Math 324 C - Winter 2017
Final Exam v.B
Wednesday, March 15, 2017

Name: ______________________________

Student ID Number: ______________________________

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- There are 7 problems spanning 7 pages (your last page should be numbered as 7). Please make sure your exam contains all these questions.
- You are allowed to use a scientific calculator (**no graphing calculators**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the back of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 110 minutes to complete the exam. Budget your time wisely.
Do not spend too much time on an individual problem, unless you are done with all the rest.
- You are not allowed to discuss this exam with other people until 5.00 pm today.

GOOD LUCK!

1. (8 pts.) **You do not need to explain your answers for this problem.**

(a) Mark the following sentence as **true** or **false**. Let c be the unit circle in $\mathbb{R}^2$ parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x, y)$ we have

$$\int_{-c} f(x, y)dx = -\int_c f(x, y)dx.$$

True **False**

(b) Mark the following sentence as **true** or **false**. Let S denote the unit ball in $\mathbb{R}^3$ with positive(outward) orientation and $\tilde{S}$ the unit ball with negative (inward) orientation. Then, for any vector field $\vec{F}(x, y, z)$ with continuous coefficients

$$\int_S \vec{F}(x, y, z) \cdot d\vec{S} = -\int_{\tilde{S}} \vec{F}(x, y, z) \cdot d\vec{S}.$$

True **False**

(c) Mark the following sentence as **true** or **false**. Let S denote the upper hemisphere of the unit ball centered at the origin in $\mathbb{R}^3$ (the one that satisfies $z \geq 0$), with **upward** orientation, and $\tilde{S}$ the lower hemisphere of the unit ball centered at the origin (the one that satisfies $z \leq 0$), again with **upward** orientation. Then, for any vector field $\vec{F}(x, y, z)$ with differentiable coefficients

$$\iint_S \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S} = \iint_{\tilde{S}} \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S}.$$

True **False**

2. (6 pts.) Show the following version of the product rule: Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field, where P, Q are differentiable scalar valued functions, and let $g(x,y)$ be a differentiable scalar valued function. Then

$$\operatorname{div}(g\vec{F}) = g\operatorname{div}(\vec{F}) + (\nabla g)\cdot\vec{F}.$$

Make sure that each step follows clearly from the previous one, otherwise you may not receive full credit.

$$\operatorname{div}(g\vec{F}) = \operatorname{div}(g\langle P, Q\rangle) =$$

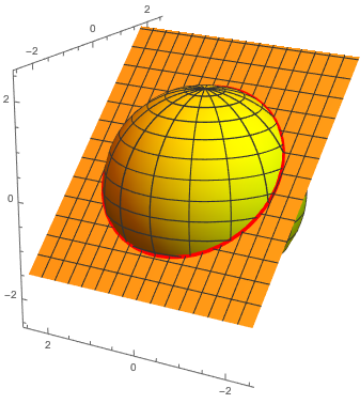
$$= \operatorname{div}(\langle gP, gQ\rangle) = \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ)$$

$$= \frac{\partial g}{\partial x}P + g\frac{\partial P}{\partial x} + \frac{\partial g}{\partial y}Q + g\frac{\partial Q}{\partial y}$$

$$= \nabla g\cdot\langle P, Q\rangle + g\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$$

$$= g\operatorname{div}\vec{F} + \nabla g\cdot\vec{F}$$

3. (Hard, messy) Find a parametrization for the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1 + x$



$$x^2+y^2+z^2 = 4$$

$$z = 1+x$$

$$r(t) = \langle x(t), y(t), 1+x(t) \rangle$$

Find $x(t), y(t)$.

eliminate z in $\begin{cases} x^2+y^2+z^2 = 4 \\ z = 1+x \end{cases}$

$$\Rightarrow \quad x^2+y^2+(1+x)^2 = 4 \Rightarrow$$

$$\Rightarrow \quad x^2+y^2+x^2+2x+1 = 4$$

$$\Rightarrow 2x^2+2x+y^2 = 3$$

$$\Rightarrow x^2+x+\frac{1}{2}y^2 = \frac{3}{2} \Rightarrow x^2+2\cdot\frac{1}{2}x+\frac{1}{4}+\left(\frac{1}{\sqrt{2}}y\right)^2 = \frac{3}{2}+\frac{1}{4}$$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}y\right)^2 = \frac{7}{4}$$

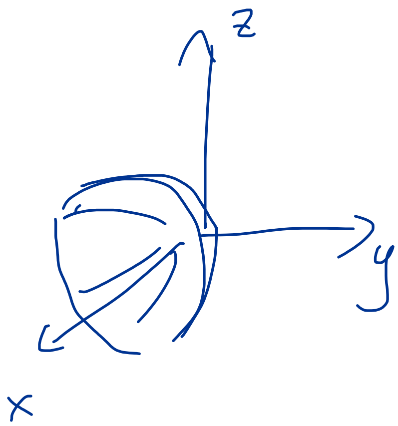
$$\Rightarrow \left[\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right)\right]^2+\left[\frac{\sqrt{2}}{\sqrt{7}}y\right]^2 = 1$$

Set $\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right) = \cos t \qquad \frac{\sqrt{2}}{\sqrt{7}}y = \sin t$

$$x = -\frac{1}{2}+\frac{\sqrt{7}}{2}\cos t \qquad y = \frac{\sqrt{7}}{\sqrt{2}}\sin t$$

$$r(t) = \left\langle -\frac{1}{2}+\frac{\sqrt{7}}{2}\cos t, \frac{\sqrt{7}}{\sqrt{2}}\sin t, \frac{1}{2}+\frac{\sqrt{7}}{2}\cos t \right\rangle, \ t \in [0, 2\pi]$$

3. Find the mass of a thin piece of aluminum foil occupying the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 4$, assuming that its density at the point (x, y, z) is



$$\rho(x, y, z) = \sqrt{\frac{x}{4x+1}}.$$

Parametrize:

$\vec{r}(u,v) = \langle u^2+v^2, u, v\rangle$, $(u,v) \in D$

To find D, project paraboloid on yz plane:

$$\left.\begin{array}{l} x = y^2+z^2 \\ x = 4 \end{array}\right\} \Rightarrow \quad y^2+z^2 = 4 \Rightarrow$$ projection is the disk of radius 2 on yz plane, so:

$$D = \{(u,v) : u^2+v^2 \leq 4\}$$

$$\vec{r}_u = \langle 2u, 1, 0\rangle, \quad \vec{r}_v = \langle 2v, 0, 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j}(-2u) + \vec{k}(-2v)$$

$$m = \iint_S \rho(x,y,z)\, dS = \iint_D \frac{\sqrt{u^2+v^2}}{\sqrt{4(u^2+v^2)+1}} \sqrt{1+4u^2+4v^2}\, dA$$

polar

$$= \int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta = 2\pi \left.\frac{r^3}{3}\right|_0^2 = \frac{8 \cdot 2\pi}{3}$$

4. (10 pts.) Let S be the onion-like surface obtained from the revolution of the graph of the function $z = \sin(y) + 1$, $-\frac{\pi}{2} \le y \le \pi$, around the y-axis (look at the picture).

Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \cancel{\langle y, y^2, x+z \rangle}$. → unit normal away from origin

$\langle 0, 1, 0 \rangle$

Parametrize the surface of revolution:

$$r(u,v) = \langle (1+\sin v)\cos u,\ v,\ (1+\sin v)\sin u \rangle \qquad u \in [0, 2\pi],\ v \in \left[-\tfrac{\pi}{2}, \pi\right]$$

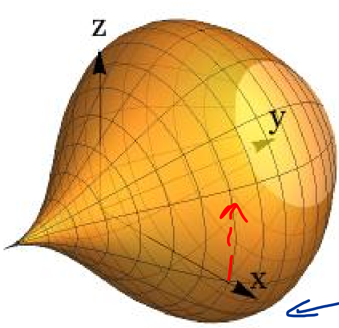
$$\vec{r}_u = \langle -(1+\sin v)\sin u,\ 0,\ (1+\sin v)\cos u \rangle$$

$$\vec{r}_v = \langle \cos v \cos u,\ 1,\ \cos v \sin u \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(1+\sin v)\sin u & 0 & (1+\sin v)\cos u \\ \cos v \cos u & 1 & \cos v \sin u \end{vmatrix} =$$

$$= \vec{i}\,(-\cos u(1+\sin v)) + \vec{j}\,((1+\sin v)\cos v \sin^2 u + (1+\sin v)\cos v \cos^2 u) + \vec{k}\,(-(1+\sin v)\sin u)$$

$$= \langle -\cos u(1+\sin v),\ (1+\sin v)\cos v,\ -(1+\sin v)\sin u \rangle$$



Plug in $u = 0$, $v = 0$,

$\vec{r}(0,0) = \langle 1, 0, 0 \rangle$, $\vec{r}_u \times \vec{r}_v(0,0) = \langle -1, 1, 0 \rangle$

it doesn't work, it's pointing inside

So:

$$\iint_S \vec{F} \cdot d\vec{S} = -\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} \vec{F} \cdot \vec{r}_u \times \vec{r}_v(u,v)\, dv\, du = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} -(1+\sin v)\cos v\, dv\, du$$

$$= -2\pi \int_{-\frac{\pi}{2}}^{\pi} \cos v + \tfrac{1}{2}\sin 2v\, dv = -2\pi \cdot \tfrac{1}{2}$$

5.

(a) Find the tangent plane to the surface described implicitly by $z^3 = x^2 - y^4 + zxy$ at $(1, 1, 1)$

Level set of $F(x,y,z) = z^3 - x^2 + y^4 - zxy$

$\nabla F = \langle -2x - zy, 4y^3 - xz, 3z^2 - xy \rangle$

$\Rightarrow \nabla F(1,1,1) = \langle -3, 3, 2 \rangle$

Therefore:

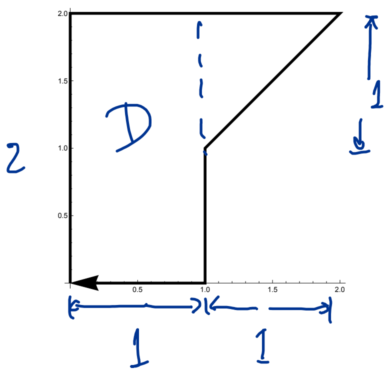
$(\langle x, y, z \rangle - \langle 1, 1, 1 \rangle) \cdot \langle -3, 3, 2 \rangle = 0$

$\Leftrightarrow -3(x-1) + 3(y-1) + 2(z-1) = 0$

7. Let c be the path on $\mathbb{R}^2$ consisting of the following line segments, as in the picture:

- A line segment from the (0,0) to (0,2).
- A line segment from (0,2) to (2,2)
- A line segment from (2,2) to (1,1)
- A line segment from (1,1) to (1,0)
- A line segment from (1,0) back to the origin.

Compute $\int_c y dx$. (**Hint:** Use Green's Theorem)



Clockwise so use $-$.

$$\boxed{\begin{matrix} P = y \\ Q = 0 \end{matrix}}$$

$$\int_c y\,dx = -\int_{-c} y\,dx = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= -\iint_D -1\, dA$$

$$= A(D) =$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

6. (10 pts) Let E be the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, is bounded below by the cone $z = -\sqrt{x^2 + y^2}$ and also bounded by the planes $y = x$ and $y = -x$, such that the y coordinate of any point in E is non-negative (look at the picture at the bottom of the page).

(a) Compute the volume of E.

Set it up in spherical coords:

$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

$\rho \le 2$

$z = -\sqrt{x^2+y^2} \Rightarrow \rho\cos\varphi = -\sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$

$\Rightarrow \rho\cos\varphi = -\rho\sin\varphi \Rightarrow \tan\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$

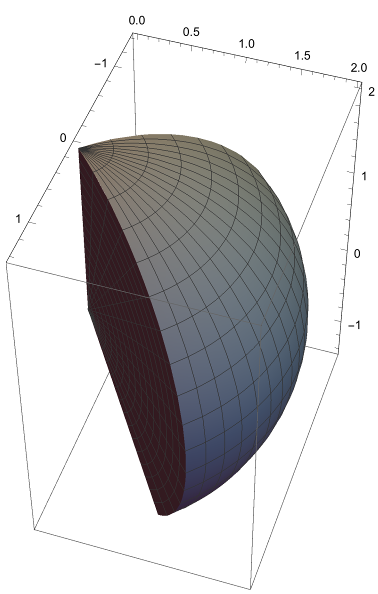
$y = x \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$y = -x \Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$

$$V = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{3\pi}{4}} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \cdot \frac{\rho^3}{3}\Big|_0^2 \cdot (-\cos\varphi)\Big|_0^{\frac{3\pi}{4}}$$

$$= \frac{4}{3}\pi\left(1 + \frac{1}{\sqrt{2}}\right) = 2\pi$$

(b) If $\vec{F}(x, y, z) = \langle x, y, z\rangle$ and S is the boundary of E (E is the same as in part (a)) with **inward orientation**, compute $\iint_S \vec{F} \cdot d\vec{S}$.



Use divergence theorem: Inward orientation, so

$$\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \operatorname{div}\vec{F} \, dV$$

$$= -\iiint_E 3 \, dV = -3\,\mathrm{Vol}(E)$$

$$= -6\pi.$$

6. (6 +3 pts) Let $\vec{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle = \langle P(x,y), Q(x,y) \rangle$, defined on

$$D = \mathbb{R}^2 \backslash \{(0,0)\}$$

(the plane without the origin). It is given that this vector field satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

(a) Compute $\int_c \vec{F} \cdot d\vec{r}$, where c is the unit circle, parametrized clockwise.

Do it for counter-clockwise, put a (−) sign.

$$\int_c \vec{F} \cdot d\vec{r} = -\int_{-c} \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} \frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\, dt$$

$$\left.\begin{array}{l} x(t) = \cos t \\ y(t) = \sin t \end{array}\right\} \quad \begin{array}{l} x'(t) = -\sin t \\ y'(t) = \cos t \end{array}$$

$$= -2\pi$$

(b) Is $\vec{F}$ conservative on D? Justify your answer.

It's not, bec. the line integral over a closed path is not zero.

(c) Bonus[1]: Find a potential function for $\vec{F}$, defined on the set

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q \Rightarrow \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$$

$$\text{so } x \neq 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}, \quad \frac{\partial f}{\partial y} = \frac{1}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$\text{So } \frac{\partial f}{\partial y} = \frac{1}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2} \Rightarrow f(x,y) = \arctan\left(\frac{y}{x}\right) + g(x)$$

$$\text{So } \frac{\partial f}{\partial x} = -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2} \Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) + g'(x) = -\frac{y}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$\Rightarrow g'(x) = 0$$

$$\text{So} \quad f(x,y) = \arctan\left(\frac{y}{x}\right) + c$$

7. (10 pts.) Let S be the unit sphere centered at the origin. Let c be the path consisting of the following curves, as in the picture at the bottom of the page:

- An arc of the intersection of S with the plane $y = x$, from (0,0,1) to $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $z = -\dfrac{1}{2}$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $y = -x$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to (0,0,1) (the one satisfying $x \geq 0$).

Let $\vec{F}(x,y,z) = \langle -yx, x^2, z \rangle$. Compute $\int_c \vec{F} \cdot d\vec{r}$ (you may do it directly, or use one of the theorems of chapter 16; if you do so, clearly state which theorem you are using).

Easier with Stokes: c is the boundary of a surface S' on the sphere. Parametrize sphere:

$$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle,$$

$$\vec{r}_u \times \vec{r}_v(u,v) = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle$$

We need correct bounds.

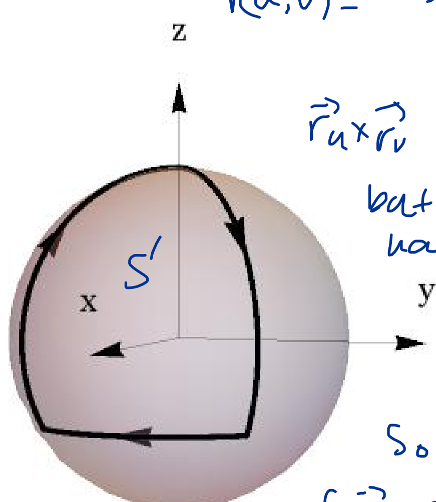
$$y = x \Rightarrow \cos v = \sin v \Rightarrow v = \frac{\pi}{4}$$

$$y = -x \Rightarrow \cos v = -\sin v \Rightarrow v = -\frac{\pi}{4}$$

$$z = -\frac{1}{2} \Rightarrow \cos u = -\frac{1}{2} \Rightarrow u = \frac{2\pi}{3}$$

So: S' can be parametrized as

$$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle, \quad u \in [0, \tfrac{2\pi}{3}], \quad v \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}]$$



$\vec{r}_u \times \vec{r}_v$ gives <u>outward</u> orientation but we need inward bec. of right hand rule.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yx & x^2 & z \end{vmatrix} = \langle 0, 0, 3x \rangle$$

So by Stokes' thm,

$$\int_c \vec{F} \cdot d\vec{r} = \iint_{S'} \text{curl } \vec{F} \cdot d\vec{S} =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} \langle 0, 0, 3 \sin u \cos v \rangle \cdot (-\langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle)\, du\, dv$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} -3 \sin^2 u \cos u \cos v \, du\, dv = \left[\sin^3 u\right]_0^{\frac{2\pi}{3}} \left[-\sin v\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\sqrt{3}}{2}\right)^3 (-\sqrt{2})$$