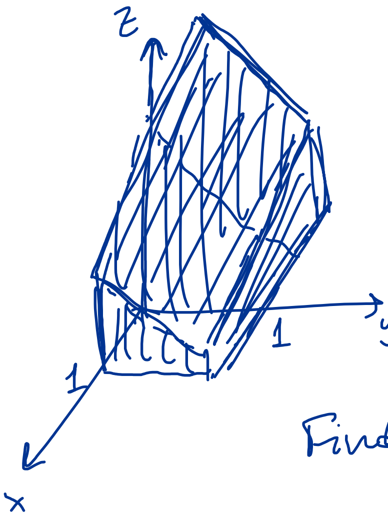
2. Set up an integral $\iiint_E f(x,y,z)dV$, where $E$ is bounded above by the plane $z = 3 - x - y$, below by the $xy$ plane, and also bounded by the planes $x = -1$, $x = 1$, $y = 0$ and $y = 1$ **in the order** $dxdzdy$.




① $x = 3 - z - y$

② $x = -1$

③ $x = 1$

④ $y = 0$

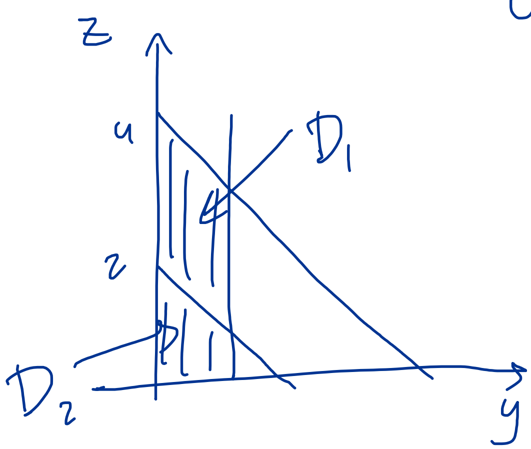
⑤ $y = 1$

Need sum of int's

Find projection on $yz$ plane

①, ② $\Rightarrow$ $z + y = 4$

①, ③ $\Rightarrow$ $z + y = 2$




Over $D_1$: $-1 \le x \le 3 - z - y$

and

$D_1 = \{(y,z) : 2 - y \le z \le 4 - y,\ 0 \le y \le 1\}$

Over $D_2$: $-1 \le x \le 1$

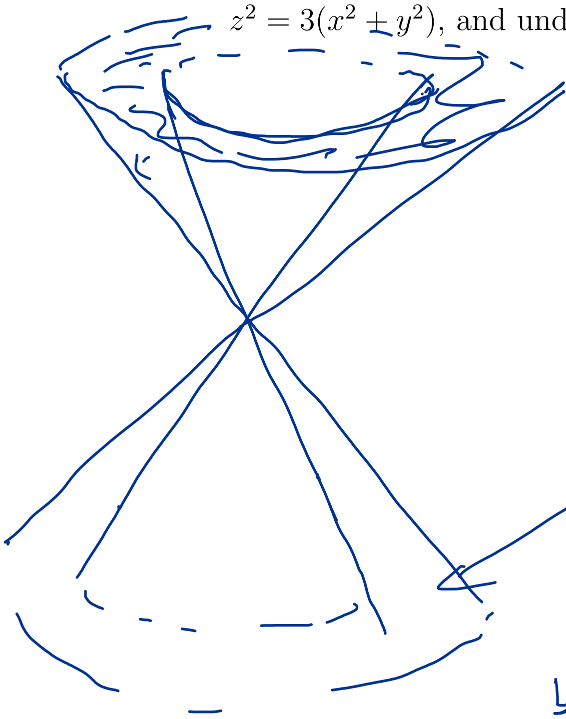
and

$D_2 = \{(y,z) : 0 \le z \le 2 - y \text{ and } 0 \le y \le 1\}$

So

$$\iiint_E f\,dV = \int_0^1 \int_0^{2-y} \int_{-1}^{1} f\,dx\,dz\,dy + \int_0^1 \int_{2-y}^{4-y} \int_{-1}^{3-z-y} f\,dx\,dz\,dy$$

3. Compute the volume of the bounded domain lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3(x^2 + y^2)$, and under the plane $z = 3$.



there is a domain here but it's not bounded.

Do it in spherical coords

$z = \sqrt{x^2+y^2} \Rightarrow$

$\rho \cos\varphi = \sqrt{\rho^2 \sin^2\varphi}$

$\Rightarrow \varphi = \frac{\pi}{4}$

$z = \sqrt{3}\sqrt{x^2+y^2} \Rightarrow$

$\rho\cos\varphi = \sqrt{3}\sqrt{\rho^2 \sin^2\varphi}$

$\Rightarrow \varphi = \frac{\pi}{6}$

So $\frac{\pi}{6} \le \varphi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$z = 3 \Rightarrow \rho\cos\varphi = 3 \Rightarrow \rho = \frac{3}{\cos\varphi}$

So

$$V = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\varphi}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\varphi \left.\frac{\rho^3}{3}\right|_0^{\frac{3}{\cos\varphi}} d\varphi \, d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\varphi \frac{9}{\cos^3\varphi} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} \cos^{-2}\varphi \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \int_0^{2\pi} \frac{9}{2}\left(\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\right) d\theta$$

$$= 9\pi\left(2 - \frac{4}{3}\right) = 6\pi$$

Done in Cylindrical coords:

3. Compute the volume of the bounded domain lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3(x^2 + y^2)$, and under the plane $z = 3$.

Write down equations solving for $z$:

① $z = \sqrt{x^2+y^2}$

② $z = \sqrt{3(x^2+y^2)}$

③ $z = 3$

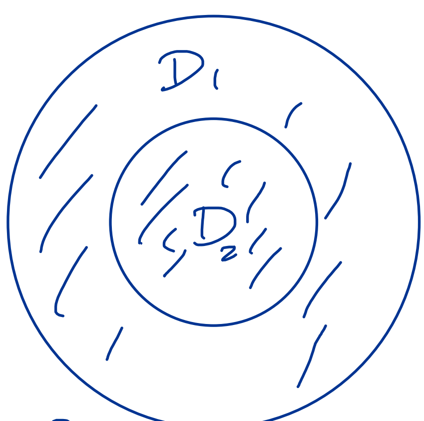
$z$ appears 3 times, so if we want $z$ to be the innermost variable we need a sum of 2 integrals

Find projection on $xy$ plane:

①,② $\Rightarrow \sqrt{x^2+y^2} = \sqrt{3}\sqrt{x^2+y^2} \Rightarrow \quad x = y = 0$ (only a point)

①,③ $\Rightarrow \sqrt{x^2+y^2} = 3 \Rightarrow \quad x^2+y^2 = 9$ (a circle)

②,③ $\Rightarrow \sqrt{3}\sqrt{x^2+y^2} = 3 \Rightarrow \quad x^2+y^2 = 3$ (circle)




Over $D_1 = \{(x,y): 3 \le x^2+y^2 \le 9\}$

$= \{(r,\theta) : \sqrt{3} \le r \le 3,\ \theta \in [0,2\pi]\}$

we have

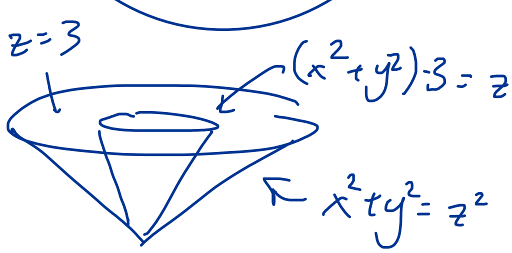
$\sqrt{x^2+y^2} \le z \le 3$ or $r \le z \le 3$

Over $D_2 = \{(x,y): 0 \le x^2+y^2 \le 3\}$

$= \{(r,\theta): 0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi\}$

we have

$\sqrt{x^2+y^2} \le z \le \sqrt{3(x^2+y^2)}$ or $r \le z \le \sqrt{3}r$




So:

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_r^{\sqrt{3}r} 1\cdot r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{3}\int_r^{3} 1\cdot r\,dz\,dr\,d\theta = 6\pi$$

4. If a transformation $T$ is written as $x = x(u,v)$ and $y = y(u,v)$ and given a point $(u_0, v_0)$ on the $uv$ plane, we can produce an affine transformation $dT$ called the **differential** or **pushforward** of $T$ at $(u_0, v_0)$ that is the best affine approximation of $T$ near $(u_0, v_0)$ and is given by

$$x = \frac{\partial x}{\partial u}_{|(u,v)=(u_0,v_0)}(u-u_0) + \frac{\partial x}{\partial v}_{|(u,v)=(u_0,v_0)}(v-v_0) + x(u_0,v_0)$$
$$y = \frac{\partial y}{\partial u}_{|(u,v)=(u_0,v_0)}(u-u_0) + \frac{\partial y}{\partial v}_{|(u,v)=(u_0,v_0)}(v-v_0) + y(u_0,v_0).$$

For the transformation $T(u,v) = (\frac{u^2}{v}, u^2 v)$ defined on $\{(u,v) : u > 0, v > 0\}$:

(a) Find the transformation $dT$ relative to the point (1,1).

(b) Find and draw the image of the box $[1,2] \times [1,2]$ under $T$ and $dT$.

a) $\frac{\partial x}{\partial u} = \frac{2u}{v}$ $\qquad$ $\frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$

$\frac{\partial y}{\partial u} = 2uv$ $\qquad$ $\frac{\partial y}{\partial v} = u^2$

So $dT_{(1,1)} = (2(u-1) - (v-1) + 1,\ 2(u-1) + (v-1) + 1)$

$= (2u - v,\ 2u + v - 2)$

or $x = 2u - v$

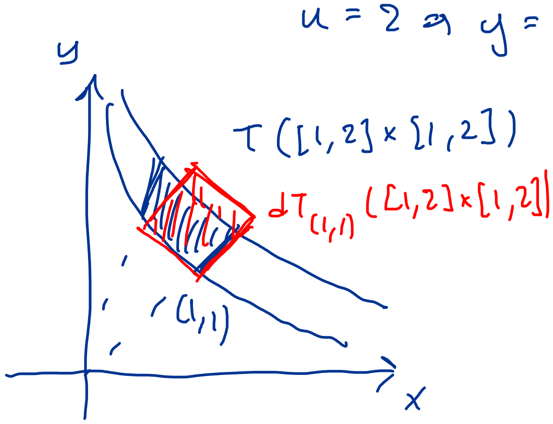
$y = 2u + v - 2$

b) Solve for $u, v$ in $T$:

$x = \frac{u^2}{v}$, $y = u^2 v$ $\Rightarrow$ $xy = u^4$, $\frac{y}{x} = v^2$ $\Rightarrow$ $u = \sqrt[4]{xy}$, $v = \sqrt{\frac{y}{x}}$

So $u = 1 \Rightarrow y = \frac{1}{x}$ $\qquad$ $v = 1 \Rightarrow y = x$

$u = 2 \Rightarrow y = \frac{16}{x}$ $\qquad$ $v = 2 \Rightarrow y = 4x$




solve for $u, v$ in $dT_{(1,1)}$

$x = 2u - v$, $y = 2u + v - 2$ $\Rightarrow$ $4u = x + y + 2$, $2v = y - x + 2$

$\Rightarrow$ $u = \frac{1}{4}(x+y) + \frac{1}{2}$

$v = \frac{1}{2}(y-x) + 1$

So $u = 1 \Rightarrow x + y = 2$

$u = 2 \Rightarrow x + y = 6$

$v = 1 \Rightarrow y = x$

$v = 2 \Rightarrow y = x + 2$

Drawn on the xy plane in red.

4. A fly flies in a room along the curve

$$c(t) = (2\sin(t), \cos(t), 2t).$$

The temperature at the point $(x, y, z)$ of the room is given by the function

$$T(x, y, z) = x^2 + 2e^z y,$$

in $^\circ$ C.

(a) Find the gradient of $T$.

(b) As the fly moves, it experiences a different temperature at each point. Find the instantaneous rate of change of the temperature that the fly experiences at time $\pi$ seconds (include units).

a) $\nabla T(x,y,z) = \langle 2x, 2e^z, 2e^z y \rangle$

b) Need: $\frac{d}{dt}(T \circ c)(t)$

Use Chain Rule: write $c(t) = (x(t), y(t), z(t))$

Then, using that $c(\pi) = (0, -1, 2\pi)$

$$\frac{d}{dt}(T \circ c)(t) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \nabla T(c(\pi)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle_{|\pi} =$$

$$= \langle 0, 2e^{2\pi}, 2e^{2\pi}(-1) \rangle \cdot \langle 2\cos t, -\sin t, 2 \rangle_{|\pi}$$

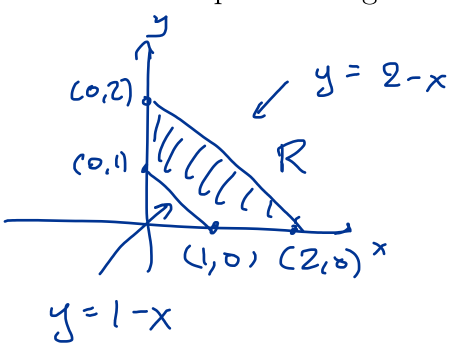
$$= \langle 0, 2e^{2\pi}, -2e^{2\pi} \rangle \cdot \langle -2, 0, 2 \rangle$$

$$= -4e^{2\pi} \ {}^\circ C/s$$

1. Make a change of variables to evaluate the integral

$$\iint_R e^{\frac{y-x}{y+x}}\, dA,$$

where $R$ is the trapezoidal region with vertices (1,0), (2,0), (0,2) and (0,1).



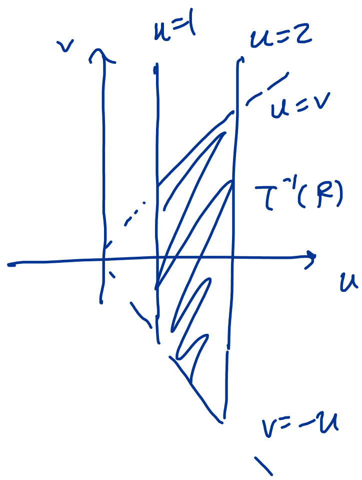

set $\left.\begin{array}{l} u = y+x \\ v = y-x \end{array}\right\} \Rightarrow x = \dfrac{u-v}{2}, \quad y = \dfrac{u+v}{2}$

$y = 2-x \Rightarrow y+x = 2 \Rightarrow u = 2$

$y = 1-x \Rightarrow y+x = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u = v$

$y = 0 \Rightarrow u = -v$




Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So

$$\iint_R e^{\frac{y-x}{y+x}}\, dA = \int_1^2 \int_{-u}^{u} \frac{1}{2} e^{\frac{v}{u}}\, dv\, du$$

$$= \int_1^2 \frac{1}{2} u e^{\frac{v}{u}} \Big|_{-u}^{u} du$$

$$= \int_1^2 \frac{1}{2} u (e - e^{-1}) du$$

$$= \frac{1}{2} \frac{u^2}{2} (e - e^{-1}) \Big|_1^2 = \frac{3}{4}\left(e - \frac{1}{e}\right)$$

6. *(Chain Rule, an interesting conceptual question) What's wrong with the following argument? Suppose we are given a function $w = f(x, y, z)$, where $z = g(x, y)$. Then, wishing to compute $\frac{\partial w}{\partial x}$, we draw a tree diagram and find that at each point $(x, y)$ we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} \implies \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0, \tag{1}$$

so at each point either $\frac{\partial w}{\partial z} = 0$ or $\frac{\partial z}{\partial x} = 0$, which can't be true: there were no assumptions on $w$ or $z$!

**Hint**: it might help you to work out a specific example.

The expressions $\frac{\partial w}{\partial x}$ in the left and right hand side are not the same! It may help to think of the function $w$ as $w = f(u, v, z)$, where $u = x$, $v = y$, $z = g(x,y)$. Then, the chain rule becomes

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial z}\frac{dz}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y},$$

which avoids misconceptions. In other words, $\frac{\partial w}{\partial x}$ on the right hand side means that we're fixing $y$ and $z$ and taking derivative wrt $x$, whereas on the left hand side the dependency of $z$ on $x$ is taking effect.