① $f_n = f_0 + f_1 + \cdots + f_{n-2} + 1$ for $n \geq 2$ by induction

Step1: When $n=2$

$$\left.\begin{aligned} f_2 &= f_0 + f_1 = 1 + 0 = 1 \\ f_2 &= f_0 + 1 = 1 \end{aligned}\right\} \text{True}$$

Step2: Assume f_n is true for $n=k$

$$f_k = f_0 + f_1 + \cdots + f_{k-2} + 1$$

Step3: we have to prove f_n is true for $n=k+1$

$$f_{k+1} = f_0 + f_1 + \cdots + f_{k-1} + 1 \text{ is true}$$

$$f_{k+1} = \underbrace{f_0 + f_1 + \cdots + f_{k-2} + 1}_{f_k} + f_{k-1}$$

$$f_{k+1} = f_k + f_{k-1} \quad \text{— from induction}$$

<u>Similarly</u>

$$f_{k+1} = f_k + f_{k-1} \text{ from RHS}$$

$\therefore f_n$ is true for $n \geq 2$

② Prove that f_{3n} is even

Step1: When $n=0$,

$F_{3\cdot 0} = f_0 = 0$ (which is even, true)

Step2: Assume F_{3n} is even for some $K \geq 0$.

ie. F_{3K} is even

Step3: we have to prove $F_{3(K+1)}$ is true for $n=K+1$

$$F_{3(K+1)} = F_{3K+3}$$

$$\begin{aligned} F_{3K+3} &= F_{3K+3-1} + F_{3K+3-2} \\ &= F_{3K+2} + F_{3K+1} \\ &= F_{3K+1} + F_{3K} + F_{3K+1} \\ &= 2 \cdot F_{3K+1} + F_{3K} \end{aligned}$$

Since $3(K+1) \geq 3$, F_{3K} is even (from assumption), and $2 \cdot F_{3K+1}$ is even, because it is divisible by 2. $\therefore f_{3n}$ is even.

③ Prove that $1.1! + 2.2! + \cdots + n.n! = (n+1)!-1$ for $n \geq 1$

Step 1: when $n=1$

$P(1): 1.1! = (1+1)!-1$

$P(1): 1 = 2-1$ (True)

Step 2: Assume $P(n)$ is true for $n=k$. i.e;

$P(k) = 1.1! + 2.2! + \cdots + k.k! = (k+1)!-1$

Step 3: we have to prove $P(n)$ is true, when $n=k+1$ ie

$P(k+1) : 1.1! + 2.2! + \cdots + (k+1)(k+1)! = (k+2)!-1$

$P(k+1) : \underbrace{1.1! + 2.2! + \cdots + k.k!} + (k+1)(k+1)! = (k+2)!-1$

$P(k+1) : (k+1)!-1 + (k+1)(k+1)! = (k+2)!-1$

$P(k+1) : (k+1+1)(k+1)! - 1 = (k+2)!-1$

$P(k+1) : (k+2)(k+1)! - 1 = (k+2)!-1$

$P(k+1) : (k+2)!-1 = (k+2)!-1$

$\therefore P(n)$ is true for $n \geq 1$.

(4) Prove that the ith Fibonacci number satisfies the equality

$$f_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

$$\phi^i = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi}^i = \frac{1-\sqrt{5}}{2}$$

Step1: When $i=0$

$$f_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \text{ (True)}$$

Step2: Assume f_i is true, when $i=k$ i.e,

$$f_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$$

Step3: we have to prove f_{k+1} is true. i.e, $f_{k+1} = \frac{\phi^{k+1} - \hat{\phi}^{k+1}}{\sqrt{5}}$

we know,

$$f_{k+1} = f_k + f_{k-1}$$

$$= \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} + \frac{\phi^{k-1} - \hat{\phi}^{k-1}}{\sqrt{5}}$$

$$= \frac{\phi^k - \hat{\phi}^k + \phi^{k-1} - \hat{\phi}^{k-1}}{\sqrt{5}}$$

$$= \frac{\phi^k - \hat{\phi}^k + \phi^k \cdot \frac{1}{\phi} - \hat{\phi}^k \cdot \frac{1}{\hat{\phi}}}{\sqrt{5}}$$

$$= \frac{\phi^k + \phi^k \cdot \frac{1}{\phi} - \hat{\phi}^k - \hat{\phi}^k \cdot \frac{1}{\hat{\phi}}}{\sqrt{5}}$$

$$= \frac{\phi^k \left(1 + \frac{1}{\phi}\right) - \hat{\phi}^k \left(1 + \frac{1}{\hat{\phi}}\right)}{\sqrt{5}}$$

$$= \frac{\phi^k \left(\frac{\phi+1}{\phi}\right) - \hat{\phi}^k \left(\frac{\hat{\phi}+1}{\hat{\phi}}\right)}{\sqrt{5}}$$

$$= \frac{\phi^k \cdot \frac{\phi^2}{\phi} - \hat{\phi}^k \cdot \frac{\hat{\phi}^2}{\hat{\phi}}}{\sqrt{5}} \qquad (\because \phi^2 = \phi + 1)$$

$$= \frac{\phi^k \cdot \phi - \hat{\phi}^k \cdot \hat{\phi}}{\sqrt{5}}$$

$$= \frac{\phi^{k+1} - \hat{\phi}^{k+1}}{\sqrt{5}}$$

(5) Prove that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n}$ for $n \geq 1$.

Step 1: When $n=1$

$P(n) = \frac{1}{\sqrt{1}} = 1 \geq \sqrt{1}$ (True)

Step 2: Assume $P(n)$ is true for $n=k$ ie

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}} \geq \sqrt{k}$$

Step 3: we have to prove $P(n)$ is true for $n=k+1$ ie

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k+1}}$$

$$= \underbrace{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}}+\frac{1}{\sqrt{k+1}}$$

$$= \sqrt{k}+\frac{1}{\sqrt{k+1}}$$

$$= \frac{\sqrt{k}\cdot\sqrt{k+1}+1}{\sqrt{k+1}}$$

$$= \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$

$$= \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}}$$

Since $k \geq 1$

$$\frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} \geq \frac{\sqrt{k^2}+1}{\sqrt{k+1}}$$

$$\frac{\sqrt{k^2}+1}{\sqrt{k+1}}$$

$$= \frac{k+1}{\sqrt{k+1}}$$

$$= \sqrt{k+1}$$

$\therefore \sqrt{k+1} \geq \sqrt{k+1}$ (Ans)

$\therefore$ By induction $P(n)$ is true for $n \geq 1$

⑥ Prove that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} < 2-\frac{1}{n}$ for $n \geq 2$

Step 1: when $n=2$,
$$P(2) = 1+\frac{1}{4} < 2-\frac{1}{2}$$
$$= \frac{5}{4} < \frac{3}{2} \quad \text{(True)}$$

Step 2: Assume $P(n)$ is true for $n=k$ i.e.;
$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2} < 2-\frac{1}{k} \quad \text{for } k \geq 2$$

Step 3: we have to show $P(n)$ is true for $n=k+1$ ie;
$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2} < 2-\frac{1}{k+1}$$

From Step 2:
$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2} < 2-\frac{1}{k}$$

Adding $\frac{1}{(k+1)^2}$ to both sides
$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

Lhs of Step 3 is
$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} = \sum_{i=1}^{k}\frac{1}{i^2}+\frac{1}{(k+1)^2}$$

Rhs of Step 3 is $2-\frac{1}{k}+\frac{1}{(k+1)^2}$

or,
$$\sum_{i=1}^{k}\frac{1}{i^2}+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$
$$< 2-\left\{\frac{(k+1)^2-k}{k(k+1)^2}\right\}$$
$$< 2-\frac{k^2+2k+1-k}{k(k+1)^2}$$
$$< 2-\frac{k^2+k+1}{k(k+1)^2}$$

$$< 2 - \frac{k^2+k+1}{k(k+1)^2}$$

~~[illegible]~~

Since $2 - \frac{k^2+k}{k(k+1)^2} > 2 - \frac{k^2+k+1}{k(k+1)^2}$

when $k \geq 2$

Ok,

$$2 - \frac{k^2+k+1}{k(k+1)^2}$$

$$< 2 - \frac{k^2+k}{k(k+1)^2}$$

$$< 2 - \frac{\cancel{k(k+1)}}{\cancel{k}(k+1)^{\cancel{2}}}$$

~~[illegible]~~

$$< 2 - \frac{1}{k+1}$$

$\therefore \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$\therefore$ $P(n)$ is true for $n \geq 2$ (By Induction)

① Prove that 21 divides $4^{n+1}+5^{2n-1}$ for $n \geq 1$

Step1: When $n=1$ $\quad P(n): 4^{1+1}+5^{2\cdot 1-1} = 4^2+5 = 16+5 = 21$ $\quad \therefore$ 21 divides $P(1)$ (True)

Step 2: Assume $P(n)$ is true for $n=k$

i.e 21 divides $4^{k+1}+5^{2k-1}$

Step 3: we have to prove $P(n)$ is true when $n=k+1$

i.e. 21 divides $4^{(k+1)+1}+5^{2(k+1)-1}$

$$4^{(k+1)+1}+5^{2(k+1)-1}$$

$$= 4^{k+2}+5^{2k+2-1}$$

$$= 4^{k+2}+5^{2k+1}$$

$$= 4\cdot 4^{k+1}+5^{2k-1+2}$$

$$= 4\cdot 4^{k+1}+5^2\cdot 5^{2k-1}$$

$$= 4\cdot 4^{k+1}+25\cdot 5^{2k-1}$$

$$= 4\cdot 4^{k+1}+(21+4)\cdot 5^{2k-1}$$

$$= 4\cdot 4^{k+1}+4\cdot 5^{2k-1}+21\cdot 5^{2k-1}$$

$$= 4(4^{k+1}+5^{2k-1})+21\cdot 5^{2k+1}$$

we already know from Step 2

$(4^{k+1}+5^{2k-1})$ is divided by 21

$\therefore$ $4(4^{k+1}+5^{2k-1})$ is divided by 21

Similarly $21\cdot 5^{2k+1}$ is divided by 21.

$\therefore$ $P(n)$ is ~~prove~~ true for $n \geq 1$ (By Induction)

Pramod Nepal   Replacement for 8

Homework 1 Substitute Problem: Exchange for one problem on Homework 1

For each function category and time value given below, find the largest input size that can be computed on a machine that runs 1,000,000,000 instructions per second in the allotted time – for example, how much input can be processed by an n^2 algorithm in 1 minute?

function	1 second	1 minute 60s*	3 hours 10800s	3 days 259,200s	3 months 23,328,000s	3 years 94,608,000s	100,000,000,000s (which is about 3200 years)
n	1,000,000,000	60,000,000,000	10.8×10^{12}	259.2×10^{12}	23.328×10^{15}	94.608×10^{15}	1×10^{20}
n^2	31622	244948	3286335	16099689	152735064	307584134	1×10^{10}
n^3	1000	3914	22104	63759	285732	455661	4641588
2^n	29	35	43	47	54	56	66
$n!$	12	13	15	16	18	18	21

9) Use induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

Step 1: When $n=1$,

$P(n)$ is a set of $1+1$ positive integers, none exceeding 2×1

i.e $P(1) = \{1,2\}$. Here 1 divides 2, therefore true

Step 2: Assume $P(n)$ is true, when $n=k$. i.e;

$P(n)$ is a set of $k+1$ positive integers, none exceeding $2k$, there is at least one integer in this set that divides another integer in the set.

Step 3: we have to prove $P(n)$ is true, when $n=k+1$.

When $n=k+1$, three cases may arise

Case 1: If $P(k+1)$ does not contain the elements $2k+1$, and $2k+2$.
Then set is similar to Step 2. Thus we know there is at least one integer in this set that divides another integer in the set

Case 2: If either of $2k+1$ or $2k+2$ element is added to the new set, but not both (because the new set contains only $n+2$ elements).

- If our set adds $2k+1$ integer.

a) - If we discard $2k+1$ integer, our set still contains one integer that divides another integer (from step 2)

- If our set adds $2k+2$ integer, there is one integer $k+1$ that divides $2(k+1)$.

Case 3: If our set contains both $2k+1$ and $2k+2$ integers and removes one of the existing integers (because the new set contains only $n+2$ elements). In this case we have to replace $2(k+1)$ with $k+1$ ($\because$ $k+1$ divides $2(k+1)$) so that our set becomes like above (case 2), thus valid for our case (at least one integer divides another).

$\therefore$ $P(n)$ is true, by induction.

(10) Use the definition of big Oh to show that $3n^2+2n+7 \in O(n^3)$ i.e, you will need to find a $\underline{c}$ and $\underline{k}$ and do small proof.

→ we know from the definition of big-Oh

$$f(n) \leq c \cdot g(n)$$

∴ we have $3n^2+2n+7 \leq c \cdot n^3$

when $c = 3+2+7 = 12$

$$3n^2+2n+7 \leq 3n^2+2n^3+7n^3$$

when $n >= 1$

$$3+2+7 \leq 3+2+7$$

∴ The equation is true, when $n \geq 1$ & $c = 12$

11) 3-3(a) Ranking

1. 1, $n^{\frac{1}{\lg n}}$
2. $\ln \ln n$
3. $\sqrt{\lg n}$
4. $\ln n$
5. $\lg^2 n$
6. $\sqrt{n}$, $(\sqrt{2})^{\lg n}$
7. n, $2^{\lg n}$
8. $\lg(n!)$, $n \lg n$
9. n^2, $4^{\lg n}$, $2^{2\lg n}$
10. n^3
11. $n^{\lg \lg n}$, $(\lg n)^{\lg n}$
12. $(3/2)^n$
13. 2^n
14. $n \cdot 2^n$
15. e^n
16. $(\lg n)!$
17. $n!$
18. $(n+1)!$
19. 2^{2^n}
20. $2^{2^{n+1}}$

11) 3-3(c) Ranking.

① $\ln\ln n > 1$: $\lim_{n\to\infty} \frac{1}{\ln\ln n} = 0$

② $\sqrt{\lg n} > \ln\ln n$: $\lim_{n\to\infty} \frac{\ln\ln n}{\sqrt{\lg n}}$

$= \frac{\ln\ln n}{\sqrt{\frac{\ln n}{\ln 2}}}$

$\lim_{n\to\infty} = \frac{\ln\ln n}{\ln n^{1/2}} \times \sqrt{\ln 2}$

$\lim_{n\to\infty} = \frac{\frac{1}{\ln n} \times \sqrt{\ln 2} \times \frac{1}{n}}{\frac{1}{2} \times \frac{1}{n}}$

$= \frac{2\sqrt{\ln 2}}{\ln n} = 0$

③ $\ln n > \sqrt{\lg n}$: $\lim_{n\to\infty} \frac{\sqrt{\lg n}}{\ln n}$

$= \lim_{n\to\infty} \frac{\sqrt{\ln n}}{\sqrt{\ln 2}\cdot \ln n} = \lim_{n\to\infty} \frac{1}{\sqrt{\ln 2}\cdot\sqrt{\ln n}} = 0$

Q.4 $\lg^2 n > \ln n$: $\lim_{n\to\infty} \frac{\ln n}{\lg^2 n} = \lim_{n\to\infty} \frac{\ln n}{\left(\frac{\ln n}{\ln 2}\right)^2} = \lim_{n\to\infty} \frac{\ln n \times (\ln 2)^2}{(\ln n)^2} = \lim_{n\to\infty} \frac{(\ln 2)^2}{\ln n} = 0$

Q.5 $(\sqrt{2})^{\lg n} > \lg^2 n$: $\lim_{n\to\infty} \frac{\lg^2 n}{(\sqrt{2})^{\lg n}} = \lim_{n\to\infty} \frac{\lg^2 n}{2^{1/2 \lg n}} = \lim_{n\to\infty} \frac{\lg^2 n}{2^{\lg n^{1/2}}}$

$$= \lim_{n\to\infty} \frac{\lg^2 n}{n^{1/2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{(\ln 2)^2 \times n^{1/2}} = \lim_{n\to\infty} \frac{2 \ln n \times \frac{1}{n}}{(\ln 2)^2 \times \frac{1}{2} n^{-1/2}}$$

$$= \lim_{n\to\infty} \frac{4 \ln n}{(\ln 2)^2 \times n^{-1/2+1}} = \lim_{n\to\infty} \frac{4 \ln n}{(\ln 2)^2 \times n^{1/2}}$$

$$= \lim_{n\to\infty} \frac{4 \times \frac{1}{n}}{(\ln 2)^2 \times \frac{1}{2} n^{-1/2}} = \lim_{n\to\infty} \frac{8}{(\ln 2)^2 n^{1-1/2}} = \lim_{n\to\infty} \frac{8}{(\ln 2)^2 \times n^{1/2}} = 0$$

Q.6 $2^{\lg n} > (\sqrt{2})^{\lg n}$: $\lim_{n\to\infty} \frac{(\sqrt{2})^{\lg n}}{2^{\lg n}}$

$$= \lim_{n\to\infty} \frac{\sqrt{2}^{\lg n}}{\sqrt{2}^{\lg n} \cdot \sqrt{2}^{\lg n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2}^{\lg n}} = 0$$

Q.7 $n \lg n > n$: $\lim_{n\to\infty} \frac{n}{n \lg(n)} = \lim_{n\to\infty} \frac{n \times \ln 2}{n \ln n}$

$$= \lim_{n\to\infty} \frac{\ln 2}{\ln n} = 0$$

1) 3-3(a) Ranking

Q.8 $n^2 > n\lg n$: $\lim_{n\to\infty} \frac{n\lg n}{n^2}$

$$= \lim_{n\to\infty} \frac{\lg n + n\cdot\frac{1}{n}}{2n}$$

$$= \lim_{n\to\infty} \frac{\lg n + 1}{2n}$$

$$= \lim_{n\to\infty} \frac{\frac{\ln n}{\ln 2} + 1}{2n}$$

$$= \lim_{n\to\infty} \frac{\ln n + \ln 2}{2n \times \ln 2}$$

$$= \lim_{n\to\infty} \frac{n\ln n}{2\ln 2} \times \frac{1}{n^2}$$

$$= \lim_{n\to\infty} \frac{\ln n}{\ln 2 \times n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\ln 2}$$

$$= \lim_{n\to\infty} \frac{1}{n\ln 2} = 0$$

Q.9 $n^3 > n^2$: $\lim_{n\to\infty} \frac{n^2}{n^3}$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

Q.10 $n^{\lg\lg n} > n^3$: no proof

Q.11 $(3/2)^n > n^{\lg\lg n}$: no proof

Q.12 $2^n > (3/2)^n$: no proof

Q.13 $n\cdot 2^n > 2^n$: $\lim_{n\to\infty} \frac{2^n}{n2^n} = \frac{1}{n} = 0$

Q.14 $e^n > n2^n$: $\lim_{n\to\infty} \frac{n2^n}{e^n}$

$$= \lim_{n\to\infty} \frac{\ln(n2^n)}{\ln(e^n)}$$

$$= \lim_{n\to\infty} \frac{\ln(n2^n)}{n}$$

$$\boxed{\begin{aligned} &\cancel{= \lim_{n\to\infty} \frac{n\ln(2n)}{n}} \\ &\cancel{= \lim_{n\to\infty} \frac{\ln(2n) + \frac{1}{2n}}{n}} \\ &\cancel{= \lim_{n\to\infty}} \end{aligned}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n} \ln 2n}{\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{2}{2n} = 0$$

Q.15 $(n+1)! > n!$: $\lim_{n\to\infty} \frac{n!}{(n+1)!}$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)\, n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Q.16 $2^{2^{n+1}} > 2^{2^n}$: $\lim_{n\to\infty} \frac{2^{2^n}}{2^{2^{n+1}}} = \lim_{n\to\infty} \frac{2^{2^n}}{2^{2^n} \times 2^2}$

$= 4$ (no-prog)

(12) $A = \langle a_1, a_2, \ldots, a_n \rangle$

How many elements of the input sequence need to be checked on average, assuming that the element being searched is equally likely to be any element in the array? How about in the worst case? What are the average-case and worst-case running times of linear search in Θ notation?

→ Assuming that the element being searched is equally likely to be any element in the array, a linear search may have to search through $n/2$ items. In the worst case it has to search through the entire array i.e, n items.

The average case runtime is $\Theta(n/2) = \Theta(n)$

& The worst case runtime is $\Theta(n)$