

# Numerical analysis of superconducting phases in the extended Hubbard model with non-local pairing

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## Abstract

[To be continued...]

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### **List of symbols and abbreviations**

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AF	Anti-Ferromagnetic
BCS	Bardeen-Cooper-Schrieffer (theory)
DoF	Degree of Freedom
HF	Hartree-Fock
MFT	Mean-Field Theory
SC	Superconductor
$T_c$	Critical temperature

# **Part I**

## **Mean-Field-Theory analysis**



# Chapter 1

## Superconducting instability

This chapter is devoted to studying the superconducting phase of the system. The only symmetry we assume to break is the  $U^c(1)$  charge symmetry, thus allowing for superconducting fluctuations. As is described thoroughly in Sec. ??, the hopping amplitude is renormalized because of the non-local attraction. The symmetry structure of the pairing mechanism determines the contributing Cooper fluctuations: for  $s$ -wave and  $d$ -wave superconductivity, only the o.s. Cooper term contributes; for  $p_\ell$ -wave superconductivity, the s.s. term contributes as well. In the following sections, a derivation containing both Cooper terms is proposed.

[ To be continued: separate singlet and triplet pairing channels, and describe them separately by the means of four-components Nambu spinors. Use selection rules to set  $\Delta^{(p_\ell)} = 0$  in the singlet channel, in order to justify results obtained by a pure space-even simulation containing just the o.s. terms. ]

### 1.1 Cooper fluctuations in the EHM

Let us start once again from the general EHM of Eq. (??),

$$\hat{H} = \underbrace{-t \sum_{\langle ij \rangle} \sum_{\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma}}_{\hat{H}_t} + \underbrace{U \sum_i \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\downarrow} \hat{c}_{i\uparrow}}_{\hat{H}_U} - V \underbrace{\sum_{\langle ij \rangle} \sum_{\sigma\sigma'} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma'}^\dagger \hat{c}_{j\sigma'} \hat{c}_{i\sigma}}_{\hat{H}_V}$$

As is discussed in Sec. ??, when applying Wick's theorem the resulting terms break the natural symmetries of the model. The superconducting symmetry we study breaks just the  $U^c(1)$  charge conservation. Now, when dealing with superconducting Cooper pairing we need to account also for the spatial structure of the Cooper pair itself. Consider the generic Cooper fluctuation

$$\langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma'}^\dagger \rangle \quad \text{with } i, j \text{ NN}$$

From basic Quantum Mechanics we know the summation rules of the spin algebra  $\mathfrak{su}(2)$ ,

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

The two pairing channels are, at this level, the singlet channel associated to total spin 0 and the triplet channel associated to total spin 1. If we impose a specific spatial symmetry on the hamiltonian the ground state wavefunction will follow naturally, and the pairing channel will be the one providing a total anti-symmetry to the full wavefunction. This gives a selection rule over the relevant pairings: if we work with space symmetric structures – say,  $s^*$ -wave or  $d$ -wave – the pairing will happen in singlet channel, allowing us to eliminate the triplet pairing. This concept is summarized in Tab. 1.1.

[ Add: Cooper pairing considerations in the pure Hubbard model. ]

### 1.2 Cooper fluctuations in the opposite-spin sector

This section deals with Cooper fluctuations induced by the o.s. part of the non-local hamiltonian –referring to the notation of Eq. (??)– somewhat the simplest form of Cooper pairing. This sector

Spatial structure	Pairing channel	Relevant pairing
Symmetric wave function	Singlet pairing	Just $\langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \rangle$
Anti-symmetric wave function	Triplet pairing	Both $\langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\bar{\sigma}}^\dagger \rangle$ and $\langle \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma}^\dagger \rangle$

Table 1.1 | Relation of the Cooper pairing channel with the wavefunction spatial symmetry (intended as the inversion  $(x, y) \rightarrow (-x, -y)$ ).

contributes both to singlet pairing and triplet pairing. Considering Cooper fluctuations in the singlet channel, we need to break  $U^c(1)$  symmetry imposing space inversion symmetry, while for the triplet anti-symmetry is required. Let us now break down the MFT discussion for the local and non-local interactions.

**Local interaction  $U$ .** Consider first the local part,

$$\hat{H}_U = U \sum_{i \in S} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \simeq U \sum_{i \in S} [\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\downarrow}^\dagger \rangle \hat{c}_{i\downarrow} \hat{c}_{i\uparrow} + \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\downarrow}^\dagger \langle \hat{c}_{i\downarrow} \hat{c}_{i\uparrow} \rangle]$$

and use the result of Eq. (??),

$$\hat{H}_U \simeq \frac{U}{L_x L_y} \sum_{\mathbf{K}, \mathbf{k}, \mathbf{k}'} [\langle \hat{c}_{\mathbf{K}+\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{K}-\mathbf{k}\downarrow}^\dagger \rangle \hat{c}_{\mathbf{K}-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{K}+\mathbf{k}'\uparrow} + \hat{c}_{\mathbf{K}+\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{K}-\mathbf{k}\downarrow}^\dagger \langle \hat{c}_{\mathbf{K}-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{K}+\mathbf{k}'\uparrow} \rangle]$$

We are not breaking translational invariance, thus only Cooper fluctuations with net zero total momentum are allowed. This means only  $\mathbf{K} = \mathbf{0}$  contributes. Define the pairing operator

$$\hat{\phi}_{\mathbf{k}} \equiv \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} \quad \hat{\phi}_{\mathbf{k}}^\dagger \equiv \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger$$

Then the non local repulsion reduces to the ordinary BCS-like interaction,

$$\hat{H}_U \simeq \sum_{\mathbf{k}} [\mathcal{U}_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} + \mathcal{U}_{\mathbf{k}}^* \hat{\phi}_{\mathbf{k}}^\dagger] \tag{1.1}$$

where the MFT parameter  $\mathcal{U}_{\mathbf{k}}$  must satisfy the self-consistency equation

$$\mathcal{U}_{\mathbf{k}} \equiv \frac{U}{L_x L_y} \sum_{\mathbf{k}} \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle \tag{1.2}$$

Note that  $\mathcal{U}_{\mathbf{k}}$  is actually momentum independent. This is due to the fact that the repulsion is completely localized.

**Non-local interaction  $V$ .** The non-local attraction in the opposite-spin sector of  $\hat{H}_V$  is given by

$$\hat{H}_V^{(\text{o.s.})} = -V \sum_{\langle ij \rangle} \sum_{\sigma} \hat{n}_{i\sigma} \hat{n}_{j\bar{\sigma}}$$

which, using Eq. (??) and performing Wick's contractions, reduces to:

$$\begin{aligned} \hat{H}_V^{(\text{o.s.})} &= -V \sum_{i \in S} \sum_{\ell=x,y} \sum_{\delta=\pm\delta_\ell} \hat{n}_{i\uparrow} \hat{n}_{i+\delta\downarrow} \\ &\simeq -V \sum_{i \in S} \sum_{\ell=x,y} \sum_{\delta=\pm\delta_\ell} [\langle \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta\downarrow}^\dagger \rangle \hat{c}_{i+\delta\downarrow} \hat{c}_{i\uparrow} + \hat{c}_{i\uparrow}^\dagger \hat{c}_{i+\delta\downarrow}^\dagger \langle \hat{c}_{i+\delta\downarrow} \hat{c}_{i\uparrow} \rangle] \end{aligned}$$

Using Eq. (??) we can move to reciprocal space,

$$\begin{aligned} \hat{H}_V^{(\text{o.s.})} &\simeq -\frac{V}{L_x L_y} \sum_{\mathbf{K}, \mathbf{k}, \mathbf{k}'} [\cos(\delta k_x) + \cos(\delta k_y)] \\ &\quad \times [\langle \hat{c}_{\mathbf{K}+\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{K}-\mathbf{k}\downarrow}^\dagger \rangle \hat{c}_{\mathbf{K}-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{K}+\mathbf{k}'\uparrow} + \hat{c}_{\mathbf{K}+\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{K}-\mathbf{k}\downarrow}^\dagger \langle \hat{c}_{\mathbf{K}-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{K}+\mathbf{k}'\uparrow} \rangle] \end{aligned}$$

Identical considerations as above hold, and just the  $\mathbf{K} = \mathbf{0}$  term contributes. We have finally

$$\hat{H}_V^{(\text{o.s.})} \simeq - \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left[ \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle \hat{\phi}_{\mathbf{k}'} + \langle \hat{\phi}_{\mathbf{k}} \rangle \hat{\phi}_{\mathbf{k}'}^\dagger \right]$$

where the two-body potential was defined

$$V_{\mathbf{k}\mathbf{k}'} = \frac{V}{L_x L_y} [\cos(\delta k_x) + \cos(\delta k_y)]$$

Making use of the decomposition of Eq. (??), the two-body potential becomes

$$\begin{aligned} V_{\mathbf{k}\mathbf{k}'} &= \frac{V}{2L_x L_y} \sum_{\gamma} \varphi_{\mathbf{k}}^{(\gamma)} \varphi_{\mathbf{k}'}^{(\gamma)*} \\ &= \sum_{\gamma} V^{(\gamma)} \varphi_{\mathbf{k}}^{(\gamma)} \varphi_{\mathbf{k}'}^{(\gamma)*} \end{aligned}$$

being  $\gamma \in \{s^*, p_x, p_y, d_{x^2-y^2}\}$  and  $\varphi_{\mathbf{k}}^{(\gamma)}$  the reciprocal-space expressions for the form factors of Tab. ??, listed explicitly in Tab. ??, and  $V_{\mathbf{k}\mathbf{k}'}^{(\gamma)}$  the symmetry-resolved components of the non-local attraction. Then the two-body potential has been decomposed in its planar symmetry components, each of which will naturally couple only to identically structured parameters in the full hamiltonian.

Define now the non-local gap function

$$\mathcal{V}_{\mathbf{k}} \equiv \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle \hat{\phi}_{\mathbf{k}'}^\dagger \rangle \quad (1.3)$$

one gets immediately

$$\hat{H}_V \simeq - \sum_{\mathbf{k}} \left[ \mathcal{V}_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} + \mathcal{V}_{\mathbf{k}}^* \hat{\phi}_{\mathbf{k}}^\dagger \right] \quad (1.4)$$

To assume symmetry is broken in a specific symmetry channel  $\gamma$  means precisely to assume  $\langle \hat{\phi}_{\mathbf{k}} \rangle \propto \varphi_{\mathbf{k}}^{(\gamma)}$ . Of course, in Eq. (1.3) only the  $\gamma$  component of the potential survives, implying the gap function acquires the same symmetry,

$$\mathcal{V}_{\mathbf{k}} \propto \sum_{\mathbf{k}'} \varphi_{\mathbf{k}}^{(\gamma)} \varphi_{\mathbf{k}'}^{(\gamma)*} \varphi_{\mathbf{k}'}^{(\gamma)} \propto \varphi_{\mathbf{k}}^{(\gamma)}$$

where orthonormality of the  $\varphi_{\mathbf{k}}^{(\gamma)}$  functions of Tab. ?? was used. Thus, assuming to have superconductivity in a given sector  $\gamma$ , the self consistency equation reads

$$\forall \gamma \in \{s^*, p_x, p_y, d_{x^2-y^2}\} \quad \mathcal{V}_{\mathbf{k}}|_{\gamma} \equiv \frac{V}{2L_x L_y} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{(\gamma)} \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle \quad (1.5)$$

Now we merge the two interaction in a single gap function.

### 1.2.1 Full gap function and self-consistency equations

Define the full gap function as the sum of both contributions,

$$\Delta_{\mathbf{k}} \equiv \mathcal{V}_{\mathbf{k}} - \mathcal{U}_{\mathbf{k}} \quad (1.6)$$

The full self-consistency equation is given by the simple combination of Eqns. (1.2) and (1.5),

$$\Delta_{\mathbf{k}} \equiv \sum_{\mathbf{k}'} \left[ V^{(s)} + V_{\mathbf{k}\mathbf{k}'} \right] \langle \hat{\phi}_{\mathbf{k}'}^\dagger \rangle \quad \text{with} \quad V^{(s)} = -\frac{U}{2L_x L_y} \quad (1.7)$$

The gap function decomposes in symmetry channels as well,

$$\Delta_{\mathbf{k}} = \sum_{\gamma} \Delta^{(\gamma)} \varphi_{\mathbf{k}}^{(\gamma)}$$

Structure	Self-consistency equation	Graph
$s$ -wave	$\Delta^{(s)} = -\frac{U}{2L_x L_y} \sum_{\mathbf{k}} \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$	Fig. ??
Extended $s$ -wave	$\Delta^{(s^*)} = \frac{V}{L_x L_y} \sum_{\mathbf{k}} (c_x + c_y) \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$	Fig. ??
$p_x$ -wave	$\Delta^{(p_x)} = -i\sqrt{2} \frac{V}{L_x L_y} \sum_{\mathbf{k}} s_x \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$	Fig. ??
$p_y$ -wave	$\Delta^{(p_y)} = -i\sqrt{2} \frac{V}{L_x L_y} \sum_{\mathbf{k}} s_y \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$	Fig. ??
$d_{x^2-y^2}$ -wave	$\Delta^{(d)} = \frac{V}{L_x L_y} \sum_{\mathbf{k}} (c_x - c_y) \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$	Fig. ??

**Table 1.2** Symmetry resolved self-consistency equations for the MFT parameters  $\Delta^{(\gamma)}$ , based on Eq. (1.7) and (1.8). By computing  $\langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle$ , it is possible to reconstruct the various components of the gap function.

If SC arises in a specific symmetry channel,  $\Delta_{\mathbf{k}}$  will show the same symmetry. It follows, due to orthonormality and using Eq. (1.7),

$$\begin{aligned}
\Delta^{(\gamma)} &= \frac{1}{L_x L_y} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{(\gamma)*} \Delta_{\mathbf{k}} \\
&= \frac{1}{L_x L_y} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{(\gamma)*} \sum_{\mathbf{k}'} \left[ V^{(s)} + V_{\mathbf{k}\mathbf{k}'} \right] \langle \hat{\phi}_{\mathbf{k}'}^\dagger \rangle \\
&= \frac{1}{L_x L_y} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{(\gamma)*} \sum_{\mathbf{k}'\gamma'} V^{(\gamma')} \varphi_{\mathbf{k}'}^{(\gamma')} \varphi_{\mathbf{k}'}^{(\gamma')*} \langle \hat{\phi}_{\mathbf{k}'}^\dagger \rangle \\
&= V^{(\gamma)} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^{(\gamma)*} \langle \hat{\phi}_{\mathbf{k}}^\dagger \rangle
\end{aligned} \tag{1.8}$$

This result provides a set of self-consistency equations for each symmetry channel, listed in Tab. 1.2. Notice that to reconstruct self-consistently the full  $s$ -wave phase transition, the actual gap function is given by

$$\Delta^{(s)} + \Delta^{(s^*)}(c_x + c_y)$$

The  $s$ -wave transition is the only one equipped of both the local and the non-local parts. Within this structure, we are finally able to move to Nambu formalism.

### 1.3 Superconducting solution in the translationally invariant sector

As a first approach to anisotropic SC in the Hubbard model, let us discuss the superconducting solutions in the simple scenario where crystal translational invariance is preserved and thus the system can be treated by the means of common BCS. As will be discussed, the non-local attraction acts as a source of SC in all symmetry sectors.

#### 1.3.1 Nambu formalism and Bogoliubov transform

Define the Nambu spinor<sup>1</sup> as in BCS

$$\hat{\Psi}_{\mathbf{k}} \equiv \begin{bmatrix} \hat{c}_{\mathbf{k}\uparrow} \\ \hat{c}_{-\mathbf{k}\downarrow}^\dagger \end{bmatrix}$$

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<sup>1</sup>Notice that the spinor is here differently defined with respect to App. ??, where because of the HF prevalence in mean-field decoupling the spinor components were homogeneously fermions creations or destructions.

Evidently,

$$\phi_{\mathbf{k}} = \hat{\Psi}_{\mathbf{k}}^\dagger \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{\Psi}_{\mathbf{k}} \quad \phi_{\mathbf{k}}^\dagger = \hat{\Psi}_{\mathbf{k}}^\dagger \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hat{\Psi}_{\mathbf{k}} \quad (1.9)$$

The full hamiltonian is then given by:

$$\hat{H} = \sum_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}} h_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}} \quad h_{\mathbf{k}} \equiv \begin{bmatrix} \epsilon_{\mathbf{k}} & -\Delta_{\mathbf{k}}^* \\ -\Delta_{\mathbf{k}} & -\epsilon_{\mathbf{k}} \end{bmatrix} \quad (1.10)$$

Let  $\tau^\alpha$  for  $\alpha = x, y, z$  be the Pauli matrices. Define:

$$\hat{s}_{\mathbf{k}}^\alpha \equiv \hat{\Psi}_{\mathbf{k}}^\dagger \tau^\alpha \hat{\Psi}_{\mathbf{k}} \quad \text{for } \alpha = x, y, z$$

As can be shown easily, these operators realize spin-1/2 algebra.  $\hat{H}$  represents an ensemble of  $L_x L_y$  independent spins subject to pseudo-magnetic fields. Note that, differently from App. ?? where the chemical potential is inserted later (because in Nambu formalism it accounts for a diagonal term) here the chemical potential is part of the  $z$  component of the pseudo-magnetic field, since

$$\begin{aligned} \hat{n}_{\mathbf{k}\uparrow} + \hat{n}_{-\mathbf{k}\downarrow} &= \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}\uparrow} + \hat{c}_{-\mathbf{k}\downarrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow} \\ &= \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}\uparrow} - \hat{c}_{-\mathbf{k}\downarrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow} + \mathbb{I} \\ &= \hat{\Psi}_{\mathbf{k}}^\dagger \tau^z \hat{\Psi}_{\mathbf{k}} + \mathbb{I} \end{aligned} \quad (1.11)$$

and then it follows

$$\begin{aligned} -\mu \hat{N} &= -\mu \sum_{\mathbf{k} \in \text{BZ}} [\hat{n}_{\mathbf{k}\uparrow} + \hat{n}_{-\mathbf{k}\downarrow}] \\ &= -\mu \sum_{\mathbf{k} \in \text{BZ}} \hat{\Psi}_{\mathbf{k}}^\dagger \tau^z \hat{\Psi}_{\mathbf{k}} - \mu L_x L_y \end{aligned}$$

Then, adding a term  $-\mu \hat{N}$  to  $\hat{H}$ , apart from an irrelevant total energy increase, changes the pseudo-field whose explicit form becomes

$$\mathbf{b}_{\mathbf{k}} \equiv \begin{bmatrix} -\text{Re}\{\Delta_{\mathbf{k}}\} \\ -\text{Im}\{\Delta_{\mathbf{k}}\} \\ \epsilon_{\mathbf{k}} - \mu \end{bmatrix} \quad (1.12)$$

This hamiltonian behaves as an ensemble of spins in local magnetic fields precisely as in Eq. (??),

$$\hat{H} - \mu \hat{N} = \sum_{\mathbf{k} \in \text{BZ}} \mathbf{b}_{\mathbf{k}} \cdot \hat{\mathbf{s}}_{\mathbf{k}} \quad \text{where} \quad \hat{\mathbf{s}}_{\mathbf{k}\sigma} = \begin{bmatrix} \hat{s}_{\mathbf{k}}^x \\ \hat{s}_{\mathbf{k}}^y \\ \hat{s}_{\mathbf{k}}^z \end{bmatrix} \quad (1.13)$$

Proceed as in App. ?? and diagonalize via a rotation,

$$d_{\mathbf{k}} \equiv \begin{bmatrix} -E_{\mathbf{k}} \\ E_{\mathbf{k}} \end{bmatrix} \quad \text{being} \quad E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$$

and  $\xi_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu$ . Given the pseudoangles

$$\tan(2\theta_{\mathbf{k}}) \equiv \frac{|\Delta_{\mathbf{k}}|}{\epsilon_{\mathbf{k}}} \quad \tan(2\zeta_{\mathbf{k}}) \equiv \frac{\text{Im}\{\Delta_{\mathbf{k}}\}}{\text{Re}\{\Delta_{\mathbf{k}}\}}$$

the general diagonalizer will be an orthogonal rotation matrix

$$\begin{aligned} W_{\mathbf{k}} &= e^{i(\theta_{\mathbf{k}} - \frac{\pi}{2})\tau^y} e^{i\zeta_{\mathbf{k}}\tau^z} \\ &= \begin{bmatrix} -\sin \theta_{\mathbf{k}} & -\cos \theta_{\mathbf{k}} \\ \cos \theta_{\mathbf{k}} & -\sin \theta_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} e^{i\zeta_{\mathbf{k}}} & \\ & e^{-i\zeta_{\mathbf{k}}} \end{bmatrix} \\ &= \begin{bmatrix} -\sin \theta_{\mathbf{k}} e^{i\zeta_{\mathbf{k}}} & -\cos \theta_{\mathbf{k}} e^{-i\zeta_{\mathbf{k}}} \\ \cos \theta_{\mathbf{k}} e^{i\zeta_{\mathbf{k}}} & -\sin \theta_{\mathbf{k}} e^{-i\zeta_{\mathbf{k}}} \end{bmatrix} \end{aligned} \quad (1.14)$$

given by a rotation of angle  $\zeta_{\mathbf{k}}$  around the  $z$  axis, to align the  $x$  axis with the field projection onto the  $xy$  plane, followed by a rotation around the  $y$  axis to anti-align with the pseudo-field. The MFT-BCS solution is given by a degenerate Fermi gas at ground state, whose quasi-particles occupy two bands  $\pm E_{\mathbf{k}}$  and their fermionic operators are given by

$$\hat{\gamma}_{\mathbf{k}}^{(-)} \equiv [W_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}}]_1 \quad \hat{\gamma}_{\mathbf{k}}^{(+)} \equiv [W_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}}]_2$$

The diagonalization operators are given by

$$\hat{\Gamma}_{\mathbf{k}} \equiv W_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}} \quad \text{where} \quad \hat{\Gamma}_{\mathbf{k}} = \begin{bmatrix} \hat{\gamma}_{\mathbf{k}}^{(-)} \\ \hat{\gamma}_{\mathbf{k}}^{(+)} \end{bmatrix}$$

then, using Eq. (??),

$$\langle [\hat{\Psi}_{\mathbf{k}}^{\dagger}]_i [\hat{\Psi}_{\mathbf{k}}]_j \rangle = [W_{\mathbf{k}}]_{1i} [W_{\mathbf{k}}^{\dagger}]_{j1} f(-E_{\mathbf{k}}; \beta, 0) + [W_{\mathbf{k}}]_{2i} [W_{\mathbf{k}}^{\dagger}]_{j2} f(E_{\mathbf{k}}; \beta, 0)$$

where in the Fermi-Dirac function chemical potential was set to zero, because it already was included in the diagonalized hamiltonian. Recalling Eq. (??), it follows

$$\langle \phi_{\mathbf{k}}^{\dagger} \rangle = [W_{\mathbf{k}}]_{11} [W_{\mathbf{k}}^{\dagger}]_{21} f(-E_{\mathbf{k}}; \beta, 0) + [W_{\mathbf{k}}]_{21} [W_{\mathbf{k}}^{\dagger}]_{22} f(E_{\mathbf{k}}; \beta, 0) \quad (1.15)$$

$$= \frac{1}{2} \sin(2\theta_{\mathbf{k}}) e^{i2\zeta_{\mathbf{k}}} \tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) \quad (1.16)$$

The last passage has been obtained by computing the matrix element from the explicit form of  $W_{\mathbf{k}}$  of Eq. (1.14) and by the simple relation

$$\begin{aligned} \frac{1}{e^{-x} + 1} - \frac{1}{e^x + 1} &= \frac{e^x - 1}{e^x + 1} \\ &= \tanh\left(\frac{x}{2}\right) \end{aligned}$$

Eqns. (1.15), (1.16) give us both the algorithmic formula (first row) and its theoretical counterpart (second row) to compute the order parameters in the HF approach at each point in  $k$ -space ( $k_x, k_y$ ). We can finally derive the BCS self-consistency equation

$$\begin{aligned} \Delta_{\mathbf{k}} &\equiv \frac{1}{2} \sum_{\mathbf{k}'} \left[ V^{(s)} + V_{\mathbf{k}\mathbf{k}'} \right] \frac{|\Delta_{\mathbf{k}'}|}{\sqrt{\xi_{\mathbf{k}'}^2 + |\Delta_{\mathbf{k}'}|^2}} \exp\left\{ i \arctan\left(\frac{\text{Im}\{\Delta_{\mathbf{k}'}\}}{\text{Re}\{\Delta_{\mathbf{k}'}\}}\right) \right\} \tanh\left(\frac{\beta}{2} \sqrt{\xi_{\mathbf{k}'}^2 + |\Delta_{\mathbf{k}'}|^2}\right) \\ &= \frac{1}{2} \sum_{\mathbf{k}'} \left[ V^{(s)} + V_{\mathbf{k}\mathbf{k}'} \right] \frac{\Delta_{\mathbf{k}'}}{\sqrt{\xi_{\mathbf{k}'}^2 + |\Delta_{\mathbf{k}'}|^2}} \tanh\left(\frac{\beta}{2} \sqrt{\xi_{\mathbf{k}'}^2 + |\Delta_{\mathbf{k}'}|^2}\right) \end{aligned} \quad (1.17)$$

The whole point of the HF algorithm is to find an iterative solution for each symmetry channel, using the self-consistency equation projection of Tab. 1.2.

Notice that the  $z$  component of the spin operators is related to density: using Eq. (??),

$$\langle \hat{\Psi}_{\mathbf{k}}^{\dagger} \tau^z \hat{\Psi}_{\mathbf{k}} \rangle = \langle [\hat{\Psi}_{\mathbf{k}}^{\dagger}]_1 [\hat{\Psi}_{\mathbf{k}}]_1 \rangle - \langle [\hat{\Psi}_{\mathbf{k}}^{\dagger}]_2 [\hat{\Psi}_{\mathbf{k}}]_2 \rangle$$

I proceed as previously, and from Eq. (1.11),

$$\begin{aligned} \langle \hat{n}_{\mathbf{k}\uparrow} \rangle + \langle \hat{n}_{-\mathbf{k}\downarrow} \rangle &= 1 + \langle \hat{\Psi}_{\mathbf{k}}^{\dagger} \tau^z \hat{\Psi}_{\mathbf{k}} \rangle \\ &= 1 + \left( |[W_{\mathbf{k}}]_{11}|^2 - |[W_{\mathbf{k}}]_{12}|^2 \right) f(-E_{\mathbf{k}}; \beta, 0) \\ &\quad + \left( |[W_{\mathbf{k}}]_{21}|^2 - |[W_{\mathbf{k}}]_{22}|^2 \right) f(E_{\mathbf{k}}; \beta, 0) \end{aligned} \quad (1.18)$$

$$= 1 - \cos(2\theta_{\mathbf{k}}) \tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) \quad (1.19)$$

The expectation value for the density is needed in order to extract the optimal chemical potential  $\mu$  for the target density we aim to simulate at the given parametrization. This is numerically obtained by using Eq. (1.18) directly on the diagonalization matrix of  $h_{\mathbf{k}}$ .

## 1.4 Results of the HF algorihtm

[To be continued...]



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