Worksheet 2

November 14, 2017

1. Among the following vector fields, one has constant upwards pointing curl. Which one?

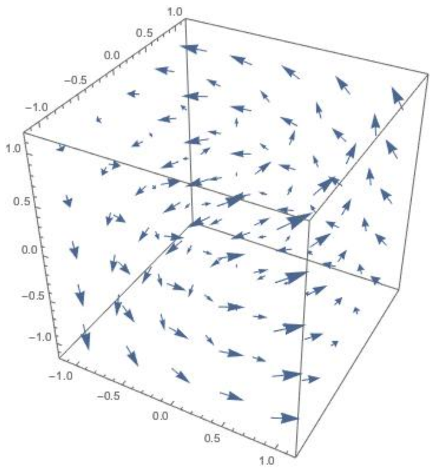


Figure 1: Plot A

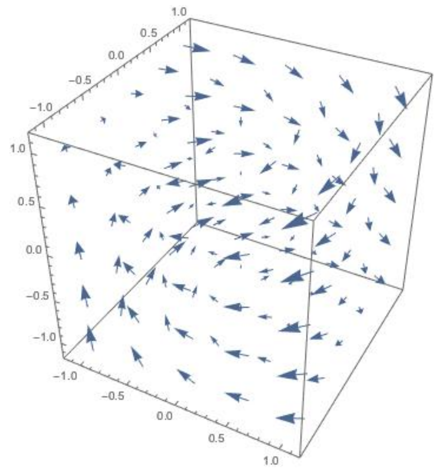


Figure 2: Plot B

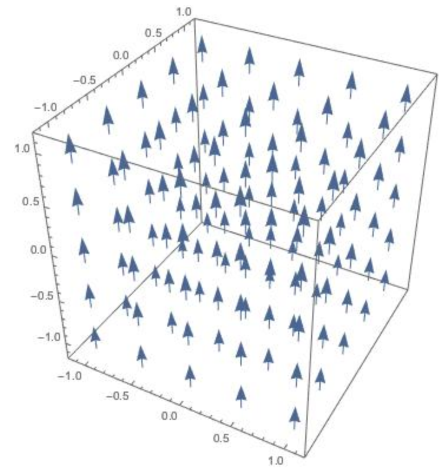


Figure 3: Plot C

2. One of the vector fields below has always negative divergence. Which one?

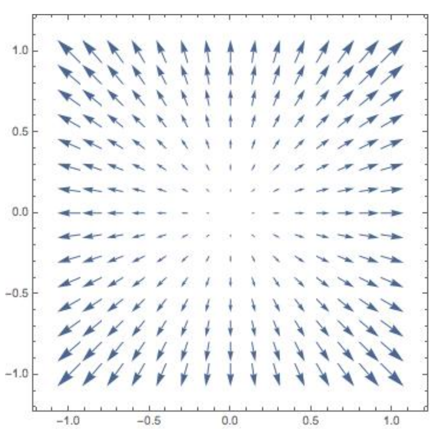


Figure 4: Plot A

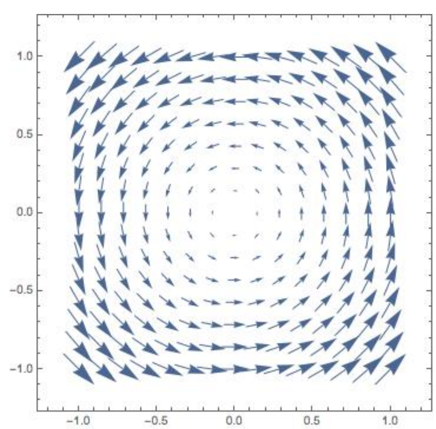


Figure 5: Plot B

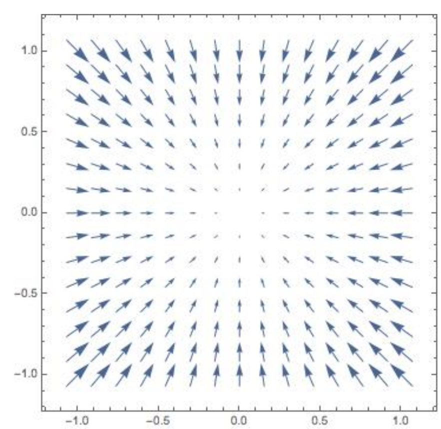
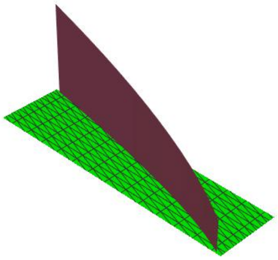


Figure 6: Plot C

3. A fence lies over the curve $y = x^2$ for $x \in [1, 3]$ and under the graph of the function $f(x, y) = x$, where x, y are in meters. Find the area of the fence, including units.



Parametrize curve:

$$c(t) = (x(t), y(t)) = (t, t^2), \ t \in [1,3]$$

$$\text{Area} = \int_c f(x,y)\,ds = \int_1^3 t\sqrt{1+(2t)^2}\,dt$$

$$= \int_1^3 t\sqrt{1+4t^2}\,dt =$$

$u = 1+4t^2$

$du = 8t\,dt$

$t=1 \Rightarrow u=5$

$t=3 \Rightarrow u=37$

$$= \int_5^{37} \frac{1}{8}\sqrt{u}\,du = \frac{1}{8}\,\frac{2}{3}\,u^{\frac{3}{2}}\Big|_5^{37}$$

$$= \frac{1}{12}\left((37)^{\frac{3}{2}} - 5^{3/2}\right)\ m^2$$

5. You are given the vector field

$$\vec{F}(x,y,z) = \langle 2xy + 2e^z, x^2, 2xe^z \rangle,$$

defined on $\mathbb{R}^3$. Show that it is conservative and find a potential function for it.

Find curl $\vec{F}$:

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy+2e^z & x^2 & 2xe^z \end{vmatrix} = \langle 0, 2e^z - 2e^z, 2x - 2x \rangle$$

$= 0$, and it's defined on all of $\mathbb{R}^3$, so conservative

Write $\vec{F} = \langle P, Q, R \rangle = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$.

Then $\partial_x f = P = 2xy + 2e^z \Rightarrow$

$\Rightarrow f(x,y,z) = x^2 y + 2xe^z + \varphi(y,z)$

Then

$\partial_y f = Q = x^2 \Rightarrow x^2 + \partial_y \varphi = x^2 \Rightarrow \varphi(y,z) = \psi(z)$

which means that $f(x,y,z) = x^2 y + 2xe^z + \psi(z)$

Then $\partial_z f = 2xe^z \Rightarrow 2xe^z + \partial_z \psi = 2xe^z \Rightarrow \psi = c$

Thus a potential function for $\vec{F}$ is

$f(x,y,z) = x^2 y + 2xe^z + c$

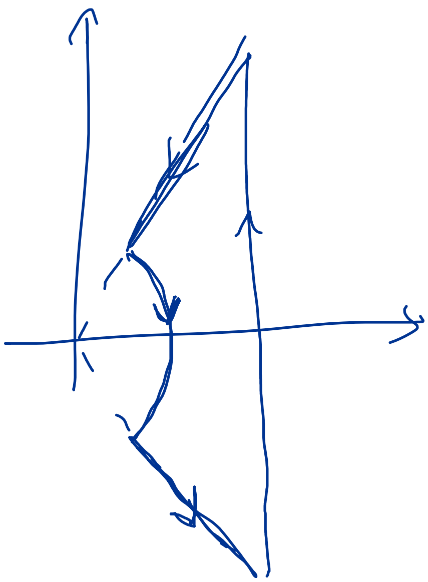
6. An object moves along the boundary of the set

$$D = \{(x, y) : x^2 + y^2 \geq 1,\ |y| \leq \sqrt{3}x,\ x \leq 3\}$$

in counterclockwise direction. Find the work produced by the force field

$$\vec{F}(x, y) = \langle 3, \ln(x^2 + y^2)\rangle$$

during the movement of the object.



Let C be the boundary of the domain. Then, by Green's theorem,

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right) - \frac{\partial}{\partial y}(3)\, dA$$

$$= \iint_D \frac{2x}{x^2+y^2}\, dA$$

Write D in polar: $x = 3 \Rightarrow r = \frac{3}{\cos\theta}$

$y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3}$

$y = -\sqrt{3}x \Rightarrow \theta = -\frac{\pi}{3}$

$x^2+y^2 = 1 \Rightarrow r = 1$

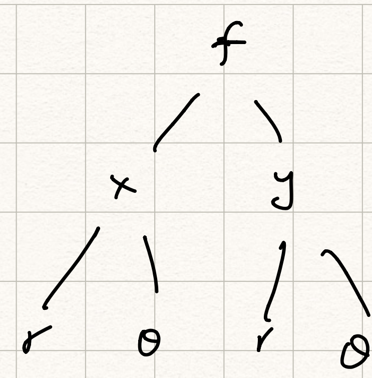
So $D = \{(r, \theta) : 1 \leq r \leq \frac{3}{\cos\theta},\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\}$ and

$$W = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{\frac{3}{\cos\theta}} \frac{2r\cos\theta}{r^2}\, r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos\theta\left(\frac{3}{\cos\theta} - 1\right) d\theta$$

$$= 6 \cdot \frac{2\pi}{3} - 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\, d\theta = 4\pi - 2\sqrt{3}$$

6.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$



$$\partial_r f = \partial_x f \, \partial_r x + \partial_y f \, \partial_r y$$

$$= \cos\theta \, \partial_x f + \sin\theta \, \partial_y f$$

$$\partial_r^2 f = \partial_r(\partial_x f)\, \partial_r x + \partial_x f \, \partial_r^2 x$$

$$+ \partial_r(\partial_y f)\, \partial_r y + \partial_y f \, \partial_r^2 y$$

$$= \partial_x^2 f \, (\partial_r x)^2 + \partial_{yx} f \, \partial_r x \, \partial_r y + \partial_x f \, \overset{0}{\cancel{\partial_r^2 x}}$$

$$+ \partial_{xy}^2 f \, \partial_r x \, \partial_r y + \partial_y^2 f \, (\partial_r y)^2 + \partial_y f \, \overset{0}{\cancel{\partial_r^2 y}}$$

$$= \partial_x^2 f \, (\cos\theta)^2 + \partial_{yx} f \, \cos\theta \sin\theta$$

$$+ \partial_{xy}^2 \cos\theta \sin\theta + \partial_y^2 f \, (\sin\theta)^2$$

$$\partial_\theta f = \partial_x f \, \partial_\theta x + \partial_y f \, \partial_\theta y$$

$$\Rightarrow \partial_\theta^2 f = \partial_x^2 f (\partial_\theta x)^2 + \partial_{xy}^2 f\, \partial_\theta x\, \partial_\theta y$$
$$+ \partial_x f\, \partial_\theta^2 x$$
$$+ \partial_{xy}^2 f\, \partial_\theta y\, \partial_\theta x + \partial_y^2 f (\partial_\theta y)^2$$
$$+ \partial_y f\, \partial_\theta^2 y$$
$$= \partial_x^2 f (-r\sin\theta)^2 + \partial_{xy}^2 f (-r\sin\theta)(r\cos\theta)$$
$$+ \partial_x f (-r\cos\theta)$$
$$+ \partial_{xy}^2 f (-r\sin\theta)(r\cos\theta) + \partial_y^2 f (r\cos\theta)$$
$$+ \partial_y f (-r\sin\theta)$$

So

$$\frac{1}{r^2}\partial_\theta^2 f = \sin^2\theta\, \partial_x^2 f + \cos^2\theta\, \partial_y^2 f$$
$$-2\cos\theta\sin\theta\, \partial_{xy}^2 f$$
$$-\frac{1}{r}\cos\theta\, \partial_x f - \frac{1}{r}\sin\theta\, \partial_y f \quad (1)$$

$$\frac{1}{r}\partial_r f = \frac{1}{r}\cos\theta\, \partial_x f + \frac{1}{r}\sin\theta\, \partial_y f \quad (2)$$

$$\partial_r^2 f = \partial_x^2 f (\cos\theta)^2 + 2\partial_{yx} f \cos\theta\sin\theta$$
$$+ \partial_y^2 f (\sin\theta)^2 \quad (3)$$

Add ①, ②, ③ to find $\partial_x^2 f + \partial_y^2 f$