Math 324 H - Autumn 2017
Midterm 2
Friday, November 17, 2017

Name: ____________________

Student ID Number: ____________________

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- There are 6 problems spanning 6 pages (your last page should be numbered as 6). Please make sure your exam contains all these questions.
- You are allowed to use a TI-30x IIS Calculator and one double sided hand-written 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam. Budget your time wisely.
Do not spend too much time on an individual problem, unless you are done with all the rest.

GOOD LUCK!

1. (5 pts) You are given the vector field

$$\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$$

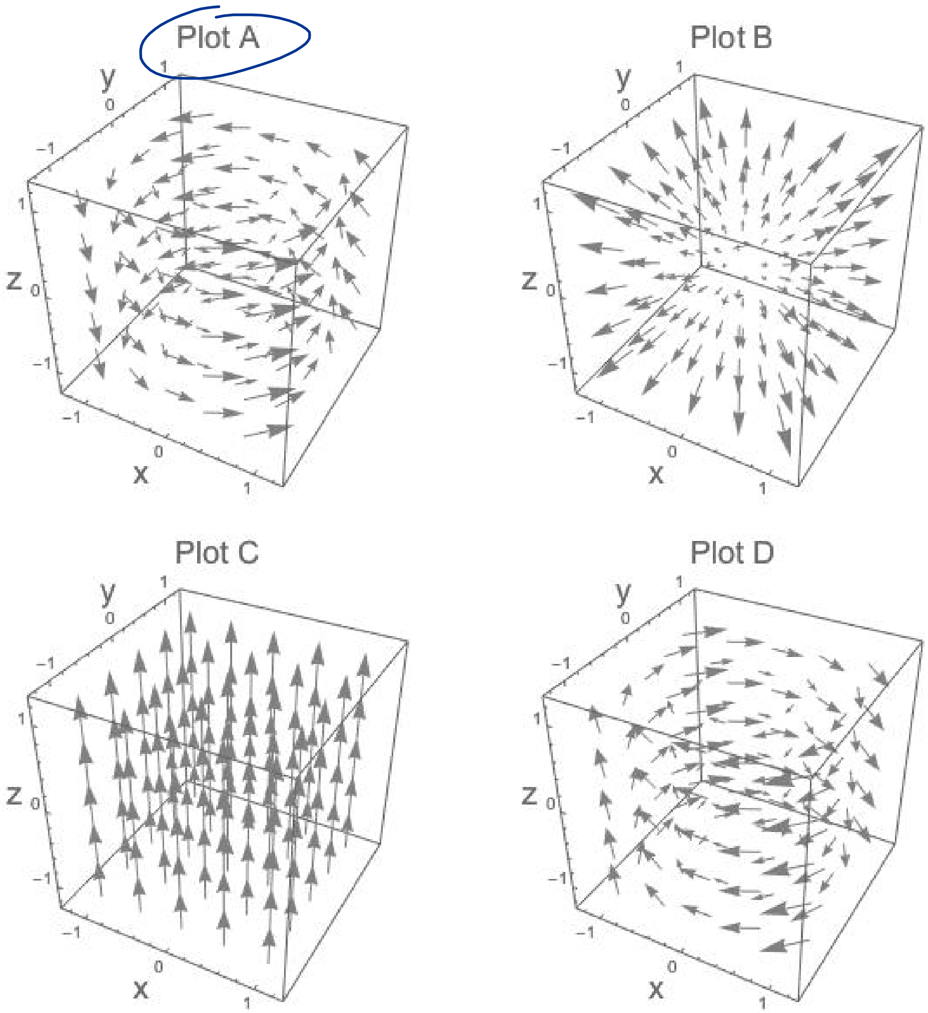
(a) Find div $\vec{F}$.

$$\operatorname{div} \vec{F} = \partial_x(-y) + \partial_y x + \partial_z 0 = 0$$

(b) Find curl $\vec{F}$.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

(c) Decide which vector plot corresponds to the vector field $\vec{F}$ (your answers from parts (a) and (b) might be helpful).



2. (8 pts) **You do not need to explain your answers to this problem.**

(a) Mark the following sentence as **true** or **false**. Let c be the unit circle in $\mathbb{R}^2$ parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x, y)$ we have

$$\int_{-c} f(x,y)ds = -\int_{c} f(x,y)ds.$$

True **False**

(b) Mark the correct answer: Let $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field in $\mathbb{R}^3$, where P, Q, R have continuous second partial derivatives. Then

$$\vec{F} \times (\operatorname{div}(\operatorname{curl} \vec{F}))$$

is

a. A vector field **b.** A scalar function **c.** Undefined (nonsense)

(c) Mark the following sentence as **true** or **false**. If $\vec{F}(x, y, z)$ is any vector field with continuous coefficients and c_1, c_2 are two curves in $\mathbb{R}^3$ such that they both start at the same point A and they both end at the same point B then

$$\int_{c_1} \vec{F}(x,y,z) \cdot d\vec{r} = \int_{c_2} \vec{F}(x,y,z) \cdot d\vec{r}.$$

True **False**

(d) Mark the following statement as **true** or **false**. If $\vec{F}$ and $\vec{G}$ are two vector fields in $\mathbb{R}^3$ satisfying $\operatorname{div} \vec{F} = \operatorname{div} \vec{G}$ **and** $\operatorname{curl} \vec{F} = \operatorname{curl} \vec{G}$ everywhere in $\mathbb{R}^3$ then $\vec{F} = \vec{G}$ everywhere in $\mathbb{R}^3$.

True **False**

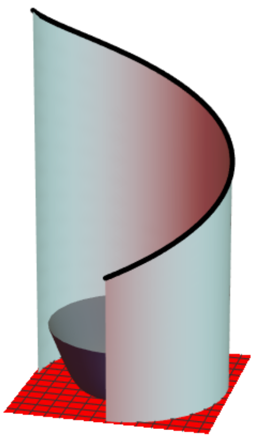
3. (6 pts) Nadine bought a new fancy curtain for her bathroom. The curtain rod (black in the picture) lies on the graph of the function

$$f(x, y) = 3 + \arctan(\frac{y}{x}),$$

where x and y are measured in feet, over the curve

$$c(t) = (2\cos(t), 2\sin(t)), t \in [0, \frac{3\pi}{2}].$$

If the curtain touches the floor, which is represented as the xy plane, as in the picture, find the area of the curtain. **Make sure to include units.**



$f(x,y) = 3 + \arctan \frac{y}{x}$

$c(t) = (x(t), y(t)) = (2\cos(t), 2\sin(t))$

$x'(t) = -2\sin(t), \quad y'(t) = 2\cos(t)$

$A = \int_c f(x,y)\, ds =$

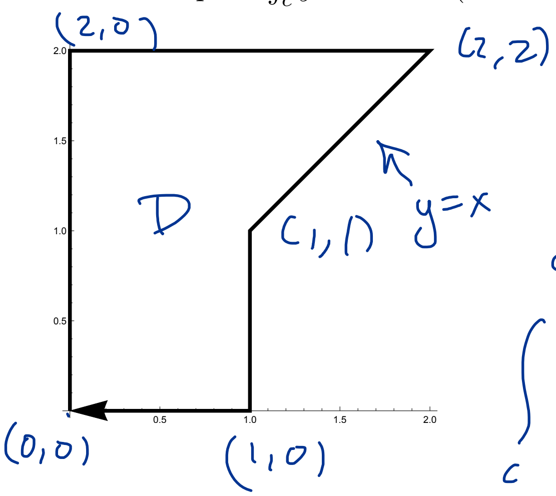
$= \int_0^{\frac{3\pi}{2}} \left(3 + \arctan\left(\frac{2\sin t}{2\cos t}\right)\right) \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$

$= \int_0^{\frac{3\pi}{2}} 2 \cdot (3 + t)\, dt = 6 \cdot \frac{3\pi}{2} + \left(\frac{3\pi}{2}\right)^2 ft^2$

4. (6 pts.) Let c be the path on $\mathbb{R}^2$ consisting of the following line segments, as in the picture:

- A line segment from the (0,0) to (0,2).
- A line segment from (0,2) to (2,2)
- A line segment from (2,2) to (1,1)
- A line segment from (1,1) to (1,0)
- A line segment from (1,0) back to the origin.

Compute $\int_c y^2 + x^3 dx$. (**Hint:** Use Green's Theorem)



Use Green's, c negatively oriented.

$$\int_c y^2+x^3 dx = -\iint_D -\frac{\partial}{\partial y}(y^2+x^3)\, dA$$

$$= \iint_D 2y\, dA = \int_0^1\int_0^2 2y\, dy\, dx + \int_1^2\int_x^2 2y\, dy\, dx$$

$$= 4 + \int_1^2 y^2\Big|_x^2 dx = 4 + \int_1^2 4 - x^2 dx$$

$$= 4 + \left[4x - \frac{x^3}{3}\right]_1^2 =$$

$$= 4 + \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)$$

$$= 8 - \frac{7}{3}$$

5. (9 pts) The shape of a hill is given by

$$z = f(x,y) = \sin\left(\frac{x^2}{2} - y^2 + 3\right) + 6,$$

where x, y and z are measured in meters, the positive x axis is pointing east and the positive y axis is pointing north. So, the altitude over the point (x,y) is $f(x,y)$.

(a) Find a vector pointing towards the direction of **minimum net rate of change** of the altitude, at the point $(2, 1, \sin(4)+6)$ (that is, the direction in which the altitude **decreases** as fast as possible).

Find $-\nabla f(2,1)$: $\nabla f = \langle x\cos\left(\frac{x^2}{2} - y^2 + 3\right), -2y\cos\left(\frac{x^2}{2} - y^2+3\right)\rangle$

So at $(2,1)$:

$-\nabla f(2,1) = -\langle 2\cos(4), -2\cos 4\rangle$

(b) Find the rate of change of the altitude function in direction **south-west**, at the point $(2, 1, \sin(4)+6)$. **Include units**.

south west is given by $\langle -1,-1\rangle$. Divide by magnitude, $\vec{u} = \frac{\langle -1,-1\rangle}{\sqrt{2}}$ and take directional der:

$D_{\vec{u}} f(2,1) = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle \cdot \nabla f(2,1) = -\frac{1}{\sqrt{2}}\cdot 2\cos 4 - \frac{1}{\sqrt{2}}\cdot(-2\cos 4) = 0$ m/m

(c) A bobcat is running on the hill, along a path $c(t) = (x(t), y(t), z(t))$, for which we know that $c(1) = (2, 1, \sin(4)+6)$, $x'(1) = 1$ and $y'(1) = -2$ and t is measured in seconds. Find the rate of change of the bobcat's altitude at time $t = 1$ (that is, find $\frac{dz}{dt}_{|t=1}$). **Include units**.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(2,1)\, x'(1) + \frac{\partial f}{\partial y}(2,1)\, y'(1)$$

$$= 2\cos 4 \cdot 1 + (-2\cos 4)\cdot(-2) \text{ m/s}$$

6. (6 +3 pts) Let $\vec{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle = \langle P(x,y), Q(x,y) \rangle$, defined on

$$D = \mathbb{R}^2 \setminus \{(0,0)\}$$

(the plane without the origin). It is given that this vector field satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

(a) Compute $\int_c \vec{F} \cdot d\vec{r}$, where c is the unit circle, parametrized clockwise.

Do it for counter-clockwise, put a (−) sign.

$$\int_c \vec{F} \cdot d\vec{r} = -\int_{-c} \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} \frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\, dt$$

$$\left.\begin{array}{l} x(t) = \cos t \\ y(t) = \sin t \end{array}\right\} \quad \begin{array}{l} x'(t) = -\sin t \\ y'(t) = \cos t \end{array}$$

$$= -2\pi$$

(b) Is $\vec{F}$ conservative on D? Justify your answer.

It's not, bec. the line integral over a closed path is not zero.

(c) Bonus[1]: Find a potential function for $\vec{F}$, defined on the set

$$D' = \{(x,y) : (x-2)^2 + y^2 \le 1\} \subset D.$$

$$\nabla f = \vec{F} \Rightarrow \partial_x f = \frac{-y}{x^2+y^2} \Rightarrow f = \int \frac{-y}{x^2+y^2}\, dx + \varphi(y)$$

$$\Rightarrow f = \int -\frac{1}{\left(\frac{x}{y}\right)^2 + 1} \cdot \frac{1}{y}\, dx + \varphi(y) \Rightarrow f = -\arctan\left(\frac{x}{y}\right) + \varphi(y)$$

$$\partial_y f = \frac{x}{x^2+y^2} \Rightarrow -\frac{1}{\left(\frac{x}{y}\right)^2 + 1} \cdot \left(-\frac{x}{y^2}\right) + \varphi'(y) = \frac{x}{x^2+y^2} \Rightarrow \varphi'(y) = 0$$

So a pot. fct is

$$f(x,y) = -\arctan\left(\frac{x}{y}\right) + c$$

[1]No partial credit will be given for the bonus question, highest possible score in this exam is 40.