Math 324 H - Autumn 2017
Final Exam
Monday, 12/11/2017

Name: ______________________________

Student ID Number: ______________________________

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IMPORTANT: THIS EXAM IS DOUBLE SIDED

- There are 8 problems spanning 8 pages (your last page should be numbered as 8). Please make sure your exam contains all these questions.
- You are allowed to use a TI-30x IIS Calculator and one double sided hand-written 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 1 hour and 50 minutes to complete the exam. Budget your time wisely. **Do not spend too much time on an individual problem, unless you are done with all the rest.**

GOOD LUCK!

1. (10 pts) **You do not need to explain your answers for this problem.**

   (a) You are given the following parametrization for a surface S:

$$\vec{r}(u, v) = \langle (3 + \cos(v))\cos(u), (3 + \cos(v))\sin(u), \sin(v)\rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi.$$

   Choose the surface corresponding to this parametrization (the first coordinate corresponds to x, the second one to y and the third one to z).

set $v=0$

$\vec{r}(u,0) = \langle 4\cos u, 4\sin u, 0\rangle$

circle on xy plane

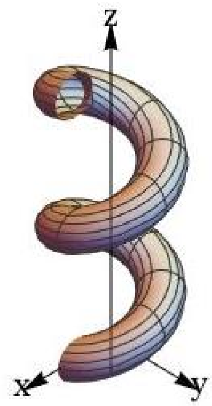


Figure 1: Surface A

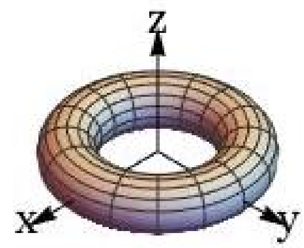


Figure 2: Surface B

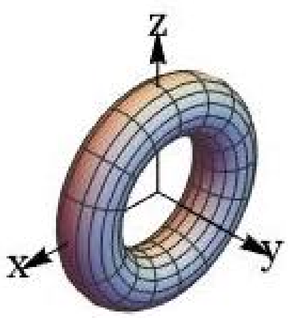


Figure 3: Surface C

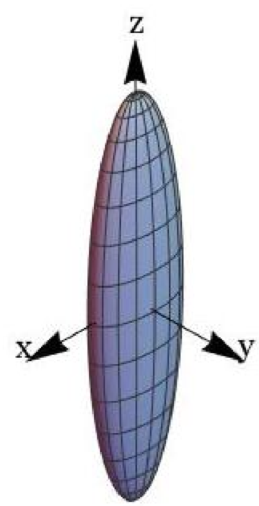


Figure 4: Surface D

THIS PROBLEM IS CONTINUED IN THE NEXT PAGE

(b) Mark the following sentence as **true** or **false**. Let S_1 be the unit sphere centered at the origin in $\mathbb{R}^3$ with **outward** orientation and S_2 be the unit sphere centered at the origin in $\mathbb{R}^3$ with **inward** orientation. Then for any continuous function $f(x, y, z)$ we have

$$\int_{S_1} f(x, y, z) dS = - \int_{S_2} f(x, y, z) dS.$$

True **False**

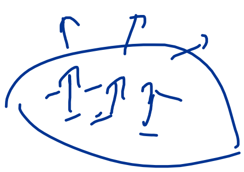
(c) If $\vec{F}$ and $\vec{G}$ are two vector fields in $\mathbb{R}^3$, then

$$\operatorname{curl}(\vec{F} \cdot \vec{G}) = \operatorname{curl} \vec{F} \cdot \operatorname{curl} \vec{G}.$$

doesn't make sense

True **False**

(d) Mark the following sentence as **true** or **false**. Let S denote the upper hemisphere of the unit sphere in $\mathbb{R}^3$ (the one that satisfies $z \geq 0$), with **upward** orientation, and $\tilde{S}$ the unit disk on the xy plane, again with **upward** orientation. Then, for any vector field $\vec{F}(x, y, z)$ with differentiable coefficients



$$\iint_S \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S} = \iint_{\tilde{S}} \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S}.$$

True **False**

(e) Mark the following sentence as **true** or **false**. Let S denote the positively oriented unit sphere in $\mathbb{R}^3$, centered at the origin. Then, for any vector field $\vec{F}(x, y, z)$ with differentiable coefficients

$$\iint_S \operatorname{curl} \vec{F}(x, y, z) \cdot d\vec{S} = 0.$$

by div. thm

True **False**

2. (6 pts) Show that the following equation is true: Let $u(x, y)$ and $v(x, y)$ be a real valued functions with continuous second partial derivatives. Then

$$\operatorname{div}(v\nabla u) = v\Delta u + \nabla v \cdot \nabla u,$$

where Δ denotes the Laplace operator, that is, $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ for any real valued twice differentiable function f.

Make sure that each one of the steps you make clearly follows from the previous one, otherwise you may not receive full credit.

$$\operatorname{div}(v\nabla u) = \operatorname{div}\left(\left\langle v\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y}\right\rangle\right)$$

$$= \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= \nabla v \cdot \nabla u + v\Delta u$$

3. (9 pts.) Let S be the surface obtained from revolution of the graph of the function $z = y^2$, $1 \le y \le 2$, around the y-axis (look at the picture).

(a) Write down a parametrization of the form $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for S. **Make sure to specify the domain of u and v!**

$$\vec{r}(u,v) = \langle v^2\cos u, v, v^2 \sin u\rangle$$

$$u \in [0, 2\pi]$$

$$v \in [1, 2]$$

(b) Compute $\iint_S \frac{1}{y} dS$.

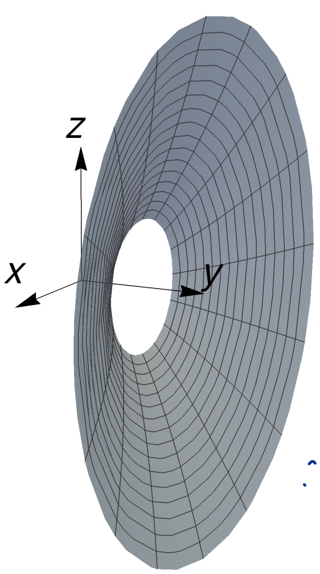
$$\iint_S \frac{1}{y}\, dS = \int_0^{2\pi}\int_1^2 \frac{1}{v} |\vec{r}_u \times \vec{r}_v|\, dv\, du$$

so find $\vec{r}_u \times \vec{r}_v$:

$$\vec{r}_u = \langle -v^2 \sin u, 0, v^2\cos u\rangle$$

$$\vec{r}_v = \langle 2v\cos u, 1, 2v\sin u\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -v^2\sin u & 0 & v^2\cos u \\ 2v\cos u & 1 & 2v\sin u \end{vmatrix} = \langle -v^2\cos u, 2v^3\cos^2 u + 2v^3\sin^2 u, -v^2 \sin u\rangle$$



so

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4\cos^2 u + 4v^6 + v^4\sin^2 u}$$

$$= \sqrt{v^4 + 4v^6}$$

$$= v^2\sqrt{1+4v^2}.$$

$$\therefore \iint_S \frac{1}{y}\, dS = \int_0^{2\pi}\int_1^2 \frac{1}{v}\cdot v^2\sqrt{1+4v^2}\, dv\, du =$$

$$= 2\pi \int_1^2 v\sqrt{1+4v^2}\, dv$$

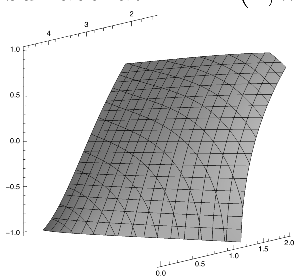
$s = 1+4v^2$
$=$
$ds = 8v\,dv$
$v = 1 \Rightarrow s = 5$
$v = 2 \Rightarrow s = 17$

$$2\pi \int_5^{17} \frac{1}{8}\sqrt{s}\, ds = \frac{1}{4}\pi\, \frac{2}{3} s^{\frac{3}{2}}\Big|_5^{17} = \frac{2\pi}{12}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

4. (6 pts.) Let S be the surface consisting of the points (x, y, z) in $\mathbb{R}^3$ that satisfy

$$yx^2 + z\sin(y) - \pi e^z = 0.$$

In the picture you see S plotted near the point $P = (1, \pi, 0)$. Find a **normal vector** to the surface at $P = (1, \pi, 0)$ and use it to find the **equation of the tangent plane** to S at P.



Level set of function

$f(x,y,z) = yx^2 + z\sin y - \pi e^z$

$\nabla f(1,\pi,0)$ normal vector at $P = (1,\pi,0)$

$\nabla f(x,y,z) = \langle 2xy, x^2 + z\cos y, \sin y - \pi e^z \rangle$

$\nabla f(1,\pi,0) = \langle 2\pi, 1, -\pi \rangle \leftarrow$ normal vector

Tang. plane:

$(x-1)\cdot 2\pi + (y-\pi)\cdot 1 + (z-0)\cdot(-\pi) = 0.$

5. (6 pts) It is given that the vector field $\vec{F}(x, y) = \langle ye^{xy} + 2xy, x^2 + xe^{xy} + 1 \rangle$ is conservative. Find a potential function for it.

$\vec{F} = \nabla f \Rightarrow \begin{cases} \partial_x f = ye^{xy} + 2xy \\ \partial_y f = x^2 + xe^{xy} + 1 \end{cases}$ so: $f(x,y) = \int ye^{xy} + 2xy \, dx + g(y)$

$\Rightarrow f(x,y) = e^{xy} + x^2 y + g(y)$

$\left.\begin{array}{l} \Rightarrow \partial_y f = xe^{xy} + x^2 + g'(y) \\ \text{But also} \\ \partial_y f = x^2 + xe^{xy} + 1 \end{array}\right\} \Rightarrow g'(y) = 1 \Rightarrow g(y) = y + c$

Potential fct: $f(x,y) = e^{xy} + x^2 y + y + c$

6. (10 pts) Let E be the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, is bounded below by the cone $z = -\sqrt{x^2 + y^2}$ and also bounded by the planes $y = x$ and $y = -x$, such that the y coordinate of any point in E is non-negative (look at the picture at the bottom of the page).

(a) Compute the volume of E.

Set it up in spherical coords:

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$\rho \leq 2$

$z = -\sqrt{x^2 + y^2} \Rightarrow \rho\cos\varphi = -\sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}$

$\Rightarrow \rho\cos\varphi = -\rho\sin\varphi \Rightarrow \tan\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$

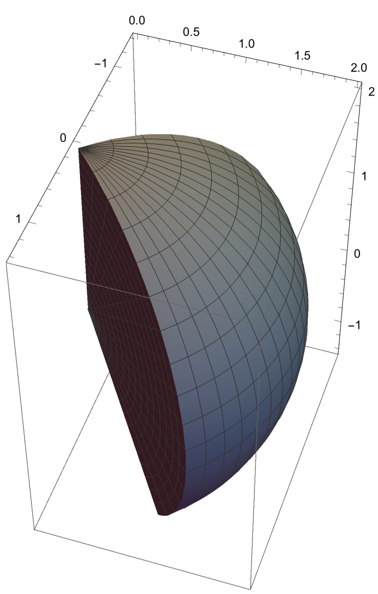
$y = x \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$y = -x \Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$

$$V = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{3\pi}{4}} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \cdot \frac{\rho^3}{3}\Big|_0^2 \cdot (-\cos\varphi)\Big|_0^{\frac{3\pi}{4}}$$

$$= \frac{4}{3}\pi\left(1 + \frac{1}{\sqrt{2}}\right) = 2\pi$$

(b) If $\vec{F}(x, y, z) = \langle x, y, z\rangle$ and S is the boundary of E (E is the same as in part (a)) with **inward orientation**, compute $\iint_S \vec{F} \cdot d\vec{S}$.



Use divergence theorem: Inward orientation, so

$$\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \operatorname{div}\vec{F} \, dV$$

$$= -\iiint_E 3 \, dV = -3\,\mathrm{Vol}(E)$$

$$= -6\pi.$$

7. (8+3 pts.) Let S be the hemisphere of the unit sphere $x^2+y^2+z^2=1$ that satisfies $y \geq 0$, oriented towards the origin and let $\vec{F}(x,y,z) = \langle 0, y, 0\rangle$.

(a) Compute $\iint_S \vec{F} \cdot d\vec{S}$.

$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$u \in [0,\pi] \quad v \in [0,\pi]$

! Note: If you want to use the Divergence Theorem you need to make it into closed surface

$\vec{r_u} \times \vec{r_v} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u \rangle$

At $u = \frac{\pi}{2}, v = \frac{\pi}{2}$

$\vec{r_u} \times \vec{r_v} = \langle 0, 1, 0 \rangle$ so not towards origin.

work w/ $-\vec{r_u} \times \vec{r_v}$

$$\iint_S \vec{F} \cdot d\vec{S} = -\int_0^\pi \int_0^\pi \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u \rangle \, du\, dv$$

$$= -\int_0^\pi \int_0^\pi \sin^3 u \sin^2 v \, du\, dv = -\int_0^\pi \frac{1-\cos 2v}{2} dv \int_0^\pi \sin u (1-\cos^2 u) du$$

$$= \frac{\pi}{2}\left[\cos u \Big|_0^\pi + \int_0^\pi \cos^2 u \sin u \, du \right] = \frac{\pi}{2}\left(-2 - \frac{\cos^3 u}{3}\Big|_0^\pi\right) = \frac{\pi}{2}\left(-2+\frac{2}{3}\right)$$

$= \frac{\pi}{2} \cdot -\frac{4}{3}$

$-\frac{2\pi}{3}$

(b) Bonus[1] (return to this question after you are done with everything else): The surface S has a boundary that is fairly easy to describe: a circle on the xz plane. Your friend suggests that you should use Stokes' theorem to turn the surface integral into a line integral on the boundary. Why would this not work?

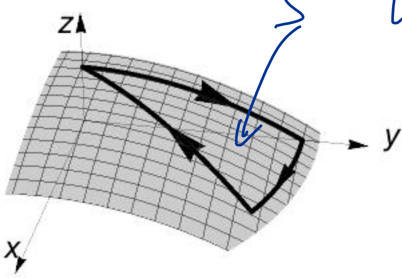
We'd have to write $\vec{F} = \text{curl}\, \vec{G}$ and that's not possible since $\text{div}\, \vec{F} = 1 \neq 0$.

[1] No partial credit will be given for the bonus question, highest possible score in this exam is 65.

8. (10 pts.) Let S be the paraboloid $z = 1 - x^2 - y^2$. Let c be the path consisting of the following curves, as in the picture:

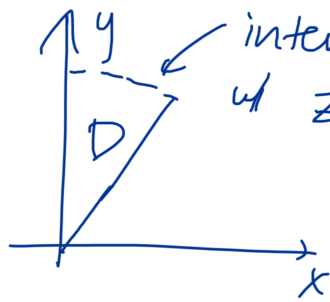
- The part of the intersection of S with the plane $x = 0$, from (0,0,1) to $(0, 1, 0)$.
- An arc of the intersection of S with the plane $z = 0$, from (0,1,0) to $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ (the one satisfying $x \geq 0$).
- The part of the intersection of S with the plane $y = \sqrt{3}x$, from $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ to $(0, 0, 1)$.

Let $\vec{F}(x, y, z) = \langle -yx, x^2, z\rangle$. Compute $\int_c \vec{F} \cdot d\vec{r}$ (you may do it directly, or use one of the theorems of chapter 16; if you do so, clearly state which theorem you are using).



S

Use Stokes.



intersection of paraboloid w/ $z = 0$: $x^2 + y^2 = 1$

The surface S w/ downward orientation has c as its positively oriented boundary.

Parametrize S:

$$r(u,v) = \langle u, v, 1-u^2-v^2\rangle \quad (u,v) \in D$$

$$\vec{r}_u \times \vec{r}_v = \langle -2u, -2v, 1\rangle \Rightarrow \text{use } -\vec{r}_u \times \vec{r}_v$$

$$\text{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yx & x^2 & z \end{vmatrix} = \langle 0, 0, 2x+x\rangle = \langle 0, 0, 3x\rangle$$

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S} = -\iint_D \langle 0, 0, 3u\rangle \cdot \langle -2u, -2v, 1\rangle \, dA$$

$$= \iint_D -3u \, dA \overset{\text{polar}}{=} \int_0^1 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} -3r\cos\theta \, r \, dr \, d\theta = -(\sin\theta)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} (r^3)\Big|_0^1$$

$$= -\left(1 - \frac{\sqrt{3}}{2}\right)$$