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GROUPS OF HOMEOMORPHISMS OF A SIMPLY CONNECTED SPACE

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1. Introduction

The theorem proved in this paper aims to justify a technique, often applied by classical writers on discontinuous groups, for deriving abstract definitions of a group by studying the transformations which map a fundamental region on its neighbours. The method is the basis, for instance, of the complete classification given by Fricke and Klein [3] of fuchsian groups with compact orbit space and certain other finitely generated fuchsian groups. An account of the method can be found in [2] where several applications are given. There seems to be some doubt how far the method is applicable to spaces of higher dimension than two and, for this reason, it seemed a good idea to attempt a rigorous analysis. Since the groups concerned include fundamental groups of compact manifolds, the main theorem is closely related to the isomorphism of the topological and combinatorial fundamental groups of a complex, and the idea behind the proof is a form of simplical approximation.

The most interesting applications of the result appear to be to discontinuous groups, in cases when the presentation obtained is finite. In its most general form, however, applied to a "G-covering" instead of a fundamental region, the theorem is valid for any group of homeomorphisms, discontinuous or not. For discontinuous groups of isometries of a metric space, the G-covering can be replaced by a fundamental region which is normal in the sense defined by Siegel [8]. The assumption that such a fundamental region exists does, of course, restrict the class of groups, but this restriction does not appear to exclude any of the classical applications.

A different method of justifying the classical technique of obtaining abstract definitions is due to Gerstenhaber [12] and Behr [11]. Behr states his result in terms of closed sets instead of open sets which are G-coverings, and his results are very similar to those in § 6. He makes a stronger finiteness restriction, but he does not assume that there is an invariant metric, and his set of defining relations is larger. His method of proof, based on the theory of covering spaces, is quite different from ours. I am grateful to the referee, Professor Martin Kneser, for drawing my attention to Behr's work, and for other helpful comments.

The formal algebra is similar to that considered by Smith in [9], and, in an earlier draft of this paper, Smith's notation was used. Then those generators of the group which were involutions and therefore their own inverses were in a special position and required separate treatment. It was found later that a treatment based on semigroups was formally simpler and avoids this kind of special case.

There is also a close connection between this paper and part of Weil's paper [10] on discrete subgroups of Lie groups. In § 8 we prove that if G is a discrete subgroup, with compact coset-space, of a Lie group, then G has a neighbourhood in the topological space of subgroups defined, for example, by Chabauty [1], consisting entirely of subgroups isomorphic to G. In § 9, this isomorphism is combined with Weil's results to establish a local homeomorphism between the space of subgroups and the space of representations of a given abstract group into the Lie group.

2. Semigroups and word-semigroups

If E is any set, define an E-word to be a finite non-empty ordered set of elements of E. Later E will be a subset of a group so we shall denote the word given by the ordered n-tuple $e_1, e_2, \cdots, e_n$ by $e_1 \cdot e_2 \cdot \cdots \cdot e_n$, inserting dots to distinguish the word from the group product. If we have two E-words

$$w_1 = e_1 \cdot e_2 \cdot \cdots \cdot e_n , \qquad w_2 = f_1 \cdot f_2 \cdot \cdots \cdot f_m ,$$

then $w_1 \cdot w_2$ is defined by the relation

$$w_1 \cdot w_2 = e_1 \cdot e_2 \cdot \cdots \cdot e_n \cdot f_1 \cdot f_2 \cdot \cdots \cdot f_m$$

obtained by writing the two words in succession. With this operation the set of E-words forms a semigroup $W(E)$[1].

An *E-relation* is an unordered pair of E-words $[w_1, w_2]$. A relation should, of course, be regarded as the identification of a pair of words. Taking this view, we say, if $u, v \in W(E)$, that the relation $[u \cdot w_1 \cdot v, u \cdot w_2 \cdot v]$ *is implied by* the relation $[w_1, w_2]$. Suppose now that R is a set of E-relations. Two words $x, y \in W(E)$ are said to be *R-equivalent* if there is a finite sequence of E-words

$$x = w_0, w_1, \cdots, w_n = y ,$$

where each of the relations

$$[w_{r-1}, w_r] \qquad (r = 1, 2, \cdots, n)$$

is implied by a relation in R. It is easily verified that R-equivalence is an

[1] The empty word can be added to $W(E)$ without loss of the semigroup property, but this is not convenient for our purpose.

equivalence relation, and that, if x, y are R-equivalent to x', y' respectively, then $x \cdot y$ is R-equivalent to $x' \cdot y'$. Thus it makes sense to define the product of the class containing x with the class containing y to be the class containing $x \cdot y$, and the R-equivalence-classes form a semigroup called the *semigroup defined by generators E and relations R*, and denoted by $W(E)/R$.

Suppose in particular that E is a subset of a group G. To each word $w = e_1 \cdot e_2 \cdot \cdots \cdot e_n \in W(E)$ we associate the group element $\varphi(w) = e_1 e_2 \cdots e_n \in G$. An E-relation $[w_1, w_2]$ is said to be *valid in G* if $\varphi(w_1) = \varphi(w_2)$. If R is a set of E-relations valid in G, then φ will map all the elements of any R-equivalence-class on the same element of G, and will thus define a homomorphism of $W(E)/R$ into G. If this homomorphism is an epimorphism, we say that *E generates G*; and if it is also an isomorphism, we say that R is a *complete set of defining relations* of G in terms of E. The pair $\{E, R\}$ is then called a *presentation* of G. This is not quite the terminology usual in group theory, since E is not considered here to generate G unless it generates G by multiplication alone without the taking of inverses, and similarly the set of relations is considered to be incomplete unless all valid relations follow from them by multiplication alone.

Consider the subset F of $E \times E$ consisting of all pairs (e, e') such that $ee' \in E$. The family of E-relations

$$\big([e \cdot e', ee']; (e, e') \in F\big)$$

is called the *local multiplication table* of E. E is called a *determining subset* of G if E generates G, and the local multiplication table of E is a complete set of defining relations.

3. Statement of the theorem

Let (G, M) be a topological transformation group as defined by Montgomery and Zippin [6, p. 40]. A subset $V \subset M$ is called a *G-covering* if

$$M = \bigcup\{gV \colon g \in G\} .$$

For each V, define $E(V)$ to be the subset of G consisting of those elements $g \in G$ for which $V \cap gV$ is not empty. Let F' denote the subset of $E(V) \times E(V)$ consisting of all pairs (g, g') such that

$$V \cap gV \cap gg'V \neq \varnothing , \tag{1}$$

and let $R(V)$ be the set of $E(V)$-relations

$$\{[g \cdot g', gg'] : (g, g') \in F'\} .$$

$R(V)$ is a subset of the local multiplication table of $E(V)$. Our main theorem is as follows.

THEOREM 1. *Let* (G, M) *be a topological transformation group and let* V *be a non-empty open set which is a G-covering. Then*

(i) *if* M *is connected,* $E(V)$ *generates* G,

(ii) *if* V *is path-connected and* M *is connected and simply connected,* $\{E(V), R(V)\}$ *is a presentation of* G.

COROLLARY. *In case* (ii) $E(V)$ *is a determining subset of* G.

4. Dissection of the unit square

By I, as usual, we denote the real unit interval $\{t: 0 \leqq t \leqq 1\}$, and by I^2 the unit square $I \times I$. We regard I as embedded in I^2 by the mapping $t \rightarrow (t, 0)$. Let D_n denote the simplicial dissection of the unit square obtained by subdividing it into n^2 equal squares by lines parallel to its sides, and then sub-dividing each square into two triangles by the diagonal of slope 1. We define an *edge-path* in D_n as a sequence of *at least two*[2] vertices in which each vertex and the succeeding one belong to a common simplex. An *allowable operation* on an edge-path is the replacement, where x, y, z are vertices belonging to a single simplex, of a triple xyz of successive vertices in the path by the pair xz, or *vice versa*, the replacement of the pair by the triple. If P_0 is the path obtained by taking the vertices on the frontier of D_n in order anti-clockwise, beginning and ending at $v_0 = (0, 0)$, then P_0 can be reduced to a constant path v_0v_0 by a succession of allowable operations [4], [7].

Let G be a group, E a subset of G, and R a set of E-relations, valid in G. A function $f: v \rightarrow \mathbf{v} = f(v)$ from the set of vertices of D_n into G is said to be *E-R-admissible* if the following conditions hold. For simplicity in stating the conditions, we denote the f-image of any vertex by the same symbol in bold type.

(i) If x, y are vertices, not necessarily distinct, of the same simplex, then $\mathbf{x}^{-1}\mathbf{y} \in E$.

(ii) If x, y, z are vertices, not necessarily distinct, of the same simplex, then

[2] The terminology is almost the same as in Hilton and Wylie [4], except that we require at least two vertices (not necessarily distinct) in a path, and that in an allowable operation the vertices need not be distinct vertices of a 2-simplex, but a contraction can take place if three successive vertices belong to a common 1-simplex. This change was made in order to avoid using the empty word, but in fact it leads to a (very slight) simplification of the conventional theory of the fundamental group.

$$[\mathbf{x}^{-1}\mathbf{y}\cdot\mathbf{y}^{-1}\mathbf{z},\ \mathbf{x}^{-1}\mathbf{z}] \in R\ .$$

Given an admissible map f, with any edge-path $P = v_1 v_2 \cdots v_{r+1}$ in D_n we associate the word

$$f(P) = \mathbf{v}_1^{-1}\mathbf{v}_2\cdot\mathbf{v}_2^{-1}\mathbf{v}_3\cdot \cdots \cdot\mathbf{v}_r^{-1}\mathbf{v}_{r+1}\ .$$

This is an E-word by condition (i). If Q is another edge-path whose first vertex is the last vertex of P, then PQ is defined and $f(PQ) = f(P)\cdot f(Q)$.

If P can be derived by an allowable operation from P', then P is of the form $AxyzB$ and P' is $AxzB$ or *vice versa*, and the relation

$$[f(P), f(P')] = [f(A)\cdot\mathbf{x}^{-1}\mathbf{y}\cdot\mathbf{y}^{-1}\mathbf{z}\cdot f(B), f(A)\cdot\mathbf{x}^{-1}\mathbf{z}\cdot f(B)]$$

is then implied by a relation which belongs to R, by (ii). Thus if one path can be derived from another by a succession of allowable operations, the associated E-words are R-equivalent. In particular, since P_0 can be reduced to the constant path v_0v_0 by a succession of allowable operations, the E-word $f(P_0)$ is R-equivalent to the single-letter E-word 1. Without real risk of confusion, the same symbol 1 denotes the unit element of the group and the single-letter E-word.

5. Proof of Theorem 1

(i) Let G_0 be the subgroup generated by $E(V)$ and let $H = G - G_0$. Then $G_0V \cap HV = \emptyset$; for from $g_0V \cap hV \neq \emptyset (g_0 \in G_0, h \in H)$ follows $g_0^{-1}h \in E(V)$, $h \in g_0E(V) \subset g_0G_0 \subset G_0$, a contradiction. On the other hand the union of the open sets G_0V, HV is the connected space M, so they cannot both be non-empty. Thus H is empty and $G_0 = G$.

(ii) In the proof of Theorem 1 (ii) we use E, R for $E(V)$, $R(V)$, and the notation $w_1 \equiv w_2$ to mean that the two E-words w_1, w_2 are R-equivalent. From the definition, $w_1 \equiv w_2$ implies $u\cdot w_1\cdot v \equiv u\cdot w_2\cdot v$ for all $u, v \in W(E)$. If $g \in E$, then $g^{-1}(V \cap gV) \neq \emptyset$, so $g^{-1} \in E$. Also if $g \in E$, the sets

$$V \cap gV \cap gg^{-1}V, \quad V \cap gV \cap g1V, \quad V \cap 1V \cap 1gV$$

are all non-empty, so the relations

$$[g\cdot g^{-1}, 1]\ , \quad [g\cdot 1, g]\ , \quad [1\cdot g, g]$$

all belong to R. If $w = g_1\cdot g_2\cdot \cdots \cdot g_m$, we define w^* to be $g_m^{-1}\cdot g_{m-1}^{-1}\cdot \cdots \cdot g_2^{-1}\cdot g_1^{-1}$, and it follows easily from the above that $w\cdot 1 \equiv w \equiv 1\cdot w$ and that $w\cdot w^* \equiv 1$.

To prove the theorem we must show that if $\varphi(w_1) = \varphi(w_2)$, then $w_1 \equiv w_2$. This follows from

(A) *If* $\varphi(w) = 1$, *then* $w \equiv 1$.

For if (A) is true and $\varphi(w_1) = \varphi(w_2)$, then $\varphi(w_2^*) = \varphi(w_2)^{-1}$, so $\varphi(w_1 \cdot w_2^*) = \varphi(w_1)\varphi(w_2)^{-1} = 1$. By (A), $w_1 \cdot w_2^* \equiv 1$, so $w_1 \equiv w_1 \cdot 1 \equiv w_1 \cdot (w_2^* \cdot w_2) \equiv (w_1 \cdot w_2^*) \cdot w_2 \equiv 1 \cdot w_2 \equiv w_2$, so it is sufficient to prove (A).

Let then $w = g_1 \cdot g_2 \cdot \cdots \cdot g_m$, and let $\varphi(w) = g_1 g_2 \cdots g_m = 1$, with $g_i \in E$. We have to show that $w \equiv 1$. For $r = 1, \cdots, m$ define $h_r = g_1 g_2 \cdots g_r$ and define $h_0 = 1$. Then also $h_m = 1$. The sequence of open sets

$$h_0 V, h_1 V, \cdots, h_m V$$

begins and ends with the same set $V = h_0 V = h_m V$ and any two successive sets in the sequence intersect, for $h_r V \cap h_{r+1} V = h_r(V \cap g_{r+1} V) \neq \varnothing$ since $g_{r+1} \in E$. For $r = 1, \cdots, m$, let x_r be a point in $h_{r-1} V \cap h_r V$, and let $x_0 = x_m$. For $r = 0, \cdots, m$, let ξ_r denote the point r/m in the unit interval I, and for $r = 0, \cdots, m-1$ let J_r be the closed interval $[\xi_r, \xi_{r+1}]$. Since x_r and x_{r+1} both belong to $h_r V$ $(r = 0, \cdots, m-1)$, and since $h_r V$ is path-connected, there is a continuous map e_r: $J_r \to h_r V$ such that $e_r(\xi_r) = x_r$, $e_r(\xi_{r+1}) = x_{r+1}$. Joining these arcs together for $r = 0, \cdots, m-1$, we have a map e: $I \to M$ such that $e(\xi_r) = x_r$ for $r = 0, \cdots, m$, and $e(J_r) \subset h_r V$ for $r = 0, \cdots, m-1$. Still regarding I as part of $\dot{I}^2$, denote by L the closure of $\dot{I}^2 - I$. Extend e to a map e': $\dot{I}^2 \to M$ by setting $e'(L) = x_0$, the e and e' images of every point of I being the same. Finally, since M is simply connected, it is possible to extend the map e', defined on the frontier of I^2, to the whole of I^2. Let e'' be such an extension.

Now the set $J_0 \cup L$ is compact and is contained in the open subset $e''^{-1}(V)$ of I^2. Thus there is a positive number ε' such that any subset of I^2 which meets $J_0 \cup L$ and whose diameter does not exceed ε', is contained in $e''^{-1}(V)$. Again for $r = 1, \cdots, m-1$, J_r is compact, so there is a positive number ε_r such that any subset of I^2 which meets J_r and whose diameter does not exceed ε_r, is contained in $e''^{-1}(h_r V)$. Finally the compact set I^2 is covered by the sets $\{e''^{-1}(g V) \colon g \in G\}$. Thus there is a Lebesgue number $\varepsilon'' > 0$ such that any subset of I^2 of diameter not exceeding ε'' is contained in some single set $e''^{-1}(g V)$. Let $\varepsilon = \min(\varepsilon', \varepsilon'', \varepsilon_1 \cdots, \varepsilon_{m-1})$. Let N be an integer so large that the diameter of the star of any vertex in the simplicial dissection D_{mN} does not exceed ε.

Now define a function f from the set of vertices of D_{mN} into G. First on $\dot{I}^2$, set $f(v) = 1$ if $v \in L$, and $f(v) = h_r$ if $\xi_{r+1} \neq v \in J_r$ $(r = 0, \cdots, m-1)$. Since the diameter of the star of any vertex does not exceed ε', ε_1, $\cdots$ or ε_{m-1}, we shall then have, for $v \in \dot{I}^2$,

(B) $e''(\mathrm{st}\, v) \subset f(v) V$.

Since the diameter of the star of any vertex does not exceed the Lebesgue number ε'', it is possible to choose a function f defined on the vertices interior to I^2 so that condition (B) is satisfied for the interior vertices also. Suppose f so chosen.

Condition (B) implies that f is E-R-admissible. For, if x, y are vertices of a simplex, then (using $\mathbf{x}, \mathbf{y}, \cdots$ for $f(x), f(y), \cdots$ as in § 4) $\mathbf{x}V \cap \mathbf{y}V \supset e''(\mathrm{st}x \cap \mathrm{st}y) \neq \emptyset$, so $V \cap \mathbf{x}^{-1}\mathbf{y}V \neq \emptyset$, and $\mathbf{x}^{-1}\mathbf{y} \in E$. And if x, y, z are vertices of a simplex, then $\mathbf{x}V \cap \mathbf{y}V \cap \mathbf{z}V \neq \emptyset$; so, operating with $\mathbf{x}^{-1}$, $V \cap \mathbf{x}^{-1}\mathbf{y}V \cap \mathbf{x}^{-1}\mathbf{z}V \neq \emptyset$, and $[\mathbf{x}^{-1}\mathbf{y}\cdot\mathbf{y}^{-1}\mathbf{z}, \mathbf{x}^{-1}\mathbf{z}] \in R$.

Now the vertices in the path P_0 are, in order,

(1) ξ_0 followed by $N-1$ vertices in J_0,

(2) ξ_1 followed by $N-1$ vertices in J_1,

. . . .

(m) ξ_{m-1} followed by $N-1$ vertices in J_{m-1},

(m + 1) ξ_m followed by $3mN-1$ vertices in L followed by ξ_0.

For each vertex v listed under (k + 1) above, we have $\mathbf{v} = h_k$. Thus if we use 1^p to denote the E-word consisting of p 1's,

$$f(P_0) = 1^{N-1}\cdot g_1\cdot 1^{N-1}\cdot g_2\cdot \cdots \cdot g_m\cdot 1^{3mN} .$$

By repeated use of the relations $1\cdot g \equiv g$ and $g\cdot 1 \equiv g$, we derive $f(P_0) \equiv w$. Since, however, f is E-R-admissible, it follows from § 4 that $f(P_0) \equiv 1$. Thus $w \equiv 1$, and Theorem 1 is proved.

6. Transformations of metric spaces and complexes. Application to the symmetric group

In classical theory of discontinuous groups much use is made of the not-too-clearly defined concept of "fundamental region". The closure of a fundamental region is a G-covering, but its interior is not usually a G-covering, so it is useful sometimes to extend Theorem 1 to *closed* coverings. The argument given below seems to apply only to groups of isometries of metric spaces. One would wish to weaken this restriction, but we have not been able to do so.

Suppose that, in addition to the restrictions given in the statement of Theorem 1, M is a metric space and G is a group of *isometries*, so that, if d denotes the distance, $d(x, y) = d(gx, gy)$ for all $g \in G$. A closed G-covering K is called *normal* if each $x \in M$ has a neighbourhood U such that $gK \cap U$ is empty for all but a finite set of $g \in G$. For each subset $T \subset M$, the *intersection pattern* $I(T)$ is defined to be the family of all finite subsets $A \subset G$ with the property that $\bigcap\{gT: g \in A\} \neq \emptyset$. Theorem 1 explains how to derive a presentation of G from the intersection pattern of an open G-covering. The same can be done from the intersection pattern of a normal closed G-covering as a result of the following theorem.

THEOREM 2. *Suppose that (G, M) is a topological transformation group in which M is a locally path-connected metric space and G is a group of isometries. If K is a path-connected normal closed G-covering, then there is a path-connected open G-covering $V \supset K$ such that $I(V) = I(K)$.*

COROLLARY. *If K is a normal closed G-covering then*

(i) *if M is connected, $E(K)$ generates G,*

(ii) *if K is path-connected and M is connected and simply connected, $\{E(K), R(K)\}$ is a presentation of G.*

(For if V is the open covering of Theorem 2, then $E(V) = E(K)$ and $R(V) = R(K)$, and the corollary follows from Theorem 1).

PROOF. For each $x \in M$, let $\Gamma(x)$ be the set of $g \in G$ such that $x \in gK$. Let U be a neighbourhood of x such that $U \cap gK = \varnothing$ except for those g in a finite set $\Gamma(U)$. Clearly $\Gamma(U) \supset \Gamma(x)$. Since K is closed, so is the set $C = \bigcup\{gK: g \in \Gamma(U) - \Gamma(x)\}$. Thus $U_1 = U - C$ is open, and $U_1 \cap gK \neq \varnothing$ if and only if $g \in \Gamma(x)$. Let $2\rho(x)$ be the radius of an open ball centre x contained in U_1, and let $V(x)$ be a path-connected neighbourhood of x contained in the open ball centre x and radius $\rho(x)$. Let $V = \bigcup\{V(x): x \in K\}$. Clearly V is path-connected. Since $V \supset K$, V is an open G-covering and $I(V) \supset I(K)$. To complete the proof we need only show conversely that $I(K) \supset I(V)$. Suppose then, that $A \in I(V)$. Then there is a point a, say, in the intersection $\bigcap\{gV: g \in A\}$. For each $g \in A$, there is a point $x_g \in K$ such that $a \in gV(x_g)$, so $d(a, gx_g) \leqq \rho(x_g)$. Since A is finite, there is an element $g_0 \in A$ for which $\rho(x_g)$ takes its maximum value ρ_0. Then

$$d(gx_g, g_0x_{g_0}) \leqq d(gx_g, a) + d(g_0x_{g_0}, a) < 2\rho_0 .$$

Thus $g_0x_{g_0} \in gK$ for each $g \in A$. Thus $\bigcap\{gK: g \in A\}$ contains $g_0x_{g_0}$ and is non-empty, so $A \in I(K)$. This proves Theorem 2.

Now let M be a simplicial complex whose underlying space is a possibly non-compact manifold of dimension n. Let G be a group of one-to-one simplicial transformations of M with the property that no simplex of dimension n is mapped on itself by a transformation of G other than the identity. Each transformation g of G induces a homeomorphism of $|M|$ which we also denote by g. Let K be a family of n-simplexes such that $|\bar{K}|$ is connected, and such that K contains exactly one n-simplex congruent to a given one under the action of G. Then $|\bar{K}|$ is closed and is a G-covering, for if σ_k is a simplex of dimension $k < n$, then σ_k is a face of some σ_n; for some $g \in G$, $g\sigma_n \in K$, $g\sigma_k \in \bar{K}$. Moreover $|M|$ is a metric space [4, p. 48] and G is a group of isometries.

If $M^{(r)}$ denotes the r-dimension skeleton of M, G maps $|M| - |M^{(r)}|$ onto itself. In relation to the transformation group $(G, |M| - |M^{(r)}|)$, $|\bar{K}| - |\bar{K}^{(r)}|$ is a normal closed G-covering. If the subset E and the relations R are constructed as in Theorem 1 for this transformation group and covering, we have

$$E(|\bar{K}| - |\bar{K}^{(r)}|) = \{g\colon \dim(\bar{K} \cap g\bar{K}) \geqq r + 1,\ g \in G\}\ ,$$
$$R(|\bar{K}| - |\bar{K}^{(r)}|) = \{[g \cdot g', gg']\colon \dim(\bar{K} \cap g\bar{K} \cap gg'\bar{K}) \geqq r + 1,\ g, g' \in G\}\ .$$

For brevity, we denote the set of elements by $E_{r+1}(K)$, and the set of relations by $R_{r+1}(K)$.

Let M' denote the first derived complex of M and N the dual cell-complex, so that $M' = N'$. The number of components of N (or M), being the zero-dimensional Betti number, is the number of components of $N^{(1)}$ and the fundamental group of N, being calculated from the two-dimensional skeleton only, is the same as that of $N^{(2)}$. Thus, if M is connected, $N^{(1)}$ is connected, and if M is simply connected, $N^{(2)}$ is simply connected.

Every simplex of M' has vertices $P_1, \cdots, P_k$, where each P is the centre of a cell of M, the dimensions of the cells being all different. Since the centre of an r-cell of M is the centre of the corresponding $(n - r)$-cell of N, the vertices of the simplex can be divided into two classes—those which are centres of M-cells of dimension $n - r - 1$ or less belonging to the first class, and those which are centres of N-cells of dimension r or less belonging to the second class. Every point x of $|M|$ can accordingly be written in the form

$$x = \lambda y + (1 - \lambda)z = \varphi(y, z, \lambda)\ ,$$

where $y \in |M^{(n-r-1)}|$, $z \in |N^{(r)}|$, $0 \leqq \lambda \leqq 1$. This expression is unique unless $x \in |M^{(n-r-1)}|$, when $\lambda = 1$, $y = x$ and z is undetermined, or unless $x \in |N^{(r)}|$, when $\lambda = 0$, $z = x$ and y is undetermined. A deformation retraction of $|M| - |M^{(n-r-1)}|$ on $|N^{(r)}|$ is defined by the homotopy

$$f(x, t) = \varphi(y, z, \lambda t) \qquad (x \in |M| - |M^{(n-r-1)}| - |N^{(r)}|)\ ,$$
$$f(x, t) = x \qquad (x \in |N^{(r)}|)\ .$$

Taking $r = 1$, we see that $|M| - |M^{(n-2)}|$ has the same homotopy type as $|N^{(1)}|$ and is connected if M is connected. Taking $r = 2$, we see that $|M| - |M^{(n-3)}|$ has the same homotopy type as $|N^{(2)}|$ and is simply connected if M is simply connected. Applying Theorem 2, Corollary, to the transformation groups $(G, |M| - |M^{(n-r)}|)$, $(r = 2, 3)$, we derive

THEOREM 3. (i) *If M is connected, $E_{n-1}(K)$ generates G.*

(ii) *If M is connected and simply connected, then $\{E_{n-2}(K), R_{n-2}(K)\}$ is a presentation of G.*

The kind of presentation derived from Theorem 3 can be of interest even in the case of finite complexes, and we shall illustrate this by using it to obtain a well-known presentation of the symmetric group S_{n+2} of degree $n + 2$. We shall use both parts of Theorem 3, replacing the elements of $E_{n-2}(K)$ in the relations $R_{n-2}(K)$ by expressions involving the elements of $E_{n-1}(K)$ only.

For M we take the first derived complex of the frontier of the $(n + 1)$-simplex in R^{n+2} spanned by the points

$$(1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 1) .$$

The symmetric group S_{n+2} acts on M in the natural fashion, permuting the coordinates and transitively permuting the faces of dimension n. Any one of these faces may be taken to comprise K, let us say the face consisting of the part of M defined by the inequalities

$$0 = x_0 < x_1 < \cdots < x_{n+1} . \tag{2}$$

The set of points $|\bar{K}|$ will be the part of M defined by the inequalities

$$0 = x_0 \leqq x_1 \leqq \cdots \leqq x_{n+1} . \tag{3}$$

The set of n-simplexes which have an $(n - 1)$-face in common with K consists of those simplexes defined by reversing precisely one of the inequalities in (2). These faces are

$$t_0K, t_1K, \cdots, t_nK ,$$

where t_r is the transposition $(r, r + 1)$. The elements $t_0, t_1, \cdots, t_n$ constitute $E_{n-1}(K)$ and generate G. An $(n - 2)$-simplex which is a face of K is the part of M defined by inequalities like (2) but with precisely two of the inequalities replaced by equality. Two cases can arise:

Case (i) The $(n - 2)$-face may be defined by

$$x_{i-1} = x_i = x_{i+1} .$$

There are six elements of G, including the identity, which map K into a simplex with which it has this face in common. These are

$$1, t_{i-1}, t_i, t_{i-1}t_i, t_it_{i-1}, t_it_{i-1}t_i .$$

It happens that these elements form a subgroup (S_3), and the relations given by their local multiplication table are all implied by the three relations

$$t_i^2 = t_{i-1}^2 = (t_it_{i-1})^3 = 1 . \tag{4}$$

Case (ii) The $(n - 2)$-face may be defined (if $n \geqq 2$) by

$$x_i = x_{i+1}, \quad x_j = x_{j+1} , \qquad (j \geqq i + 2) .$$

There are four elements of G, including the identity, which map K into a simplex with which it has this face in common. These are $1, t_i, t_j, t_it_j$. Again they form a subgroup, and the non-trivial relations between them are implied by

$$t_i^2 = t_j^2 = (t_it_j)^2 = 1 \; . \tag{5}$$

The relations (4), (5) combine to give a presentation of the symmetric group. It can, of course, be derived by algebraic methods (see, e.g., [2]).

7. Discrete subgroups of Lie groups

Let Λ be a connected Lie group. If H is a maximal compact subgroup, the coset space Λ/H is a euclidean space R. Let p denote the natural map $p: \Lambda \to R$ so that $p(\lambda) = \lambda H$ $(\lambda \in \Lambda)$. If G is a discrete subgroup of Λ, G acts on R as a transformation group. We use $g \cdot x$ to denote the action of G on R, so that, if x is the coset λH, $g \cdot x$ is the coset $g\lambda H$. We use $\mathbf{N}$ to denote the system of neighbourhoods of the identity in Λ, and the letter W always denotes a symmetric neighbourhood of the identity with compact closure.

If K is a compact subset of Λ, if $U \in \mathbf{N}$, and if G is a discrete subgroup, we define the family $\mathrm{Nd}(G, K, U)$ to consist of those subgroups G' such that

$$GU \supset K \cap G' \text{ and } G'U \supset K \cap G \; .$$

With this system of neighbourhoods, the set of all discrete subgroups of Λ forms a topological space, which we denote by $S(\Lambda)$. Our main result is as follows.

THEOREM 4. *If Λ/G is compact, $G \in S(\Lambda)$, then there is a neighbourhood of G in $S(\Lambda)$ consisting entirely of groups isomorphic to G.*

A few preliminary lemmas are required, some of them implicit in earlier work [5]. Many of these lemmas apply to more general situations—Λ is often required to be only a locally compact and compactly generated topological group. In the final part of the argument, however, the two characteristic properties of a Lie group are used—that Λ/H is simply connected, and that Λ does not have arbitrarily small subgroups.

LEMMA 7.1. *If Λ/G is compact, V open and $GV = \Lambda$, then there is a compact subset C of V such that $GC = \Lambda$.*

PROOF. Since Λ is locally compact, each point $x \in V$ has a neighbourhood U_x with compact closure $\bar{U}_x \subset V$. The covering of Λ/G by the open sets $p(U_x)$ has a finite subcovering $p(U^1), \cdots, p(U^k)$. The union C of the closures $\bar{U}^1, \cdots, \bar{U}^k$ has the required property.

LEMMA 7.2. *If* $GC = \Lambda$, C *compact and* $W \in \mathbf{N}$, *then* $G'WC = \Lambda$ *for all* $G' \in \mathrm{Nd}(G, G_0, W)$ *where* $G_0 = G \cap WCC^{-1}W^{-1}$.

PROOF. Since $C^{-1}W^{-1}$ is an open set containing C^{-1}, there is a compact neighbourhood D of unity such that $DC^{-1} \subset C^{-1}W^{-1}$. Since Λ is connected, D generates Λ and $\bigcup D^n = \Lambda$. Then

$$GC = \Lambda \supset WCD .$$

If $g \in G - G_0$, $g \notin WCC^{-1}W^{-1}$, so $g \notin WCDC^{-1}$, $gC \cap WCD = \varnothing$. Thus $G_0C \supset WCD$. Since W, C have compact closure, G_0 is finite. Now $G'W \supset G_0$ so $G'WC \supset G_0C \supset WCD$. Then $G'WC = G'^2WC \supset G'WCD \supset WCD^2$. Continuing we derive $G'WC \supset WCD^n$, so $G'WC \supset \bigcup WCD^n = \Lambda$.

A corollary to Lemma 7.2 which we shall not need, but is of some interest in itself:

LEMMA 7.2a. *If* V *is an open set with compact closure, the set* Σ *of* $G \in S(\Lambda)$ *such that* $GV = \Lambda$ *is an open subset of* $S(\Lambda)$.

PROOF. If $G \in \Sigma$, choose C compact, $C \subset V$, $GC = \Lambda$ and choose W such that $WC \subset V$. By Lemma 7.2, $\mathrm{Nd}(G, G_0, W) \subset \Sigma$, so G is an interior point of Σ.

LEMMA 7.3. *Let* $G \in S(\Lambda)$, *let* W *be a symmetric neighbourhood of unity such that* $W^3 \cap G = \{1\}$, *and let* K *be a compact set containing* W^3. *Then, if* $G' \in \mathrm{Nd}(G, K, W)$, $G' \cap W$ *is a subgroup.*

PROOF. Since $W = W^{-1}$, it suffices to show that $G' \cap W$ is closed under multiplication. Suppose, then, that $g_1', g_2' \in G' \cap W$. Then $g_1'g_2' \in WW \subset K$, so, by the definition of $\mathrm{Nd}(G, K, W)$, $g_1'g_2' \in gW$, where $g \in G$. Let $g_1'g_2' = gw$. Then $g = g_1'g_2'w^{-1} \in WWW$, so, by hypothesis, $g = 1$. Thus $g_1'g_2' \in W$.

LEMMA 7.4. *Let* $G \in S(\Lambda)$, $W = W^{-1} \in \mathbf{N}$, $W^6 \cap G = \{1\}$. *Let* K *be a compact set containing* W^6. *If* W^2 *contains no non-trivial subgroup and* $G' \in \mathrm{Nd}(G, K, W)$, *then, for each* $g \in G \cap K$, *there is exactly one element of* G' *in* gW.

PROOF. Since $g \in G'W$ for each $g \in K \cap G$, it follows that there is at least one element g' such that $g = g'w$, $w \in W$. Then $g' \in gW$, since $W = W^{-1}$. If there were two such elements g_1', g_2', then $g_1'^{-1}g_2' \in W^2$. By Lemma 7.3 with W^2 for W, $g_1'^{-1}g_2'$ belongs to a subgroup contained in W^2, so it must be the unit element.

8. Proof of Theorem 4

Now let G be any discrete subgroup, Λ/G compact. Let C be a compact set such that $GC = \Lambda$. Since CH is also compact, we may assume without loss of generality that C is a union of left H-cosets, that is $CH = C$. Let

$G_0 = G \cap CC^{-1}$. The complement of $G - G_0$ in Λ is open and contains CC^{-1}, so there is $W_1 \in N$ such that $(G - G_0) \cap W_1CC^{-1}W_1^{-1} = \varnothing$. Let $W_2W_2 \subset W_1$. Then $(G - G_0)W_2 \cap W_2CC^{-1}W_2^{-1} = \varnothing$. Write V for W_2C and K for the closure of VV^{-1}.

The following facts are easily checked:

(i) $K \cap G = VV^{-1} \cap G = G_0$, a finite set.

(ii) $p(V)$ is a G-covering for the transformation group (G, R).

(iii) Since $VH = V$, we have $p(V) \cap g \cdot p(V) \neq \varnothing$ if and only if $V \cap gV \neq \varnothing$ or, if and only if $g \in VV^{-1}$. Thus by Theorem 1, G_0 is a determining subset of G.

Now choose W_3 so small that

(iv) W_3^2 contains no non-trivial subgroup,

(v) $W_3^6 \cap G = 1$,

(vi) $W_3^6 \subset K$.

Choose W_4, W_5, W_6 with the following properties:

(vii) $W_4W_4 \subset W_3$, $W_5 = \bigcap\{gW_4g^{-1}: g \in G_0\}$, $W_6 = W_5 \cap W_2$.

Let U denote $\mathrm{Nd}(G, K, W_6)$, and let $G' \in U$. By Lemma 7.2, $G'V = \Lambda$. By Lemma 7.4, to each $g \in G_0$ there is exactly one $g' \in G' \cap gW_6$. Let G_0' be the set of all g' obtained in this way. Then $G_0' = G' \cap VV^{-1}$ since, if $g' \in VV^{-1}$, then $g' \in GW_6$. But $(G - G_0)W_6 \subset (G - G_0)W_2$ which does not meet VV^{-1}, so $g' \in G_0W_6$; i.e., $g_0' \in G_0'$. (It follows that $U = \mathrm{Nd}(G, G_0, W_6)$). By the same argument as for G_0 ((ii), (iii) above), G_0' is a determining subset of G'. The relation $g' \leftrightarrow g$ defined by $g' \in gW_6$ gives a one-to-one correspondence between G_0 and G_0'. If we can show that the local multiplication tables correspond, it will follow that the groups are isomorphic.

Let g_1', g_2', g_3' correspond to g_1, g_2, g_3. If $g_1'g_2' = g_3'$, then $g_1W_6g_2W_6$ meets g_3W_6. By (vii), $W_6g_2 \subset W_5g_2 \subset g_2W_4$. Thus $g_1g_2W_4W_6 \subset g_1g_2W_3$ meets g_3W_3, and $g_3^{-1}g_1g_2 \in W_3^2$. So by (v), $g_3^{-1}g_1g_2 = 1$, $g_1g_2 = g_3$.

On the other hand, if $g_1g_2 = g_3$, then $g_1'g_2' \in g_1W_6g_2W_6 \subset g_1g_2W_4W_6 \subset g_1g_2W_3 = g_3W_3$. By Lemma 7.4, there is a unique element of G' in g_3W_3, so it follows that $g_1'g_2' = g_3'$. Thus G and G' are isomorphic, and Theorem 4 follows.

9. Connection with Weil's result

Let $S_0(\Lambda)$ consist of the subset of $S(\Lambda)$ consisting of those groups G with compact coset-space. In [10], Weil considers the space of homomorphisms of an abstract group G into a connected Lie group Λ, with the topology of compact convergence. Now if $G \in S_0(\Lambda)$, the set of those groups in $S_0(\Lambda)$ which are isomorphic to G is open in $S_0(\Lambda)$ by Theorem 4. Moreover, the set of groups in $S_0(\Lambda)$ which are isomorphic to other groups is a union of open sets. Thus the set isomorphic to G is open and closed in $S_0(\Lambda)$. Denote this set by $S(\Lambda, G)$. If we use Weil's notation, $R(G, \Lambda)$ is the set of all homomorphisms of G into Λ and $R_0 = R_0(G, \Lambda)$ is the set of all isomorphisms r of G into Λ such that $r(G)$ is discrete and $\Lambda/r(G)$ compact. By Weil's theorem $R_0(G, \Lambda)$ is an open subset of $R(G, \Lambda)$.

If $A(G)$ denotes the group of automorphisms of G, then $A(G)$ acts on R_0 by right translation, for if $a \in A(G)$, $r \in R_0$ then $ra \in R_0$. Moreover, since r is one-to-one, $ra = r$ if and only if $a = 1$, so $A(G)$ acts on R_0 without fixed points. Let $g_1, \cdots, g_n$ be a set of generators of G. Then, if $a \in A(G)$, $a = 1$ if and only if $a(g_i) = g_i$ for $i = 1, \cdots, n$. Let $r_0 \in R_0$, and let V be a neighbourhood of unity in Λ such that

$$r(g_i)V \cap r(g_j)V = \varnothing \qquad \text{unless } i = j .$$

Let $U_1 \subset R_0$ denote the neighbourhood of r_0 consisting of those r for which

$$r(g_i) \in r_0(g_i)V \qquad (i = 1, \cdots, n) .$$

Then $U_1 a \cap U_1 = \varnothing$ for $1 \neq a \in A(G)$, so that $A(G)$ acts as a properly discontinuous group of transformations of R_0.

The relation $r \to r(G)$ defines a mapping of R_0 onto $S(\Lambda, G)$. Denote this mapping by j, so that $j(r) = r(G)$. Clearly $j(r_1) = j(r_2)$ if and only if $r_2^{-1}r_1 \in A(G)$. Thus the elements of $S(\Lambda, G)$ are in one-to-one correspondence with the orbit-space $R_0/A(G)$.

THEOREM 5. *The mapping*

$$j\colon r \to r(G)$$

is locally homeomorphic. Thus $S(\Lambda, G)$ is homeomorphic to $R_0/A(G)$.

PROOF. In the proof of Theorem 4 it was shown that there is a finite subset $G_0 \subset G$ and a neighbourhood W_6 of unity satisfying conditions (i)-(vii) of § 8, and also

(viii) if $G' \in \mathrm{Nd}(G, G_0, W_6)$, there is a unique isomorphism $r\colon G \to G'$ such that $r(g) \in gW_6$ for $g \in G_0$.

Let G_1 be a finite subset of G and let $V \in \mathbf{N}$. Let $\mathrm{Nd}(r_0, G_1, V)$ denote the set of representations r such that

$$r(g) \in r_0(g)V \qquad \text{for } g \in G_1 .$$

The set of all $\mathrm{Nd}(r_0, G_1, V)$ forms, by definition, a basis at r_0 in R. Since, however, every $g \in G_1$ is a finite product of elements of G_0 and the group operations are continuous, it follows that the smaller family of sets $\mathrm{Nd}(r_0, G_0, V)$ forms a basis at r_0.

Let id denote the identity map of G into Λ, and let $W_7 \subset W_6$ be so small that every representation r of $\mathrm{Nd}(\mathrm{id}, G_0, W_7)$ is faithful, and $r(G)$ is discrete. Such a W_7 exists by Weil's theorem [10]. By (viii), j defines a one-to-one correspondence between Nd(G, G_0, W_7) and Nd(id, G_0, W_7). We must show that this is a homeomorphism.

If $r: G \to G'$ and $r_1: G \to G'_1$ are the isomorphisms mentioned in (viii) above, then $r' = r_1 \circ r^{-1}$ is an isomorphism of G' onto G'_1.

If $g' \in r(G_0)$, then $g' = r(g)$ where $g' \in g W_6$ and $r'(g') = r_1(g) \in g W_6$. Thus $r'(g') \in g' W_6 W_6 \subset g' W_3$. Now there cannot be two elements of G'_1 in $g' W_3$ for, if these were $r_1(g_1)$, $r_1(g_2)$, we should have $r_1(g_1) \in g_1 W_6$, $r_1(g_2) \in g_2 W_6$, so $g_1 W_6 \cap g' W_3 \neq \emptyset$, $g_2 W_6 \cap g' W_3 \neq \emptyset$, giving $g_1 W_6 W_3 \cap g_2 W_6 W_3 \neq \emptyset$, $g_1^{-1} g_2 \in W_3 W_6^2 W_3$, implying $g_1 = g_2$ by (v). Thus r' is uniquely defined by the relation

$$r'(g') \in G'_1 \cap g' W_3 \qquad \text{for } g' \in r(G_0) .$$

Now let $G'_1, G'_2, \cdots, G'_n, \cdots$ be a sequence of groups in Nd(G, G_0, W_7) tending to a limit G' in the same neighbourhood. Let $r_k: G \to G'_k$, $r: G \to G'$ be the isomorphisms mentioned in (viii) above. Each element $r(g)$ of $r(G_0)$ is a limit of a sequence

$$r_1(g_1),\ r_2(g_2),\ \cdots,\ r_n(g_n),\ \cdots$$

where $g_n \in G'_n$. The elements of this sequence must belong to $r(g) W_3$ for large n, so the sequence must coincide, apart from a finite number of terms, with the sequence

$$r_1(g),\ r_2(g),\ \cdots,\ r_n(g),\ \cdots .$$

Thus $r_n(g) \to r(g)$ for $g \in G_0$. Since G_0 generates G_1, $r_n(g) \to r(g)$ for all $g \in G$, and the sequence r_n in Nd(id, G_0, W_7) converges to r. Conversely, it is easy to see that if $r_n \to r$, then $r_n(G) \to r(G)$. Thus j does indeed define a homeomorphism between Nd(id, G_0, W_7) and Nd(G, G_0, W_7). Since this holds for every $G \in S_0(\Lambda)$, Theorem 5 follows.

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REFERENCES

1. C. CHABAUTY, *Limites d'ensembles et geometrie des nombres*, Bull. Soc. Math. France. 78 (1950), 143–151.
2. H. S. M. COXETER and W. O. J. MOSER, *Generators and relations for discrete groups*, Ergebnisse der Math. (Neue Folge), 14 (1957).
3. R. FRICKE and F. KLEIN, Vorlesungen ueber die Theorie der automorphen Funktionen, Teubner, Leipzig, (1897–1912).
4. P. J. HILTON and S. WYLIE, Homology Theory, Cambridge, 1960.
5. A. M. MACBEATH and S. SWIERCZKOWSKI, *Limits of lattices in a compactly generated group*, Canad. J. of Math., 12 (1959), 426–436.
6. D. MONTGOMERY and L. ZIPPIN, Topological Transformation Groups, Interscience Tracts, New York, 1955.

7. H. SEIFERT and W. THRELFALL, Lehrbuch der Topologie. Teubner, Berlin, 1934. Reprinted by Chelsea, New York, 1947.
8. C. L. SIEGEL, *Discontinuous groups*, Ann. of Math., 44 (1943), 674-678.
9. P. A. SMITH, *The complex of a group relative to a set of generators*: I, Ann. of Math., 54 (1951), 371-402.
10. A. WEIL, *On discrete subgroups of Lie groups*. Ann. of Math., 72 (1960), 369-384.
11. H. BEHR, *Über die eindliche Definierbarkeit von Gruppen*, Crelles J., 211 (1962). 116-122.
12. M. GERSTENHABER, *On the algebraic structure of discontinuous groups*, Proc. Amer. Math. Soc., 4 (1953), 745-750.