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Polyhedral Groups and Related Polytopes in Quaternions

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이 논문을 석사학위 논문으로 제출함

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최 지 현의 석사학위 논문을 인준함

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ABSTRACT

There are 5 types of finite subgroups of SL(n,$\mathbb{C}$) conjugate to finite subgroups of SU(2). Such finite subgroups are called binary polyhedral groups and 3 of those finite subgroups of SL(n,$\mathbb{C}$) are double cover of rotation group of certain 3 dimensional polyhedrons. Since SU(2) and the set of unit quaternions Sp(1) are isomorphic, we can write the elements of binary polyhedral groups in terms of quaternions.

The aim of this paper is to give an intuitive way to understand binary polyhedral groups via 3 dimensional and 4 dimensional polytopes. I will introduce the construction method to verify pure quaternions which correspond to vertices of 3 dimensional regular polyhedrons by applying barycentric subdivision. All together the dual compound of the polyhedron and barycenters of the edges are pure imaginary quaternions which are projected images onto Im($\mathbb{H}$). The convex hull of all pure quaternions form a special polyhedra and the original unit quaternions of the vertices of the special polyhedra form 4 dimensional polytope. The union of 4 dimensional polytopes and its duals is binary polyhedral groups. This result helps us to understand binary polyhedral groups intuitively.

1 Introduction

The classification of finite subgroups of SL(n,$\mathbb{C}$) was initiated by Felix Klein in late 19th century. At that time, mathematicians started to give strong relations between geometry, group theory and representation theory. It is known as Erlangen program which was named after the University Erlangen-N$\ddot{u}$rnberg, where Klein worked.

The finite subgroups of SL(n,$\mathbb{C}$) are enumerable upto conjugacy. In 1884, Felix Klein gave the classification and the finite subgroups were binary polyhedral groups or a cyclic group of odd order. As any reader soon discovers, these groups are related with the regular polyhedrons([4]). More precisely, it is related with the rotation group of 3 dimensional regular polyhedron which preserves orientation of the object.

The set of unit quaternions Sp(1) is double cover of the 3 dimensional rotation group SO(3). By using this fact, we can derive the elements of binary polyhedral group directly from the given regular polyhedrons.

The aim of this paper is to find relations between binary polyhedral groups and 3 dimensional polytopes by using algebra of quaternions. I will introduce construction scheme to derive appropriate polytopes directly from a given regular polyhedron. And then, we can find the appropriate 4 dimensional polytopes of the binary polyhedral group.

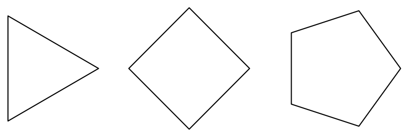
2 Preliminaries

2.1 Polygons, Polyhedrons and Polytopes

Let $\{A_i | i = 1, \cdots p\}$ be a set of p points. By connecting pair of points, we can obtain segments A_iA_j for $i \neq j$. We define a *p-gon* as a circuit of p line segments A_1A_2, A_2A_3, $\cdots$, A_pA_1([1]). The points A_i are called vertices of the polygon, the segments A_iA_{i+1} are called sides of the polygon. If all the points are coplanar, then the resulting polygon is called plane polygon, otherwise, it is called skew polygon.

Definition 1. *A plane p-gon is called* regular *if it is both equilateral and equiangular. A regular p-gon is denoted by* $\{p\}$*.*

For instance, $\{3\}$ is a equilateral triangles, $\{4\}$ is a square and $\{5\}$ is a regular pentagon.



Note that the internal angle of the regular p-gon is $(1-\dfrac{2}{p})\pi$. In the same fashion, we can consider a 3 dimensional object. A polyhedron can be defined as a finite connected set of plane polygons([1]). At a vertex, the polygons surrounding each vertex form a circuit. For a polyhedron, polygons are called faces and the sides are called edges. In this paper, a polyhedron always means a convex polyhedron.

Definition 2. *A polyhedron is called regular if its faces are regular and same.*

If the face of the polyhedron is $\{p\}$ and q faces meet at every vertex, the polyhedron is denoted by $\{p, q\}$. It is well known that there are only 5 regular polyhedrons which are also called Platonic solids.

Remark 1. *A polyhedron associated into pairs called dual where the vertices of one polyhedra correspond to the faces of the other. Regular polyhedrons have dual relations for each cases.*

$$\begin{array}{ccc} \text{Tetrahedron} & \leftrightarrow & \text{Tetrahedron(self-dual)} \\ \text{Octahedron} & \leftrightarrow & \text{Cube} \\ \text{Icosahedron} & \leftrightarrow & \text{Dodecahedron} \end{array}$$

The rotation of a regular polyhedron is an interesting subject. For instance, the tetrahedron has 2 different types of axis of rotations. One is the line passing through a vertex and the center of the opposite face, the other is the line connecting center of the edge and the center of the opposite edge. Those two lines are mutually perpendicular.

Define point symmetry to be a orientation-preserving symmetry which send a polyhedron to itself fixing one vertex. For instance, tetrahedron, at a vertex, has of order 3 point symmetries. Similarly, we can define edge symmetry and face symmetry.

polyhedron	Tetrahedron	Octahedron	Icosahedron
Edge Symmetry	2	2	2
Face Symmetry	3	3	3
Point Symmetry	3	4	5

A *polyhedron compound* is an arrangement of a number of polyhedras having same center. The compound is said to be *vertex-regular* if the vertices of its components are together the vertices of a single regular polyhedron. The compound is said to be *face-regular* if the face-planes of its components are face plans of a single regular polyhedron ([1]).

Definition 3. *A dual compound is a polyhedron compound which is consists of a polyhedron and its dual.*

A polyhedron compound is not convex. Hence we have 2 kinds of vertices; the outer vertices and the inner vertices. The outer vertices of a compound can be connected to form a convex polyhedron called the convex hull. Another convex polyhedron is formed by the central space common to all members of the compound. It is called a core of the polyhedron compound.

Example 1. *A regular tetrahedron is self-dual regular polyhedron. It can be inscribed in the cube whose consecutive diagonals of faces are the edges of tetrahedron. All of these diagonals of a cube form a compound, a tetrahedron 2-compound. It is also known as stella octangula. Its convex hull is a cube and core is an octahedron.*

A *polytope* is a generalized version of polygons and polyhedrons. It is bounded by (n-1) dimensional facets and these facets are also polytopes. 0-dimensional facets are called vertices, 1-dimensional facets are called edges, 2-dimensional facets are called faces, and 3-dimensional faces are called cells.

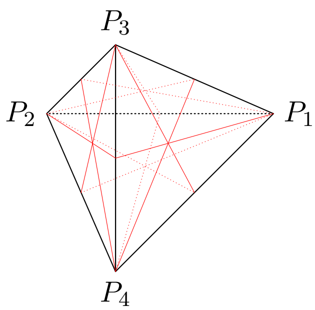
Definition 4. *A polytope is called regular if its facets and vertex figures are regular.*

Definition 5. *A polytope is called convex, for each pair of distinct points a, b in the polytope, the segment with endpoints a and b is contained within the polytope.*

2.2 Barycentric Subdivision

The barycenter or centroid of an n dimensional polytope is the arithmetic average position of all the vertices. The barycentric subdivision is a standard way of dividing an arbitrary convex polygon into triangles, a convex polyhedron into tetrahedra by connecting the barycenters of their faces in a specific way. For a point, barycenter is itself and the barycenter of a segment is the midpoint of the segment.

For instance, barycentric subdivision of tetrahedron is following



2.3 The Algebra of Quaternions

Definition 6. *The algebra of quaternions $\mathbb{H}$ is the 4 dimensional vector space over $\mathbb{R}$ defined by*

$$\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$$

which satisfies the multiplication law

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k, jk = i, ki = j$$

$$ji = -k, kj = -i, ik = -j$$

The quaternion conjugate of q is q^*.

$$q = a + bi + cj + dk \mapsto q^* = a - bi - cj - dk$$

As like the complex numbers, quaternion conjugate gives *quaternion norm*;

$$\| q \| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

For unit quaternions, $(pq)^* = q^* p^*$. A quaternion is called *invertible* if the quaternion norm is nonzero, *i.e.*, $\| q \| \neq 0$. A quaternion is called *real* if $b = c = d = 0$ and is called *imaginary* if $a = 0$. From this point of view, we can divide $\mathbb{H}$ into 2 parts;

$$\mathbb{H} \cong \mathbb{R}^4 \cong Re(\mathbb{H}) \oplus Im(\mathbb{H}) = \mathbb{R} \oplus \mathbb{R}^3$$

Lemma 1. *The multiplication of 2 pure quaternions can be written as an addition of cross product and inner product;*

$$uv = -u \cdot v + u \times v$$

Proof. Let $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$ be the pure quaternions where $u_i's$, $v_i's \in \mathbb{R}$. Then

$$\begin{aligned} uv &= (u_1 i + u_2 j + u_3 k)(v_1 i + v_2 j + v_3 k) \\ &= -(u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k \\ &= -u \cdot v + u \times v \end{aligned}$$

□

2.4 Regular Convex 4 polytopes

Definition 7. [6] *The set of quaternions*

$$H = \left\{ a + bi + cj + dk | a, b, c, d \in \mathbb{Z} \text{ } or \text{ } a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}$$

is called Hurwitz integers.

The Hurwitz unit is a Hurwitz integer of norm 1. There are 24 Hurwitz units. For a 4 dimensional polytope, we can identify vertices as a quaternion;

$$(a, b, c, d) \mapsto a + bi + cj + dk$$

Here, we will consider 4 polytopes whose vertices are Hurwitz units.

Definition 8. *8 cell is the convex regular 4-polytope which is a convex hull of 16 vertices*

$$\begin{array}{cccc}
(+1, +1, +1, +1), & (-1, -1, -1, -1), & (-1, +1, +1, +1), & (+1, -1, -1, -1) \\
(+1, -1, +1, +1), & (-1, +1, -1, -1), & (+1, +1, -1, +1), & (-1, -1, +1, -1) \\
(+1, +1, +1, -1), & (-1, -1, -1, +1), & (-1, -1, +1, +1), & (+1, +1, -1, -1) \\
(-1, +1, -1, +1), & (+1, -1, +1, -1), & (-1, +1, +1, -1), & (+1, -1, -1, +1)
\end{array}$$

8 cell consists of 8 cubes, 24 squares, 32 edges, and 16 vertices. If we normalize edge length of the 8 cell, vertices become Hurwitz units, *i.e.* its vertices are of the form $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2} \right)$

Definition 9. *16 cell is the convex regular 4-polytope which is a convex hull of 8 vertices*

$$(\pm 1, 0, 0, 0) \quad (0, \pm 1, 0, 0) \quad (0, 0, \pm 1, 0) \quad (0, 0, 0, \pm 1)$$

16 cell is bounded by 16 regular tetrahedrons. And it has 32 triangular faces, 24 edges, and 8 vertices. Note that the vertices of 8 cell and 16 cells are Hurwitz units.

Definition 10. *24 cell is the convex regular 4-polytope which is a convex hull of 24 Hurwitz units.*

Definition 11. *600 cell is a convex regular 4-polytope whose vertices are*

1. *The 8 permutations of* $(\pm 1, 0, 0, 0)$ *; the vertices of 16 cell*
2. *The 16 permutations of* $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$*; the vertices of 8 cell*
3. *The 96 even permutations of* $(\pm\alpha, \pm 1, \pm\frac{1}{\alpha}, 0)$ *where* $\alpha = \dfrac{1+\sqrt{5}}{2}$.

600 cell is bounded by 600 tetrahedral cells with 20 meeting at each vertex. There are 1200 triangular faces, 720 edges, and 120 vertices.

2.5 Special Linear Group SL(2, $\mathbb{C}$)

The special linear group SL(2,$\mathbb{C}$) is the group of 2×2 matrices which satisfy $det(A) = 1$ and ij entries $A_{ij} \in \mathbb{C}$ where $A \in$SL(2,$\mathbb{C}$).

$$SL(2,\mathbb{C}) = \Big\{A \in GL(2,\mathbb{C}) | A_{ij} \in \mathbb{C}, det(A) = 1\Big\}$$

Note that SL(2,$\mathbb{C}$) may be considered as a set of an action on $\mathbb{C}^2$.

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ (z_1, z_2) & \mapsto & (z_1', z_2') = (az_1 + bz_2, cz_1 + dz_2) \end{array}$$

Refer to [4], a finite subgroup G of SL(2,$\mathbb{C}$) is one of the following cases;

1. a cyclic group
2. a binary dihedral group
3. binary group corresponding to one of the regular polyhedron; tetrahedron, octahedron, icosahedron

Recall that there is a dual identification for regular polyhedrons. Dual polyhedrons have the same set of axis of rotations which fixes a polyhedron. Hence we have 3 finite subgroups and it suffices to study those of 3 regular polyhedrons.

We will see the details of classifications of finite subgroups G of SL(2,$\mathbb{C}$) by finding relations of SL(2,$\mathbb{C}$), SU(2) and SO(3).

The special unitary group SU(2) is the subgroup of SL(2,$\mathbb{C}$).

$$SU(2) = \left\{ A \in SL(2,\mathbb{C}) | A^*A = AA^* = I_2 \right\}$$

where A^* denotes the conjugate transpose of A.

Lemma 2. *Any elements of* SU(2) *is of the form*

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where $a, b \in \mathbb{C}$ which satisfy $|a|^2 + |b|^2 = 1$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a element of SU(2) where $a, b, c, d \in \mathbb{C}$. Since $A^*A = AA^* = I_2$, $A^* = A^{-1}$.

$$AA^* = \begin{pmatrix} a\bar{a} + b\bar{b} & a\bar{c} + b\bar{d} \\ c\bar{a} + d\bar{b} & c\bar{c} + d\bar{d} \end{pmatrix}$$

$$A^*A = \begin{pmatrix} a\bar{a} + c\bar{c} & b\bar{a} + d\bar{c} \\ c\bar{a} + a\bar{b} & b\bar{b} + d\bar{d} \end{pmatrix}$$

Since $det(A) = 1$ and $A^*A = AA^* = I_2$, we have following equations;

$$ad - bc = 1$$

$$a\bar{a} + b\bar{b} = c\bar{c} + d\bar{d} = 1$$

$$a\bar{a} + c\bar{c} = b\bar{b} + d\bar{d} = 1$$

$$a\bar{b} + c\bar{d} = c\bar{a} + d\bar{b} = 0$$

We have $c = -\bar{b}$ and $d = \bar{a}$ and $|a|^2 + |b|^2 = 1$. □

Proposition 1. [4] *Every finite subgroup of* SL(2,ℂ) *is conjugate to a finite subgroup of* SU(2).

Refer to above proposition, classifications of the finite subgroup of SL(2,ℂ) reduced to classifications of the finite subgroup of SU(2).

Definition 12. *The orthogonal group* O(n) *over* ℝ *is the set of* $n \times n$ *matrices whose elements satisfy* $A^tA = I$.

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^tA = I\}$$

The special orthogonal group SO(n) *is a subgroup of* O(n) *whose elements have determinant 1.*

$$SO(n) = \{A \in O(n) | det(A) = 1, A^tA = I\}$$

Remark 2. SO(3) *is also called the rotation group of* $\mathbb{R}^3$*. Its elements can be de-*

scribed, in a suitable basis, as one of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

This matrix describes rotations around the axis through the origin.

Proposition 2. [4] SU(2) *is a 2-to-1 cover of* SO(3)*. More precisley, there exists a surjective group homomorphism* $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ *with* $Ker(\pi) = \{\pm I_2\}$

Hence, the problem of classifying finite subgroup of SL(2,$\mathbb{C}$) up to conjugacy reduced to classifications of finite subgroups of SO(3). After the classification, we may obtain the desire results by lifting these finite groups back to SL(2,$\mathbb{C}$).

Lemma 3. [4] SU(2)$\cong S^3 \cong$ Sp(1) *where* Sp(1) *is the set of unit quaternions.*

Proof. Let $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ be a element of SU(2). Since $a, b \in \mathbb{C}$, they can be written as tuples, $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

$$\begin{aligned} |a|^2 &= a_1^2 + a_2^2 \\ |b|^2 &= b_1^2 + b_2^2 \end{aligned}$$

Thus, we have a map

$$\begin{array}{ccc} \mathit{SU(2)} & \to & S^3 \\ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} & \mapsto & (a_1, a_2, b_1, b_2). \end{array}$$

Define a map ϕ by

$$\begin{array}{cccc} \phi : & \mathit{SU(2)} & \to & \mathit{Sp(1)} \\ & \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} & \mapsto & a + bj = a_1 + a_2 i + b_1 j + b_2 k \end{array}$$

□

Define an action of Sp(1) on $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ by

$$\begin{array}{cccccc} \mathrm{Sp}(1) & \to & \mathrm{SO}(3) & & & \\ x & \mapsto & \rho_x : & \mathrm{Im}(\mathbb{H}) & \to & \mathrm{Im}(\mathbb{H}) \\ & & & v & \mapsto & \rho_x(v) = xvx^* \end{array}$$

Note that this action is linear, *i.e.* $xvx^* + xwx^* = x(v + w)x^*$. Now, let $v = ai+bj+jk$ for $a, b, c \in \mathbb{R}$. Since $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$, we can identify $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$. Then for some $x = x_0+x_1i+x_2j+x_3k \in \mathbb{H}$. Denote $\overrightarrow{v} = (a, b, c)$ and $\overrightarrow{A} = (x_1, x_2, x_3)$ as identified 3 dimensional vectors of pure imaginary quaternions.

Then we may obtain $\rho_x(v)$ as

$$\begin{aligned}\rho_x(v) &= xvx^* \\ &= (x_0 + \overrightarrow{A})v(x_0 - \overrightarrow{A}) \\ &= (x_0^2 - |\overrightarrow{A}|^2)\overrightarrow{v} + 2 < \overrightarrow{A}, \overrightarrow{v} > \overrightarrow{A} + 2x_0\overrightarrow{A} \times \overrightarrow{v}\end{aligned}$$

where $< *, * >$ is an Euclidean inner product.

Remark 3. *Consider 2 cases of possible $\overrightarrow{v}$'s.*

Case 1. $\overrightarrow{v} \perp \overrightarrow{A}$

Since those two vectors are perpendicular, their Euclidean inner product is zero. Hence we have

$$\begin{aligned}\rho_x(v) &= (x_0^2 - |\overrightarrow{A}|^2)\overrightarrow{v} + 2x_0\overrightarrow{A} \times \overrightarrow{v} \\ &= (x_0^2 - |\overrightarrow{A}|^2)\overrightarrow{v} + 2x_0|\overrightarrow{A}|(\frac{\overrightarrow{A}}{|\overrightarrow{A}|} \times \overrightarrow{v})\end{aligned}$$

Now we observe that

$$\begin{aligned}(x_0 - |\overrightarrow{A}|^2)^2 + (2x_0|\overrightarrow{A}|)^2 &= x_0^2 + 2x_0|\overrightarrow{A}|^2 + |\overrightarrow{A}|^4 \\ &= (x_0 + |\overrightarrow{A}|^2)^2 \\ &= 1\end{aligned}$$

Then we can identify

$$\cos\theta = x_0^2 - |\overrightarrow{A}|^2$$

$$\sin\theta = 2x_0|\overrightarrow{A}|$$

for some $\theta \in [0, 2\pi)$.

Case 2. $\overrightarrow{v} // \overrightarrow{A}$

Since two vectors are parallel, we may denote $\overrightarrow{v} = t\overrightarrow{A}$ *for some* $t \in \mathbb{R}$.

$$\begin{aligned}
\rho_x(v) &= (x_0^2 - |\overrightarrow{A}|^2)\overrightarrow{v} + 2 < \overrightarrow{A}, \overrightarrow{v} > \overrightarrow{A} \\
&= \left(t(x_0^2 - |\overrightarrow{A}|^2) + 2 < \overrightarrow{A}, \overrightarrow{v} > \right)\overrightarrow{A} \\
&= t(x_0^2 + |\overrightarrow{A}|^2)\overrightarrow{A} \\
&= t\overrightarrow{A}
\end{aligned}$$

Hence, $\overrightarrow{A}$ *means the axis of rotation. Thus, if we have a set of quaternions which are rotations of an objects, pure quaternion provides axis and the angle around the axis.*

Refer to [4], we can see an element of SU(2) as $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ and map this to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$ which is an element of SO(3). We will use this map to obtain quaternions corresponde to vertices of polyhedrons.

3 Main study

3.1 Construction

Let $\{P_i\}$ be the set of vertices of a regular polyhedron. These vertices can be interpreted in terms of 3 dimensional vectors in $\mathbb{R}$. Then we can compute the barycenters of edges and faces. See Table 1.

	Barycenters
vertices	P_i
edges	$\frac{P_i + P_j}{2}$
faces	$\frac{P_i + P_j + P_k}{3}$

Table 1: Barycenter of vertices, edges, faces of a polyhedron whose vertices are P_i's

The degree of the vertex, edge and face,respectively, is the number of rotations fixing the vertex, edge and face, respectively. For instance, tetrahedron, octahedron and icosahedron have equilateral triangle faces, degree of face is all equal to 3.

Remark 4. *For a given quaternion* $q = a + bi + cj + dk = a + \overrightarrow{A}$*,*

$$e^q = e^{a+\overrightarrow{A}} = e^a(\cos|\overrightarrow{A}| + \frac{\overrightarrow{A}}{|\overrightarrow{A}|}\sin|\overrightarrow{A}|)$$

For a pure quaternions, we have

$$e^q = (\cos|\overrightarrow{A}| + \frac{\overrightarrow{A}}{|\overrightarrow{A}|}\sin|\overrightarrow{A}|)$$

We will use this remark to compute quaternions correspond to barycenters of the polyhedrons.

I will introduce the algorithm which gives a way to find the polytopes correspond to binary polyhedral groups. At first, We are going to focus on finding 3 dimensional polyhedra which gives information of corresponding 4-polytope.

Step 1: Let $\{p, q\}$ be a regular polyhedron. Find the barycenters of vertices, edges and faces.

Step 2: The elements of binary polyhedral group corresponds to $\{p, q\}$ can be written in term of quaternions, for instance, of the form $\cos\frac{\pi}{q} + \sin\frac{\pi}{q}\frac{P_i}{|P_i|}$. By using degrees and barycenters, find ρ_B^{-1}'s, the quaternions correspond to the barycenter B. This quaternion presents the elements of binary polyhedral group.

Step 3: Take an orthogonal projection ρ_B^{-1} onto $\mathrm{Im}(\mathbb{H})$. Then we have elements of the form $\sin\frac{\pi}{n}\frac{B}{|B|}$ where n is a degree of the barycenter, B is a barycenter.

Step 4: Find appropriate projected quaternions in $\mathrm{Im}(\mathbb{H})$ which form a polyhedron $\{p, q\}$. For instance, tetrahedron case, the elements which are the vertices of a regular tetrahedron are

$$P_1 = \frac{i+j+k}{2} \quad P_2 = \frac{i-j-k}{2} \quad P_3 = \frac{-i+j-k}{2} \quad P_4 = \frac{-i-j+k}{2}$$

Since we know barycenters of faces, we may have vertices of the dual polyhedron. By taking union of original polyhedron and dual polyhedron, we obtain a 3 dimensional dual compound and it gives us way to verify the elements of binary polyhedral group which form 4-polytope.

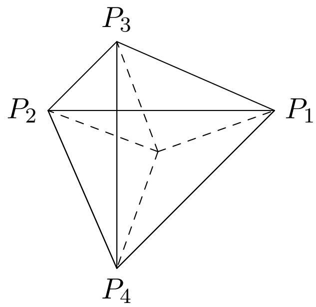
Step 5: Compute the original quaternions of the pure imaginary quaternions and find the 4 polytope whose vertices are the original quaternions. The remainders are also form 4 polytope. And then, take union of two polytopes. This polytope gives an intuitive way to understand the binary polyhedral group.

All each steps, we obtain the result to fill out the construction table. See Table 2.

	Degree	Barycenters	$\rho_{P_i}^{-1} \in \mathbb{H}$	$\mathrm{Im}(\mathbb{H})$
vertices	q	P_i	$\cos\frac{\pi}{q} + \sin\frac{\pi}{q}\frac{P_i}{\lvert P_i\rvert}$	$\sin\frac{\pi}{q}\frac{P_i}{\lvert P_i\rvert}$
edges	2	$\frac{P_i+P_j}{2}$	$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$	$\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$
faces	3	$\frac{P_i+P_j+P_k}{3}$	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$	$\frac{\sqrt{3}}{2}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$

Table 2: Construction Table

3.2 Binary Tetrahedral group



	Degree	Barycenters	$\rho_{P_i}^{-1} \in \mathbb{H}$	$\mathrm{Im}(\mathbb{H})$
vertices	3	P_i	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{P_i}{\lvert P_i \rvert}$	$\frac{\sqrt{3}}{2}\frac{P_i}{\lvert P_i \rvert}$
edges	2	$\frac{P_i + P_j}{2}$	$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$	$\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$
faces	3	$\frac{P_i + P_j + P_k}{3}$	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$	$\frac{\sqrt{3}}{2}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$

Table 3: Construction Table of Binary tetrahedral group

The binary tetrahedral group BT_{24} is the group of 24 elements with generators i, j and $\frac{1}{2}(1 + i + j + k)$. Take P_i's as following;

$$P_1 = \frac{i+j+k}{2} \quad P_2 = \frac{i-j-k}{2} \quad P_3 = \frac{-i+j-k}{2} \quad P_4 = \frac{-i-j+k}{2}$$

The convex hull of P_i's form a regular tetrahedron. Then the imaginary part of quaternion elements become

$$
\begin{aligned}
\frac{\sqrt{3}}{2}\frac{P_i}{|P_i|} &= P_1 \\
\frac{\frac{P_i+P_j}{2}}{|\frac{P_i+P_j}{2}|} &= P_i+P_j \\
\frac{\sqrt{3}}{2}\frac{\frac{P_i+P_j+P_k}{3}}{|\frac{P_i+P_j+P_k}{3}|} &= P_i+P_j+P_k
\end{aligned}
$$

Note that $P_i+P_j+P_k=-P_l$ where P_l is the vertex opposite to the face bounded by vertices P_i, P_j and P_k. The length of $|P_i|$ is equal to $\frac{\sqrt{3}}{2}$ and the length of $|P_i+P_j|$ is equal to 1 . See Table 4.

Let O be the center of regular tetrahedron. Since

$$P_1+P_2+P_3+P_4=0$$

O is an origin.

Remark 5. $\cos(\angle P_iOP_j)$ *is* $-\frac{1}{3}$.

For vertices, $\sin\frac{\pi}{3}\frac{P_i}{|P_i|}=\frac{\sqrt{3}}{2}\frac{P_i}{r}=P_i$ and for edges, $\sin\frac{\pi}{2}\frac{\frac{P_i+P_j}{2}}{|\frac{P_i+P_j}{2}|}=P_i+P_j$. Lastly, for barycenters of faces, $\sin\frac{\pi}{3}\frac{\frac{P_i+P_j+P_k}{3}}{|\frac{P_i+P_j+P_k}{3}|}=-P_l$.

	$\mathrm{Im}(\mathbb{H})$	Length	Number of elements
vertices	P_i	$\dfrac{\sqrt{3}}{2}$	4
edges	$P_i + P_j$	1	6
faces	-P_i	$\dfrac{\sqrt{3}}{2}$	4

Table 4: Pure imaginary part of elements of BT_{24} and its lengths

Note that the the barycenters of faces projected onto $\mathrm{Im}(\mathbb{H})$ are conjugate of vertices P_i. Hence we have two sets of pure quaternions forming interpenetrating tetrahedrons of edge length $\dfrac{\sqrt{3}}{2}$.

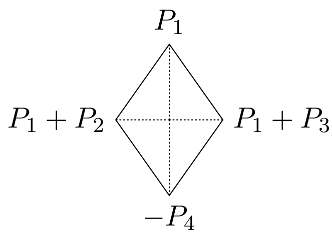
$$T : \{ \text{ a convex hull formed by vertices } P_1, P_2, P_3 \text{ and } P_4 \text{ } \}$$
$$\tilde{T} : \{ \text{ a convex hull formed by vertices } -P_1, -P_2, -P_3 \text{ and } -P_4 \text{ } \}$$

Note that the tetrahedron is self-dual and $\{-P_i\}$'s and the barycenters of the faces of the original tetrahedron T are parallel. The centroids of T and $\tilde{T}$ are coincides. Hence, T and $\tilde{T}$ are dual.

Now, take an union of T and $\tilde{T}$. Then we have first polyhedron compound; tetrahedral 2 compound. In summary, we may have following;

Proposition 3. *Let P_i's be the pure quaternions which can be obtained by projecting elements of BT_{24} onto $Im(\mathbb{H})$. Let T be the convex hull whose vertices are P_1, P_2, P_3 and P_4. Then, $T \cup \tilde{T}$ is a tetrahedral 2 compound where $\{-P_i\}$'s are the set of vertices of the dual of the tetrahedron $\tilde{T}$.*

Now, we are going to consider the vertices of $T \cup \tilde{T}$ and the barycenters of edges of the form $P_i + P_j$. For instance, there is a face whose vertices are P_1, $P_1 + P_2$, $P_1 + P_3$ and $-P_4$.



The lengths of the edges are all same, $\dfrac{\sqrt{3}}{2}$. Let us check the angle between the diagonals.

$$
\begin{aligned}
(P_1 + P_4) \cdot ((P_1 + P_2) - (P_1 + P_3)) &= (P_1 + P_4) \cdot (P_2 - P_3) \\
&= 0
\end{aligned}
$$

Hence, two diagonals are perpendicular to each other. Lastly, consider the midpoint of the diagonals.

$$P_1 + P_2 + P_1 + P_3 = (P_1 + P_2 + P_3) + P_1 = -P_4 + P_1$$

Hence the mid points of two diagonals coincides and thus 4 vertices are coplanar. Thus, this face is a rhombus.

Now, we have the set of pure quaternions which are vertices of 2-tetrahedron compound. The elements provides information of vertices and faces. Note that this polyhedra is not convex. In summary, we have following proposition.

Proposition 4. *The convexhull of the pure imaginary parts of the quaternions in BT_{24} is a rhombic dodecahedron.*

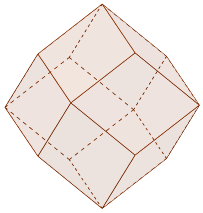


Figure 1: Rhombic dodecahedron

Remark 6. *Rhombic dodecahedron has 2 kinds of vertices; the vertices meeting 3 edges and the vertices meeting 4 edges. See Figure 1.*

Remark 7. *The dual of rhombic dodecahedron is cuboctahedron. A cuboctahedron is a polyhedron with 8 triangular faces and 6 square faces. See Figure 2.*

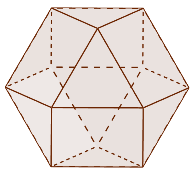


Figure 2: Cuboctahedron

Since the original quaternions of the vertices of rhombic dodecahedron are unit quaternions, we can compute real part of the given pure imaginary quaternions. First of all, the vertices of $T \cup \tilde{T}$ are of the form $\frac{1}{2}(\pm i \pm j \pm k)$. Hence, the original quaternions are of the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. Note that these quaternions are the vertices of 8 cell. Lastly, the original quaternion of the edges are

$$\pm 1, \quad \pm i, \quad \pm j, \quad \pm k$$

Recall that those are the vertices of 16 cell.

Theorem 1. *The 4 dimensional polytope whose vertices are the elements of binary tetrahedral group is a convex hull of an union of 8 cell and its dual 16 cell, i.e. 24 cell.*

Proof. Consider the original quaternions of pure imaginary quaternions which are the vertices of tetrahedron 2 compound $T \cup \tilde{T}$. Let $a \pm P_i$ be of the form of original quaternion of $P_i \in \mathrm{Im}(\mathbb{H})$. Since $a \pm P_i \in \mathrm{Sp}(1)$, $a = \pm\frac{1}{2}$. Thus, the original quaternions are

$$\left\{(x_1, x_2, x_3, x_4) | -\frac{1}{2} \leq x_i \leq \frac{1}{2}, i = 1, 2, 3, 4\right\}$$

Hence, we have

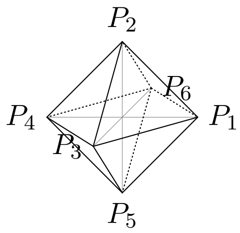
$$\begin{gathered}
\pm\frac{1}{2}(1 + i - j + k), \pm\frac{1}{2}(-1 + i + j + k) \\
\pm\frac{1}{2}(1 - i + j + k), \ \pm\frac{1}{2}(1 + i + j - k) \\
\pm\frac{1}{2}(1 - i + j - k), \ \pm\frac{1}{2}(1 + i - j - k) \\
\pm\frac{1}{2}(1 - i - j + k), \ \pm\frac{1}{2}(1 + i + j + k)
\end{gathered}$$

There are 8 facets which are cubes. Now consider the original quaternion of the barycenters of the edges. Then these are of the form

$$\pm 1, \quad \pm i, \quad \pm j, \quad \pm k$$

These unit quaternions form 16 cell. Hence, we obtain 8 cell from the dual compound and its dual from barycenters of the edges. The convex hull of the union of 8 cell and 16 cell is 24 cell. $\square$

3.3 Binary Octahedral group



	Degree	Barycenter	$\rho_B^{-1} \in \mathbb{H}$	$\mathrm{Im}(\mathbb{H})$
vertices	4	P_i	$\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\frac{P_i}{\lvert P_i\rvert}$	$\frac{\sqrt{2}}{2}\frac{P_i}{\lvert P_i\rvert}$
	2	P_i	$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\frac{P_i}{\lvert P_i\rvert}$	$\frac{P_i}{\lvert P_i\rvert}$
edges	2	$\frac{P_i+P_j}{2}$	$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$	$\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$
faces	3	$\frac{P_i+P_j+P_k}{3}$	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$	$\frac{\sqrt{3}}{2}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$

Table 5: Construction Table of Binary Octahedral Group in terms of quaternions

The binary octahedral group BO_{48} is group of 48 elements with generators $\frac{1+i}{\sqrt{2}}, j$ and $\frac{1}{2}(1+i+j+k)$. In the construction table, there are 2 kinds of degree of the vertices; degree 4 and degree 2. Take P_i's as following;

$$P_1 = i \qquad P_2 = j \qquad P_3 = k$$

$$P_4 = -i \quad P_5 = -j \quad P_6 = -k$$

The convexhull of P_i's form a regular octahedron. The lengths of $|P_i|$ and $|P_i+P_j|$

are $|P_i| = 1$ and $|P_i + P_j| = \sqrt{2}$. Then the imaginary part of quaternion elements become

$$\frac{\sqrt{2}}{2}\frac{P_i}{|P_i|} = \frac{\sqrt{2}}{2}P_i$$

$$\frac{\frac{P_i + P_j}{2}}{|\frac{P_i + P_j}{2}|} = \frac{1}{\sqrt{2}}(P_i + P_j)$$

$$\frac{\sqrt{3}}{2}\frac{\frac{P_i + P_j + P_k}{3}}{|\frac{P_i + P_j + P_k}{3}|} = \frac{1}{2}(P_i + P_j + P_k)$$

Let O be the centroid of regular octahedron. Since

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 0$$

O is an origin.

Remark 8. $\cos(\angle P_i O P_j)$ *is 0 where* $|i - j| \neq 3$.

	$\mathrm{Im}(\mathbb{H})$	Length	Number of elements
vertices	P_i	1	6
edges	$\frac{P_i + P_j}{\sqrt{2}}$	1	12
faces	$\frac{P_i + P_j + P_k}{2}$	$\frac{\sqrt{3}}{2}$	8

Table 6: Pure imaginary part of elements of BO_{48} and its lengths

The barycenters of faces form a cube of edge length 1. This cube is dual of the regular octahedron whose vertices are $\pm i$, $\pm j$ and $\pm k$.

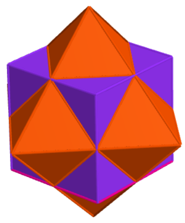


Figure 3: Cube-Octahedron Compound

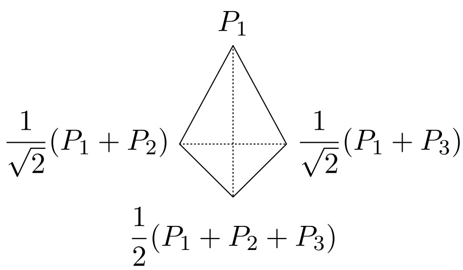
Hence we have following proposition;

Proposition 5. *Let P_i's be the pure quaternions which can be obtained by projecting elements of BO_{48} onto* $\mathrm{Im}(\mathbb{H})$. *Let O be the regular octahedron whose vertices are P_i $(1 \leq i \leq 6)$. Then, $O \cup \tilde{O}$ is a cube-octahedron compound where $\tilde{O}$ is a convexhull of vertices of the form $\dfrac{P_i + P_j + P_k}{2}$.*

Note that

$$\frac{P_i + P_j + P_k}{\sqrt{3}} = \frac{\frac{P_i + P_j + P_k}{2}}{\left|\frac{P_i + P_j + P_k}{2}\right|}$$

Now, we are going to consider the vertices of $O \cup \tilde{O}$ and the barycenter of edges projected onto $\mathrm{Im}(\mathbb{H})$ of the form $\dfrac{P_i + P_j}{\sqrt{2}}$. Then, there is a face whose vertices are P_1, $\dfrac{1}{\sqrt{2}}(P_1 + P_2)$, $\dfrac{1}{\sqrt{2}}(P_1 + P_3)$ and $\dfrac{1}{2}(P_1 + P_2 + P_3)$.



Unlike the case of tetrahedral group, this face have 2 different edge lengths.

$$|P_1 - \frac{1}{\sqrt{2}}(P_1 + P_2)| = \sqrt{2 - \sqrt{2}}$$

$$|\frac{1}{2}(P_1 + P_2 + P_3) - \frac{1}{\sqrt{2}}(P_1 + P_2)| = \frac{\sqrt{7 - 4\sqrt{2}}}{2}$$

Hence, this face is a kite.

Even edge lengths are not same, diagonals are perpendicular.

$$(P_1 - \frac{1}{2}(P_1 + P_2 + P_3)) \cdot (\frac{1}{\sqrt{2}}(P_1 + P_2) - \frac{1}{\sqrt{2}}(P_1 + P_3)) = 0$$

Hence, two diagonals are perpendicular to each other. Lastly, we need to check those two diagonals are meet at a point or not. It gives the information of the kite, it is either folded or not.

The equation of a plane formed by P_1, $\frac{1}{\sqrt{2}}(P_1 + P_2)$ and $\frac{1}{\sqrt{2}}(P_1 + P_3)$ is

$$\frac{1}{2}(x - 1) - (\frac{\sqrt{2} - 1}{2\sqrt{2}})y + (\frac{\sqrt{2} - 1}{2\sqrt{2}})z = 0$$

The equation of a plane formed by $\frac{1}{2}(P_1 + P_2 + P_3)$, $\frac{1}{\sqrt{2}}(P_1 + P_2)$ and $\frac{1}{\sqrt{2}}(P_1 + P_3)$ is

$$\frac{2 - \sqrt{2}}{2}(x - \frac{1}{2}) - (\frac{\sqrt{2} - 1}{2\sqrt{2}})(y - \frac{1}{2}) - \frac{1}{2}(z - \frac{1}{2}) = 0$$

The centroid of the triangle bounded by $\frac{1}{\sqrt{2}}(P_1 + P_2)$, $\frac{1}{\sqrt{2}}(P_1 + P_3)$ and $\frac{1}{\sqrt{2}}(P_2 +$

$P_3)$ is

$$(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$$

and its length is $\frac{\sqrt{6}}{3}$. Since the length of $\frac{1}{2}(P_1 + P_2 + P_3)$ is $\frac{\sqrt{3}}{2}$, the kite is folded along the line connecting $\frac{1}{\sqrt{2}}(P_1 + P_2)$ and $\frac{1}{\sqrt{2}}(P_1 + P_3)$.

Hence, in summary, we have following proposition.

Proposition 6. *The polytope whose vertices are pure imaginary parts of the quaternions in BO_{48} is a polyhedra with 2 different kinds of 48 isosceles triangles; 24 isosceles triangles whose 2 equal side lengths are $\sqrt{2-\sqrt{2}}$, and 24 isosceles triangles whose 2 equal side lengths are $\frac{\sqrt{7-4\sqrt{2}}}{2}$.*

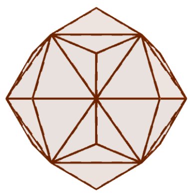


Figure 4: BO_{48} Polytope whose vertices are pure imaginary parts of the quaternions

It looks like a little bit stretched version of cube-octahedron compound.

Theorem 2. *The 4 dimensional polytope whose vertices are the elements of binary octahedral group is an union of 24 cell and its dual, another 24 cell.*

Proof. Consider the original quaternions of the pure imaginary quaternions which form BO_{48} polytope in previous proposition. In previous section, we have already shown that the original quaternions of the pure imaginary quaternions of the form P_i's and $\frac{1}{2}(P_i + P_j + P_k)$ are vertices of 24 cells. All edges are length 1.

Since the length of $\frac{1}{\sqrt{2}}(P_i + P_j)$ is equal to 1, pure quaternions $\frac{1}{\sqrt{2}}(P_i + P_j)$ are original quaternions. There are 12 of such quaternions. Together with the 12 quaternions of the form $\pm\frac{1}{\sqrt{2}}(1 + P_i)$, it form dual 24 cell whose edge lengths are $\sqrt{2}$. Hence we obtain the result.

$\square$

4 Conclusion

There are 5 types of the classification of finite subgroups of SL(2, $\mathbb{C}$) upto conjugacy. The binary polyhedral groups are double cover of rotation groups of corresponding regular polyhedron. By using the construction algorithm, we can find the appropriate 4-polytopes of binary polyhedral groups, rather compute all the elements in BT_{24} and BO_{48}. The elements in BT_{24} and BO_{48} can be derive directly b using degree of vertices, edges and faces of corresponding regular polyhedrons. In fact, BT_{24} is the group of 24 elements;

$$\pm 1, \pm i, \pm j, \pm k$$

$$\pm\frac{1}{2}(1+i-j+k), \pm\frac{1}{2}(-1+i+j+k)$$

$$\pm\frac{1}{2}(1-i+j+k),\ \pm\frac{1}{2}(1+i+j-k)$$

$$\pm\frac{1}{2}(1-i+j-k),\ \pm\frac{1}{2}(1+i-j-k)$$

$$\pm\frac{1}{2}(1-i-j+k),\ \pm\frac{1}{2}(1+i+j+k)$$

The elements in the first line form 16 cell and the others form 8 cell. This facts coincides our results. In the same fashion, BO_{48} can be divided into 2 subgroups of quaternions where each of the subgroup forms 24 cell.

5 Further Studies

Naturally, it arises questions;

- Can we explain binary Icosahedral group by applying same algorithm?
- Can we explain such 4 dimensional compounds in terms of Coxeter-Dynkin Diagram?

For the first question, it seems possible. Table 7 shows the construction table of the binary icosahedral group.

	Degree	Barycenter	$\rho_B^{-1} \in \mathrm{Sp}(1)$	$\mathrm{Im}(\mathbb{H})$
vertices	5	P_i	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{P_i}{\lvert P_i\rvert}$	$\frac{\sqrt{3}}{2}\frac{P_i}{\lvert P_i\rvert}$
edges	2	$\frac{P_i+P_j}{2}$	$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$	$\frac{\frac{P_i+P_j}{2}}{\lvert\frac{P_i+P_j}{2}\rvert}$
faces	3	$\frac{P_i+P_j+P_k}{3}$	$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$	$\frac{\frac{P_i+P_j+P_k}{3}}{\lvert\frac{P_i+P_j+P_k}{3}\rvert}$

Table 7: Construction Table of Binary Icosahedral group in terms of quaternions

It is known that the the order of binary icosahedral group so 120 and its elements form 600 cell. But it is not easy to find the vertices of dual compound, icosahedron-dodecahedron compound.

Our expectations are following;

1. In BI_{120}, there are quaternions whose pure imaginary parts form icosahedron in $\mathrm{Im}(\mathbb{H})$. And by barycentric subdivision, we can find vertices of its dual, a

dodecahedron. This will lead us to verify the pure quaternions which are the vertices of icosahedron-dodecahedron compound.

2. By using informations of icosahedron-dodecahedron compound, we can compute the set of original quaternions of the pure quaternions since original quaternions are in Sp(1). In the same fashion, part the binary icosahedral group into 2 subgroups, one from the icosahedron-dodecahedron compound, the other obtained by finding barycenter of the cell. But the polytope in $\mathbb{H}$ corresponds to icosahedron-dodecahedron compound may not seem regular polytope. We have 62 pure quaternions by projecting all the elements in BI_{120} onto $\mathrm{Im}(\mathbb{H})$ where $\alpha = \dfrac{1+\sqrt{5}}{2}$, the golden ratio.

$$\pm i, \pm j, \pm k$$

$$\pm\frac{1}{\sqrt{3}}(i+j+k), \pm\frac{1}{\sqrt{3}}(-i+j+k)$$
$$\pm\frac{1}{\sqrt{3}}(i-j+k),\ \pm\frac{1}{\sqrt{3}}(i+j-k)$$

$$\pm\frac{1}{\sqrt{3}}(\alpha i+\alpha^{-1}j),\ \pm\frac{1}{\sqrt{3}}(\alpha i-\alpha^{-1}j)$$
$$\pm\frac{1}{\sqrt{3}}(\alpha^{-1}i+\alpha k)\ \pm\frac{1}{\sqrt{3}}(\alpha^{-1}i-\alpha k)$$
$$\pm\frac{1}{\sqrt{3}}(\alpha j+\alpha^{-1}k), \pm\frac{1}{\sqrt{3}}(\alpha j-\alpha^{-1}k)$$

$$\pm\frac{1}{1+\alpha^{-1}}(\alpha^{-1}i+j),\ \pm\frac{1}{1+\alpha^{-1}}(\alpha^{-1}i-j)$$
$$\pm\frac{1}{1+\alpha^{-1}}(i+\alpha^{-1}k),\ \pm\frac{1}{1+\alpha^{-1}}(i-\alpha^{-1}k)$$
$$\pm\frac{1}{1+\alpha^{-1}}(\alpha^{-1}j+k),\pm\frac{1}{1+\alpha^{-1}}(\alpha^{-1}j-k)$$

$$\pm\frac{1}{2}(i+\alpha^{-1}j+\alpha k),\pm\frac{1}{2}(-i+\alpha^{-1}j+\alpha k),\pm\frac{1}{2}(i-\alpha^{-1}j+\alpha k),\pm\frac{1}{2}(i+\alpha^{-1}j-\alpha k)$$
$$\pm\frac{1}{2}(\alpha i+j+\alpha^{-1}k),\pm\frac{1}{2}(-\alpha i+j+\alpha^{-1}k),\pm\frac{1}{2}(\alpha i-j+\alpha^{-1}k),\pm\frac{1}{2}(\alpha i+j-\alpha^{-1}k)$$
$$\pm\frac{1}{2}(\alpha^{-1}i+\alpha j+k),\pm\frac{1}{2}(-\alpha^{-1}i+j+\alpha k),\pm\frac{1}{2}(\alpha^{-1}i-j+\alpha k),\pm\frac{1}{2}(\alpha^{-1}i+j-\alpha k)$$

If we can apply the construction and obtain the desired results, it provides good way to understand the nature of binary polyhedral groups.

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이 논문은 복소수 체 위에서 정의된 2×2 특수 선형군의 유한 부분군을 3 차원 정다면체를 이용하여 구하는 방법을 설명한다. 유한 부분군은 2×2 정사각 행렬로 이루어진 특수 유니터리군의 유한 부분군과 켤레를 이루며 3차원 유한회전군의 이중덮개이다. 3차원 정다면체의 회전은 유한 회전군을 이루며, 이 때 쌍대성을 갖는 정다면체의 회전은 같은 회전축을 가지므로, 동일한 회전군으로 나타난다. 이 3 차원 다면체의 유한 회전군은 특수 유니터리군에서 유한개의 원소를 갖는 바이너리 다면체군과 대응한다. 특수 유니터리군의 원소는 사원수와 대응한다는 사실을 이용하여, 사원수의 허수부가 이루는 3차원 다면체를 구하는 방법을 소개하고, 이들이 바이너리 다면체군과 어떠한 관계를 갖는지 설명하고자 한다.