1. (10 pts.) The two parts are not related.

(a) Use an integral to compute the volume of a ball of radius R in the xyz space.

Center at $(0,0,0)$. Use spherical coords

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= 2\pi \left.\frac{\rho^3}{3}\right|_0^R \left(-\cos\varphi\right)\Big|_0^{\pi} = 2\pi \cdot \frac{R^3}{3} \cdot 2 =$$

$$= \frac{4}{3}\pi R^3$$

2. (8 pts.) Suppose we have a lamina that lies on the subset

$$D = \{(x,y) : x^2 + y^2 \geq 1,\ |y| \leq \sqrt{3}x,\ x \leq 3\}$$

of the xy plane and has density function $\rho(x,y) = \frac{1}{x^2+y^2}$. Find its moment about the y axis (the M_y)

$-\sqrt{3}x \leq y \leq \sqrt{3}x$

Write D in polar:

$x^2+y^2 \geq 1 \Rightarrow r \geq 1$

$|y| \leq \sqrt{3}x \Rightarrow$

$|r\sin\vartheta| \leq \sqrt{3}\, r\cos\vartheta$

$\Rightarrow |\tan\theta| \leq \sqrt{3}$

$\Rightarrow \theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$x \leq 3 \Rightarrow r\cos\vartheta \leq 3 \Rightarrow r \leq \frac{3}{\cos\vartheta}$

So $M_y = \iint_D x\,\rho(x,y)\,dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{\frac{3}{\cos\vartheta}} r\cos\vartheta \frac{1}{r^2}\, r\,dr\,d\vartheta$

$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\vartheta\left(\frac{3}{\cos\vartheta} - 1\right) d\vartheta =$

$= 3 \cdot \frac{2\pi}{3} - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\vartheta = 2\pi - \sin\vartheta \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= 2\pi - \left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi - 1$

2. (10 pts.) Write down the equations used to change from Cartesian coordinates to Spherical coordinates (ρ, θ, ϕ) (that is, $x = x(\rho, \theta, \phi)$, $y = y(\rho, \theta, \phi)$ and $z = z(\rho, \theta, \phi)$) and then set up **but do not evaluate** the integral

$$\iiint_E yz\,dV$$

in **spherical coordinates**, where E is the set below:

E lies inside the sphere $x^2 + y^2 + z^2 = -6z$, under the half cone $z = -\sqrt{x^2+y^2}$ and satisfies $y \leq 0$.

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$x^2+y^2+z^2 = -6z \Rightarrow \rho^2 = -6\rho\cos\varphi \Rightarrow \rho = -6\cos\varphi$

So $0 \leq \rho \leq -6\cos\varphi$

$y \leq 0 \Rightarrow \pi \leq \theta \leq 2\pi$

$z = -\sqrt{x^2+y^2} \Rightarrow r\cos\varphi = -\rho\sin\varphi \Rightarrow \varphi = \frac{3\pi}{4}$

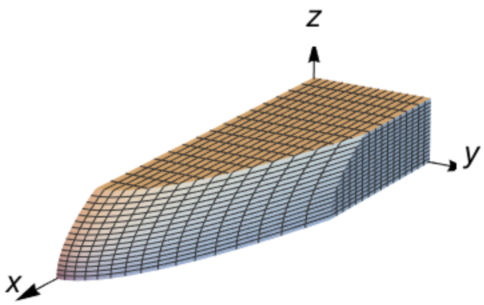
So

$$\iiint_E yz\,dV = \int_{\pi}^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^{-6\cos\varphi} (\rho\sin\varphi\sin\theta)(\rho\cos\varphi)\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

4. (12 pts.) Set up an integral of

$$\iiint_E f(x,y,z)dV$$

in the order $dydxdz$, where E is the solid satisfying $0 \le x \le 7-y^2-z^2$, $0 \le z \le 1$ and $0 \le y \le 2$ (the set E can be seen in the picture).



Equations, solve for y:

(1) $x = 7 - y^2 - z^2 \Rightarrow y^2 = 7 - x - z^2$

$\Rightarrow y = \sqrt{7-x-z^2}$

(2) $x = 0$

(3) $y = 0$

(4) $y = 2$

(5) $z = 0$

(6) $z = 1$

y appears 3 times $\Rightarrow$ sum of int's.

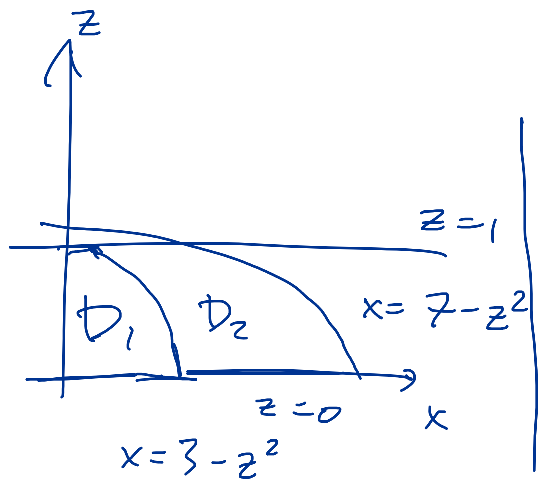
write projection on xz plane:

(1),(3) $\Rightarrow 7 = x + z^2 \Rightarrow x = 7 - z^2$

(1),(4) $\Rightarrow 2 = \sqrt{7-x-z^2} \Rightarrow$

$x = 3 - z^2$

So



Over $D_1 = \{(x,z): 0 \le x \le 3 - z^2, 0 \le z \le 1\}$

we have $0 \le y \le 2$

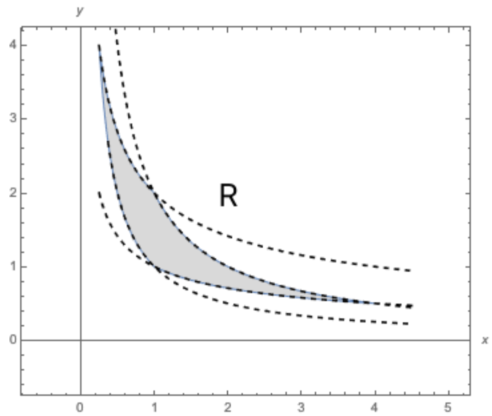
Over $D_2 = \{(x,z): 3 - z^2 \le x \le 7 - z^2, 0 \le z \le 1\}$

$0 \le y \le \sqrt{7-x-z^2}$

So:

$$\iiint_E f\,dV = \int_0^1 \int_0^{3-z^2} \int_0^2 f(x,y,z)\,dy\,dx\,dz + \int_0^1 \int_{3-z^2}^{7-z^2} \int_0^{\sqrt{7-x-z^2}} f(x,y,z)\,dy\,dx\,dz$$

5. (10 pts.) Use the change of variables $x = \frac{u^2}{v^2}$, $y = \frac{v^2}{u}$ defined on $\{(u,v) : u > 0, v > 0\}$, to evaluate $\iint_R 4xy^3 dA$, where R is the region in the first quadrant bounded by $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = \frac{1}{\sqrt{x}}$ and $y = \frac{2}{\sqrt{x}}$ (the domain R can be seen in the picture). You must show your work clearly, but you don't need to show that the transformation is invertible.



$T:\quad x = \frac{u^2}{v^2}$

$y = \frac{v^2}{u}$

Find $T^{-1}(R)$:

$y = \frac{1}{x} \Rightarrow \frac{v^2}{u} = \frac{v^2}{u^2} \Rightarrow u = 1$

$y = \frac{2}{x} \Rightarrow \frac{v^2}{u} = \frac{v^2}{u^2} \cdot 2 \Rightarrow u = 2$

$y = \frac{1}{\sqrt{x}} \Rightarrow \sqrt{x}\, y = 1 \Rightarrow \frac{u}{v}\frac{v^2}{u} = 1 \Rightarrow v = 1$

$y = \frac{2}{x} \Rightarrow v = 2$

so $T^{-1}(R) = [1,2] \times [1,2]$.

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2u}{v^2} & -2\frac{u^2}{v^3} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = \frac{4}{v} - 2\frac{1}{v} = \frac{2}{v} \neq 0$$

By theorem

$$\iint_R 4xy^3 dA = \int_1^2 \int_1^2 4\frac{u^2}{v^2}\frac{v^6}{u^3}\frac{2}{v}\, dv\, du =$$

$$= 2\int_1^2 \frac{1}{u}\, du \int_1^2 4v^3 dv = 2\ln 2(16-1)$$