

Math 324 C - Winter 2017

Final Exam v.B

Wednesday, March 15, 2017

Name: _____

Student ID Number: _____

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- There are 7 problems spanning 7 pages (your last page should be numbered as 7). Please make sure your exam contains all these questions.
- You are allowed to use a scientific calculator (**no graphing calculators**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the back of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 110 minutes to complete the exam. Budget your time wisely.
Do not spend too much time on an individual problem, unless you are done with all the rest.
- You are not allowed to discuss this exam with other people until 5.00 pm today.

GOOD LUCK!

1. (8 pts.) You do not need to explain your answers for this problem.

- (a) Mark the following sentence as **true** or **false**. Let c be the unit circle in \mathbb{R}^2 parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x, y)$ we have

$$\int_{-c} f(x, y) dx = - \int_c f(x, y) dx.$$

True False

- (b) Mark the following sentence as **true** or **false**. Let S denote the unit ball in \mathbb{R}^3 with positive(outward) orientation and \tilde{S} the unit ball with negative (inward) orientation. Then, for any vector field $\vec{F}(x, y, z)$ with continuous coefficients

$$\int_S \vec{F}(x, y, z) \cdot d\vec{S} = - \int_{\tilde{S}} \vec{F}(x, y, z) \cdot d\vec{S}.$$

True False

- (c) Mark the following sentence as **true** or **false**. Let S denote the upper hemisphere of the unit ball centered at the origin in \mathbb{R}^3 (the one that satisfies $z \geq 0$), with **upward** orientation, and \tilde{S} the lower hemisphere of the unit ball centered at the origin (the one that satisfies $z \leq 0$), again with **upward** orientation. Then, for any vector field $\vec{F}(x, y, z)$ with differentiable coefficients

$$\iint_S \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S} = \iint_{\tilde{S}} \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S}.$$

True False

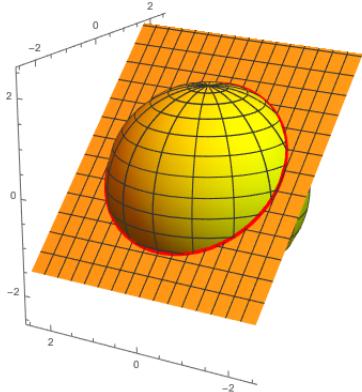
2. (6 pts.) Show the following version of the product rule: Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field, where P, Q are differentiable scalar valued functions, and let $g(x, y)$ be a differentiable scalar valued function. Then

$$\operatorname{div}(g\vec{F}) = g \operatorname{div}(\vec{F}) + (\nabla g) \cdot \vec{F}.$$

Make sure that each step follows clearly from the previous one, otherwise you may not receive full credit.

$$\begin{aligned}
 \operatorname{div}(g\vec{F}) &= \operatorname{div}(g \langle P, Q \rangle) = \\
 &= \operatorname{div}(\langle gP, gQ \rangle) = \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ) \\
 &= \frac{\partial g}{\partial x}P + g \frac{\partial P}{\partial x} + \frac{\partial g}{\partial y}Q + g \frac{\partial Q}{\partial y} \\
 &= \nabla g \cdot \langle P, Q \rangle + g \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \\
 &= g \operatorname{div}\vec{F} + \nabla g \cdot \vec{F}
 \end{aligned}$$

3. (Hard, messy) Find a parametrization for the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1 + x$



$$x^2 + y^2 + z^2 = 4$$

$$z = 1 + x$$

$$\mathbf{r}(t) = \langle x(t), y(t), 1+x(t) \rangle$$

Find $x(t), y(t)$.

eliminate z in $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = 1 + x \end{cases}$

$$\Rightarrow x^2 + y^2 + (1+x)^2 = 4 \Rightarrow$$

$$\Rightarrow x^2 + y^2 + x^2 + 2x + 1 = 4$$

$$\Rightarrow 2x^2 + 2x + y^2 = 3$$

$$\Rightarrow x^2 + x + \frac{1}{2}y^2 = \frac{3}{2} \Rightarrow x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} + \left(\frac{1}{\sqrt{2}}y\right)^2 = \frac{3}{2} + \frac{1}{4}$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}y\right)^2 = \frac{7}{4}$$

$$\Rightarrow \left[\frac{2}{\sqrt{7}}\left(x + \frac{1}{2}\right)\right]^2 + \left[\frac{\sqrt{2}}{\sqrt{7}}y\right]^2 = 1$$

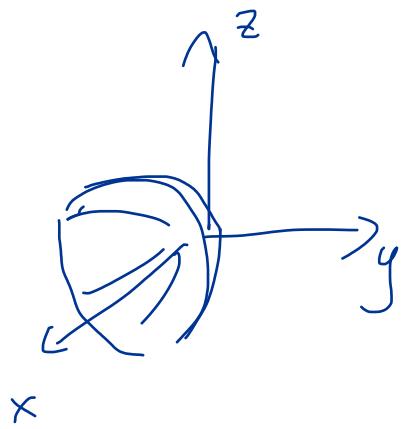
Set $\frac{2}{\sqrt{7}}\left(x + \frac{1}{2}\right) = \cos t \quad \frac{\sqrt{2}}{\sqrt{7}}y = \sin t$

$$x = -\frac{1}{2} + \frac{\sqrt{7}}{2}\cos t \quad y = \frac{\sqrt{2}}{\sqrt{7}}\sin t$$

$$\mathbf{r}(t) = \left\langle -\frac{1}{2} + \frac{\sqrt{7}}{2}\cos t, \frac{\sqrt{2}}{\sqrt{7}}\sin t, \frac{1}{2} + \frac{\sqrt{7}}{2}\cos t \right\rangle$$

3. Find the mass of a thin piece of aluminum foil occupying the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 4$, assuming that its density at the point (x, y, z) is

$$\rho(x, y, z) = \sqrt{\frac{x}{4x+1}}.$$



Parametrize:

$$\vec{r}(u, v) = \langle u^2 + v^2, u, v \rangle, (u, v) \in D$$

To find D , project paraboloid
on $y=0$ plane:

$$\begin{aligned} x &= y^2 + z^2 \\ x &= 4 \end{aligned} \quad \Rightarrow \quad y^2 + z^2 = 4 \Rightarrow \text{projection is the disk of radius 2 on } y=0 \text{ plane,}$$

so:

$$D = \{(u, v) : u^2 + v^2 \leq 4\}$$

$$\vec{r}_u = \langle 2u, 1, 0 \rangle, \quad \vec{r}_v = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j}(-2u) + \vec{k}(-2u)$$

$$m = \iint_S \rho(x, y, z) dS = \iint_D \frac{\sqrt{u^2 + v^2}}{\sqrt{4(u^2 + v^2) + 1}} \sqrt{1 + 4u^2 + 4v^2} dA$$

$$\text{polar} = \int_0^{2\pi} \int_0^2 r^2 dr d\theta = 2\pi \left[\frac{r^3}{3} \right]_0^2 = \frac{8 \cdot 2\pi}{3}$$

4. (10 pts.) Let S be the onion-like surface obtained from the revolution of the graph of the function $z = \sin(y) + 1$, $-\frac{\pi}{2} \leq y \leq \pi$, around the y -axis (look at the picture).

Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle y, y^2, x+z \rangle$. \rightarrow unit normal away from origin

Parametrize the surface of revolution:

$$\vec{r}(u, v) = \langle (1+\sin v) \cos u, v, (1+\sin v) \sin u \rangle \quad u \in [0, 2\pi] \\ v \in [-\frac{\pi}{2}, \pi]$$

$$\vec{r}_u = \langle -(\cos v) \sin u, 0, (\cos v) \cos u \rangle$$

$$\vec{r}_v = \langle \cos v \cos u, 1, \cos v \sin u \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(\cos v) \sin u & 0 & (\cos v) \cos u \\ \cos v \cos u & 1 & \cos v \sin u \end{vmatrix} =$$

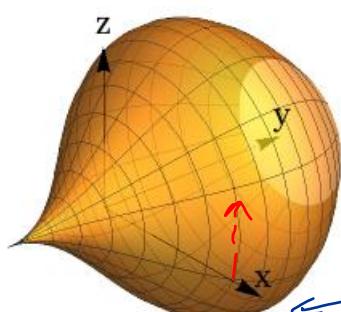
$$= \vec{i} (-\cos u (1+\sin v)) + \vec{j} ((1+\sin v) \cos v \sin^2 u \\ + (1+\sin v) \cos v \cos^2 u) \\ + \vec{k} (-(\cos v) \sin u)$$

$$= \langle \cos u (1+\sin v), (1+\sin v) \cos v, -(\cos v) \sin u \rangle$$

Plug in $u=0, v=0$,

$$\vec{r}(0, 0) = \langle 1, 0, 0 \rangle, \vec{r}_u \times \vec{r}_v(0, 0) = \langle -1, 1, 0 \rangle$$

it doesn't work, it's pointing inside



So:

$$\iint_S \vec{F} \cdot d\vec{S} = - \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} \vec{F} \cdot \vec{r}_u \times \vec{r}_v(u, v) dv du = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} -(\cos v) \sin u dv du \\ = -8\pi \int_{-\frac{\pi}{2}}^{\pi} \cos v + \frac{1}{2} \sin 2v dv = -2\pi \cdot \frac{1}{2}$$

5.

- (a) Find the tangent plane to the surface described implicitly by $z^3 = x^2 - y^4 + zxy$ at $(1, 1, 1)$

Level set of $F(x, y, z) = z^3 - x^2 + y^4 - zxy$

$$\nabla F = \langle -2x - zy, 4y^3 - xz, 3z^2 - xy \rangle$$

$$\Rightarrow \nabla F(1, 1, 1) = \langle -3, 3, 2 \rangle$$

Therefore:

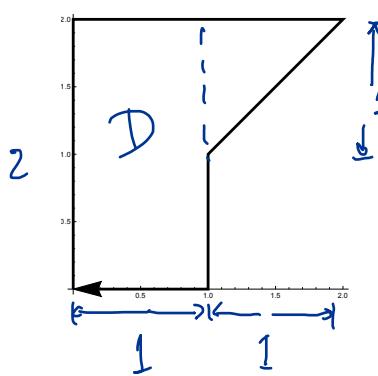
$$(\langle x, y, z \rangle - \langle 1, 1, 1 \rangle) \cdot \langle -3, 3, 2 \rangle = 0$$

$$\Leftrightarrow -3(x-1) + 3(y-1) + 2(z-1) = 0$$

7. Let c be the path on \mathbb{R}^2 consisting of the following line segments, as in the picture:

- A line segment from the $(0,0)$ to $(0,2)$.
- A line segment from $(0,2)$ to $(2,2)$
- A line segment from $(2,2)$ to $(1,1)$
- A line segment from $(1,1)$ to $(1,0)$
- A line segment from $(1,0)$ back to the origin.

Compute $\int_c ydx$. (**Hint:** Use Green's Theorem)



$$\begin{aligned}
 & \text{Clockwise so use } - . \quad \boxed{\begin{array}{l} P = y \\ Q = 0 \end{array}} \\
 \int_C y dx &= - \int_C y dx = - \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\
 &= - \iint_D -1 dA \\
 &= A(D) = \\
 &= 2 + \frac{1}{2} = \frac{5}{2}
 \end{aligned}$$

6. (10 pts) Let E be the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, is bounded below by the cone $z = -\sqrt{x^2 + y^2}$ and also bounded by the planes $y = x$ and $y = -x$, such that the y coordinate of any point in E is non-negative (look at the picture at the bottom of the page).

(a) Compute the volume of E .

Set it up in spherical coords:

$$\rho \leq 2$$

$$z = -\sqrt{x^2 + y^2} \Rightarrow \rho \cos \varphi = -\sqrt{\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta}$$

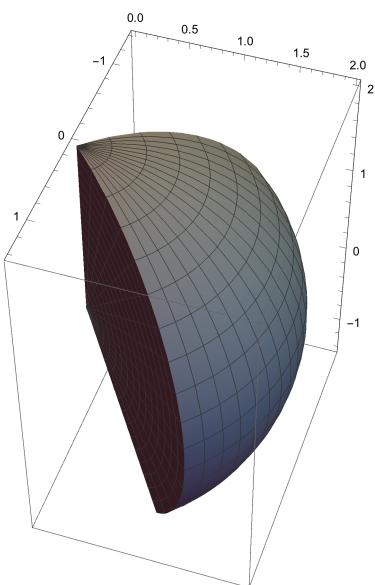
$$\Rightarrow \rho \cos \varphi = -\rho \sin \varphi \Rightarrow \tan \varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$$

$$y = x \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$y = -x \Rightarrow \tan \theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$$

$$V = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{3\pi}{4}} \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \cdot \frac{\rho^3}{3} \Big|_0^2 \cdot (-\cos \varphi) \Big|_0^{\frac{3\pi}{4}} = \frac{4}{3}\pi \left(1 + \frac{1}{\sqrt{2}}\right) = 2\pi$$

- (b) If $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and S is the boundary of E (E is the same as in part (a)) with **inward orientation**, compute $\iint_S \vec{F} \cdot d\vec{S}$.



Use divergence theorem: inward orientation,

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= - \iiint_E \operatorname{div} \vec{F} \cdot dV \\ &= - \iiint_E 3 \, dV = -3 \operatorname{Vol}(E) \\ &= -6\pi. \end{aligned}$$

6. (6 + 3 pts) Let $\vec{F}(x, y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle = \langle P(x, y), Q(x, y) \rangle$, defined on

$$D = \mathbb{R}^2 \setminus \{(0, 0)\}$$

(the plane without the origin). It is given that this vector field satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D .

- (a) Compute $\int_c \vec{F} \cdot d\vec{r}$, where c is the unit circle, parametrized clockwise.

Do it for counter-clockwise, put a (-) sign.

$$\int_c \vec{F} \cdot d\vec{r} = - \int_{-c} \vec{F} \cdot d\vec{r} = - \int_0^{2\pi} \frac{-\sin t}{\cos^2 t + \sin^2 t} (-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t} (\cos t) dt$$

$$\left. \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \right\} \quad \left. \begin{cases} x'(t) = -\sin t \\ y'(t) = \cos t \end{cases} \right\} = -2\pi$$

- (b) Is \vec{F} conservative on D ? Justify your answer.

It's not, bcs. the line integral over a closed path is not zero.

- (c) Bonus¹: Find a potential function for \vec{F} , defined on the set

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q \Rightarrow \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$$

$$\text{so } x \neq 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{y}{x^2} \frac{1}{1+(\frac{y}{x})^2}, \quad \frac{\partial f}{\partial y} = \frac{1}{x} \frac{1}{1+(\frac{y}{x})^2}$$

$$\text{so } \frac{\partial f}{\partial y} = \frac{1}{x} \frac{1}{1+(\frac{y}{x})^2} \Rightarrow f(x, y) = \arctan(\frac{y}{x}) + g(x)$$

$$\text{so } \frac{\partial f}{\partial x} = -\frac{y}{x^2} \frac{1}{1+(\frac{y}{x})^2} \Rightarrow \frac{1}{1+(\frac{y}{x})^2} \left(-\frac{y}{x^2} \right) + g'(x) = -\frac{y}{x} \frac{1}{1+(\frac{y}{x})^2}$$

$$\Rightarrow g'(x) = 0$$

$$\text{so } f(x, y) = \arctan(\frac{y}{x}) + c$$

7. (10 pts.) Let S be the unit sphere centered at the origin. Let c be the path consisting of the following curves, as in the picture at the bottom of the page:

- An arc of the intersection of S with the plane $y = x$, from $(0,0,1)$ to $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $z = -\frac{1}{2}$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $y = -x$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to $(0,0,1)$ (the one satisfying $x \geq 0$).

Let $\vec{F}(x, y, z) = \langle -yx, x^2, z \rangle$. Compute $\int_c \vec{F} \cdot d\vec{r}$ (you may do it directly, or use one of the theorems of chapter 16; if you do so, clearly state which theorem you are using).

Easier with Stokes: c is the boundary of a surface S' on the sphere. Parametrize sphere:

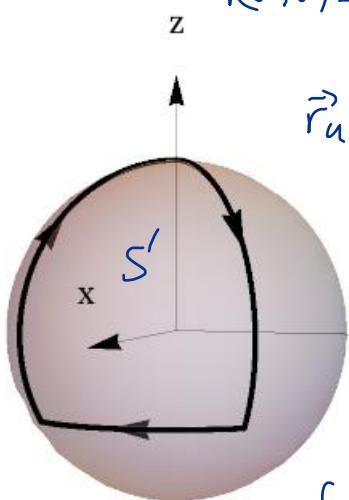
$$\vec{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle,$$

$$\vec{r}_u \times \vec{r}_v (u, v) = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle$$

We need correct bounds. $y = x \Rightarrow \cos v = \sin v \Rightarrow v = \frac{\pi}{4}$
 $y = -x \Rightarrow \cos v = -\sin v \Rightarrow v = -\frac{\pi}{4}$
 $z = -\frac{1}{2} \Rightarrow \cos u = -\frac{1}{2} \Rightarrow u = \frac{2\pi}{3}$

So: S' can be parametrized as

$$\vec{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle, \quad u \in [0, \frac{2\pi}{3}], \quad v \in [-\frac{\pi}{4}, \frac{\pi}{4}]$$



$\vec{r}_u \times \vec{r}_v$ gives outward orientation

but we need inward b.c. of right hand rule.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yx & x^2 & z \end{vmatrix} = \langle 0, 0, 3x \rangle$$

So by Stokes' thm,

$$\int_c \vec{F} \cdot d\vec{r} = \iint_{S'} \text{curl } \vec{F} \cdot d\vec{S} =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} \langle 0, 0, 3 \sin u \cos v \rangle \cdot (-\langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \rangle) du dv$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} -3 \sin^3 u \cos u \cos v du dv = \left[\sin^3 u \right]_0^{\frac{\pi}{4}} \left[-\sin v \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\sqrt{3}}{2} \right)^3 (\sqrt{2})$$