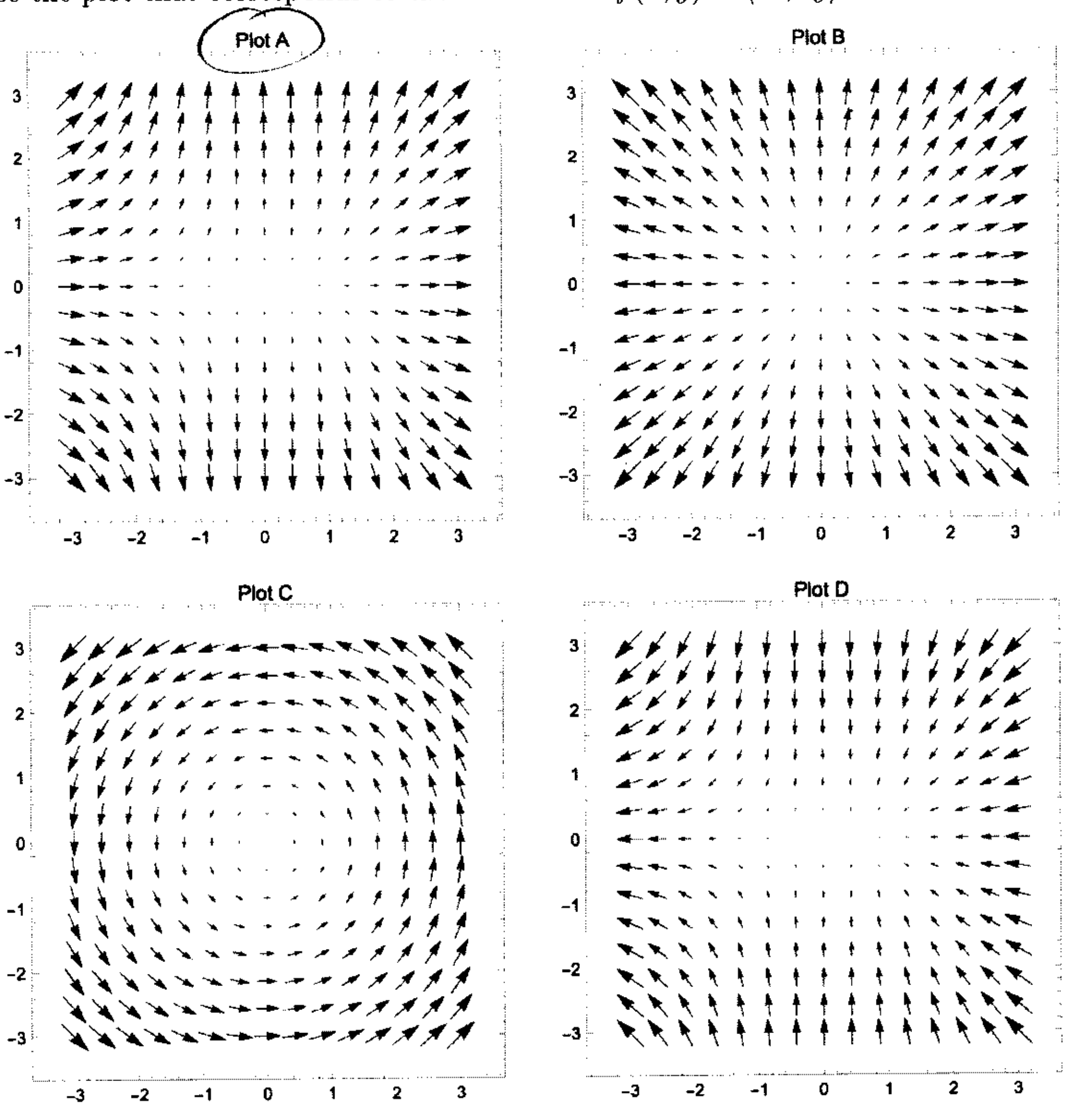
1. (8 pts) **You do not need to explain your answers for this problem.**

(a) Choose the plot that corresponds to the vector field $f(x, y) = \langle x^2, 3y \rangle$.



(b) Mark the following sentence as **true** or **false**. Let c be the unit circle in $\mathbb{R}^2$ parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x, y)$ we have

$$\int_{-c} f(x, y) ds = -\int_{c} f(x, y) ds.$$

True **False**

THIS PROBLEM IS CONTINUED IN THE NEXT PAGE

(c) Mark the correct answer: Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field in $\mathbb{R}^3$, where P, Q, R have continuous second partial derivatives. Then

$$(\text{div}(\text{curl}\,\vec{F}))\vec{F}$$

is

a. A vector field   **b.** A scalar field   **c.** Undefined (nonsense)

(d) Mark the correct answer: Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field in $\mathbb{R}^3$, where P, Q, R have continuous third partial derivatives. Then

$$\vec{F}\cdot\text{curl}(\nabla(\text{div}\,\vec{F}))$$

is

a. A vector field   **b.** A scalar field   **c.** Undefined (nonsense)

2. (6 pts) The mass of a wire lying on a curve c on the xy−plane and having density function $\rho(x,y)$ is given by

$$m = \int_c \rho(x,y)ds.$$

If c is the part of the ellipse $9x^2 + 4y^2 = 36$ between the points $(\sqrt{2}, \frac{3\sqrt{2}}{2})$ and $(-2, 0)$ satisfying $y \geq 0$, as in the picture, and $\rho(x,y) = y$, **set up but do not evaluate** an integral calculating the mass of the wire.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \begin{cases} x(t) = 2\cos t \\ y(t) = 3\sin t \end{cases} \Rightarrow \begin{cases} x'(t) = -2\sin t \\ y'(t) = 3\cos t \end{cases}$$

Get bounds

$$\begin{cases} x(t) = \sqrt{2} \\ y(t) = \frac{3\sqrt{2}}{2} \end{cases} \Rightarrow \begin{cases} 2\cos t = \sqrt{2} \\ 3\sin t = \frac{3\sqrt{2}}{2} \end{cases} \Rightarrow \begin{matrix} \cos t = \frac{\sqrt{2}}{2} \\ \sin t = \frac{\sqrt{2}}{2} \end{matrix}$$

$$\Rightarrow t = \frac{\pi}{4}$$

$$\begin{cases} x(t) = -2 \\ y(t) = 0 \end{cases} \Rightarrow t = \pi$$

Therefore

$$m = \int_{\frac{\pi}{4}}^{\pi} 3\sin t \sqrt{4\sin^2 t + 9\cos^2 t}\, dt$$

3. (10 pts) You are given the vector field

$$\vec{F}(x,y) = \langle 6\cos(y) + 2xy^3, -6x\sin(y) + 3x^2y^2 + 1\rangle$$

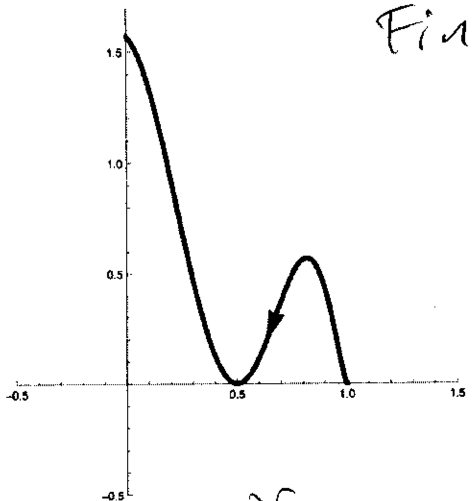
in $\mathbb{R}^2$ and the curve c paramatrized by $\vec{r}(t) = \langle \cos(t), t\sin^2(3t)\rangle$, $t \in [0, \frac{\pi}{2}]$ as in the picture.

a) Show that $\vec{F}$ is conservative.

$$\frac{\partial P}{\partial y} = -6\sin y + 6xy^2$$
$$\frac{\partial Q}{\partial x} = -6\sin y + 6xy^2$$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so conservative

b) Evaluate $\int_c \vec{F} \cdot d\vec{r}$ (show your work clearly).



Find pot. fct., say $f(x,y)$

$\frac{\partial f}{\partial x} = P \Rightarrow$

$\frac{\partial f}{\partial x} = 6\cos y + 2xy^3$

$f(x,y) = 6x\cos y + x^2y^3 + g(y)$

$\frac{\partial f}{\partial y} = Q \Rightarrow -6x\sin y + 3x^2y^2 + g'(y)$

$= -6x\sin y + 3x^2y^2 + 1$

$\Rightarrow g(y) = y + c$

So $f(x,y) = 6x\cos y + x^2y^3 + y + c$

$\vec{r}(0) = (1, 0)$, $\vec{r}(\frac{\pi}{2}) = \langle 0, \frac{\pi}{2}\rangle$. By FTC

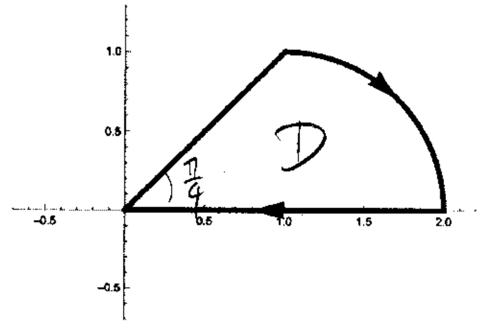
$\int_c \vec{F} \cdot d\vec{r} = f(\vec{r}(\frac{\pi}{2})) - f(\vec{r}(0)) =$

$= \frac{\pi}{2} + c - 6 - c = \frac{\pi}{2} - 6$

4. (9 pts) Let c be the curve of the picture, consisting of the following pieces:

- A line segment from (0,0) to (1,1)
- An arc of the circle $(x-1)^2 + y^2 = 1$ from (1,1) to (2,0) (the one that satisfies $y \geq 0$).
- A line segment from (2,0) to (0,0).

Evaluate $\int_c f(x,y)dx$, where $f(x,y) = xy^2$.



Use Green's thm

Orientation is negative:

$$\int_C f(x,y)dx = -\iint_D -\frac{\partial P}{\partial y}\, dA$$

$$= +\iint_D 2xy\, dA$$

Polar coords: $(x-1)^2+y^2=1 \Rightarrow x^2-2x+1+y^2=1$

$\Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

So

$$\iint_D 2xy\,dA = \int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} 2r\cos\theta\, r\sin\theta\, r\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 2\frac{(2\cos\theta)^4}{4} \cos\theta\sin\theta\,d\theta =$$

$$= 8\int_0^{\frac{\pi}{4}} \cos^5\theta\sin\theta\,d\theta = \frac{-8}{6}\cos^6\theta\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{-8}{6}\left(\left(\frac{\sqrt{2}}{2}\right)^6 - 1\right) = \frac{7}{6}$$

5. (8 pts) The temperature at a point (x, y) of the plane is given in degrees Celcius by

$$T(x, y) = x^2y^3 + 2\cos(3x\pi + y\pi),$$

where x and y are in meters. You are standing at the point (1,2) and you want to move towards the direction in which the temperature **drops** most rapidly (that is, the direction in which you have the minimum net rate of change of temperature).

(a) **Find a vector** that gives this direction.

Use $-\nabla T(1,2)$

$\nabla T(x,y) = \langle 2xy^3 - 2\sin(3x\pi + y\pi)3\pi,\ 3x^2y^2 - 2\sin(3x\pi + y\pi)\cdot\pi \rangle$

So

$\boxed{-\nabla T(1,2) = -\langle 16, 12 \rangle}$

(b) **Find the directional derivative of** T in the direction determined by this vector. **Make sure to include units in your answer.**

$|-\nabla T(1,2)| = \sqrt{16^2 + 12^2} = \sqrt{400} = 20$

So

$D_{\frac{-\nabla T(1,2)}{|\nabla T(1,2)|}} = -\frac{\nabla T(1,2)}{|\nabla T(1,2)|} \cdot \nabla T(1,2) = -20\ °C/m$

6. (9 pts) Let $z = z(x, y)$ be a twice differentiable function with continuous second partial derivatives and $x = x(t)$, $y = y(t)$ be differentiable with continuous first partial derivatives. You are given the following table with data. Use the chain rule to evaluate $\frac{d^2z}{dt^2}(0)$.

$x(0) = 1$	$y(0) = -1$	$z(1, -1) = -1$
$\frac{\partial z}{\partial x}(1, -1) = -2$	$\frac{\partial z}{\partial y}(1, -1) = 3$	$\frac{dx}{dt}(0) = 1$
$\frac{dy}{dt}(0) = 0$	$\frac{d^2x}{dt^2}(0) = 0$	$\frac{d^2y}{dt^2}(0) = 2$
$\frac{\partial^2 z}{\partial x^2}(1, -1) = -2$	$\frac{\partial^2 z}{\partial y^2}(1, -1) = -6$	$\frac{\partial^2 z}{\partial x \partial y}(1, -1) = 6$

Hint: Note that some of the partial derivatives given are 0; this may simplify your calculation.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

At $t=0$, $\frac{dy}{dt} = 0$ and $\frac{d^2x}{dt^2} = 0$

Find $\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)$:

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}$$

At $t=0$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = -2 \cdot 1 + 6 \cdot 0 = -2$$

So $\frac{d^2z}{dt^2}(0) = -2 \cdot 1 + 3 \cdot 2 = 4$