Leibniz University Hannover
Institut of Differential Geometry
Winter Semester 2023/24
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Differential Topology
Exercise Sheet 4

Sumbission Deadline: Wednesday, 08.11., 23:59

Hinweise:

1. There are 12 exercise sheets in total. 40% of all points guarantee the exercise certificate.
2. Group submissions are not allowed.
3. Please include your name and matriculation number to your submission.
4. Please upload your solution to Stud.IP, in the folder

   `Übung zu Differentialtopologie→Dateien→Übungsblatt 4 Abgabe`

   or turn it in during the tutorial on 07.10..

Important Concepts: Homotopy, isotopy. Homotopy inverses, homotopy equivalent topological spaces. Retractions and deformation retractions. Topological sum, cells, gluing of two topological spaces by means of a continuous function. Smooth homotopies, orientation preserving diffeomorphisms. Homogeneity lemma. Degree modulo 2 of a map.

Satz 1 (Whitney Approximation Theorem)**.** *Every continuous map between differentiable manifolds M, N (without boundary) is homotopic to a $C^1(M,N)$ map.*

Definition 1. *A manifold M is called* simply connected *if it is connected and every continuous map $\gamma : \mathbb{S}^1 \to M$ is homotopic to a constant map.*

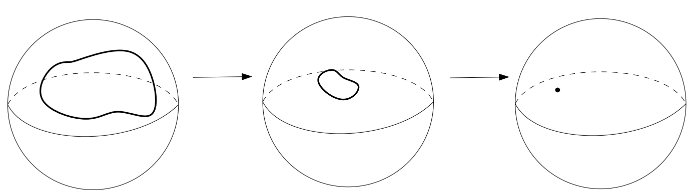

Figure 1: $\mathbb{S}^2$ is simply connected.

Exercise 4.1 (8+8 points)**.**

(i) Suppose that $f \in C^0_c(\mathbb{R}^n)$. Consider $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and let

$$f_\varepsilon(x) := \varepsilon^{-n} \int_{\mathbb{R}^n} \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)dy, \quad \varepsilon > 0.$$

Show that the map

$$H : [0,1] \times \mathbb{R}^n \to \mathbb{R}, \quad H(t,x) = \begin{cases} f_t(x), & t \in (0,1] \\ f(x), & t = 0 \end{cases},$$

is a homotopy satisfying $H(0,\cdot) = f$ *and* $H(t,\cdot) \in C^1(\mathbb{R}^n, \mathbb{R})$ *for all* $t > 0$.

(ii) Use Part i, in order to prove Whitney's Approximation Theorem in the case M *="arbitrary compact manifold" and* $N = \mathbb{S}^k$.

Hint for Part ii: Use a partition of unity.

Exercise 4.2 (4+4+2+4 points). *Using the following steps prove that every orientation-preserving automorphism of a 2-dimensional vector space* V *is isotopic to the identity (this also holds when* $\dim(V) > 2$*, but the proof is a bit more complicated).*

(i) Consider $a, b \in \mathbb{R}$ *with* $ab > 0$ *and let* $z \in \mathbb{C} \setminus \{0\}$. *Consider the matrices* $P_1, P_2 \in \mathrm{GL}(2,\mathbb{R})$

$$P_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad P_2 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Show that for $j = 1, 2$ *there exist continuous maps* $\gamma_j : [0,1] \to \mathrm{GL}(2,\mathbb{R})$ *such that*

$$\gamma_j(0) = P_j, \quad \gamma_j(1) = Id, \quad \det(\gamma_j(t)) > 0 \quad \forall t \in [0,1].$$

(ii) Consider the subgroup $\mathsf{U} \subseteq \mathrm{GL}(2,\mathbb{C})$, *where*

$$\mathsf{U} = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} : z \in \mathbb{C} \setminus \{0\} \right\}.$$

Prove that for every $R \in \mathsf{U}$ *there exists a continuous map* $\gamma_R : [0,1] \to \mathsf{U}$ *such that* $\gamma_R(0) = R$ *and* $\gamma_R(1) = Id$.

(iii) let V *be a* n*-dimensional real vector space, and let* $A \in \mathrm{Aut}(V)$ *be a linear automorphism of* V. *By choosing a basis for* V*, one can represent* A *as a matrix* $\tilde{A} \in \mathrm{GL}(n,\mathbb{R})$. *The* determinant *of* A *is defined as* $\det(A) := \det(\tilde{A})$. *Explain why* $\det(A)$ *is well defined for* $A \in \mathrm{Aut}(V)$ *(that is, why it is independent of the choice of basis).*

(iv) An automorphism $A \in \mathrm{Aut}(V)$ *is called* orientation-preserving *if* $\det(A) > 0$. *If* V *is 2-dimensional and* $A \in \mathrm{Aut}(V)$ *is orientation-preserving, use parts i-ii to show that* A *is isotopic to the identity.*

Exercise 4.3 (6+4 points). *Prove the following statements:*

(i) The sphere $\mathbb{S}^k$ *is simply connected when* $k > 1$.

(ii) The circle $\mathbb{S}^1$ *is not simply connected.*

Hint for part (i)*:* use Whitney's Approximation Theorem.