ON GROUPS OF UNIT ELEMENTS OF CERTAIN QUADRATIC FORMS

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Abstract. It is shown that, for the groups of unit elements of certain integral quadratic forms of signature $(n, 1)$, there exist subgroups of finite index, generated by reflections; and the generators and relations of these subgroups are found.
Figures: 11. Bibliography: 4 items.

The group Θ of integral automorphisms ("unit elements") of the quadratic form

$$f(x) = -x_0^2 + x_1^2 + \dots + x_n^2, \tag{1}$$

mapping each of the two connected components of the set $\{x: f(x) < 0\}$ into itself, finds a natural interpretation as a discrete group of motions of n-dimensional Lobačevskiĭ space Λ^n. The present paper shows that, with $n \leq 17$, the group Θ splits up into semidirect product

$$\Theta = \Gamma \cdot \mathrm{H}, \tag{2}$$

where Γ is a group generated by reflections and H is a finite group. The generators and fundamental polyhedron P of the group Γ are found explicitly, while H is shown to be the same as the symmetry group of the polyhedron P. With $n \leq 14$, the same is done for the form

$$f(x) = -2x_0^2 + x_1^2 + \dots + x_n^2. \tag{3}$$

The first two sections are auxiliary, and apart from §§1.4, 2.4 and 2.5 are devoted to necessary definitions and earlier results.

§3 gives an algorithm whereby, given any discrete group Θ of motions of the space Λ^n, the fundamental polyhedron of the group generated by all reflections belonging to Θ can be found.

The algorithm is applied in §4 to the groups of unit elements of quadratic forms (1) and (3). The results obtained are quoted in Tables 4–7.

The group of unit elements of form (1) with $n \geq 18$, or of form (3) with $n \geq 15$, does not appear to contain a subgroup of finite index generated by reflections.

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The author's examination of other quadratic forms suggests that the situation, in which the group of unit elements of an integral quadratic form of signature $(n, 1)$ contains a subgroup of finite index generated by reflections, is the exception rather than the rule, and can only occur when the value of n is comparatively small and when the discriminant of the quadratic form is small.

Throughout, Λ^n denotes n-dimensional Lobačevskiĭ space, and μ the invariant measure in it.

§1. Convex polyhedra in Lobačevskiĭ space

1. Let V be an $(n+1)$-dimensional vector space over the real number field $\mathbf{R}$, supplied with a scalar product of signature $(n, 1)$. Denote by $\mathfrak{C}$ one of the connected components of the set $\{v \in V: (v, v) < 0\}$. It is an open convex cone, and

$$\{v \in V: (v, v) < 0\} = \mathfrak{C} \cup (-\mathfrak{C}) \tag{4}$$

The space Λ^n will be identified with a set of rays in $\mathfrak{C}$ in such a way that motions of Λ^n are induced by cone-$\mathfrak{C}$-preserving orthogonal transformations of V. Rays on the boundary of $\mathfrak{C}$ may be regarded as points at infinity in Λ^n.

Every k-dimensional plane Π of the space Λ^n is a set of rays lying in the intersection of the cone $\mathfrak{C}$ with a $(k+1)$-dimensional hyperbolic subspace U of V. We shall call U the continuation of the plane Π and write it as $\operatorname{cn} \Pi$.

Denote by Π^+ and Π^- the closed halfspaces bounded by a hyperplane Π of the space Λ^n. Denote by $\operatorname{cn} \Pi^+$ and $\operatorname{cn} \Pi^-$ the closed halfspaces of V bounded by the subspace $\operatorname{cn}\Pi$ and arranged in such an order that rays corresponding to points of Π^- lie in $\operatorname{cn} \Pi^-$.

An intersection

$$P = \bigcap_{i \in I} \Pi_i^- \tag{5}$$

of a family of halfspaces will be called a *convex polyhedron* if 1) every bounded set is cut by only a finite number of hyperplanes Π_i and 2) P contains a nonempty open set.

It can always be assumed that no halfspace Π_i^- contains the intersections of the remainder. This will be assumed henceforth without special proviso. Under this condition, the halfspaces Π_i^- are uniquely defined by the polyhedron P. We shall speak of the hyperplanes Π_i as each *bounding* P.

The convex cone

$$\operatorname{cn} \Pi = \bigcap_{i \in I} \operatorname{cn} \Pi_i^- . \tag{6}$$

will be called the *continuation* of the convex polyhedron P defined by (5).

The polyhedron P will be called *nondegenerate* if 1) the hyperplanes Π_i have no common point (including a point at infinity) and 2) there is no hyperplane orthogonal to all the Π_i.

These conditions may easily be seen to be equivalent to strict convexity of the cone cn P.

Given any vector $e \in V$ satisfying the condition

$$(e, e) > 0, \tag{7}$$

denote by Π_e the hyperplane of Λ^n whose continuation is an n-dimensional subspace orthogonal to e. The notation for the halfspaces bounded by it is such that

$$\operatorname{cn} \Pi_e^- = \{v \in V: (v, e) \leqslant 0\}. \tag{8}$$

Let P be the convex polyhedron given by (5), and e_i, $i \in \mathrm{I}$, vectors of V such that 1) $(e_i, e_i) = 1$ and 2) $\Pi_i^- = \Pi_{e_i}^-$.

The Gram matrix of the vector system $\{e_i\}$ will be called the *Gram matrix of the polyhedron* P and denoted by $G = G(P)$. Its elements g_{ij} satisfy the condition $g_{ij} \leq 1$ and have the following geometric significance:

1) if $g_{ij} \leq -1$, Π_i and Π_j do not intersect; the distance ρ_{ij} between them is given by

$$\cosh \rho_{ij} = -g_{ij}; \tag{9}$$

2) if $|g_{ij}| < 1$, Π_i and Π_j intersect; the angle ϕ_{ij} between them is given by

$$\cos \varphi_{ij} = -g_{ij}. \tag{10}$$

It was shown in [4] that *if no dihedral angle of the polyhedron P exceeds $\pi/2$, hyperplanes corresponding to nonadjacent faces of P will not intersect.* In this case, therefore,

$$g_{ii} = 1; \quad g_{ij} \leqslant 0 \text{ for } i \neq j. \tag{11}$$

If P is nondegenerate, G is a symmetric matrix of signature $(n, 1)$. On the other hand, *every symmetric matrix of signature $(n, 1)$ satisfying conditions* (11) *is the Gram matrix of a convex polyhedron in Λ^n, uniquely defined up to motion* [3].

A convex polyhedron is called *finite* if it is the intersection of a finite number of halfspaces. Obviously every bounded polyhedron is finite.

A finite convex polyhedron $P \subset \Lambda^n$ is bounded if and only if

$$\operatorname{cn} P \subset \mathfrak{C} \cup \{0\}, \tag{12}$$

and has a finite volume if and only if

$$\operatorname{cn} P \subset \overline{\mathfrak{C}}. \tag{13}$$

2. **Definition.** The *complex*, denote it $\mathfrak{F}P$, *of a convex polyhedron* P is the set of its (closed) faces, partially ordered with respect to inclusion. The complex of a polyhedron describes its combinatorial structure.

The complex of a bounded convex polyhedron P in n-dimensional space (Euclidean or Lobačevskiĭ will be called an *n-dimensional closed complex.*

It may easily be shown that *a proper subset of an n-dimensional closed complex cannot be an n-dimensional closed complex*. This fact is useful when determining the combinatorial structure of a polyhedron.

Let P be a nondegenerate finite convex polyhedron in space Λ^n. Then cn P is a strictly convex polyhedral angle, and a hyperplane $H \subset V$ exists such that

$$P_s = \mathrm{cn}\, P \cap H \tag{14}$$

is a bounded convex polyhedron in H. The complex of the polyhedron P_s (independent of the choice of H) will be called the *supercomplex* of P and denoted by $\mathfrak{F}_s P$. It is an n-dimensional closed complex, containing $\mathfrak{F}P$.

A ray L lying on the boundary of the cone $\mathfrak{C}$ will be called a *vertex at infinity* of the polyhedron P if it is a rib of the angle cn P and there exists a neighborhood $O(Q)$ of the point $Q = L \cap H$ in space H such that $P_s \cap O(Q) \subset \mathfrak{C} \cap H$. The partially ordered set which is obtained by adding vertices at infinity to the complex $\mathfrak{F}P$ will be called the *extended complex* of P and denoted by $\mathfrak{F}_w P$. Thus

$$\mathfrak{F}P \subset \mathfrak{F}_w P \subset \mathfrak{F}_s P. \tag{15}$$

A polyhedron P is bounded (has finite volume) if and only if $\mathfrak{F}_s P = \mathfrak{F}P$ ($\mathfrak{F}_s P = \mathfrak{F}_w P$), while this is in turn equivalent to the fact that $\mathfrak{F}P$ ($\mathfrak{F}_w P$) is an n-dimensional closed complex.

Note that, if no dihedral angle of P exceeds $\pi/2$, the complexes $\mathfrak{F}P$ and $\mathfrak{F}_w P$ can be found quite easily from the Gram matrix [3]. Some further notation will be introduced before stating this result precisely.

First, given any face F of the polyhedron P_s, let

$$\sigma(F) = \{i \in \mathbf{I} : e_i \text{ orthogonal to } F\} \tag{16}$$

Obviously σ is a one-to-one mapping of the complex $\mathfrak{F}_s P$ onto some set of subsets of the the set $\mathbf{I}$. In addition, σ is an anti-isomorphism with respect to inclusion. Hence the complexes $\mathfrak{F}P$, $\mathfrak{F}_w P$ and $\mathfrak{F}_s P$ are anti-isomorphic to the partially ordered sets

$$\mathfrak{f}P = \sigma(\mathfrak{F}P), \quad \mathfrak{f}_w P = \sigma(\mathfrak{F}_w P), \quad \mathfrak{f}_s P = \sigma(\mathfrak{F}_s P), \tag{17}$$

formed from subsets of the set $\mathbf{I}$.

Next, a matrix A will be called the *direct sum* of matrices $A_1, \cdots, A_k$ if it is reducible to the form

$$\begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ & & \ddots \\ 0 & & A_k \end{pmatrix}$$

by a permutation of its rows and the same permutation of its columns. In this case we write $A = A_1 \oplus \cdots \oplus A_k$. A matrix which is not representable as the direct sum of two nonempty matrices will be termed *irreducible*.

Every symmetric matrix A is uniquely expandable into a direct sum of irreducible matrices, which we shall call its *components*. Denote by A^+ (A^0) the direct sum of all the positive definite (degenerate nonnegative definite) components of A, and by A^- the direct sum of all components which are not nonnegative definite.

Finally, given any matrix $A = (a_{ij})$ $(i, j \in \mathbf{I})$ and any set $S \subset \mathbf{I}$, denote by A_S the principal submatrix of the matrix A formed by elements a_{ij}, $i, j \in S$.

Using this notation, we have

Theorem. *If no dihedral angle of the polyhedron P exceeds $\pi/2$, then*

1) $S \in \mathfrak{f}P \Leftrightarrow G_S = G_S^+$,

2) $S \in \mathfrak{f}_w P \setminus \mathfrak{f}P \Leftrightarrow G_S = G_S^0$ *and the rank of* $G_S = n - 1$.

If the matrix G_S is a positive definite, the face of P corresponding to the set S has $n - s$ dimensions, where s is the number of elements in S.

3. Let G be an irreducible symmetric matrix satisfying condition (11). It may be either nonnegative definite or indeterminate. It is well known (see, for example, [1]) that, if G is positive definite, all the elements of the matrix G^{-1} must be positive. If G is nonnegative definite and degenerate, its rows are connected by a linear dependence with positive coefficients, and all its proper principal submatrices are positive definite.

A symmetric matrix G satisfying (11) will be termed *critical* if it is not positive definite, yet every proper principal submatrix of it is positive definite. A critical matrix must obviously be irreducible. It may be either degenerate nonnegative definite, or indeterminate. The value of the concept of critical matrix will be clear from the next subsection.

4. Let $P \subset \Lambda^n$ be a nondegenerate finite convex polyhedron, no dihedral angle of which exceeds $\pi/2$, and let G be its Gram matrix.

A polyhedron P can be shown to be bounded or of finite volume without completely determining its combinatorial structure. This may be seen as follows.

Recalling the notation of subsection 2, let

$$K = \operatorname{cn} P \tag{18}$$

and for any set $S \subset \mathbf{I}$, let

$$K_S = \{v \in K : (e_i, v) = 0 \text{ for } i \in S\}. \tag{19}$$

Since

$$K = \{v \in V : (e_i, v) \leqslant 0 \text{ for all } i \in \mathbf{I}\}, \tag{20}$$

the vectors e_i cannot be connected by a nontrivial linear dependence with nonnegative coefficients. Hence, if $G_S = G_S^0$ and a_i, $i \in S$, are positive numbers, representing the coefficients of the linear dependence between the rows of matrix G_S, the vector

$$e^S = \sum_{i \in S} a_i e_i \tag{21}$$

must be nonzero. On the other hand, it is clear that

$$(e^S, e^S) = 0. \tag{22}$$

Proposition 1. *The necessary and sufficient condition for the polyhedron P to have finite volume is that, for any critical (see subsection 3) principal submatrix G_S of the matrix G, either*

(a) *if $G_S = G_S^0$, then a $T \supset S$ exists such that $G_T = G_T^0$ and* rank $G_T = n - 1$, *or*

(b) *if $G_S = G_S^-$, then $K_S = \{0\}$.*

The necessary and sufficient condition for P to be bounded is that the matrix G contain no degenerate nonnegative definite principal submatrices, while any indeterminate critical principal submatrix of G satisfies condition (b).

Proof. Necessity. Let $\mu(P) < \infty$ and let G_S be a critical principal submatrix of G. If G_S is semidefinite, consider the vector e^S defined by (21). Clearly $e^S \in K$, so that e^S corresponds to a vertex Q at infinity of the polyhedron P. Take $T = \sigma(Q)$. Obviously $T \supset S$. By the theorem of subsection 2, $G_T = G_T^0$ and rank $G_T = n - 1$. If the polyhedron P is bounded, this case is generally impossible. Now take the case when G_S is indeterminate. Then G_S cannot be contained in a semidefinite principal submatrix, i. e. K_S cannot contain a face of the angle K other than $\{0\}$. Hence $K_S = \{0\}$.

Sufficiency. Let conditions (a) and (b) be satisfied. Recalling the notation of subsection 2, let Q be a vertex of the polyhedron P_s not lying in $\mathfrak{C} \cap H$. In that case the matrix $G_{\sigma(Q)}$ is not positive definite, and consequently contains a critical principal submatrix G_S. Since $K_S \neq \{0\}$, the matrix G_S is semidefinite. Hence a $T \supset S$ exists such that $G_T = G_T^0$ and rank $G_T = n - 1$. We have $T = \sigma(\tilde{Q})$, where $\tilde{Q}$ is a vertex at infinity of the polyhedron P. Any face of P_s which contains $\tilde{Q}$ and is different from $\tilde{Q}$ belongs to $\mathfrak{F}P$, and hence corresponds to a positive definite principal submatrix. Since G_S cannot be contained in a positive definite principal submatrix, $K_S = K_T$. Obviously, on the other hand, $Q \in K_S$. Hence $Q = \tilde{Q}$.

In short, all the vertices of the polyhedron P_s lie in $\overline{\mathfrak{C}} \cap H$. Hence $\mu(P) < \infty$. But if G contains no semidefinite critical principal submatrices, the previous argument shows that $P_s \subset \mathfrak{C} \cap H$, i. e. the polyhedron P is bounded. QED.

Note the fact, proved in [3], that the matrix G must be irreducible in order for P to have finite volume.

Verification of condition (b) in Proposition 1 is sometimes simplified by

Proposition 2. *Let the Gram matrix G of the polyhedron P be irreducible. If S and $T \subset$ I are such that*

$$G_{S \cup T} = G_S \oplus G_T, \qquad G_T = G_T^+, \tag{23}$$

then

$$K_{S \cup T} = \{0\} \Rightarrow K_S = \{0\}. \tag{24}$$

Proof. It may be assumed that the matrix G_T is irreducible. The equation $K_{S \cup T} = \{0\}$ means that there exists a linear dependence $\sum c_i e_i = 0$ in which $c_i > 0$ for $i \notin S \cup T$. It is easily seen that

$$\sum_{i \in T} c_i (e_i, e_j) \geqslant 0 \quad \text{for all} \quad j \in T, \tag{25}$$

where the sign of equality cannot hold for all $j \in T$, since otherwise G_T would be separable as a direct summation term in G. Since all the elements of the matrix G_T^{-1} are positive, it follows from (25) that $c_i > 0$ for $i \in T$. Hence $c_i > 0$ for all $i \notin S$, i. e. $K_S = \{0\}$.

§2. Discrete groups generated by reflections

1. Let X^n be an n-dimensional simply-connected space of constant curvature, i. e. Euclidean space E^n, the sphere S^n or Lobačevskiĭ space Λ^n.

Further, let Γ be a discrete group of motions of X^n generated by reflections (in hyperplanes). The mirrors of all the reflections belonging to Γ divide X^n into convex polyhedra, which will be termed the *cells* of the group Γ. The cells are moved transitively by Γ, and every cell is a fundamental region for it. Let P be a cell, let P_i $(i \in \mathrm{I})$ be its $(n-1)$-dimensional faces, and let R_i be the reflection in the hyperplane containing P_i. The angle between any pair of adjacent faces P_i, P_j of P is the form π/n_{ij}, where $n_{ij} \in \mathbf{Z}$. If the faces P_i and P_j are not adjacent, we put $n_{ij} = \infty$. In this notation the group Γ is generated by reflections R_i $(i \in \mathrm{I})$, with the defining relations

$$R_i^2 = 1, \quad (R_iR_j)^{n_{ij}} = 1, \tag{26}$$

i. e. it is an abstract Coxeter group with exponents n_{ij}.

Conversely, let $P \subset X^n$ be a convex polyhedron all the dihedral angles of which are unit fractions of π. Then the group Γ generated by reflections in the hyperplanes of the $(n-1)$-dimensional faces of P is discrete, and P is a cell of the group.

All these assertions may be proved in the same way as in [1] for the space E^n, with minor complications in the case when the polyhedron P is infinite.

2. The Coxeter group Γ with generators R_i $(i \in \mathrm{I})$ and exponents n_{ij} is decribed by a *Coxeter graph*, which is constructed as follows. For each $i \in \mathrm{I}$ the graph has a corresponding node v_i. If $n_{ij} < \infty$, the nodes v_i and v_j are joined either by an $(n_{ij}-2)$-tuple branch, or by a simple branch marked n_{ij}. If $n_{ij} = \infty$, v_i and v_j are joined by a boldface branch, or by a simple branch marked ∞.

When Γ is a discrete group of motions of Λ^n generated by reflections, let us stipulate the following modification of the Coxeter graph used in [3]. If $n_{ij} = \infty$, nodes v_i and v_j will be joined by a boldface branch or by a simple branch with the ∞ mark only if $g_{ij} = -1$; otherwise they will be joined by a broken-line branch, marked $-g_{ij}$ if necessary.

3. With the abstract Coxeter group Γ with exponents n_{ij} will be associated the *cosine matrix*

$$\operatorname{Cos}\Gamma = \left(-\cos\frac{\pi}{n_{ij}}\right), \tag{27}$$

where we take $n_{ii} = 1$ and $\pi/n_{ij} = 0$ when $n_{ij} = \infty$.

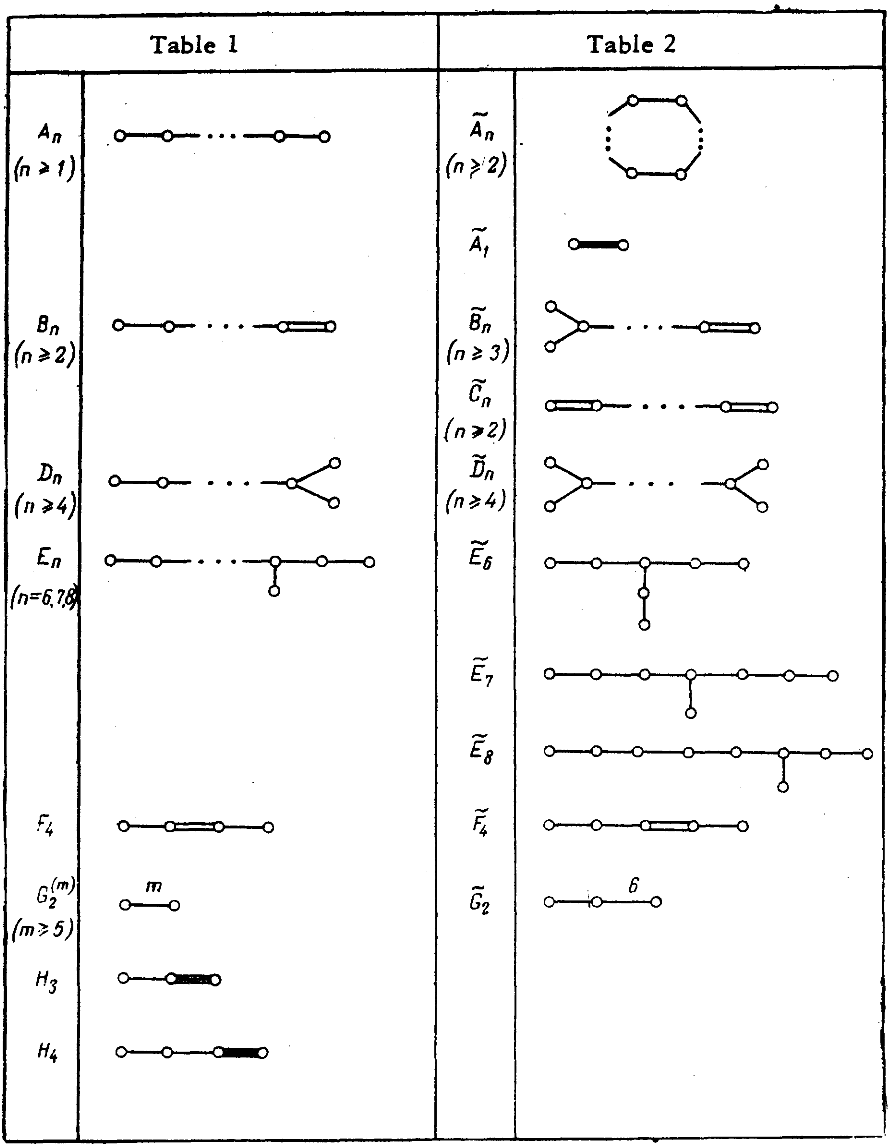
The group Γ is well known to be finite if and only if the matrix $\operatorname{Cos}\Gamma$ is positive definite, i. e., in our notation,

$$\operatorname{Cos}\Gamma = (\operatorname{Cos}\Gamma)^+. \tag{28}$$

The Coxeter group Γ will be called *parabolic* if it is isomorphic (as a group with the indicated system of generators) with a discrete group of motions of E^n generated by reflections and having a bounded cell. The group Γ is well known to be parabolic if and only if

$$\operatorname{Cos}\Gamma = (\operatorname{Cos}\Gamma)^0. \tag{29}$$

Every finite or parabolic Coxeter group is the direct product of a certain number of irreducible groups of the same type. The graphs of irreducible finite and parabolic Coxeter groups are given in Tables 1 and 2. The index in the graph notation is equal to the rank of the cosine matrix, i. e., to the number of generators in the case of a finite group or to this number minus one in the case of a parabolic group.

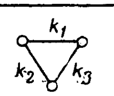
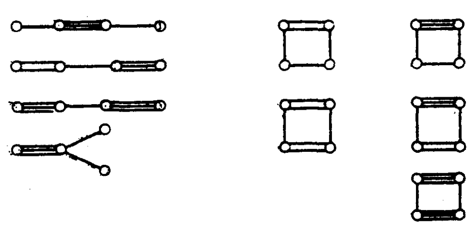
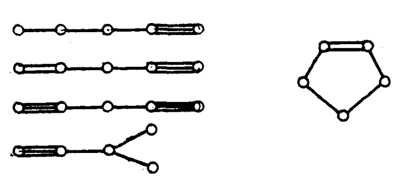


The graph of a finite (parabolic) Coxeter group will be referred to as an *elliptic* (*parabolic*) *graph*. It is clear from the above that a graph is elliptic (parabolic) if and only if all its connected components are included in Table 1 (Table 2). The difference between the number of nodes and number of connected components of a parabolic graph, i. e. the sum of the indices in thr notation of its connected components, will be termed the *rank* of the graph.

A Coxeter group Γ group will be termed a *Lannér* group if it is isomorphic with a discrete group of motions of Λ^n, generated by reflections and having a bounded simplex as its cell. These groups are due to Lannér [2]; they only exist when $n \leq 4$.

The Coxeter group Γ is easily seen to be a Lannér group if and only if $\operatorname{Cos}\Gamma$ is an indeterminate critical matrix (see §1.3). The graphs of Lannér groups (to be referred to as *Lannér graphs*) are given in Table 3.

Table 3

$n=2$	 $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}<1$
$n=3$	
$n=4$	

4. Let Γ be a discrete group of motions of Λ^n, generated by reflections, let P be a cell, P_i $(i \in \mathrm{I})$ the $(n-1)$-dimensional faces of P, $G=(g_{ij})$ the Gram matrix of P, R_i a reflection in the hyperplane of face P_i, and Σ the graph of Γ.

Given any subset $S \subset \mathrm{I}$, denote by Γ_S the subgroup of Γ generated by reflections R_i, $i \in S$. Its graph is the subgraph Σ_S of Σ formed by the nodes v_i, $i \in S$. If Σ_S contains no broken-line branches, then

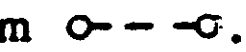
$$G_S=\operatorname{Cos}\Gamma_S. \tag{30}$$

In view of this, $G_S=G_S^+$ $(G_S=G_S^0)$ if and only if Σ_S is an elliptic (parabolic) graph. If $G_S=G_S^0$, the rank of G_S is equal to the rank of Σ_S.

The matrix G_S is critical in the following three cases:

1) Σ_S is a connected parabolic graph,

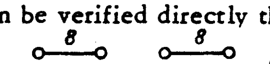
2) Σ_S is a Lannér graph,

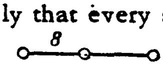
3) Σ_S is a graph of the form .

In case 1) the matrix G_S is semidefinite, while it is indeterminate in the other two.

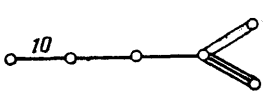
5. Examples. 1. Let Γ be the Coxeter group defined by the graph

(31)

Put $G = \text{Cos}\,\Gamma$. It can be verified directly that $\det G = 0$. Since the graph (31) contains the elliptic subgraph , the positive index of inertia of the matrix G is not less than 4. On the other hand, since graph (31) is not parabolic, G cannot be semidefinite. Consequently it has the signature (4,1) and is the Gram matrix of a convex polyhedron P in Λ^4. The group Γ is realizable as a discrete group of motions of Λ^4, generated by reflections and having the polyhedron P as a cell.

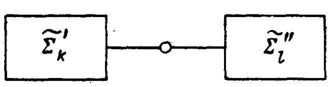
It can be seen immediately that every subgraph of graph (31) containing neither of the two subgraphs of the form  is elliptic. The partially ordered set formed by such subgraphs is anti-isomorphic to the complex of the direct product of two triangles. The polyhedron P is thus bounded and has the combinatorial structure of a direct product of two triangles (see § 1.2).

All the above is applicable, with obvious modifications, to the Coxeter group defined by the graph



(32)

2. Let $\tilde{\Sigma}'_k$ and $\tilde{\Sigma}''_l$ be connected parabolic graphs. Consider the Coxeter group Γ whose graph is

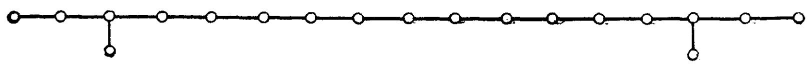


(33)

where the central node is joined by a simple branch with just one node each of the subgraphs $\tilde{\Sigma}'_k$ and $\tilde{\Sigma}''_l$. It can be verified immediately that the matrix $G = \text{Cos}\,\Gamma$ is degenerate and has the signature $(k + l + 1,\ 1)$. Hence Γ is realizable as a discrete group generated by reflections in Λ^{k+l+1}. Denote its cell by P.

The subgraph Σ_0 of graph (33), obtained by discarding the central node, corresponds to a vertex Q_0 at infinity of the polyhedron P. Corresponding to the ribs issuing from Q_0 we have subgraphs of Σ_0 which are obtained by discarding a node from each of its connected components. The polyhedron P_s (see (14)) is a pyramid with vertex at Q_0, constructed on the direct product of k-dimensional and l-dimensional simplexes. The subgraphs corresponding to the vertices other than Q_0 of P_s are obtained by discarding a node from each of $\tilde{\Sigma}'_k$ and $\tilde{\Sigma}''_l$ in the graph (33).

It is easy to devise examples of graphs (33) for which $\mu(P) < \infty$. The record graph from the point of view of the sum $k + l$ is

 , (34)

where $\widetilde{\Sigma}_k' = \widetilde{\Sigma}_l'' = \widetilde{E}_8$, $k + l + 1 = 17$.

§3. Maximum subgroups generated by reflections

1. Let Θ be a discrete group of motions of Λ^n. Denote by Γ the subgroup generated by all reflections belonging to Θ, by P a cell of this subgroup (see §2.1), and by Sym P the symmetry group of P.

Proposition 3. *In the above notation, the group Θ decomposes into a semidirect product*

$$\Theta = \Gamma \cdot \mathrm{H}, \tag{35}$$

where $\mathrm{H} \subset \mathrm{Sym}\ P$.

Proof. Clearly Γ is a normal divisor in Θ and the system of mirrors of Γ is invariant under Θ. Given any $\theta \in \Theta$, the polyhedron $\theta(P)$ is a cell of Γ. Hence an element $\gamma \in \Gamma$ exists such that $\theta(P) = \gamma(P)$. Hence

$$\theta = \gamma\eta, \text{ where } \gamma \in \Gamma,\ \eta \in \mathrm{Sym}\, P. \tag{36}$$

It remains to observe that the expansion (36) is unique, since $\Gamma \cap \mathrm{Sym}\ P = \{\mathrm{id}\}$.

2. An algorithm will be devised, in the notation of subsection 1, for constructing cells of group Γ'.

The model of Λ^n described in §1.1 will be used for this purpose. In addition to the notation of §1, given any vector $e \in V$ satisfying the condition $(e, e) > 0$, denote by R_e a reflection in the hyperplane Π_e of space Λ^n.

Take an arbitrary point $p_0 \in \Lambda^n$ and denote by Γ_0 the subgroup of Γ generated by all reflections whose mirrors pass through p_0. This subgroup will be a finite group with $k < n$ generators. Let P_0 be a cell of the subgroup.

There is a unique cell of Γ which is the same as P_0 in a neighborhood of the point p_0. Call it P.

Consider the vector set

$$\mathfrak{R} = \{e \in V\colon (e, e) > 0 \ \&\ R_e \in \Theta\} \tag{37}$$

and form a sequence

$$e_1, e_2, \ldots \in \mathfrak{R} \tag{38}$$

in accordance with the following rules:

1) $e_1, \cdots, e_k$ are found from the condition

$$P_0 = \bigcap_{i=1}^{k} \Pi_{e_i}^{-}\,. \tag{39}$$

2) When $l > k$, we select e_l from among the vectors $e \in \mathfrak{R}$ satisfying the condition

$$(e, e_i) \leqslant 0 \quad \text{for all} \quad i<l, \tag{40}$$

in such a way as to minimize the distance $\rho(p_0, \Pi_{e_l})$ from the point p_0 to the hyperplane Π_{e_l}.

3) For all i, the vector e_i is oriented in such a way that

$$p_0 \in \Pi_{e_l}^{-}. \tag{41}$$

Note that if v_0 is a vector along the ray corresponding to the point p_0, the condition $p_0 \in \Pi_e^-$ implies that

$$(e, v_0) \leqslant 0, \tag{42}$$

while the distance $\rho(p_0, \Pi_e)$ is given by

$$\sinh^2 \rho(p_0, \Pi_e) = -\frac{(e, v_0)^2}{(e, e)(v_0, v_0)}, \tag{43}$$

so that $\rho(p_0, \Pi_e)$ and

$$\nu(e) = \frac{(e, v_0)^2}{(e, e)} \tag{44}$$

are minimized simultaneously.

Proposition 4. $P = \bigcap_{i=1} \Pi_{e_i}$. *(The upper limit of* i *is equal to the length of the sequence* (38); *in particular, it may be infinite.)*

Proof. It will be shown first that P is bounded by each of the hyperplanes Π_{e_i}. Assume the contrary, and let e_l be the first vector in the sequence (38) such that Π_{e_l} does not bound P. By construction, $\Pi_{e_l} \not\ni p_0$. Denote by q the projection of the point p_0 on Π_{e_l}.

Let Π_e be a hyperplane bounding P and having at least one point in common with the segment $p_0 q$. It will be assumed that $P \subset \Pi_e^-$. Clearly

$$\rho(p_0, \Pi_e) < \rho(p_0, \Pi_{e_l}). \tag{45}$$

This means that the vector e does not satisfy condition (49); but by hypothesis the hyperplanes Π_{e_i}, $i < l$, bound P, so that this is only possible if $\Pi_e = \Pi_{e_j}$ for some $j < l$. Hence

$$(e, e_l) \leqslant 0. \tag{46}$$

Take the two-dimensional plane Π_0 passing through the points p_0 and q and orthogonal to Π_e. From (46), the section

$$\Sigma = \Pi_e^- \cap \Pi_{e_l}^- \cap \Pi_0 \tag{47}$$

is either an acute angle or a strip between nonintersecting straight lines in the plane Π_0. Hence $q \in \Pi_e^-$; and if $p_0 \notin \Pi_e$, then $q \notin \Pi_e$ also. This conclusion also holds trivially in the case when Π_e has no points in common with the segment p_0q.

In view of this, $q \in P$; and a neighborhood $O(q)$ of the point q exists such that

$$O(q) \cap P_0 \subset P. \tag{48}$$

Hence Π_{e_l} contains interior points of P; but this contradicts the fact that P is a cell of the group Γ.

In short, P is bounded by the hyperplanes Π_{e_i}. Now let Π_e be any hyperplane bounding P, where $P \subset \Pi_e^-$. If Π_e is not the same as one of the hyperplanes Π_{e_i}, then $(e, e_i) \leq 0$ for all i, and the sequence (38) cannot break off, since e can always be taken as the next vector. On the other hand, we must have in this case

$$\rho(p_0, \Pi_{e_i}) \leqslant \rho(p_0, \Pi_e) \quad \text{for all} \quad i. \tag{49}$$

This contradicts the discrete nature of the group Θ. Hence $\Pi_e = \Pi_{e_i}$ for some i. This completes the proof.

Note. It is clear from the proof that Proposition 4 remains in force if, when constructing the sequence (38), we confine ourselves in condition (40) only to those i for which $\rho(p_0, \Pi_{e_i}) < \rho(p_0, \Pi_e)$.

Retaining the above notation, put

$$P^{(m)} = \bigcap_{i=1}^{m} \Pi_{e_i}^- \tag{50}$$

for any m not exceeding the length of sequence (38).

Proposition 5. *If* $\mu(P^{(m)} < \infty$, *the sequence* (38) *breaks off at the mth term, and hence* $P^{(m)} = P$.

Proof. Suppose that sequence (38) does not break off at the mth term. Then the Gram matrix of the polyhedron $P^{(m)}$ is a principal submatrix of the Gram matrix of $P^{(m+1)}$. But the principal submatrices of the Gram matrices of $P^{(m)}$ and $P^{(m+1)}$ are characterized by their own internal properties when they correspond to vertices, including those at infinity; so that every vertex of $P^{(m)}$ is at the same time a vertex of $P^{(m+1)}$. Hence $P^{(m+1)} = P^{(m)}$, which is obviously impossible.

§4. Groups of unit elements of quadratic forms (1) and (3)

1. Let $\{v_0, \cdots, v_n\}$ be a basis of the space V, in which the scalar square is expressed by the quadratic form (1). Also, let L be a lattice stretched over $v_0, \cdots, v_n$, and Θ a group of orthogonal tranformations of V preserving the lattice L and mapping the cone $\mathfrak{C}$ into itself. Then the group of unit elements of the form (1) is the direct product $\Theta \times \{1, -1\}$.

The algorithm described in §3.2 will be applied to the group Θ.

First consider the conditions to be satisfied by a vector $e \in V$ in order for $R_e \in \Theta$. Obviously it must be proportional to a vector with rational components. Further, it can be normalized in such a way that its components are integers and relatively prime.

Let

$$e=\sum_{i=0}^{n} k_i v_i, \quad k_i \in \mathbf{Z}, \quad \text{g. c. d. } \{k_i\}=1. \tag{51}$$

Then

$$R_e v_j = v_j \pm \frac{2k_j}{(e,\, e)}\, e, \tag{52}$$

and the condition $R_e \in \Theta$ is equivalent to the fact that

$$\frac{2k_j}{(e,\, e)} \in \mathbf{Z} \quad \text{for all} \quad j. \tag{53}$$

Since the k_j are relatively prime, (53) only holds when

$$(e,\, e)=1 \quad \text{or} \quad 2. \tag{54}$$

Take as p_0 the point of Λ^n corresponding to the vector v_0. The group Γ_0 will consist of all possible permutations of the vectors $v_1, \cdots, v_n$, with some of them multiplied by -1. Take as P_0 the polyhedral angle whose continuation is given by the inequalities

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0. \tag{55}$$

The first vectors in the sequence (38) will be

$$\begin{aligned} e_i &= -v_i + v_{i+1}, \quad i=1, \ldots, n-1, \\ e_n &= -v_n. \end{aligned} \tag{56}$$

The subsequent vectors are chosen from among those of type (51), while condition (54) written as the equation

$$k_1^2 + k_2^2 + \ldots + k_n^2 = k_0^2 + \varepsilon, \text{ where } \varepsilon = 1 \quad \text{or} \quad 2, \tag{57}$$

and conditions (42) and $(e,\, e_i) \leq 0$, $i=1, \cdots, n$, as

$$k_0 \geqslant 0, \quad k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n \geqslant 0. \tag{58}$$

Expression (44) for $\nu(e)$ becomes

$$\nu(e) = k_0^2/\varepsilon. \tag{59}$$

Construction of the sequence (38) thus reduces to writing down those solutions of the Diophantine equation (57) that satisfy conditions (58) in increasing order of the quantity (59), and in checking conditions (49) for each solution as it is written down. Table 4 gives the results for $n \leq 17$. (It will be seen below that the sequence (38) is in fact exhausted by the vectors written.)

Let m be the number of the last vector e_i written in Table 4 (for a fixed n). The polyhedron $P^{(m)}$ defined by (50) is a cell of the group $\Gamma^{(m)}$, generated by reflections $R_{e_1}, \cdots, R_{e_m}$. Table 5 illustrates the graph of $\Gamma^{(m)}$ for each $n \leq 17$ (see §2.2); the

Table 4

i	e_i	ε	n	$\frac{k_0^2}{\varepsilon}$
$n+1$	$v_0+v_1+v_2$	1	2	1
	$v_0+v_1+v_2+v_3$	2	$\geqslant 3$	0,5
$n+2$	$v_0+v_1+\ldots+v_{10}$	1	10	9
	$v_0+v_1+\ldots+v_{11}$	2	$\geqslant 11$	4,5
$n+3$	$4v_0+2v_1+v_2+\ldots+v_{14}$	1	14	16
	$4v_0+2v_1+v_2+\ldots+v_{15}$	2	$\geqslant 15$	8
$n+4$	$6v_0+2(v_1+\ldots+v_7)+v_8+\ldots+v_{16}$	1	16	36
	$4v_0+v_1+\ldots+v_{17}$	1	17	16
$n+5$	$6v_0+2(v_1+\ldots+v_7)+v_8+\ldots+v_{17}$	2	17	18

number of the corresponding vector e_i is written next to each node. Corresponding to the vectors $e_1, \cdots, e_n$ we have the type B_n elliptic subgraph

(60)

The results of §§1.4 and 2.4 will be used to show that $\mu(P^{(m)}) < \infty$. To this end, note first the graph of group $\Gamma^{(m)}$ contains no Lannér subgraphs and no broken-line branches. The conditions of Proposition 1 thus reduce to the requirement that every connected parabolic subgraph be included (as a connected component) in a parabolic subgraph of rank $n-1$. This may be verified by inspection for each graph of Table 5. Hence $\mu(P^{(m)}) < \infty$, and it can be concluded from Proposition 5 that $P^{(m)} = P$ and $\Gamma^{(m)} = \Gamma$.

With $n \leq 9$, P is a simplex, while with $10 \leq n \leq 13$ it is a pyramid, constructed on the direct product of 8-dimensional and $(n-9)$-dimensional simplexes; the base of the pyramid corresponds to node number 9 of the graph (see Example 2 of §2.5).

By Propostion 3, the group Θ may be decomposed into the semidirect product (35). Obviously the group Sym P is naturally isomorphic with the symmetry group of a Coxeter graph. This means, in particular, that the group H is trivial when $n \leq 13$. Let us show that for all $n \leq 17$

$$\mathrm{H} = \mathrm{Sym}\, P. \tag{61}$$

It is clear from (56) and Table 4 that the lattice L is generated by the vectors $e_1, \cdots, e_m$. Given any $\eta \in \mathrm{Sym}\, P$,

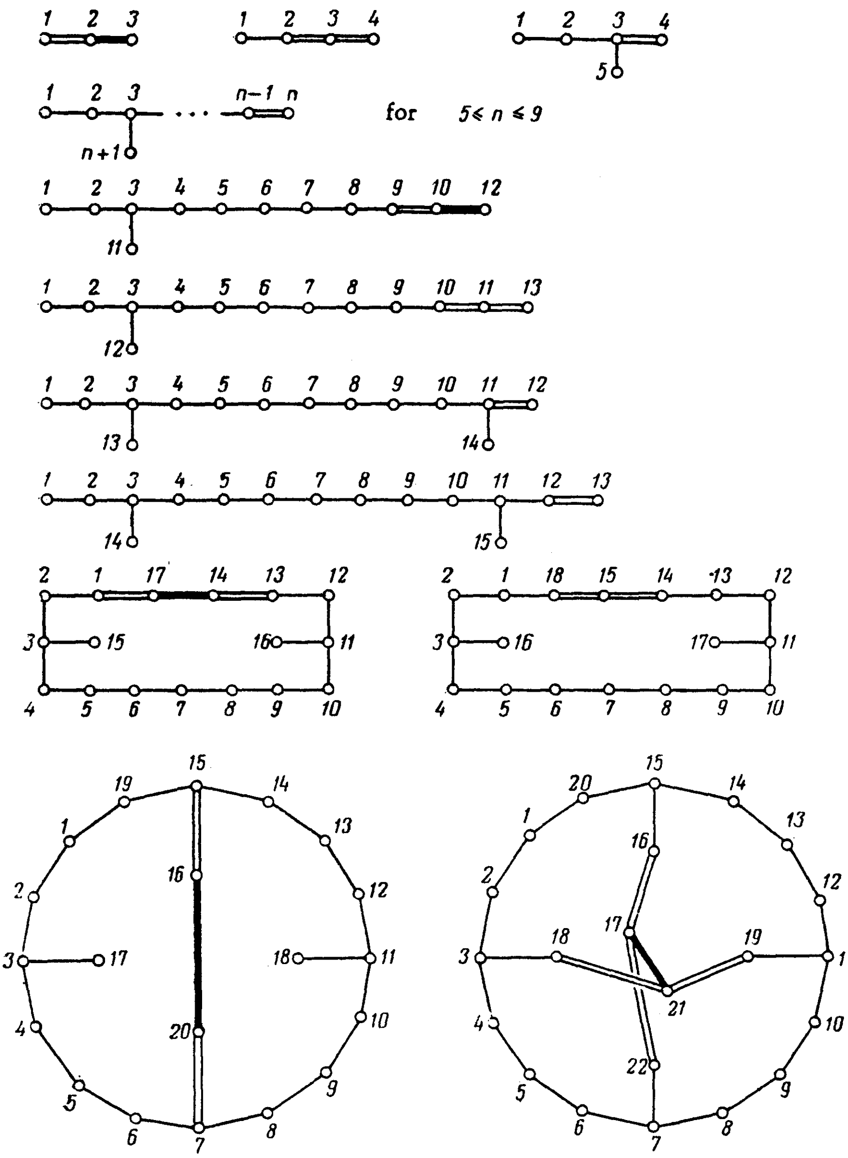
$$\eta(e_i) = c_i e_{\sigma(i)}, \quad c_i > 0, \tag{62}$$

where σ is a permutation. By direct inspection,

$$(e_{\sigma(i)}, e_{\sigma(i)}) = (e_i, e_i). \tag{63}$$

Hence $c_i = 1$ for all i. Consequently $\eta(L) \subset L$, i. e. $\eta \in \mathrm{H}$.

Table 5



2. A similar procedure may be followed for the group of unit elements of the quadratic form (3). Instead of repeating the entire discussion of subsection 1, we shall merely indicate the modifications needed when the form (1) is replaced throughout by form (3).

Table 6

i	e_i	ε	n	$\frac{k_0^2}{\varepsilon}$
$n+1$	v_0+2v_1	2	$\geqslant 2$	0,5
$n+2$	$v_0+v_1+v_2+v_3$	1	3	1
	$v_0+v_1+v_2+v_3+v_4$	2	$\geqslant 4$	0,5
$n+3$	$2v_0+v_1+\ldots+v_9$	1	9	4
	$2v_0+v_1+\ldots+v_{10}$	2	$\geqslant 10$	2
$n+4$	$3v_0+3v_1+v_2+\ldots+v_{11}$	1	11	9
	$3v_0+3v_1+v_2+\ldots+v_{12}$	2	$\geqslant 12$	4,5
$n+5$	$3v_0+2(v_1+v_2)+v_3+\ldots+v_{13}$	1	13	9
	$3v_0+2(v_1+v_2)+v_3+\ldots+v_{14}$	2	14	4,5
$n+6$	$5v_0+2(v_1+\ldots+v_{13})$	2	$\geqslant 13$	12,5

Equation (52) is replaced by

$$R_e v_j = \begin{cases} v_j - \dfrac{2k_j}{(e,e)}e & \text{for } j>0, \\ v_0 + \dfrac{4k_0}{(e,e)}e & \text{for } j=0, \end{cases} \tag{64}$$

and we get the new possibility that $(e, e) = 4$ provided that all the k_j, $j > 0$, are even. This possibility is not actually realized, however, since, if all the k_j are even, then k_0 is odd, and hence (e, e) is not divisible by 4. Hence (54) remains in force.

The Diophantine equation (57) becomes

$$k_1^2+k_2^2+\ldots+k_n^2 = 2k_0^2+\varepsilon, \text{ where } \varepsilon = 1 \text{ or } 2. \tag{65}$$

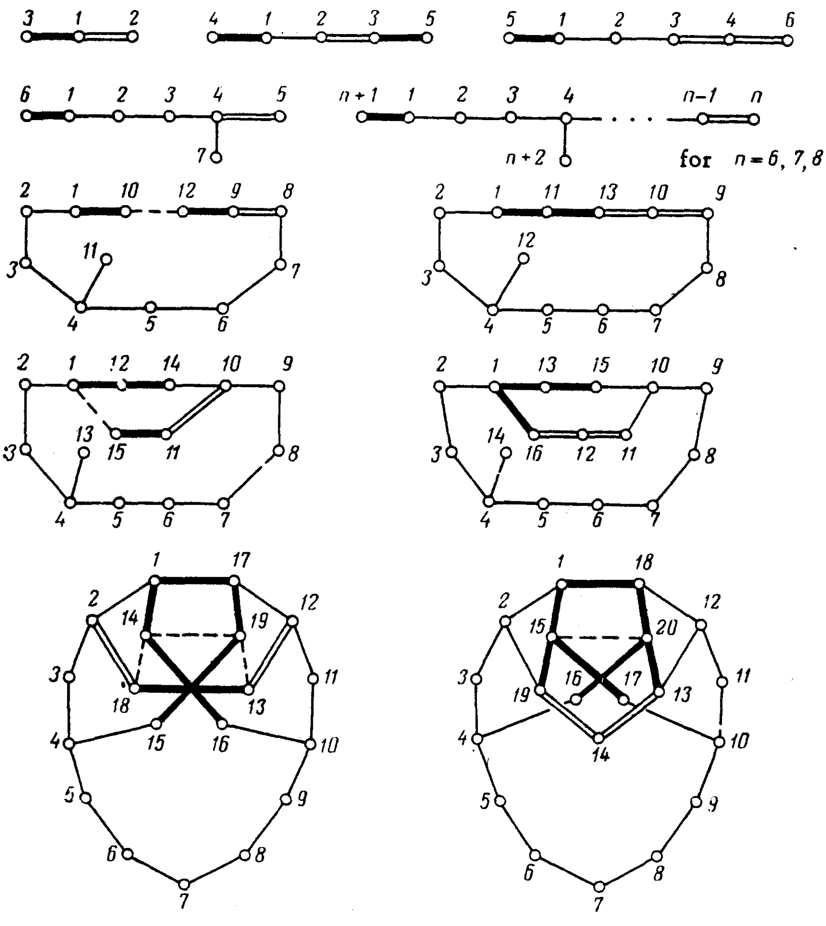
The unimportant factor 4 appears in expression (59) for $\nu(e)$.

Table 6 gives the terms of sequence (38), with subscripts $i > n$, evaluated for the group of unit elements of the quadratic form (3) with $n \leq 14$. Table 7 illustrates, for every $n \leq 14$, the graph of group $\Gamma^{(m)}$, where m is the number of the last of the vectors e_i written in Table 6.

When $n \leq 8$, or $n = 10$ or 12, the proof that $\mu(P^{(m)}) < \infty$ is exactly the same as for the form (1). When $n = 9$, 11, 13 or 14, further analysis is needed, due to the presence of broken-line branches in the graph. To every broken-line branch there corresponds an indeterminate critical principal submatrix of the Gram matrix (see §2.4), for which condition (b) of Proposition 1 has to be checked. Proposition 2 may be used for this purpose.

Let the ith and jth nodes of the graph be joined by a broken-line branch. Put $S = \{i, j\}$, and denote by T the set of all the k such that the kth node is joined with neither the ith nor the jth. In all the cases in question, Σ_T (see notation of §2.4) is an elliptic graph, i. e. $G_T = G_T^+$. By Proposition 2, to show that $K_S = \{0\}$, it only needs to be verfied that $K_{S \cup T} = \{0\}$.

Table 7 *



With $n = 9$ and 11, and in the two cases with $n = 13$, T contains $n - 1$ elements. In these cases, therefore,

$$\operatorname{rank} G_{S \cup T} = \operatorname{rank} G_S + \operatorname{rank} G_T = n + 1, \tag{66}$$

i. e., $K_{SUT} = \{0\}$.

With $n = 13$ and 14, the set T contains only $n - 2$ elements in the case $S = \{n + 1, n + 6\}$. Consider the set $U = \{n + 1\} \cup T$. Obviously Σ_U is an elliptic subgraph. Hence $U \in \mathfrak{f}K$, while $\dim K_U = 2$. It can easily be seen that when any node numbered 13 or $n + 4$ is added to Σ_U, an elliptic subgraph is obtained. Consequently the sets corresponding to the two ribs of face K_U are $U \cup \{13\}$ and $U \cup \{n + 4\}$. Since neither of these sets contains $S \cup T = \{n + 6\} \cup U$, we have $K_{S \cup T} = \{0\}$.

Hence $P^{(m)} = P$ and $\Gamma^{(m)} = \Gamma$.

When $n = 2$, P is a triangle, while if $3 \leq n \leq 8$ it is a pyramid constructed on a simplicial prism, with its base corresponding to graph node number 2.

Equation (61) is proved in the same way as in subsection 1. When $n \leq 12$, the group H is trivial.

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