Parametrizations of a few curves that show up frequently

October 30, 2017

- If the curve is part of a circle of radius r and center (x_0, y_0): Use a parametrization of the form
$$c(t) = (x_0 + r\cos(t), y_0 + r\sin(t)), t \in [a, b]$$
for appropriate choices of bounds a, b.

- For an entire ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, you can use $c(t) = (a\cos(t), b\sin(t))$, $t \in [0, 2\pi]$. If you have part of it, modify the bounds accordingly.

- If you have the graph of a function $y = f(x)$, set $x(t) = t$ and $y(t) = f(t)$.

- If you have a straight line from (a, b) to (c, d), set $x(t) = a + (c-a)t$ and $y(t) = b + (d-b)t$, $t \in [0, 1]$.

- If the curve is more complicated but can be split into simpler parts, write a parametrization for each one of them and write a sum of integrals: if a curve c can be written as the union of two curves c_1 and c_2,
$$\int_c f ds = \int_{c_1} f ds + \int_{c_2} f ds.$$

- A helix in $\mathbb{R}^3$ about the z axis can be parametrized as
$$c(t) = (a\cos(t), b\sin(t), ct),$$
for appropriate bounds on t.

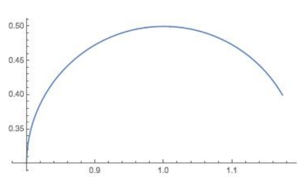

Figure 1: Part of a circle, parametrized as $c(t) = (1 + 0.2\cos(t), 0.3 + 0.2\sin(t))$, $t \in [\pi/6, \pi]$

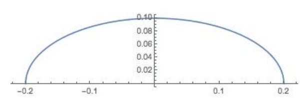

Figure 2: Part of an ellipse, parametrized as $c(t) = (0.2\cos(t), 0.1\sin(t))$, $t \in [0, \pi]$.

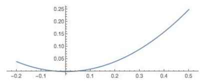

Figure 3: Part of the graph of $f(x) = x^2$, parametrized as $c(t) = (t, t^2)$, $t \in [-0.2, 0.5]$.

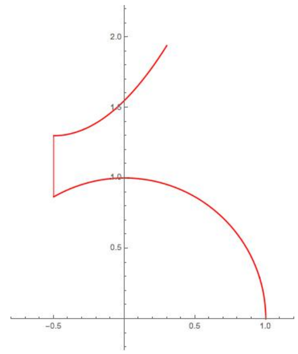

Figure 4: A curve consisting of several simpler curves

- If you have to parametrize the intersection of two surfaces in $\mathbb{R}^3$, something that can work is to write $c = (x, y, z)$ and use the equations of the surfaces to eliminate two of the variables. Then, set the third one to be t. Alternatively, use one equation to eliminate one variable and then once you end up with a something that contains two variables, use one of the above parametrizations of 2 dimensional objects to write them in terms of t (see example in Figure 6).

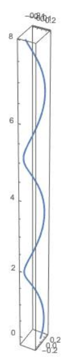

Figure 5: A helix, parametrized as $c(t) = (0.2\cos(t), 0.3\sin(t), 0.5t)$, $t \in [-0.2, 5\pi]$.

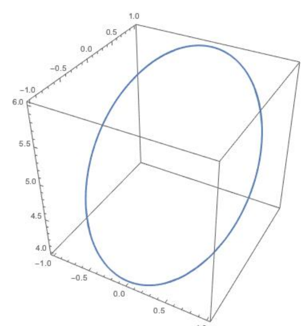

Figure 6: The intersection of a $x^2 + y^2$ and $z = 5 + y$, parametrized as $c(t) = (\cos(t), \sin(t), 5 + \sin(t)$, $t \in [0, 2\pi]$.

General remark: The definition of the line integral with respect to arc length involves a pretty nasty root expression. This means that for a generic curve its calculation is difficult or impossible. In many cases (and in exams) nice simplifications happen and the integrals can be calculated, so try to be looking for those: in an exam, if the integrals look too complicated you have probably done something wrong in setting them up.