Notes for Math 606: Proofs and Confirmations

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Note to students: With these notes I've attempted to clarify the exposition in the textbook and to fill in some of the details. Read these notes as a companion to the textbook. Observe that many conjectures and theorems are stated without proof. Some of the proofs are supplied later in the book. Be mindful as you are reading about which statements require a proof and which statements you should accept on faith. If a theorem is cited without proof, make sure you understand the statement of the theorem and how it is being applied to obtain new results.

Students should also understand that this book is about much more than just counting alternating sign matrices. It covers a large swath of the branch of mathematics known as *Algebraic Combinatorics.* This include such topics as generating functions, recursive formulae, partitions, plane partitions, lattice paths, determinant evaluations, and basic hypergeometric series. These ideas were developed well before the arrival of the Alternating Sign Matrix Conjecture, and all these ideas come into play in the proof of this conjecture.

Chapter 1: The Conjecture

Preliminary remarks

General remarks about conjectures. Definition of alternating sign matrix.

Section 1.1: How many are there?

Topics to review: Permutations, combinations, binomial coefficients, Pascal's triangle, combinatorial proofs of binomial coefficient identities.

The first and last non-zero entry in any row or column of an alternating sign matrix must be 1, otherwise there is no way to produce 1 as an alternating sum. Hence the four borders of an alternating sign matrix must each contain a single 1 and the other entries must be 0. $A_{n,k}$ = number of alternating sign matrices with 1 in row 1, column k.

It turns out that if you arrange the numbers $\frac{A_{n,k}}{A_{n,k+1}}$ into a triangle, and express these fractions in just the right form, the triangle of numerators looks like the sum of two Pascal's Triangles and the triangle of denominators looks like the sum of two other Pascal's Triangles. This leads to the conjecture

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-k}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

Rearranging, we have

$$A_{n,k+1} = \frac{(n-k)(n+k-1)}{k(2n-k-1)} A_{n,k}.$$

Therefore

$$\begin{aligned}
A_{n,2} &= \frac{(n-1)n}{1(2n-2)} A_{n,1} \\
A_{n,3} = \frac{(n-2)(n+1)}{2(2n-3)} A_{n,2} &= \frac{(n-2)(n-1)n(n+1)}{(1)(2)(2n-3)(2n-2)} A_{n,1} \\
A_{n,4} = \frac{(n-3)(n+2)}{3(2n-4)} A_{n,3} &= \frac{(n-3)(n-2)(n-1)n(n+1)(n+2)}{(1)(2)(3)(2n-4)(2n-3)(2n-2)} A_{n,1} \\
&\vdots \\
A_{n,k} &= \frac{(n+k-2)!}{(n-k)!} \frac{1}{(k-1)!} \frac{(2n-k-1)!}{(2n-2)!} A_{n,1}.
\end{aligned}$$

This formula is valid even for $k = 1$. Observe that

$$\frac{(n+k-2)!}{(n-k)!} \frac{1}{(k-1)!} \frac{(2n-k-1)!}{(2n-2)!} =$$

$$\frac{(n-1)!(n-1)!}{(2n-2)!} \binom{n+k-2}{n-1} \binom{2n-k-1}{n-1}.$$

Therefore

$$A_n = \sum_{k=1}^{n} A_{n,k} = \frac{(n-1)!(n-1)!}{(2n-2)!} \sum_{k=1}^{n} \binom{n+k-2}{n-1} \binom{2n-k-1}{n-1} A_{n,1}.$$

With a change of variables, we can write

$$\sum_{k=1}^{n} \binom{n+k-2}{n-1} \binom{2n-k-1}{n-1} = \sum_{j=n}^{2n-1} \binom{j-1}{n-1} \binom{3n-2-j}{n-1}.$$

The product $\binom{j-1}{n-1}\binom{3n-2-j}{n-1}$ can be interpreted as the number of binary strings of length $3n-2$ which contain $2n-2$ ones and n zeros, in which the n^{th} one appears in position j. After placing the n^{th} one in position j, there are $\binom{j-1}{n-1}$ to position the first $n-1$ ones in the first $j-1$ positions and there are $\binom{3n-2-j}{n-1}$ ways to position the remaining $n-1$ ones in the remaining $3n-2-j$ positions. The n^{th} one can appear in position $j \in \{n, n+1, \ldots, 2n-1\}$. Hence the sum is equal to the total number of such binary strings, namely $\binom{3n-2}{2n-1}$. Therefore

$$A_n = \frac{(n-1)!(n-1)!}{(2n-2)!} \binom{3n-2}{2n-1} A_{n,1} =$$

$$\frac{(n-1)!(n-1)!}{(2n-2)!} \binom{3n-2}{2n-1} A_{n-1} = \frac{(n-1)!(3n-2)!}{(2n-2)!(2n-1)!} A_{n-1}.$$

Now we can compute

$$\begin{aligned}
A_2 = \frac{1!4!}{2!3!} A_1 &= \frac{1!4!}{2!3!} \\
A_3 = \frac{2!7!}{4!5!} \frac{1!4!}{2!3!} &= \frac{1!4!7!}{3!4!5!} \\
A_4 = \frac{3!10!}{6!7!} \frac{1!4!7!}{3!4!5!} &= \frac{1!4!7!10!}{4!5!6!7!} \\
&\vdots \\
A_n &= \frac{1!4!7! \cdots (3n-2)!}{n!(n+1)!(n+2)! \cdots (2n-1)!}.
\end{aligned}$$

This is an amazing formula. Of course, its derivation depends on the original conjecture regarding the ratios $\frac{A_{n,k}}{A_{n,k+1}}$, which we have not proved to be true for all n and k.

Exercises: Section 1.1, problems 11, 12, 13, 14

Section 1.2: Plane Partitions

Read the chronology in my notes for Section 1.3 to see the connection between alternating sign matrices and plane partitions.

Plane Partition: see Figure 1.3, page 10. A more rigorous definition is given on page 13: Regard the partition $\mathcal{P}$ as consisting of points in space with positive integer coordinates. If $(r, s, t) \in \mathcal{P}$ and $1 \le r' \le r$, $1 \le s' \le s$, and $1 \le t' \le t$, then $(r', s', t') \in \mathcal{P}$ also.

One way to visualize these: Let $\mathcal{P}$ be a plane partition. Let

$$\mathcal{P}_k^{(z)} = \{(x, y) : (x, y, k) \in \mathcal{P}\}$$

denote the slice of $\mathcal{P}$ consisting of all (x, y) coordinates corresponding to $z = k$. We can visualize this as a horizontal slice hovering k units above the xy plane. If $(x, y, k) \in \mathcal{P}$ and $k > 1$ then we know that $(x, y, k-1) \in \mathcal{P}$ also. Therefore $\mathcal{P}_k^{(z)} \subseteq \mathcal{P}_{k-1}^{(z)}$. That is, the slice at $z = k$ fits nicely inside the slice at $z = k - 1$. We can define $\mathcal{P}_k^{(x)}$ and $\mathcal{P}_k^{(y)}$ similarly, slices parallel to the yz-plane and the xz-plane respectively:

$$\mathcal{P}_k^{(x)} = \{(y, z) : (k, y, z) \in \mathcal{P}\}$$

and

$$\mathcal{P}_k^{(y)} = \{(x, z) : (x, k, z) \in \mathcal{P}\}.$$

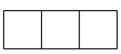
We have the containments

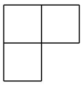
$$\mathcal{P}_1^{(x)} \supseteq \mathcal{P}_2^{(x)} \supseteq \mathcal{P}_3^{(x)} \supseteq \cdots,$$

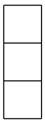
$$\mathcal{P}_1^{(y)} \supseteq \mathcal{P}_2^{(y)} \supseteq \mathcal{P}_3^{(y)} \supseteq \cdots,$$

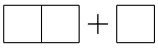
$$\mathcal{P}_1^{(z)} \supseteq \mathcal{P}_2^{(z)} \supseteq \mathcal{P}_3^{(z)} \supseteq \cdots.$$

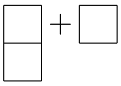
Another way to represent plane partitions is to use 2-dimensional diagrams which describe the z-slices. For example, there are 6 plane partitions of the number 3:

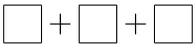
1.  $\mathcal{P}_1^{(z)} = \{(1, 1), (1, 2), (1, 3)\}$

2.  $\mathcal{P}_1^{(z)} = \{(1,1),(1,2),(2,1)\}$

3.  $\mathcal{P}_1^{(z)} = \{(1,1),(2,1),(3,1)\}$

4.  $\mathcal{P}_1^{(z)} = \{(1,1),(1,2)\}$, $\mathcal{P}_2^{(z)} = \{(1,1)\}$

5.  $\mathcal{P}_1^{(z)} = \{(1,1),(2,1)\}$, $\mathcal{P}_2^{(z)} = \{(1,1)\}$

6.  $\mathcal{P}_1^{(z)} = \{(1,1)\}$, $\mathcal{P}_2^{(z)} = \{(1,1)\}$, $\mathcal{P}_3^{(z)} = \{(1,1)\}$.

If we denote by $pp(n)$ the number of plane partitions with n cubes in it, then a challenging problem is to provide a formula or some other systematic way of computing $pp(n)$. There is a table of the first few values of $pp(n)$ on page 10. Properties of plane partitions were known to people working with alternating sign matrices, and these researchers discovered numerical evidence suggesting a link between alternating sign matrices and a special class of plane partitions. So we will digress at this point to describe a standard technique for counting plane partitions.

Method of counting plane partitions: generating functions

A generating function for the sequence of numbers a_0, a_1, a_2, $\ldots$ is the power series

$$g = a_0 + a_1 q + a_2 q^2 + \cdots = \sum_{n=0}^{\infty} a_n q^n.$$

Example 1: Let $a_n = 1$ for all $n \geq 0$. Then g is the geometric series

$$g = 1 + q + q^2 + \cdots = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

Example 2: Let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

Then g is the geometric series

$$g = 1 + q^2 + q^4 + \cdots = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}.$$

Example 3: For each $n \geq 0$ let $a_n = n + 1$. Then

$$g = \sum_{n=0}^{\infty} (n+1)q^n.$$

We can simplify this:

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1)q^n &= 1 + 2q + 3q^2 + 4q^3 + \cdots \\
&= (1 + q + q^2 + q^3 + q^4 + \cdots)' \\
&= \left(\frac{1}{1-q}\right)' \\
&= \frac{1}{(1-q)^2}.
\end{aligned}$$

Therefore

$$g = \frac{1}{(1-q)^2}.$$

Example 4: Let $a_0 = 1$, and for each $n \geq 1$ let $a_n =$ the number of ways to express the integer n as a sum of the form

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$$

and the numbers add up to n. For example, $a_3 = 3$ because there are 3 ways to express the number 3:

$$3,\ 2+1,\ 1+1+1.$$

Note the correspondence between these expressions and the products

$$q^3,\ q^2q^1,\ (q^1)^3,$$

which represent all the ways to produce q^3 when expanding the product

$$\left(1+q^3+(q^3)^2+\cdots\right)\left(1+q^2+(q^2)^2+\cdots\right)\left(1+q^1+(q^1)^2+\cdots\right)=$$

$$\frac{1}{(1-q^3)(1-q^2)(1-q^1)}.$$

Therefore a_3 is the coefficient of q^3 in

$$\frac{1}{(1-q^3)(1-q^2)(1-q)}.$$

Similarly, a_4 is the coefficient of q^4 in

$$\frac{1}{(1-q^4)(1-q^3)(1-q^2)(1-q)}.$$

Observe that a_3 is also the coefficient of q^3 in

$$\frac{1}{(1-q^4)(1-q^3)(1-q^2)(1-q)},$$

because the terms in $\frac{1}{1-q^4}$ to not make any contribution to q^3. Generalizing further, we can say that a_3 is the coefficient of q^3 and a_4 is the coefficient of q^4 in the same series

$$\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots}=\prod_{j=1}^{\infty}\frac{1}{1-q^j}.$$

In fact, a_n is the coefficient of q^n in this series for each $n \geq 1$. Therefore the generating function for the sequence of numbers a_0, a_1, a_2, $\ldots$ is

$$g=\prod_{j=1}^{\infty}\frac{1}{1-q^j}.$$

This is the context for Theorem 1.1, page 11: If $a_n = pp(n)$ for each n, then the generating function is

$$g = pp(0) + pp(1)q + pp(2)q^2 + \cdots = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j},$$

where by convention we set $pp(0) = 1$ (there is one empty partition). Given this generating function representation, we can compute the numbers $pp(n)$ recursively by Theorem 1.2: for $n \geq 2$ we have

$$pp(n) = \frac{1}{n}(\sigma_2(1)pp(n-1) + \sigma_2(2)pp(n-2) + \cdots + \sigma_2(n)pp(0)),$$

where

$$\sigma_2(j) = \sum_{d|j} d^2.$$

As mentioned above, there is a connection between alternating sign matrices and a special class of plane partitions. We will spend some time wrapping our minds around these objects.

Classes of plane partitions:

a. Restricted plane partitions, page 13. $\mathcal{B}(r,s,t)$ consists of all points (i,j,k) such that $1 \leq i \leq r$, $1 \leq j \leq s$, and $1 \leq k \leq t$. These points form a cubic lattice of points with one vertex at $(1,1,1)$ and another vertex at (r,s,t). Let $rpp(n) =$ the number of plane partitions of n which are subsets of $\mathcal{B}(r,s,t)$. Note that $rpp(n) = 0$ if $n > rst$. By Theorem 1.3, the generating function for the number of such partitions is

$$\sum_{n=0}^{rst} rpp(n)q^n = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}} = \prod_{\eta \in \mathcal{B}(r,s,t)} \frac{1-q^{1+\mathrm{ht}(\eta)}}{1-q^{\mathrm{ht}(\eta)}},$$

where the height of $\eta = (i,j,k) \in \mathcal{B}(r,s,t)$ is $i+j+k-2$.

Note that if we let $q \to 1$ in

$$\sum_{n=0}^{rst} rpp(n)q^n$$

we get

$$\sum_{n=0}^{rst} rpp(n),$$

which is the total number of plane partitions inside $\mathcal{B}(r,s,t)$ (including the empty partition). Therefore, we should get the same answer if we let $q \to 1$ in

$$\prod_{\eta\in\mathcal{B}(r,s,t)} \frac{1-q^{1+\mathrm{ht}(\eta)}}{1-q^{\mathrm{ht}(\eta)}}.$$

Note that

$$\frac{1-q^a}{1-q^b} = \frac{(1-q)(1+q+\cdots+q^{a-1})}{(1-q)(1+q+\cdots+q^{b-1})} = \frac{1+q+\cdots+q^{a-1}}{1+q+\cdots+q^{b-1}},$$

hence letting $q \to 1$ we obtain $\frac{a}{b}$. Therefore the total number of plane partitions inside $\mathcal{B}(r,s,t)$ is

$$\prod_{\eta\in\mathcal{B}(r,s,t)} \frac{1+\mathrm{ht}(\eta)}{\mathrm{ht}(\eta)}.$$

b. Symmetric plane partitions, page 13. If (i,j,k) is a point in a symmetric plane partition, then so is (j,i,k). That is, if $(x,y) \in \mathcal{P}_k^{(z)}$ then $(y,x) \in \mathcal{P}_k^{(z)}$ also. There is therefore symmetry about the line $y=x$ in every slice of $\mathcal{P}$ which is parallel to the xy-plane. We can see this in Figure 1.5 on page 13.

Note that if (i,j,k) belongs to $\mathcal{B}(r,r,t)$ then so does (j,i,k). Moreover,

$$\mathrm{ht}(i,j,k) = \mathrm{ht}(j,i,k) = i+j+k-2.$$

The symmetric orbit of (i,j,k) is $\{(i,j,k),(j,i,k)\}$, and the size of this orbit is 2 or 1 depending on whether $i \neq j$ or $i = j$. Therefore the generating function for plane partitions restricted to $\mathcal{B}(r,r,t)$ can be expressed as

$$\sum_{n=0}^{r^2t} rpp(n)q^n = \prod_{\eta\in\mathcal{B}(r,r,t)/\mathcal{S}_2} \left(\frac{1-q^{1+\mathrm{ht}(\eta)}}{1-q^{\mathrm{ht}(\eta)}}\right)^{|\eta|},$$

where $|\eta|$ is the size of the symmetric orbit of η and $\mathcal{B}(r,r,t)/\mathcal{S}_2$ is a set consisting of one η per orbit. Let $rspp(n)$ = the number of symmetric plane partitions restricted to $\mathcal{B}(r,r,t)$. The MacMahon Conjecture (subsequently proved to be true) is that the generating function for symmetric plane partitions restricted to $\mathcal{B}(r,r,t)$ is

$$\sum_{n=0}^{r^2t} rspp(n)q^n = \prod_{\eta\in\mathcal{B}(r,r,t)/\mathcal{S}_2} \frac{1-q^{|\eta|(1+\mathrm{ht}(\eta))}}{1-q^{|\eta|\mathrm{ht}(\eta)}}.$$

Letting $q \longrightarrow 1$, we find that the number of symmetric plane partitions living in $\mathcal{B}(r,r,t)$ is equal to

$$\sum_{n=0}^{r^2t} rspp(n) = \prod_{\eta\in\mathcal{B}(r,r,t)/\mathcal{S}_2} \frac{1+\mathrm{ht}(\eta)}{\mathrm{ht}(\eta)}.$$

c. Cyclic plane partitions, page 15. If (i,j,k) is in the partition, then so are (j,k,i) and (k,i,j). How to visualize these: imagine you are looking at the point $(0,10,5)$ in the yz-plane. If you draw a line from $(0,0,0)$ to $(0,10,5)$, you will see that the line has slope equal to $\frac{1}{2}$ in the yz-plane. When you cyclically permute the coordinates of $(0,10,5)$ you produce the point $(10,5,0)$. If you draw a line from $(0,0,0)$ to $(10,5,0)$, you will see that the line has slope equal to $\frac{1}{2}$ in the xy-plane. It is easy to see that when you cyclically permute all the coordinates on the line $z=\frac{1}{2}y$ in the yz-plane you produce all the points on the line $y=\frac{1}{2}x$ in the xy-plane. Geometrically, the effect of cyclically permuting the coordinates of the yz-plane is to rotate the yz-plane into the position of the xy-plane.

Continuing this example, if you cyclically permute the coordinates of $(10,5,0)$ you produce the point $(5,0,10)$. This is the point that results by rotating the xy-plane into the position of the xz-plane. Another cyclic permutation produces $(0,10,5)$, the point we started with.

If we repeat this exercise starting with the point $(k,10,5)$, and regarding this as a point in the $x=k$ slice of a cyclically symmetric plane partition $\mathcal{P}$, we can see that $(10,5,k)$ is the corresponding point in the $z=k$ slice of $\mathcal{P}$ and $(5,k,10)$ is the corresponding point in the $y=k$ slice of $\mathcal{P}$. To summarize, when you cyclically permute the coordinates of the $x=k$ slice of $\mathcal{P}$ you obtain the $z=k$ slice of $\mathcal{P}$, when you cyclically permute the coordinates of the $z = k$ slice of $\mathcal{P}$ you obtain the $y = k$ slice of $\mathcal{P}$, and when you cyclically permute the coordinates of the $y = k$ slice of $\mathcal{P}$ you obtain the $x = k$ slice of $\mathcal{P}$.

Analogous to our remarks regarding symmetric plane partitions restricted to $\mathcal{B}(r, r, t)$, we can make the following observations about cyclically symmetric plane partitions restricted to $\mathcal{B}(r, r, r)$. Note that if (i, j, k) belongs to $\mathcal{B}(r, r, r)$ then so does (j, k, i) and (k, i, j). Moreover,

$$\mathrm{ht}(i, j, k) = \mathrm{ht}(j, k, i) = \mathrm{ht}(k, i, j) = i + j + k - 2.$$

The cyclic orbit of (i, j, k) is $\{(i, j, k), (j, k, i), (k, i, j)\}$, and the size of this orbit is 3 or 1 depending on i, j and k. Therefore the generating function for plane partitions restricted to $\mathcal{B}(r, r, r)$ can be expressed as

$$\sum_{n=0}^{r^3} rpp(n)q^n = \prod_{\eta \in \mathcal{B}(r,r,r)/\mathcal{C}_3} \left(\frac{1 - q^{1+\mathrm{ht}(\eta)}}{1 - q^{\mathrm{ht}(\eta)}} \right)^{|\eta|},$$

where $|\eta|$ is the size of the cyclic orbit of η and $\mathcal{B}(r, r, t)/\mathcal{C}_3$ is a set consisting of one η per orbit. Let $rcspp(n) =$ the number of cyclically symmetric plane partitions which live in $\mathcal{B}(r, r, r)$. Another conjecture due to MacMahon (also subsequently proved to be true) is that the generating function for symmetric plane partitions restricted to $\mathcal{B}(r, r, r)$ is

$$\sum_{n=0}^{r^3} rcspp(n)q^n = \prod_{\eta \in \mathcal{B}(r,r,r)/\mathcal{C}_3} \frac{1 - q^{|\eta|(1+\mathrm{ht}(\eta))}}{1 - q^{|\eta|\mathrm{ht}(\eta)}}.$$

Letting $q \to 1$, we find that the number of cyclically symmetric plane partitions living in $\mathcal{B}(r, r, t)$ is equal to

$$\sum_{n=0}^{r^3} rcspp(n) = \prod_{\eta \in \mathcal{B}(r,r,r)/\mathcal{C}_3} \frac{1 + \mathrm{ht}(\eta)}{\mathrm{ht}(\eta)}.$$

d. Totally symmetric plane partitions, page 15. These are partitions with the following property: if (p, q, r) is in the partition, then so is every permutation of these three coordinates. Note: If a plane partition is totally symmetric, then it must be both symmetric and cyclically symmetric. So it will have both of the geometric properties we stated above. Moreover, it is easy to see that if a plane partition is symmetric and cyclically symmetric, then it must be totally symmetric. So the totally symmetric plane partitions are precisely those with symmetry about the line $x = y$ and whose slices can be rotated into each other by cyclic permutations.

Let $rtspp(n)$ = the total number of totally symmetric plane partitions living in $\mathcal{B}(r, r, r)$. It would be tempting to say that the generating function for totally symmetric plane partitions restricted to $\mathcal{B}(r, r, r)$ is

$$\sum_{n=0}^{r^3} rtspp(n)q^n = \prod_{\eta \in \mathcal{B}(r,r,r)/\mathcal{S}_3} \frac{1 - q^{|\eta|(1+\mathrm{ht}(\eta))}}{1 - q^{|\eta|\mathrm{ht}(\eta)}},$$

where we define the totally symmetric orbit of (i, j, k) to be the set of all permutations of the coordinates i, j, k. It turns that this conjecture is false (in problem 10 you will prove that the right-hand side of this formula is not a polynomial for $r = 3$, while the left-hand side must be a polynomial). However, the conjecture suggests the formula

$$\sum_{n=0}^{r^3} rtspp(n) = \prod_{\eta \in \mathcal{B}(r,r,r)/\mathcal{S}_3} \frac{1 + \mathrm{ht}(\eta)}{\mathrm{ht}(\eta)},$$

and this formula is correct (but must be proved by other means).

Exercises: 1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 15

Section 1.3: Descending Plane Partitions

There is a very tangled narrative here which I will attempt to unravel. It is a chronology which shows how researchers made a connection between alternating sign matrices and descending plane partitions.

1. MacMahon derived the generating function for plane partitions restricted to $\mathcal{B}(r, s, t)$:

$$\prod_{\eta \in \mathcal{B}(r,s,t)} \frac{1 - q^{1+\mathrm{ht}(\eta)}}{1 - q^{\mathrm{ht}(\eta)}}.$$

2. MacMahon conjectured (correctly) that the generating function for symmetric plane partitions restricted to $\mathcal{B}(r,r,t)$ is

$$\prod_{\eta\in\mathcal{B}(r,r,t)/\mathcal{S}_2}\left(\frac{1-q^{1+\mathrm{ht}(\eta)}}{1-q^{\mathrm{ht}(\eta)}}\right)^{|\eta|}.$$

3. Macdonald conjectured (incorrectly) that the generating function for totally symmetric plane partitions restricted to $\mathcal{B}(r,r,r)$ is

$$\prod_{\eta\in\mathcal{B}(r,r,r)/\mathcal{S}_3}\frac{1-q^{|\eta|(1+\mathrm{ht}(\eta))}}{1-q^{|\eta|\mathrm{ht}(\eta)}}.$$

4. Macdonald conjectured (correctly) that the generating function for cyclically symmetric plane partitions restricted to $\mathcal{B}(r,r,r)$ is

$$\prod_{\eta\in\mathcal{B}(r,r,r)/\mathcal{C}_3}\frac{1-q^{|\eta|(1+\mathrm{ht}(\eta))}}{1-q^{|\eta|\mathrm{ht}(\eta)}}.$$

5. The generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ can be expressed as a determinant:

$$\det\left(\delta_{ij}+q^{3i-2}\begin{bmatrix}i+j-2\\ i-1\end{bmatrix}_{q^3}\right)_{i,j=1}^{r}.$$

6. Andrews was unable to prove Macdonald's conjecture by evaluating the determinant in (5). But he was able to prove the $q\to 1$ version:

$$\det\left(\delta_{ij}+\binom{i+j-2}{j-1}\right)=\prod_{\eta\in\mathcal{B}(r,r,r)/\mathcal{C}_3}\frac{|\eta|(1+\mathrm{ht}(\eta))}{|\eta|\mathrm{ht}(\eta)}.$$

7. Andrews found a generating function for descending plane partitions with largest part $\leq r$, expressed in determinant form:

$$\det\left(\delta_{ij}+q^{i+1}\begin{bmatrix}i+j\\ j-1\end{bmatrix}_{q}\right).$$

Every cyclically symmetric plane partition corresponds in a unique way to a special class of strict shifted plane partitions; strict shifted plane partitions are a special class of descending plane partitions.

8. Andrews conjectured that his generating function for descending plane partitions (7) has a product form:

$$\prod_{1\le i\le j\le r} \frac{1-q^{r+i+j-1}}{1-q^{2i+j-1}}.$$

9. Andrews was unable to evaluate the determinant representing the generating function for descending plane partitions, just as he was unable to evaluate the determinant representing the generating function for cyclically symmetric plane partitions. But he was able to prove the $q \to 1$ version, which yields a conjectured formula for descending plane partitions with largest part $\le r$:

$$\det\left(\delta_{ij} + \binom{i+j}{j-1}\right) = \prod_{1\le i\le j\le r} \frac{r+i+j-1}{2i+j-1}.$$

10. Stanley observed that Andrews result (9) seems to count the number of alternating sign matrices of size r. Andrews' formula can be expressed in the form

$$\prod_{1\le i\le j\le r} \frac{r+i+j-1}{2i+j-1} = \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j)!},$$

which we have already encountered as the conjectured formula for A_r.

11. This discovery motivated Mills, Robbins, and Rumsey to find a one-to-one correspondence between descending plane partitions with largest part $\le r$ and $r \times r$ alternating sign matrices. They didn't find one, and this is still an open question.

12. Alternating sign matrices of size r can be divided into classes according to the position of the 1 in the first row. This suggested to Mills, Robbins, and Rumsey that there should be a way to divide descending plane partitions with largest part $\le r$ into classes of the same size. Class k of alternating sign matrices conjectured to correspond to descending plane partitions with largest part $\le r$ and the part r appears exactly $k-1$ times.

13. This method of dividing descending plane partitions into classes led to techniques for evaluating the determinant expressing total number of descending plane partitions. These techniques were eventually used to prove Macdonald's conjecture (6), Andrews' conjecture (8), and the conjecture of Mills, Robbins, and Rumsey (12).

There are a lot of new ideas here: strict shifted plane partition, the correspondence between cyclically symmetric plane partitions and a special class of strict shifted plane partitions, and descending plane partitions. They all seem to run together and I get dizzy trying to keep track of them all. The only way to remedy this is to make them real by defining them carefully and investigating their properties. So we will make a digression here and take a close look at these objects. We will begin by looking at matrix formulation of plane partition.

Let $\mathcal{P}$ be a plane partition. We can encode the information in $\mathcal{P}$ by a matrix of non-negative integers $M = (z_{ij})$ as follows: If $(i, j, 1)$ belongs to $\mathcal{P}$, then we set z_{ij} equal to the maximum z-coordinate such that (i, j, z_{ij}) belongs to $\mathcal{P}$. If $(i, j, 1) \notin \mathcal{P}$ then we set $z_{ij} = 0$. We can interpret z_{ij} as the number of cubes of the partition at x coordinate i and y-coordinate j. The matrix which represents the plane partition of 75 on page 10 is

$$M = \begin{bmatrix} 6 & 5 & 5 & 4 & 3 & 3 \\ 6 & 4 & 3 & 3 & 1 & \\ 6 & 4 & 3 & 1 & 1 & \\ 4 & 2 & 2 & 1 & & \\ 3 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \end{bmatrix},$$

where we have omitted the zero entries.

Observe that the entries of each row of the matrix weakly decrease from left to right and the entries of each column of the matrix weakly decrease from top to bottom. In order to prove that this will always be the case, consider z_{ij}, the entry in row i and column j. Assume $z_{ij} > 0$. If $i > 1$ we must prove that $z_{ij} \leq z_{i-1,j}$, and if $j > 1$ we must prove that $z_{ij} \leq z_{i,j-1}$.

Consider $i > 1$. We have $(i, j, z_{ij}) \in \mathcal{P}$, therefore $(i-1, j, z_{ij}) \in \mathcal{P}$. Therefore the maximum z-coordinate such that $(i-1, j, z)$ belongs to $\mathcal{P}$ is greater than or equal to z_{ij}. This implies $z_{i-1,j} \geq z_{ij}$.

Consider $j > 1$. We have $(i, j, z_{ij}) \in \mathcal{P}$, therefore $(i, j-1, z_{ij}) \in \mathcal{P}$. Therefore the maximum z-coordinate such that $(i, j-1, z)$ belongs to $\mathcal{P}$ is greater than or equal to z_{ij}. This implies $z_{i,j-1} \geq z_{i,j}$.

A natural question to ask is this: beginning with an arbitrary matrix of non-negative integers

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

such that each row and each column weakly decreases, is there a corresponding plane partition $\mathcal{P}$ whose stack heights are represented by M? Yes: set

$$\mathcal{P} = \{(i, j, k) : i \geq 1,\ j \geq 1,\ 1 \leq k \leq a_{ij}\}.$$

We must show that if $(i, j, k) \in \mathcal{P}$ and if $1 \leq i' \leq i$, $1 \leq j' \leq j$, and $1 \leq k' \leq k$, then $(i', j', k') \in \mathcal{P}$ also.

Let $(i, j, k) \in \mathcal{P}$ be given. Then $1 \leq k \leq a_{ij}$, so we know that $a_{ij} \geq 1$. Now let $1 \leq i' \leq i$, $1 \leq j' \leq j$, and $1 \leq k' \leq k$ be given. We must show that $(i', j', k') \in \mathcal{P}$, that is $1 \leq k' \leq a_{i'j'}$. We have

$$k' \leq k \leq a_{ij} \leq a_{i'j} \leq a_{i'j'},$$

the third and fourth inequalities due to the fact that the row entries of M weakly increase and the column entries of M weakly decrease, respectively.

We will now investigate what properties the matrix representing a plane partition $\mathcal{P}$ must possess if $\mathcal{P}$ is cyclically symmetric. In order to do this, we need to understand how to extract information about the z-slices of a plane partition from its matrix representation $M = (z_{ij})$.

For each $k \geq 1$, $\mathcal{P}_k^{(z)}$ is represented by $M_k^{(z)}$, the matrix whose i, j-entry is

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } z_{ij} \geq k \\ 0 & \text{otherwise.} \end{cases}$$

The positions of the 1s record positions of cubes with $z = k$. For the partition on page 10, we have

$$M_1^{(z)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \end{bmatrix}, \; M_2^{(z)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & & & & & \\ & & & & & \end{bmatrix},$$

$$M_3^{(z)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & & & \\ 1 & & & & & \\ 1 & & & & & \\ & & & & & \end{bmatrix}, \; M_4^{(z)} = \begin{bmatrix} 1 & 1 & 1 & 1 & & \\ 1 & 1 & & & & \\ 1 & 1 & & & & \\ 1 & & & & & \\ & & & & & \\ & & & & & \end{bmatrix},$$

$$M_5^{(z)} = \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & & & & & \\ 1 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}, \; M_6^{(z)} = \begin{bmatrix} 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}.$$

Observe that

$$M = M_1^{(z)} + M_2^{(z)} + M_3^{(z)} + M_4^{(z)} + M_5^{(z)} + M_6^{(z)},$$

and that each z-slice fits inside the one below it. This explains why we refer to $\mathcal{P}$ as a plane partition.

The plane partition on page 10 happens to be cyclically symmetric. The following theorem tells us how to recognize a cyclically symmetric plane partition by its z slices.

Theorem: *Let $\mathcal{P}$ be an arbitrary plane partition, and let $M = (z_{ij})$ be its matrix representation. Let $M_1^{(z)}$, $M_2^{(z)}$, $M_3^{(z)}$, … be the matrices representing the z-slices. Then $\mathcal{P}$ is cyclically symmetric if and only for each i and j the number z_{ij} is equal to the number of 1s in row j of $M_i^{(z)}$ and is also equal to the number of 1s in column i of $M_j^{(z)}$. In brief, for each p the row sums in $M_p^{(z)}$ are recorded in row p of M, and the column sums in $M_p^{(z)}$ are recorded in column p of M. (Verify this for the plane partition on page 10.)*

Proof. This is really just a restatement of what it means to be cyclically symmetric. Consider the vertical column of cubes in a plane partition $\mathcal{P}$ with x-coordinate i and y-coordinate j. There are z_{ij} of these cubes, and their coordinates are

$$(i, j, 1),\ (i, j, 2),\ , (i, j, 3),\ , \ldots, (i, j, z_{ij}).$$

In order for there to be cyclic symmetry, each of the coordinates

$$(j, 1, i),\ (j, 2, i),\ , (j, 3, i),\ , \ldots, (j, z_{ij}, i)$$

must correspond to cubes in $\mathcal{P}$, and they all belong to the $z = i$ slice. Therefore there are at least z_{ij} positions in the j^{th} row of $M_i^{(z)}$. Again, in order for there to be cyclic symmetry, each of the coordinates

$$(1, i, j),\ (2, i, j),\ , (3, i, j),\ , \ldots, (z_{ij}, i, j)$$

must correspond to cubes in $\mathcal{P}$, and they all belong to the $z = j$ slice. Therefore there are at least z_{ij} positions in the i^{th} column of $M_j^{(z)}$. Another cyclic permutation of coordinates brings us back to the original coordinates. Therefore $\mathcal{P}$ is cyclically symmetric if and only if z_{ij} is exactly equal to the number of 1s in row j of $M_i^{(z)}$ and is also equal to the number of 1s in column i of $M_j^{(z)}$. $\square$

The chronology above makes reference to a correspondence between cyclically symmetric plane partitions and a special class of strict shifted plane partitions. We will now describe this correspondence.

Let $M = (z_{ij})$ be the matrix representing a cyclically symmetric plane partition $\mathcal{P}$. We know that the $x = 1$, $y = 1$, and $z = 1$ slices are all isomorphic to each other in the sense that each can be rotated to overlap each other. Therefore, if we throw away all cubes in these slices, then replace each remaining (i, j, k) by $(i-1, j-1, k-1)$, we obtain a smaller cyclically symmetric plane partition. If we continue to do this we obtain a finite sequence of cyclically symmetric plane partitions $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots$, where $\mathcal{P}_1 = \mathcal{P}$. We can use this idea to obtain an alternative matrix representation of $\mathcal{P}$. Here's how:

Let M_1, M_2, M_3, $\ldots$ be the matrices which represent $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots$. For each $k > 1$ the matrix M_k is obtained from the matrix M_{k-1} is obtained by subtracting 1 from all the non-zero entries (throw out the $z = 1$ slice), then deleting the first row and first column (throw out the $x = 1$ and $y = 1$ slices). Let R_k represent the first row of M_k. The new matrix representation, which we call R, is obtained by assembling the rows R_1, R_2, R_3, $\ldots$ into a single matrix.

Example: The plane partition on page 10 is cyclically symmetric.

$$M_1 = \begin{bmatrix} 6 & 5 & 5 & 4 & 3 & 3 \\ 6 & 4 & 3 & 3 & 1 & \\ 6 & 4 & 3 & 1 & 1 & \\ 4 & 2 & 2 & 1 & & \\ 3 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \end{bmatrix}, \ R_1 = \begin{bmatrix} 6 & 5 & 5 & 4 & 3 & 3 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 3 & 2 & 2 & & \\ 3 & 2 & & & \\ 1 & 1 & & & \end{bmatrix}, \ R_2 = \begin{bmatrix} 3 & 2 & 2 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 \end{bmatrix}, R_3 = \begin{bmatrix} 1 \end{bmatrix}.$$

Therefore

$$R = \begin{bmatrix} 6 & 5 & 5 & 4 & 3 & 3 \\ & 3 & 2 & 2 & & \\ & & 1 & & & \end{bmatrix}.$$

See pp. 19-20.

In general, beginning with the matrix $M = (z_{ij})$ we obtain the matrix

$$R = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & \cdots \\ & z_{22} - 1 & z_{23} - 1 & z_{24} - 1 & \cdots \\ & & z_{33} - 2 & z_{34} - 2 & \cdots \\ & & & z_{44} - 3 & \cdots \\ & & & & \ddots \end{bmatrix}.$$

The rows of R are weakly decreasing from left to right and strictly decreasing from top to bottom. R is an example of a **strict shifted plane partition.** We will refer to R as the strict shifted representation of $\mathcal{P}$.

It is not an easy task to construct a cyclically symmetric plane partition from scratch, let alone count all of those containing a given number of cubes.

The next theorem spells out the correspondence between cyclically symmetric matrices and a special class of strict shifted plane partitions. It is much easier to generate strict shifted plane partitions than to generate cyclically symmetric plane partitions.

Theorem:

(1) Let $\mathcal{P}$ is a cyclically symmetric plane partition with strict shifted representation R. Then R has the following property: the largest entry in each row is equal to the number of positive entries in that row. (See for example the cyclically symmetric plane partition on page 10.)

(2) For each strict shifted plane partition R there is a unique cyclically symmetric plane partition $\mathcal{P}$ which generates it.

Proof. We will first prove (1). Let $\mathcal{P}$ be a cyclically symmetric plane partition, and let R be its strict shifted matrix representation. We will prove that the first entry of each row is equal to the number of positive entries in that row by strong induction on the size n of $\mathcal{P}$.

Base Case: $n = 1$. In this case $M = [1] = R$, and the property is true.

Induction Hypothesis: For every cyclically symmetric plane partition $\mathcal{P}$ of size n or less cubes, the matrix R has property (1).

Now consider any cyclically symmetric plane partition $\mathcal{P}$ which has size $n+1$ cubes. Form the sequence $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, …, and the corresponding strict shifted representation R. Observe that if we delete the first row and first column of R, and call the result R', then R' is the strict shifted representation of $\mathcal{P}_2$. Since $\mathcal{P}_2$ has size n or less, R' satisfies property (1) by the induction hypothesis. So we just need to show that the first row of R satisfies property (1). It does: the largest entry in the first row of R is z_{11}, and by the previous theorem we know that z_{11} is equal to the number of 1s in the first row of $M_1^{(z)}$. This is equal to the number of positive entries in the first row of M, which is equal to the number of positive entries in the first row of R. This completes the induction proof of (1).

We will now prove (2) by induction on n, the number of rows of R.

Base Case: $n = 1$. Let $R = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \end{bmatrix}$ be a strictly shifted plane partition with one row, and assume $a_{11} = k$. We will construct a matrix M as follows:

Step 1: put R in the first row.

Step 2: By the previous theorem, we know what $M_1^{(z)}$ should look like: there should be a_{1j} 1s in row j.

Step 3: By the previous theorem, we should set z_{j1} equal to the number of 1s in column j of $M_1^{(z)}$.

Step 4: We have a possible conflict when we define z_{11}: it should equal a_{11}, and it should also equal the number of 1s in column 1 of $M_1^{(z)}$. The number of 1s in column 1 of $M_1^{(z)}$ is equal to the number of rows in $M_1^{(z)}$, and there is one row per positive number in R. The number of positive numbers in R is by hypothesis equal to a_{11}.

Step 5: There cannot be a cube in position $(2, 2, 2)$, otherwise M cannot be represented by R. So the entries of M not in row 1 or column 1 must be equal to 1, and their positions are determined by $M_1^{(z)}$.

Step 6: M is cyclically symmetric: since we used property (1) to construct the $y = 1$ slice from the $x = 1$ slice, we know the $x = 1$ slice can be rotated into the $y = 1$ slice. Also, we constructed the $z = 1$ slice in such a way that the $x = 1$ slice can be rotated into it. So M is cyclically symmetric.

Step 7: We must verify that the rows and columns of M are weakly decreasing, otherwise we cannot be sure that it represents a valid plane partition. It is easy to see that all of its rows are weakly decreasing. It is also easy to see that columns 2 through k are weakly decreasing. We must verify that column 1 is weakly decreasing.

Row i of $M_1^{(z)}$ consists of 1s in positions $(i, 1), (i, 2), \ldots, (i, a_{1i})$. Therefore row i contains a 1 in column j if and only if $j \leq a_{1i}$. Therefore the number of rows that have an entry in column j is equal to the number of indices i such that $j \leq a_{1i}$. Note that if $j \leq a_{1i}$ then $j \leq a_{1,i-1}$ because $a_{1,i-1} \geq a_{1i}$. Therefore the number of 1s in column j of $M_1^{(z)}$ is equal to $\max\{i : j \leq a_{1i}\}$. Therefore

$$z_{j1} = \max\{i : j \leq a_{1i}\}.$$

Compare this to

$$z_{j-1,1} = \max\{i : j - 1 \leq a_{1i}\}.$$

Every index i which satisfies $j \leq a_{1i}$ will also satisfy $j - 1 \leq a_{1i}$. Therefore the maximum value in the larger set will be greater than or equal to the maximum value in the smaller set. That is,

$$z_{j-1,1} \geq z_{j1}.$$

This proves that the first column of M is weakly decreasing.

Hence we have uniquely constructed a valid cyclically symmetric plane partition which is represented by R. This completes the base case.

Induction Hypothesis: Every strict shifted plane partition R with property (1) and n rows represents a unique cyclically symmetric plane partition.

Now consider R with property (1) and $n+1$ rows. We need to construct a cyclically symmetric plane partition $\mathcal{P}$ which is represented by R, and show that no other cyclically symmetric plane partition is represented by R. Let R_1 be the first row of R. Construct a cyclically symmetric plane partition $\mathcal{A}$ represented by R_1 using the algorithm above. Let R' be what's left of R after deleting the first row and column. R' is a strict shifted plane partition which satisfies property (1) and which has n rows. Construct a cyclically symmetric plane partition $\mathcal{B}$ represented by R' using the induction hypothesis. The choices of $\mathcal{A}$ and $\mathcal{B}$ are unique. Let $\mathcal{B}'$ be the set of points that result from adding $(1,1,1)$ to all the points in $\mathcal{B}$. Glue together $\mathcal{A}$ and $\mathcal{B}'$ together to form $\mathcal{P}$. Note that since $\mathcal{B}$ is cyclically symmetric, so is $\mathcal{B}'$. We must verify that $\mathcal{P}$ is a valid cyclically symmetric plane partition which is represented by R.

There is no question that $\mathcal{P}$ is cyclically symmetric, because $\mathcal{A}$ and $\mathcal{B}'$ are. To check that $\mathcal{P}$ is a valid plane partition we need to verify that the $x=2$, $y=2$, and $z=2$ slices of $\mathcal{P}$ are dominated by the $x=1$, $y=1$, and $z=1$ slices of $\mathcal{A}$. By cyclic symmetry it will suffice to check that the $x=2$ slice is dominated by the $x=1$ slice.

Let $A=(a_{ij})_{i,j\geq 1}$ represent $\mathcal{A}$, let $B=(b_{ij})_{i,j\geq 2}$ represent $\mathcal{B}$, and let $A_1^{(z)}=(c_{ij})_{i,j\geq 1}$ be the $z=1$ slice of A. Then the matrix representing $\mathcal{P}$ is $M=(z_{ij})_{i,j\geq 1}$ defined by

$$z_{ij}=\begin{cases} a_{ij} & \text{if } i=1 \text{ or } j=1 \\ b_{ij}+c_{ij} & \text{if } i\geq 2 \text{ and } j\geq 2. \end{cases}$$

Note how the indexing of the matrix B has to be altered to make this formula work. Interpret b_{ij} as the height of cubes in $\mathcal{B}$ over x-coordinate $i-1$ and y-coordinate $j-1$. Note that $a_{1j}=r_{1j}$ for $j\geq 1$ and $b_{2j}=r_{2j}$ for $j\geq 2$. We know that $r_{1j}\geq r_{2j}+1$ for $j\geq 2$, therefore

$$z_{1j}=a_{1j}=r_{1j}\geq r_{2j}+1=b_{2j}+1\geq b_{2j}+c_{2j}=z_{2j}.$$

This proves that the $x = 2$ slice of $\mathcal{P}$ is dominated by the $x = 1$ slice of $\mathcal{A}$.

Finally, we must show that $\mathcal{P}$ is unique. If $\mathcal{Q}$ is a cyclically symmetric plane partition which is represented by R, then $\mathcal{Q}$ can be decomposed into $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where $\mathcal{Q}_1$ consists of the bottom shell of $\mathcal{Q}$ and $\mathcal{Q}_2$ contains all the other cubes in $\mathcal{Q}$. Then $\mathcal{Q}$ must be represented by R_1 and $\mathcal{Q}_2$ must be represented by R'. Therefore, by uniqueness we must have $\mathcal{Q}_1 = \mathcal{A}$ and $\mathcal{Q}_2 = \mathcal{B}$. Therefore $\mathcal{Q} = \mathcal{P}$. This completes the induction proof of (2).

□

According to the theorem we have just proved, the strict shifted plane partitions which result from cyclically symmetric plane partitions must have the following additional property that the first positive number in any given row is equal to the number of positive numbers in that row. Not all strict shifted plane partitions have this property, so the cyclically symmetric plane partitions correspond to a proper subset of all strict shifted plane partitions.

Another class of strict shifted plane partition is the set of descending plane partitions. These have the property that the number of parts in a row is strictly less than the first part in the row and is greater than or equal to the first part in the next row down. For example, see Figure 1.8 on page 21.

As stated above, Stanley observed that there appear to be the same number of descending plane partitions with largest part $\leq r$ as there are alternating sign matrices of size r. There is an example of this on page 21. This observation was subsequently proved to be true. But to date one has found a direct correspondence between these and alternating sign matrices, analogous to the direct correspondence between cyclically symmetric plane partitions and the special class of strict shifted plane partitions we have exhibited.

Conjecture 8, page 24, is a first step towards finding a correspondence between alternating sign matrices of size r and descending plane partitions with largest part $\leq r$. This conjecture is still open. Conjecture 8 implies Conjecture 9, page 24, a formula for the number of descending plane partitions with largest part r having the additional property that r appears exactly $k - 1$ times. Conjecture 9 turns out to be true, and can be taken as evidence that Conjecture 8 might also be true.

Exercises: 1, 2, 3, 4, 5, 6, 7, 10, 12, 13

Chapter 2: Fundamental Structures

We will be taking a closer look at generating functions, integer partitions, recursive formulas, and determinants in this chapter. Symmetric functions are introduced on pp. 33–35, but are not studied until Chapter 4. So we will skip over this material for now.

Section 2.1: Generating Functions

We gave a primer on generating functions on pp. 5–7 of these notes. Example 4 on pp. 6–7 illustrates how we can use generating functions to count integer partitions of a given size.

We use the notation $p(n)$ to denote the number of partitions of the positive integer n. By convention we set $p(0) = 1$. The generating function for partitions is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

The first few terms of this generating function are given on page 35.

We will prove that this formula is correct. See page 43 for the definition of the Ferrers graph representation of a partition. We can express each of the geometric series

$$\frac{1}{1-q}, \frac{1}{1-q^2}, \frac{1}{1-q^3}, \dots$$

in terms of Ferrers graphs:

$$\frac{1}{1-q} = \frac{1}{1-q^{\bullet}} = 1 + q^{\bullet} + q^{\bullet\bullet} + q^{\bullet\bullet\bullet} + q^{\bullet\bullet\bullet\bullet} + \cdots$$

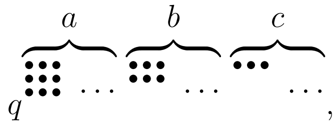
$$\frac{1}{1-q^2} = \frac{1}{1-q^{\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}}} = 1 + q^{\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet\\\bullet&\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet&\bullet\end{smallmatrix}} + \cdots$$

$$\frac{1}{1-q^3} = \frac{1}{1-q^{\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}}} = 1 + q^{\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet\\\bullet&\bullet\\\bullet&\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{smallmatrix}} + q^{\begin{smallmatrix}\bullet&\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet&\bullet\end{smallmatrix}} + \cdots$$

The typical term in

$$\frac{1}{(1-q^3)(1-q^2)(1-q)}$$

is



which corresponds to the partition with $a \geq 0$ copies of 3, $b \geq 0$ copies of 2, and $c \geq 0$ copies of 1. Therefore

$$\frac{1}{(1-q^3)(1-q^2)(1-q)}$$

is the generating function for all partitions whose parts are of size ≤ 3. Generalizing this argument,

$$\frac{1}{(1-q^n)(1-q^{n-1})\cdots(1-q)}$$

is the generating function for all partitions whose parts are of size $\leq n$, and

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

is the generating function for all partitions (no upper limit to size of largest part).

It should be clear now that

$$\prod_{k=1}^{\infty} \frac{1}{1-q^{2k}}$$

is the generating function for all partitions whose parts are all even,

$$\prod_{k=1}^{\infty} \frac{1}{1-q^{2k+1}}$$

is the generating function for partitions whose parts are all odd,

$$\prod_{k=1}^{\infty} (1+q^k)$$

is the generating function for all partitions whose parts are distinct,

$$\prod_{k=1}^{\infty} (1+q^k+q^{2k})$$

is the generating function for all partitions whose parts occur at most twice, and so on.

We can use generating functions to prove that the number of partitions of n with all odd parts is equal to the number of partitions of n with all distinct parts. All we have to do is prove that their generating functions are equal. See Theorem 2.2, pp. 38–39. There are many variations on this argument. See Corollaries 2.3 and 2.4, pp. 39–40.

The product $\prod_{k=1}^{\infty}(1-q^k)$ cannot easily be recognized as a generating function, but when you multiply it out you get the expression on page 36. Every non-zero coefficient is 1 or -1, and the exponents corresponding to these coefficients are

$$1,\ 5=1+4,\ 12=1+4+7,\ 22=1+4+7+10,\ \ldots.$$

These are called **pentagonal numbers.** Hence we obtain Euler's pentagonal number theorem, top of page 37 (proof deferred to Section 2.1).

When expand the product

$$(1-q-q^2+q^5+q^7-\cdots)(p(0)+p(1)q+p(2)q^2+p(3)q^3+\cdots)$$

you obtain

$$p(0)+(p(1)-p(0))q+(p(2)-p(1)-p(0))q^2+(p(3)-p(2)-p(1))q^3+\cdots.$$

The coefficient of q^n is equal to

$$p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-\cdots.$$

However, this product is equal to

$$\prod_{k=1}^{\infty}(1-q^k)\cdot\prod_{k=1}^{\infty}\frac{1}{1-q^k}=1.$$

Therefore for $n\geq 1$ the coefficient of q^n in the product should be equal to 0. This gives rise to a recurrence relation for $p(n)$:

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots,$$

where the terms drop back according to the signed pentagonal numbers

$$-1,-2,5,7,-12,-15,\cdots.$$

Exercises: 7, 8, 9, 12, 13, 16, 17.

Section 2.2: Partitions.

The convention in this book for depicting an integer partition by a Ferrers graph is to let the number of dots in row i represent the i^{th} part of the partition. The bijection on page 44 between partitions of 7 into 3 parts and partitions of 7 whose largest part is 3 can be described as follows: take a partition, represent it by a Ferrers diagram, then use the column sums to create the new partition. For example,

$$5+1+1 \to \begin{array}{lllll}\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&&&&\\ \bullet&&&&\end{array} \to 3+1+1+1+1.$$

This process is called conjugation. The mystery is why the column sums are always weakly decreasing. But we have actually proved this property already: In Step 1 of the proof on page 19 of these notes, we used a set of weakly decreasing numbers $a_1 \geq a_2 \geq a_3 \geq \cdots$ (the first row of a strict shifted plane partition) to create the matrix $M_1^{(z)}$ with a_i 1s in row i. In Step 3 the numbers $z_{11}, z_{21}, z_{31}, \ldots$ were found by setting z_{j1} equal to the number of 1s in column j of $M_1^{(z)}$. In Step 7 we proved that these numbers are weakly decreasing. This is exactly the process we use to create a conjugate partition. Geometrically, conjugation is reflection across the main diagonal.

Sylvester's bijection between partitions with odd parts and partitions with distinct parts is given on pp. 44–46. We will not give a proof that the correspondence is a bijection, but you will familiarize yourself with it in the exercises.

The Jacobi triple product can be used to prove the pentagonal number theorem.

Jacobi Triple Product:

$$\prod_{i=1}^{\infty}(1+xq^i)(1+x^{-1}q^{i-1}) = \prod_{j=1}^{\infty}\frac{1}{1-q^j}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^n.$$

Proof. Write

$$f(x) = \prod_{i=1}^{\infty}(1+xq^i)(1+x^{-1}q^{i-1}) = \sum_{n=-\infty}^{\infty} a_n(q)x^n.$$

Then

$$f(xq) = \prod_{i=1}^{\infty} (1 + xq^{i+1})(1 + x^{-1}q^{i-2}) = \frac{1 + x^{-1}q^{-1}}{1 + xq} f(x) = x^{-1}q^{-1}f(x),$$

$$\sum_{n=-\infty}^{\infty} a_n(q)q^n x^n = \sum_{n=-\infty}^{\infty} a_{n+1}(q)q^{-1}x^n,$$

therefore

$$a_{n+1}(q) = q^{n+1}a_n(q)$$

for all n. This recurrence relation implies that

$$a_n(q) = q^{n(n+1)/2}a_0(q)$$

for all integers n. Therefore

$$f(x) = a_0(q) \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^n.$$

Now we just need to compute $a_0(n)$, the coefficient of x^0 in $f(x)$.

Let $DP(n, k)$ denote the number of partitions of n into k distinct positive parts. Then we have

$$\prod_{i=1}^{\infty} (1 + xq^i) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} DP(n, k)x^k q^n.$$

Let $DNN(n, k)$ denote the number of partitions of n into k distinct non-negative parts. Then we have

$$\prod_{i=1}^{\infty} (1 + x^{-1}q^{i-1}) = 1 + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} DNN(n, k)x^{-k}q^n.$$

Then

$$f(x) = \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} DP(n, k)x^k q^n\right)\left(1 + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} DNN(n, k)x^{-k}q^n\right).$$

The coefficient of x^0 in $f(x)$ is therefore

$$\sum_{k=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} DP(n, k)q^n\right)\left(\sum_{n=0}^{\infty} DNN(n, k)q^n\right)\right].$$

This can be interpreted as the generating function for the number of partitions which can be decomposed into $k \geq 1$ distinct positive parts and an equal number of distinct non-negative parts. The diagram on page 50 indicates that every partition has a unique decomposition of this form. Therefore this is the generating function for all partitions, and

$$a_0(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

$\square$

The pentagonal number theorem can now be proved by replacing q by q^3 in Jacobi triple product identity, then setting $x = -q^{-1}$. See page 51.

Exercises: 1, 3–5, 6–7, 8–10

Section 2.3: Recursive Formulae.

The generating function for plane partitions is given by

$$\sum_{n=0}^{\infty} pp(n)q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}.$$

The proof appears later in the book. However, if you differentiate the generating function equation, you get a recurrence relation for $pp(n)$.

Differentiating $\sum_{n=0}^{\infty} pp(n)q^n$, we obtain

$$\sum_{n=0}^{\infty} npp(n)q^{n-1}.$$

Differentiating

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = e^{\ln \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}} = e^{\sum_{k=1}^{\infty} \ln \frac{1}{(1-q^k)^k}} = e^{\sum_{k=1}^{\infty} -k \ln(1-q^k)}$$

we obtain

$$\left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} \right)' = \left(\sum_{k=1}^{\infty} -k \ln(1-q^k) \right)' e^{\sum_{k=1}^{\infty} -k \ln(1-q^k)}$$

$$= \left(\sum_{k=1}^{\infty} \frac{k^2 q^{k-1}}{1-q^k}\right) \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}.$$

Therefore

$$\sum_{n=0}^{\infty} pp(n) q^n = \left(\sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^k}\right) \prod_{n=1}^{\infty} pp(n) q^n.$$

The coefficient of q^j in

$$\frac{k^2 q^k}{1-q^k} = k^2(q^k + q^{2k} + q^{3k} + \cdots)$$

is

$$\begin{cases} k^2 & \text{if } j|k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the coefficient of q^j in

$$\sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^k}$$

is

$$\sum_{k|j} k^2 = \sigma_2(j).$$

This implies

$$pp(n) = \sum_{j=1}^{n} \sigma_2(j) pp(n-j).$$

Counting alternating sign matrices

Starting with an $n \times n$ matrix A, let B be the matrix whose i^{th} row is the sum of the first i rows of A, $1 \leq i \leq n$. Then B turns out to have i 1s in row i for each i. The monotone triangle records the column positions of the 1s. The rows of the monotone triangle are strictly increasing, the ascending diagonals are weakly increasing, and the descending diagonals art weakly decreasing. It turns out that there is a 1:1 correspondence between alternating sign matrices and monotone triangles, although we will not give a proof. A proof would consist of showing that each monotone triangle gives rise to a unique alternating sign matrix that generates it. You will gain some insight into this proof by doing exercises 14 and 15.

Exercises: 4, 5, 6, 12, 13, 14, 15, 16, 18, 19.

Section 2.4: Determinants

Alternating polynomial in n variables: every transposition in two of the variables results in a change of sign.

Example 1: $f(x_1, x_2) = x_1 - x_2$ is an alternating polynomial in 2 variables. Reason:

$$f(x_2, x_1) = x_2 - x_1 = -f(x_1, x_2).$$

Example 2:

$$f(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_3)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

is an alternating polynomial in 4 variables. Reason: let's look at what happens to each factor when you swap x_a and x_b where $a < b$.

$$\begin{array}{rcll} x_a - x_b & \to & -(x_a - x_b) & \\ (x_i - x_a)(x_i - x_b) & \to & (x_i - x_b)(x_i - x_a) & \text{for } i < a \\ (x_a - x_i)(x_i - x_b) & \to & (x_b - x_i)(x_i - x_a) & \text{for } a < i < b \\ (x_a - x_i)(x_b - x_i) & \to & (x_b - x_i)(x_a - x_i) & \text{for } i < b. \end{array}$$

The only sign change is $x_a - x_b \to x_b - x_a$. Therefore swapping x_a and x_b results in $-f(x_1, x_2, x_3, x_4)$.

The argument in Example 2 proves that

$$f(x_1, x_2, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_i)$$

is an alternating polynomials in n variables for all $n \ge 2$.

Note: if $f(x_1, \ldots, x_{n-1}, x_n)$ is alternating and you evaluate at

$$(x_1, \ldots, x_n) = (a_1, \ldots, a_n),$$

where $a_i = a_j = a$ but $i < j$, then $f(a_1, \ldots, a_n) = 0$ (assuming characteristic $\neq 0$). Reason: evaluating $f(x_1, \ldots, x_n)$ at $(a_1, \ldots, a_n)$ is the same as evaluating $f(\ldots, x_j, \ldots, x_i, \ldots)$ at $(a_1, \ldots, a_n)$. The first evaluation produces

$$f(a_1, \ldots, a, \ldots, a, \ldots, a_n)$$

and the second evaluation produces

$$-f(a_1, \ldots, a, \ldots, a, \ldots, a_n).$$

Since these must be the same, it must true that

$$f(a_1, \ldots, a_n) = 0.$$

Symmetric polynomial in n variables: every transposition in two of the variables results in the same polynomial.

Example 3: $f(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3$ is a symmetric polynomial in 3 variables.

Theorem: *Let $n \geq 2$ be given. Every alternating polynomial $f(x_1, x_2, \ldots, x_n)$ can be factored into*

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \prod_{i<j} (x_i - x_i)$$

for some symmetric polynomial $g(x_1, x_2, \ldots, x_n)$.

Proof. First suppose that we have already factored $f(x_1, \ldots, x_n)$ into

$$f(x_1, \ldots, x_n) = g(x_1, x_2, \ldots x_n) \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

for some polynomial $g(x_1, \ldots, x_n)$. Then $g(x_1, \ldots, x_n)$ must be symmetric, because swapping x_p and x_q for any $i < j$ changes the sign in both $f(x_1, \ldots, x_n)$ and $\prod_{1 \leq i < j \leq n} (x_i - x_j)$. Therefore swapping x_p and x_q must leave $g(x_1, \ldots, x_n)$ unchanged. So now we just have to show that is possible to factor $f(x_1, \ldots, x_n)$ in this way.

For each i let $h_i(t)$ be the polynomial obtained by setting $x_i = t$ in $f(x_1, \ldots, x_n)$. Then we can say

$$h_i(t) = \sum_{p \geq 0} g_{ip}(x_1, \ldots, x_n) t^p,$$

where the variable x_i does not appear in any of the polynomials $g_{ip}(x_1, \ldots, x_n)$. Then we have

$$h_i(x_j) = f(\ldots, x_i, \ldots, x_i, \ldots) = 0$$

for each $j > i$. Therefore each of the expressions $t - x_j$ is a factor of $h_i(t)$ for $j > i$. Since $f(x_1, \ldots, x_n) = h_i(x_i)$, $x_i - x_j$ is a factor of $f(x_1, \ldots, x_n)$ for each $j > i$. Therefore $f(x_1, \ldots, x_n)$ can be factored as above. $\square$

Corollary:

$$\det \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j).$$

Proof. Let

$$f(x_1, \ldots, x_n) = \det \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

By properties of determinants, $f(x_1, \ldots, x_n)$ is an alternating polynomial. Therefore, by the theorem we have

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \prod_{1 \le i < j \le n} (x_i - x_j)$$

for some polynomial $g(x_1, \ldots, x_n)$. Since the leading term of both $f(x_1, \ldots, x_n)$ and $\prod_{1 \le i < j \le n} (x_i - x_j)$ is equal to $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, we must have

$$g(x_1, \ldots, x_n) = 1.$$

□

The same idea is used to prove Krattenthaler's formula on page 67. It will be used later to derive the generating function

$$\sum_{n=0}^{\infty} pp(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^k}.$$

The Weyl denominator formula

The formula for $\mathbf{B_n}$ will used later in the book to derive the generating function for symmetric plane partitions.

Exercises: 7, 8, 9

Chapter 3: Lattice Paths and Plane Partitions

A lattice path from $(0, 0)$ to (m, n) can be encoded by a binary string consisting of m 1s and n 0s in some order. Every 0 encodes a step north of 1 unit, and every 1 encodes a step east of 1 unit. For example, the lattice path on page 74 is encoded by the binary string

$$01101011001.$$

We will use the notation $S(m, n)$ to denote the set of lattice paths from $(0, 0)$ to (m, n). Since there are $\binom{m+n}{n}$ possible binary strings of this description (choose 0 locations) there are this many lattice paths from $(0, 0)$ to (m, n).

We can classify the lattice paths in $S(m, n)$ as those ending in 0 and those ending in 1. The number of lattice paths which end in 0 is equal to the number of lattice paths in $S(m, n-1)$. The number of lattice paths which end in 1 is equal to the number of lattice paths in $S(m-1, n)$. Therefore we have a combinatorial proof of the identity

$$\binom{m+n}{n} = \binom{m+n-1}{n-1} + \binom{m+n-1}{n}.$$

Section 3.1: Lattice Paths

Note to students: we will skip the alternative proof of Jacobi's triple product, pp. 78–79.

A weighted lattice path from $(0, 0)$ to (m, n) is a lattice path with every square in $m \times n$ board which is above the path filled with a dot. See the figure at the bottom of page 74. The weight of the path is the number of dots. We will use the notation $\mathcal{I}(\sigma)$ to denote the weight of the path σ. Using this notation, we have $\mathcal{I}(01101011001) = 15$. We will use the notation $S(m, n)$ to denote the set binary strings with m 1s and n 0s, which we can identify with the set of all lattice paths from $(0, 0)$ to (m, n). We will also set

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\sigma \in S(m,n)} q^{\mathcal{I}(\sigma)}.$$

This can be considered a q-analogue of $\binom{m+n}{m}$, because if we let $q \to 1$ we obtain

$$\lim_{q\to 1} \begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\sigma \in S(m,n)} 1^{\mathcal{I}(\sigma)} = |S(m,n)| = \binom{m+n}{n}.$$

The expression $\mathcal{I}(\sigma)$ is called the inversion number of σ. It can be computed from σ as follows: write

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k,$$

where each σ_i is 0 or 1. For each i let $\mathcal{I}_i(\sigma)$ = number of characters to the left of σ_i which are strictly larger than σ_i. For example, in $\sigma = 01101011001$ we have

$$(\mathcal{I}_1(\sigma), \mathcal{I}_2(\sigma), \mathcal{I}_3(\sigma), \mathcal{I}_4(\sigma), \mathcal{I}_5(\sigma), \mathcal{I}_6(\sigma), \mathcal{I}_7(\sigma), \mathcal{I}_8(\sigma), \mathcal{I}_9(\sigma), \mathcal{I}_{10}(\sigma), \mathcal{I}_{11}(\sigma)) =$$

$$(0,0,0,2,0,3,0,0,5,5,0).$$

If you look at weighted lattice path on page 74, you can see that the number of dots in each row (from bottom to top) is recorded by the positive entries in the vector. The inversion number of σ is therefore

$$\mathcal{I}(\sigma) = \sum_{i=1}^{k} \mathcal{I}_i(\sigma).$$

For example, the inversion number of 01101011001 is

$$\mathcal{I}(01101011001) = 0+0+0+2+0+3+0+0+5+5+0 = 15.$$

We say that $\mathcal{I}_i(\sigma)$ is the number of inversions caused by σ_i.

A recurrence relation for $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$

In the remarks above we showed that

$$S(m,n) = \{\sigma 0 : \sigma \in S(m,n-1)\} \bigcup \{\sigma 1 : \sigma \in S(m-1,n)\}.$$

Moreover,

$$\sigma \in S(m,n-1) \Rightarrow \mathcal{I}(\sigma 0) = \mathcal{I}(\sigma) + m$$

and

$$\sigma \in S(m-1,n) \Rightarrow \mathcal{I}(\sigma 1) = \mathcal{I}(\sigma).$$

Therefore

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\sigma \in S(m,n)} q^{\mathcal{I}(\sigma)} = \sum_{\sigma \in S(m,n-1)} q^{\mathcal{I}(\sigma)+m} + \sum_{\sigma \in S(m-1,n)} q^{\mathcal{I}(\sigma)} =$$

$$q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q .$$

Proposition 3.2:

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^{m+n})}{(1-q)\cdots(1-q)^m(1-q)\cdots(1-q^n)}.$$

In the book this is proved by showing that both sides of the equation satisfy the same recurrence relation. I will give an alternative proof of this in the next section.

Note that $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ can be regarded as the generating function for partitions into at most n parts with each part less than or equal to m. Reason: There is an exact correspondence between partitions of this sort and lattice paths in $S(m,n)$. Each part in the partition is equal to the number of dots in each column above the lattice path. See the figure at the bottom of page 74.

q-Binomial Theorem: *For any positive integer n,*

$$(1+xq)(1+xq^2)\cdots(1+xq^n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q q^{i(i+1)/2} x^i.$$

Proof. Let $\mathcal{DP}(i,n)$ denote the set of partitions with i distinct parts, all of which are $\leq n$. Then

$$(1+xq)(1+xq^2)\cdots(1+xq^n) = 1 + \sum_{i=1}^{n} \sum_{\lambda \in \mathcal{DP}(i,n)} q^{|\lambda|} x^i.$$

We must show that

$$\sum_{\lambda \in \mathcal{DP}(i,n)} q^{|\lambda|} = \begin{bmatrix} n \\ i \end{bmatrix}_q q^{i(i+1)/2}.$$

We have

$$\begin{bmatrix} n \\ i \end{bmatrix}_q q^{i(i+1)/2} = \begin{bmatrix} (n-i)+i \\ i \end{bmatrix}_q q^{i(i+1)/2} = \sum_{\sigma \in S(n-i,i)} q^{\mathcal{I}(\sigma)} q^{i(i+1)/2}.$$

Therefore it will suffice to exhibit a bijection between $S(n-i,i)$ and $\mathcal{DP}(i,n)$ with the following property: if $\sigma \leftrightarrow \lambda$ then

$$\mathcal{I}(\sigma) + i(i+1)/2 = |\lambda|.$$

Let $\sigma \in S(n-i,i)$ be given. Then σ can be interpreted as partition into i non-negative parts, each of which is $\leq n-i$. Call this partition $\alpha_1 + \alpha_2 + \cdots + \alpha_i$, where $0 \leq \alpha_1 \leq \cdots \leq \alpha_i \leq n-i$. Then

$$(\alpha_1 + 1) + (\alpha_2 + 2) + \cdots + (\alpha_i + i)$$

is a partition with i distinct positive parts, each of which is $\leq n$. If we set

$$\lambda = (\alpha_1 + 1, \alpha_2 + 2, \ldots, \alpha_i + i),$$

then

$$|\lambda| = \alpha_1 + \cdots + \alpha_i + (1 + 2 + \cdots + i) = \mathcal{I}(\sigma) + i(i+1)/2.$$

Therefore $\sigma \leftrightarrow \lambda$ is the desired correspondence.

□

Exercises: 18–22.

Section 3.2: Inversion numbers

Note to students: we will skip the material on page 88 for now, but come back to it later.

We can compute the inversion number of a permutation using the same definition as above for binary strings. For example, we have

$$\mathcal{I}(13524) = 0 + 0 + 0 + 2 + 1 = 3.$$

Before we prove Proposition 3.2, we will prove Corollary 3.5, page 86.

Corollary 3.5: *If we let $\mathcal{S}_n$ denote the set of permutations on n letters, then*

$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathcal{I}(\sigma)} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}.$$

Proof. First note that the formula is equivalent to

$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathcal{I}(\sigma)} = (1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

Reason:

$$\frac{1-q^i}{1-q} = 1+q+\cdots+q^{i-1}.$$

It will suffice to prove the recurrence relation

$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathcal{I}(\sigma)} = (1+q+q^2+\cdots+q^{n-1}) \sum_{\sigma\in\mathcal{S}_{n-1}} q^{\mathcal{I}(\sigma)}.$$

Consider the typical permutation in $\mathcal{S}_n$. It is a list of the numbers 1 through n written in some order. If the number n appears in position k, then it contributes 1 to the number of inversions caused by σ_{k+1} through σ_n. Therefore, if we let σ' denote the permutation of 1 through $n-1$ obtained by deleting n from σ, we have

$$\mathcal{I}(\sigma) = n-k+\mathcal{I}(\sigma').$$

Therefore

$$\sum_{\substack{\sigma\in\mathcal{S}_n,\\ \sigma_k=n}} q^{\mathcal{I}(\sigma)} = q^{n-k} \sum_{\sigma'\in\mathcal{S}_{n-1}} q^{\mathcal{I}(\sigma')}.$$

Summing over all possible values of k we obtain

$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathcal{I}(\sigma)} = \sum_{k=1}^{n} \left(\sum_{\substack{\sigma\in\mathcal{S}_n,\\ \sigma_k=n}} q^{\mathcal{I}(\sigma)} \right) = \sum_{k=1}^{n} q^{n-k} \sum_{\sigma'\in\mathcal{S}_{n-1}} q^{\mathcal{I}(\sigma')} =$$

$$(1+q+q^2+\cdots+q^{n-1})\sum_{\sigma\in\mathcal{S}_{n-1}} q^{\mathcal{I}(\sigma)}.$$

□

We are now in a position to prove Proposition 3.2.

Proof. It will be sufficient to prove the identity

$$(1-q)(1-q^2)\cdots(1-q^{m+n}) = (1-q)\cdots(1-q)^m(1-q)\cdots(1-q^n)\begin{bmatrix} m+n \\ n \end{bmatrix}_q.$$

Dividing both sides by $(1-q)^{m+n}$, this is equivalent to

$$\prod_{i=1}^{m+n}(1+q+\cdots+q^{i-1}) = \prod_{i=1}^{m}(1+q+\cdots+q^{i-1})\prod_{i=1}^{n}(1+q+\cdots+q^{i-1})\begin{bmatrix} m+n \\ n \end{bmatrix}_q.$$

By Corollary 3.5, the left-hand side of this identity is equal to

$$\sum_{\sigma\in\mathcal{S}_{m+n}} q^{\mathcal{I}(\sigma)}.$$

We must show that we can factor this to look like the right-hand side of the identity.

Let $\sigma \in \mathcal{S}_{m+n}$ be given. Then σ is a list of the numbers 1 through $m+n$ in some order. Let α be the corresponding binary string with

$$\alpha_i = \begin{cases} 0 & \text{if } \sigma_i \le m \\ 1 & \text{if } \sigma_i > m. \end{cases}$$

For example, if $m = 6$ and $n = 5$ and

$$\sigma = 28(10)1(11)397546$$

then

$$\alpha = 01101011001.$$

We can decompose σ into two permutations and a binary string,

$$\sigma \to (\sigma', \sigma'', \alpha),$$

as follows: to obtain σ', delete all the terms in σ which are $> m$. To obtain σ'', delete all the terms in σ which are $\leq m$. Let α be the binary string which records the positions where $\sigma_i \leq m$ and $\sigma_i > m$. For example,

$$28(10)1(11)397546 \to (213546, 8(10)(11)97, 01101011001).$$

It is easy to see that all permutations of $m+n$ can be uniquely decomposed in this form. We can reconstruct σ from $(\sigma', \sigma'', \alpha)$ by using α to tell us how to combine σ' and σ''. Note that

$$\mathcal{I}(\sigma) = \mathcal{I}(\sigma') + \mathcal{I}(\sigma'') + \mathcal{I}(\alpha).$$

Reason: if $\sigma_i > m$ then it causes the same number of inversions in σ'' as it does in σ. But if $\sigma_i \leq m$ then some of the inversions it causes in σ do not appear in σ', namely instances where a number to the left of σ_i is larger than σ_i and also larger than m. This deficit is made up by counting the 1s to the left of α_i (note $\alpha_i = 0$ because $\sigma_i \leq m$).

Note that σ'' is a permutation of the numbers $\{m+1, m+2, \cdots m+n\}$. If we subtract m from each of the terms in σ'', we will obtain a permutation of the numbers 1 through n having the same inversion number. So we can regard σ'' as belonging to $\mathcal{S}_n$.

We now have

$$\sum_{\sigma \in \mathcal{S}_{m+n}} q^{\mathcal{I}(\sigma)} = \sum_{\substack{\sigma' \in \mathcal{S}_m, \\ \sigma'' \in \mathcal{S}_n, \\ \alpha \in S(m,n)}} q^{\mathcal{I}(\sigma', \sigma'', \alpha)} = \sum_{\substack{\sigma' \in \mathcal{S}_m, \\ \sigma'' \in \mathcal{S}_n, \\ \alpha \in S(m,n)}} q^{\mathcal{I}(\sigma') + \mathcal{I}(\sigma'') + \mathcal{I}(\alpha)} =$$

$$\left(\sum_{\sigma' \in \mathcal{S}_m} q^{\mathcal{I}(\sigma')} \right) \left(\sum_{\sigma'' \in \mathcal{S}_n} q^{\mathcal{I}(\sigma'')} \right) \left(\sum_{\alpha \in S(m,n)} q^{\mathcal{I}(\alpha)} \right).$$

However, by Corollary 3.5 we have

$$\sum_{\sigma \in \mathcal{S}_p} q^{\mathcal{I}(\sigma)} = \prod_{i=1}^{p} (1 + q + \cdots + q^{i-1})$$

and by definition we have

$$\sum_{\alpha \in S(m,n)} q^{\mathcal{I}(\alpha)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q,$$

so we're done.

$\square$

Proposition 3.4, page 86, is the natural generalization of this argument. We will omit the proof, but suffice it to say that involves decomposing a permutation

$$\sigma \in \mathcal{S}_{m_1+\cdots+m_n}$$

into

$$(\tau_1, \ldots, \tau_n, \alpha),$$

where $\tau_i \in \mathcal{S}_i$ for each i and α is string consisting of m_1 1s, m_2 2s, $\ldots$, m_n ns which describes how the τs are woven together to produce σ.

Exercises: 5, 7.

Additional Exercise #1: Let $S(m_1, m_2, m_3)$ denote the set of strings consisting of m_1 1s, m_2 2s, and m_3 3s. Prove that

$$\sum_{\alpha \in S(m_1,m_2,m_3)} q^{\mathcal{I}(\alpha)} =$$

$$\frac{(1-q)\cdots(1-q^{m_1+m_2+m_3})}{(1-q)\cdots(1-q^{m_1})(1-q)\cdots(1-q^{m_2})(1-q)\cdots(1-q^{m_3})}.$$

Additional Exercise #2: Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of 1 through n. Prove that if you swap two adjacent terms in this list then the number of inversions either increases or decreases by 1.

Section 3.3: Plane partitions

Topic to review: definition of the determinant, sign-reversing involutions, involution principle.

In this section we construct a generating function for plane partitions restricted to $\mathcal{B}(r, s, t)$. This generating function will have the form of a determinant of q-binomial coefficients. We will then use Krattenthaler's formula (page 67) to express this determinant as a product. Finally, we will use the generating function for restricted plane partitions to derive the generating function for all plane partitions. I have reworked the proofs and exercises to make them more clear and to correct some minor mistakes in the textbook.

Determinant of $A = (a_{ij})_{i,j=1}^{n}$:

$$\det A = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Sign-reversing involution: *Let X be a finite set. Let $w(x)$ be a scalar for each $x \in X$. Let $\theta : X \to X$ be a function for X to X. Then θ is a sign-reversing involution with respect to w if*

$$w(\theta(x)) = -w(x)$$

and

$$\theta(\theta(x)) = x$$

for all $x \in X$.

Involution Principle: *Let X be a finite set. Let $w(x)$ be a scalar for each $x \in X$. Let $\theta : X \to X$ be a sign-reversing involution with respect to w. Then*

$$\sum_{x \in X} w(x) = 0.$$

Proof. Every $x \in X$ can be represented as $\theta(y)$ for some $y \in X$. Reason: given $x \in X$, set $y = \theta(x)$. Then $\theta(y) = \theta(\theta(x)) = x$. Moreover, the choice of y is unique: if $\theta(y') = x$ then $\theta(\theta(y')) = \theta(x)$, which implies $y' = \theta(x)$. Now write

$$Z = \sum_{x \in X} w(x).$$

Then we have

$$Z = \sum_{y \in X} w(\theta(y)) = -\sum_{y \in X} w(y) = -Z.$$

Therefore $Z = 0$. □

The generating function for plane partitions is derived using a sign-reversing involution.

Theorem: *Let $pp(n)$ denote the number of plane partitions in $\mathcal{B}(r, s, t)$. Then*

$$\sum_{n=0}^{rst} pp(n) q^n = \det \left(q^{i(i-j)} \begin{bmatrix} t+s \\ s-i+j \end{bmatrix} \right).$$

Proof. We will evaluate the determinant, then use a sign-reversing involution to show that a large number of terms cancel each other out. What survives will be the generating function for restricted plane partitions.

We have

$$\det\left(q^{i(i-j)}\begin{bmatrix} t+s \\ s-i+j \end{bmatrix}\right) =$$

$$\sum_{\sigma\in\mathcal{S}_r}(-1)^{\mathcal{I}(\sigma)}q^{1(1-\sigma(1))+\cdots+n(n-\sigma(n))}\begin{bmatrix} s+t \\ s-1+\sigma(1) \end{bmatrix}_q\cdots\begin{bmatrix} s+t \\ s-r+\sigma(r) \end{bmatrix}_q.$$

For each $\sigma\in\mathcal{S}_r$ let $\mathcal{L}_\sigma(r,s,t)$ denote the set of r-tuples of lattice paths of the form $(P_1,\dots,P_r)$, where P_i begins at vertex $(-\sigma(i)+1,\sigma(i)-1)$ and ends at vertex $(s-i+1,t+i-1)$. The example depicted in Figure 3.3, page 96, belongs to $\mathcal{L}_{4162735}(7,6,6)$. Since P_i takes $s-i+\sigma(i)$ steps north and $t+i-\sigma(i)$ steps east, it corresponds to an integer partition having $\le s-i+\sigma(i)$ parts and largest part $\le t+i-\sigma(i)$. We will denote by $|P_i|$ size of this partition. We will also write $|P|=|P_1|+\cdots+|P_r|$. The generating function for $\mathcal{L}_\sigma(r,s,t)$ is

$$\sum_{(P_1,\dots,P_r)\in\mathcal{L}_\sigma(r,s,t)}q^{|P_1|+\cdots+|P_r|}=\left(\sum_{P_1}q^{|P_1|}\right)\cdots\left(\sum_{P_r}q^{|P_r|}\right)=$$

$$\begin{bmatrix} s+t \\ s-1+\sigma(1) \end{bmatrix}_q\cdots\begin{bmatrix} s+t \\ s-r+\sigma(r) \end{bmatrix}_q.$$

Therefore, if we set

$$X=\{(\sigma,P_1,\dots,P_r):\sigma\in S_r,(P_1,\dots,P_r)\in\mathcal{L}_\sigma(r,s,t)\}$$

and

$$w(\sigma,P_1,\dots,P_r)=(-1)^{\mathcal{I}(\sigma)}q^{1(1-\sigma(1))+\cdots+n(n-\sigma(n))}q^{|P_1|+\cdots+|P_r|},$$

then

$$\det\left(q^{i(i-j)}\begin{bmatrix} t+s \\ s-i+j \end{bmatrix}\right)=\sum_{(\sigma,P_1,\dots,P_r)\in X}w(\sigma,P_1,\dots,P_r).$$

We will now identify a subset X_0 of X and a sign-reversing involution θ on X_0. We set X_0 equal to all $(\sigma,P_1,\dots,P_r)$ such that two or more of the lattice paths intersect each other. Given $(\sigma,P_1,\dots,P_r)\in X_0$, let P_a and P_b be the two intersecting paths forming the northern-most, eastern-most intersection point (with $a < b$). We will think of P_a and P_b as being represented by binary strings. These can be factored into $P_a^{(1)}P_a^{(2)}$ and $P_b^{(1)}P_b^{(2)}$, the subpaths before and after the point of intersection. We will set

$$\theta(\sigma, P_1, \ldots, P_r) = (\sigma', P_1', \ldots, P_r'),$$

where σ' is obtained from σ by swapping σ_a and σ_b, setting

$$P_a' = P_a^{(1)}P_b^{(2)}$$

and

$$P_b' = P_b^{(1)}P_a^{(2)},$$

and leaving the other lattice paths unchanged. It is easy to see that

$$\theta(\theta(\sigma, P_1, \ldots, P_r)) = (\sigma, P_1, \ldots, P_r).$$

We need to verify that

$$w(\sigma', P_1', \ldots, P_r') = -w(\sigma, P_1, \ldots, P_r).$$

Assume $(\sigma, P_1, \ldots, P_n)$ and $(\sigma', P_1', \ldots, P_n')$ be related by a tail switch of paths P_a and P_b as above. Claim: $b = a + 1$. Reason: Suppose $a < c < b$. Consider the region between P_a and P_b to the north and east of the intersection point we identified. There can be no intersections in this region. Since P_c terminates between P_a and P_b, it must be above P_a and below P_b throughout this region. Therefore P_c has to intersect both P_a and P_b at the point where P_a and P_b intersect. This is impossible, because there are only two ways to exit the intersection point, north and east, but P_c can't proceed in either direction without intersecting either P_a or P_b. So the claim is true.

If we write

$$\sigma = \sigma_1 \cdots \sigma_r$$

and

$$\sigma' = \sigma_1' \cdots \sigma_r',$$

then $\sigma_a' = \sigma_b$ and $\sigma_b' = \sigma_a$ and all the other terms are unchanged. Therefore

$$\mathcal{I}(\sigma') = \mathcal{I}(\sigma) \pm 1$$

(additional exercise #2, Section 3.2). This implies $(-1)^{\mathcal{I}(\sigma')} = -(-1)^{\mathcal{I}(\sigma)}$.

We also have

$$|P'| = |P| - |P_a| - |P_b| + |P'_a| + |P'_b| = |P| + \sigma(a) - \sigma(a+1).$$

See exercises 1 and 2 at the end of this section.

Finally, we have

$$\sum_{i=1}^{r} i(i - \sigma'(i)) = \sum_{i=1}^{r} i(i - \sigma(i)) + \sigma(a+1) - \sigma(a).$$

See exercise 3 at the end of this section.

Putting everything together we have

$$w(\sigma', P'_1, \dots, P'_r) = -w(\sigma, P_1, \dots, P_r).$$

Therefore θ is a sign-reversing involution on X_0 and we have

$$\sum_{(\sigma, P_1, \dots, P_r) \in X_0} w(\sigma, P_1, \dots, P_r) = 0.$$

Therefore

$$\det\left(q^{i(i-j)} \begin{bmatrix} t+s \\ s-i+j \end{bmatrix}\right) = \sum_{(\sigma, P_1, \dots, P_r) \in X \setminus X_0} w(\sigma, P_1, \dots, P_r).$$

However, it is easy to see that $(\sigma, P_1, \dots, P_r)$ belongs to $X \setminus X_0$ if and only if σ is the identity permutation e, each P_i begins at $(-i+1, i-1)$ and ends at $(s-i+1, t+i-1)$, and no two of the lattice paths intersect each other. Therefore each P_i represents an integer partition with $\leq s$ parts, all of which are $\leq t$, and if we arrange these partitions into an $r \times s$ matrix then the rows and columns will be weakly decreasing. That is, $(e, P_1, \dots, P_r)$ corresponds to a plane partition in $\mathcal{B}(r, s, t)$ with $|P_1| + \cdots + |P_r|$ cubes. Note also that $\mathcal{I}(e) = 0$ and $\sum_{i=1}^{r} i(i - e(i)) = 0$. Therefore

$$w(e, P_1, \dots, P_r) = q^{|P_1| + \cdots + |P_r|}.$$

Hence

$$\sum_{(\sigma, P_1, \dots, P_r) \in X \setminus X_0} w(\sigma, P_1, \dots, P_r) = \sum_{P_1, \dots, P_r} q^{|P_1| + \cdots + |P_r|},$$

where the sum is taking over all families of lattice paths which represent plane partitions in $\mathcal{B}(r,s,t)$. Therefore our determinant is the generating function for these plane partitions.

$\square$

Theorem:

$$\det\left(q^{i(i-j)}\begin{bmatrix} t+s \\ s-i+j \end{bmatrix}\right) = \prod_{(a,b,c)\in\mathcal{B}(r,s,t)} \frac{1-q^{a+b+c-1}}{1-q^{a+b+c-2}}.$$

Proof. We will use Krattenhaler's formula, page 67. We have

$$(q;q)_{s-i+j}(q^{s-i+j+1};q)_{r-j} = (q;q)_{s-i+r}$$

and

$$(q;q)_{t-j+i}(q^{t-j+i+1};q)_{j-1} = (q;q)_{t+i-1}.$$

Therefore

$$\begin{aligned} q^{i(i-j)}\begin{bmatrix} t+s \\ s-i+j \end{bmatrix}_q &= \frac{q^{i(i-j)}(q;q)_{t+s}}{(q;q)_{s-i+j}(q;q)_{t-j+i}} \\ &= \frac{q^{i(i-j)}(q;q)_{t+s}(q^{s-i+j+1};q)_{r-j}(q^{t-j+i+1};q)_{j-1}}{(q;q)_{s-i+r}(q;q)_{t+i-1}}. \end{aligned}$$

We have

$$\begin{aligned} (q^{s-i+j+1};q)_{r-j} &= (1-q^{s-i+j+1})\cdots(1-q^{s-i+r}) \\ &= q^{-i(r-j)}(q^i-q^{s+j+1})\cdots(q^i-q^{s+r}) \\ &= q^{-i(r-j)}(x_i+a_r)\cdots(x_i+a_{j+1}) \end{aligned}$$

and

$$\begin{aligned} (q^{t-j+i+1};q)_{j-1} &= (1-q^{t-j+i+1})\cdots(1-q^{t+i-1}) \\ &= (-1)^{j-1}q^{(j-1)t-\binom{j}{2}}(q^i-q^{-t+j-1})\cdots(q^i-q^{-t+2-1}) \end{aligned}$$

$$= \quad (-1)^{j-1} q^{(j-1)t-\binom{j}{2}} (x_i + b_j) \cdots (x_i + b_2),$$

where for each $1 \le k \le r$ we have $x_k = q^k$, $a_k = -q^{s+k}$, $b_k = -q^{-t+k-1}$ for $1 \le k \le r$.

Therefore

$$q^{i(i-j)} \begin{bmatrix} t+s \\ s-i+j \end{bmatrix}_q =$$

$$\frac{(q;q)_{t+s} q^{-i(r-i)}}{(q;q)_{s-i+r}(q;q)_{t+i-1}} (-1)^{j-1} q^{(j-1)t-\binom{j}{2}} (x_i + a_r) \cdots (x_i + a_{j+1})(x_i + b_j) \cdots (x_i + b_2).$$

Therefore we can factor

$$\frac{(q;q)_{t+s} q^{-i(r-i)}}{(q;q)_{s-i+r}(q;q)_{t+i-1}}$$

from each row of the determinant,

$$(-1)^{j-1} q^{(j-1)t-\binom{j}{2}}$$

from each column of the determinant, leaving Krattenthaler's formula

$$\det((x_i + a_r) \cdots (x_i + a_{j+1})(x_i + b_j) \cdots (x_i + b_2))_{i,j=1}^r =$$

$$\prod_{1 \le i < j \le r} (x_i - x_j) \prod_{2 \le i \le j \le r} (b_i - a_j) =$$

$$\prod_{1 \le i < j \le r} (q^i - q^j) \prod_{2 \le i \le j \le r} (-q^{-t+i-1} + q^{s+j}).$$

Therefore our determinant is equal to

$$\prod_{i=1}^r \frac{(q;q)_{t+s} q^{-i(r-i)}}{(q;q)_{s-i+r}(q;q)_{t+i-1}} \prod_{j=1}^r (-1)^{j-1} q^{(j-1)t-\binom{j}{2}} \prod_{1 \le i < j \le r} (q^i - q^j) \prod_{2 \le i \le j \le r} (-q^{-t+i-1} + q^{s+j}).$$

To complete the proof, do exercises 4 through 7. □

Corollary: *The generating function for all plane partitions is*

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j}.$$

Proof. We will compute $pp(n)$, the total number of plane partitions of size n. All of these plane partitions appear in $\mathcal{B}(n,n,n)$. Therefore we need the coefficient of q^n in

$$\prod_{(i,j,k)\in\mathcal{B}(n,n,n)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

Fixing i and k we have

$$\prod_{k=1}^{n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \frac{(1-q^{i+j})\cdots(1-q^{i+j+n-1})}{(1-q^{i+j-1})\cdots(1-q^{i+j+n-2})} = \frac{1-q^{i+j+n-1}}{1-q^{i+j-1}}.$$

Therefore

$$\prod_{(i,j,k)\in\mathcal{B}(n,n,n)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{i=1}^{n}\prod_{j=1}^{n} \frac{1-q^{i+j+n-1}}{1-q^{i+j-1}}.$$

The coefficient of q^n in this last expression is equal to the coefficient of q^n in

$$\prod_{i=1}^{n}\prod_{j=1}^{n} \frac{1}{1-q^{i+j-1}}.$$

The coefficient of q^n in this is equal to the coefficient of q^n in

$$\prod_{\substack{1\le i,j\le n\\ i+j-1\le n}} \frac{1}{1-q^{i+j-1}}.$$

If $p \geq n$, then there are exactly p solutions to $i+j-1=p$ using $1\le i\le n$ and $1\le j\le n$:

$$(i,j)\in\{(1,p),(2,p-1),\ldots,(p,1)\}.$$

Therefore

$$\prod_{\substack{1\le i,j\le n\\ i+j-1\le n}} \frac{1}{1-q^{i+j-1}} = \frac{1}{(1-q)(1-q^2)^2\cdots(1-q^n)^n}.$$

Therefore $pp(n)$ is the coefficient of q^n in

$$\prod_{j=1}^{n} \frac{1}{(1-q^j)^j},$$

which is the same as the coefficient of q^n in

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j}.$$

$\square$

Exercises:

1. Let f_0 and f_1 denote the functions which count the number of zeros and number of ones in a binary string. Let α and β be binary strings. Prove that

$$\mathcal{I}(\alpha\beta) = \mathcal{I}(\alpha) + \mathcal{I}(\beta) + f_1(\alpha) f_0(\beta).$$

2. Let $(P_1, \dots, P_r)$ and $(P'_1, \dots, P'_r)$ be related by a tail switch of P_a and P_b as in the theorem proved above. Show that

$$|P'| - |P| = |P'_a| + |P'_b| - |P_a| - |P_b| =$$
$$\mathcal{I}(P_a^{(1)} P_b^{(2)}) + \mathcal{I}(P_b^{(1)} P_a^{(2)}) - \mathcal{I}(P_a^{(1)} P_a^{(2)}) - \mathcal{I}(P_b^{(1)} P_b^{(2)}) =$$
$$(f_1(P_a^{(1)}) - f_1(P_b^{(1)}))(f_0(P_b^{(2)}) - f_0(P_a^{(2)})) =$$
$$(b-a)(\sigma(a) - \sigma(b)).$$

3. Let σ' be obtained from σ by swapping $\sigma(a)$ and $\sigma(b)$. Show that

$$\sum_{i=1}^{r} i(i - \sigma'(i)) - \sum_{i=1}^{r} i(i - \sigma(i)) = (b-a)(\sigma(b) - \sigma(a)).$$

4. Prove that

$$\prod_{i=1}^{r} q^{-i(r-i)} \prod_{1 \le i < j \le r} (q^i - q^j) = \prod_{k=1}^{r-1} (q;q)_k.$$

5. Prove that

$$\prod_{j=1}^{r} (-1)^{j-1} q^{(j-1)t-\binom{j}{2}} \prod_{2 \le i \le j \le r} (-q^{-t+i-1} + q^{s+j}) = \prod_{k=1}^{r-1} \frac{(q;q)_{s+k} (q^{s+k+1};q)_t}{(q;q)_{s+t}}.$$

6. Prove that

$$\prod_{i=1}^{r} \frac{(q;q)_{t+s}}{(q;q)_{s-i+r} (q;q)_{t+i-1}} \prod_{k=1}^{r-1} (q;q)_k \prod_{k=1}^{r-1} \frac{(q;q)_{s+k} (q^{s+k+1};q)_t}{(q;q)_{s+t}} = \prod_{i=1}^{r} \frac{(q^{i+s};q)_t}{(q^i;q)^t} =$$

$$\prod_{i=1}^{r} \prod_{j=1}^{t} \frac{1-q^{i+j+s-1}}{1-q^{i+j-1}}.$$

7. Prove that

$$\prod_{i=1}^{r} \prod_{j=1}^{t} \frac{1-q^{i+j+s-1}}{1-q^{i+j-1}} = \prod_{(a,b,c) \in \mathcal{B}(r,s,t)} \frac{1-q^{a+b+c-1}}{1-q^{a+b+c-2}}.$$

Section 3.4: Cyclically symmetric plane partitions

Topic to review: inversion number of a permutation

Preliminaries

Theorem: *Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of the numbers 1 through n. Let σ' be the permutation obtained by swapping two terms of σ. Then $\mathcal{I}(\sigma') \equiv \mathcal{I}(\sigma) \pmod 2$.*

Proof. We will first prove this is true when we swap two consecutive terms of σ, $\sigma_a \leftrightarrow \sigma_{a+1}$. We have

$$\mathcal{I}(\sigma') = \begin{cases} \mathcal{I}(\sigma) + 1 & \text{if } \sigma_{a+1} > \sigma_a \\ \mathcal{I}(\sigma) - 1 & \text{if } \sigma_{a+1} < \sigma_a, \end{cases}$$

therefore $\mathcal{I}(\sigma') \equiv \mathcal{I}(\sigma) \pmod 2$.

For the general case, we can swap σ_b with σ_a, $b > a$, by performing the sequence of $b - a$ swaps

$$\sigma_b \leftrightarrow \sigma_{b-1}, \sigma_b \leftrightarrow \sigma_{b-2}, \ldots, \sigma_b \leftrightarrow \sigma_a,$$

resulting in the sequence of permutations

$$\sigma = \sigma^{(0)} \to \sigma^{(1)} \to \cdots \to \sigma^{(b-a)},$$

followed by the sequence of $b - a + 1$ swaps

$$\sigma_a \leftrightarrow \sigma_{a+1}, \sigma_a \leftrightarrow \sigma_{a+2}, \ldots, \sigma_a \leftrightarrow \sigma_{b+1},$$

resulting in the sequence of permutations

$$\sigma^{(b-a)} \to \sigma^{(b-a+1)} \to \cdots \to \sigma^{(2b-2a+1)} = \sigma'.$$

In every case we are swapping consecutive terms of a permutation, hence we have

$$\mathcal{I}(\sigma) \equiv \mathcal{I}(\sigma^{(1)})+1 \equiv \mathcal{I}(\sigma^{(2)})+2 \equiv \cdots \equiv \mathcal{I}(\sigma^{(2b-2a+1)})+2b-2a+1 \equiv \mathcal{I}(\sigma')+1.$$

□

Corollary: *If the permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ can be transformed into the permutation $12\cdots n$ by k swaps, then $(-1)^{\mathcal{I}(\sigma)} = (-1)^k$.*

Proof. This follows from $0 = \mathcal{I}(12\cdots n) \equiv \mathcal{I}(\sigma) + k \pmod 2$. □

Corollary: *Let (z_{ij}) be an $r \times r$ matrix. Given $0 < a_1 < \cdots < a_m \leq r$, let $(z_{a_i a_j})$ denote the $m \times m$ matrix obtained by using rows a_1 through a_m and columns a_1 through a_m of (z_{ij}). For each i and j between 1 and r let*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then

$$\det(\delta_{ij} + z_{ij}) = 1 + \sum_{m=1}^{r} \sum_{0<a_1<\cdots<a_m\leq r} \det(z_{a_i a_j}).$$

Proof. We have

$$\det(\delta_{ij} + z_{ij}) = \sum_{\sigma \in \mathcal{S}_r} (-1)^{\mathcal{I}(\sigma)} (\delta_{1\sigma(1)} + z_{1\sigma(1)}) \cdots (\delta_{n\sigma(n)} + z_{n\sigma(n)}).$$

The typical term in the product produces

$$\delta_{b_1\sigma(b_1)} \cdots \delta_{b_{r-m}\sigma(b_{r-m})} z_{a_1\sigma(a_1)} \cdots z_{a_m\sigma(a_m)},$$

where $\{a_1, \ldots, a_m\}$ is an arbitrary subset of $\{1, \ldots, r\}$ and $\{b_1, \ldots, b_{r-m}\}$ contains the other indices. This term is non-zero if and only if $\sigma(b_i) = b_i$ for each i. Therefore

$$\det(\delta_{ij} + z_{ij}) =$$

$$1 + \sum_{r=1}^{m} \sum_{0 < a_1 < \cdots < a_m \leq r} \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma(b_i) = b_i \ \forall i}} (-1)^{\mathcal{I}(\sigma)} z_{a_1\sigma(a_1)} \cdots z_{a_m\sigma(a_m)}.$$

The last summation in this expression resembles a determinant. Say that the permutation $\tau = \sigma(a_1) \cdots \sigma(a_m)$ requires k swaps to bring it into the form $a_1 \cdots a_m$. Then σ requires k swaps to bring it into the form $12 \cdots r$, since the numbers b_1 through b_{r-m} are already in their natural positions in σ. Therefore

$$(-1)^{\mathcal{I}(\tau)} = (-1)^k = (-1)^{\mathcal{I}(\sigma)}.$$

Hence

$$\sum_{\substack{\sigma \in \mathcal{S} \\ \sigma(b_i) = b_i \ \forall i}} (-1)^{\mathcal{I}(\sigma)} z_{a_1\sigma(a_1)} \cdots z_{a_m\sigma(a_m)} =$$

$$\sum_{\tau \in \mathcal{S}(a_1, \ldots, a_m)} (-1)^{\mathcal{I}(\tau)} z_{a_1\tau(a_1)} \cdots z_{a_m\tau(a_m)} =$$

$$\det(z_{a_i a_j}).$$

□

A generating function for cyclically symmetric plane partitions in $\mathcal{B}(r, s, t)$

Recall that every cyclically symmetric plane partition can be represented by a strict shifted plane partition. The correspondence is

$$M = (z_{ij}) \leftrightarrow R = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & \cdots \\ & z_{22}-1 & z_{23}-1 & z_{24}-1 & \cdots \\ & & z_{33}-2 & z_{34}-2 & \cdots \\ & & & z_{44}-3 & \cdots \\ & & & & \ddots \end{bmatrix}.$$

If we write $a_i = z_{ii} - i + 1$ for each i then the row leaders of R are the numbers $a_1 > a_2 > \cdots$. Row i of R has exactly a_i parts, each of which is $\leq a_i$. If we delete the row leader of row i and subtract 1 from the remaining positive entries of row i, we obtain a partition with $\leq a_i - 1$ positive parts and each part $\leq a_i - 1$. This partition can be represented by a lattice path containing $a_i - 1$ northward steps and $a_i - 1$ eastward steps. These can be drawn as a non-intersecting family as follows:

Assume the plane partition is restricted to $\mathcal{B}(r, r, r)$. In the section 3.3 we represented the i^{th} row of M by a lattice path which begins in position $(-i+1, i-1)$ and ends in position $(r-i+1, r+i-1)$. Subtracting i from every entry in the i^{th} row of M is equivalent to throwing away every portion of every lattice path that has an x-coordinate ≤ 1 and counting squares to the right of x-coordinate 1. Deleting the first $i-1$ entries in the i^{th} row of the resulting matrix is equivalent to throwing away every portion of every lattice path that has a y-coordinate $\geq r$. So we can take the lattice paths corresponding to M and just use what survives to the right of $x = 1$ and below $y = r$. In general, if there are m row leaders in R, then this procedure will define m lattice paths $(P_1, \ldots, P_m)$. Each P_i begins in position $(1, r - a_i + 1)$ and ends in position (a_i, r). See for example Figure 3.5, page 103. It has been obtained from Figure 3.2, page 95, by taking everything that survives to the right of $x = 1$ and below $y = 6$, then shifting everything down 1 unit and to the right 1 unit.

We now have a one-to-one correspondence between cyclically symmetric plane partitions restricted to $\mathcal{B}(r, r, r)$ and all lists of non-intersecting lattice paths $(P_1, \ldots, P_m)$ with $1 \leq m \leq r$ as described above. As the textbook explains on pp. 103-104, the number of cubes in the i^{th} shell of the cyclically symmetric plane partition represented by $(P_1, \ldots, P_m)$ having row leaders $a_1 > \cdots > a_m$) is

$$3a_i - 2 + 3|P_i|,$$

where $|P_i|$ is the number of cubes in the partition represented by P_i, equivalently the inversion number of the binary string which represents P_i. Therefore the generating function for cyclically symmetric plane partitions with m shells and row leaders $a_1 > \cdots > a_m$ is

$$\sum_{(P_1,\ldots,P_m)} q^{(3a_1-2+3|P_1|)+\cdots+(3a_m-2+3|P_m|)},$$

where the lattice paths P_1 through P_m are constructed as above.

Theorem: *The generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ having m shells and row leaders $a_1 > \cdots > a_m$ is*

$$\det\left(q^{3a_i-2}\begin{bmatrix}a_i+a_j-2\\ a_i-1\end{bmatrix}_{q^3}\right).$$

Proof. As before, we will evaluate the determinant, then use a sign-reversing involution to show that a large number of terms cancel each other out. What survives will be the generating function for restricted cyclically symmetric plane partitions with row leaders $r \geq a_1 > \cdots > a_m \geq 1$.

We have

$$\det\left(q^{3a_i-2}\begin{bmatrix}a_i+a_j-2\\ a_i-1\end{bmatrix}_{q^3}\right) =$$

$$\sum_{\sigma\in\mathcal{S}_m}(-1)^{\mathcal{I}(\sigma)}q^{(3a_1-2)+\cdots+(3a_m-2)}\begin{bmatrix}a_1+a_{\sigma(1)}-2\\ a_1-1+\sigma(1)\end{bmatrix}_{q^3}\cdots\begin{bmatrix}a_m+a_{\sigma(m)}-2\\ a_m-1+\sigma(r)\end{bmatrix}_{q^3}.$$

For each $\sigma \in \mathcal{S}_r$ let $\mathcal{L}_\sigma(r;a_1,\ldots,a_m)$ denote the set of m-tuples of lattice paths of the form $(P_1,\ldots,P_m)$, where P_i begins at vertex $(1, r-a_{\sigma(i)}+1)$ and ends at vertex (a_i, r). Since P_i takes a_i-1 steps north and $a_{\sigma(i)}-1$ steps east, it corresponds to an integer partition having a_i-1 parts and largest part $\leq a_{\sigma(i)}-1$. We will denote by $|P_i|$ size of this partition. We will also write $|P| = |P_1|+\cdots+|P_m|$. The generating function for $\mathcal{L}_\sigma(r;a_1,\ldots,a_m)$ is

$$\sum_{(P_1,\ldots,P_m)\in\mathcal{L}_\sigma(r;a_1,\ldots,a_m)} q^{3|P_1|+\cdots+3|P_m|} = \left(\sum_{P_1} q^{3|P_1|}\right)\cdots\left(\sum_{P_m} q^{3|P_m|}\right) =$$

$$\begin{bmatrix} a_1 + a_{\sigma(1)} - 2 \\ a_1 - 1 \end{bmatrix}_{q^3} \cdots \begin{bmatrix} a_m + a_{\sigma(m)} - 2 \\ a_m - 1 \end{bmatrix}_{q^3}.$$

Therefore, if we set

$$X = \{(\sigma, P_1, \ldots, P_m) : \sigma \in S_m, (P_1, \ldots, P_m) \in \mathcal{L}_\sigma(r; a_1, \ldots, a_m)\}$$

and

$$w(\sigma, P_1, \ldots, P_m) = (-1)^{\mathcal{I}(\sigma)} q^{(3a_1-2)+\cdots+(3a_m-2)} q^{3|P_1|+\cdots+3|P_m|},$$

then

$$\det\left(q^{3a_i-2)} \begin{bmatrix} a_i + a_{\sigma(i)} - 2 \\ a_i - 1 \end{bmatrix}_{q^3} \right) = \sum_{(\sigma, P_1, \ldots, P_m) \in X} w(\sigma, P_1, \ldots, P_m).$$

We will now identify a subset X_0 of X and a sign-reversing involution θ on X_0. We set X_0 equal to all $(\sigma, P_1, \ldots, P_m)$ such that two or more of the lattice paths intersect each other. Given $(\sigma, P_1, \ldots, P_m) \in X_0$, let P_a and P_b be the two intersecting paths forming the northern-most, eastern-most intersection point (with $a < b$). We will think of P_a and P_b as being represented by binary strings. These can be factored into $P_a^{(1)} P_a^{(2)}$ and $P_b^{(1)} P_b^{(2)}$, the subpaths before and after the point of intersection. We will set

$$\theta(\sigma, P_1, \ldots, P_m) = (\sigma', P_1', \ldots, P_m'),$$

where σ' is obtained from σ by swapping σ_a and σ_b, setting

$$P_a' = P_a^{(1)} P_b^{(2)}$$

and

$$P_b' = P_b^{(1)} P_a^{(2)},$$

and leaving the other lattice paths unchanged. We have

$$\theta(\theta(\sigma, P_1, \ldots, P_m)) = (\sigma, P_1, \ldots, P_m).$$

We need to verify that

$$w(\sigma', P_1', \ldots, P_m') = -w(\sigma, P_1, \ldots, P_m).$$

Assume $(\sigma, P_1, \ldots, P_n)$ and $(\sigma', P_1', \ldots, P_n')$ be related by a tail switch of paths P_a and P_b as above. As in the previous theorem, we have $b = a + 1$. We also have $(-1)^{\mathcal{I}(\sigma')} = -(-1)^{\mathcal{I}(\sigma)}$.

The one major difference between this construction and the one in Section 3.3 is that $|P'| = |P|$. Reason: if we look at exercise 2 of Section 3.3, we see that

$$|P'| - |P| = (f_1(P_a^{(1)}) - f_1(P_b^{(1)}))(f_0(P_b^{(2)}) - f_0(P_a^{(2)})).$$

Since $P_a^{(1)}$ and $P_b^{(1)}$, the initial part of each tail before the intersection point, both begin at x-coordinate 1, the number of 1s in each string is the same. Therefore

$$f_1(P_a^{(1)}) - f_1(P_b^{(1)}) = 0.$$

Putting everything together we have

$$w(\sigma', P_1', \ldots, P_m') = -w(\sigma, P_1, \ldots, P_m).$$

Therefore θ is a sign-reversing involution on X_0 and we have

$$\sum_{(\sigma, P_1, \ldots, P_r) \in X_0} w(\sigma, P_1, \ldots, P_m) = 0.$$

Therefore

$$\det\left(q^{3a_i - 1}\begin{bmatrix} a_i + a_j - 2 \\ a_i - 1 \end{bmatrix}\right) = \sum_{(\sigma, P_1, \ldots, P_m) \in X \setminus X_0} w(\sigma, P_1, \ldots, P_m).$$

However, it is easy to see that $(\sigma, P_1, \ldots, P_m)$ belongs to $X \setminus X_0$ if and only if σ is the identity permutation e, each P_i begins at $(1, r - a_i + 1)$ and ends at (a_i, r), and no two of the lattice paths intersect each other. Therefore each P_i represents an integer partition with $\leq a_i - 1$ parts, all of which are $\leq a_i - 1$, and $(P_1, \ldots, P_m)$ encodes a strict shifted plane partition R having row leaders $a_1 > \cdots > a_m$. That is, $(e, P_1, \ldots, P_m)$ corresponds to a cyclically symmetric plane partition in $\mathcal{B}(r, r, r)$ with

$$(3a_1 - 2 + 3|P_1|) + \cdots + (3a_m - 2 + 3|P_m|)$$

cubes. Hence

$$\sum_{(\sigma, P_1, \ldots, P_m) \in X \setminus X_0} w(\sigma, P_1, \ldots, P_m) = \sum_{P_1, \ldots, P_m} q^{(3a_1 - 2 + 3|P_1|) + \cdots + (3a_m - 2 + 3|P_m|)}$$

is the generating function for these cyclically symmetric plane partitions. $\square$

The theorem above provides us a generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ with a fixed set of row leaders. To obtain the generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ and all possible row leaders, we have to add all the generating functions corresponding to all possible sequences of row leaders $r \geq a_1 > a_2 \cdots > a_m \geq 1$ and all m between 1 and r. Hence the generating function we are looking for is

$$\sum_{m=1}^{r} \sum_{0<a_1<\cdots<a_m\leq r} \det\left(q^{3a_i-2}\begin{bmatrix} a_i+a_j-2 \\ a_i-1 \end{bmatrix}_{q^3}\right).$$

We can express this sum as a single determinant using the theorem we proved in the preliminaries section.

Theorem: *The generating function for cyclically symmetric plane partitions contained in $\mathcal{B}(r,r,r)$ is*

$$\det\left(\delta_{ij} + q^{3i-2}\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^3}\right)_{i,j=1}^{r}.$$

Note that if a cyclically symmetric plane partition in $\mathcal{B}(r,r,r)$ has no parts equal to r, then it lives in $\mathcal{B}(r-1,r-1,r-1)$. So the generating function for these is

$$\det\left(\delta_{ij} + q^{3i-2}\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^3}\right)_{i,j=1}^{r-1}.$$

A slight modification of the two theorems we proved above allows us to construct the generating function for cyclically symmetric plane partitions with exactly $k \geq 1$ parts equal to r.

Theorem: *The generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ having m shells and row leaders $a_1 > \cdots > a_m$ and exactly $k \geq 1$ parts equal to r is $\det(z_{ij})$, where*

$$z_{ij} = \begin{cases} q^{3a_i-2}\begin{bmatrix} a_i+a_j-2 \\ a_i-1 \end{bmatrix}_{q^3} & \text{if } i \leq m-1 \\ \\ q^{(3kr-2)+(k-1)(r-1)}\begin{bmatrix} r-2+a_j-k \\ r-2 \end{bmatrix}_{q^3} & \text{if } i = m. \end{cases}$$

Proof. We must have $a_m = r$. We will modify our construction of $\mathcal{L}_\sigma(a_1, \ldots, a_m)$ so that P_m begins at $(1, r - a_{\sigma(m)} + 1)$, passes through $(r-1, r-k+1)$, then takes one step east and $k-1$ steps north. Therefore it decomposes into two parts, the first of which represents a partition with $\leq r-2$ parts and each part $\leq a_{\sigma(m)} - k$, followed by the remaining steps which contribute $(r-1)(k-1)$ cubes. The position of the tail is such (at the extreme right of the diagram) that if P_i intersects P_m then the intersection must occur to the south and west of $(r-1, r-k+1)$. Therefore our sign-reversing involution θ will preserve this feature. The generating function for $\mathcal{L}_\sigma(r; a_1, \ldots, a_m)$ is now

$$\sum_{(P_1,\ldots,P_m)\in\mathcal{L}_\sigma(r;a_1,\ldots,a_m)} q^{3|P_1|+\cdots+3|P_m|} = \left(\sum_{P_1} q^{3|P_1|}\right)\cdots\left(\sum_{P_m} q^{3|P_m|}\right) =$$

$$\begin{bmatrix} a_1 + a_{\sigma(1)} - 2 \\ a_1 - 1 \end{bmatrix}_{q^3} \cdots \begin{bmatrix} a_{m-1} + a_{\sigma(m-1)} - 2 \\ a_{m-1} - 1 \end{bmatrix}_{q^3} q^{(r-1)(k-1)} \begin{bmatrix} r - 2 + a_{\sigma(m)} - k \\ r-2 \end{bmatrix}_{q^3}.$$

The remaining details of the proof are unchanged.

□

Given this theorem, the generating for all cyclically symmetric plane partitions in $\mathcal{B}(r, r, r)$ with exactly $k \geq 1$ parts equal to r is

$$\sum_{m=0}^{r-1} \sum_{0<a_1<\cdots<a_{m-1}\leq r-1} \det(z_{ij}(a_1, \ldots, a_m))_{i,j=1}^{m+1},$$

where $a_m = r$ and

$$z_{ij}(a_1, \ldots, a_m) = \begin{cases} q^{3a_i-2} \begin{bmatrix} a_i + a_j - 2 \\ a_i - 1 \end{bmatrix}_{q^3} & \text{if } i \leq m-1 \\ q^{(3kr-2)+(k-1)(r-1)} \begin{bmatrix} r - 2 + a_j - k \\ r - 2 \end{bmatrix}_{q^3} & \text{if } i = m. \end{cases}$$

Using additional exercise #1, we can express this as a single determinant:

Theorem: *The generating function for cyclically symmetric plane partitions in $\mathcal{B}(r,r,r)$ with exactly $k \geq 1$ parts equal to r is* $\det(\widehat{\delta}_{ij} + z_{ij})$*, where*

$$z_{ij} = \begin{cases} q^{3i-2} \begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_{q^3} & \text{if } i \leq r-1 \\ \\ q^{(3kr-2)+(k-1)(r-1)} \begin{bmatrix} r-2+j-k \\ r-2 \end{bmatrix}_{q^3} & \text{if } i = r. \end{cases}$$

There is an analogous generating function for descending plane partitions (see 9. 21 of the textbook for the definition). You will prove this theorem in the exercises.

Exercises: 9–14.

Additional Exercise #1: Let (z_{ij}) be an $r \times r$ matrix. Given $0 < a_1 < \cdots < a_{m-1} \leq r-1$, let $(z_{a_i a_j})$ denote the $m \times m$ matrix obtained by using rows a_1 through a_m and columns a_1 through a_m of (z_{ij}), where $a_m = r$. For each i and j between 1 and r let

$$\widehat{\delta}_{ij} = \begin{cases} 1 & \text{if } i = j \leq m-1 \\ 0 & \text{if } i = j = m \\ 0 & \text{if } i \neq j. \end{cases}$$

Then

$$\det(\widehat{\delta}_{ij} + z_{ij}) = \sum_{m=0}^{r-1} \sum_{0<a_1<\cdots<a_{m-1}\leq r-1} \det(z_{a_i a_j}).$$

Mimic the proof in the preliminary section, changing what needs to be changed.

Section 3.5: Dodgson's Algorithm

Topic to review: the inversion number of a matrix.

The inversion number of a matrix

Let $M = (a_{ij})$ be an arbitrary $n \times n$ matrix. The inversion number of M is

$$\mathcal{I}(M) = \sum_{\substack{n \geq i > j \geq 1 \\ 1 \leq p < q \leq n}} a_{ip} a_{jq}.$$

Each term in this sum represents the product of a pair of entries, a_{jq} being to the north (smaller row index) and east (larger column index) of a_{ip}. If M is the matrix representing a permutation $\sigma = \sigma_1 \cdots \sigma_n$, in which for each $i \leq n$ the i^{th} row contains 1 in column σ_i and 0s elsewhere, then $\mathcal{I}(M) = \mathcal{I}(\sigma)$.

In this section Dodgson's algorithm for computing a determinant is introduced. The algorithm is based on the following idea: The determinant of an $n \times n$ matrix can be computed from its $(n-1) \times (n-1)$ minors and its $(n-2) \times (n-2)$ minors. The calculation (Theorem 3.12) can be expressed in the form of a fraction whose numerator can be recognized as a 2×2 determinant. Working our way backwards, each $(n-1) \times (n-1)$ minor can be computed from its $(n-3) \times (n-3)$ minors and its $(n-4) \times (n-4)$ minors. We can go all the way back to 2×2 minors. Dodgson's algorithm reverses the direction of this calculation, successively working out 2×2 determinants and storing intermediate results in a pair of matrices. You will practice this computation in Exercises 1 and 4. The λ-determinant of a matrix is obtained by replacing each 2×2 determinant in Dodgson's algorithm by its λ-analogue,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}_\lambda = ad + \lambda bc.$$

You will use this to derive the generalized Vandermonde determinant

$$|x_j^{n-i}|_\lambda = \prod_{1 \leq i < j \leq n} (x_i + \lambda x_j).$$

At the bottom of page 116 there is a general formula for $|M|_\lambda$, the λ-determinant of the $n \times n$ matrix M. It is expressed in the form

$$|M|_\lambda = \sum_{B \in \mathcal{A}_n} w(B),$$

where

$$w(B) = \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{N(B)}) \prod_{i,j=1}^{n} a_{ij}^{B_{ij}},$$

$\mathcal{A}_n$ is the set of $n \times n$ alternating matrices, $\mathcal{I}(B)$ is the inversion number of B, $N(B)$ is the number of -1s in B, and the entries of M are a_{ij} and the entries of B are B_{ij}. This was the starting point of the alternating sign matrix conjecture: how many of these things are there?

Exercises: 1, 4, 6. Note: problem 4 refers to the Vandermonde formula on page 63. Problem 6 refers to the generalized Vandermonde product on page 117.

Chapter 4: Symmetric Functions

Topic to review: properties of permutations.

$\mathcal{S}_n$ is the group of permutations of n letters. We have been using the notation $\sigma = \sigma_1 \cdots \sigma_n$. This is shorthand for the function

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

defined by $\sigma(i) = \sigma_i$ for all $i \leq n$. Permutations are multiplied together using function composition. If σ and τ are permutations, then $\sigma \cdot \tau = \sigma \circ \tau$. As a list of numbers, we have

$$\sigma\tau = \sigma(\tau(1)) \cdots \sigma(\tau(n)).$$

The inverse of σ is the function

$$\sigma^{-1} : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

which satisfies $\sigma^{-1}(\sigma_i) = i$ for all $i \leq n$. The identity permutation is

$$e : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

which satisfies $e(i) = i$ for all $i \leq n$. As a list of numbers, we have $e = 1 \cdots n$.

In the previous section we introduced the matrix representation of a permutation: If $\sigma = \sigma_1 \cdots \sigma_n$, then

$$M_\sigma = (\delta_{\sigma(i),j}),$$

where

$$\delta_{p,q} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

In particular, M_e is the $n \times n$ identity matrix. Note that $M_\sigma M_\tau = M_{\tau\sigma}$. To prove this, we must show that the i, j entry of $M_\sigma M_\tau$ is $\delta_{\tau(\sigma(i)),j}$. Here's the computation: The i, j entry of $M_\sigma M_\tau$ is

$$\sum_{k=1}^{n} \delta_{\sigma(i),k} \delta_{\tau(k),j}.$$

The $k \neq \sigma(i)$ terms are zero, leaving

$$\delta_{\sigma(i),\sigma(i)}\delta_{\tau(\sigma(i)),j} = \delta_{\tau(\sigma(i)),j}.$$

Another property to be aware of is that $\mathcal{I}(\sigma\tau) \equiv \mathcal{I}(\sigma) + \mathcal{I}(\tau) \pmod 2$, which is equivalent to

$$(-1)^{\mathcal{I}(\sigma\tau)} = (-1)^{\mathcal{I}(\sigma)}(-1)^{\mathcal{I}(\tau)}.$$

To prove this, recall that if k swaps are required to rearrange a permutation to the identity permutation, then the inversion number of the permutation is congruent to $k \pmod 2$. Say that t swaps are required to rearrange τ to e and that s swaps are required to rearrange σ to e. We can rearrange $\sigma\tau$ to e in $s + t$ swaps as follows:

$$\sigma(\tau(1)) \cdots \sigma(\tau(n)) \rightarrow \text{ (t swaps) } \rightarrow \sigma(1) \cdots \sigma(n) \rightarrow \text{ (s swaps) } \rightarrow 1 \cdots n.$$

Therefore

$$\mathcal{I}(\sigma\tau) \equiv s + t \equiv \mathcal{I}(\sigma) + \mathcal{I}(\tau).$$

Section 4.1: Schur functions

Fix variables x_1 through x_n. All partitions in this section will be of the form $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, that is a partition with $\leq n$ positive parts. For each $p \geq 1$ let V_p be the vector space over the rational numbers of all symmetric polynomials in x_1 through x_n having total degree p. Then V_p has basis $\{m_\lambda : \lambda \vdash p\}$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, $\lambda_1 + \cdots + \lambda_n = p$, and m_λ consists of all terms of the form

$$x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(n)}^{\lambda_n}$$

with all coefficients equal to 1. For example, if $n = 3$ then in V_3 we have

$$m_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

If we order the monomials in increasing lexicographic order (same as increasing alphabetical order if our alphabet is $x_1, x_2, \ldots, x_n$), then the leading (first) term of m_λ is $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. The polynomials m_λ are called monomial symmetric functions. We will describe three other classes of symmetric functions.

Elementary symmetric functions

For $1 \leq i \leq n$ the i^{th} elementary symmetric function e_i is the sum of every product of i variables:

$$e_i = \sum_{1 \leq k_1 < k_2 < \cdots < k_i \leq n} x_{k_1} x_{k_2} \cdots x_{k_i}.$$

We also set $e_0 = 1$ and $e_i = 0$ for $i > n$. The leading term of e_i is $x_1 x_2 \cdots x_i$ for $1 \leq i \leq n$. The generating function for the elementary symmetric functions is

$$E(t) = \sum_{i=0}^{n} e_i t^i = \prod_{k=1}^{n} (1 + x_k t).$$

Given a partition $\lambda \vdash p$ we set

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}.$$

Provided each $\lambda_i \leq n$, the leading term of e_λ is the product of the leading terms of the e_{λ_i}. The leading term of e_{λ_i} is $x_1 \ldots x_{\lambda_i}$. We can compute the leading term of e_λ as follows: For each $i \leq n$ replace the dots in the i^{th} row of the Ferrers graph of λ by the variables $x_1, x_2, \ldots, x_{\lambda_i}$. Then multiply all the variables in the diagram together. The exponent of x_i will be equal to the number of terms in the i^{th} column of the diagram. Therefore the leading term of e_λ is equal to

$$x_1^{\lambda'_1} x_2^{\lambda'_2} \cdots x_n^{\lambda'_n},$$

where

$$\lambda' = \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n$$

is the conjugate partition. For example, if $n = 5$ and $p = 7$ then in V_7 we have

$$e_{(5,1,1)} \to \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & & & & \\ x_1 & & & & \end{matrix} \to x_1^3 x_2 x_3 x_4 x_5,$$

and the leading term of $e_{(5,1,1)}$ is $x_1^3 x_2 x_3 x_4 x_5$.

The elementary symmetric functions of total degree n form a basis for V_n. To see this, expand e_λ in terms of the monomial symmetric functions:

$$e_\lambda = \sum_{\gamma \vdash n} c_\gamma m_\gamma.$$

The non-zero coefficients c_γ correspond to partitions which are $\leq \lambda'$ in lexicographic order. For example, we have

$$e_{(2,1)} = e_2 e_1 = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) =$$

$$x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 3x_1x_2x_3 =$$

$$m_{(2,1)} + 3m_{(1,1,1)}.$$

If we create the matrix M_n whose rows are indexed by partitions of n which decrease in lexicographic order and whose columns are indexed by partitions of n which increase in lexicographic order and whose λ^{th} row contains the coefficients which express e_λ as a linear combination of the monomial symmetric functions, then M_n will be a lower triangular matrix with 1s along the diagonal. For example, we have

$$\begin{aligned} e_{(3)} &= 1m_{(111)} \\ e_{(21)} &= 3m_{(111)} + 1m_{(21)} \\ e_{(111)} &= 6m_{(111)} + 3m_{(21)} + 1m_{(3)}, \end{aligned}$$

therefore

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{pmatrix}.$$

In general, M_n is a lower-triangular matrix with 1s down the diagonal (proof omitted). This implies that the elementary symmetric functions are linearly independent and span the vector space of all symmetric functions in x_1 through x_n having total degree n. Reason: The linear transformation $f : V_n \to V_n$ defined by

$$f(\sum_{\lambda \vdash n} \alpha_\lambda m_\lambda) = \sum_{\lambda \vdash n} \alpha_\lambda e_{\lambda'}$$

is represented by the matrix M_n. Since M_n is invertible, f is an isomorphism. Since $f(m_\lambda) = e_{\lambda'}$ for each λ, the $e_{\lambda'}$ form a basis for V_n.

Complete symmetric functions.

These are also known as the homogeneous symmetric functions. For $i \geq 0$, h_i is the sum of all monomials in the variables x_1 through x_n with total degree equal to i:

$$h_i = \sum_{\substack{0 \leq \alpha_1, \ldots, \alpha_n \leq n \\ \alpha_1 + \cdots + \alpha_n = i}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

The leading term of h_i is x_1^i. The generating function for the complete symmetric functions is

$$H(t) = \sum_{i=0}^{\infty} h_i t^i = \prod_{j=1}^{n} \frac{1}{1 - x_j t}.$$

Given a partition $\lambda \vdash p$ we set

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}.$$

The leading term of h_λ is x_1^p. If $p = n$ then $h_\lambda \in V_n$. The matrix expressing the coefficients of the h_λ expanded in terms of the m_λ is not lower triangular, so we can't prove that the complete symmetric functions of total degree n form a basis for V_n using the argument above. However, if we can prove that the complete symmetric functions of total degree n span V_n, then they must be linearly independent because there are as many of them as there are monomial symmetric functions of total degree n.

We know that the elementary symmetric functions of total degree n span V_n. If we can show that every e_i is a linear combination of complete symmetric functions, then every product of the e_i will be a linear combination of complete symmetric functions, therefore every e_λ will be a linear combination of complete symmetric functions of total degree n when $\lambda \vdash n$. We will show that e_i is a linear combination of complete symmetric functions by induction on i.

The base case is $i = 0$, and we have $e_0 = h_0 = 1$. Now assume that e_0 through e_{p-1} can be expressed as linear combinations of complete symmetric functions. Observe that we have

$$E(t)H(-t) = \prod_{j=1}^{n} (1 + x_j t) \prod_{j=1}^{n} \frac{1}{1 + x_j t} = 1.$$

Extracting the coefficient of t^p we obtain

$$e_p h_0 - e_{p-1} h_1 + e_{p-2} h_2 - \cdots = 0.$$

Since $h_0 = 1$, we can rearrange this to

$$e_p = e_{p-1} h_1 - e_{p-2} h_2 + \cdots .$$

Therefore e_p can be expressed as a linear combination of complete symmetric functions. This completes the induction proof.

Schur functions

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be a partition of p. The determinant

$$\det(x_j^{n-i+\lambda_i}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} x_{\sigma(1)}^{n-1+\lambda_1} \cdots x_{\sigma(n)}^{n-n+\lambda_n}$$

is an alternating polynomial of total degree $\binom{n}{2} + p$. The Vandermonde determinant

$$\det(x_j^{n-i}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} x_{\sigma(1)}^{n-1} \cdots x_{\sigma(n)}^{n-n+\lambda_n}$$

is an alternating polynomial of total degree $\binom{n}{2}$. Therefore the Schur function

$$s_\lambda = \frac{\det(x_j^{n-i+\lambda_i})}{\det(x_j^{n-i})}$$

is a symmetric polynomial of total degree p. In exercise 3 of Section 4.2 you will prove that the Schur functions of total degree n form a basis of V_n. One of the ingredients of the proof is the Jacobi-Trudi Identity.

Jacobi-Trudi Identity: *Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be a partition of p. Then*

$$s_\lambda = \det(h_{\lambda_i + j - i}).$$

Proof. We have

$$s_\lambda = \frac{\det(x_j^{n-i+\lambda_i})}{\det(x_j^{n-i})}.$$

We will factor the matrices (x_j^{n-1}) and $(x_j^{n-i+\lambda_i})$ as follows:

Our starting point is

$$H(t)E(-t) = 1.$$

Fix a and b between 1 and n. Multiplying both sides by $\frac{1}{1-x_b t}$ we obtain

$$H(t) \prod_{\substack{1 \le j \le n \\ j \ne b}} (1 - x_j t) = \frac{1}{1 - x_b t}.$$

Write

$$\prod_{\substack{1 \le j \le n \\ j \ne b}} (1 - x_j t) = \sum_{j=0}^{n-1} e_j^{(b)} t^j.$$

Then we have

$$H(t) \sum_{j=0}^{n-1} e_j^{(b)} t^j = \frac{1}{1 - x_b t}.$$

Extracting the coefficient of t^{n-a} we obtain

$$\sum_{k=1}^{n} h_{k-a} e_{n-k}^{(b)} = x_b^{n-a}.$$

This implies the matrix equation

$$(h_{j-i})(e_{n-i}^{(j)}) = (x_j^{n-i}).$$

If instead we extract the coefficient of $t^{\lambda_a + n - a}$ we obtain

$$\sum_{k=1}^{n} h_{\lambda_a + k - a} e_{n-k}^{(b)} = x_b^{\lambda_a + n - a}.$$

This implies the matrix equation

$$(h_{\lambda_i + j - i})(e_{n-i}^{(j)}) = (x_j^{\lambda_i + n - i}).$$

The matrix (h_{j-i}) has determinant equal to 1 because it is upper triangular with 1s along the diagonal. Evaluating determinants we obtain

$$\det(e_{n-i}^{(j)}) = \det(x_j^{n-i})$$

and

$$\det(h_{\lambda_i+j-i})\det(e^{(j)}_{n-i}) = \det(x_j^{\lambda_i+n-i}).$$

Therefore

$$\det(h_{\lambda_i+j-i}) = \frac{\det(x_j^{\lambda_i+n-i})}{\det(e^{(j)}_{n-i})} = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})} = s_\lambda.$$

□

Exercises: 1, 2, 4, 6, 7, 8

Section 4.2: Semistandard tableaux

Given a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, we derive an alternative expression for s_λ. Let $\mathcal{T}_{\lambda,n}$ denote the set of semistandard tableaux of shape λ. This is the set of all diagrams obtained by replacing the dots in the Ferrers diagram for λ with numbers selected from $\{1,\ldots,n\}$, where the numbers weakly increase along rows and strictly decrease down columns. Then

$$s_\lambda = \sum_{T\in\mathcal{T}_{\lambda,n}} x^T,$$

where

$$x^T = \prod_{i\in T} x_i.$$

See the example at the bottom of page 128.

Proof. The idea of the proof is to evaluate $\det(h_{\lambda_i+j-i})$ using families of lattice paths, then construct a sign-reversing involution which cancels out terms corresponding to families with intersections. This leaves a sum over families of lattice paths which don't intersect, which will be equal to $\sum_{T\in\mathcal{T}_{\lambda,n}} x^T$.

Consider the semistandard tableau at the bottom of page 128. For each i, the entries in row i constitute a partition with exactly λ_i parts, each of which is $\leq n$. If we subtract 1 from each entry, we obtain a partition with $\leq \lambda_i$ parts, each of which is $\leq n-1$. Therefore we can represent this row by a lattice path which has λ_i northward steps and $n-1$ eastward steps. If we let the lattice path P_i begin in position $(1, n-i)$ and end in position $(n, \lambda_n + n - i)$, then the condition that the columns strictly decrease is precisely the restriction needed to keep the lattice paths from intersecting each other.

More generally, for each $\sigma \in \mathcal{S}_n$ let $\mathcal{L}_\sigma$ represent the set of all lattice paths of the form $(P_1, \dots, P_n)$, where P_i begins in position $(1, n-\sigma(i))$ and ends in position $(n, \lambda_n + n - i)$. Next, we set

$$X = \{(\sigma, P_1, \dots, P_n) : \sigma \in \mathcal{S}_n, (P_1, \dots, P_n) \in \mathcal{L}_\sigma\}.$$

For each $(\sigma, P_1, \dots, P_n) \in X$ we set

$$w(\sigma, P_1, \dots, P_n) = (-1)^{\mathcal{I}(\sigma)} x^{P_1} \cdots x^{P_n},$$

where

$$x^{P_i} = \prod_{i=1}^{n} x_i^{(\#\text{ of northward steps at } x=i \text{ in } P_i)}.$$

Finally, we set

$$Z = \sum_{(\sigma, P_1, \dots, P_n) \in X} w(\sigma, P_1, \dots, P_n).$$

Claim: $Z = s_\lambda$. To see this, fix a permutation σ and observe that every x^{P_i} corresponding to $(\sigma, P_1, \dots, P_n) \in \mathcal{L}_\sigma$ is a monomial of total degree $\lambda_{\sigma(i)} + n - i$ in the variables x_1 through x_n. If we sum over all such P_i, we obtain $h_{\lambda_{\sigma(i)}+n-i}$. Therefore

$$\sum_{(\sigma, P_1, \dots, P_n) \in \mathcal{L}_\sigma} w(\sigma, P_1, \dots, P_n) = (-1)^{\mathcal{I}(\sigma)} h_{\lambda_{\sigma(1)}+n-1} \cdots h_{\lambda_{\sigma(n)}+n-n}.$$

Therefore

$$Z = \sum_{\sigma \in \mathcal{S}_n} \sum_{(\sigma, P_1, \dots, P_n) \in \mathcal{L}_\sigma} w(\sigma, P_1, \dots, P_n) =$$

$$\sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} h_{\lambda_{\sigma(1)}+n-1} \cdots h_{\lambda_{\sigma(n)}+n-n} = \det(h_{\sigma(i)+j-i}) = s_\lambda.$$

Let X_0 consist of all configurations $(\sigma, P_1, \dots, P_n) \in X$ which contains intersecting lattice paths. Construct an involution $\theta : X_0 \to X_0$ via tail-switches. Observe that if $\theta(\sigma, P_1, \dots, P_n) = (\sigma', P_1', \dots, P_n')$, then $(-1)^{\mathcal{I}(\sigma')} = (-1)^{\mathcal{I}(\sigma)}$ as before, and

$$x^{P_1} \cdots x^{P_n} = x^{P_1'} \cdots x^{P_n'}$$

because the total number of northward steps along any vertical line in two diagrams related by a tail switch does not change. Therefore

$$w(\sigma', P_1', \ldots, P_n') = -w(\sigma, P_1, \ldots, P_n),$$

which means that θ is a sign-reversing involution. Hence

$$\sum_{(\sigma, P_1, \ldots, P_n) \in X_0} w(\sigma, P_1, \ldots, P_n) = 0$$

and

$$s_\lambda = Z = \sum_{(\sigma, P_1, \ldots, P_n) \in X \setminus X_0} w(\sigma, P_1, \ldots, P_n) = \sum_{T \in \mathcal{T}_{\lambda,n}} x^T.$$

□

Proof of the generating function for plane partitions

We now have a very powerful tool at our disposal, because column-strict tableaux of shape λ, where $\lambda_1 = \cdots = \lambda_r = s$ and $\lambda_{r+1} = \cdots = \lambda_r + t = 0$, can be used to represent plane partitions in $\mathcal{B}(r, s, t)$. The construction is this: Fill the Ferrers diagram of this partition with the variables x_1 through x_{r+t} to create a column-strict tableau. Then evaluate $x_1 = q^{r+t}$, $x_2 = q^{r+1-1}$, $\ldots$, $x_{r+t} = q$. The exponents of the qs will be weakly decreasing in rows and strictly decreasing in columns. If we multiply each term in the first row by q^{-r}, each term in the second row by $q^{-(r-1)}$, and each term in the r^{th} row by q^{-1}, then the exponents in the rows will be weakly decreasing and the exponents in the columns will be weakly decreasing. Moreover, the largest exponent in the diagram will by $\leq t$. Hence the diagram of exponents will represent a plane partition in $\mathcal{B}(r, s, t)$. The product of all the qs in the diagram will be equal to the number of cubes in the plane partition. The upshot is that if we evaluate s_λ at $x_i = q^{r+t+1-i}$ for each i, then multiply by $q^{-rs-(r-1)s-\cdots-s} = q^{-s\binom{r+1}{2}}$, we obtain

$$\sum_{P \in \mathcal{B}(r,s,t)} q^{|P|}.$$

Therefore

$$q^{-s\binom{r+1}{2}} s_\lambda(q^{r+t}, q^{r+t-1}, \ldots, q) = \sum_{n=0}^{rst} pp(n) q^n.$$

We can evaluate $q^{-s\binom{r+1}{2}} s_\lambda(q^{r+t}, q^{r+t-1}, \dots, q)$ using the Jacobi-Trudi identity and Vandermonde's formula to obtain the product formula on page 131.

You have plenty of experience with these calculations, given the exercises on pp. 48-49 of these notes, so skip the details on pp. 132-133 of the textbook.

Exercises: 3, 4, 5, 6, 8.

Section 4.3: Proof of the MacMahon conjecture (generating function for symmetric plane partitions)

Theorem: *There is a one-one correspondence between symmetric plane partitions in $\mathcal{B}(r,r,t)$ and column-strict partitions with odd stack heights contained in $\mathcal{B}(r,t,2r-1)$.*

Proof. Let P be a symmetric plane partition in $\mathcal{B}(r,r,t)$. Then each z-slice can be interpreted as a self-conjugate partition – see Figure 4.3, page 136. Any self-conjugate partition can be decomposed into a partition with distinct odd parts – see Figure 4.4, page 136. We will represent the j^{th} slice by the partition $2\alpha_{1j} - 1 > 2\alpha_{2j} - 1 > \cdots$. There can be at most r positive parts to this partition. Since each z-slice has to fit on top of the one below it, we have $2\alpha_{i1} - 1 \geq 2\alpha_{i2} - 1 \geq \cdots$. There can be at most t z-slices. We will identify P with the $r \times t$ matrix whose non-zero entries are $2\alpha_{ij} - 1$. The largest entry is $\leq 2r - 1$. We have just made an argument that the rows are weakly decreasing and the columns are strictly decreasing. The entries are all odd and they add up to the size of $\mathcal{P}$. Hence the matrix $(2\alpha_{ij} - 1)$ represents a column-strict plane partition in $\mathcal{B}(r,t,2r-1)$. This establishes the one-to-one correspondence. □

Corollary: *The generating function for symmetric plane partitions contained in $\mathcal{B}(r,r,t)$ is*

$$\sum_{t \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0} s_\lambda(q^{2r-1}, q^{2r-3}, \dots, q).$$

Proof. Represent a symmetric plane partition P by the matrix $(2\alpha_{ij} - 1)$ as above. Then

$$\prod_{i=1}^{r} \prod_{j=1}^{r} q^{2\alpha_{ij}-1} = q^{|P|}.$$

We can form a semistandard tableau T by overwriting each non-zero entry $2\alpha_{ij}-1$ by the variable $x_{r-\alpha_{ij}+1}$. The assignment $x_i = q^{2r-2i+1}$, $1 \le i \le r$, yields the evaluation

$$x^T = \prod_{x_i \in T} x_i = \prod_{\alpha_{ij} \neq 0} x_{r-\alpha_{ij}+1} = \prod_{\alpha_{ij} \neq 0} q^{2\alpha_{ij}-1} = q^{|P|}.$$

The shape of T is $\lambda = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$, where λ_i is the number of positive entries in the i^{th} row of $(2\alpha_{ij}-1)$. Therefore we have a one-to-one correspondence between all $q^{|P|}$ and x^T, where P is a symmetric plane partition in $\mathcal{B}(r,r,t)$ and T is a semistandard tableau in the variables x_1 through x_r with shape λ, $t \ge \lambda_1 \ge \cdots \ge \lambda_r \ge 0$. This yields

$$\sum_{t \ge \lambda_1 \ge \cdots \ge \lambda_r \ge 0} s_\lambda(q^{2r-1}, q^{2r-3}, \ldots, q) = \sum_{t \ge \lambda_1 \ge \cdots \ge \lambda_r \ge 0} \sum_{T \in \mathcal{T}(\lambda, r)} x^T =$$

$$\sum_{\substack{P \in \mathcal{B}(r,r,t) \\ P \text{ is symmetric}}} q^{|P|} = \sum_{n=0}^{rrt} rspp(n) q^n.$$

□

Lemma 4.5, page 135, provides a proof of the formula

$$\sum_{t \ge \lambda_1 \ge \cdots \ge \lambda_r \ge 0} s_\lambda(x_1, x_2, \ldots, x_r) = \frac{\det(x_i^{j-1} - x_i^{t+2r-j})}{\det(x_i^{j-1} - x_i^{2r-j})}.$$

It is a proof by induction, incorporating the Jacobi-Trudi identity, Vandermonde's product, and the Weyl denominator formula, pp. 140–145. It is somewhat easier to derive the formula

$$\sum_{\lambda_1 \ge \cdots \ge \lambda_r \ge 0} s_\lambda(x_1, x_2, \ldots, x_r) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \le i < j \le n} \frac{1}{1-x_i x_j}.$$

See pp. 138-140.

In these notes we will limit ourselves to proving

$$\sum_{\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge 0} s_\lambda(x_1, x_2, x_3) = \prod_{i=1}^{3} \frac{1}{1-x_i} \prod_{1 \le i < j \le 3} \frac{1}{1-x_i x_j}.$$

If T is a semistandard tableau of shape $\lambda_1 \geq \lambda_2 \geq \lambda_3$ in the variables x_1, x_2, and x_3, then every column of T must belong to the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix}, \begin{bmatrix} x_1 \\ x_3 \\ \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \\ \end{bmatrix}, \begin{bmatrix} x_1 \\ \\ \end{bmatrix}, \begin{bmatrix} x_2 \\ \\ \end{bmatrix}, \begin{bmatrix} x_3 \\ \\ \end{bmatrix} \right\}.$$

The columns $\begin{bmatrix} x_2 \\ x_3 \\ \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ \\ \end{bmatrix}$ cannot both appear in T. There are two possibilities:

$$T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^a \begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix}^b \begin{bmatrix} x_1 \\ x_3 \\ \end{bmatrix}^c \begin{bmatrix} x_1 \\ \\ \end{bmatrix}^d \begin{bmatrix} x_2 \\ \\ \end{bmatrix}^e \begin{bmatrix} x_3 \\ \\ \end{bmatrix}^f$$

for $a, b, c, d, e, f \geq 0$, and

$$T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^a \begin{bmatrix} x_1 \\ x_2 \\ \end{bmatrix}^b \begin{bmatrix} x_1 \\ x_3 \\ \end{bmatrix}^c \begin{bmatrix} x_2 \\ x_3 \\ \end{bmatrix}^d \begin{bmatrix} x_2 \\ \\ \end{bmatrix}^e \begin{bmatrix} x_3 \\ \\ \end{bmatrix}^f$$

for $a, b, c, e, f \geq 0$ and $d \geq 1$. Therefore

$$\sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} s_\lambda(x_1, x_2, x_3) = \sum_{T \in \mathcal{T}_{\lambda,3}} x^T =$$

$$\sum_{a,b,c,d,e,f \geq 0} (x_1x_2x_3)^a (x_1x_2)^b (x_1x_3)^c x_1^d x_2^e x_3^f +$$

$$\sum_{a,b,c,e,f \geq 0, d \geq 1} (x_1x_2x_3)^a (x_1x_2)^b (x_1x_3)^c (x_2x_3)^d x_2^e x_3^f =$$

$$\frac{1}{1-x_1x_2x_3} \frac{1}{1-x_1x_2} \frac{1}{1-x_1x_3} \frac{1}{1-x_1} \frac{1}{1-x_2} \frac{1}{1-x_3} +$$

$$\frac{1}{1-x_1x_2x_3} \frac{1}{1-x_1x_2} \frac{1}{1-x_1x_3} \frac{x_2x_3}{1-x_2x_3} \frac{1}{1-x_2} \frac{1}{1-x_3} =$$

$$\frac{1}{1-x_1} \frac{1}{1-x_2} \frac{1}{1-x_3} \frac{1}{1-x_1x_2} \frac{1}{1-x_1x_3} \frac{1}{1-x_2x_3}.$$

Exercise 1: Prove that the number of symmetric plane partitions of size n which are restricted to $\mathcal{B}(3,3,\infty)$ is equal to the coefficient of q^n in

$$\sum_{n=0}^{\infty} rspp(n)q^n = \frac{1}{(1-q)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^8)}.$$

Exercise 2: Find the number of symmetric plane partitions of 50 which live in $\mathcal{B}(3,3,\infty)$.

Exercise 3: Do exercise 4.3.9 under the assumption that $n = 3$. Adapt the proof given at the end of this section.

Exercise 4: Do exercise 4.3.10 using Exercise 3 above.

Exercise 5: Do exercise 4.3.12 under the assumption that $n = 3$. Adapt the proof given at the end of this section.

Exercise 6: Do exercise 4.3.13 under the assumption $n = 3$. Use Exercise 5.

Chapter 5: Hypergeometric Series

This chapter is going to require major surgery. Andrews' conjecture (page 21 of textbook) can be proved without recourse to the theory of basic hypergeometric series. It's not clear if Macdonald's conjecture can be proved without basic hypergeometric series, but the details presented in the book are not pleasant to work through. So we are going to skip the treatment of basic hypergeometric series completely and just use the q-binomial theorem.

In this textbook we are working our way through the sequence of ideas in the chronology described on pp. 12–15 of these notes. Let's see where we are going and what we've done so far:

Conjecture 2: *The total number of $n \times n$ alternating sign matrices with a 1 in the k^{th} column of the first row is*

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

Conjecture 3: *The total number of $n \times n$ alternating sign matrices is*

$$A_n = A_{n+1,1} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Conjecture 8: *The number of descending plane partitions with largest part $\leq r$ and for which r appears exactly $k-1$ times is equal to the number of $r \times r$ alternating sign matrices with a 1 in the k^{th} column of the first row.*

Conjecture 9: *The number of descending plane partitions with largest part $\leq r$ and for which r appears exactly $k-1$ times is equal to*

$$\binom{r+k-2}{k-1} \frac{(2r-k-1)!}{(r-k)!} \prod_{j=0}^{r-2} \frac{(3j+1)!}{(r+j)!}.$$

Theorem 3.11: *Let $H_r(q)$ denote the matrix*

$$H_r(q) = \left(q^{i+1} \begin{bmatrix} i+j \\ j-1 \end{bmatrix} \right)_{i,j=1}^{r}.$$

The generating function for descending plane partitions with largest part $\leq r$ is given by

$$det(I_{r-1} + H_{r-1}(q)).$$

The generating function for descending plane partitions with exactly k parts of size r, $0 \leq k \leq r-1$, is given by

$$det(H_{k,r-1}(q)),$$

where $H_{k,r-1}(q)$ is the matrix formed by replacing the last row of $I_{r-1} + H_{r-1}$ by $(0, \ldots, 0, 1)$ when $k = 0$ and by

$$q^{kr} \left(\begin{bmatrix} r-0 \\ 1-k \end{bmatrix}, \begin{bmatrix} r+1-k \\ 2-k \end{bmatrix}, \ldots, \begin{bmatrix} r+r-1-k \\ r-1-k \end{bmatrix} \right).$$

Conjecture 7 (The Andrews conjecture): *The generating function for descending plane partitions with largest part less than or equal to r is*

$$\prod_{1 \leq i \leq j \leq r} \frac{1-q^{r+i+j-1}}{1-q^{2i+j-1}}.$$

We will denote by H_r and H_{ir} the limits as $q \to 1$ of the matrices $H_r(q)$ and $H_{ir}(q)$. We don't as yet have a formula for $\det(I_r + H_r)$. But if Conjectures 8 and 9 are correct, then combined with Theorem 3.11 they would imply that

$$\det(I_r + H_r) = \prod_{j=0}^{r} \frac{(3j+1)!}{(r+j+1)!}$$

and

$$\det(H_{ir}) = \binom{r+i}{i} \frac{(2r-i)!}{(r-i)!} \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j+1)!}.$$

We don't need to prove the conjectures to verify that these evaluations are correct. In the next section we will learn how Mills, Robbins and Rumsey proved these determinant formulas.

Section 5.1: Mills, Robbins, and Rumsey's bright idea

Let $h_r = \det(I_r + H_r)$, $h_{ir} = \det(H_{ir})$,

$$L_r = \prod_{j=0}^{r} \frac{(3j+1)!}{(r+j+1)!},$$

and

$$L_{ir} = \binom{r+i}{i} \frac{(2r-i)!}{(r-i)!} \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j+1)!}.$$

We wish to prove that $h_r = L_r$ and $h_{ir} = L_{ir}$ for all $r \geq 1$ and $0 \leq i \leq r$.

Observe that we have $\sum_{i=0}^{r} h_{ir} = h_r$, because we are counting descending plane partitions by type. Also, h_{0r} = number of descending plane partitions with largest part $\leq r$, hence

$$h_{0r} = \det(I_{r-1} + H_{r-1}) = h_{r-1}.$$

We can easily check that $\sum_{i=0}^{r} L_{ir} = L_r$ and $L_{0r} = L_{r-1}$. (See the derivation of A_n from $A_{n,k}$ on pp. 2–3 of these notes.)

We clearly have $h_1 = L_1 = 2$. To prove that $h_r = L_r$ and $h_{ir} = L_{ir}$ for all $r \geq 1$ and $0 \leq i \leq r$, we can use the following theorem. (Note that the textbook drops the r and writes $L_i = L_{i,r}$.)

Theorem: *Let $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ be two sequences of numbers with $a_1 = b_1$. Assume that for each $r \geq 1$ there are numbers a_{0r} through a_{rr} and b_{0r} and b_{rr} such that $a_r = \sum_{i=0}^{r} a_{ir}$ and $b_r = \sum_{i=0}^{r} b_{ir}$. Assume further that for each $r \geq 2$ we have $a_{0r} = a_{r-1}$ and $b_{0r} = b_{r-1}$ and there exists a non-singular $r \times r$ matrix M_r and a column vector V_r such that*

$$M_r \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{rr} \end{bmatrix} = a_{0r} V_r$$

and

$$M_r \begin{bmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{rr} \end{bmatrix} = b_{0r} V_r.$$

Then $a_r = b_r$ and $a_{ir} = b_{ir}$ for all $r \geq 1$ and $0 \leq i \leq r$.

Proof. By induction on r. We have $a_1 = b_1$ by hypothesis. Now assume that $a_{r-1} = b_{r-1}$. Since $a_{0r} = a_{r-1}$ and $b_{0r} = b_{r-1}$, this implies $a_{0r} = b_{0r}$. Therefore

$$M_r \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{rr} \end{bmatrix} = a_{0r} V_r = b_{0r} V_r = M_r \begin{bmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{rr} \end{bmatrix}.$$

Since M_r is non-singular, this implies

$$\begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{rr} \end{bmatrix} = \begin{bmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{rr} \end{bmatrix},$$

hence $a_{ir} = b_{ir}$ for $0 \leq i \leq r$. Moreover we have

$$a_r = \sum_{i=0}^{r} a_{ir} = \sum_{i=0}^{r} b_{ir} = b_r.$$

This completes the induction proof. $\square$

Corollary: *With all notation as above, replace the hypothesis*

$$M_r \begin{bmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{rr} \end{bmatrix} = b_{0r} V_r$$

by the hypothesis that the vector

$$\begin{bmatrix} b_{0r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix}$$

satisfies the matrix equation

$$K_r \begin{bmatrix} b_{0r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix} = \begin{bmatrix} b_{0r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix},$$

where K_r is any $(r+1) \times (r+1)$ matrix with V_r embedded in the lower left hand corner and $I_r - M_r$ embedded in the lower right hand corner. Then $a_r = b_r$ and $a_{ir} = b_{ir}$ for all $r \geq 1$ and $0 \leq i \leq r$.

Proof. The equation

$$K_r \begin{bmatrix} b_{0r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix} = \begin{bmatrix} b_{0r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix}$$

implies the equation

$$\begin{bmatrix} b_{1r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix} = b_{0r} V_r + (I_r - M_r) \begin{bmatrix} b_{1r} \\ b_{1r} \\ \vdots \\ b_{rr} \end{bmatrix},$$

which can be rearranged to

$$M_r \begin{bmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{rr} \end{bmatrix} = b_{0r} V_r.$$

□

This corollary can be used to prove that $a_r = b_r$ and $a_{ir} = b_{ir}$ for all r and $i \le r$, assuming that formulas for both are given. If we wish to derive the b_r and b_{ir} formulas, starting only with the a_r and a_{ir} values, we can proceed as follows:

1. Derive b_1, b_{01}, and b_{11} directly.
2. Assuming b_{r-1} and $b_{i,r-1}$ have been derived for $0 \le i \le r-1$, let

$$\begin{bmatrix} v_{0r} \\ v_{1r} \\ \vdots \\ v_{rr} \end{bmatrix}$$

be any eigenvector of K_r corresponding to eigenvalue 1. Assuming that $v_0 \neq 0$, the vector

$$\frac{b_{r-1}}{v_0} \begin{bmatrix} v_{0r} \\ v_{1r} \\ \vdots \\ v_{rr} \end{bmatrix}$$

is another eigenvalue corresponding to eigenvalue 1. Set

$$b_{ir} = \frac{b_{r-1}}{v_{0r}} v_{ir}$$

and

$$b_r = \sum_{i=0}^{r} b_{i,r}.$$

Then $b_{0r} = b_{r-1}$ by design.

3. Use step 2 to create the numbers b_r and b_{ir} for all values of r and $i \leq r$. These numbers satisfy the hypotheses of the corollary, hence $a_r = b_r$ and $a_{ir} = b_{ir}$ for all r and $i \leq r$.

Here's the bright idea of Mills, Robbins, and Rumsey: they formulated this procedure and applied it to $a_r = h_r$, $a_{ir} = h_{ir}$, $b_r = L_r$, $b_{ir} = L_{ir}$. We will carry out these steps and construct M_r , V_r, and K_r as follows: Let c_1 through c_r be the cofactors of the r^{th} row of the matrix $I_r + H_r$. These are also the cofactors of the last row of each H_{ir} because $I_r + H_r$ and H_{ir} are identical in the first $r-1$ rows. Let R_i denote the r^{th} row of H_{ir}. Let C be the column vector containing the cofactors c_1 through c_r. By properties of determinants the dot product $R_i \bullet C$ is equal to $\det(H_{ir}) = h_{ir}$. If we let R be the $r \times r$ matrix whose rows are R_1 through R_r, then we obtain

$$RC = \begin{bmatrix} h_{1r} \\ h_{2r} \\ \vdots \\ h_{rr} \end{bmatrix}.$$

Since R is an upper triangular matrix with 1s along the diagonal, it is invertible. Hence

$$C = R^{-1} \begin{bmatrix} h_{1r} \\ h_{2r} \\ \vdots \\ h_{rr} \end{bmatrix}.$$

If T is any row of H_{0r} other than the last row, then the dot product $T \bullet C$ can be interpreted as the determinant of the matrix obtained by replacing the last row of H_{0r} with T. Hence this determinant is zero. Combined with $R_0 \bullet C = h_{0r}$, we obtain

$$H_{0r}C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ h_{0r} \end{bmatrix} = h_{0r} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Therefore

$$RH_{0r}R^{-1}\begin{bmatrix} h_{1r} \\ h_{2r} \\ \vdots \\ h_{rr} \end{bmatrix} = RH_{0r}C = h_{0r}R\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

If we choose

$$M_r = RH_{0r}R^{-1}$$

and

$$V_r = R\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

then we have

$$M_r\begin{bmatrix} h_{1r} \\ h_{2r} \\ \vdots \\ h_{rr} \end{bmatrix} = h_{0r}V_r.$$

M_r is non-singular because it has determinant equal to $\det(H_{0r}) = h_{0r} \neq 0$.

R is the matrix whose rows are R_1 through R_r, where R_i is the r^{th} row of H_{ir}. Therefore

$$R = \left(\binom{r+j-i}{j-i}\right)_{1\leq i,j\leq r}.$$

We wish to compute R^{-1}. Observe that

$$(1-x)^{r+1} = \sum_{a=0}^{r}(-1)^a\binom{r+1}{a}x^a$$

and

$$\frac{1}{(1-x)^{r+1}} = \sum_{a=0}^{\infty}\binom{r+a}{a}x^a.$$

If we extract the coefficient of x^{j-i} from both sides of the equation

$$(1+x)^{r+1}\frac{1}{(1+x)^{r+1}} = 1,$$

where $1 \leq i, j \leq r$, we obtain

$$\sum_{k \geq i} (-1)^{k-i} \binom{r+1}{k-i} \binom{r+j-k}{j-k} = \delta_{i,j}.$$

The non-zero terms must satisfy $k \geq i$ and $j \geq k$. Therefore we can assume k lies in the range between 1 and r, and write

$$\sum_{k=1}^{r} (-1)^{k-i} \binom{r+1}{k-i} \binom{r+j-k}{j-k} = \delta_{i,j}.$$

This implies the matrix product

$$\left((-1)^{j-i}\binom{r+1}{j-i}\right)\left(\binom{r+j-i}{j-i}\right) = I.$$

For example, when $r = 3$ we get

$$\begin{pmatrix} 1 & -4 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, we conclude

$$R^{-1} = \left(\binom{r+j-i}{j-i}\right)^{-1} = \left((-1)^{j-i}\binom{r+1}{j-i}\right).$$

$H_{0r} = I_r + H_r^*$, where H_r^* is the same as H_r in the first $r-1$ rows and has last row 0. Therefore

$$RH_{0r}R^{-1} = I_r + RH_r^*R^{-1}.$$

We will compute the entries of $RH_r^*R^{-1}$ using the binomial theorem. The last row of $H_r^*R^{-1}$ contains zeroes. The i-j entry of $H_r^*R^{-1}$ for $i < r$ is

$$\sum_{k=1}^{r} (H_r^*)_{i,k}(R^{-1})_{k,j} = \sum_{k=1}^{r} \binom{i+k}{k-1}(-1)^{j-k}\binom{r+1}{j-k} =$$

$$\sum_{k=0}^{r-1} \binom{i+1+k}{k}(-1)^{j-1-k}\binom{r+1}{j-1-k}.$$

This is the coefficient of x^{j-1} in

$$\frac{1}{(1-x)^{i+2}}(1-x)^{r+1} = (1-x)^{r-i-1},$$

namely

$$(-1)^{j-1}\binom{r-i-1}{j-1}.$$

The i-j entry of $RH_r^* R^{-1}$ is

$$(-1)^{j-1}\sum_{k=1}^{r-1}\binom{r+k-i}{k-i}\binom{r-k-1}{j-1}.$$

The non-zero terms come from $i \leq k \leq r-j$. So we will rewrite this as

$$(-1)^{j-1}\sum_{k=i}^{r-j}\binom{r+k-i}{k-i}\binom{r-k-1}{j-1} = (-1)^{j-1}\sum_{k=0}^{r-i-j}\binom{r+k}{k}\binom{r-k-i-1}{j-1} =$$

$$(-1)^{j-1}\sum_{k=0}^{r-i-j}\binom{r+k}{k}\binom{r-k-i-1}{r-k-i-j}.$$

We recognize this as the coefficient of x^{r-i-j} in

$$(-1)^{j-1}\frac{1}{(1-x)^{r+1}}\frac{1}{(1-x)^{j}} = \frac{1}{(1-x)^{r+j+1}},$$

which is

$$(-1)^{j-1}\binom{2r-i}{r-i-j}.$$

Hence

$$I - M_r = \left((-1)^j\binom{2r-i}{r-i-j}\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq r}}$$

and

$$V_r = R\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} = \begin{bmatrix}\binom{2r-1}{r-1}\\\vdots\\\binom{2r-r}{r-r}\end{bmatrix},$$

therefore we obtain

$$K_r = \left((-1)^j \binom{2r-i}{r-i-j}\right)_{\substack{0 \le i \le r \\ 0 \le j \le r}}$$

as described on page 78 of these notes.

At this point we just need to check that

$$K_r \begin{bmatrix} L_{0r} \\ L_{1r} \\ \vdots \\ L_{rr} \end{bmatrix} = \begin{bmatrix} L_{0r} \\ L_{1r} \\ \vdots \\ L_{rr} \end{bmatrix},$$

where

$$L_{ir} = \binom{r+i}{i} \frac{(2r-i)!}{(r-i)!} \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j+1)!} = \binom{r+i}{i} \binom{2r-i}{r-i} r! \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j+1)!}.$$

Since every L_{ir} has the same factor $r! \prod_{j=0}^{r-1} \frac{(3j+1)!}{(r+j+1)!}$, it will suffice to show that the column vector

$$\begin{bmatrix} \binom{r+0}{0}\binom{2r-0}{r-0} \\ \vdots \\ \binom{r+r}{r}\binom{2r-r}{r-r} \end{bmatrix}$$

is an eigenvector of K_r with eigenvalue 1. This amounts to verifying

$$\sum_{k=0}^{r} (-1)^k \binom{2r-p}{r-p-k} \binom{r+k}{k} \binom{2r-k}{r-k} = \binom{r+p}{p} \binom{2r-p}{r-p}$$

for $0 \le p \le r$. Note that

$$\binom{2r-p}{r-p-k} \binom{r+k}{k} = \binom{2r-p}{r-p} \binom{r-p}{k} = \frac{(2r-p)!}{(r-p-k)!r!k!}$$

Therefore we must verify

$$\sum_{k=0}^{r} (-1)^k \binom{r-p}{k} \binom{2r-k}{r-k} = \binom{r+p}{p}.$$

This identity is true: it is the result of comparing the coefficient of x^r in both sides of the equation

$$(1-x)^{r-p}\frac{1}{(1-x)^{r+1}} = \frac{1}{(1-x)^{p+1}}.$$

Exercises: The following exercises will take you through the q-analogue of this process, which will lead to a proof of Andrews' conjecture,

$$\det\left(\delta_{ij} + q^{i+1}\begin{bmatrix} i+j \\ j-1 \end{bmatrix}_q\right)_{1\le i,j\le r} = \prod_{1\le i\le j\le r+1}\frac{1-q^{r+i+j}}{1-q^{2i+j-1}}.$$

The first thing to note is that now

$$R = \left(q^{i(r+1)}\begin{bmatrix} r+j-i \\ j-i \end{bmatrix}\right)_{1\le i,j\le r}.$$

Exercise 1. Using the q-binomial theorem of page 36 of these notes,

$$(1+x)(1+xq)\cdots(1+xq^{n-1}) = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_j q^{i(i-1)/2}x^i,$$

and using the other q-binomial theorem

$$\sum_{k=0}^{\infty}\begin{bmatrix} n-1+k \\ k \end{bmatrix} x^k = \frac{1}{(1-x)(1-xq)\cdots(1-xq^{n-1})},$$

prove that

$$R^{-1} = \left((-1)^{j-i}q^{\frac{(j-i)(j-i-1)}{2}-j(r+1)}\begin{bmatrix} r+1 \\ j-i \end{bmatrix}\right).$$

Exercise 2. The next step is to simplify the calculation of the i-j entry of $H_r^*R^{-1}$ for $i<r$, namely

$$\sum_{k=1}^{r}(H_r^*)_{ik}(R^{-1})_{kj} = \sum_{k=1}^{r} q^{i+1}\begin{bmatrix} i+k \\ k-1 \end{bmatrix}(-1)^{j-k}q^{\frac{(j-k)(j-k-1)}{2}-j(r+1)}\begin{bmatrix} r+1 \\ j-k \end{bmatrix}.$$

Show that this is equal to the coefficient of x^{j-1} in

$$q^{i+1-j(r+1)}(1-xq^{i+2})\cdots(1-xq^{r}),$$

namely

$$(-1)^{j-1}q^{\frac{j(j-1)}{2}-jr+ij}\begin{bmatrix} r-i-1 \\ j-1 \end{bmatrix}.$$

Exercise 3. Show that the i-j entry of $RH_r^*R^{-1}$ can be interpreted as the coefficient of x^{r-i-j} in

$$(-1)^{j-1}q^{(i-j)(r+1)+\frac{j(j+1)}{2}+ij}\frac{1}{(1-x)(1-xq)\cdots(1-xq^{r+j})},$$

hence

$$RH_r^*R^{-1} = \left((-1)^{j-1}q^{(i-j)(r+1)+\frac{j(j+1)}{2}+ij}\begin{bmatrix} 2r-i \\ r-i-j \end{bmatrix}\right).$$

Exercise 4. Explain why we can choose

$$K_r = \left((-1)^{j}q^{(i-j)(r+1)+\frac{j(j+1)}{2}+ij}\begin{bmatrix} 2r-i \\ r-i-j \end{bmatrix}\right)_{\substack{0\le i\le r \\ 0\le j\le r}}.$$

Exercise 5. For $0 \le k \le r$ let

$$v_{kr} = q^{k(r+1)}\begin{bmatrix} r+k \\ k \end{bmatrix}\begin{bmatrix} 2r-k \\ r-k \end{bmatrix}.$$

Explain why

$$K_r\begin{bmatrix} v_{0r} \\ v_{1r} \\ \vdots \\ v_{rr} \end{bmatrix} = \begin{bmatrix} v_{0r} \\ v_{1r} \\ \vdots \\ v_{rr} \end{bmatrix}.$$

Exercise 6. Explain why

$$\sum_{k=0}^{r} v_{kr} = \begin{bmatrix} 3r+1 \\ r \end{bmatrix}.$$

Exercise 7. Set

$$L_{ir}(q) = L_{r-1}(q)\frac{v_{ir}}{v_{0r}}$$

and

$$L_r(q) = \sum_{i=0}^{r} L_{ir}(q).$$

Show that this implies the recurrence relation

$$L_r(q) = \frac{\begin{bmatrix} 3r+1 \\ r \end{bmatrix}}{\begin{bmatrix} 2r \\ r \end{bmatrix}} L_{r-1}(q).$$

Exercise 8. Prove that the recurrence relation in exercise 7 implies Andrews conjecture, namely

$$\det\left(\delta_{ij} + q^{i+1}\begin{bmatrix} i+j \\ j-1 \end{bmatrix}_q\right)_{1\le i,j\le r} = L_r(q) = \prod_{1\le i\le j\le r+1} \frac{1-q^{r+i+j}}{1-q^{2i+j-1}}.$$

Chapter 6: Explorations

In this chapter we finally get to the proof of the alternating sign matrix conjecture. We count totally symmetric self-complementary plane partitions using a lattice-path argument, expressing the result as a Pfaffian (a cousin to the determinant) of binomial coefficients (pages 209–211). The Pfaffian can be expressed as the square root of a determinant, and Andrews evaluated this determinant (page 211). There is a one-to-one correspondence between TSSCPs and magog triangles. There are the same number of magog trapezoids as there are gog trapezoids. Magog trapezoids are a generalization of magog triangles, and gog trapezoids are a generalization of monotone triangles. So there are the same number of magog triangles as there are monotone triangles (proved by Zeilberger – see pages 215–220). Monotone triangles are in one-to-one correspondence with alternating sign matrices (pp. 57–58). So there are the same number of alternating sign matrices as there are TSSCPs. The formula for the number of TSSCPs is the same as the formula for the number of descending plane partitions (Andrew's determinant evaluation referenced on page 211). The formula for the number of descending plane partitions is equal to the conjectured formula for the number of alternating sign matrices.

Section 6.1: Charting the territory

There is a lot of material here. We will focus our attention on the hidden symmetry in descending plane partitions described on pages 194–195. Thinking about this symmetry motivated David Robbins to start counting totally symmetric self-complementary plane partitions (see Figure 6.1, page 196), which led him to Conjecture 12 (about which more below).

A hidden symmetry

Conjecture 8 claims that there is a correspondence between alternating sign matrices and descending plane partitions. Conjecture 10 (to date still unproven) is a more refined version this correspondence:

Conjecture 10 (Mills, Robbins, Rumsey 1983): *Let $A(n, k, m, p)$ be the number of $n \times n$ alternating sign matrices with a 1 in the kth column of the first row, with m -1s, and with inversion number equal to p. Let $D(n, k, m, p)$ be the number of descending plane partitions with largest part $\leq n$, and with exactly $k - 1$ parts of size n, with m special parts, and with a total of p parts. We then have that*

$$A(n, k, m, p) = D(n, k, m, p).$$

The inversion number of an alternating sign matrix is defined on page 59 of these notes. An example of a descending plane partition is given at the top of page 195 of the textbook:

$$\begin{array}{cccccc} 7 & 6 & 6 & 5 & 4 & 4 \\ & 5 & 5 & 3 & 3 & - \\ & & 3 & 2 & - & - \\ & & & - & - & - \\ & & & & - & - \\ & & & & & - \end{array}$$

We can see that this meets the definition of a descending plane partition because there is weak decrease across rows, strict decrease along columns, the number of parts in each row is strictly less than the first part in each row, and the first part in each row below the first is less than or equal to the number of parts in the row above.

The definition of special parts in a descending plane partition is given on page 192 of the textbook: those parts $a_{i,j}$ which satisfy $a_{i,j} \leq j - i$. We can visualize the special parts as follows: we will compare this to the corresponding diagram of $j - i$ values. We will use boldface to mark locations of special parts. We can see that the descending plane partition has 3 special parts:

$$\begin{array}{cccccc} 7 & 6 & 6 & 5 & \mathbf{4} & \mathbf{4} \\ & 5 & 5 & 3 & \mathbf{3} & - \\ & & 3 & 2 & - & - \\ & & & - & - & - \\ & & & & - & - \\ & & & & & - \end{array} \quad \text{versus} \quad \begin{array}{cccccc} 0 & 1 & 2 & 3 & \mathbf{4} & \mathbf{5} \\ & 0 & 1 & 2 & \mathbf{3} & 4 \\ & & 0 & 1 & 2 & 3 \\ & & & 0 & 1 & 2 \\ & & & & 0 & 1 \\ & & & & & 0 \end{array}$$

Since the largest part is $n = 7$ and there are exactly $k - 1 = 1$ parts of size 7 and there are $m = 3$ special parts, we would expect that there is a unique 7×7 alternating sign matrix corresponding to this having a 1 in the 2^{nd} column of the first row and containing 3 -1s.

Much earlier in the course (Exercise 2.2.1) you found a correspondence between self-conjugate integer partitions of n and partitions of n into distinct odd parts. This correspondence implies that there are the same number of each. To date nobody has identified an explicit correspondence between alternating sign matrices and descending plane partitions. The content of the alternating sign conjecture is that there are the same number of each. Anyone looking for a correspondence should make sure that it does not contradict Conjecture 10 (which undoubtedly is supported by strong numerical evidence).

Alternating sign matrices have a lot of symmetries. For example, if (a_{ij}) is an $n \times n$ alternating sign matrix with 1 in the k^{th} column of row one, m -1s, and inversion number p, then $(a_{i,n-j+1})$, its reflection about the central vertical axis, is an $n \times n$ alternating sign matrix with a 1 in column $n - k + 1$, m -1s, and inversion number $\binom{n}{2} + m - p$. (You will prove this in Exercise 5.) It stands to reason that there should be a way to reflect a descending plane partition with largest part $\leq n$, $k - 1$ parts of size n, m special parts, and p parts overall into another descending plane partition with largest part

$\leq n$, $n-k$ parts of size n, m special parts, and $\binom{n}{2}+m-p$ parts overall. Such a reflection is described on pages 194 and 195 of the textbook. You will verify its properties in the exercises. The algorithm for this reflection has a complicated description, but we can visualize it as follows:

Consider the descending plane partition

$$\begin{array}{cccccc} 7 & 6 & 6 & 5 & \mathbf{4} & \mathbf{4} \\ & 5 & 5 & 3 & \mathbf{3} & - \\ & & 3 & 2 & - & - \\ & & & - & - & - \\ & & & & - & - \\ & & & & & - \end{array}$$

which we examined above. Represent this by its matrix of entries (a_{ij}). The reflected descending plane partition is created in two steps. First create the complementary array (b_{ij}). Then create the mirror image of (b_{ij}), namely $(b_{n-j,n-i})$. Our goal is to visualize the complementary array (b_{ij}), given its description at the bottom of page 194 of the textbook. The result is the array on the right-hand side at the top of page 195.

First, we describe how to create those entries b_{ij} corresponding to the special parts of (a_{ij}):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & \mathbf{6} \\ & 1 & 2 & 3 & \mathbf{4} & 5 \\ & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix} - \begin{pmatrix} 7 & 6 & 6 & 5 & \mathbf{4} & \mathbf{4} \\ & 5 & 5 & 3 & \mathbf{3} & - \\ & & 3 & 2 & - & - \\ & & & - & - & - \\ & & & & - & - \\ & & & & & - \end{pmatrix} = \begin{pmatrix} * & * & * & * & \mathbf{1} & \mathbf{2} \\ & * & * & * & \mathbf{1} & * \\ & & * & * & * & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & & * \end{pmatrix}.$$

Next, we describe how to create those entries b_{ij} corresponding to blank positions in (a_{ij}). The formula is

$$b_{ij} = j + 1 - \beta_{ij}.$$

Therefore we are computing

$$\begin{pmatrix} - & - & - & - & - & - \\ & - & - & - & - & b_{2,6} \\ & & - & - & b_{3,5} & b_{3,6} \\ & & & b_{4,4} & b_{4,5} & {}_{4,6} \\ & & & & b_{5,5} & b_{5,6} \\ & & & & & b_{6,6} \end{pmatrix} =$$

$$\begin{pmatrix} - & - & - & - & - & - \\ & - & - & - & - & 7 \\ & & - & - & 6 & 7 \\ & & & 5 & 6 & 7 \\ & & & & 6 & 7 \\ & & & & & 7 \end{pmatrix} - \begin{pmatrix} - & - & - & - & - & - \\ & - & - & - & - & \beta_{2,6} \\ & & - & - & \beta_{3,5} & \beta_{3,6} \\ & & & \beta_{4,4} & \beta_{4,5} & {}_{4,6} \\ & & & & \beta_{5,5} & \beta_{5,6} \\ & & & & & \beta_{6,6} \end{pmatrix}.$$

To compute β_{ij}, count the number of non-negative entries in the i^{th} column of (a_{ij}) minus the column

$$\begin{bmatrix} j+2 \\ j+1 \\ \vdots \\ j-n+3 \end{bmatrix}.$$

For example, to compute $\beta_{4,5}$ we calculate

$$\begin{bmatrix} 5 \\ 3 \\ 2 \\ - \\ \\ \\ \end{bmatrix} - \begin{bmatrix} 7 \\ 6 \\ 5 \\ - \\ \\ \\ \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -3 \\ - \\ \\ \\ \end{bmatrix},$$

therefore $\beta_{4,5} = 0$. Hence $b_{4,5} = 6 - 0 = 6$.

Exercises: 2, 5, 6, 7, 8.

Section 6.2: Totally symmetric self-complementary plane partitions

We have seen examples of symmetric plane partitions (invariance with respect to reflection across the plane $x = y$), cyclically symmetric plane partitions (invariance with respect to cyclic permutation of axes), and totally symmetric plane partitions (invariance with respect to transposition of any two axes). Another type of symmetry is invariance with respect to complementation. Namely, a plane partition $\mathcal{P}$ which lives inside $\mathcal{B}(r,s,t)$ is self-complementary if and only if the set complement $\mathcal{B}(r,s,t)-\mathcal{P}$ can be interpreted as a plane partition (by rotating it into the correct position). For example, the first plane partition at the top of page 196 is

$$\begin{bmatrix} 6 & 6 & 6 & 5 & 4 & 3 \\ 6 & 6 & 5 & 3 & 3 & 2 \\ 6 & 5 & 5 & 3 & 3 & 1 \\ 5 & 3 & 3 & 1 & 1 & \\ 4 & 3 & 3 & 1 & & \\ 3 & 2 & 1 & & & \end{bmatrix},$$

which lives in $\mathcal{B}(6,6,6)$. If we subtract each entry from 6, we will be counting the number of empty spaces above each x-y coordinate. We obtain

$$\begin{bmatrix} & & & 1 & 2 & 3 \\ & & 1 & 3 & 3 & 4 \\ & 1 & 1 & 3 & 3 & 5 \\ 1 & 3 & 3 & 5 & 5 & 6 \\ 2 & 3 & 3 & 5 & 6 & 6 \\ 3 & 4 & 5 & 6 & 6 & 6 \end{bmatrix},$$

which is an inverted version of what we started with.

As we mentioned above, thinking about symmetries of descending plane partitions led David Robbins to start counting totally symmetric self complementary plane partitions. I imagine he was thunderstruck when he discovered, in every case he was able to check, that the number of totally symmetric self-complementary plane partitions in $\mathcal{B}(2n,2n,2n)$ is equal to the number of $n\times n$ alternating sign matrices. This is the content of Conjecture 12 (page 195).

In order to count totally symmetric self-complementary plane partitions, we need an efficient way to represent them. The example we considered above is totally symmetric and self complementary. Since the $x=k$ slice is equal to the $y=k$ slice and the $z=k$ slice, all the information about this plane partition is contained in its z-slices. Moreover, since the plane partition is self-complementary, it can be reconstructed from the top half of its z-slices. There is an illustration in Figure 6.5, page 205, of this idea.

The $z = 5$ through 10 slices of the totally symmetric self-complementary plane partition in Figure 6.3, page 204, are shown. Each z-slice can be viewed as the Ferrers diagram of an integer partition. Since the plane partition is totally symmetric, each Ferrers diagram is self-conjugate. We know that every self-conjugate integer partition can be represented by a partition of distinct odd numbers. In Figure 6.5, each z-slice is encoded by such a partition. These partitions are assembled into the array of numbers at the bottom of page 205.

The properties of this array of numbers are described at the top of page 206. We can summarize them as follows: upper triangular, all positive entries are odd numbers, diagonal entries are $2n-1$, $2n-3$, ..., 1, strict decrease across rows and weak increase down columns. There is a one-to-one correspondence between TSSCPP's of order n (largest part $2n-1$) and TSSCPP's living in $\mathcal{B}(2n, 2n, 2n)$.

The corresponding nest of lattice paths

A TSSCPP array of order n can be represented by a nest of lattice paths. See for example the correspondence on page 207. Row i, $1 \leq i \leq n-1$, is represented by a lattice path beginning at position $(-2(n-1-i), n-1-i)$ and ending on the line $y = -x + 1$. The lattice path corresponding to

$$(2(n-i+1)-1) + (2\lambda_2 - 1) + \cdots + (2\lambda_k - 1)$$

will pass through $k-1$ points

$$(\alpha_i, \beta_i - \lambda_2), (\alpha_i + 1, \beta_i - 1 - \lambda_3), \ldots, (\alpha_i + k - 2, \beta_i - k + 2 - \lambda_k),$$

where $\alpha_i = -2(n-1-i)$ and $\beta_i = -\alpha_i + 1$. These coordinates are respectively λ_2, λ_3, ..., λ_k units below the line $y = -x + 1$. To complete the lattice path, rise to the line $y = -x + 1$.

In order for such a lattice path to exist, we need the y-coordinates to increase, i.e.

$$\beta_i - j - \lambda_{j+2} \leq \beta_i - j - 1 - \lambda_{j+3}$$

for $0 \leq j \leq k-3$, and we need the last y-coordinate to be below the line $y = -x + 1$, i.e.

$$\beta_i - k + 2 - \lambda_k \leq -\alpha_i - k + 2 + 1.$$

But the first inequality is equivalent to

$$\lambda_{j+3} \leq \lambda_{j+2} - 1,$$

and the second inequality is equivalent to $\lambda_k \geq 1$, both of which are true.

The properties of the TSSCPP array guarantee that the lattice paths can be constructed and that none of them will intersect. As a technical matter we include an additional lattice path from $(2, -1)$ to $(2, -1)$ if necessary to ensure that there are an even number of lattice paths in the nest. There is a one-to-one correspondence between nests of lattice paths and TSSCPPs.

At this point it would be tempting to express the generating function for TSSCPPs corresponding to lattice paths which have a fixed set of ending points in the form of a determinant of q-binomial coefficients, then show by means of a sign-reversing involution that the terms corresponding to intersecting lattice paths cancel out. I suspect this can be done. I also suspect that when we try to sum all the determinants together, this expression does not easily collapse into a single determinant. The book takes another approach using Pfaffians (about which more below). Moreover, the author of the book does not derive the generating function for TSSCPPs, just the formula for the number of TSSCPPs in $\mathcal{B}(2n, 2n, 2n)$, which is found by letting $q \to 1$ in the generating function. So we are going to be dealing with a Pfaffian of binomial coefficients.

Pfaffians

1-factors are to Pfaffians as permutations are to determinants. The determinant of the $n \times n$ matrix A is

$$\det(A) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\mathcal{I}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

For an even number n, the Pfaffian (named after Gauss's teacher Pfaff) of a collection of numbers $A = \{a_{i,j} | 1 \leq i < j \leq n\}$ is

$$\mathrm{Pf}(A) = \sum_{F \in \mathcal{F}_n} (-1)^{\mathcal{X}(F)} \prod_{(i,j) \in F} a_{i,j}.$$

The set $\mathcal{F}_n$ consists of all partitions of the set $\{1, 2, \ldots, n\}$ into disjoint subsets of size 2. For example, $\mathcal{F}_3 = \{F_1, F_2, F_3\}$, where $F_1 = \{(1,2), (3,4)\}$, $F_2 = \{(1,3), (2,4)\}$, and $F_3 = \{(1,4), (2,3)\}$. The total number of 1-factors of a $2k$-element set is

$$f_{2k} = (2k-1)f_{2k-2} = (2k-1)(2k-3)f_{2k-4} = \cdots = \prod_{i=1}^{k} (2i-1).$$

The logic behind this counting argument is that there are $2k-1$ numbers that 1 can be paired with, and then we have to pair off the remaining $2k-2$ numbers.

Given a 1-factor $F \in \mathcal{F}_n$, the crossing number of F is $\mathcal{X}(F) =$ the number of pairs (a,b), (c,d) in F such that $a < c < b < d$. The Pfaffian of

$$A = \{a_{i,j} | 1 \leq i < j \leq 4\}$$

will have three terms, corresponding to the three 1-factors F_1, F_2, F_3. Since $\mathcal{X}(F_1) = \mathcal{X}(F_3) = 0$ and $\mathcal{X}(F_2) = 1$, we have

$$\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

It is a fact that the square of a Pfaffian can be expressed as the determinant of an appropriately defined skew-symmetric matrix. You will prove this in Exercise 11. See the example at the bottom of page 208. So now we are anticipating that the number of TSSCPP's will be expressed as the square root of a determinant.

TSSCPP as Pfaffian

We are tracing through the description on page 209. I am going to slightly modify the proof to make it clearer. Let $\mathcal{N}_n$ denote the set of nests of lattice paths of order n beginning at the the coordinates $(2-2i, 2-1)$ for a certain range of values of i (see page 209) and ending on the line $y = 1 - x$ *in which no two paths end in the same vertex.* (In the book, lattice paths in $\mathcal{N}_n$ are allowed to arrive at the same vertex. But the weight of any nest with two paths arriving at the same vertex is set to zero. So it is not necessary to include these. The proof still works.) For example, let $n = 5$. Then there will be 4 lattice paths. Path 1 begins at $(0,0)$, Path 2 begins at $(-2,1)$, Path 3 begins at $(-4,2)$, Path 4 begins at $(-6,3)$. The paths must end on the line $y = -x + 1$. An example of a nest is

$$N = (P_1, P_2, P_3, P_4) = (0, 00, 111, 1001),$$

where we have encoded the paths by binary strings. If we denote by

$$(x_k, 1 - x_k)$$

the last vertex of path P_k, then we have

$$(x_1, x_2, x_3, x_4) = (0, -2, -1, -4).$$

There are 3 perfect matchings (1-factors) of these 4 indices:

$$F_1 = \{(1,2), (3,4)\},$$

$$F_2 = \{(1,3), (2,4)\},$$
$$F_3 = \{(1,4), (2,3)\}.$$

This induces a perfect matching of x_1 through x_4:

$$f_1 = \{(x_1, x_2), (x_3, x_4)\} = \{(0, -2), (-1, -4)\},$$

$$f_2 = \{(x_1, x_3), (x_2, x_4)\} = \{(0, -1), (-2, -4)\},$$
$$f_3 = \{(x_1, x_4), (x_2, x_3)\} = \{(0, -4), (-2, -1)\}.$$

Note that when $i < j$, P_i begins to the right of P_j. If they cross each other an even number of times, then P_i ends to the right of P_j (same relative positions). But if they cross each other an odd number of times, then P_i ends to the left of P_j. We can detect an odd number of crossings by the relationship $x_i < x_j$.

According to the book, the inversion number of (F, N) is the number of pairs $(x_i, x_j) \in f$ such that $i < j$ and $x_i < x_j$. So by my count we have

$$\mathcal{I}(F_1, N) = 0,$$

$$\mathcal{I}(F_2, N) = 0,$$
$$\mathcal{I}(F_3, N) = 1.$$

It is easy to see that

$$(-1)^{\mathcal{I}(F,N)} = \prod_{(i,j)\in F} (-1)^{\chi(x_i < x_j)},$$

where

$$\chi(x_i < x_j) = \begin{cases} 1 & \text{if } x_i < x_j \\ 0 & \text{if } x_i > x_j. \end{cases}$$

Having defined $\mathcal{I}(F,N)$, we set

$$w(F,N) = (-1)^{\chi(F)+\mathcal{I}(F,N)}.$$

Having defined $w(F,N)$ for each 1-factor F of the nest of lattice paths, we define the weight of the entire nest as

$$w(N) = \sum_F w(F,N).$$

Since $\mathcal{X}(F_1) = 0$, $\mathcal{X}(F_2) = 0$, $\mathcal{X}(F_3) = 1$, $\mathcal{I}(F_1,N) = 0$, $\mathcal{I}(F_2,N) = 0$, $\mathcal{I}(F_3,N) = 1$, and no two of the lattice paths have the same ending point, we have

$$w(N) = (-1)^{0+0} + (-1)^{0+0} + (-1)^{1+1} = 3.$$

In general, if a nest N has no crossing lattice paths, then $\mathcal{I}(F,N) = 0$ for all 1-factors F, hence $w(F,N) = 1$ for all F and

$$w(N) = \sum_F (-1)^{\chi(F)} = 1,$$

since the Pfaffian of a set of 1s is equal to 1 by Exercise 9. We know that there is a one-to-one correspondence between non-crossing nests of Pfaffians of order n and TSSCPPs of order n. Therefore the number of TSSCPPs is

$$\sum_{N \in \mathcal{N}_n \text{ is non-crossing}} w(N).$$

In what follows we will show that if $N \in \mathcal{N}_n$ is a nest of lattice paths which contains at least one pair of crossing paths, then there is a unique nest $N' \in \mathcal{N}_n$ which also contains a pair of crossing paths and such that $w(N') = -w(N)$. In other words, there is a sign-reversing involution which maps crossing nests to other crossing nests. Therefore

$$\sum_{N\in\mathcal{N}_n} w(N) = \sum_{N \in \mathcal{N}_n \text{ is non-crossing}} w(N)$$

because the terms corresponding to crossing nests cancel out.

The advantage of passing from

$$\sum_{N \in \mathcal{N}_n \text{ is non-crossing}} w(N)$$

to

$$\sum_{N \in \mathcal{N}_n} w(N)$$

is that the latter expression can be recognized as a Pfaffian. We have

$$\sum_{N \in \mathcal{N}_n} w(N) = \sum_{N \in \mathcal{N}_n} \left(\sum_F w(F, N) \right) = \sum_F \left(\sum_{N \in \mathcal{N}_n} w(F, N) \right).$$

We will show later that there exists a collection of numbers

$$H = \{H(i, j) : i < j\}$$

such that

$$\sum_{N \in \mathcal{N}_n} w(F, N) = (-1)^{\chi(F)} \prod_{(i,j) \in F} H(i, j)$$

for each matching F. This will imply that

$$\sum_F \left(\sum_{N \in \mathcal{N}_n} w(F, N) \right) = \sum_F (-1)^{\chi(F)} \prod_{(i,j) \in F} H(i, j) = \mathrm{Pf}(H).$$

Hence the number of TSCPPs of order n is $\mathrm{Pf}(H)$.

We will use our example above to illustrate the sign-reversing involution on nests with crossing lattice paths. Given a nest N which contains crossing paths, we must pick out one of the intersection points in a unique way, then switch tails as we have before. This intersection point is found as follows (see pp 209-210): "If any two paths intersect, then we choose the point of intersection that is closest to one of the starting points, and among those points at the same distance we choose the one with x coordinate closest to 0."

Our example has exactly one intersection point, so it is too small to illustrate the tie-breaking rule described above. But suffice it to say that the method by which we find the intersection point guarantees that the intersecting paths will be of the form P_i and P_{i+1} – we made a similar argument on page 44 of these notes. Starting with the nest N and the 1-factor F, we create the new nest N' and the new 1-factor F' by switching the tails in P_i and P_{i+1} and switching i and $i+1$ in F.

Let $N = (0, 00, 111, 1001)$ as above. We can see that paths 2 and 3 intersect after 1 step along path 2 and after 2 steps along path 3, and this is the unique intersection point. We have

$$P_2 = 0/0$$

and

$$P_3 = 11/1,$$

where / marks the the dividing line before and after the point of intersection. The tails occur after /. Switching these tails, and leaving the other paths unchanged, we arrive at

$$N' = (0, 01, 110, 1001).$$

When N is paired with F, we send (F, N) to (F', N') by swapping the indices 2 and 3 in F to produce F'. Hence

$$((0, 00, 111, 1001), \{(1, 2), (3, 4)\}) \longrightarrow ((0, 01, 110, 1001), \{(1, 3), (2, 4)\}),$$

$$((0, 00, 111, 1001), \{(1, 3), (2, 4)\}) \longrightarrow ((0, 01, 110, 1001), \{(1, 2), (3, 4)\}),$$

$$((0, 00, 111, 1001), \{(1, 4), (2, 3)\}) \longrightarrow ((0, 01, 110, 1001), \{(1, 4), (2, 3)\}).$$

We will now argue that that $w(N', F') = -w(N, F)$ in every case. This is the same as checking that

$$(-1)^{\chi(F')+\mathcal{I}(N',F')} = -(-1)^{\chi(F)+\mathcal{I}(N,F)}.$$

This will occur provided we can show that either the matching number or the inversion number changes by 1 when we perform a tail switch, while the other value remains the same. There are two cases to consider. We will give the general argument, not just the argument for our example.

Case 1: i and $i+1$, the indices corresponding to crossing paths P_i and P_{i+1}, occur in the same pair in F. Then the inversion number changes by 1, while the crossing number remains the same. This corresponds to the example

$$((0, 00, 111, 1001), \{(1,4), (2,3)\}) \longrightarrow ((0, 01, 110, 1001), \{(1,4), (2,3)\}).$$

We can see that $F' = F$ and $\chi(F') = \chi(F)$. To prove that $\mathcal{I}(N', F') = \mathcal{I}(N, F) \pm 1$, we just note that the only difference between $\mathcal{I}(N, F)$ and $\mathcal{I}(N', F')$ is in the terms $\chi(x_i < x_{i+1})$ and $\chi(x'_i < x'_{i+1}) = \chi(x_{i+1} < x_i)$. Therefore

$$\mathcal{I}(N', F') = \begin{cases} \mathcal{I}(N, F) - 1 & \text{if } x_i < x_{i+1} \\ \mathcal{I}(N, F) + 1 & \text{if } x_i > x_{i+1} \end{cases}$$

Case 2: i and $i+1$ occur in different pairs in F. Say that a and i occur in one pair and b and $i+1$ occur in a second pair in F. In principle we do not know which index is the larger one in each pair, so there are 6 subcases to check.

Case 2.1: $a < i < i+1 < b$. An example of this is $i = 2$, $i+1 = 3$, $a = 1$, $b = 4$ in

$$((0, 00, 111, 1001), \{(1,2), (3,4)\}) \longrightarrow ((0, 01, 110, 1001), \{(1,3), (2,4)\}).$$

Then

$$(a, i) \to (a, i+1),$$
$$(i+1, b) \to (i, b).$$

The pair (a, i), $(i+1, b)$ contributes 0 to the crossing number of F and the pair $(a, i+1)$, (i, b) contributes 1 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) + 1$. Also,

$$\chi(x'_a < x'_{i+1}) = \chi(x_a < x_i)$$

and

$$\chi(x'_i < x'_b) = \chi(x_{i+1} < x_b),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Case 2.2: $b < i < i+1 < a$. An example of this is $i = 2$, $i+1 = 3$, $a = 4$, $b = 1$ in

$$((0, 00, 111, 1001), \{(1,3), (2,4)\}) \longrightarrow ((0, 01, 110, 1001), \{(1,2), (3,4)\}).$$

Then

$$(i, a) \to (i+1, a),$$

$$(b, i+1)) \to (b, i).$$

The pair $(b, i+1)$, (i, a) contributes 1 to the crossing number of F and the pair $(b, i))$, $(i+1, a)$ contributes 0 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) - 1$. Also,

$$\chi(x'_{i+1} < x'_a) = \chi(x_i < x_a)$$

and

$$\chi(x'_b < x'_i) = \chi(x_b < x_{i+1}),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Case 2.3: $a < b < i < i+1$. Then

$$(a, i) \to (a, i+1),$$

$$(b, i+1) \to (b, i).$$

The pair (a, i), $(b, i+1)$ contributes 1 to the crossing number of F and the pair $(a, i+1)$, (b, i) contributes 0 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) - 1$. Also,

$$\chi(x'_a < x'_{i+1}) = \chi(x_a < x_i)$$

and

$$\chi(x'_b < x'_i) = \chi(x_b < x_{i+1}),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Case 2.4: $b < a < i < i+1$. Then

$$(a, i), \to (a, i+1),$$

$$(b, i+1) \to (b, i).$$

The pair $(b, i+1)$, (a, i) contributes 0 to the crossing number of F and the pair (b, i), $(a, i+1)$ contributes 1 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) + 1$. Also,

$$\chi(x'_a < x'_{i+1}) = \chi(x_a < x_i)$$

and

$$\chi(x'_b < x'_i) = \chi(x_b < x_{i+1}),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Case 2.5: $i < i+1 < a < b$. Then

$$(i, a), \to (i+1, a),$$

$$(i+1, b)) \to (i, b).$$

The pair (i, a), $(i+1, b)$ contributes 1 to the crossing number of F and the pair (i, b), $(i+1, a)$ contributes 0 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) - 1$. Also,

$$\chi(x'_{i+1} < x'_a) = \chi(x_i < x_a)$$

and

$$\chi(x'_i < x'_b) = \chi(x_{i+1} < x_b),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Case 2.6: $i < i+1 < b < a$. Then

$$(i, a), \to (i+1, a),$$

$$(i+1, b) \to (i, b).$$

The pair (i, a), $(i+1, b)$ contributes 0 to the crossing number of F and the pair (i, b), $(i+1, a)$ contributes 1 to the crossing number of F'. No other pairs of ordered pairs are affected. Therefore $\chi(F') = \chi(F) + 1$. Also,

$$\chi(x'_{i+1} < x'_a) = \chi(x_i < x_a)$$

and

$$\chi(x'_i < x'_b) = \chi(x_{i+1} < x_b),$$

therefore $\mathcal{I}(F', N') = \mathcal{I}(F, N)$.

Our mapping sends (F, N) to (F', N'). We can see that this mapping is an involution because the set of intersection points in N' are the same as the set of intersection points in N. Applying the mapping to (F', N'), we switch the tails of P_i and P_{i+1} in N' and swap i and $i+1$ in F'. This brings us back to (F, N). We have

$$w(N) = \sum_F w(F, N) = \sum_{F'} (-w(F', N')) = -w(N').$$

Sum of weights as Pfaffian

We return to the demonstration that

$$\sum_{N \in \mathcal{N}_n} w(F, N) = (-1)^{\chi(F)} \prod_{(i,j) \in F} H(i, j)$$

for a fixed matching F and an appropriate H. This is essential a counting problem. For convenience we will just treat the case where n is odd. Then we know that lattice paths begin in the $n-1$ vertices $(0,0)$, $(-2,1)$, $(-4,2)$, ..., $(-2n+4, n-2)$ and end in $n-1$ vertices in the range $(0,1)$, $(-1,2)$, $(-2,3)$, ..., $(-2n+4, 2n-3)$. We will denote by $D(x_1, \ldots, x_{n-1})$ the set of lattice paths in which P_i ends in the vertex $(x_i, 1-x_i)$ for $1 \le i \le n-1$. Any lattice path which begins in position $(-2i+2, i-1)$ and ends in position $(x_i, 1-x_i)$ can be represented by a binary string with $x_i + 2i - 2$ 1s and $2 - x_i - i$ 0s. The number of such binary strings is $\binom{i}{2-x_i-i}$. In the book, the notation for this is $h(i, 2-x_i)$ where $h(i, r) = \binom{i}{r-i}$. Therefore, the number of lattice paths in $D(x_1, \ldots, x_{n-1})$ is

$$\prod_{i=1}^{n-1} h(i, 2-x_i) = \prod_{(i,j) \in F} h(i, 2-x_i) h(j, 2-x_j).$$

Given $N \in D(x_1, \ldots, x_{n-1})$, the weight of the pair (F, N) is

$$w(F, N) = (-1)^{\chi(F)} \prod_{(i,j) \in F} (-1)^{\chi(x_i < x_j)}.$$

This weight is the same for all $N \in D(x_1, \ldots, x_{n-1})$. Therefore

$$\sum_{N \in D(x_1, \ldots, x_{n-1})} w(F, N) =$$

$$(-1)^{\chi(F)} \prod_{(i,j)\in F} (-1)^{\chi(x_i<x_j)} h(i, 2-x_i) h(j, 2-x_j).$$

Letting x_i range through all possible values for each i, we obtain

$$\sum_{N\in\mathcal{N}_n} w(F,N) = (-1)^{\chi(F)} \prod_{(i,j)\in F} \left(\sum_{r<s} h(i,r)h(j,s) - h(i,s)h(i,r) \right).$$

If we set

$$H(i,j) = \sum_{r<s} h(i,r)h(j,s) - h(i,s)h(i,r),$$

then we have

$$\sum_{N\in\mathcal{N}_n} w(F,N) = (-1)^{\chi(F)} \prod_{(i,j)\in F} H(i,j).$$

Therefore the number of TSSCPPs of order n is $\mathrm{Pf}(H)$ when n is odd.

The determinant evaluation

The Pfaffian $\mathrm{Pf}(H)$ can be expressed as the square root of an associated determinant. You will demonstrate this in Exercise 11. George Andrews evaluated this determinant. Apparently this was not easy, or, as Andrews put it, "inordinately complicated." See page 211 of the textbook. The proof is omitted. But Andrews determinant evaluation proved the following theorem:

Theorem: *The number of TSSCPPs in $\mathcal{B}(2r, 2r, 2r)$ is equal to the number of descending plane partitions with largest part $\leq r$. (See page 199.)*

Exercises: 1, 2, 3, 4, 5, 7, 9, 10, 11, 12

Section 6.3: Proof of the ASM conjecture

Each TSCPP of order in $\mathbf{B}(2n, 2n, 2n)$ is uniquely encoded by a nest of lattice paths, which is uniquely encoded by collection of shapes (Figure 6.6, page 216), which is uniquely encoded by a magog triangle of order n (Figure 6.7, page 217) whose properties are given on page 216. The last row of a magog triangle contains weakly increasing integers in which the j^{th} entry is $\leq j$. Alternating sign matrices are uniquely encoded by monotone triangles of order n (see page 58 and bottom of page 217). The north-west edge of numbers are weakly increasing and whose j^{th} entry is also $\leq$ j. This suggests that there might be a one-to-one correspondence between magog triangles and monotone triangles. More generally, we can define (n,k)-magog trapezoids (last k rows of an order n magog triangle) and (n,k)-gog trapezoids (first k northwest rows of an order n monotone triangle). Doron Zeilberger proved that there are always the same number of each. This proved the alternating sign matrix conjecture: the number of alternating sign matrices is equal to the number of monotone triangles is equal to the number of magog triangles is equal to the the number of TSSCPPs, the formula for which is the same as the formula for the number of descending plane partitions, the formula for which is equal to the conjectured formula for the number of alternating sign matrices. You will do an exercise to familiarize yourself with gog and magog trapezoids.

Constant term identities

Apparently Zeilberger did not provide a bijective proof that there are the same number of magog and gog trapezpoids. Instead, he counted each by identifying the numbers in question as constant terms of Laurent series. See pp. 218-219. You will do some exercises to familiarize yourself with constant term identities.

Exercises: 1, 2, 3, 4.

Chapter 7: Square Ice

Section 7.1: Insights from statistical mechanics

An 5×5 sheet of square ice is depicted at the top of page 225 in Figure 7.1. This is a 5×5 array of oxygen atoms, each of which is attached to exactly two hydrogen atoms. Any configuration of bonds is possible, subject to the following restrictions: no O in the first row is attached to an H above it, no O in the last row is attached to an H below it, every O in the first column is attached to an H to its left, and every O in the last column is attached to an H to its right.

It is easier to think about square ice if it is represented by the directed graph below it in Figure 7.1. Represent each O by a vertex. Think of an O as having four compass points around it: N, S, E, W (north, south, east, west). Each O will be bonded with H's in exactly 2 of these four compass points. A bond is represented by an arrow directed from the compass point towards the O. There are exactly 6 ways to choose 2 compass points out of 4, hence there are exactly 6 types of vertex in this figure: NS, NE, NW, SE, SW, EW. Hence the name six vertex model.

There is a one-to-one correspondence between $n \times n$ 6-vertex models and $n \times n$ alternating sign matrices. The book gives one description on pages 224-225, but I found this difficult use and have another way to think about it.

Look at the 6-vertex model on page 225. I will classify all vertices into two types: consistent and inconsistent. Consistent vertices have both horizontal arrows pointing in the same direction and both vertical arrows pointing in the same direction. Inconsistent vertices have both the horizontal arrows pointing in opposite directions and the vertical arrows pointing in the opposite directions. Vertices NE, NW, SE, and SW are all consistent. Vertices NS and EW are inconsistent.

We are going to create a 5×5 matrix $A = (a_{ij})$ which corresponds to the 6-vertex model on page 225. The rule is

$$a_{ij} = \begin{cases} 0 & \text{if vertex } v_{ij} \text{ is consistent} \\ \\ 1 & \text{if vertex } v_{ij} \text{ is inconsistent of type } EW \\ \\ -1 & \text{if vertex } v_{ij} \text{ is inconsistent of type } NS. \end{cases}$$

Then A must be an alternating sign matrix for the following reason: As you read the arrows across any row, direction changes alternate. The same is true down any column. Inconsistent vertices record direction changes. Therefore 1s alternate with -1s. The boundary conditions (direction of arrows along west and north face of the diagram) imply that the first non-zero entry in every row and column is equal to 1. Since the direction of the arrow at the end of every row and column is the opposite of the direction of the arrow at the beginning, there are an odd number of direction changes. This implies that there is one more 1 than -1 in any row and column. Therefore the row sums and column sums are all equal to 1.

Every alternating sign matrix corresponds to exactly one 6-vertex model. To create the 6-vertex model, first create inconsistent vertices corresponding to the 1s and the -1s. All the other vertices must be consistent, and their arrow directions are determined by the arrow directions of the inconsistent vertices and by the boundary conditions. The fact that we are starting with an alternating sign matrix guarantees that it is possible to complete the diagram, since direction changes alternate and the first and last non-zero entry in each row and column is equal to 1.

There are other representations of an alternating sign matrix, including nested lattice paths that are allowed to touch at corners, and 3-colorings of squares.

The weights

Each vertex in the 6-vertex model has an associated weight. These weights are given on page 228. Inconsistent vertices of type EW have weight z. Inconsistent vertices of type NS has weight z^{-1}. Consistent vertices of type NW and SE have weight

$$[z] = \frac{z - z^{-1}}{a - a^{-1}}.$$

Consistent vertices of type NE and SW have weight

$$[az] = \frac{az - (az)^{-1}}{a - a^{-1}}.$$

The total weight of a 6-vertex model is the product of the weights of all the vertices appearing in the model. Having defined the weight of a 6-vertex model, we set $Z_n(z, a)$ equal to the sum of the weights of all possible $n \times n$ models. That is,

$$Z_n(z, a) = \sum_M z^{n_{EW}} z^{-n_{NS}} [z]^{n_{NW}+n_{SE}} [az]^{n_{NE}+n_{SW}} =$$

$$z^n \sum_M [z]^{n_{NW}+n_{SE}} [az]^{n_{NE}+n_{SW}},$$

where the sum is taken over all $n \times n$ 6-vertex models M. $Z_n(z, a)$ can be viewed as a generating function for $n \times n$ alternating sign matrices.

We can obtain information about alternating sign matrices by choosing z and a carefully. First we need more information about how many of each type of vertex occurs in any given model. Let n_{XY} be the number of vertices of type XY. In the first row of edges, all the edges are pointing up. Since each row sum in the alternating sign matrix is equal to 1, the sum of the entries in the first $k-1$ rows is equal to $k-1$. Hence there are a net of $k-1$ direction changes along the edges from the first row to the k^{th} row. Therefore there are exactly $k-1$ edges pointing down in the k^{th} row of edges for each k. The total number of edges pointing down in the model is $1+2+\cdots+n$. Similarly, there are $1+2+\cdots+n$ edges pointing up in the model. Playing the same game with the column sums, there are $1+2+\cdots+n$ edges $1+2+\cdots+n$ edges pointing left and the same number pointing right.

All edges pointing north are incident to vertices of type XY, where $N \notin \{X, Y\}$. All edges pointing south are incident to vertices of type XY, where $S \notin \{X, Y\}$. All edges pointing east are incident to vertices of the type XY, where $E \in \{X, Y\}$. All edges pointing west are incident to vertices of type XY, where $W \in \{X, Y\}$. Since every row of the alternating sum matrix has row sum 1, the sum of all the entries in the alternating sign matrix is n. Therefore there are n more vertices of type EW than of type NS. This information leads to the system of equations

$$n_{SE} + n_{SW} + n_{EW} = \frac{n(n+1)}{2},$$

$$n_{NE} + n_{NW} + n_{EW} = \frac{n(n+1)}{2},$$

$$n_{NE} + n_{SE} + n_{EW} = \frac{n(n+1)}{2},$$

$$n_{NW} + n_{SW} + n_{EW} = \frac{n(n+1)}{2},$$

$$n_{EW} - n_{SW} = n.$$

The solution to this system of equations is

$$\begin{bmatrix} n_{NS} \\ n_{NE} \\ n_{NW} \\ n_{SE} \\ n_{SW} \\ n_{EW} \end{bmatrix} = \begin{bmatrix} s-n \\ r \\ \binom{n+1}{2} - r - s \\ \binom{n+1}{2} - r - s \\ r \\ s \end{bmatrix}$$

where r and s are integers which vary depending on the model. The weight of the 6-vertex model will be

$$z^{n_{EW}} z^{-n_{NS}} [z]^{n_{NW}+n_{SE}} [az]^{n_{NE}+n_{SW}} = z^n [z]^{n(n+1)-2r-2s} [az]^{2r}.$$

Note that $s-n$ can be interpreted as the number of -1s in the alternating sign matrix which corresponds to the model.

If we substitute $a = z^{-2}$ then we have $[z] = -\frac{1}{z+z^{-1}}$ and $[az] = \frac{1}{z+z^{-1}}$. Therefore

$$Z_n(z, z^{-2}) = z^n \sum_{B \in \mathcal{A}_n} \left(\frac{1}{z+z^{-1}} \right)^{n(n+1)-2N(B)-2n} =$$

$$\frac{z^n}{(z+z^{-1})^{n^2-n}} \sum_{B \in \mathcal{A}_n} (z+z^{-1})^{2N(B)}.$$

Let $\omega = e^{\frac{2\pi i}{3}}$. Then $\omega^3 = 1$ and $w^2 + w + 1 = 0$. Therefore $\omega^{-2} = \omega$ and $\omega + \omega^{-1} = -1$. Setting $z = \omega$ we have

$$Z_(\omega, \omega) = \omega^n A_n,$$

where A_n is the number of $n \times n$ alternating sign matrices. There is a direct path from these observations to a formula for A_n: see exercises 2.4.10, 7.2.6, 7.2.7, and 7.2.8.

In order to prove the refined alternating sign matrix conjecture, namely the formula for $A_{n,k}$, it is necessary to work with the expression $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$, whose formula is given in Theorem 7.1, page 229. $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$ is obtained by replacing z by $\frac{x_i}{y_j}$ when computing the weight of the vertex in row i, column j of the 6-vertex model. We think of $\frac{x_i}{y_j}$ as the label of this vertex. This more general formula can be used to derive $A_{n,k}$. For example, the weight of the 6-vertex model on page 225 is $R_1 R_2 R_3 R_4 R_5$, where

$$\begin{aligned}
R_1 &= \left[\frac{ax_1}{y_1}\right] \frac{x_1}{y_2} \left[\frac{x_1}{y_3}\right] \left[\frac{x_1}{y_4}\right] \left[\frac{x_1}{y_5}\right], \\
R_2 &= \frac{x_2}{y_1} \frac{y_2}{x_2} \left[\frac{ax_2}{y_3}\right] \frac{x_2}{y_4} \left[\frac{x_2}{y_5}\right], \\
R_3 &= \left[\frac{x_3}{y_1}\right] \frac{x_3}{y_2} \left[\frac{x_3}{y_3}\right] \frac{y_4}{x_3} \frac{x_3}{y_5},
\end{aligned}$$

$$
\begin{aligned}
R_4 &= \left[\frac{x_4}{y_1}\right]\left[\frac{x_4}{y_2}\right]\left[\frac{ax_4}{y_3}\right]\frac{x_4}{y_4}\left[\frac{ax_4}{y_5}\right], \\
R_5 &= \left[\frac{x_5}{y_1}\right]\left[\frac{x_5}{y_2}\right]\frac{x_5}{y_3}\left[\frac{ax_5}{y_4}\right]\left[\frac{ax_5}{y_5}\right].
\end{aligned}
$$

Theorem 7.1:

$$
Z_n(\overrightarrow{x};\overrightarrow{y};a) = \frac{\prod_{i=1}^n \frac{x_i}{y_i}\prod_{1\le i,j\le n}\left[\frac{x_i}{y_j}\right]\left[\frac{ax_i}{y_j}\right]}{\prod_{1\le i<j\le n}\left[\frac{x_i}{x_j}\right]\left[\frac{y_i}{y_j}\right]}\det\left(\frac{1}{\left[\frac{x_i}{y_j}\right]\left[\frac{ax_i}{y_j}\right]}\right)_{i,j=1}^n .
$$

We can use this formula to derive $A_{n,k}$ using the following argument. Evaluating at $x_1 = \omega t$, $x_2 = \cdots = x_n = \omega$, $y_1 = \cdots = y_n = 1$, $a = \omega$ we obtain

$$
\left(\frac{\omega^n t}{(\omega-\omega^{-1})^{n-1}}\right)^{-1} Z_n(\overrightarrow{x};\overrightarrow{y};a) = \sum_{k=1}^n (-1)^{k-1}A_{n,k}P_{n,k}(t),
$$

where

$$
P_{n,k} = (t\omega^2 - t^{-1}\omega^{-2})^{k-1}(t\omega - t^{-1}\omega^{-1})^{n-k}.
$$

See equation 7.4, page 230 of the textbook. The expressions $P_{n,k}(t)$ are linearly independent in the sense that

$$
\sum_{k=1}^n \alpha_k P_{n,k}(t) = 0 \Longrightarrow \alpha_1 = \cdots = \alpha_n = 0
$$

when α_1 through α_n are complex numbers. You will prove this in exercise 7.1.9. This implies

$$
\sum_{k=1}^n \alpha_k P_{n,k}(t) = \sum_{k=1}^n \beta_k P_{n,k}(t) \Longrightarrow \alpha_1 = \beta_1,\ \cdots,\ \alpha_n = \beta_n.
$$

Therefore, if we can show that

$$
\left(\frac{\omega^n t}{(\omega-\omega^{-1})^{n-1}}\right)^{-1} Z_n(\overrightarrow{x};\overrightarrow{y};a) =
$$

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} P_{n,k}(t),$$

then we will have a proof that

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

for each k.

Exercises: 1, 2, 7, 8, 9, 10, 11, 12, 13, 17

Section 7.2: Baxter's triangle-to-triangle relation

The key to proving Theorem 7.1 is to first prove that $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$ is a symmetric function in x_1 through x_n and is also a symmetric function in y_1 through y_n. In other words, the value of $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$ does not change when we swap x_i with x_j or when we swap y_i with y_j. If we swap x_2 with x_3 then the weight of the example on page 225 is changed to $R_1 R_2' R_3' R_4 R_5$, where

$$\begin{aligned}
R_2' &= \frac{x_3}{y_1} \frac{y_2}{x_3} \left[\frac{ax_3}{y_3}\right] \frac{x_3}{y_4} \left[\frac{x_3}{y_5}\right], \\
R_3' &= \left[\frac{x_2}{y_1}\right] \frac{x_2}{y_2} \left[\frac{x_2}{y_3}\right] \frac{y_4}{x_2} \frac{x_2}{y_5}.
\end{aligned}$$

We are going to illustrate the idea behind the proof that $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$ is a symmetric function in the x_i's and in the y_i's without working through all the details. Let $Z_n^{x_i,x_j}(\overrightarrow{x}; \overrightarrow{y}; a)$ denote the expression obtained by swapping x_i and x_j in $Z_n(\overrightarrow{x}; \overrightarrow{y}; a)$. We wish to show that

$$Z_n^{x_i,x_j}(\overrightarrow{x}; \overrightarrow{y}; a) = Z_n(\overrightarrow{x}; \overrightarrow{y}; a).$$

We will argue that

$$\left[\frac{ax_{i+1}}{x_i}\right] Z_n^{x_i,x_j}(\overrightarrow{x}; \overrightarrow{y}; a) = \left[\frac{ax_{i+1}}{x_i}\right] Z_n(\overrightarrow{x}; \overrightarrow{y}; a).$$

Dividing through by $\left[\frac{ax_{i+1}}{x_i}\right]$ we will obtain the desired result.

$Z_n(\overrightarrow{x};\overrightarrow{y};a)$ is the sum of weights of $n\times n$ 6-vertex models. We will introduce a new vertex v_0 of type SW to each model, attached to the vertices in positions $(i,1)$ and $(i+1,1)$ as in the diagram on page 233, and label it $\frac{x_{i+1}}{x_i}$. The weight of this modified vertex model is $\left[\frac{ax_{i+1}}{x_i}\right]$ times the weight of the original model. Therefore $\left[\frac{ax_{i+1}}{x_i}\right] Z_n(\overrightarrow{x};\overrightarrow{y};a)$ is the sum of weights of all the modified vertex models which contain v_0. It can be shown (see the calculation at the top of page 233) that the weight of any modified vertex model is equal to the sum of the weights of two other modified vertex models. To create these two models, swap the labels x_i/y_1 and x_{i+1}/y_1, remove v_0 and restore the edges which were incident to v_0 to their original positions, then introduce a vertex v_1 labeled $\frac{x_{i+1}}{x_i}$ in the middle of the square created by the edges connecting vertices $(i,1)$, $(i,2)$, $(i+1,2)$, $(i+1,1)$. Relocate the edge connecting vertices $(i,1)$ and $(i,2)$ so that it connects v_1 and $(i,2)$, retaining the original direction of the edge. Relocate the edge connecting vertices $(i+1,1)$ and $(i+1,2)$ so that it connects v_1 and $(i+1,2)$, retaining the original direction of the edge. There are now two ways to connect vertices $(i,1)$ and $(i,2)$ to v_1 so that v_1 has two edges directed into it and two edges directed out of it.

It can be shown (Theorem 7.2, the triangle-to-triangle relation) that the sum of the weights of these two modified models is equal to the sum of the weights of yet two more modified models, obtained by first swapping the labels $\frac{x_i}{y_2}$ and $\frac{x_{i+1}}{y_2}$ in the the model containing v_1, removing v_1, restoring the edges which were incident to v_1 to their original positions, then introducing the vertex v_2 in the middle of the next square over and directing 4 edges into it analogous to what we did with v_1. If we keep on going, we will find that the sum of the weights of all models containing v_0 is equal to the sum of the weights of all models containing v_1, which is equal to the sum of the weights of all the models containing v_2, ..., which is equal to the sum of the weights of all the models containing v_n, where v_n appears on the right hand side of the model attached to vertices (n,i) and $(n,i+1)$. The label of each v_k is $\frac{x_{i+1}}{x_i}$. The vertex v_n will always be of type NE. The weight of v_n is $\left[\frac{ax_{i+1}}{x_i}\right]$, and in all the models containing v_n the labels $\frac{x_i}{y_j}$ have been swapped with $\frac{x_{i+1}}{y_j}$ for all j. Therefore the sum of the weights in the the models containing

v_n is $\left[\frac{ax_{i+1}}{x_i}\right] Z_n^{x_i,x^j}(\overrightarrow{x};\overrightarrow{y};a)$. Hence we have shown

$$\left[\frac{ax_{i+1}}{x_i}\right] Z_n^{x_i,x_j}(\overrightarrow{x};\overrightarrow{y};a) = \left[\frac{ax_{i+1}}{x_i}\right] Z_n(\overrightarrow{x};\overrightarrow{y};a).$$

One can play the same game to show that

$$\left[\frac{y_{i+1}}{y_i}\right] Z_n^{y_i,y_j}(\overrightarrow{x};\overrightarrow{y};a) = \left[\frac{y_{i+1}}{y_i}\right] Z_n(\overrightarrow{x};\overrightarrow{y};a),$$

where we use vertices w_0 through w_n attached between columns i and $i+1$, where w_0 is of type SE and w_n is of type NW. So $Z_n(\overrightarrow{x};\overrightarrow{y};a)$ is invariant with respect to swapping any adjacent pair x_i, x_{i+1} and any adjacent pair y_i, y_{i+1}. Therefore it is invariant with respect to any arbitrary permutation of the x_i's and any arbitrary permutation of the y_i's.

In the additional exercise (see below), you will verify that

$$\left[\frac{ax_3}{x_2}\right] Z_5^{x_2,x_3}(\overrightarrow{x};\overrightarrow{y};a) = \left[\frac{ax_3}{x_2}\right] Z_5(\overrightarrow{x};\overrightarrow{y};a)$$

for the 5×5 6-vertex model on page 225 of the textbook by mimicking this calculation.

Now that we know that $Z_n(\overrightarrow{x};\overrightarrow{y};a)$ is a symmetric function in the x_i's and the y_i's, Theorem 7.1 can be proved using this fact and an induction argument. See pp. 238–240.

Exercises 7.1.17, 7.2.6, 7.2.7, 7.2.8, 2.4.9. and 2.4.10 supply a proof of the Alternating Sign Matrix conjecture (Conjecture 3).

Exercises: 6, 7, 8, 2.4.9, and 2.4.10.

Additional Exercise: verify that

$$\left[\frac{ax_3}{x_2}\right] Z_5^{x_2,x_3}(\overrightarrow{x};\overrightarrow{y};a) = \left[\frac{ax_3}{x_2}\right] Z_5(\overrightarrow{x};\overrightarrow{y};a)$$

for the 5×5 6-vertex model on page 225 of the textbook.