Surface Integrals

In this section we'll make sense of integrals over surfaces. In a sense, the content of this section is very analogous to the one discussing line integrals, except here we'll be working with two dimensional objects instead of one dimensional.

Before actually discussing integrals, recall that when we talked line integrals, some of them depended on the parameterization of the curve, and more specifically on the direction the curve was transversed, whereas others didn't. It's therefore natural to expect that we might have to deal with some similar behavior when discussing surface integrals.

Orientations

The surface analogue of choosing a "direction" is called a choice of orientation, and is essentially a choice of a preferred side of a surface. For example, we might want to consider the interior or the exterior of a sphere to be its preferred side.

Note that choosing a preferred side might not be possible for some surfaces. The mathematicians' favorite such surface is called a Möbius strip, a surface that has only one side.

- A nice animation on Möbius strip, here.
- A video explaining the proper way to cut a bagel, along a Möbius strip, here.

Definition 1. *The surfaces for which we can choose an orientation (preferred side) are called **orientable**.*

More rigorously, orientations are described in terms of unit normal vector fields. At each point of a parametric surface that is sufficiently regular (i. e. has no cusps) we can find 2 vectors that have unit length and are normal to the tangent plane of t surface at the given point. By choosing one of them at each point, we can form a rector field on the surface (since we are making sense of a function that takes points on the surface as input and outputs vectors). Now we want to choose a unit normal vector at each point so that our vector field becomes continuous (for example, in a sphere we should choose vectors pointing only inwards or only outwards)

An orientation on an orientable surface is a choice of a continuous unit normal vector field.

In Practice:

How can we find a unit normal vector field for a given parametric surface? From the parametrization we can automatically find 2 continuous unit vector fields, namely

Usually, a problem will tell us which orientation we are expected to use. So:

- We compute one of the unit normal vector fields to the surface.
- We pick a point on the surface* and check if the normal vector given by our vector field agrees with the given orientation.
- If yes, we work with our vector field, otherwise we work we use its negative.

*As a guideline for picking a point to check on a parametrized surface, it is better to avoid choosing points on the image the boundary of the domain D where the parametrization is defined. For example, in a sphere parametrized as

$$\vec{r}(u,v) = \langle R\sin(u)\cos(v), R\sin(u)\cos(v), R\cos(u)\rangle, (u,v) \in [0,\pi] \times [,2\pi]$$

it is safer to not choose points that correspond to