Math 324 A - Winter 2018
Midterm Exam
Wednesday, February 7, 2018

Name: ____________________

Student ID Number: ____________________

Problem 1	10	
Problem 2	10	
Problem 3	10	
Problem 4	10	
Problem 5	10	
Total	40	

- There are 5 problems spanning 5 pages (your last page should be numbered as 5). Please make sure your exam contains all these questions.
- You are allowed to use a TI-30x IIS Calculator and one double sided hand-written 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam. Budget your time wisely.
Do not spend too much time on an individual problem, unless you are done with all the rest.

GOOD LUCK!

1. (10 pts) The shape of a valley is given by

$$z = f(x, y) = \sin\left(\frac{x^2}{2} - y^2 + 3\right) + 6,$$

where x, y and z are measured in meters, the positive x axis is pointing east and the positive y axis is pointing north. So, the altitude over the point (x, y) is $f(x, y)$.

(a) Find a (2 dimensional) vector pointing towards the direction of **minimum net rate of change** of the altitude, when you're standing at the point with coordinates $(2, 1, \sin(4) + 6)$ (that is, find the direction in which the altitude **decreases** as fast as possible).

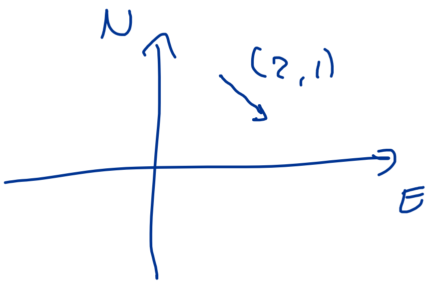
Find $-\nabla f$:

$\nabla f(x, y) = \langle \cos(\frac{x^2}{2} - y^2 + 3) \cdot x, \cos(\frac{x^2}{2} - y^2 + 3) \cdot (-2y) \rangle$

At $(2, 1)$:

$-\nabla f(x, y) = -\langle 2\cos 4, -2\cos(4) \rangle$

(b) Find the rate of change of the altitude function in direction **south-east**, when you're standing at the point with coordinates $(2, 1, \sin(4) + 6)$. In other words, find the directional derivative of the altitude function in direction south-east at the point $(2, 1, \sin(4) + 6)$. **Include units**.



$\langle 1, -1 \rangle$ is pointing SE

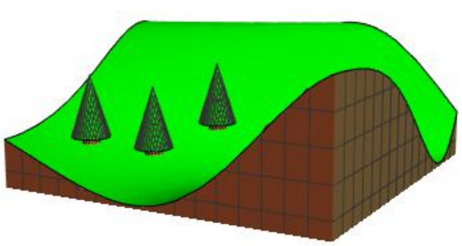
turn into unit

$\vec{u} = \frac{\langle 1, -1 \rangle}{\sqrt{1+1}} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$

$D_{\vec{u}} f = \nabla f(2,1) \cdot \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle \cdot \langle 2\cos 4, -2\cos 4 \rangle$

$= \sqrt{2}\cos 4 + \sqrt{2}\cos 4 =$

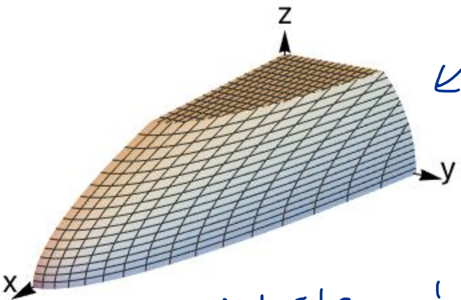
$2\sqrt{2}\cos 4$ m/m



2. (10 pts.) Set up an integral

$$\iiint_E f(x,y,z)dV$$

in the order $dydzdx$, where E is the solid satisfying $0 \le x \le 9 - y^2 - z^2$, $y \ge 0$ and $0 \le z \le 2$ (the solid E can be seen in the picture).



$x = 9 - y^2 - z^2$

Write eq'ns ① $x = 9 - y^2 - z^2 \Rightarrow y = \sqrt{9 - x - z^2}$

② $x = 0$

③ $y = 0$

④ $z = 0$

⑤ $z = 2$

So

$0 \le y \le \sqrt{9 - x - z^2}$

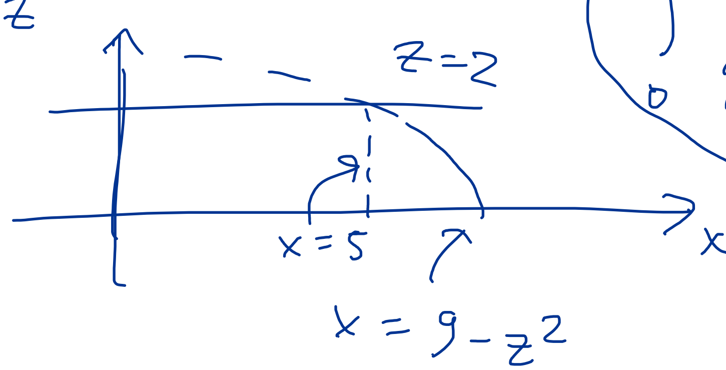
Draw projection, on xz plane.

①,③ $\Rightarrow$ $9 - x - z^2 = 0 \Rightarrow x = 9 - z^2 \Rightarrow z = \sqrt{9 - x}$

②: $x = 0$

④: $z = 0$

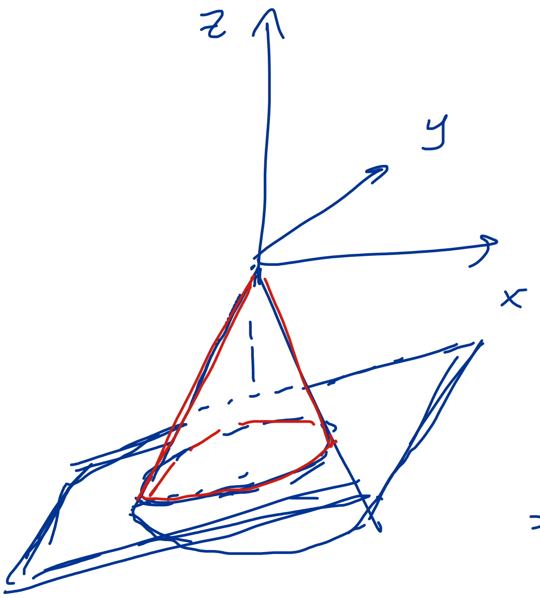
⑤ $z = 2$



So $\iiint_E f\, dV =$

$$\int_0^5 \int_0^2 \int_0^{\sqrt{9-x-z^2}} f\, dy\,dz\,dx + \int_5^9 \int_0^{\sqrt{9-x}} \int_0^{\sqrt{9-x-z^2}} f\, dy\,dz\,dx$$

3. (10 pts.) Set up **but do not evaluate** an integral **in spherical coordinates** computing the volume of the solid that lies under the half cone $z = -\sqrt{x^2+y^2}$ and above the plane $z = -1+0.5x$.



$z = -\sqrt{x^2+y^2} \Rightarrow$

$\Rightarrow \rho\cos\varphi = -\sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$

$\Rightarrow \rho\cos\varphi = -\rho\sin\varphi$

$\Rightarrow \tan\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$

$z = -1+0.5x \Rightarrow \rho\cos\varphi = -1 + 0.5\rho\sin\varphi\cos\theta$

$\Rightarrow \rho = \dfrac{-1}{\cos\varphi - 0.5\sin\varphi\cos\theta}$

$$V = \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{-1}{\cos\varphi-0.5\sin\varphi\cos\theta}} \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta.$$

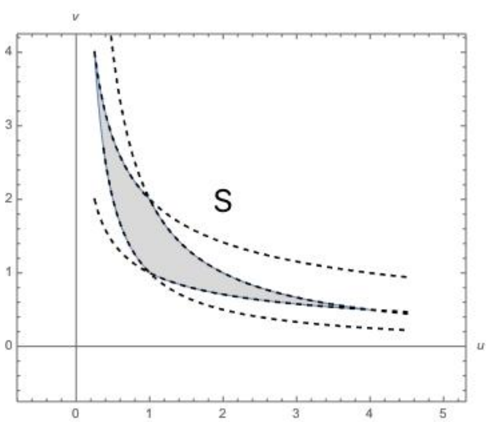
4. (10 pts.) You are given the transformation $(x, y) = T(u, v)$ on the first quadrant, defined by

$$\begin{cases} x = v \\ y = uv \end{cases} \tag{1}$$

(a) Compute the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v$$

(b) Let S be the set on the first quadrant of uv plane bounded by $v = \frac{1}{u}$, $v = \frac{2}{u}$, $v = \frac{1}{\sqrt{u}}$ and $v = \frac{2}{\sqrt{u}}$. Find the **boundary curves** of the image $R := T(S)$ of T (which will be a subset of the xy plane) and **draw a picture of** R.



Invert T:

$x = v \Rightarrow v = x$

$y = uv \Rightarrow u = \frac{y}{x}$

Bdry curves:

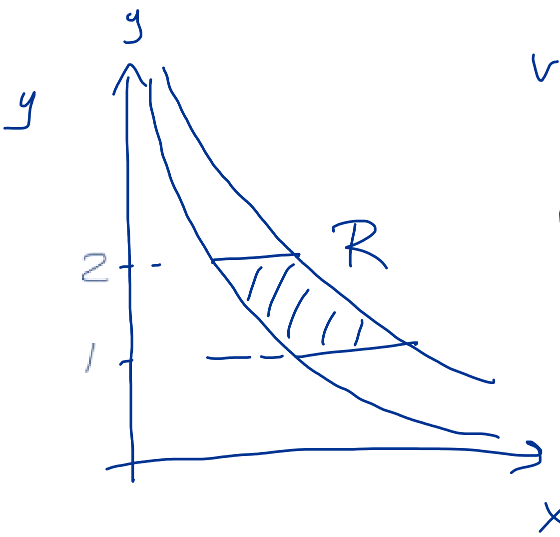
$v = \frac{1}{u} \Rightarrow uv = 1 \Rightarrow \frac{y}{x} \cdot x = 1 \Rightarrow y = 1$

$v = \frac{2}{u} \Rightarrow uv = 2 \Rightarrow \frac{y}{x} \cdot x = 2 \Rightarrow y = 2$

$v = \frac{1}{\sqrt{u}} \Rightarrow v\sqrt{u} = 1 \Rightarrow x \cdot \sqrt{\frac{y}{x}} = 1 \Rightarrow \sqrt{xy} = 1$

$\Rightarrow y = \frac{1}{x}$

$v = \frac{2}{\sqrt{u}} \Rightarrow x = \frac{2\sqrt{x}}{\sqrt{y}} \Rightarrow y = \frac{4}{x}$



5. (10 pts.) You are given the following functions

(a) $w = w(x, y, z, s) = x^2y + ze^x + s$

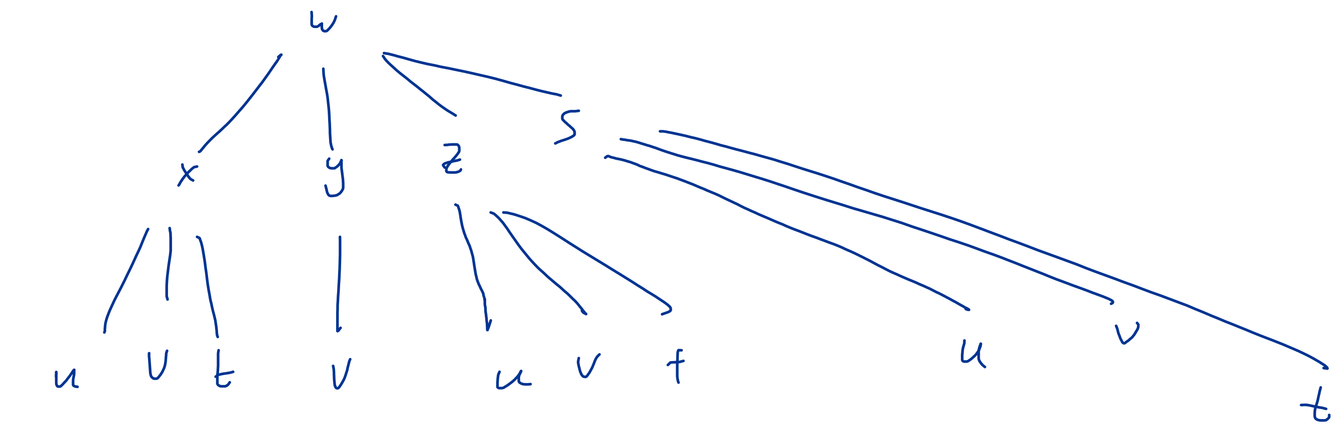
(b) $x = x(u, v, t) = u + v^2 + t$

(c) $y = y(v) = 2v$

(d) $z = z(u, v, t) = 3ut\cos(v)$

(e) $s = s(u, v, t) = u^2vt$

Compute $\frac{\partial w}{\partial u}$ when $u = 1$, $v = 0$ and $t = 1$.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial u}$$

$$= \left(2xy + ze^x\right) \cdot 1 + e^x \cdot 3t\cos(v) + 1 \cdot 2uvt$$

$u = 1,\ v = 0,\ t = 1 \Rightarrow \quad x = 2$

$y = 0$

$z = 3$

So

$$\frac{\partial w}{\partial u} = 3e^2 + 3e^2 + 0 = 6e^2$$