1 (a) $\sum_{i=1}^{20} (3i^2 - 2i + 2)$ Homework 2 [illegible]

$= 3\sum_{i=1}^{20} i^2 - 2\sum_{i=1}^{20} i + \sum_{i=1}^{20} 2$

$= \dfrac{3 \times n(n+1)(2n+1)}{6} - \dfrac{2 \times 20 \times (20+1)}{2} + 20 \times 2$

$\because n = 20$

$= \dfrac{3 \times 20 \times 21 \times 41}{6} - \dfrac{2 \times 20 \times 21}{2} + 20 \times 2$

$= 8610 - 420 + 40$

$= 8230$ //

(b) $\sum_{i=0}^{n} (3i^2 - 2i + 2)$

$= \sum_{i=1}^{n} (3i^2 - 2i + 2) - \sum_{i=0}^{0} (3i^2 - 2i + 2)$

$= \sum_{i=1}^{n} (3i^2 - 2i + 2) - 2$

$= 3\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 - 2$

$= \dfrac{3 \times n \times (n+1)(2n+1)}{6} - \dfrac{2 \times n \times (n+1)}{2} + 2n - 2$

$= \dfrac{n(n+1)(2n+1)}{2} - n(n+1) + 2n - 2$

$\dfrac{(n^2+n)(2n+1)}{2} - n^2 - n + 2n - 2$

$\dfrac{2n^3 + 2n^2 + n^2 + n}{2} - n^2 - n + 2n - 2$

$= \dfrac{2n^3 + \cancel{2n^2} + n^2 + n - \cancel{2n^2} - 2n + 4n - 4}{2}$

$= \dfrac{2n^3 + n^2 + n - 2n + 4n - 4}{2}$

$= \dfrac{2n^3 + n^2 + 3n - 4}{2}$ //

(c) $\sum_{i=0}^{\lg n} 2^i = \dfrac{2^{\lg n + 1} - 1}{2 - 1}$ $\left(\because \sum_{i=0}^{n} c^i = \dfrac{c^{n+1} - 1}{c - 1}\right)$

$= 2^{\lg n + 1} - 1$

$= 2 \cdot 2^{\lg n} - 1$

$= 2 \cdot n - 1$ //

(d) $\sum_{i=4}^{n} (3^i) = \sum_{i=0}^{n} 3^i - \sum_{i=0}^{3} 3^i$

$= \frac{3^{n+1}-1}{3-1} - \frac{3^{3+1}-1}{3-1}$

$= \frac{3^{n+1}-1}{2} - \frac{80}{2}$

$= \frac{3^{n+1}-1}{2} - 40$

4.2-1

Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

$P_1 = A_{11} \cdot B_{12} - A_{11} B_{22} = 1 \cdot 8 - 1 \cdot 2 = 6$

$P_2 = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} = 1 \cdot 2 + 3 \cdot 2 = 8$

$P_3 = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} = 7 \cdot 6 + 5 \cdot 6 = 42 + 30 = 72$

$P_4 = A_{22} \cdot B_{21} - A_{22} B_{11} = 5 \cdot 4 - 5 \cdot 6 = 20 - 30 = -10$

$P_5 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} B_{11} + A_{22} \cdot B_{22} = 1 \cdot 6 + 1 \cdot 2 + 5 \cdot 6 + 5 \cdot 2 = 6 + 2 + 30$

$P_6 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} = 3 \cdot 4 + 3 \cdot 2 - 5 \cdot 4 - 5 \cdot 2 = 12 + 6 - 20 - 10 =$

$P_7 = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} = 1 \cdot 6 + 1 \cdot 8 - 7 \cdot 6 - 7 \cdot 8 = 6 + 8 - 42 - 56 = -84$

$C_{11} = P_5 + P_4 - P_2 + P_6 = 48 + (-10) - 8 + (-12) = 18$

$C_{12} = P_1 + P_2 = 6 + 8 = 14$

$C_{21} = P_3 + P_4 = 72 + (-10) = 62$

$C_{22} = P_5 + P_1 - P_3 - P_7 = 48 + 6 - 72 - (-84) = 48 + 6 - 72 + 84 = 66$

$$\therefore C = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$$

4-1 Recurrence examples ($n \leq 2$)

(a) $T(n) = 2T(n/2) + n^4$

$a = 2,\ b = 2,\ f(n) = n^4$

$n^{\log_b a} = n^{\log_2 2} = n$

Here $f(n) > n$,

Case 3 applies. we need to show

$af(n/b) \leq cf(n)$ for some $c < 1$

$af(n/b) = 2f\left(\frac{n}{2}\right) = 2\left(\frac{n}{2}\right)^4 = \frac{n^4}{2^3} = \frac{n^4}{8} = \frac{1}{8}f(n)$

$\therefore\ T(n) = \Theta(n^4)$ //

(b) $T(n) = T(7n/10) + n$

$a = 1,\ b = 10/7,\ f(n) = n$

$n^{\log_b a} = n^{\log_{10/7} 1} = 0$

here, $f(n) > 0$

case 3 applies,

we need to show

$af(n/b) \leq cf(n)$ for some $c < 1$

$af(n/b) = 1(7n/10) = \frac{7n}{10} = \frac{7}{10}f(n)$

$\therefore\ T(n) = \Theta(n)$ //

(c) $T(n) = 16T(n/4) + n^2$

$a = 16,\ b = 4,\ f(n) = n^2$

$n^{\log_b a} = n^{\log_4 16} = n^2$

Here, $f(n) = n^2$

Case 2 applies.

$T(n) = \Theta(n^{\log_b a} \lg n)$

$= \Theta(n^{\log_4 16} \lg n)$

$= \Theta(n^2 \lg n)$ //

(d) $T(n) = 7T(n/3) + n^2$

$a = 7, b = 3, f(n) = n^2$

$n^{\log_b a} = n^{\log_3 7}$ $[\because 1 < \log_3 7 < 2]$

$\therefore f(n) = n^2 > n^{\log_3 a}$

Case 3 applies.

we need to show

$af(n/3) \leq cf(n)$ for some $c<1$

$af(n/3) = 7f(n/3) = 7(n/3)^2 = 7\frac{n^2}{9} = \frac{7}{9}f(n)$

$\therefore T(n) = \Theta(n^2)$ //

(e) $T(n) = 7T(n/2) + n^2$

$a = 7, b = 2, f(n) = n^2$

$n^{\log_b a} = n^{\log_2 7}$ $[\because 2 < \log_2 7 < 3]$

$\therefore f(n) = n^2 < \log_2 7$

Case 1 applies

$\therefore T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_2 7})$ //

(f) $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2, b = 4, f(n) = n^{1/2} = n^{0.5}$

$n^{\log_b a} = n^{\log_4 2} = n^{1/2} = n^{0.5}$

here $f(n) = n^{0.5}$

case 2 applies

$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(n^{0.5} \lg n) = \Theta(\sqrt{n} \lg n)$

4-1 g

$$T(n) = T(n-2) + n^2$$

$$T(n-2) = T(n-4) + (n-2)^2$$

$$T(n) = T(n-4) + (n-2)^2 + n^2$$

$$T(n-4) = T(n-6) + (n-4)^2$$

$$\therefore T(n) = T(n-6) + (n-4)^2 + (n-2)^2 + n^2$$

$$= \sum_{i=0}^{n-1} (n-2i)^2$$

$$= \sum_{i=0}^{n-1} (n^2 - 4ni + 4i^2)$$

$$= \sum_{i=0}^{n-1} n^2 - 4n \sum_{i=0}^{n-1} i + 4 \sum_{i=0}^{n-1} i^2$$

Considering the highest polynomial

$$4 \sum_{i=0}^{n-1} i^2$$

$$= 4 \sum_{i=1}^{n} i^2$$

$$= \frac{4 \times n(n+1) \times (2n+1)}{6}$$

$$= \Theta(n^3)$$

4-3 Recurrence Examples

(a) $T(n) = 4T(n/3) + n\lg n$

$a = 4,\ b = 3,\ f(n) = n\lg n$

$n^{\log_b a} = n^{\log_3 4} \approx n^{1.26}$

$f(n) = n\lg n < n^{1.26}$

Case 1 applies

$T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_3 4})$

(b) $T(n) = 3T(n/3) + n/\lg n$

$a = 3,\ b = 3,\ f(n) = \frac{n}{\lg n}$

$n^{\log_b a} = n^{\log_3 3} = n$

$f(n) = \frac{n}{\lg n} < n$

Case 1 applies

$T(n) = \Theta(n^{\log_3 3})$

$= \Theta(n)$

(c) $T(n) = 4T(n/2) + n^2\sqrt{n}$

$a = 4,\ b = 2,\ f(n) = n^{2+\frac{1}{2}} = n^{5/2}$

$n^{\log_b a} = n^{\log_2 4} = n^2$

$f(n) = n^{5/2} > n^2$

Case 3 applies

we have to show

$af(n/b) \le cf(n)$ for some $c<1$

$af(n/b) = 4(n/2)^{5/2} = 4\frac{n^{5/2}}{2^{2\sqrt{2}}} = \frac{1}{\sqrt{2}}n^{5/2}$

$T(n) = \Theta(n^{5/2})$

4-3 (d) $T(n) = 3T(n/3 - 2) + n/2$

$$T(n) = 3T\left(\frac{n-6}{3}\right) + n/2$$

$$= 3T(n/3) + n/2$$

$$a = 3,\ b = 3,\ f(n) = n/2$$

$$n^{\log_b a} = n^{\log_3 3} = n$$

$$\therefore f(n) = \frac{n}{2} \approx n$$

Case 2 applies

$$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(n \lg n),$$

4.3 (e) $T(n) = 2T(n/2) + n/\lg n$

$a = 2,\ b = 2,\ f(n) = n/\lg n$

$n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = \dfrac{n}{\lg n}$

$\lim_{n \to \infty} \dfrac{n}{\lg n}$

(apply L'Hopital's Rule)

$\dfrac{1}{1/n} = n$

$f(n) = n = n$

Case 2 applies

$T(n) = \Theta(n^{\log_b a} \lg n)$

$= \Theta(n \lg n)$

4-3-6 $T(n) = T(n/2) + T(n/4) + T(n/8) + n$

Our tree will grow as

$$n \rightarrow n/2, \frac{n}{4}, \frac{n}{8} = \frac{4n+2n+n}{8} = \frac{7n}{8} \rightarrow$$

Third will be $\frac{7}{8}$th of the second $= \left(\frac{7}{8}\right)^2 n$

$\therefore$

$$T(n) = n + (7/8)n + \left(\frac{7}{8}\right)^2 n + \left(\frac{7}{8}\right)^3 n + \cdots$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$ (Geometric Series)

$$T(n) \leq \frac{1}{1-7/8} n = 8n$$

If we double the left most branch which is n

$$T(n) \geq 2n$$

$$\therefore \quad T(n) = \Theta(n)$$

4-3 (g) $T(n) = T(n-1) + 1/n$

$T(n-1) = T(n-2) + \frac{1}{n-1}$

$T(n-2) = T(n-3) + \frac{1}{n-2}$

$T(n) = T(n-2) + \frac{1}{n-1} + \frac{1}{n}$

$T(n) = T(n-3) + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$

$T(n) = \sum_{i=0}^{n-1} \frac{1}{n-i}$

$= \sum_{i=1}^{n} \frac{1}{i}$ (harmonic series)

$= \Theta(\lg n)$

4-3 (h) $T(n) = T(n-1) + \lg n$

$T(n-1) = T(n-2) + \lg(n-1)$

$T(n-2) = T(n-3) + \lg(n-2)$

$T(n) = T(n-2) + \lg(n-1) + \lg n$

$T(n) = T(n-3) + \lg(n-2) + \lg(n-1) + \lg n$

$= \sum_{i=0}^{n-1} \lg(n-i)$

$= \sum_{i=1}^{n} \lg i \simeq n\lg n - 1.5 = \Theta(n\lg n)$

4-3 (i)

$$T(n) = T(n-2) + \frac{1}{\lg n}$$

$$T(n-2) = T(n-4) + \frac{1}{\lg(n-2)}$$

$$T(n-4) = T(n-6) + \frac{1}{\lg(n-6)}$$

$$\therefore T(n) = \sum_{K=1}^{n/2} \frac{1}{\lg(2-2K)}$$

$$= \sum_{K=1}^{n/2} \frac{1}{\lg(2K)}$$

$$= \sum_{K=1}^{\infty} \frac{1}{\lg K}$$

$$= \Theta(\lg \lg n)$$

4-3j $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

Suppose $n = 2^{\lg n}$

$n^{\frac{1}{2^k}} = 2^{\frac{\lg n}{2^k}}$

$2^{\frac{\lg n}{2^k}} = 2$

$\lg \frac{n}{2^k} = 1$

$\lg n = 2^k$

$\lg \lg n = k$

Assuming $T(n) \le Cn \lg\lg n$

$T(n) \le \sqrt{n}\, C \sqrt{n} \lg\lg\sqrt{n} + n$

$= cn \lg \frac{\lg n}{2} + n$

$= cn \lg\lg n - cn \lg 2 + n$

$= cn \lg\lg n + (1-c) n$ $\quad (c > 1)$

$= \Theta(n \lg\lg n)$

4-5

(a) If there are $n/2$ bad chips and $n/2$ good chips equal no. of both good & bad chips can identify themselves as good and others as bad. Although the goods chips are not lying, it is insufficient to find the good chips

(b) Assuming more than $n/2$ chips are good. If we pair the chips and keep only the chips that report both as good. When we discard other chips, we are discarding chip pairs in which at least one is bad. What we are left is a combination which has more good chips than bad chips.

(c) Assuming $n/2$ chips are good, we can compute a good chip. When we test the chip with others it have n-1 tests. Therefore the recurrence relation to find good chip is

$$T(n) = T(n/2) + n/2$$

According to master method

$$a = 1, b = 2, f(n) = n/2$$

$$n^{\log_b a} = 0 < f(n)$$

case 3 applies

we have

$$a f(n/b) \le c f(n)$$

$$1 \left(\frac{n}{4}\right) = \frac{1}{2}(n/4)$$

$$T(n) = \Theta(n)$$

(13) 1 ⓐ Source code submitted separately

1 ⓑ 48

ⓒ $f_n = f_{n-1} + f_{n-2}$

ⓓ $f_0 = 0$

$f_1 = 1$

$f_2 = f_0 + f_1 = 0+1 = 1$

$f_3 = f_1 + f_2 = 1+2 = 2$

$f_4 = f_3 + f_2 = 2+1 = 3$

we want to solve

$f_n = c_1 f_{n-1} + c_2 f_{n-2}$, where c_1 & c_2 are real numbers

The solution depends on the roots of the equation $x^2 - c_1 x - c_2 = 0$

The general solution for the recurrance realtion is $f_n = c_1 + c_2$

When we consider fibonacii series for f_0 and f_1, $c_1 = 1$ & $c_2 = 1$

Hence $f_n = f_{n-1} + f_{n-2}$

(13) 2 ⓐ Submitted separately

ⓑ To find, ion no. of steps $= n-1+1+1 = n+1 = 1001$

ⓒ, ⓓ

$T(0) = C$

$T(n) = C + T(n-1)$

If we know $T(n-1)$, we can find $T(n)$

Similarly

$T(n-1) = C + T(n-2)$

$T(n-2) = C + T(n-3)$

$T(n) = T(n-1) + C$

$= T(n-2) + C + C$

$= T(n-3) + C + C + C$

$= T(n-3) + 3C$

$T(n) = T(n-k) + kC$ for $k = 1$ to $n-1$

When $k = n$ $T(n) = T(n-k) + nC$

$= T(0) + nC$

$= C + nC = \Theta(n)$ //

13 (3) (a) Submitted separately

(b) To find 1^{1000} no. of steps $= \frac{N}{2} + 1 = 501$

(c) & (d)

$T(0) = c$

$T(1) = c$

$T(n) = T(n/2) + c$

$T(n/2) = T(n/4) + c$

$T(n/4) = T(n/8) + c$

$T(n) = T(n/4) + c + c$

$T(n) = T(n/8) + c + c + c$

$= T(n/8) + 3c$

$= T(n/2^3) + 3c$

$T(n) = T\left(\frac{n}{2^k}\right) + kc$

for $T(1)$

$\frac{n}{2^k} = 1$

$n = 2^k$

$\lg n = \lg 2^k$

$= k$

$\therefore T(n) = T\left(\frac{n}{2^{\lg n}}\right) + \lg n \, c$

$= T(n/n) + \lg n \, c$

$= T(1) + \lg n \, c$

(13) 4 (a) Submitted separately

(b)

$\frac{N}{4}+1=250+1=251$

(c) & (d)

$T(0)=c$

$T(1)=c$

$T(n)= T(n/2)+T(n/2)+c$

$= 2T(n/2)+c$

$T(n/2)= 2T(n/4)+c$

$T(n/4) = 2T(n/8)+c$

$T(n) = 2\{2T(n/4)+c\}+c$

$= 4T(n/4) + 3c$

$= 4\{2T(n/8)+c\}+3c$

$= 8T(n/8) + 7c$

$= 2^3 T(\frac{n}{2^3}) + (2^3-1)c$

$= 2^k T(n/2^k) + (2^k-1)c$

$T(0)= c$

$T(1) = c$

$T(n) = 2^k T(n/2^k) + (2^k-1)c$

Select k, such as $n/2^k = 1$

$\therefore \frac{n}{2^k}=1$

$\lg n = k$

$$T(n) = 2^{\lg n} T(n/2^{\lg n}) + (2^{\lg n} - 1)c$$

$$= nT\left(\frac{n}{n}\right) + (n-1)c$$

$$= nT(1) + (n-1)c$$

$$= nc + (n-1)c$$

$$= \Theta(n)$$

$$T(n) = 2^{\lg n} T\left(\frac{n}{2^{\lg n}}\right) + (2^{\lg n} - 1)c$$

$$= nT\left(\frac{n}{n}\right) + (n-1)c$$

$$= nT(1) + (n-1)c$$

$$= nc + (n-1)c$$

$$= \Theta(n)$$