# Worksheet 2

November 14, 2017

1. Among the following vector fields, one has constant upwards pointing curl. Which one?

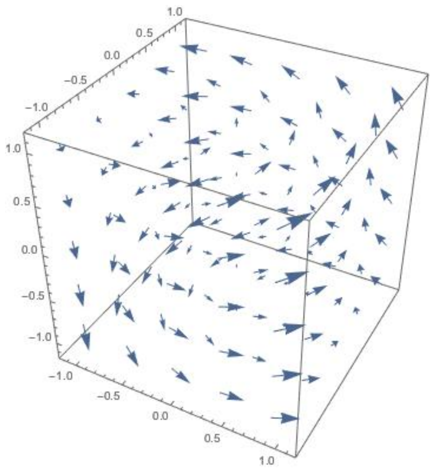


Figure 1: Plot A

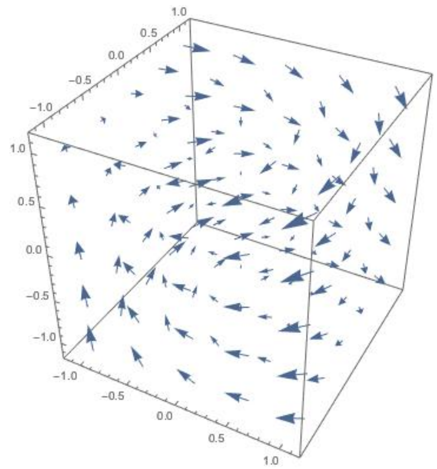


Figure 2: Plot B

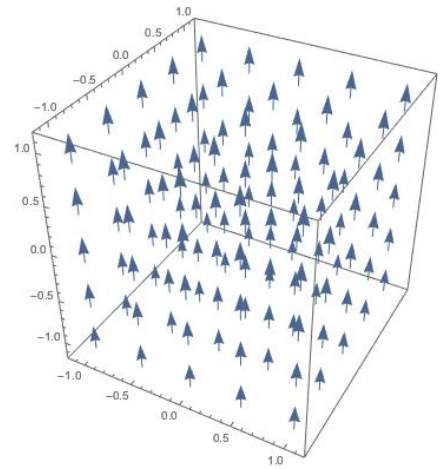


Figure 3: Plot C

2. One of the vector fields below has always negative divergence. Which one?

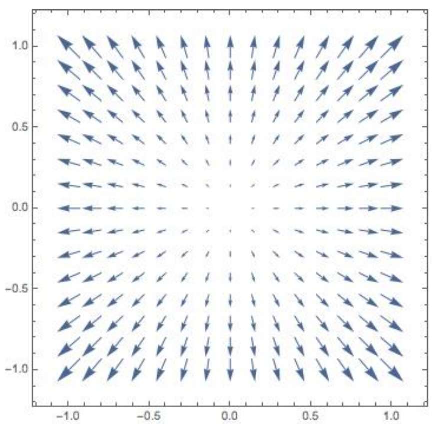


Figure 4: Plot A

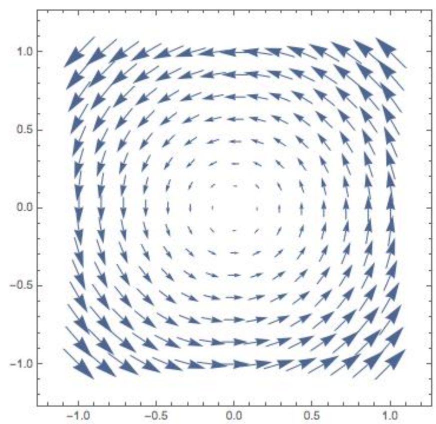


Figure 5: Plot B

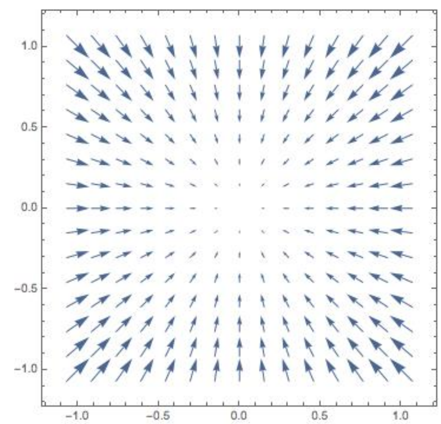
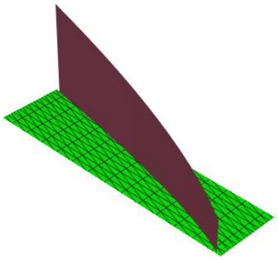


Figure 6: Plot C

3. A fence lies over the curve $y = x^2$ for $x \in [1, 3]$ and under the graph of the function $f(x, y) = x$, where $x$, $y$ are in meters. Find the area of the fence, including units.



Parametrize curve:

$$c(t) = (x(t), y(t)) = (t, t^2), \; t \in [1,3]$$

$$\text{Area} = \int_C f(x,y)\, ds = \int_1^3 t \sqrt{1+(2t)^2}\, dt$$

$$= \int_1^3 t \sqrt{1+4t^2}\, dt =$$

$u = 1+4t^2$

$du = 8t\,dt$

$t=1 \Rightarrow u=5$

$t=3 \Rightarrow u=37$

$$= \int_5^{37} \frac{1}{8} \sqrt{u}\, du = \frac{1}{8} \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{37}$$

$$= \frac{1}{12}\left((37)^{\frac{3}{2}} - 5^{3/2}\right) m^2$$

4. A fly flies in a room along the curve

$$c(t) = (2\sin(t), \cos(t), 2t).$$

The temperature at the point $(x, y, z)$ of the room is given by the function

$$T(x, y, z) = x^2 + 2e^z y,$$

in $^\circ$ C.

(a) Find the gradient of $T$.

(b) As the fly moves, it experiences a different temperature at each point. Find the instantaneous rate of change of the temperature that the fly experiences at time $\pi$ seconds (include units).

a) $\nabla T(x,y,z) = \langle 2x, 2e^z, 2e^z y \rangle$

b) Need: $\frac{d}{dt}(T \circ c)(t)$

Use Chain Rule: write $c(t) = (x(t), y(t), z(t))$

Then, using that $c(\pi) = (0, -1, 2\pi)$

$$\frac{d}{dt}(T \circ c)(t) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \nabla T(c(\pi)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle\Big|_{\pi} =$$

$$= \langle 0, 2e^{2\pi}, 2e^{2\pi}(-1) \rangle \cdot \langle 2\cos t, -\sin t, 2 \rangle\Big|_{\pi}$$

$$= \langle 0, 2e^{2\pi}, -2e^{2\pi} \rangle \cdot \langle -2, 0, 2 \rangle$$

$$= -4e^{2\pi} \ {}^\circ C/s$$

5. You are given the vector field

$$\vec{F}(x,y,z) = \langle 2xy + 2e^z, x^2, 2xe^z \rangle,$$

defined on $\mathbb{R}^3$. Show that it is conservative and find a potential function for it.

Find curl $\vec{F}$:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy+2e^z & x^2 & 2xe^z \end{vmatrix} = \langle 0, 2e^z - 2e^z, 2x - 2x \rangle$$

$= 0$, and it's defined on all of $\mathbb{R}^3$, so conservative

Write $\vec{F} = \langle P, Q, R \rangle = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$.

Then $\partial_x f = P = 2xy + 2e^z \Rightarrow$

$\Rightarrow f(x,y,z) = x^2 y + 2xe^z + \varphi(y,z)$

Then $\partial_y f = Q = x^2 \Rightarrow x^2 + \partial_y \varphi = x^2 \Rightarrow \varphi(y,z) = \psi(z)$

which means that $f(x,y,z) = x^2 y + 2xe^z + \psi(z)$

Then $\partial_z f = 2xe^z \Rightarrow 2xe^z + \partial_z \psi = 2xe^z \Rightarrow \psi = c$

Thus a potential function for $\vec{F}$ is

$f(x,y,z) = x^2 y + 2xe^z + c$

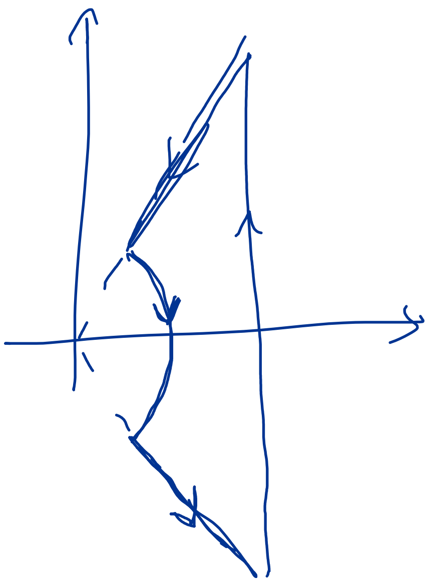
6. An object moves along the boundary of the set

$$D = \{(x,y) : x^2+y^2 \geq 1,\ |y| \leq \sqrt{3}x,\ x \leq 3\}$$

in counterclockwise direction. Find the work produced by the force field

$$\vec{F}(x,y) = \langle 3, \ln(x^2+y^2)\rangle$$

during the movement of the object.



Let $c$ be the boundary of the domain. Then, by Green's theorem,

$$\text{Work} = \int_c \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right) - \frac{\partial}{\partial y}(3)\, dA$$

$$= \iint_D \frac{2x}{x^2+y^2}\, dA$$

Write $D$ in polar:

$$x = 3 \Rightarrow r = \frac{3}{\cos\theta}$$

$$y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3}$$

$$y = -\sqrt{3}x \Rightarrow \theta = -\frac{\pi}{3}$$

$$x^2+y^2 = 1 \Rightarrow r = 1$$

So $D = \left\{(r,\theta) : 1 \leq r \leq \frac{3}{\cos\theta},\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\right\}$ and

$$W = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{\frac{3}{\cos\theta}} \frac{2r\cos\theta}{r^2}\, r\,dr\,d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos\theta\left(\frac{3}{\cos\theta} - 1\right) d\theta$$

$$= 6 \cdot \frac{2\pi}{3} - 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\, d\theta = 4\pi - 2\sqrt{3}$$