The Divergence Theorem

In the last section we saw a theorem about closed curves. In this one we'll see a theorem about closed surfaces (you can imagine bubbles). As we've mentioned before, closed surfaces split $\mathbb{R}^3$ two domains, one bounded and one unbounded.

Theorem 1. *(Divergence) Suppose we have a* ***closed*** *parametric surface with* ***outward orientation*** *that is the boundary of the bounded domain E in $\mathbb{R}^3$. Also, let $\vec{F}$ be a vector field in $\mathbb{R}^3$ with differentiable coefficients. Then*

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV.$$

Remarks:

1. It is very important that it is a **closed** surface that we're finding the flux of $\vec{F}$ across. To understand this, find the flaw in the following argument:

   "Let c be a simple closed curve in $\mathbb{R}^3$ and $\vec{F}$ be a vector field. Then:

   $$\int_c \vec{F} \cdot d\vec{r} \overset{\text{Stokes}}{=} \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} \overset{\text{Divergence}}{=} \iiint_E \operatorname{div} \operatorname{curl} \vec{F} dV = 0,$$

   since $\operatorname{div} \operatorname{curl} \vec{F} = 0$ for any vector field $\vec{F}$, as we've already seen."

2. As with Stokes' Theorem, we often denote the positively oriented boundary of a solid E by ∂E and write the Divergence Theorem as

   $$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV.$$

3. If a closed surface S that is the boudary of a domain E is **negatively oriented**, we have

   $$\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \operatorname{div} \vec{F} dV.$$

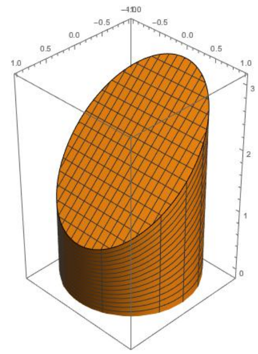
4. When is the divergence theorem useful? Usually whenever we would like to compute a flux across a closed surface, especially when the surface consists of several smooth smaller surfaces and we'd need to write a sum of integrals. See the following example:

   Example 1. *Find the flux $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle x, -1, 2y \rangle$ and S is the positively oriented boundary of the solid E in $\mathbb{R}^3$ bounded by the xy plane, the cylinder $x^2 + y^2 = 1$ and the plane $z = 2 - x$.*

   Solution. We could find the flux by computing three surface integrals, but it's easier to use the Divergence Theorem instead. We have:

   $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\langle x, -1, 2y \rangle dV = \iiint_E 1 dV = \int_0^{2\pi} \int_0^1 \int_0^{2-r\cos(\theta)} r dz dr d\theta$$

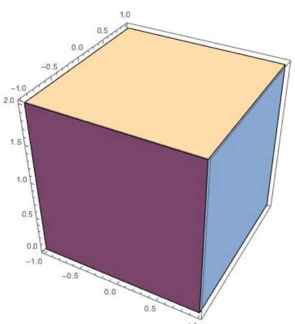
   □



Another situation where the divergence theorem can be useful is when we don't have a closed surface, but we can complete into a closed surface to make a computation easier.

Let's see an example:

Example 2. *Compute the flux* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = \langle x, -1, 2y \rangle$ *and* S *consists of the five sides other than the lower one of the cube* E *with vertices (-1,-1,0), (1,-1,0), (1,1,0), (-1,1,0), (1,1,2), (-1,1,2), (1,-1,2), (-1,-1,2) and outward orienation.*



Solution. Normally we'd have to compute 5 surface integrals and add them. Note that this is **not** a closed surface, so we can't apply the Divergence Theorem directly. What we can do is attach an extra face to the cube with appropriate orientation and make into a closed surface. That is, we attach the face S' parametrized as

$$\vec{r}(u, v) = \langle u, v, 0 \rangle, \text{ for } (u, v) \in [-1, 1] \times [-1, 1].$$

We give it **downward** orientation, so that $S' \cup S$ ends up being a closed surface with outward orientation. Then, by Divergence theorem,

$$\iint_{S \cup S'} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV = \iiint_E 1 dV = 2^3 = 8.$$

Since

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S' \cup S} \vec{F} \cdot d\vec{S} - \iint_{S'} \vec{F} \cdot d\vec{S},$$

we only need to compute $\iint_{S'} \vec{F} \cdot d\vec{S}$. Since S' is a surface on the xy plane with downward orientation, its unit normal vector field is $\vec{n} = -\langle 0, 0, 1\rangle$ and we have

$$\iint_{S' \cup S} \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_{-1}^{1} \langle u, -1, 2v\rangle \cdot \langle 0, 0, -1\rangle du dv = 0$$

and so

$$\iint_{S} \vec{F} \cdot d\vec{S} = 8.$$

□

5. What does the Divergence Theorem say from a physical point of view? When we discussed divergence, we said that divergence describes the extent to which the vector field behaves like a source (positive divergence) or as a sink (negative divergence) near a point p. Then, the Divergence Theorem roughly says that summing up all the infinitesimally small sources and subtracting all the sinks inside a domain of $\mathbb{R}^3$ gives the total flux on its boundary.

6. The Divergence Theorem holds in any dimension, and in dimension 2 it is equivalent Green's Theorem (this means that you can derive it from Green's Theorem and you can derive Green's Theorem from the Divergence Theorem).

Green's First Identity

We can use use the Divergece Theorem to derive the following useful formula. Let E be a domain in $\mathbb{R}^3$ and ∂E its positively oriented boundary with unit normal $\vec{n}$. Then, if u, v are twice differentiable scalar valued functions, we have

$$\iint_{\partial E} v D_{\vec{n}} u - u D_{\vec{n}} v dS = \iiint_{E} v \Delta u - u \Delta v dV.$$

We have

$$\begin{aligned} \operatorname{div}(u \nabla v) =& \partial_x (u \partial_x v) + \partial_y (u \partial_y v) + \partial_z (u \partial_z v) && (1) \\ =& \partial_x u \partial_x v + \partial_y u \partial_y v + \partial_z u \partial_z v + u(\partial_x^2 v + \partial_y^2 v + \partial_z^2 v) && (2) \\ =& \nabla u \cdot \nabla v + u \Delta v && (3) \end{aligned}$$

Similarly,

$$\operatorname{div}(v \nabla u) = \nabla v \cdot \nabla u + v \Delta u \qquad (4)$$

Subtract (3) from (4) and find

$$\operatorname{div}(v \nabla u - u \nabla v) = v \Delta u - u \Delta v.$$

Then e integrate over E and use Divergence Theorem:

$$\begin{aligned} \iiint_{E} v \Delta u - u \Delta v dV &= \iiint_{E} \operatorname{div}(v \nabla u - u \nabla v) dV \\ &= \iint_{\partial E} (v \nabla u - u \nabla v) \cdot d\vec{S} \\ &= \iint_{\partial E} (v \nabla u - u \nabla v) \cdot \vec{n} dS \\ &= \iint_{\partial E} v D_{\vec{n}} u - u D_{\vec{n}} v dS \end{aligned}$$