sigma_0\tau\sigma_2\sigma_0\tau\sigma_2)^q = (\sigma_0\sigma_1\sigma_3\sigma_2)^q \in A(\mathscr{L}_q) = \langle \sigma_0, ..., \sigma_3 \rangle,$$

the map $M_{2q}/2$ is not the Petrie–Coxeter polyhedron of a self-dual regular 4-incidence-polytope, since the respective generators defined by (20) do not have the intersection property (2).

Now, to determine the structure of the skew polyhedron $\mathscr{P}_q := \mathscr{L}_q^\mu$ of $\mathscr{L}_q$ we rewrite, for $k = 2q$, the vectors a in $\hat{V}$ in the form $a = (a', a'')$ with $a' := (a_0, a_2, ..., a_{2q-2})$ and $a'' := (a_1, a_3, ..., a_{2q-1})$. Then, by (10), (20), and (21), the basic generators τ_0, τ_1, τ_2 (say) of $A(\mathscr{P}_q)$ are given by

$$\begin{aligned} \tau_0((a', a'')) &= ((a_2, a_0, a_{2q-2}, ..., a_4), (a_1, a_{2q-1}, ..., a_3)),\\ \tau_1((a', a'')) &= ((-a_0, a_{2q-2}, ..., a_2), (a_{2q-1}, ..., a_1)), \\ \tau_2((a', a'')) &= ((a_0, a_2, ..., a_{2q-2}), (-a_1, a_3, ..., a_{2q-1})) \end{aligned} \tag{22}$$

for $(a', a'') \in \hat{V}$. Note that (by an obvious abuse of notation) $\tau_i((a', a'')) = (\tau_i(a'), \tau_i(a''))$ for all i and (a', a''). This way $A(\mathscr{P}_q)$ becomes a subgroup of the direct product $(C_2 \wr D_q) \times (C_2 \wr D_q)$, with the two factors acting on the even or odd components, respectively. In terms of the basic generators $\varphi_0, \varphi_1, \varphi_2$ and ψ_0, ψ_1, ψ_2 of these factors (given by (21)) we can write

$$\begin{aligned} \tau_0 &= (\varphi_1, \psi_2), \\ \tau_1 &= (\varphi_0\varphi_2, \psi_2\psi_1\psi_2), \\ \tau_2 &= (1, \psi_0). \end{aligned} \tag{23}$$

The projection of $A(\mathscr{P}_q)$ onto the second factor $C_2 \wr D_q$ is obviously surjective, since all conjugates of ψ_0, ψ_1, ψ_2 by ψ_2 are involved. On the other hand, $\gamma_q := (\tau_0\tau_1)^q = ((\varphi_1\varphi_0\varphi_2)^q, 1) = ((\varphi_0\varphi_1\varphi_2)^q, 1)$ is a central involution of $A(\mathscr{P}_q)$, for which the projection of $A(\mathscr{P}_q)/\langle\gamma_q\rangle$ onto the second factor $C_2 \wr D_q$ is in fact an isomorphism; this can either be seen directly or by checking the relations (5) (with $r_j = 4$) for the images of τ_0, τ_1, τ_2 under the projection (but with the roles of τ_0 and τ_2 interchanged). It is easy to see that the centre of $A(\mathscr{P}_q)$ is generated by γ_q and

$$(\tau_0\tau_1\tau_2)^q = ((\varphi_1\varphi_0\varphi_2)^q, (\psi_1\psi_2\psi_0)^q) = ((\varphi_0\varphi_1\varphi_2)^q, (\psi_0\psi_1\psi_2)^q) = \alpha_{2q}.$$

All these considerations imply that the contraction c_q of $\mathscr{P}_q$ induced by γ_q gives a map isomorphic to M_q^*. Note that c_q halves the order of $A(\mathscr{P}_q)$ (which is $2^{q+1} \cdot 2q$) and changes the type $\{2q, 4\}$ of $\mathscr{P}_q$ to the type $\{q, 4\}$ of M_q^*.

The situation is summarized by the following diagram involving μ, κ^{-1}, c_q and the duality operation δ:

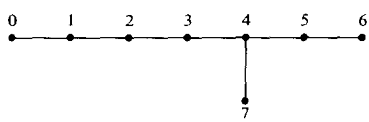
$$M_{2q} \xmapsto[\kappa^{-1}]{} \mathscr{L}_q \xmapsto[\mu]{} \mathscr{P}_q \xmapsto[c_q]{} M_q^* \xmapsto[\delta]{} M_q. \tag{24}$$

Now, if q is even, the diagram can be extended to the right until we finally reach a map M_l with l odd and $l \mid q$. Clearly there is also an 'infinite' extension to the left connecting M_{2q} with M_{2k}, $\mathscr{L}_k$, $\mathscr{P}_k$ and M_k^* for $k = 2^n q$, $n \geqslant 2$.

7. Non-linear Diagrams

Let us also give some examples of mixing applied to Coxeter groups with non-linear diagrams. While these do not in fact yield new groups (but rather new realizations of old ones), they are of interest in explaining various connexions between polytopes of different dimensions. (Such connexions have been pointed out by, for example, Coxeter [7]; Monson [24] has independently made observations very similar to ours, and has discussed their geometric significance.)

We begin with the group $E_8 = [3^{4,2,1}]$; its generators $\sigma_0, ..., \sigma_7$ are indicated on the Coxeter diagram



(cf. [6]). We generate a subgroup by mixing these generators in pairs:

$$(\sigma_0, ..., \sigma_7) \mapsto (\sigma_0\sigma_6, \sigma_1\sigma_5, \sigma_2\sigma_4, \sigma_3\sigma_7) =: (\rho_0, ..., \rho_3).$$

Then $\langle \rho_0, ..., \rho_3 \rangle \cong [3, 3, 5]$. Among the various ways of applying Wythoff's construction, one leads to an isomorphic copy of $\{3, 3, 5\}$ with half the 240 vertices of the Gosset polytope 4_{21} (see [6, p. 200]), and this projects onto orthogonal 4-dimensional subspaces of $\mathbb{E}^8$ to give $\{3, 3, 5\}$ itself and $\{3, 3, \frac{5}{2}\}$.

The subgroup $\langle \rho_1, \rho_2, \rho_3 \rangle \cong [3, 5]$ similarly derives $\{3, 5\}$ and $\{3, \frac{5}{2}\}$ from the 6-cross-polytope $\{3, 3, 3, 3, 4\}$. Related constructions applied to the infinite groups $T_7 = [3^{2,2,2}]$ and $T_8 = [3^{3,3,1}]$ yield $[3, 4, 3, 3]$ and $[3, 3, 4, 3]$, respectively.

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