1. (12 pts) The two parts are not related.

(a) Determine whether the following statement is **true** of **false**, and explain your answer: The set in $\mathbb{R}^3$ described in cartesian coordinates as $A = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}$ is the same as the set in $\mathbb{R}^3$ described in spherical coordinates as $B = \{(\rho, \theta, \phi) : \phi = \frac{3\pi}{4}\}$, under the usual convention $\rho \geq 0$, $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$.

False, B is the lower half of a cone but A is a full cone.

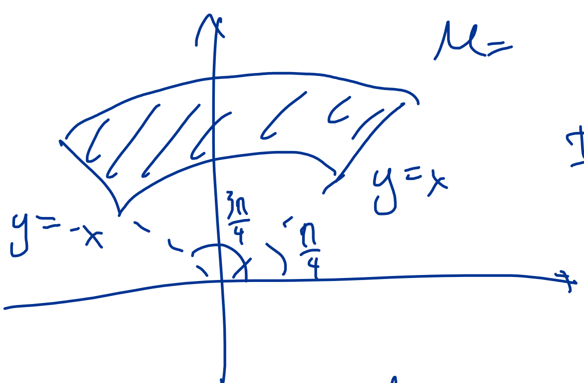
(b) A thin lamina occupies the region

$$D = \{(x, y) : 1 \leq x^2 + y^2 \leq 16 \text{and } y \geq |x|\}.$$

If the density function ρ at each point (x, y) is inversely proportional to the square of the distance of the point to the origin, find the y coordinate of the center of mass of the lamina (the $\bar{y}$).

$$\rho(x,y) = \frac{k}{(\sqrt{x^2+y^2})^2} = \frac{k}{x^2+y^2} = \frac{k}{r^2} \text{ in polar}$$

$$D = \{(r,\theta) : \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4},\ 1 \leq r \leq 4\}$$



$$M = \iint_D \rho(x,y)\,dA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^4 \frac{k}{r^2}\, r\,dr\,d\theta$$

$$= \frac{\pi}{2} k \ln r \Big|_1^4 = \frac{\pi}{2} k \ln 4$$

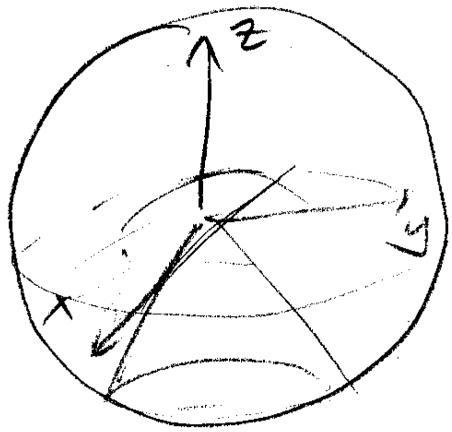
$$M_x = \iint_D y\rho(x,y)\,dA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^4 r\sin\theta \frac{k}{r^2}\, r\,dr\,d\theta$$

$$= 3k\left[-\cos\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= 3k\sqrt{2}$$

So $\boxed{\bar{y} = \frac{3k\sqrt{2}}{\frac{\pi}{2}k\ln 4} = \frac{6\sqrt{2}}{\pi \ln 4}}$

2. (8 pts) Let $f(x, y, z) = xy$. Set up **but do not evaluate** $\iiint_E f(x, y, z)dV$ in **cylindrical coordinates**, where E is the solid that lies above the sphere $x^2 + y^2 + z^2 = 9$, under the cone $z = -\sqrt{x^2 + y^2}$ and satisfies $y \leq 0$.



$x^2 + y^2 + z^2 = 9 \Rightarrow$

$r^2 + z^2 = 9$

$\Rightarrow z = \pm\sqrt{9 - r^2}$

$z = -\sqrt{9 - r^2}$ bec. we want lower part

$z = -\sqrt{x^2 + y^2} \Rightarrow z = -\sqrt{r^2} \Rightarrow z = -r$

So $-\sqrt{9 - r^2} \leq z \leq -r$

$y \leq 0 \Rightarrow \theta \in [\pi, 2\pi]$.

Find projection:

$$\begin{cases} r^2 + z^2 = 9 \\ z = -r \end{cases} \Rightarrow 2r^2 = 9 \Rightarrow r = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

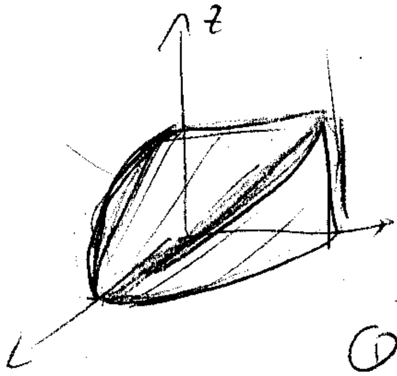
So

$$\iiint_E f \, dV = \int_{\pi}^{2\pi} \int_{0}^{\frac{3\sqrt{2}}{2}} \int_{-\sqrt{9-r^2}}^{-r} r\cos\theta \, r\sin\theta \, r \, dz \, dr \, d\theta$$

5.43

3. (16 pts) [You should be able to answer each part regardless of whether you have answered the other one]
Let $f(x, y, z) = z$.

(a) Set up **but do not evaluate** $\iiint_E f(x,y,z)dV$ **in the order** $dxdydz$, where E is the solid **in the first octant** bounded by the coordinate planes, the circular cylinder $x^2 + y^2 = 4$ and circular cylinder $x^2 + z^2 = 4$. (make sure to involve the given function in your formula!)



Write $u_1(y,z) \le x \le u_2(y,z)$

(1) $\Rightarrow x = \sqrt{4-y^2}$

(2) $\Rightarrow x = \sqrt{4-z^2}$

(3) $x = 0$

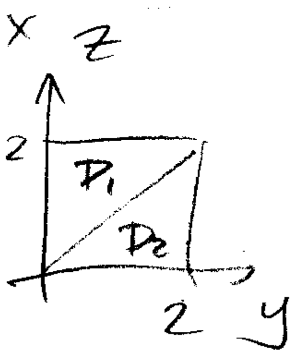
(1) $x^2+y^2=4$
(2) $x^2+z^2=4$
(3) $x=0$
(4) $y=0$
(5) $z=0$

Projection.

(1),(2) $\to y^2 = z^2 \Rightarrow y = \pm z \Rightarrow \boxed{y=z}$, $(y, z \ge 0)$

(4) $y=0$ (5) $z=0$

(1),(3) $\Rightarrow y = 2$ (2),(3) $\Rightarrow z = 2$ inters. 3



On D_1 $0 \le x \le \sqrt{4-z^2}$

D_2 $0 \le x \le \sqrt{4-y^2}$

$D_1 = \{(y,z): 0 \le z \le 2,\ 0 \le y \le z\}$

$D_2 = \{(y,z): 0 \le z \le 2,\ z \le y \le 2\}$

So

$$\iiint_E f(x,y,z)dV = \int_0^2 \int_0^z \int_0^{\sqrt{4-z^2}} z\,dxdydz + \int_0^2 \int_z^2 \int_0^{\sqrt{4-y^2}} z\,dxdydz$$

5.55 (b) **Evaluate** $\iiint_E f(x,y,z)dV$ using cylindrical coordinates.

$0 \le z \le \sqrt{4-x^2} \Rightarrow$

$\Rightarrow 0 \le z \le \sqrt{4-r^2\cos^2\theta}$

$0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$

$$\int_0^2 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{4-r^2\cos^2\theta}} z\,r\,dz\,d\theta\,dr =$$

$$= \int_0^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(4-r^2\cos^2\theta)\,r\,d\theta\,dr = \int_0^2 \int_0^{\frac{\pi}{2}} 2r\,d\theta\,dr - \frac{1}{2}\int_0^2\int_0^{\frac{\pi}{2}} r^3\cos^2\theta\,d\theta\,dr$$

$$= 4 \cdot \frac{\pi}{2} - \frac{1}{2}\frac{2^4}{4}\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 2\pi - 2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta = 2\pi - 2\frac{\pi}{4}$$

3

455.

4. (12 pts) Let R be the trapezoid in the xy plane defined by the points (1,1), (2,2), (2,0) and (4,0), as in the picture, and you are given the transformation $x = u + v$ and $y = u - v$.

(a) Compute the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

(b) Find the inverse transformation T^{-1} (that is, $u = u(x, y)$ and $v = v(x, y)$).

$$\left.\begin{array}{l} x = u+v \\ y = u-v \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y = 2u \\ x-y = 2v \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u = \frac{x+y}{2} \\ v = \frac{x-y}{2} \end{array}\right.$$

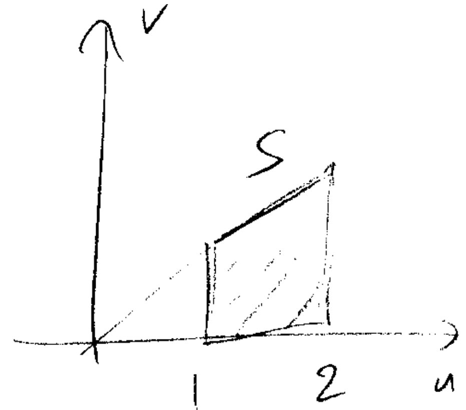
(c) Find the image S of R under T^{-1} in the uv plane (that is, the set $S = T^{-1}(R)$) and **draw a picture of it.**

$y = x \Rightarrow 2v = 0 \Rightarrow v = 0$

$y = 4 - x \Rightarrow u = 2$

$y = 0 \Rightarrow u = v$

$y = 2 - x \quad u = 1$

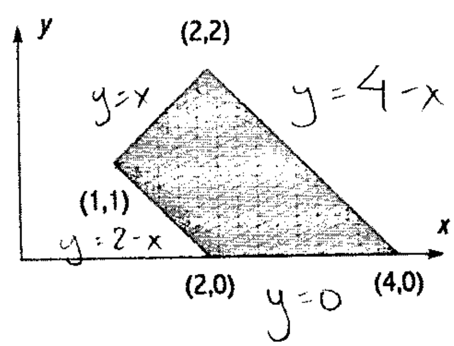


(d) Use your work in the parts (a)-(c) to calculate $\iint_R e^{\frac{x-y}{x+y}} dA$ (you can use the back of the page if you run out of space).

$$\int_1^2 \int_0^u e^{\frac{2v}{2u}} \cdot |-2| \, dv\, du =$$

$$= \int_1^2 2 \left[u e^{\frac{v}{u}} \right]_0^u du = \int_1^2 2u(e-1)\, du$$

$$= 3(e-1)$$



5. (8 pts) The temperature at a point (x, y) of the plane is given in degrees Celcius by

$$T(x, y) = x^2 y^3 + 2\cos(3x\pi + y\pi),$$

where x and y are in meters. You are standing at the point (1,2) and you want to move towards the direction in which the temperature **drops** most rapidly (that is, the direction in which you have the minimum net rate of change of temperature).

(a) **Find a vector** that gives this direction.

Use $-\nabla T(1,2)$

$\nabla T(x,y) = \langle 2xy^3 - 2\sin(3x\pi + y\pi)3\pi,\ 3x^2y^2 - 2\sin(3x\pi + y\pi)\cdot\pi \rangle$

So

$\boxed{-\nabla T(1,2) = -\langle 16, 12 \rangle}$

(b) **Find the directional derivative of** T in the direction determined by this vector. **Make sure to include units in your answer.**

$|-\nabla T(1,2)| = \sqrt{16^2 + 12^2} = \sqrt{400} = 20$

So

$D_{-\frac{\nabla T(1,2)}{|\nabla T(1,2)|}} = -\frac{\nabla T(1,2)}{|\nabla T(1,2)|} \cdot \nabla T(1,2) = -20\ ^\circ C/m$

6. (9 pts) Let $z = z(x, y)$ be a twice differentiable function with continuous second partial derivatives and $x = x(t)$, $y = y(t)$ be differentiable with continuous first partial derivatives. You are given the following table with data. Use the chain rule to evaluate $\frac{d^2z}{dt^2}(0)$.

$x(0) = 1$	$y(0) = -1$	$z(1,-1) = -1$
$\frac{\partial z}{\partial x}(1,-1) = -2$	$\frac{\partial z}{\partial y}(1,-1) = 3$	$\frac{dx}{dt}(0) = 1$
$\frac{dy}{dt}(0) = 0$	$\frac{d^2x}{dt^2}(0) = 0$	$\frac{d^2y}{dt^2}(0) = 2$
$\frac{\partial^2 z}{\partial x^2}(1,-1) = -2$	$\frac{\partial^2 z}{\partial y^2}(1,-1) = -6$	$\frac{\partial^2 z}{\partial x \partial y}(1,-1) = 6$

Hint: Note that some of the partial derivatives given are 0; this may simplify your calculation.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

At $t=0$, $\frac{dy}{dt} = 0$ and $\frac{d^2x}{dt^2} = 0$

Find $\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)$:

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}$$

At $t=0$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = -2 \cdot 1 + 6 \cdot 0 = -2$$

So $\frac{d^2z}{dt^2}(0) = -2 \cdot 1 + 3 \cdot 2 = 4$