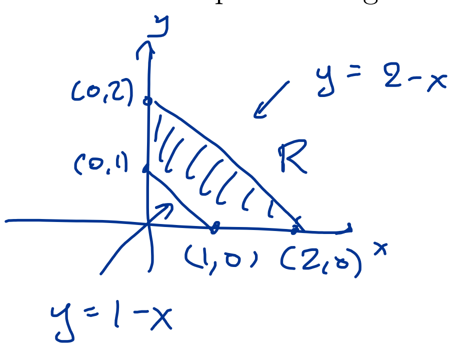
Worksheet

October 16, 2017

1. Make a change of variables to evaluate the integral

$$\iint_R e^{\frac{y-x}{y+x}}\, dA,$$

where R is the trapezoidal region with vertices (1,0), (2,0), (0,2) and (0,1).



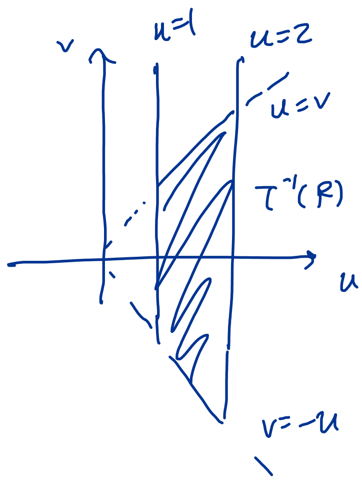
set $\left.\begin{array}{l} u = y+x \\ v = y-x \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{u-v}{2} \\ y = \frac{u+v}{2} \end{array}$

$y = 2-x \Rightarrow y+x = 2 \Rightarrow u = 2$

$y = 1-x \Rightarrow y+x = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u = v$

$y = 0 \Rightarrow u = -v$



Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So

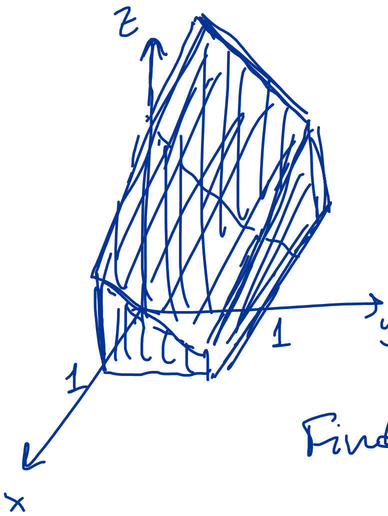
$$\iint_R e^{\frac{y-x}{y+x}}\, dA = \int_1^2 \int_{-u}^{u} \frac{1}{2} e^{\frac{v}{u}}\, dv\, du$$

$$= \int_1^2 \frac{1}{2} u e^{\frac{v}{u}} \Big|_{-u}^{u} du$$

$$= \int_1^2 \frac{1}{2} u \left(e - e^{-1}\right) du$$

$$= \frac{1}{2} \frac{u^2}{2} \left(e - e^{-1}\right) \Big|_1^2 = \frac{3}{4}\left(e - \frac{1}{e}\right)$$

2. Set up an integral $\iiint_E f(x,y,z)dV$, where E is bounded above by the plane $z = 3 - x - y$, below by the xy plane, and also bounded by the planes $x = -1$, $x = 1$, $y = 0$ and $y = 1$ **in the order** $dxdzdy$.



① $x = 3 - z - y$

② $x = -1$

③ $x = 1$

④ $y = 0$

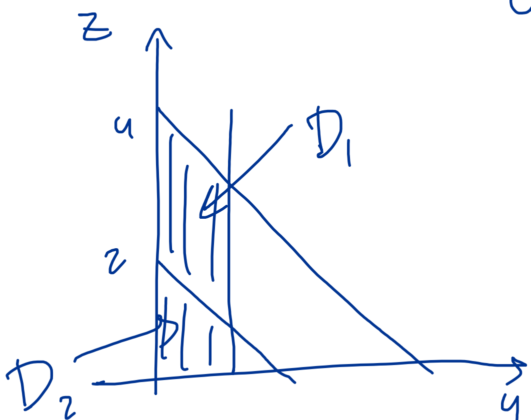
⑤ $y = 1$

Need sum of int's

Find projection on yz plane

①, ② $\Rightarrow$ $z + y = 4$

①, ③ $\Rightarrow$ $z + y = 2$



Over D_1: $-1 \le x \le 3 - z - y$

and

$D_1 = \{(y,z) : 2 - y \le z \le 4 - y,\ 0 \le y \le 1\}$

Over D_2: $-1 \le x \le 1$

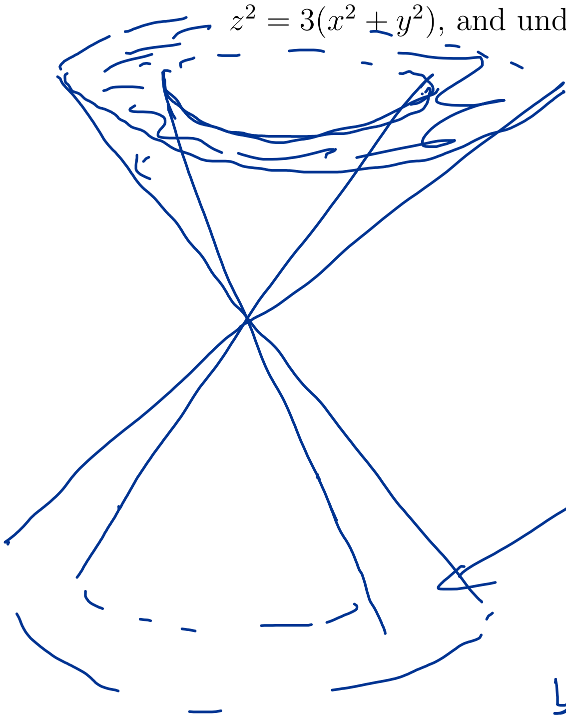
and

$D_2 = \{(y,z) : 0 \le z \le 2 - y \text{ and } 0 \le y \le 1\}$

So

$$\iiint_E f\,dV = \int_0^1 \int_0^{2-y} \int_{-1}^{1} f\,dx\,dz\,dy + \int_0^1 \int_{2-y}^{4-y} \int_{-1}^{3-z-y} f\,dx\,dz\,dy$$

3. Compute the volume of the bounded domain lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3(x^2 + y^2)$, and under the plane $z = 3$.



there is a domain here but it's not bounded.

Do it in spherical coords

$z = \sqrt{x^2 + y^2} \Rightarrow$

$\rho \cos\varphi = \sqrt{\rho^2 \sin^2\varphi}$

$\Rightarrow \varphi = \frac{\pi}{4}$

$z = \sqrt{3}\sqrt{x^2 + y^2} \Rightarrow$

$\rho \cos\varphi = \sqrt{3}\sqrt{\rho^2 \sin^2\varphi}$

$\Rightarrow \varphi = \frac{\pi}{6}$

So $\frac{\pi}{6} \le \varphi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$z = 3 \Rightarrow \rho\cos\varphi = 3 \Rightarrow \rho = \frac{3}{\cos\varphi}$

So

$$V = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos\varphi}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\varphi \left. \frac{\rho^3}{3} \right|_0^{\frac{3}{\cos\varphi}} d\varphi \, d\theta = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\varphi \frac{9}{\cos^3\varphi} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} \cos^{-2}\varphi \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \int_0^{2\pi} \frac{9}{2}\left(\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} \right) d\theta$$

$$= 9\pi\left(2 - \frac{4}{3}\right) = 6\pi$$

Done in Cylindrical coords:

3. Compute the volume of the bounded domain lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3(x^2 + y^2)$, and under the plane $z = 3$.

Write down equations solving for z:

① $z = \sqrt{x^2+y^2}$

② $z = \sqrt{3(x^2+y^2)}$

③ $z = 3$

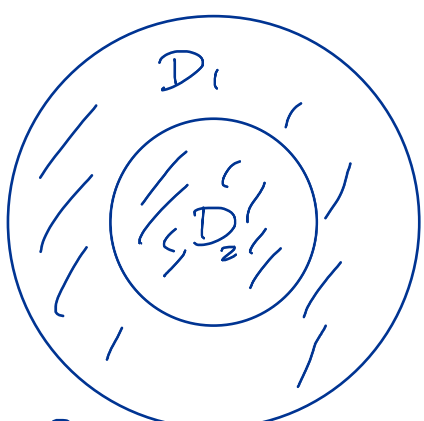
z appears 3 times, so if we want z to be the innermost variable we need a sum of 2 integrals

Find projection on xy plane:

①,② $\Rightarrow \sqrt{x^2+y^2} = \sqrt{3}\sqrt{x^2+y^2} \Rightarrow x = y = 0$ (only a point)

①,③ $\Rightarrow \sqrt{x^2+y^2} = 3 \Rightarrow x^2+y^2 = 9$ (a circle)

②,③ $\Rightarrow \sqrt{3}\sqrt{x^2+y^2} = 3 \Rightarrow x^2+y^2 = 3$ (circle)



Over $D_1 = \{(x,y) : 3 \le x^2+y^2 \le 9\}$

$= \{(r,\theta) : \sqrt{3} \le r \le 3,\ \theta \in [0,2\pi]\}$

we have

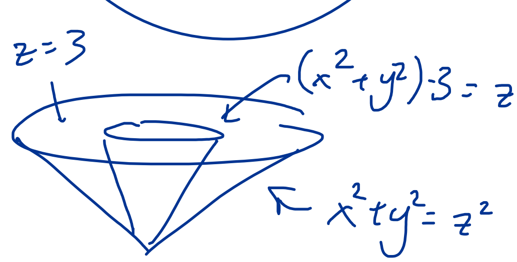
$\sqrt{x^2+y^2} \le z \le 3$ or $r \le z \le 3$

Over $D_2 = \{(x,y) : 0 \le x^2+y^2 \le 3\}$

$= \{(r,\theta) : 0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi\}$

we have

$\sqrt{x^2+y^2} \le z \le \sqrt{3(x^2+y^2)}$ or $r \le z \le \sqrt{3}r$



So:

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_r^{\sqrt{3}r} 1 \cdot r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{3}\int_r^{3} 1 \cdot r\,dz\,dr\,d\theta = 6\pi$$

4. If a transformation T is written as $x = x(u, v)$ and $y = y(u, v)$ and given a point (u_0, v_0) on the uv plane, we can produce an affine transformation dT called the **differential** or **pushforward** of T at (u_0, v_0) that is the best affine approximation of T near (u_0, v_0) and is given by

$$x = \frac{\partial x}{\partial u}_{|(u,v)=(u_0,v_0)}(u - u_0) + \frac{\partial x}{\partial v}_{|(u,v)=(u_0,v_0)}(v - v_0) + x(u_0, v_0)$$
$$y = \frac{\partial y}{\partial u}_{|(u,v)=(u_0,v_0)}(u - u_0) + \frac{\partial y}{\partial v}_{|(u,v)=(u_0,v_0)}(v - v_0) + y(u_0, v_0).$$

For the transformation $T(u, v) = (\frac{u^2}{v}, u^2v)$ defined on $\{(u, v) : u > 0, v > 0\}$:

(a) Find the transformation dT relative to the point (1,1).

(b) Find and draw the image of the box $[1, 2] \times [1, 2]$ under T and dT.

a) $\frac{\partial x}{\partial u} = \frac{2u}{v}$ $\qquad \frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$

$\frac{\partial y}{\partial u} = 2uv$ $\qquad \frac{\partial y}{\partial v} = u^2$

So $dT_{(1,1)} = (2(u-1) - (v-1) + 1,\ 2(u-1) + (v-1) + 1)$

$= (2u - v,\ 2u + v - 2)$

or $x = 2u - v$

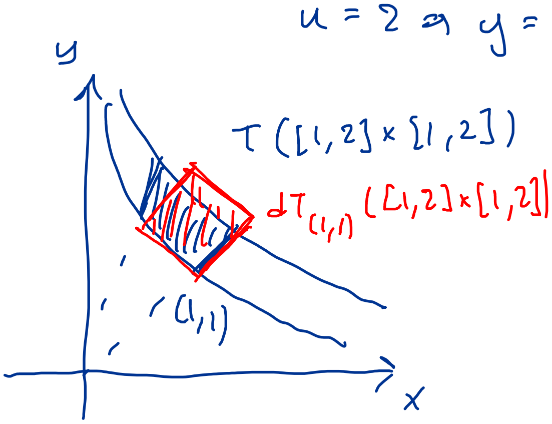
$y = 2u + v - 2$

b) Solve for u, v in T:

$$\left.\begin{aligned} x &= \frac{u^2}{v} \\ y &= u^2 v \end{aligned}\right| \Rightarrow \left.\begin{aligned} xy &= u^4 \\ \frac{y}{x} &= v^2 \end{aligned}\right| \Rightarrow \begin{aligned} u &= \sqrt[4]{xy} \\ v &= \sqrt{\frac{y}{x}} \end{aligned}$$

So $u = 1 \Rightarrow y = \frac{1}{x}$ $\qquad v = 1 \Rightarrow y = x$

$u = 2 \Rightarrow y = \frac{16}{x}$ $\qquad v = 2 \Rightarrow y = 4x$



Solve for u, v in $dT_{(1,1)}$

$$\left.\begin{aligned} x &= 2u - v \\ y &= 2u + v - 2 \end{aligned}\right| \Rightarrow \begin{aligned} 4u &= x + y + 2 \\ 2v &= y - x + 2 \end{aligned}$$

$$\Rightarrow \begin{aligned} u &= \tfrac{1}{4}(x + y) + \tfrac{1}{2} \\ v &= \tfrac{1}{2}(y - x) + 1 \end{aligned}$$

So $u = 1 \Rightarrow x + y = 2$

$u = 2 \Rightarrow x + y = 6$

$v = 1 \Rightarrow y = x$

$v = 2 \Rightarrow y = x + 2$

Drawn on the xy plane in red.