

Numerical analysis of superconducting phases in the extended Hubbard model with non-local pairing

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Abstract

[To be continued...]

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List of symbols and abbreviations

AF	Anti-Ferromagnetic
BCS	Bardeen-Cooper-Schrieffer (theory)
DoF	Degree of Freedom
HF	Hartree-Fock
SC	Superconductor
T_c	Critical temperature

Introduction

This thesis project is about my favorite ice cream flavor. [To be continued...]

Chapter 1

Theoretical introduction

[To be continued...]

1.1 Antiferromagnetic ordering in the Hubbard model

Consider the ordinary Hubbard model:

$$\hat{H} = -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \quad t, U > 0 \quad (1.1)$$

The two competing mechanisms are site-hopping of amplitude t and local repulsion of amplitude U . For this model defined **on a bipartite lattice at half filling** and fixed electron number, it is well known [6] that, below a certain critical temperature T_c and above some (small) critical repulsion U_c/t , the ground-state acquires antiferromagnetic (AF) long-range ordering. schematically depicted in Fig. 1.1a. The mechanism for the formation of the AF phase takes advantage of virtual hopping, as described in App. ??; the Mean-Field Theory treatment of ferromagnetic-antiferromagnetic orderings in 2D Hubbard lattices is discussed in App. ??.

In this chapter the discussion is limited to the two-dimensional square lattice Hubbard model. The lattice considered has L_{ℓ} sites on side $\ell = x, y$, thus a total of $L_x L_y$ sites. The total number of single electron states is given by $D = 2L_x L_y$. All theoretical discussion neglects border effects, thus considering $D \rightarrow +\infty$.

1.2 The Extended Fermi-Hubbard model

The Extended Fermi-Hubbard model is defined by:

$$\hat{H} = -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - V \sum_{\langle ij \rangle} \sum_{\sigma\sigma'} \hat{n}_{i\sigma} \hat{n}_{j\sigma'} \quad (1.2)$$

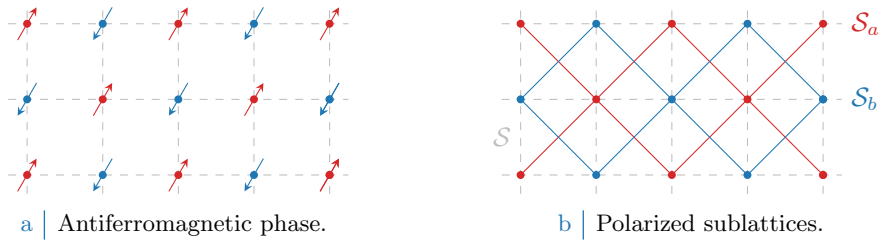


Figure 1.1 | Schematic representation of the AF phase. Fig. 1.1a shows a portion of the square lattice with explicit representation of the spin for each site. Fig. 1.1b divides the square lattice \mathcal{S} in two polarized sublattices $\mathcal{S}_a, \mathcal{S}_b$. The AF phase results from the interaction of two inversely polarized “ferromagnetic” square lattices.

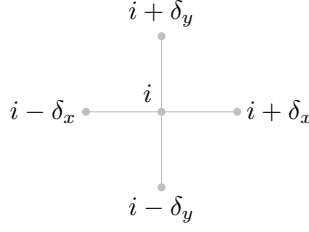


Figure 1.2 | Schematic representation of the four NNs of a given site i for a planar square lattice.

The last term represents an effective attraction between neighboring electrons, of amplitude V . Such an interaction is believed [1] necessary to describe the insurgence of high- T_c superconductivity in cuprate SCs. [To be continued...]

1.2.1 Experimental insight on NN attraction

Todo:

- High T_c SC in cuprates;
- Experimental evidence of topological SC;
- Insertion of the non-local attraction;

1.3 Mean-Field theory description

This section is devoted to develop a rough mean field approximation of the Extended Hubbard model of Eq. (1.2). First part focuses on the non-local interaction V , expected to be source of superconductivity; the second part on the local interaction U , known to be source of Slater-like anti-ferromagnetism, as described in App. ??.

1.3.1 Mean-field treatment of the non-local term

Consider the non-local term,

$$\hat{H}_V \equiv -V \sum_{\langle ij \rangle} \sum_{\sigma\sigma'} \hat{n}_{i\sigma} \hat{n}_{j\sigma'} \quad (1.3)$$

Since the relevant values for V are $\mathcal{O}(t)$, in this model $V \ll U$. Evidently the hamiltonian can be decomposed in various spin terms,

$$\begin{aligned} \hat{H}_V &= \sum_{\sigma\sigma'} \hat{H}_V^{\sigma\sigma'} \\ &= \underbrace{\hat{H}_V^{\uparrow\uparrow} + \hat{H}_V^{\downarrow\downarrow}}_{\text{Ferromagnetic}} + \underbrace{\hat{H}_V^{\uparrow\downarrow} + \hat{H}_V^{\downarrow\uparrow}}_{\text{Anti-ferromagnetic}} \end{aligned}$$

Evidently, to carry out a summation over nearest neighbors $\langle ij \rangle$ of a square lattice means precisely to sum over all links of the lattice. Then we can identify the generic AF term $\hat{H}_V^{\sigma\bar{\sigma}}$ as the one collecting the σ operators of sublattice \mathcal{S}_a and $\bar{\sigma}$ operators of sublattice \mathcal{S}_b . At half-filling, as described in Sec. 1.1, the ground-state leading contribution will be the antiferromagnetic state, with the square lattice decomposed in two oppositely polarized square lattices with spacing increased by a factor $\sqrt{2}$. Then, on this configuration the ferromagnetic contributions are suppressed¹. This leads to the approximation

$$\hat{H}_V \simeq \hat{H}_V^{\uparrow\downarrow} + \hat{H}_V^{\downarrow\uparrow}$$

¹This is also due to superexchange stabilization: the triplet contribution to hamiltonian is suppressed, and this cancels out the ferromagnetic terms $\hat{H}_V^{\sigma\sigma}$ while privileging the singlet configuration of the anti-ferromagnetic terms $\hat{H}_V^{\sigma\bar{\sigma}}$.

The AF non-local interactions can be written as a sum of terms over just one of the two sublattices \mathcal{S}_a and \mathcal{S}_b , oppositely polarized in the AF configuration (see Fig. 1.1b)

$$\begin{aligned}\hat{H}_V &\simeq \overbrace{\sum_{i \in \mathcal{S}_a} \hat{h}_V^{(i)}}^{\hat{H}_V^{\uparrow\downarrow}} + \overbrace{\sum_{i \in \mathcal{S}_b} \hat{h}_V^{(i)}}^{\hat{H}_V^{\downarrow\uparrow}} \\ &= \sum_{i \in \mathcal{S}} \hat{h}_V^{(i)}\end{aligned}\quad \hat{h}_V^{(i)} = -V \sum_{\ell=x,y} (\hat{n}_{i\uparrow} \hat{n}_{i+\delta_\ell\downarrow} + \hat{n}_{i\uparrow} \hat{n}_{i-\delta_\ell\downarrow})$$

Here the notation of Fig. 1.1b is used. The two-dimensional lattice is regular-square. For each site i in a given sublattice, the nearest neighbors sites are four – all in the other sublattice. The notation used is $i \pm \delta_x$, $i \pm \delta_y$ as in Fig. 1.2. The non-local interaction contribution to energy, as a function of the $T = 0$ full hamiltonian ground-state² $|\Psi\rangle$, is given by

$$\begin{aligned}E_V[\Psi] &= \langle \Psi | \hat{H}_V | \Psi \rangle \\ &= -V \sum_{i \in \mathcal{S}} \sum_{\ell=x,y} \langle \hat{n}_{i\uparrow} \hat{n}_{i+\delta_\ell\downarrow} + \hat{n}_{i\uparrow} \hat{n}_{i-\delta_\ell\downarrow} \rangle\end{aligned}$$

Shorthand notation has been used: $\langle \Psi | \cdot | \Psi \rangle = \langle \cdot \rangle$. Consider one specific term, say, $\hat{n}_{i\uparrow} \hat{n}_{i+\delta_x\downarrow}$. Wick's Theorem states that, if the expectation value is performed onto a coherent state,

$$\begin{aligned}\langle \hat{n}_{i\uparrow} \hat{n}_{i+\delta_x\downarrow} \rangle &= \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta_x\downarrow}^\dagger \hat{c}_{i+\delta_x\downarrow} \hat{c}_{i\uparrow} \rangle \\ &= \underbrace{\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta_x\downarrow}^\dagger \rangle \langle \hat{c}_{i+\delta_x\downarrow} \hat{c}_{i\uparrow} \rangle}_{\text{Cooper}} - \underbrace{\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta_x\downarrow} \rangle \langle \hat{c}_{i+\delta_x\downarrow}^\dagger \hat{c}_{i\uparrow} \rangle}_{\text{Fock}} + \underbrace{\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\uparrow} \rangle \langle \hat{c}_{i+\delta_x\downarrow}^\dagger \hat{c}_{i+\delta_x\downarrow} \rangle}_{\text{Hartree}}\end{aligned}$$

As a first approximation, the theorem is assumed to hold (which, in a BCS-like fashion, is equivalent to assuming for the ground-state to be a coherent state). The last two terms account for single-particle interactions with a background field; they are relevant in the Hartree-Fock scheme, being direct-exchange contributions to single particle energies. The first term accounts for non-local electrons pairing, mimicking the Cooper term of BCS theory. **The core assumption, here, is that only one of the HF and Bogoliubov pairings survive. Appendix ?? shows a situation dominated by HF terms. Here, I assume the symmetry to be broken by a Cooper term.** Energy then is cast to the form

$$E_V[\Psi] = -V \sum_{i \in \mathcal{S}} \sum_{\ell=x,y} \left[\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta_\ell\downarrow}^\dagger \rangle \langle \hat{c}_{i+\delta_\ell\downarrow} \hat{c}_{i\uparrow} \rangle + \langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i-\delta_\ell\downarrow}^\dagger \rangle \langle \hat{c}_{i-\delta_\ell\downarrow} \hat{c}_{i\uparrow} \rangle \right]$$

The ground-state must realize the condition

$$\frac{\delta}{\delta \langle \Psi |} E[\Psi] = 0$$

being $E[\Psi]$ the total energy (made up of the three terms of couplings t , U and V). **[Expand derivation?]** The functional derivative must be carried out in a variational fashion including a Lagrange multiplier, the latter accounting for state-norm conservation, as is done normally in deriving the Hartree-Fock approximation for the eigenenergies of the electron liquid [4, 5]. This approach leads to the conclusion that the (coherent) ground-state of the system must be an eigenstate of the mean-field effective hamiltonian:

$$\begin{aligned}\hat{H}^{(e)} &= -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U \sum_{i \in \mathcal{S}} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \\ &\quad - V \sum_{i \in \mathcal{S}} \sum_{\ell=x,y} \sum_{\delta=\pm\delta_\ell} \left[\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta\downarrow}^\dagger \rangle \hat{c}_{i+\delta\downarrow} \hat{c}_{i\uparrow} + \text{h.c.} \right]\end{aligned}\tag{1.4}$$

²Extensions to finite temperatures is simple: minimization must be carried out on free energy, while expectation values must be taken in a thermodynamic fashion.

The pairing correlation function is defined across each bond as the pairing expectation

$$g_{ij\sigma} \equiv \langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \rangle$$

The effective hamiltonian reads:

$$\hat{H}^{(e)} = -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U \sum_{i \in S} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - V \sum_{\langle ij \rangle} \sum_{\sigma} \left[g_{ij\sigma} \hat{c}_{j\bar{\sigma}} \hat{c}_{i\sigma} + g_{ij\sigma}^* \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \right] \quad (1.5)$$

As in standard BCS theory, this hamiltonian – being quadratic in the electronic operators – can be diagonalized via a Bogoliubov rotation. Superconducting pairing can arise both from the local U term and from the non-local V term. In next sections it is assumed the V term generates dominant superconductivity via its weak non-local pairing.

1.3.2 Mean-field treatment of the local term

The mean-field description of the local (on-site) U interaction is given in detail in App. ??, along with a simple numerical analysis of the insurgence of antiferromagnetic ordering in a Hartree-Fock approximation scheme. Here the Cooper pairing is likewise assumed to dominate. Performing an analysis analogous to the one carried out in last section, we get the decoupling

$$U \sum_{i \in S} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \simeq \frac{U}{2} \sum_{i\sigma} \left[f_{i\sigma} \hat{c}_{i\bar{\sigma}} \hat{c}_{i\sigma} + f_{i\sigma}^* \hat{c}_{i\sigma}^\dagger \hat{c}_{i\bar{\sigma}}^\dagger \right]$$

being

$$f_{i\sigma} \equiv \langle \hat{c}_{i\sigma}^\dagger \hat{c}_{i\bar{\sigma}}^\dagger \rangle$$

Collect f and g in the unique function of two variables:

$$C_\sigma(i, j) = \begin{cases} f_{i\sigma} & \text{if } i = j \\ g_{ij\sigma} & \text{if } |i - j| = 1 \\ (\dots) & \text{otherwise} \end{cases}$$

which expresses the generic correlator $\langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \rangle$. The correlator for $|i - j| > 1$ is left unexpressed, and supposed to be subdominant. The decoupled hamiltonian, apart from pure energy shifts and suppressed terms, is given by

$$\begin{aligned} \hat{H}^{(e)} = & -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \frac{U}{2} \sum_{i\sigma} \left[f_{i\sigma} \hat{c}_{i\bar{\sigma}} \hat{c}_{i\sigma} + f_{i\sigma}^* \hat{c}_{i\sigma}^\dagger \hat{c}_{i\bar{\sigma}}^\dagger \right] \\ & - V \sum_{\langle ij \rangle} \sum_{\sigma} \left[g_{ij\sigma} \hat{c}_{j\bar{\sigma}} \hat{c}_{i\sigma} + g_{ij\sigma}^* \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \right] \end{aligned} \quad (1.6)$$

The last form of the full hamiltonian contains explicitly spin DoF. The dissertation that follows includes explicitly spin DoF.

1.3.3 Topological superconducting parameters over the square lattice

Consider Eq. (1.6), and for simplicity neglect the spin DoF in the last term.

The correlator g_{ij} is a function of position, specifically of its variables difference $r \equiv j - i$. Over the square lattice with NN interaction, the latter can assume four values: $\pm\delta_x, \pm\delta_y$. For a function of space defined over the four rim sites $i \pm \delta_\ell$ of Fig. 1.2, various symmetry structures can be defined under the planar rotations group $SO(2)$. In other words, the function g_r can be decomposed in planar harmonics (which are simply the sine-cosine basis). Fig. 1.3 shows the first four spatial structures for the NN term. For each graph, the relative weight in the decomposition of g_r is given by a constant correlation $g^{(\ell)}$ times a sum of signs and zeros as indicated in the various subgraphs of Fig. 1.3. Tab. 1.1 reports said gap function contributions explicitly.

SC is established with a given symmetry – which means, symmetry breaking in the phase transition proceeds in a specific channel. Conventional BCS superconductivity arises from the only

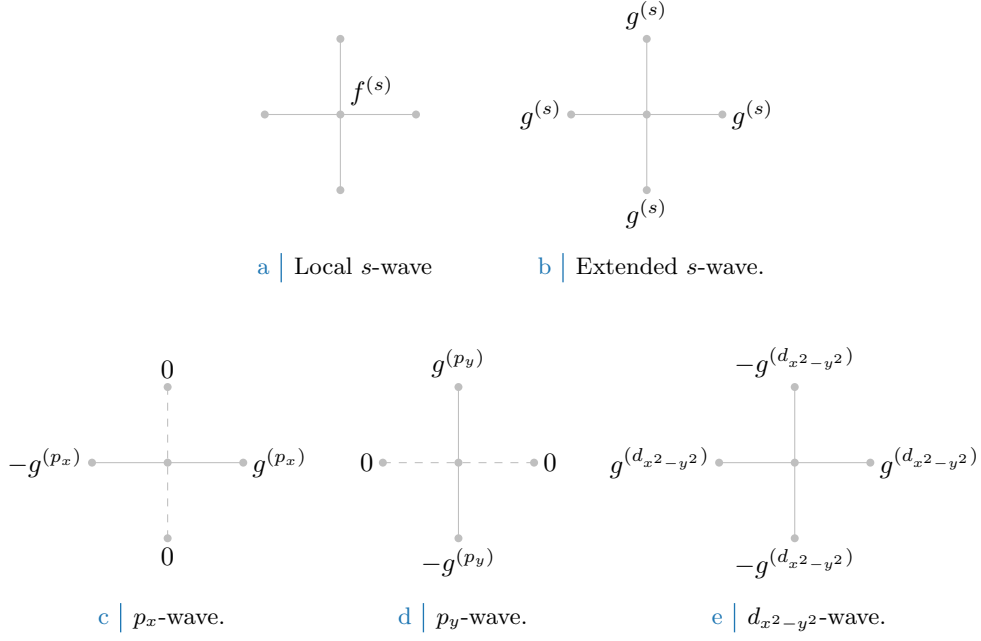


Figure 1.3 | Different spatial structures of the correlation function under planar rotations. The figures show the first five local and NN contributions to the spatial structure of $C_\sigma(i, j)$. Each graph is a different contribution to the expansion of C_σ . Under each graph, each rim site contributes by a weight given by ± 1 or 0 times a constant dependent of the specific symmetry (f or g in figures), as specified in text.

Structure	Correlation function	Graph
s -wave	$f_i^{(s)} = f^{(s)}$	Fig. 1.3a
Extended s -wave	$g_{ij}^{(s)} = g^{(s)} (\delta_{j=i+\delta_x} + \delta_{j=i-\delta_x} + \delta_{j=i+\delta_y} + \delta_{j=i-\delta_y})$	Fig. 1.3b
p_x -wave	$g_{ij}^{(p_x)} = g^{(p_x)} (\delta_{j=i+\delta_x} - \delta_{j=i-\delta_x})$	Fig. 1.3c
p_y -wave	$g_{ij}^{(p_y)} = g^{(p_y)} (\delta_{j=i+\delta_y} - \delta_{j=i-\delta_y})$	Fig. 1.3d
$d_{x^2-y^2}$ -wave	$g_{ij}^{(d_{x^2-y^2})} = g^{(d_{x^2-y^2})} (\delta_{j=i+\delta_x} + \delta_{j=i-\delta_x} - \delta_{j=i+\delta_y} - \delta_{j=i-\delta_y})$	Fig. 1.3e

Table 1.1 | First four spatial structures for the correlation function $C(i, j)$ (spin DoF is neglected). In the middle column, all spatial dependence is included in the δ s, while $f^s, g^{(\ell)} \in \mathbb{C}$. The last column indicates the graph representation of each contribution given in Fig. 1.3.

possible spatial structure of the local pairing, s -wave – here appearing as a local term (Fig. 1.3a) and extended on a non-local term (Fig. 1.3b). Cuprates exhibit a tendency towards $d_{x^2-y^2}$ SC, while other materials towards p -wave types – eventually with some chirality, as is the case for $p_x \pm ip_y$ SCs.

In this BCS-like approach, a self-consistent equation for the gap function must be retrieved in order to further investigate the model and extract the conditions for the formation of a superconducting phase with a given pairing topology.

1.3.4 Reciprocal space transformation of hamiltonian

Let me take a step back and perform explicitly the Fourier-transform of the various terms of Eq. 1.2. Mean-field theory will be implemented later.

Kinetic term

The kinetic part is trivial to transform. Calculation is carried out in App. ???. Let

$$\epsilon_{\mathbf{k}} \equiv -2t [\cos(k_x \delta_x) + \cos(k_y \delta_y)]$$

then we have

$$\begin{aligned} -t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma} &= \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^{\dagger} \hat{c}_{\mathbf{k}\sigma} \\ &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\hat{c}_{\mathbf{k}\uparrow}^{\dagger} \hat{c}_{\mathbf{k}\uparrow} + \hat{c}_{\mathbf{k}\downarrow}^{\dagger} \hat{c}_{\mathbf{k}\downarrow} \right] \\ &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left[\hat{c}_{\mathbf{k}\uparrow}^{\dagger} \hat{c}_{\mathbf{k}\uparrow} - \hat{c}_{-\mathbf{k}\downarrow}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow} \right] \end{aligned}$$

In last passage I used fermionic anti-commutation rules and reversed the sign of the mute variable. This will become useful later.

Non-local interaction

Consider a generic bond, say, the one connecting sites j and $j \pm \delta_{\ell}$ (variable i is here referred to as the imaginary unit to avoid confusion). \mathbf{x}_j is the 2D notation for the position of site j , while δ_{ℓ} is the 2D notation for the lattice spacing previously indicated as δ_{ℓ} . Fourier transform it according to the convention

$$\hat{c}_{j\sigma} = \frac{1}{\sqrt{L_x L_y}} \sum_{\mathbf{k} \in \text{BZ}} e^{-i\mathbf{k} \cdot \mathbf{x}_j} \hat{c}_{\mathbf{k}\sigma}$$

Then:

$$\begin{aligned} \hat{n}_{j\uparrow} \hat{n}_{j\pm\delta_{\ell}\downarrow} &= \hat{c}_{j\uparrow}^{\dagger} \hat{c}_{j\pm\delta_{\ell}\downarrow}^{\dagger} \hat{c}_{j\pm\delta_{\ell}\downarrow} \hat{c}_{j\uparrow} \\ &= \frac{1}{(L_x L_y)^2} \sum_{\nu=1}^4 \sum_{\mathbf{k}_{\nu} \in \text{BZ}} e^{i[(\mathbf{k}_1+\mathbf{k}_2)-(\mathbf{k}_3+\mathbf{k}_4)] \cdot \mathbf{x}_j} e^{\pm i(\mathbf{k}_2-\mathbf{k}_3) \cdot \delta_{\ell}} \hat{c}_{\mathbf{k}_1\uparrow}^{\dagger} \hat{c}_{\mathbf{k}_2\downarrow}^{\dagger} \hat{c}_{\mathbf{k}_3\downarrow} \hat{c}_{\mathbf{k}_4\uparrow} \end{aligned}$$

It follows,

$$\begin{aligned} \hat{h}_V^{(j)} &= -\frac{V}{(L_x L_y)^2} \sum_{\ell=x,y} \sum_{\nu=1}^4 \sum_{\mathbf{k}_{\nu} \in \text{BZ}} e^{i[(\mathbf{k}_1+\mathbf{k}_2)-(\mathbf{k}_3+\mathbf{k}_4)] \cdot \mathbf{x}_j} \\ &\quad \times \left(e^{i(\mathbf{k}_2-\mathbf{k}_3) \cdot \delta_{\ell}} + e^{-i(\mathbf{k}_2-\mathbf{k}_3) \cdot \delta_{\ell}} \right) \hat{c}_{\mathbf{k}_1\uparrow}^{\dagger} \hat{c}_{\mathbf{k}_2\downarrow}^{\dagger} \hat{c}_{\mathbf{k}_3\downarrow} \hat{c}_{\mathbf{k}_4\uparrow} \\ &= -\frac{2V}{(L_x L_y)^2} \sum_{\ell=x,y} \sum_{\nu=1}^4 \sum_{\mathbf{k}_{\nu} \in \text{BZ}} e^{i[(\mathbf{k}_1+\mathbf{k}_2)-(\mathbf{k}_3+\mathbf{k}_4)] \cdot \mathbf{x}_j} \cos[(\mathbf{k}_2 - \mathbf{k}_3) \cdot \delta_{\ell}] \hat{c}_{\mathbf{k}_1\uparrow}^{\dagger} \hat{c}_{\mathbf{k}_2\downarrow}^{\dagger} \hat{c}_{\mathbf{k}_3\downarrow} \hat{c}_{\mathbf{k}_4\uparrow} \end{aligned}$$

The full non-local interaction is given by summing over all sites of \mathcal{S} . This gives back momentum conservation,

$$\frac{1}{L_x L_y} \sum_{j \in \mathcal{S}} e^{i[(\mathbf{k}_1+\mathbf{k}_2)-(\mathbf{k}_3+\mathbf{k}_4)] \cdot \mathbf{x}_j} = \delta_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4}$$

Let $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 = \mathbf{K}$, and define \mathbf{k}, \mathbf{k}' such that

$$\mathbf{k}_1 \equiv \mathbf{K} + \mathbf{k} \quad \mathbf{k}_2 \equiv \mathbf{K} - \mathbf{k} \quad \mathbf{k}_3 \equiv \mathbf{K} - \mathbf{k}' \quad \mathbf{k}_4 \equiv \mathbf{K} + \mathbf{k}' \quad \delta \mathbf{k} \equiv \mathbf{k} - \mathbf{k}'$$

Sums over these variable must be intended as over the Brillouin Zone (BZ). Then, finally

$$\begin{aligned} \hat{H}_V &\simeq \sum_{j \in \mathcal{S}} \hat{h}_V^{(j)} \\ &= -\frac{2V}{L_x L_y} \sum_{\ell=x,y} \sum_{\mathbf{K}, \mathbf{k}, \mathbf{k}'} \cos(\delta \mathbf{k} \cdot \delta_{\ell}) \hat{c}_{\mathbf{K}+\mathbf{k}\uparrow}^{\dagger} \hat{c}_{\mathbf{K}-\mathbf{k}\downarrow}^{\dagger} \hat{c}_{\mathbf{K}-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{K}+\mathbf{k}'\uparrow} \\ &= -2V \sum_{\ell=x,y} \sum_{\mathbf{k}, \mathbf{k}'} \cos(\delta \mathbf{k} \cdot \delta_{\ell}) \hat{c}_{\mathbf{k}\uparrow}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow}^{\dagger} \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow} \\ &= -2V \sum_{\mathbf{k}, \mathbf{k}'} [\cos(\delta k_x \delta_x) + \cos(\delta k_y \delta_y)] \hat{c}_{\mathbf{k}\uparrow}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow}^{\dagger} \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow} \end{aligned}$$

Eventually, in the second passage the prefactor 2 can be absorbed by reintroducing the spin DoF³. Then in the third passage a sum over \mathbf{K} has been absorbed recognizing that it generates $L_x L_y$ identical terms.

Taking in the mean-field approximation (with Cooper pair symmetry breaking), we get

$$\hat{H}_V \simeq -2V \sum_{\mathbf{k}, \mathbf{k}'} [\cos(\delta k_x \delta_x) + \cos(\delta k_y \delta_y)] \left[\langle \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \rangle \hat{\phi}_{\mathbf{k}'} + \langle \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} \rangle \hat{\phi}_{\mathbf{k}}^\dagger \right]$$

having I defined the pairing operator

$$\hat{\phi}_{\mathbf{k}} \equiv \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow}$$

Defining appropriately the gap function,

$$\Delta_{\mathbf{k}} \equiv 2V \sum_{\mathbf{k}'} [\cos((k_x - k'_x)\delta_x) + \cos((k_y - k'_y)\delta_y)] \langle \hat{c}_{\mathbf{k}'\uparrow}^\dagger \hat{c}_{-\mathbf{k}'\downarrow}^\dagger \rangle \quad (1.7)$$

one gets immediately

$$\hat{H}_V \simeq - \sum_{\mathbf{k}} [\Delta_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} + \Delta_{\mathbf{k}}^* \hat{\phi}_{\mathbf{k}}^\dagger] \quad (1.8)$$

[To be continued...]

Local interaction

A very similar argument can be carried out for the local U term. Without delving in too many details, the local gap $\Sigma_{\mathbf{k}\sigma}$ is given by

$$\Sigma_{\mathbf{k}} \equiv U \sum_{\mathbf{k}'} \langle \hat{c}_{\mathbf{k}'\uparrow}^\dagger \hat{c}_{-\mathbf{k}'\downarrow}^\dagger \rangle$$

evidently independent of \mathbf{k} , correctly. The local part of the hamiltonian then gets

$$\hat{H}_U = \sum_{\mathbf{k}} [\Sigma_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} + \Sigma_{\mathbf{k}}^* \hat{\phi}_{\mathbf{k}}^\dagger] \quad (1.9)$$

Within this structure, we are finally able to move to Nambu formalism.

1.3.5 Nambu formalism and Bogoliubov transform

Define the Nambu spinor⁴ as in BCS

$$\hat{\Psi}_{\mathbf{k}} \equiv \begin{bmatrix} \hat{c}_{\mathbf{k}\uparrow} \\ \hat{c}_{-\mathbf{k}\downarrow}^\dagger \end{bmatrix}$$

Evidently,

$$\phi_{\mathbf{k}} = \hat{\Psi}_{\mathbf{k}}^\dagger \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{\Psi}_{\mathbf{k}} \quad \phi_{\mathbf{k}}^\dagger = \hat{\Psi}_{\mathbf{k}}^\dagger \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hat{\Psi}_{\mathbf{k}}$$

The full hamiltonian is then given by:

$$\hat{H} = \sum_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}} h_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}} \quad h_{\mathbf{k}} \equiv \begin{bmatrix} \epsilon_{\mathbf{k}} & \Sigma_{\mathbf{k}} + \Delta_{\mathbf{k}} \\ \Sigma_{\mathbf{k}}^* + \Delta_{\mathbf{k}}^* & -\epsilon_{\mathbf{k}} \end{bmatrix} \quad (1.10)$$

Proceed as in App. ?? and diagonalize via a rotation,

$$d_{\mathbf{k}} \equiv \begin{bmatrix} -E_{\mathbf{k}} & \\ & E_{\mathbf{k}} \end{bmatrix} \quad \text{being} \quad E_{\mathbf{k}} \equiv \sqrt{\epsilon_{\mathbf{k}}^2 + |\Sigma_{\mathbf{k}} + \Delta_{\mathbf{k}}|^2}$$

³Justification can be given in two ways: either commuting appropriately the \hat{c} operators, or by carrying out the previous space sums independently over the two sublattices.

⁴Notice that the spinor is here differently defined with respect to App. ??, where because of the HF prevalence in mean-field decoupling the spinor components were homogeneously fermions creations or destructions.

Structure	Gap function	Graph
Extended s -wave	$\Delta_{\mathbf{k}\sigma}^{(s)} = \Delta^{(s)} [\cos(k_x) + \cos(k_y)]$	Fig. 1.3b
p_x -wave	$\Delta_{\mathbf{k}\sigma}^{(p_x)} = i\Delta^{(p_x)} \sin(k_x)$	Fig. 1.3c
p_y -wave	$\Delta_{\mathbf{k}\sigma}^{(p_y)} = i\Delta^{(p_y)} \sin(k_y)$	Fig. 1.3d
$d_{x^2-y^2}$ -wave	$\Delta_{\mathbf{k}\sigma}^{(d)} = \Delta^{(d)} [\cos(k_x) - \cos(k_y)]$	Fig. 1.3e

Table 1.2 | Gap structures for the four spatial symmetries of Fig. 1.3.

1.3.6 Non-local gap structures

Let's have a closer look to Eq. (1.7): performing explicitly

$$\begin{aligned}
\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{-\mathbf{k}\bar{\sigma}}^\dagger \rangle &= \frac{1}{L_x L_y} \sum_{ij} e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} C_\sigma(i, j) \\
&= \frac{1}{L_x L_y} \sum_{i \in \mathcal{S}} f_{i\sigma} + \frac{1}{L_x L_y} \sum_{\langle ij \rangle} e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} g_{ij\sigma} + (\text{subdominant})
\end{aligned}$$

Contributions to $C_\sigma(i, j)$ from $|i - j| > 1$ are assumed to be subdominant and then neglected. The above equation needs to be equipped with some structure for $f_{i\sigma}$ and $g_{ij\sigma}$. The main idea is to try different channels of symmetry breaking starting from the gap function structures of Tab. 1.1 and follow the general HF procedure sketched in detail in App. ???. An iterative algorithm will be carried out to estimate, rather roughly, the creation of an instability towards a precise topological superconducting pairing. Before doing so, notice that $g_{ij\sigma}$ is actually a function of $r = j - i$ (in vectorial notation: $\delta_\ell = \mathbf{x}_j - \mathbf{x}_i$, with $\ell \in \{x, y\}$) in the symmetries we are considering, listed in Tab. 1.1. Thus

$$\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{-\mathbf{k}\bar{\sigma}}^\dagger \rangle = f^{(s)} + \sum_{\ell=x,y} [e^{i\mathbf{k} \cdot \delta_\ell} g_{\delta_\ell\sigma} + e^{-i\mathbf{k} \cdot \delta_\ell} g_{-\delta_\ell\sigma}] \quad (1.11)$$

By using the simple structures of Fig. 1.3 this expression simplifies a lot. Moreover, considering the terms of Eq. (1.7),

$$\begin{aligned}
\cos(\delta k_x \delta_x) + \cos(\delta k_y \delta_y) &= \cos(k_x \delta_x) \cos(k'_x \delta_x) + \sin(k_x \delta_x) \sin(k'_x \delta_x) \\
&\quad + \cos(k_y \delta_y) \cos(k'_y \delta_y) + \sin(k_y \delta_y) \sin(k'_y \delta_y)
\end{aligned}$$

For the sake of readability, the notations

$$c_\ell \equiv \cos(k_\ell \delta_\ell) \quad s_\ell \equiv \sin(k_\ell \delta_\ell) \quad c'_\ell \equiv \cos(k'_\ell \delta_\ell) \quad s'_\ell \equiv \sin(k'_\ell \delta_\ell)$$

are used. Group the four terms above,

$$\underbrace{(c_x c'_x + c_y c'_y)}_{\text{Symmetric}} + \underbrace{(s_x s'_x + s_y s'_y)}_{\text{Anti-symmetric}} \quad (1.12)$$

Since sums on \mathbf{k}, \mathbf{k}' are carried out, based on the specific $g_{\mathbf{k}}$ symmetry structures one can eliminate one of the two contributions. Take e.g. the p_x gap structure of Fig. 1.3c, listed in Tab. 1.1: since it exhibits spatial anti-symmetry on the x direction and (trivial) symmetry on the y direction, of the above terms only the x anti-symmetric part will contribute – which is, $s_x s'_x$. This property becomes useful in the next paragraphs, where detailed derivations on the spatial structures of the gap function are carried out. Results are listed in Tat. 1.2.

Extended s -wave gap structure

Fix from here...

The standard s -wave structure of the gap function in BCS theory is given by the local interaction. Consider its NN extension of Fig. 1.3b, we will refer to also as “ s^* -wave”. It is completely symmetric

under rotations and inversions, thus it couples trivially to the symmetric part of the decoupling (1.12). Eq. (1.11) becomes:

$$\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{-\mathbf{k}\bar{\sigma}}^\dagger \rangle = 2g^{(s)}(c_x + c_y)$$

Performing the computation of Eq. (1.7) within this approximation, we get:

$$\Delta_{\mathbf{k}\sigma}^{(s)} = 2g^{(s)}V \sum_{\mathbf{k}'} (c_x c'_x + c_y c'_y)(c'_x + c'_y)$$

Take a continuum approximation,

$$\begin{aligned} \frac{1}{L_x L_y} \sum_{\mathbf{k}'} c_\ell (c'_\ell)^2 &\simeq c_\ell \int_{-\pi}^{\pi} \frac{ds}{2\pi} \cos^2 s = \frac{c_\ell}{2} \\ \frac{1}{L_x L_y} \sum_{\mathbf{k}'} c_\ell (c'_\ell c'_\ell) &\simeq c_\ell \left[\int_{-\pi}^{\pi} \frac{ds}{2\pi} \cos s \right]^2 = 0 \end{aligned}$$

It follows:

$$\Delta_{\mathbf{k}\sigma}^{(s)} = \Delta^{(s)}(c_x + c_y) \quad (1.13)$$

where $\Delta^{(s)} \equiv L_x L_y V g^{(s)}$. The quantity $\Delta^{(s)}$ is the order parameter in our HF scheme.

p_ℓ -wave gap structure

Take $\ell \in \{x, y\}$ and consider the p_ℓ -wave correlation in Tab. 1.1. Due to its anti-symmetry, it couples only to the antisymmetric part of (1.12). From Eq. (1.11),

$$\langle \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{-\mathbf{k}\bar{\sigma}}^\dagger \rangle = 2ig^{(p_\ell)} s_\ell$$

and then from Eq. (1.7)

$$\Delta_{\mathbf{k}\sigma}^{(p_\ell)} = 2ig^{(p_\ell)}V \sum_{\mathbf{k}'} (s_x s'_x + s_y s'_y) s'_\ell$$

Using identical reasoning as above,

$$\Delta_{\mathbf{k}\sigma}^{(p_\ell)} = i\Delta^{(p_\ell)} s_\ell$$

where $\Delta^{(p_\ell)} \equiv L_x L_y V g^{(p_\ell)}$.

d -wave gap structure

Consider a $d_{x^2-y^2}$ -wave structured gap function as in Fig. 1.3e. The discourse here is perfectly analogous to Sec. 1.3.6, with the only difference

$$c_x + c_y \rightarrow c_x - c_y$$

Then the d -wave gap has the structure

$$\Delta_{\mathbf{k}\sigma}^{(d)} = \Delta^{(d)}(c_x - c_y) \quad (1.14)$$

where $\Delta^{(d)} \equiv L_x L_y V g^{(d)}$. Subscript $x^2 - y^2$ has been omitted for reasons of clarity.

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