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A finiteness property and an automatic structure for Coxeter groups

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1. Root systems of Coxeter groups

Let (W, R) be a Coxeter system, as defined in [6], and for all $r, s \in R$ let m_{rs} be the order of rs in W. Let $\Pi = \{\alpha_r \mid r \in R\}$ denote the set of *simple roots* and V the **R**-vector space with basis Π. Let $\langle\ ,\ \rangle : V \times V \to \mathbf{R}$ be a symmetric bilinear form which satisfies $\langle\alpha_r, \alpha_s\rangle = -\cos(\pi/m_{rs})$ for all $r, s \in R$ such that m_{rs} is finite, and $\langle\alpha_r, \alpha_s\rangle \leq -1$ if m_{rs} is infinite; then $r \cdot v = v - 2\langle v, \alpha_r\rangle\alpha_r$ (for all $r \in R$ and $v \in V$) determines a faithful action of W on V which preserves $\langle\ ,\ \rangle$. We refer to such representations as *standard geometric realizations* of W.

The set $\Phi = \{w \cdot \alpha_r \mid w \in W,\ r \in R\}$ is called the *root system* of W in V. The subsets $\Phi^+ = \{\sum_{r\in R} \lambda_r\alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r \in R\}$ and $\Phi^- = \{\alpha \in V \mid -\alpha \in \Phi^+\}$ are the sets of *positive roots* and *negative roots* respectively. Define the *support* of $\alpha = \sum_{r\in R} \lambda_r\alpha_r \in \Phi$ to be the set $\mathrm{supp}_R(\alpha) = \{r \in R \mid \lambda_r \neq 0\}$. For $w \in W$ define the *length* of w to be $l(w) = \min\{l \in \mathbb{N} \mid w = r_1 \cdots r_l \text{ for some } r_1, \ldots, r_l \in R\}$ (where $\mathbb{N}$ is the set of nonnegative integers), and let $N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}$.

We start with a proposition which lists an assortment of well known facts (see [6], for example).

Proposition 1.1 (i) $\Phi = \Phi^+ \cup \Phi^-$.
(ii) $|N(w)| = l(w)$ *for all* $w \in W$.
(iii) *Let* $r_1, \ldots, r_n, s \in R$ *with* $l(r_1 \cdots r_n s) < l(r_1 \cdots r_n)$. *There exists* $i \in \{1, \ldots, n\}$ *such that* $r_1 \cdots r_n s = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$.
(iv) *For all* $w \in W$ *and* $r \in R$,

$$l(wr) = \begin{cases} l(w) + 1 & \text{if } w \cdot \alpha_r \in \Phi^+, \\ l(w) - 1 & \text{if } w \cdot \alpha_r \in \Phi^-. \end{cases}$$

(v) *For all* $\alpha \in \Phi$ *there is a uniquely defined element* $r_\alpha \in W$ *such that* $r_\alpha = wrw^{-1}$ *for all* $w \in W$, $r \in R$ *with* $\alpha = w \cdot \alpha_r$. *Moreover,* $r_\alpha \cdot v = v - 2\langle\alpha, v\rangle\alpha$ *for all* $v \in V$.

(vi) *Let* $\alpha, \beta \in \Phi^+$ *with* $\langle \alpha, \beta \rangle \leq -1$, *and let* $(r_\alpha r_\beta)^n \cdot \alpha = \lambda_n \alpha + \mu_n \beta$ *for each* $n \in \mathbb{N}$. *Then* $\lambda_n \geq \mu_n + 1$ *and* $\mu_{n+1} \geq \lambda_n + 1$ *(for all* $n \in \mathbb{N}$*).*

The following lemma is a straightforward consequence of Proposition 1.1 (ii).

Lemma 1.2 *Let* $v, w \in W$. *Then* $l(vw^{-1}) = l(v) + l(w)$ *if and only if* $N(v) \cap N(w) = \emptyset$.

The elements r_α defined in Proposition 1.1 (v) are called the *reflections* in W, and the elements of $R = \{ r_\alpha \mid \alpha \in \Pi \}$ are called *simple reflections*.

For each set J of simple reflections, the subgroup W_J generated by J is called a *parabolic subgroup* of W. Each W_J is itself a Coxeter group, as can be seen from its action on the subspace of V spanned by $\{ \alpha_r \mid r \in J \}$. The following result appears as an exercise in [1], but we include a proof for completeness' sake.

Proposition 1.3 *If* H *is a finite subgroup of* W *then there exists* $w \in W$ *and* $J \subseteq R$ *such that* W_J *is finite and* $wHw^{-1} \subseteq W_J$.

Proof Use induction on $|R|$. Choose any $v_0 \in V$ such that $\langle v_0, \alpha_r \rangle > 0$ for all $r \in R$; such a v_0 clearly exists since $\{ \alpha_r \mid r \in R \}$ is a linearly independent set and $\langle \alpha_r, \alpha_r \rangle \neq 0$ for all $r \in R$. If $w \in W$ is arbitrary and $\alpha \in \Phi^+$ then $\langle w \cdot v_0, \alpha \rangle = \langle v_0, w^{-1} \cdot \alpha \rangle$ is negative if and only if $\alpha \in N(w^{-1})$. So if H is a finite subgroup of W and $v = \sum_{h \in H} h \cdot v_0$ then $\langle v, \alpha \rangle > 0$ for all but (at most) finitely many $\alpha \in \Phi^+$. Now if there exists $\alpha \in \Phi^+$ with $\langle v, \alpha \rangle < 0$ then there must exist $r \in R$ such that $\langle v, \alpha_r \rangle < 0$, and then

$$|\{ \alpha \in \Phi^+ \mid \langle r \cdot v, \alpha \rangle < 0 \}| < |\{ \alpha \in \Phi^+ \mid \langle v, \alpha \rangle < 0 \}|$$

(since if $\alpha \in \Phi^+$ satisfies $\langle r \cdot v, \alpha \rangle < 0$, then $\langle v, r \cdot \alpha \rangle < 0$ and $r \cdot \alpha \in \Phi^+ - \{\alpha_r\}$). Repeating this argument yields a $g \in W$ such that $\langle g \cdot v, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$.

If $\langle g \cdot v, \alpha \rangle = 0$ for all $\alpha \in \Phi^+$ then $\langle v, \alpha \rangle = 0$ for all $\alpha \in \Phi$, and hence Φ is finite. This implies that W is finite (since, by Proposition 1.1 (ii), only the identity element of W fixes all roots), and so the desired conclusion holds with $J = R$. So suppose that there exist roots α with $\langle g \cdot v, \alpha \rangle \neq 0$, and let $K = \{ r \in R \mid \langle g \cdot v, \alpha_r \rangle = 0 \}$, a proper subset of R. If $x \in W$ satisfies $x \cdot (g \cdot v) = g \cdot v$, then for any $r \in R$ such that $l(rx) < l(x)$ we have

$$0 \leq \langle g \cdot v, \alpha_r \rangle = \langle x \cdot (g \cdot v), \alpha_r \rangle = \langle g \cdot v, x^{-1} \cdot \alpha_r \rangle \leq 0$$

since $x^{-1} \cdot \alpha_r \in \Phi^-$. So $r \in K$, and $(rx) \cdot (g \cdot v) = g \cdot v$. Repeating this argument we deduce that $x \in W_K$. But since $h \cdot v = v$ for all $h \in H$, it follows that $gHg^{-1} \subseteq W_K$. But $|K| < |R|$, and so the inductive hypothesis guarantees that a conjugate of gHg^{-1} is contained in a finite parabolic subgroup of W_K, whence the result.

The authors are indebted to Prof. M. J. Dyer for drawing their attention to the above result and its application to Proposition 2.6 below, which replaces their original much longer argument.

Proposition 1.4 (Dyer [4]) *If* $\alpha, \beta \in \Phi$ *and* $|\langle \alpha, \beta \rangle| < 1$, *then* $\langle \alpha, \beta \rangle = \cos(p\pi/q)$ *for some integers* p *and* q, *and the subgroup of* W *generated by* r_α *and* r_β *is a finite dihedral group.*

Proof Since $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$ the condition $|\langle \alpha, \beta \rangle| < 1$ implies, by an easy calculation, that the restriction of the form $\langle\ ,\ \rangle$ to the subspace V_0 spanned by α and β is positive definite.

Replacing α and β by $w \cdot \alpha$ and $w \cdot \beta$ for a suitable $w \in W$ we may assume that $\beta = \alpha_s$ for some $s \in R$, and replacing α by $-\alpha$ if need be, we may assume that $\alpha \in \Phi^+$. Let ν be the coefficient of α_s in α, and let $v = \alpha - \nu\alpha_s$. Then certainly $v \neq 0$, since α_s is the only positive root with support $\{s\}$, and $\alpha \neq \alpha_s$. If $\lambda v + \mu\alpha_s \in \Phi$, then by Proposition 1.1 (i) it follows that λ and μ are either both nonnegative or both nonpositive. But if $\cos^{-1}(\langle\alpha, \alpha_s\rangle)$ is not a rational multiple of π then there exist powers of $r_\alpha s$ which act on the Euclidean plane V_0 as rotations through arbitrarily small angles. Thus some power of $r_\alpha s$ will rotate α into the (skew) quadrant $\{\, \lambda v + \mu\alpha_s \mid \lambda > 0 \text{ and } \mu < 0 \,\}$, contradicting the fact that this set contains no roots. So $\cos^{-1}(\langle\alpha, \alpha_s\rangle)$ is a rational multiple of π, as required. Furthermore, the subgroup D generated by r_α and s acts on V_0 as a finite dihedral group. But since V_0 is Euclidean we see that V is the orthogonal direct sum of V_0 and $V_0^\perp$, and since D acts trivially on $V_0^\perp$ it follows that the representation of D on V_0 is faithful, so that D is finite dihedral.

Definition 1.5 (i) For each $\alpha \in \Phi^+$ define the *depth* of α (relative to R) to be $\mathrm{dp}(\alpha) = \min\{\, l \in \mathbb{N} \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in W \text{ with } l(w) = l \,\}$.

(ii) For $\alpha, \beta \in \Phi^+$ define $\alpha \preceq \beta$ if and only if the following condition holds: there exists $w \in W$ such that $\beta = w \cdot \alpha$ and $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = l(w)$. We write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

Lemma 1.6 *$\preceq$ is a partial order on Φ^+.*

Proof Suppose that $\alpha, \beta \in \Phi^+$ with $\alpha \preceq \beta$ and $\alpha \neq \beta$. Then there exists $w \in W$ with $w \neq 1$ such that $\beta = w \cdot \alpha$ and $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = l(w) \geq 1$. Thus $\mathrm{dp}(\beta) > \mathrm{dp}(\alpha)$, and so $\preceq$ must be antisymmetric. It remains to show that $\preceq$ is transitive.

Let $\alpha, \beta, \gamma \in \Phi^+$ with $\alpha \preceq \beta$ and $\beta \preceq \gamma$. Then there exist $v, w \in W$ such that $\beta = w \cdot \alpha$ and $\gamma = v \cdot \beta$, where $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = l(w)$ and $\mathrm{dp}(\gamma) - \mathrm{dp}(\beta) = l(v)$. Then $\gamma = vw \cdot \alpha$ and

$$\mathrm{dp}(\gamma) - \mathrm{dp}(\alpha) = \mathrm{dp}(\gamma) - \mathrm{dp}(\beta) + \mathrm{dp}(\beta) - \mathrm{dp}(\alpha) = l(v) + l(w),$$

and so it suffices to prove $l(vw) = l(v) + l(w)$. Let $u \in W$ such that $l(u) = \mathrm{dp}(\alpha)$ and $u \cdot \alpha \in \Phi^-$. Then $(uw^{-1}v^{-1}) \cdot \gamma = u \cdot \alpha \in \Phi^-$, and hence $l(uw^{-1}v^{-1}) \geq \mathrm{dp}(\gamma)$. Furthermore,

$$\begin{aligned} l(uw^{-1}v^{-1}) &\leq l(u) + l(w) + l(v) \\ &= \mathrm{dp}(\alpha) + \mathrm{dp}(\beta) - \mathrm{dp}(\alpha) + \mathrm{dp}(\gamma) - \mathrm{dp}(\beta) \\ &= \mathrm{dp}(\gamma). \end{aligned}$$

Hence $l(uw^{-1}v^{-1}) = l(u) + l(w) + l(v)$, and so $l(vw) = l(v) + l(w)$.

Lemma 1.7 *Let $r \in R$ and $\alpha \in \Phi^+ - \{\alpha_r\}$. Then*

$$\mathrm{dp}(r \cdot \alpha) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } \langle\alpha, \alpha_r\rangle > 0, \\ \mathrm{dp}(\alpha) & \text{if } \langle\alpha, \alpha_r\rangle = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } \langle\alpha, \alpha_r\rangle < 0. \end{cases}$$

Proof If $\langle\alpha, \alpha_r\rangle = 0$ then $r \cdot \alpha = \alpha - 2\langle\alpha, \alpha_r\rangle\alpha_r = \alpha$; hence trivially $\mathrm{dp}(r \cdot \alpha) = \mathrm{dp}(\alpha)$.

Suppose next that $\langle\alpha, \alpha_r\rangle > 0$. It suffices to show $\mathrm{dp}(r\cdot\alpha) < \mathrm{dp}(\alpha)$ since it is trivial that $\mathrm{dp}(\alpha) \leq \mathrm{dp}(r\cdot\alpha)+1$. To do so we construct a $w \in W$ with $w\cdot(r\cdot\alpha) \in \Phi^-$ and $l(w) < \mathrm{dp}(\alpha)$. Choose $v \in V$ such that $v\cdot\alpha \in \Phi^-$ and $l(v) = \mathrm{dp}(\alpha)$. If $v\cdot\alpha_r \in \Phi^-$ we choose $w = vr$; then $l(w) = l(v)-1$ by Proposition 1.1 (iv), and $w\cdot(r\cdot\alpha) = v\cdot\alpha \in \Phi^-$, as required. Hence we may assume that $v\cdot\alpha_r \in \Phi^+$. Now

$$v\cdot(r\cdot\alpha) = v\cdot(\alpha - 2\langle\alpha, \alpha_r\rangle\alpha_r) = v\cdot\alpha - 2\langle\alpha, \alpha_r\rangle v\cdot\alpha_r$$

is negative and has at least two simple roots in its support, since $v\cdot\alpha$ and $-2\langle\alpha, \alpha_r\rangle v\cdot\alpha_r$ are both negative linear combinations of simple roots (and not scalar multiples of each other). Now there exist $s \in R$, $w \in W$ with $v = sw$ and $l(v) = l(w)+1$, and it follows that $w\cdot(r\cdot\alpha) = s\cdot(v\cdot(r\cdot\alpha)) \in \Phi^-$ by the above, as s is a simple reflection and the negative root $v\cdot(r\cdot\alpha)$ cannot equal $-\alpha_s$.

Finally, suppose that $\langle\alpha, \alpha_r\rangle < 0$. Then $\langle r\cdot\alpha, \alpha_r\rangle = -\langle\alpha, \alpha_r\rangle > 0$; so the preceding paragraph shows that $\mathrm{dp}(\alpha) = \mathrm{dp}(r\cdot(r\cdot\alpha)) = \mathrm{dp}(r\cdot\alpha) - 1$.

Corollary 1.8 *Let $\alpha, \beta \in \Phi^+$ with $\alpha \preceq \beta$. Let $\alpha = \sum_{r\in R}\lambda_r\alpha_r$ and $\beta = \sum_{r\in R}\mu_r\alpha_r$. Then $\lambda_r \leq \mu_r$ for all $r \in R$.*

Proof If $\beta = s\cdot\alpha \succ \alpha$ then $\mu_r = \lambda_r$ for all $r \neq s$, and $\mu_s > \lambda_s$ by Lemma 1.7. A straightforward induction on depth completes the proof.

2. A partial order on the positive roots

Definition 2.1 For $\alpha, \beta \in \Phi^+$ we say that α *dominates* β with respect to W (we write $\alpha \operatorname{dom}_W \beta$) if and only if for all $w \in W$, if $\alpha \in N(w)$ then $\beta \in N(w)$. Define also $\Delta_W = \{\alpha \in \Phi^+ \mid \exists\beta \in \Phi^+ - \{\alpha\},\ \alpha \operatorname{dom}_W \beta\}$.

Our principal result is that if $|R| < \infty$ then $|\Phi^+ - \Delta_W| < \infty$. It can be shown that this is equivalent to the Parallel Wall Theorem of the preprint [3], in which the proof given is incomplete.

The next lemma gives some basic properties of dominance.

Lemma 2.2 *Let $\alpha, \beta \in \Phi^+$ with $\alpha \operatorname{dom}_W \beta$. Then*

(i) *$\langle\alpha, \beta\rangle > 0$;*
(ii) *$(w\cdot\alpha) \operatorname{dom}_W (w\cdot\beta)$ for all $w \in W$ with $w\cdot\beta \in \Phi^+$;*
(iii) *if $\alpha' \in \Phi^+$ with $\alpha' \succ \alpha$, then $\alpha' \in \Delta_W$;*
(iv) *$\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$, with equality if and only if $\alpha = \beta$.*

Proof Since $r_\alpha\cdot\alpha = -\alpha \in \Phi^-$ it follows that $r_\alpha\cdot\beta \in \Phi^-$; that is, $\beta - 2\langle\alpha, \beta\rangle\alpha \in \Phi^-$. Hence $\langle\alpha, \beta\rangle > 0$, and (i) is proved. Part (ii) is trivial.

For (iii) it is clearly sufficient to consider the case $\mathrm{dp}(\alpha') = \mathrm{dp}(\alpha) + 1$. Then $\alpha' = r\cdot\alpha$ for some $r \in R$, and $\langle\alpha, \alpha_r\rangle < 0$, by Lemma 1.7. Hence $\beta \neq \alpha_r$ (by (i)); thus $r\cdot\beta \in \Phi^+$, and (ii) finishes the proof.

It is clear that $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$. Let $w \in W$, $r \in R$ such that $w\cdot\alpha = -\alpha_r$ and $l(w) = \mathrm{dp}(\alpha)$. Then $l(rw) = l(w) - 1$ by Proposition 1.1 (iv), and also $w\cdot\beta \in \Phi^-$ as $\alpha \operatorname{dom}_W \beta$. If $(rw)\cdot\beta \in \Phi^+$, then $(rw)\cdot\beta = \alpha_r$ (since $r\cdot(rw\cdot\beta) \in \Phi^-$), and this gives $\beta = \alpha$. If $(rw)\cdot\beta \in \Phi^-$ then $\mathrm{dp}(\beta) \leq l(rw) < \mathrm{dp}(\alpha)$.

It is clear that dom_W is transitive, and Lemma 2.2 (iv) shows that dom_W is antisymmetric; hence dom_W is partial order on Φ^+. The set $\Phi^+ - \Delta_W$ consists of the minimal elements in this partial order. Note furthermore that if W is finite and w_0 is the element of maximal length in W, then $\alpha \mapsto -w_0(\alpha)$ is a depth preserving and dominance reversing permutation of the positive roots. Thus Lemma 2.2 (iv) shows that there is no nontrivial dominance in a finite Coxeter group.

The following lemma provides us with an alternative characterization of dominance.

Lemma 2.3 *Let α, $\beta \in \Phi^+$ be arbitrary. Then α dom_W β if and only if $\langle\alpha, \beta\rangle \geq 1$ and $\operatorname{dp}(\alpha) \geq \operatorname{dp}(\beta)$.*

Proof We may assume that $\alpha \neq \beta$, since both implications are trivial otherwise.

Suppose first that α dom_W β. By Lemma 2.2 (iv) we need only show that $\langle\alpha, \beta\rangle \geq 1$. If $\langle\alpha, \beta\rangle < 1$ then by Lemma 2.2 (i) and Proposition 1.4 it follows that $\langle\alpha, \beta\rangle = \cos(p\pi/q)$ for some integers p and q, and the subgroup D generated by r_α and r_β is a finite dihedral group. Since there is no dominance in D there exists $w \in D$ with $w \cdot \alpha \in \Phi^-$ and $w \cdot \beta \in \Phi^+$. Since $D \subseteq W$, this contradicts our hypothesis that α dom_W β. So $\langle\alpha, \alpha_r\rangle \geq 1$, as required.

For the converse, assume that $\langle\alpha, \beta\rangle \geq 1$ and $\operatorname{dp}(\alpha) \geq \operatorname{dp}(\beta)$, and consider first the case $\beta \in \Pi$; say $\beta = \alpha_r$. Then $r \cdot \alpha \in \Phi^+$, since $\alpha \neq \beta$. Furthermore,

$$\langle\alpha, r \cdot \alpha\rangle = \langle\alpha, \alpha\rangle - 2\langle\alpha, \alpha_r\rangle^2 = 1 - 2\langle\alpha, \alpha_r\rangle^2 \leq -1,$$

and by Proposition 1.1 (vi) there are infinitely many roots of the form $\lambda\alpha + \mu r \cdot \alpha$ with λ, $\mu > 0$. Suppose that α does not dominate β, and choose $w \in W$ such that $w \cdot \alpha \in \Phi^-$ and $w \cdot \alpha_r \in \Phi^+$. Then

$$w \cdot (r \cdot \alpha) = w \cdot \alpha + 2\langle\alpha, \alpha_r\rangle(-w \cdot \alpha_r)$$

is a positive linear combination of negative roots, and must therefore be negative. So $N(w)$ contains α and $r \cdot \alpha$, and hence also contains all roots of the form $\lambda\alpha + \mu r \cdot \alpha$ with λ, $\mu > 0$. This contradicts the finiteness of $N(w)$ (see Proposition 1.1 (ii)).

Proceeding by induction on $\operatorname{dp}(\beta)$, suppose now that $\operatorname{dp}(\beta) > 1$, and choose $r \in R$ such that $\beta \succ r \cdot \beta \in \Phi^+$. Since $\operatorname{dp}(\alpha) \geq \operatorname{dp}(\beta) > 1$, clearly $r \cdot \alpha \in \Phi^+$. Now $\langle r \cdot \alpha, r \cdot \beta\rangle \geq 1$ and

$$\operatorname{dp}(r \cdot \alpha) \geq \operatorname{dp}(\alpha) - 1 \geq \operatorname{dp}(\beta) - 1 = \operatorname{dp}(r \cdot \beta).$$

Hence by induction $(r \cdot \alpha)$ dom_W $(r \cdot \beta)$, and by Lemma 2.2 (ii) this implies that α dom_W β.

Corollary 2.4 *Let α, $\beta \in \Phi^+$ with $\beta \preceq \alpha$ and $\alpha \notin \Delta_W$; let furthermore $r \in R$ such that $\langle\beta, \alpha_r\rangle \leq -1$. Then the coefficient of α_r in α equals the coefficient of α_r in β.*

Proof Let $\gamma \in \Phi^+$ such that $\beta \preceq \gamma \preceq \alpha$ and γ is maximal (with respect to $\preceq$) subject to having the same coefficient of α_r as β has. If $\gamma \neq \alpha$ then $\gamma \prec s \cdot \gamma \preceq \alpha$ for some $s \in R$, and by maximality of γ we must have $s = r$. But $\gamma - \beta = \sum_{t \in R - \{r\}} \lambda_t \alpha_t$ for some $\lambda_t \geq 0$, and hence

$$\langle r \cdot \gamma, \alpha_r\rangle = -\langle\gamma, \alpha_r\rangle = -\langle\beta, \alpha_r\rangle - \sum_{t \in R - \{r\}} \lambda_t \langle\alpha_t, \alpha_r\rangle \geq 1.$$

Hence $r \cdot \gamma \operatorname{dom}_W \alpha_r$, and by Lemma 2.2 (iii) it follows that $\alpha \in \Delta_W$, which is the desired contradiction.

Our proof that $\Phi^+ - \Delta_W$ is finite (if R is finite) depends on the fact that the set of real numbers $\{ \langle \alpha, \alpha_r \rangle \mid \alpha \in \Phi^+ - \Delta_W \text{ and } r \in R \}$ is finite. The next definition facilitates the statement of the relevant facts.

Definition 2.5 For $J \subseteq R$ define $C(J)$ to be the set of all real numbers of the form $\cos(a\pi/m)$ with $a \in \mathbb{N}$ and $m = m_{rs}$ for some $r, s \in J$.

Proposition 2.6 *Let $\alpha, \beta \in \Phi$, and suppose that $\langle \alpha, \beta \rangle \in [-1, 1]$. Then $\langle \alpha, \beta \rangle \in C(R)$.*

Proof If $\langle \alpha, \beta \rangle = \pm 1$ then the result is trivial since $m_{rr} = 1$ for all r. If $\langle \alpha, \beta \rangle \in (-1, 1)$ then by Proposition 1.4 it follows that the subgroup of W generated by r_α and r_β is finite, whence, by Proposition 1.3, there exists $w \in W$ such that $w \cdot \alpha$ and $w \cdot \beta$ are in the root system of some finite parabolic subgroup W_J of W. It follows by well known properties of finite Coxeter groups that $\langle w \cdot \alpha, w \cdot \beta \rangle \in C(J)$; indeed, by a similar argument to that used in the proof of Proposition 1.3, there exist $t \in W_J$ and $r, s \in J$ such that $(tw) \cdot \alpha$ and $(tw) \cdot \beta$ are in the root system of the parabolic subgroup generated by r and s, whence the angle between them is an integral multiple of π/m_{rs}.

The following technical lemma, though trivial, provides the key for our proof of the main theorem.

Lemma 2.7 *Let $\alpha = \sum_{r \in R} \lambda_r \alpha_r$ and $\beta = \sum_{r \in R} \mu_r \alpha_r$ be positive roots. Let further $R = R_1 \cup R_2$ such that*
(i) *$\langle \alpha, \alpha_r \rangle = \langle \beta, \alpha_r \rangle$ for all $r \in R_1$,*
(ii) *$\lambda_r = \mu_r$ for all $r \in R_2$.*
Then $\langle \alpha, \beta \rangle = 1$.

Proof We have that $\alpha - \beta = \sum_{r \in R_1} (\lambda_r - \mu_r) \alpha_r$. Thus

$$\langle \alpha, \alpha - \beta \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r) \langle \alpha, \alpha_r \rangle = \sum_{r \in R_1} (\lambda_r - \mu_r) \langle \beta, \alpha_r \rangle = \langle \beta, \alpha - \beta \rangle.$$

Since $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$, this becomes $1 - \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle - 1$, and the result follows.

Theorem 2.8 *If R is finite, then so is $\Phi^+ - \Delta_W$.*

Proof Let $c = |C(R)|$; then $c \leq \Sigma_{r,s \in R} m_{r,s} < \infty$, as R is finite. Since every root of depth d can be expressed as $(r_d r_{d-1} \cdots r_2) \cdot \alpha_{r_1}$ with each $r_i \in R$, there are no more than $|R|^d$ roots of depth d. If we can show that no root in $\Phi^+ - \Delta_W$ can have depth exceeding $c^{|R|}(|R| + 1)$ then the proof will be complete. So let $\beta \in \Phi^+ - \Delta_W$ have depth d, and let $\beta_1 \prec \cdots \prec \beta_d = \beta$ be a sequence of roots. Note that $\beta_i \in \Phi^+ - \Delta_W$ for each $i \in \{1, \ldots, d\}$. For $i \in \{1, \ldots, d\}$ define $J_i = \{ r \in R \mid \langle \beta_i, \alpha_r \rangle \geq -1 \}$. If $r \notin J_i$ then by Corollary 2.4 the coefficient of α_r in β_j is constant for all $j \geq i$. Since $\langle \alpha_s, \alpha_r \rangle \leq 0$ for all $s \in R - \{r\}$ it follows from Corollary 1.8 that $\langle \beta_j, \alpha_r \rangle \leq \langle \beta_i, \alpha_r \rangle$ for $j \geq i$, and hence $\alpha_r \notin J_j$, for all $j \geq i$. Thus the sets J_i form a decreasing chain.

Suppose $J_i = \cdots = J_j = J$ for $i \leq j$. If $k \in \{i, \ldots, j\}$ and $r \in J$ then $-1 \leq \langle \beta_k, \alpha_r \rangle \leq 1$, since $r \in J_k$ and β_k does not dominate α_r (unless $\beta_k = \alpha_r$

in which case $\langle\beta_k,\alpha_r\rangle = 1$). Hence $\langle\beta_k,\alpha_r\rangle \in C(R)$, by Proposition 2.6. So if $j - i > c^{|R|}$, then there will exist $m, n \in \{i, i+1, \dots, j\}$ with $n > m$ and $\langle\beta_n,\alpha_r\rangle = \langle\beta_m,\alpha_r\rangle$ for all $r \in J$. But if $r \notin J$, then α_r has the same coefficient in β_m as in β_n, and it follows by Lemma 2.7 that $\langle\beta_n,\beta_m\rangle = 1$. This contradicts Lemma 2.3, since $\beta_n \notin \Delta_W$. We conclude that if $j - i \geq c^{|R|}$ then J_j is strictly smaller than J_i. Since $J_1 \subseteq R$ it follows that the chain $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_d$ can have length at most $c^{|R|}(|R|+1)$ and this finishes the proof.

3. Finitely generated Coxeter groups are automatic

The aim of this section is to show that Coxeter groups on finite generating sets are automatic groups (as defined in [2] or [5], for example). This is proved in [3] under the assumption that the Parallel Wall Theorem is valid. In our proof, the concept of root dominance introduced above replaces the parallel wall property.

Let (W, R) be a Coxeter system with R finite. Let R^* be the free monoid on R, and let $\pi\colon R^* \to W$ be the natural homomorphism. To avoid confusion with multiplication in W, we shall write $x * y$ for the product of elements $x, y \in R^*$. Let ℓ be the length function on R^*, defined by $\ell(r_1 * \cdots * r_l) = l$ whenever $r_1, \dots, r_l \in R$. Furthermore, let $\preceq$ be the lexicographical order on R^* for some (arbitrary) ordering of R. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$.

In the present context, any subset L of R^* will be called a *language*, R being the *alphabet* and the elements of L the *words* of the language. Later in this section we will consider languages on other alphabets; however, our first objective is to describe a regular language L on R such that the restriction of π is a bijection from L to W.

A language is *regular* if and only if there exists a finite state automaton which accepts the words of the language and rejects words which are not in the language. The automaton has a finite number of states, one of which is prescribed as the starting state; it reads the letters of a word one at a time, starting from the left, and its state after reading a letter is completely determined by the letter and the state it was in before reading the letter. Finally, certain states are designated as "accept" states, the others as "reject" states; the automaton accepts the word if it is in an accept state after reading the final letter, and rejects it otherwise.

Recall that if $w \in W$, then $l(w) = \min\{\,\ell(x) \mid x \in \pi^{-1}(w)\,\}$. An element $x \in R^*$ is called a *reduced word* if $\ell(x) = l(\pi(x))$. Let L' be the set of all reduced words. Clearly for each $w \in W$ there exists a unique $\nu(w) \in \pi^{-1}(w)$ such that $\nu(w) \in L'$ and $\nu(w) \preceq x$ for all $x \in \pi^{-1}(w) \cap L'$. We define the language L to consist of all these lexicographically minimal reduced words for the various elements of W:

$$L = \{\,\nu(w) \mid w \in W\,\} = \{\,y \in L' \mid y \preceq x \text{ for all } x \in L' \text{ with } \pi(x) = \pi(y)\,\}.$$

We shall describe a finite state automaton $\mathcal{W}$ of which L is the regular language.

Let $\widetilde{\Delta}_W = \Phi^+ - \Delta_W$ denote the (finite) complement of Δ_W in Φ^+. The accept states of the automaton $\mathcal{W}$ will be the subsets of $\widetilde{\Delta}_W$, and there will be one reject state. The starting state is the empty subset of $\widetilde{\Delta}_W$. The transition function $\mu\colon \mathrm{S} \times R \to \mathrm{S}$ (where S is the set of all states) is given by $(S, r) \mapsto S'$, where

(i) if S is the reject state then so is S',
(ii) if $S \subseteq \widetilde{\Delta}_W$ and $\alpha_r \in S$ then S' is the reject state,

(iii) if $S \subseteq \widetilde{\Delta}_W$ and $\alpha_r \notin S$ then $S' = S'' \cap \widetilde{\Delta}_W$, where

$$S'' = \{ r \cdot \alpha \mid \alpha \in S \} \cup \{\alpha_r\} \cup \{ r \cdot \alpha_s \mid s \in R \text{ and } s \prec r \}.$$

The proof that L is the regular language of $\mathcal{W}$ depends on the following lemma:

Lemma 3.1 *Let $w \in W$ and $r_1, r_2, \ldots, r_l \in R$ with $\nu(w) = r_1 * r_2 * \cdots * r_l$. Then for arbitrary $r \in R$,*

(i) *$\nu(w) * r$ is not reduced if and only if there exists $i \in \{1, 2, \ldots, l\}$ such that $\alpha_r = (r_l r_{l-1} \cdots r_{i+1}) \cdot \alpha_{r_i}$,*

(ii) *if $\nu(w) * r$ is reduced, then $\nu(w) * r \notin L$ if and only if there exists $i \in \{1, 2, \ldots, l\}$ and $s \in R$ with $s \prec r_i$ such that $(r_l r_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_r$; if this holds then $\nu(wr) = r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l$.*

Proof If $\alpha_r = r_l r_{l-1} \cdots r_{i+1} \cdot \alpha_{r_i}$, then $r = v^{-1} r_i v$, where $v = r_{i+1} \cdots r_l$, and so $wr = r_1 r_2 \cdots r_{i-1} r_i v r = r_1 r_2 \cdots r_{i-1} v = r_1 \cdots r_{i-1} r_{i+1} \cdots r_l$ has length less than l. If there is no i such that $\alpha_r = (r_l r_{l-1} \cdots r_{i+1}) \cdot \alpha_{r_i}$, then clearly $\alpha_r \notin N(w)$, and $l(wr) = l(w) + 1$ by Proposition 1.1. Thus (i) is proved.

Suppose now that $l(wr) = l + 1$, and let $\nu(wr) = s_1 * \cdots * s_{l+1}$. Then

$$\nu(wr) = s_1 * \cdots * s_{l+1} \preceq r_1 * \cdots * r_l * r \tag{1}$$

since the right hand side also has length $l + 1$ and is in $\pi^{-1}(wr)$. Since $l(w) = l$ and $w = (wr)r = s_1 s_2 \cdots s_{l+1} r$, it follows by the exchange condition (see Proposition 1.1) that $w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{l+1}$ for some $i \leq l + 1$, and so

$$\nu(w) = r_1 * \cdots * r_l \preceq s_1 * \cdots * s_{i-1} * s_{i+1} * \cdots * s_{l+1}. \tag{2}$$

It is immediate from Eqs. (1) and (2) that $r_j = s_j$ for all $j \in \{1, 2, \ldots i - 1\}$. So $w = r_1 \cdots r_{i-1} s_{i+1} \cdots s_{l+1}$, and we deduce that $s_{i+1} \cdots s_{l+1} = r_i \cdots r_l$. Now if $r_i * \cdots * r_l \prec s_{i+1} * \cdots * s_{l+1}$, then $s_1 * \cdots * s_i * r_i * \cdots * r_l \prec s_1 * \cdots * s_{l+1} = \nu(wr)$, which is impossible, and, similarly, $s_{i+1} * \cdots * s_{l+1} \prec r_i * \cdots * r_l$ would give $r_1 * \cdots * r_{i-1} * s_{i+1} * \cdots * s_{l+1} \prec r_1 * \cdots * r_l = \nu(w)$, which is also impossible. Hence $r_j = s_{j+1}$ for all $j \in \{i, i + 1, \ldots, l\}$. Thus we have shown that $\nu(wr) = r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l$, where $s = s_i$.

By Eq. (1) we have either that $i \leq l$ and $s \prec r_i$, or else $i = l + 1$ and $s = r$. If $\nu(w) * r \notin L$ the former case obtains, and since $s r_i \cdots r_l = r_i \cdots r_l r$ it follows readily that $(r_l r_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_r$. Conversely, whenever there exists $i \in \{1, 2, \ldots, l\}$ and $s \prec r_i$ such that $(r_l r_{l-1} \cdots r_i) \cdot \alpha_s = \alpha_r$, then $r_1 \cdots r_{i-1} s r_i \cdots r_l = r_1 \cdots r_l r$ and $r_1 * \cdots * r_{i-1} * s * r_i * \cdots * r_l \prec r_1 * \cdots * r_l * r$. Hence $\nu(wr) \neq r_1 * \cdots * r_l * r$; that is, $\nu(w) * r \notin L$.

Lemma 3.2 *Suppose that $\beta \in \Delta_W$ and u, $v \in W$ satisfy $u \cdot \beta$, $v^{-1} \cdot \beta \in \Pi$. Then $l(uv) \neq l(u) + l(v)$.*

Proof Let $u \cdot \beta = \alpha_r$ and $v^{-1} \cdot \beta = \alpha_s$, where r, $s \in R$. Since $r \cdot \alpha_r$ and $s \cdot \alpha_s$ are negative, $\beta \in N(ru) \cap N(sv^{-1})$. Since $\beta \in \Delta_W$ there exists $\gamma \in \Phi^+$, with $\gamma \neq \beta$, such that $\beta \operatorname{dom}_W \gamma$. By the definition of dominance we see that $ru \cdot \gamma$ and $sv^{-1} \cdot \gamma$ are both negative. Since $u \cdot \gamma \neq u \cdot \beta = \alpha_r$, it follows that $u \cdot \gamma \notin N(r)$; hence $u \cdot \gamma \in \Phi^-$. Similarly, $v^{-1} \cdot \gamma \in \Phi^-$, since $v^{-1} \cdot \gamma \neq \alpha_s$. Hence $N(u) \cap N(v^{-1}) \neq \emptyset$, and so $l(uv) \neq l(u) + l(v)$ (by Lemma 1.2).

Proposition 3.3 *The language L is the language recognized by the automaton $\mathcal{W}$ described above.*

Proof Suppose first of all that the automaton rejects the word $r_1 * \cdots * r_n$, and that rejection occurs when reading r_{l+1}. For convenience of notation, let $\alpha_i = \alpha_{r_i}$ for all $i \in \{1, 2, \ldots, n\}$; for $i \leq l$ let $R_i = \{\, s \in R \mid s \prec r_i \,\}$, and let $S_i \subseteq \widetilde{\Delta}_W$ be the state of the automaton after reading r_i. Then for all $i < l$,

$$S_{i+1} \subseteq r_{i+1} \cdot S_i \cup \{\alpha_{i+1}\} \cup r_{i+1} \cdot R_{i+1}.$$

Thus S_l is contained in

$$\{\, (r_l \ldots r_{i+1}) \cdot \alpha_i \mid 1 \leq i \leq l \,\} \cup r_l \cdot R_l \cup (r_l r_{l-1}) \cdot R_{l-1} \cup \cdots \cup (r_l r_{l-1} \cdots r_1) \cdot R_1.$$

Since rejection occurs when r_{l+1} is read, α_{l+1} is in this set. Now by Lemma 3.1 (i), if $\alpha_{l+1} \in T = \{\, (r_l \ldots r_{i+1}) \cdot \alpha_i \mid 1 \leq i \leq l \,\}$ then $r_1 * \cdots * r_{l+1}$ is not reduced, hence not in L. If $\alpha_{l+1} \notin T$ then $r_1 * \cdots * r_{l+1}$ is reduced, but since α_{l+1} is in some $(r_l r_{l-1} \cdots r_i) \cdot R_i$ it now follows by Lemma 3.1 (ii) that $r_1 * \cdots * r_{l+1} \notin L$. Hence $r_1 * \cdots * r_n \notin L$, as required.

For the converse, suppose that $r_1 * \cdots * r_n \notin L$, and choose $l < n$ maximal such that $r_1 * \cdots * r_l \in L$. Using the same notation as above, our aim is to show that $\alpha_{l+1} \in S_l$, so that the automaton rejects $r_1 * \cdots * r_{l+1}$. Assume, for a contradiction, that $\alpha_{l+1} \notin S_l$.

Suppose first of all that $r_1 * \cdots * r_{l+1}$ is not reduced. By Lemma 3.1 (i) we have that $\alpha_{l+1} = (r_l \cdots r_{i+1}) \cdot \alpha_i$ for some $i \in \{1, 2, \ldots, l\}$. Thus l is in the set $\{\, k \in \{i, i+1, \ldots, l\} \mid (r_k r_{k-1} \cdots r_{i+1}) \cdot \alpha_i \notin S_k \,\}$; let j be the least element of this set, and let $\beta = (r_j r_{j-1} \cdots r_{i+1}) \cdot \alpha_i$. Since $\alpha_i \in S_i$, certainly $j > i$. Thus $(r_{j-1} \cdots r_{i+1}) \cdot \alpha_i \in S_{j-1}$, by minimality of j. So $\beta \in r_j \cdot S_{j-1}$. Since $r_j \cdot S_{j-1} \cap \widetilde{\Delta}_W \subseteq S_j$ it follows that $\beta \notin \widetilde{\Delta}_W$. Observe that $r_{i+1} r_{i+2} \cdots r_j \cdot \beta = \alpha_i$ and $(r_{j+1} r_{j+2} \cdots r_l)^{-1} \cdot \beta = \alpha_{l+1}$ are both in Π. By Lemma 3.2 it follows that $l(r_{i+1} \cdots r_l) \neq l - i$; but this contradicts the fact that $r_1 \cdots r_l \in L$. Hence $r_1 * \cdots * r_{l+1}$ is reduced.

Since $r_1 * \cdots * r_{l+1} \notin L$ it follows by Lemma 3.1 (ii) that $\alpha_{l+1} = (r_l \cdots r_i) \cdot \alpha_s$ for some $i \in \{1, 2, \ldots l\}$ and some $s \in R_i$. Let $j \in \{i, i+1, \ldots, l\}$ be minimal such that $\beta = (r_j r_{j-1} \cdots r_i) \cdot \alpha_s \notin S_j$. If $j = i$, then since $r_i \cdot R_i \cap \widetilde{\Delta}_W \subseteq S_i$ it follows that $\beta \notin \widetilde{\Delta}_W$. If $j > i$ then minimality of j forces $(r_{j-1} \cdots r_i) \cdot \alpha_s \in S_{j-1}$, and since $r_j \cdot S_{j-1} \cap \widetilde{\Delta}_W \subseteq S_j$ it again follows that $\beta \notin \widetilde{\Delta}_W$. But $r_i r_{i+1} \cdots r_j \cdot \beta = \alpha_s$ and $(r_{j+1} r_{j+2} \cdots r_l)^{-1} \cdot \beta = \alpha_{l+1}$ are both in Π, and the same argument as used in the previous paragraph again yields a contradiction.

For each $r \in R$ define

$$L_r = \{\, (x_1, x_2) \in R^* \times R^* \mid x_1, x_2 \in L \text{ and } \pi(x_1) r = \pi(x_2) \,\}.$$

We shall describe a finite state automaton $\mathcal{M}_r$ which recognizes whether or not a pair $(x_1, x_2) \in R^* \times R^*$ is in L_r. These "multiplier automata" $\mathcal{M}_r$ together with the "word acceptor" $\mathcal{W}$ form an automatic structure for W. (Strictly speaking, a multiplier automaton is also required for the identity element of W, recognizing (x_1, x_2) if and only if x_1, $x_2 \in L$ and $\pi(x_1) = \pi(x_2)$. However, since the words of our language L correspond bijectively with the elements of W, the required automaton is a trivial modification of $\mathcal{W}$.) We refer the reader to [2] or [5] for further discussion of automatic groups.

The automaton $\mathcal{M}_r$ is required to read x_1 and x_2 simultaneously, one letter from each at a time, from left to right. If x_1 and x_2 are of unequal lengths then, of course, the end of the shorter will be encountered with part of the longer still unread. To cope with this, a *padding symbol*, which we will denote by \$, is used; the automaton appends as many of these to the shorter of x_1 and x_2 as is necessary to make the lengths equal. This turns L_r into a language over the alphabet $\mathrm{R} = (R \cup \{\$\}) \times (R \cup \{\$\}) - \{(\$, \$)\}$.

Lemma 3.1 enables us to give an explicit description of the language L_r. Note first that, trivially, $(y, z) \in L_r$ if and only if $(z, y) \in L_r$.

Proposition 3.4 *Suppose that y, $z \in R^*$ with $\ell(z) \leq \ell(y)$, and let $y = s_1 * \cdots * s_l$, where $s_1, s_2, \ldots, s_l \in R$. Then (y, z), $(z, y) \in L_r$ if and only if $y \in L$ and the following conditions are satisfied for some $j \in \{1, 2, \ldots, l\}$:*
(a) $z = s_1 * \cdots * s_{j-1} * s_{j+1} * \cdots * s_l$, *and*
(b) $\alpha_r = (s_l s_{l-1} \cdots s_{j+1}) \cdot \alpha_{s_j}$.
Furthermore, if these conditions hold, then either $j = l$ or $s_j \prec s_{j+1}$.

Proof If $(y, z) \in L_r$ then $y = \nu(\pi(z)r)$, and it follows from Lemma 3.1 that (a) and (b) are satisfied for some $j \in \{1, 2, \ldots, l\}$, and that $s_j \prec s_{j+1}$ if $j \neq l$.

Conversely, suppose that $y \in L$ and that conditions (a) and (b) are satisfied; we must show that $z = \nu(\pi(y)r)$. Clearly $\pi(z) = \pi(y)r$, and by Lemma 3.1 it follows that $\nu(\pi(z))$ is obtained from y by deleting some s_k. If $k \neq j$ then we have that $s_1 \cdots s_{j-1} s_{j+1} \cdots s_l = s_1 \cdots s_{k-1} s_{k+1} \cdots s_l$, and it follows easily that s_j and s_k can be cancelled from $s_1 \cdots s_l$, contradicting the fact that y is reduced.

We now describe a suitable automaton $\mathcal{M}_r$. It has one accept state, $\mathcal{Y}$, all other states being reject states. There is a "failure" state, $\mathcal{F}$ (from which there are no transitions to other states). All subsets of $\widetilde{\Delta}_W$ are states, and the remaining states are the elements of the Cartesian product $\mathrm{P}(\widetilde{\Delta}_W) \times \widetilde{\Delta}_W \times R \times \{\pm 1\}$, where $\mathrm{P}(\widetilde{\Delta}_W)$ is the power set of $\widetilde{\Delta}_W$. Let S_r be the set of all these states, and let $\emptyset \subset \widetilde{\Delta}_W$ be the initial state.

Note that the subset $\mathrm{P}(\widetilde{\Delta}_W) \cup \{\mathcal{F}\}$ of S_r can be identified with the set of states of $\mathcal{W}$. Let μ be the transition function, described above, for $\mathcal{W}$. The transition function $\mu_r \colon \mathrm{S}_r \times \mathrm{R} \to \mathrm{S}_r$ for the automaton $\mathcal{M}_r$, is defined by the rules listed below. Let $S \in \mathrm{S}_r$ and s, $t \in R^+$, and for brevity let $S' = \mu_r(S, (s, t))$.

Case 1: $\mathcal{X} \subseteq \widetilde{\Delta}_W$.

(i) If $s = t \in R$ then $\mathcal{X}' = \mu(\mathcal{X}, s)$.

(ii) If either s or t is \$, then $\mathcal{X}' = \begin{cases} \mathcal{Y} & \text{if } \{s,t\} = \{r, \$\} \text{ and } \mu(\mathcal{X}, r) \neq \mathcal{F}, \\ \mathcal{F} & \text{otherwise.} \end{cases}$

(iii) If $s \prec t \in R$, then $\mathcal{X}' = \begin{cases} (\mu(\mathcal{X}, s), \alpha_s, t, +1) & \text{if } \mu(X, s) \neq \mathcal{F}, \\ \mathcal{F} & \text{if } \mu(X, s) = \mathcal{F}. \end{cases}$

(iv) If $t \prec s \in R$, then $\mathcal{X}' = \begin{cases} (\mu(\mathcal{X}, t), \alpha_t, s, -1) & \text{if } \mu(X, t) \neq \mathcal{F}, \\ \mathcal{F} & \text{if } \mu(X, t) = \mathcal{F}. \end{cases}$

Case 2: $\mathcal{X} = (X, \beta, u, +1) \in P(\widetilde{\Delta}_W) \times \widetilde{\Delta}_W \times R^+ \times \{\pm 1\}$.

Let $X' = \mu(X, s)$ and $\gamma = s \cdot \beta$.

(i) If $s \neq u$, then $\mathcal{X}' = \mathcal{F}$.

(ii) If $s = u$ and $t \in R$, then $\mathcal{X}' = \begin{cases} (X', \gamma, t, +1) & \text{if } X' \neq \mathcal{F} \text{ and } \gamma \in \widetilde{\Delta}_W, \\ \mathcal{F} & \text{if } X' = \mathcal{F} \text{ or } \gamma \notin \widetilde{\Delta}_W. \end{cases}$

(iii) If $(s, t) = (u, \$)$ and $X' \neq \mathcal{F}$, then $\mathcal{X}' = \begin{cases} \mathcal{Y} & \text{if } \gamma = \alpha_r, \\ \mathcal{F} & \text{if } \gamma \neq \alpha_r. \end{cases}$

Case 3: $\mathcal{X} = (X, \beta, u, -1) \in P(\widetilde{\Delta}_W) \times \widetilde{\Delta}_W \times R^+ \times \{\pm 1\}$.

Let $X' = \mu(X, t)$ and $\gamma = t \cdot \beta$.

(i) If $t \neq u$ then $\mathcal{X}' = \mathcal{F}$.

(ii) If $t = u$ and $s \in R$, then $\mathcal{X}' = \begin{cases} (X', \gamma, s, -1) & \text{if } X' \neq \mathcal{F} \text{ and } \gamma \in \widetilde{\Delta}_W, \\ \mathcal{F} & \text{if } X' = \mathcal{F} \text{ or } \gamma \notin \widetilde{\Delta}_W. \end{cases}$

(iii) If $(s, t) = (\$, u)$ and $X' \neq \mathcal{F}$, then $\mathcal{X}' = \begin{cases} \mathcal{Y} & \text{if } \gamma = \alpha_r, \\ \mathcal{F} & \text{if } \gamma \neq \alpha_r. \end{cases}$

Case 4: $\mathcal{X} = \mathcal{Y}$ *or* $\mathcal{F}$.

$\mathcal{X}' = \mathcal{F}$ in all cases.

Proposition 3.5 *The automaton $\mathcal{M}_r$ defined above recognizes the language L_r.*

Proof For each $x \in R^*$ and $i \leq \ell(x)$, let $x(i)$ be the initial segment of x length i. That is, $\ell(x(i)) = i$ and $x = x(i) * y$ for some y. If y, $z \in R^*$ with $\ell(y) = \ell(z) = i$, we will say that (y, z) is *viable* if there exists $(x_1, x_2) \in L_r$ with $x_1(i) = y$ and $x_2(i) = z$. Note in particular that (y, z) is not viable if either y or z is not in L.

Let $x_1 = s_1 * s_2 * \cdots * s_l$ and $x_2 = t_1 * t_2 * \cdots * t_m$ be elements of R^*; we must show that $\mathcal{M}_r$ accepts (x_1, x_2) if and only if $(x_1, x_2) \in L_r$. For each $i \leq \max(l, m)$, let $\mathcal{X}_i$ be the state of $\mathcal{M}_r$ after reading $(x_1(i), x_2(i))$ (padded if necessary).

Let $j \in \mathbb{N}$ be maximal such that $s_i = t_i$ for all $i < j$. In view of rule (i) of Case 1, and the fact that $\mathcal{X}_0 \subseteq \widetilde{\Delta}_W$, we see that $\mathcal{X}_i \subseteq \widetilde{\Delta}_W$ for all $i < j$, and, furthermore, $\mathcal{X}_i$ is the state that $\mathcal{W}$ would be in after reading $s_1 * \cdots * s_i$. So if $\mathcal{X}_i = \mathcal{F}$, then $s_1 * \cdots * s_i \notin L$, in which case (x_1, x_2) is clearly not viable.

Suppose instead that $\mathcal{X}_{j-1} \neq \mathcal{F}$, so that $x_1(j-1) = x_2(j-1) = x \in L$. If $l = m = j - 1$ (so that the process terminates here) then rejection occurs, since $\mathcal{X}_{j-1}$ is a subset of $\widetilde{\Delta}_W$, and hence not equal to $\mathcal{Y}$, which is the only accept state. Clearly, in this case, $(x_1, x_2) = (x, x) \notin L_r$ since $\pi(x)r \neq \pi(x)$. Suppose now that one of l, m equals $j-1$, and the other is greater than $j-1$. Then (x_1, x_2) will be in L_r if and only if $\{x_1, x_2\} = \{x, x * r\}$ and $x * r \in L$. By rule (ii) of Case 1 we see that $\mathcal{X}_j$ is indeed the failure state if the element of R that is read at this stage is not r, or if $\mathcal{W}$ would reject $x * r$. Otherwise, $\mathcal{X}_j = \mathcal{Y}$, and acceptance will occur, as it should, provided that the end of (x_1, x_2) has been reached; the Case 4 rule shows that $\mathcal{X}_{j+1} = \mathcal{F}$ if there is another letter after $x * r$ in the longer x_i.

Suppose that $x(j-1) \in L$ and $l, m \geq j$, and suppose first of all that $s_j \prec t_j$. By Proposition 3.4 we see that $(x_1, x_2) \in L_r$ if and only if $x_1 \in L$ and $t_i = s_{i+1}$ for $j \leq i \leq m = l - 1$, and $\alpha_r = (s_l s_{l-1} \cdots s_{j+1}) \cdot \alpha_{s_j}$. Rule (iii) of Case 1 applies when (s_j, t_j) is read. If $\mathcal{X}_j = \mathcal{F}$ then $x_1(j) \notin L$, and so $(x_1(j), x_2(j))$ is not viable. Otherwise, the following conditions hold at $k = j$:

(a) $s_1 * s_2 * \cdots * s_k \in L$,
(b) $s_i = t_{i-1}$ for all i such that $j < i \leq k$,
(c) $\mathcal{X}_k = (X_k, \beta_k, t_k, +1)$, where X_k is the state $\mathcal{W}$ would be in after reading $s_1 * s_2 * \cdots * s_k$, and $\beta_k = (s_k \cdots s_{j+1}) \cdot \alpha_{s_j} \in \widetilde{\Delta}_W$.

Choose k maximal such that these conditions hold. By Case 2 we see that either $s_{k+1} \neq t_k$, or $t_{k+1} \notin R$, or $\mu(X_k, s_{k+1}) = \mathcal{F}$, or $s_{k+1} \cdot \beta \notin \widetilde{\Delta}_W$.

Suppose first that $t_{k+1} \in R$, so that either the first, the third or the fourth of the above alternatives obtains. If $s_{k+1} \neq t_k$ then Proposition 3.4 shows that $(x_1, x_2) \notin L_r$; rule (i) of Case 2 shows that failure occurs. If $\mu(X_k, s_{k+1}) = \mathcal{F}$ then $x_1(k+1) \notin L$, and so $(x_1, x_2) \notin L_r$; rule (ii) of Case 2 shows that failure occurs. If $s_{k+1} \cdot \beta \in \Delta_W$ then Lemma 3.2 shows that $l(uv) = l(u) + l(v)$ is not compatible with $us_{k+1} \cdot \beta$ and $v^{-1}s_{k+1} \cdot \beta$ being both in Π. Taking $u = s_{j+1} \cdots s_{k+1}$ and $v = s_{k+2} \cdots s_l$, it follows that either $s_{j+1} * \cdots * s_l$ is not reduced, or else $(s_l \cdots s_{j+1}) \cdot \alpha_{s_j} \neq \alpha_r$, whence $(x_1, x_2) \notin L_r$. Furthermore, rule (ii) of Case 2 shows that failure occurs.

Consider now the case $t_{k+1} = \$$; that is, $m = k$. If $s_{k+1} \neq t_k$ or if $x_1(k) * s_{k+1} \notin L$ then $(x_1, x_2) \notin L_r$; rules (i) and (iii) of Case 2 show that failure occurs. If $s_{k+1} = t_k$ and $x_1(k) * s_{k+1} \in L$ then $(x_1, x_2) \in L_r$ if and only if $l = k+1$ and $s_l \cdot \beta = \alpha_r$. Rule (iii) of Case 2 shows that $\mathcal{X}_l = \mathcal{Y}$ if and only if $s_{k+1} \cdot \beta = \alpha_r$; by the Case 4 rule, acceptance occurs if and only if the process stops at this point, that is, if and only if $l = k+1$.

We have now shown that in all cases when $s_j \prec t_j$, the automaton $\mathcal{M}_r$ accepts (x_1, x_2) if and only if $(x_1, x_2) \in L_r$. Totally analogous arguments apply if $t_j \prec s_j$, using the rules of Case 3 in place of Case 2.

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