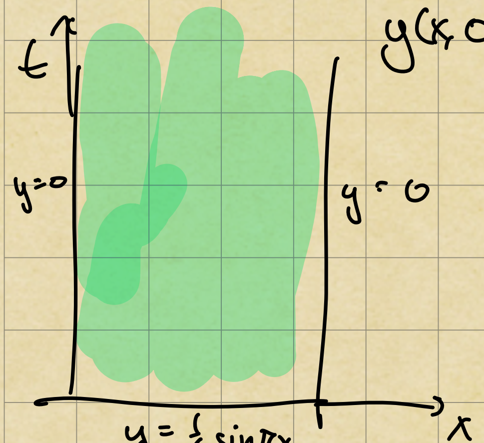
H 9.6.5
9.7 #4.
10 2009.

§ 9.6 # 5. $y_{tt} = 25 y_{xx}$ $0 < x < 3$, $t > 0$

$y(0,t) = y(3,t) = 0$

$y(x,0) = \frac{1}{4}\sin \pi x$, $y_t(x,0) = 10 \sin \pi x$



Pr. A:

$y_{tt} = 25 y_{xx}$

$y(0,t) = y(3,t) = 0$

$y(x,0) = \frac{1}{4}\sin \pi x$

$y_t(x,0) = 0$

$y_A \to$ sol'n.

Pr: B.

$y_{tt} = 25 y_{xx}$

$y(0,t) = y(3,t) = 0$

$y(x,0) = 0$

$y_t(x,0) = 10 \sin(\pi x)$

y_B.

Soln: formulas for y_A, y_B

$$y_A = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi x}{L}\right)$$

$y_A(x,0) = \frac{1}{4}\sin \pi x$

known

$$\sum_{n=1}^{\infty} \underbrace{A_n}_{} \sin\left(\frac{n\pi x}{L}\right) = \overbrace{y_A(x,0)}$$

$\hookrightarrow$ F. sine series coef. of $y_A(x,0)$.

$A_n \to$ sine series coef. of $\frac{1}{4}\sin \pi x$

Interval: $[0,3]$

Want:

$$\frac{1}{4}\sin \pi x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \ldots$$

Shortcut: when $n=3$, get a $b_3 \sin(\pi x)$ term on RHS. Matches the $\frac{1}{4}\sin(\pi x)$ on LHS

$\Rightarrow b_3 = \frac{1}{4}$, $b_n = 0$ for all $n \neq 3$

So: $y_A = \frac{1}{4}\cos(5\pi t)\sin(\pi x)$

no summation: $A_3 = \frac{1}{4}$, $A_n = 0$ for all $n \neq 3$

Problem B → similar.

$$y = y_A + y_B$$

4. $\begin{cases} u_{xx} + u_{yy} = 0 \\ u_x(0,y) = u_x(a,y) = u(x,0) = 0 \\ u(x,b) = f(x) \end{cases}$

$$u(x,y) = \sum_{n=0}^{\infty} c_n \underbrace{X_n(x)\, Y_n(y)}_{u_n}$$

$$\partial_x^2 u_n + \partial_y^2 u_n = 0 \Rightarrow X_n'' Y_n + X_n Y_n'' = 0$$

$$\Rightarrow \frac{X_n''}{X_n} = -\frac{Y_n''}{Y_n} = -\lambda$$

$$X_n'' + \lambda X_n = 0$$

$$Y_n'' - \lambda Y_n = 0$$

Endpt cond.: $\partial_x u_n(0,y) = 0 \Rightarrow X_n'(0)\, Y_n(y) = 0$

$\Rightarrow X_n'(0) = 0$

$\partial_x u_n(a,y) = 0 \rightarrow X_n'(a) = 0$

$u_n(x,0) = 0 \Rightarrow Y_n(0) = 0.$

Collect:

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n'(0) = X_n'(a) = 0 \end{cases}$$

$\leftarrow$ start w/ problem w/ 2 endpt. conditions.

$$\begin{cases} Y_n'' - \lambda Y_n = 0 \\ Y_n(0) = 0 \end{cases}$$

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n'(0) = X_n'(a) = 0 \end{cases} :$$

$\lambda = 0$: $X_0'' = 0 \Rightarrow X_0 = Ax + B.$

$X_0'(0) = X_0'(a) = 0 \Rightarrow A = 0$

so $X_0 = B$ (can take $B = 1$)

$\lambda_n = \alpha_n^2$

$$X_n'' + \alpha^2 X_n = 0$$

$$X_n(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

$$X_n'(0) = 0 \Rightarrow B = 0$$

$$X_n'(a) = 0 \Rightarrow -\alpha A \sin(\overbrace{\alpha a}^{n\pi}) = 0$$

$$\Rightarrow \alpha = \frac{1}{a} n\pi$$

So $\lambda_n = \left(\frac{n\pi}{a}\right)^2$. $\quad X_n = A\cos\left(\frac{n\pi}{a}x\right)$

(can take $A = 1$)

$$X_n = \cos\left(\frac{n\pi}{a}x\right)$$

For λ_n: $\quad Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0$

$$Y_n(0) = 0$$

$n = 0$: $\quad Y_0'' = 0 \Rightarrow Y_0 = Ay + B$

$$Y_0(0) = 0 \Rightarrow B = 0$$

can take $Y_0(y) = y$

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0 \quad , \quad n \geq 1$$

$$Y_n(y) = A\cosh\left(\frac{n\pi}{a}y\right) + B\sinh\left(\frac{n\pi}{a}y\right)$$

$$Y_n(0) = 0 \Rightarrow \quad A = 0$$

$$Y_n(y) = \sinh\left(\frac{n\pi}{a}y\right)$$

took $B = 1$.

So:

$$u(x,y) = c_0 X_0 Y_0 + \sum_{n=1}^{\infty} c_n X_n Y_n$$

$$= c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}y\right)\cos\left(\frac{n\pi}{a}x\right)$$

To use $u(x,b) = f(x)$:

$$f(x) = \underbrace{c_0 b}_{\text{const.}} + \sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi b}{a}\right)}_{\text{const.}} \cos\left(\frac{n\pi}{a}x\right)$$

$$c_0 b = \frac{a_0}{2} \quad , \quad a_0 = \frac{2}{a}\int_0^a f(x)\,dx$$

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)dx$$