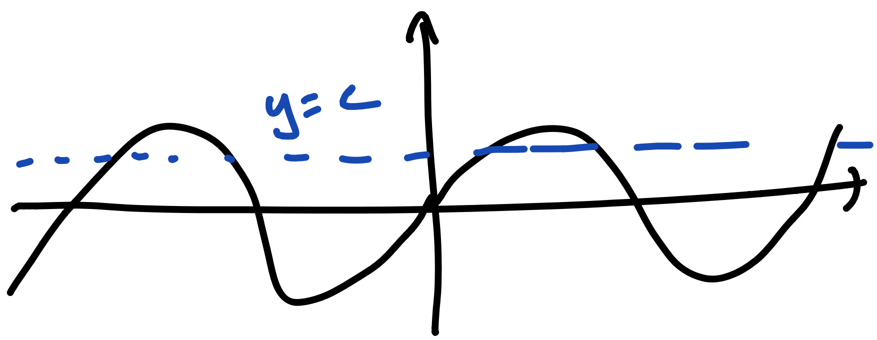
Finish Ch. 20.

Wanted: $\sin(x) = c$
$\cos(x) = c$
$\tan(x) = c$

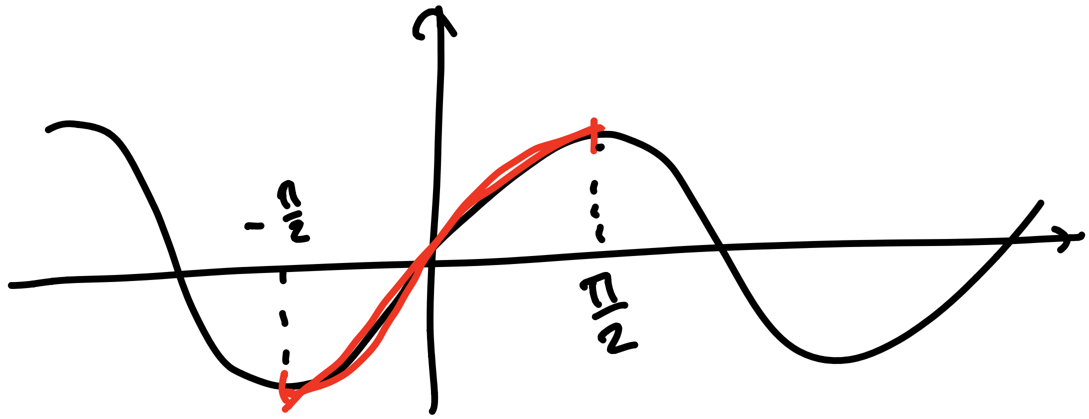
Recall: f 1-1
$f^{-1}(y)$ = sol. of eqn $f(x) = y$, x in domain of f.

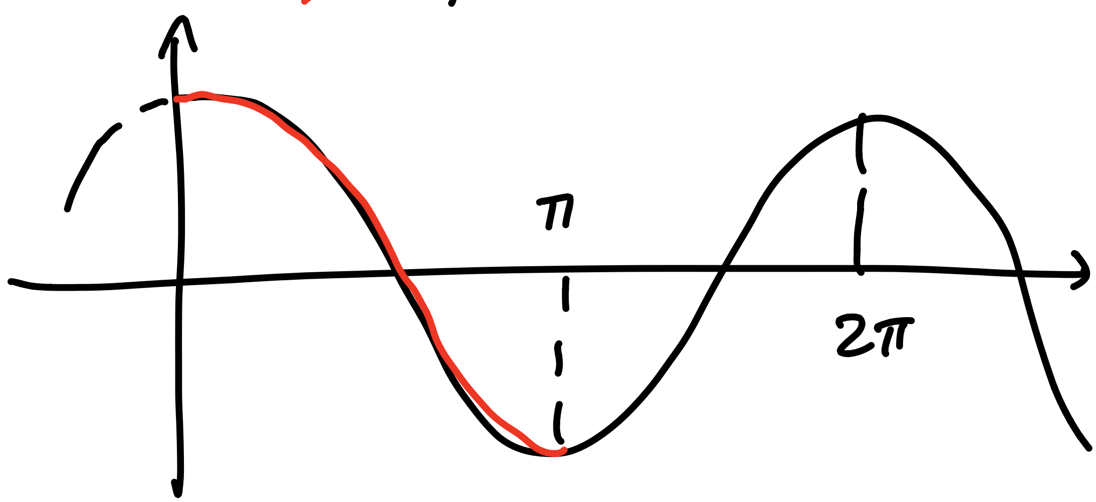


Can't find inverse of $\sin(x)$ on its entire domain.

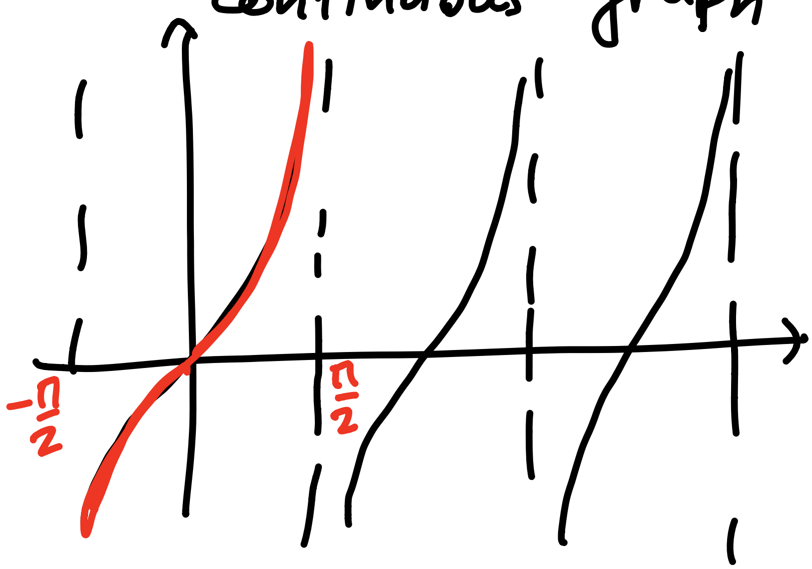
Smaller interval, where we can define inverse function.

1: On this smaller domain we achieve the entire range
2: Include acute angles $[0, \frac{\pi}{2}]$





Tangent: Want same as before,
also want to have
continuous graph



We define the <u>principal domains</u> of trig. fcts:

sin : $[-\frac{\pi}{2}, \frac{\pi}{2}]$

cos : $[0, \pi]$

tan : $(-\frac{\pi}{2}, \frac{\pi}{2})$

On those intervals the trig. fcts are 1-1. We can define their inverses.

Define:

→ $\sin^{-1}(y)$ (arcsin(y), asin(y)): inverse fct of $y = \sin(x)$

$\sin^{-1}(y)$: domain $[-1, 1]$
range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ← principal domain of sin

→ $\cos^{-1}(y)$ (arccos(y), acos(y)):

inverse of $y = \cos(x)$
restricted to $x \in [0, \pi]$

$\cos^{-1}$: domain $[-1, 1]$
range $[0, \pi] \leftarrow$ principal domain of cos

$\rightarrow \tan^{-1}(y)$ ($\arctan(y)$, $\mathrm{atan}(y)$)
inverse of $y = \tan(x)$
restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Domain: $(-\infty, \infty)$ (all real numbers)

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \leftarrow$ pr. domain of tangent.

Rmk: $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

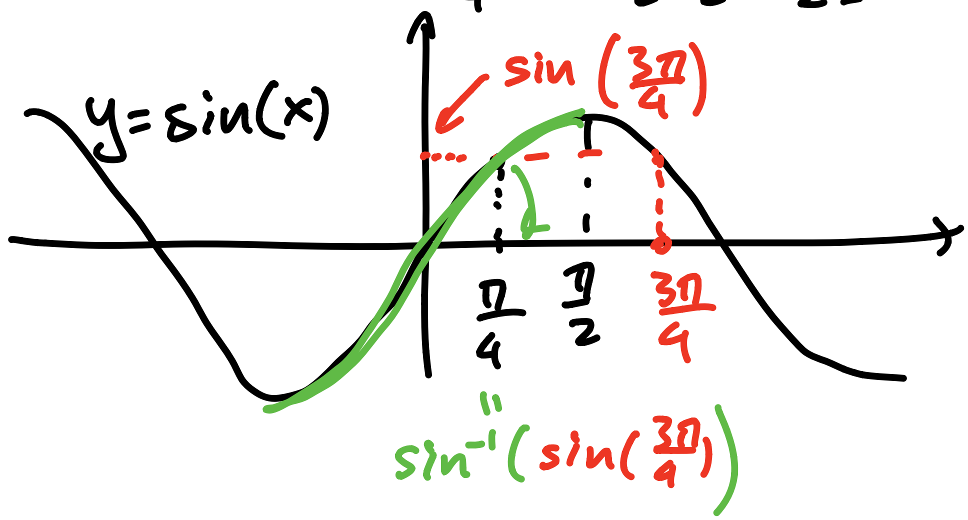
$$\sin^{-1}(\sin(x)) = x$$

Ex: $x = \frac{\pi}{4}$, $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

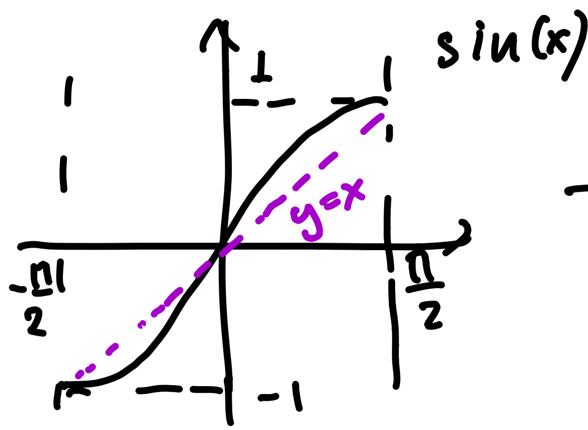
$x = \frac{3\pi}{4}$ $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

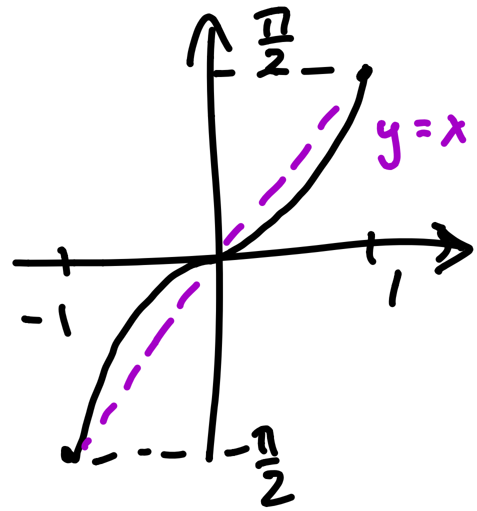
Why? $\sin^{-1}$ is the inverse of sin restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

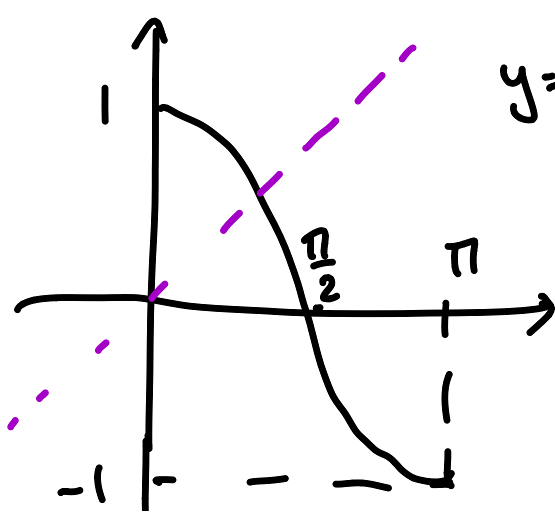
and $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

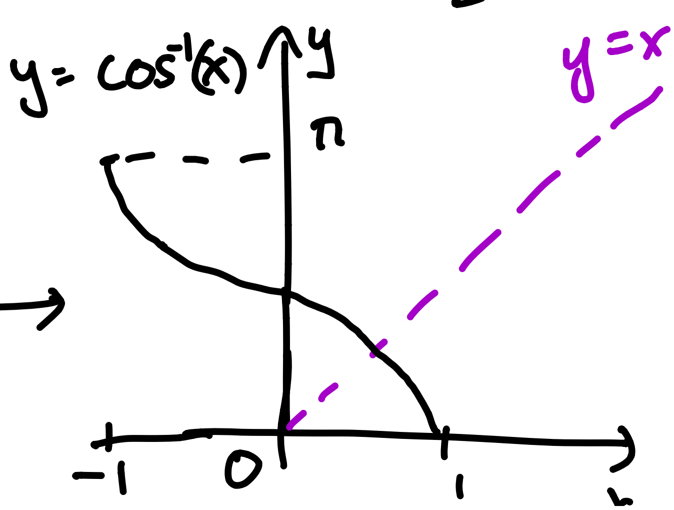


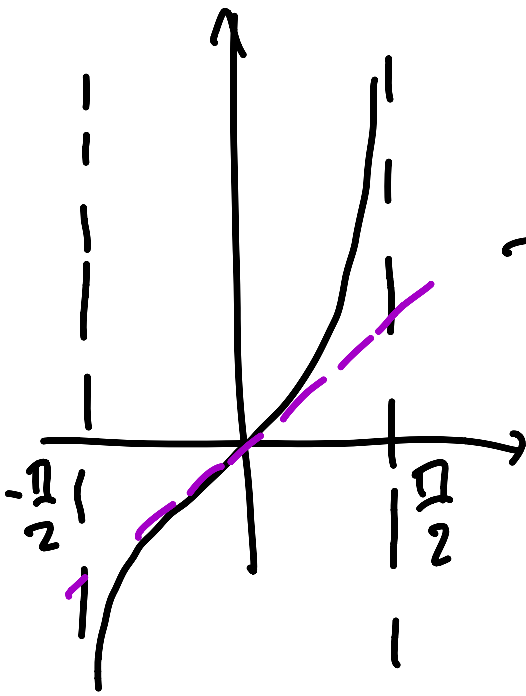
Graphs of inverse trig functions

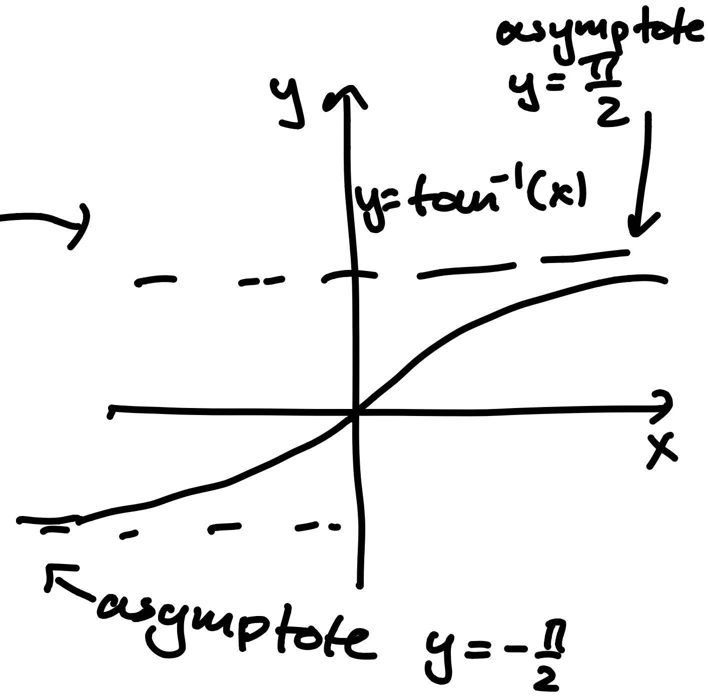




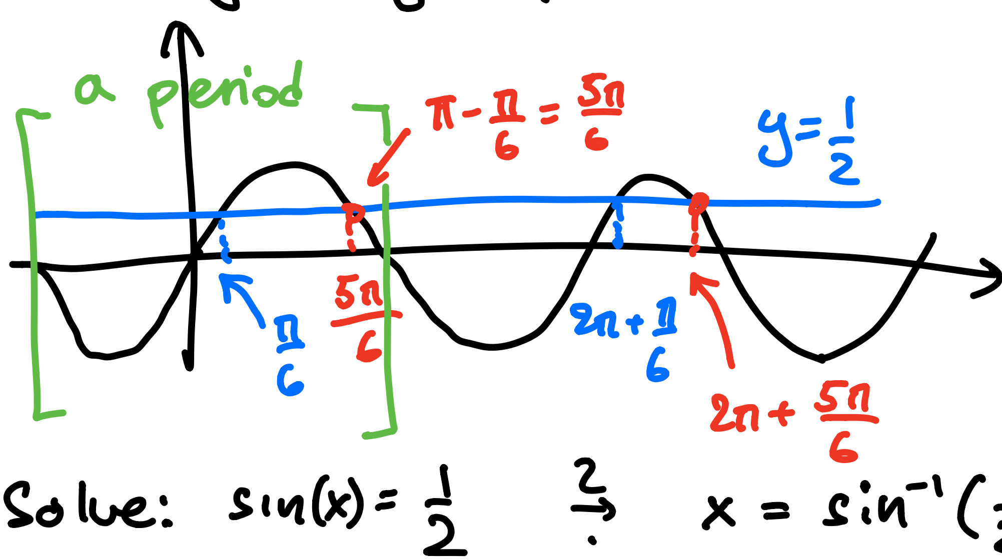








Back to the problem of solving trig. eq'ns.



Solve: $\sin(x)=\frac{1}{2} \overset{?}{\rightarrow} x=\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{6}$

This is a solution. But <u>not all</u> solutions.

Recall: $\sin(\pi - x) = \sin(x)$

If x solves eq'n $\sin(x) = \frac{1}{2}$

then $\pi - x$ also does!

$$\sin(\pi - x) = \sin(x) = \frac{1}{2}$$

So far: 2 solutions

$$x = \frac{\pi}{6}, \quad \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Those are all solutions in a period.

Add periods to get all solutions:

$$2\pi \cdot k + \frac{\pi}{6}, \quad k = 0, \pm 1, \pm 2 \ldots$$

is solution. blue points

$$2\pi \cdot k + \frac{5\pi}{6}, \quad k = 0, \pm 1, \pm 2, \ldots$$

are solutions.

red points

Summary: Solutions of $\sin(x) = c$, when $-1 \le c \le 1$ (otherwise no sol's):

$$2\pi \cdot k + \sin^{-1}(c), \quad k = 0, \pm 1, \pm 2, \ldots$$

$$2\pi \cdot k + \pi - \sin^{-1}(c), \quad k = 0, \pm 1, \pm 2, \ldots$$

Ex: Solve
$$2\sin(x^2-1) = \sqrt{3}$$
$$\sin(x^2-1) = \sqrt{3}$$

If it helps: set $a = x^2-1$
$$\sin(a) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \begin{cases} a = 2\pi\cdot k + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ a = 2\pi\cdot k + \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \end{cases} \quad k = 0, \pm 1, \ldots$$

But $a = x^2-1$
$$\begin{cases} x^2-1 = 2\pi\cdot k + \frac{\pi}{3}, & k = 0, \pm 1, \ldots \\ x^2-1 = 2\pi\cdot k + \pi - \frac{\pi}{3}, & k = 0, \pm 1, \ldots \end{cases}$$

$$* \begin{cases} x^2 = 2\pi\cdot k + \frac{\pi}{3} - 1 \\ x^2 = 2\pi\cdot k + \frac{2\pi}{3} - 1 \end{cases} \quad k = 0, \pm 1, \pm 2$$

Issue: Not all k work!
If k is large negative, say $k = -50$, right hand side is negative.

Find for what k I can solve.
Make sure that right hand side in ✱ is non-negative.

Solve $\quad 2\pi k + \frac{\pi}{3} - 1 \geq 0$

$$\Rightarrow 2\pi k \geq 1 - \frac{\pi}{3}$$

$$\Rightarrow k \geq \frac{1 - \frac{\pi}{3}}{2\pi}$$

$$\Rightarrow k \geq \frac{3-\pi}{6\pi}$$

$$\Rightarrow k \geq -.007, \ k \text{ integer.}$$

$$\Rightarrow k \geq 0$$

First set of sol's:

$$x^2 = 2\pi \cdot k + \frac{\pi}{3} - 1, \quad k \geq 0 \text{ integer}$$

$$\Rightarrow x = \pm\sqrt{2\pi \cdot k + \frac{\pi}{3} - 1}, \ k \geq 0, \text{ int.}$$

<u>2nd set of sol's:</u>

$$x^2 = 2\pi \cdot k + \frac{2\pi}{3} - 1$$

$\rightarrow$ Find for what k this makes sense

→ solve for x as before.

→ Get full set of sols.

What about cos?

$$\cos(x) = c$$

Recall: $\cos(-x) = \cos(x)$

If x solves $\cos(x) = c$, then $-x$ also solves $\cos(-x) = c$

As before, general solution of

$$\cos(x) = c, \quad -1 \leq c \leq 1$$

$$\begin{cases} x = 2\pi \cdot k + \cos^{-1}(c) \\ x = 2\pi \cdot k - \cos^{-1}(c) \end{cases} \quad k = 0, \pm 1, \pm 2, \ldots$$

For tan: Tangent has period π.

eqn $\tan(x) = c$

c any real number,

has exactly one solution in one period.

General solution:

$$x = \pi \cdot k + \tan^{-1}(c)$$

period