MA 30300 - Spring 2021
Midterm 1
Wednesday, February 16, 2022

Name: ______________________________

PUID: ______________________________

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- There are 7 problems spanning 6 pages (your last page should be numbered as 6). Please make sure your exam contains all these questions.
- You are allowed to use a double sided hand-written 8.5 by 11 inch page of notes.
- Show work for free response questions; you do not need to show work for multiple choice questions.
- If you need more space to use as scratch paper you can use the back of the page. If part of your answer ends up in the back of the page, indicate it clearly at the front.
- Any student found engaging in academic misconduct will be reported to the Office of the Dean of Students.
- You have 60 minutes to complete the exam. **Do not spend too much time on an individual problem, unless you are done with all the rest.**
- Please do not write out of the margins.
- Unless otherwise stated, primes denote derivatives with respect to t, i.e. $x' = dx/dt$.

GOOD LUCK!

PART I: Multiple Choice (spend no more than 25'). You do not need to show work. Mark your answer by shading the corresponding box.

1. (4 pts.) You are given the following system

$$\begin{cases} \dfrac{dx}{dt} = 7x - x^2 - xy \\ \dfrac{dy}{dt} = y - 4y^2 \end{cases} \tag{1}$$

describing two animal populations. Which of the following best describes the way the populations interact?

A. ☐ Predation

B. ☐ Cooperation

C. ☐ Competition

D. ☐ The two populations do not interact (none is affected by the presence of the other)

E. ■ Only one of the two populations is affected by the other while the other evolves independently.

2. (4 pts.) You are given the following nonlinear system which depends on a **positive** parameter $\epsilon > 0$:

$$\begin{cases} x' = x + \epsilon y + xy \\ y' = -2x \end{cases} . \tag{2}$$

For any $\epsilon > 0$ this system has exactly one critical point, which is $(0,0)$. What is the largest range of positive values of $\epsilon > 0$ for which the critical point $(0,0)$ of the nonlinear system (2) is a spiral source?

A. ☐ $0 < \epsilon < 1/8$

B. ☐ $0 < \epsilon \leq 1/8$

C. ☐ $1/8 \leq \epsilon$

D. ☐ $1/8 < \epsilon$.

E. ☐ There exist no positive such values of ϵ.

Hint: compute the linearization at the origin and look at its eigenvalues.

$J = \begin{bmatrix} 1+y & \varepsilon \\ -2 & 0 \end{bmatrix}$ At origin $\begin{bmatrix} 1 & \varepsilon \\ -2 & 0 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & \varepsilon \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 2\varepsilon = 0 \Rightarrow \lambda = \dfrac{1 \pm \sqrt{1-8\varepsilon}}{2}$

Want: $1 - 8\varepsilon < 0$

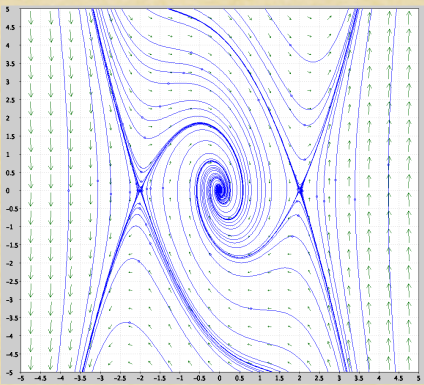
$\Rightarrow \varepsilon > 1/8$

3. (4 pts.) You are given the equation satisfied by the displacement from equilibrium $x(t)$ of a mass attached to a hard spring in the presence of damping:

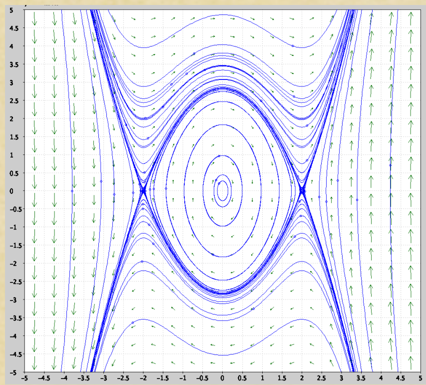
$$x''(t) = -x' - 4x - 3x^3. \tag{3}$$

Which of the following corresponds to the phase plane portrait of the first order displacement-velocity system corresponding to (3)

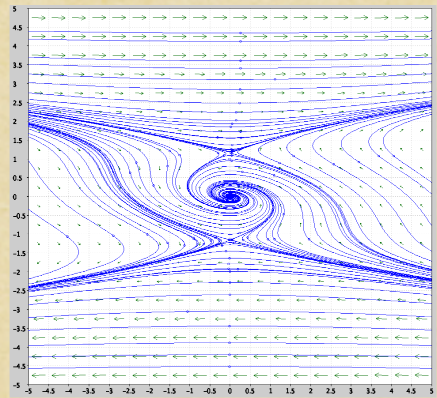
In all phase plane portraits below the horizontal axis corresponds to displacement and the vertical one to velocity, and the values on both axes range from -5 to 5.



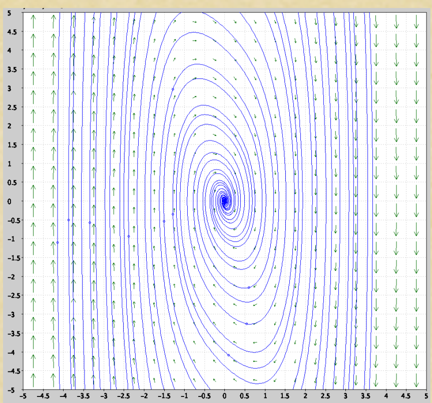
A. □



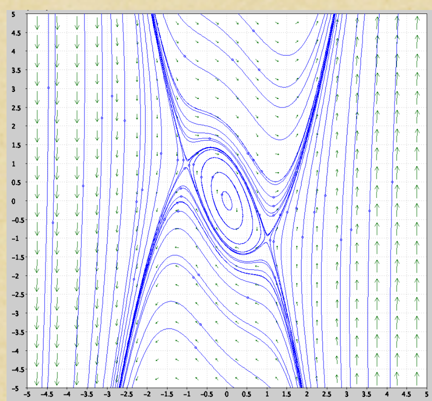
B. □



C. □



D. □



E. □

1st order system:
$$\begin{cases} x' = y \\ y' = -y - 4x - 3x^3 \end{cases}$$

C.P. : $y = 0$

$-y - 4x - 3x^3 = 0 \overset{y=0}{\Rightarrow} -4x - 3x^3 = 0$

$\Rightarrow -x(4 + 3x^2) = 0$

$\Rightarrow x = 0$

4. (4 pts.) Which of the following is the solution to the system of Ordinary Differential Equations with prescribed data at $t = 1$:

$$\begin{cases} x_1'(t) = x_1(t) + 4x_2(t) \\ x_2'(t) = 2x_1(t) + 3x_2(t) \end{cases}, \qquad x_1(1) = e^{-1}, \quad x_2(1) = 0 \tag{4}$$

A. ☐ $x_1(t) = \frac{1}{3}e^{-5+4t} + \frac{2}{3}e^{1-2t}, \quad x_2(t) = \frac{1}{3}e^{-5+5t} - \frac{1}{3}e^{1-2t}$

B. ☐ $x_1(t) = \frac{1}{3}e^{-1+5t} + \frac{2}{3}e^{-1-t}, \quad x_2(t) = \frac{1}{3}e^{-1-5t} - \frac{1}{3}e^{-1-t}$ X

C. ☐ $x_1(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-1-t}, \quad x_2(t) = \frac{1}{3}e^{-1+4t} - \frac{1}{3}e^{-1-t}$ X

D. ☒ $x_1(t) = \frac{1}{3}e^{-6+5t} + \frac{2}{3}e^{-t}, \quad x_2(t) = \frac{1}{3}e^{-6+5t} - \frac{1}{3}e^{-t}$

E. ☐ No solution exists to the given system. X

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, find e.values

$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda = \dfrac{4 \pm \sqrt{16+20}}{2}$

$\Rightarrow \lambda = \dfrac{4 \pm 6}{2} \Rightarrow \lambda = 5, -1$

PART II: Free response (About 35'). You must show work to receive full credit. Partial credit will be given for partially correct answers.

5. (6 pts.) The real 2×2 matrix $\mathbf{A}$ has an eigenvalue $2 - 3i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 5i \\ 1 - 2i \end{bmatrix}$. Find two **real** linearly independent solutions for the system of differential equations

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}.$$

That is, you should privide two linearly independent solutions of the form $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, where $x_1(t)$ and $x_2(t)$ are **real valued**. If you find the solutions using the method discussed in the class you do not need to check their linear independence explicitly.

soln: $e^{(2-3i)t} \begin{bmatrix} 5i \\ 1-2i \end{bmatrix}$

$= e^{2t}\left(\cos(3t) - i\sin(3t)\right) \begin{bmatrix} 5i \\ 1-2i \end{bmatrix}$

$\Rightarrow e^{2t} \begin{bmatrix} 5\sin(3t) \\ \cos(3t) - 2\sin(3t) \end{bmatrix} + i\, e^{2t} \begin{bmatrix} 0 \\ -2\cos(3t) - \sin(3t) \end{bmatrix}$

$x_1 = e^{2t} \begin{bmatrix} 5\sin(3t) \\ \cos(3t) - 2\sin(3t) \end{bmatrix}$

$x_2 = e^{2t} \begin{bmatrix} 0 \\ -2\cos(3t) - \sin(3t) \end{bmatrix}$

6. (10 pts.) The matrix $\mathbf{A}$ below has eigenvalue 4 with multiplicity 3.

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & -1 \\ -1 & 0 & 5 \end{bmatrix} \tag{5}$$

(a) Compute the defect of the eigenvalue $\lambda = 4$.

(b) Compute the general solution to the system $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$.

It is given that

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}^2 = \mathbf{0} \tag{6}$$

a) eigenvectors:

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -a + c = 0$$

no other restriction

so $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$ is e-vector for all evectors.

$= a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is e-vector for all $a, b \neq 0$

$\Rightarrow$ can find 2 lin. indep. e-vectors $\Rightarrow$ defect $= 3 - 2 = 1$

b) Find gen. e-vector of deg. 2

$$\left.\begin{aligned} &\cancel{(A-4I)^2 v_2 = 0} \\ &(A-4I)^2 v_2 \neq 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \underline{0}, \text{ take}$$

$\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\underline{v}_1 = (A - 4I)^2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Take $\underline{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Soln: $\underline{x}(t) = c_1 e^{4t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{4t}\left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

7. (12 pts.) *Parts (b)-(d) are independent of (a) and Part (e) is independent of all previous parts.*

You are given the following nonlinear system, for which $(-1, 0)$ is a critical point.

$$\begin{cases} \dfrac{dx}{dt} = y(2-x) \\ \dfrac{dy}{dt} = -4(1+x)(2-x). \end{cases} \tag{7}$$

(a) What other critical points does this system have besides $(-1, 0)$?

$y(2-x)=0 \Rightarrow y=0$ or $x=2$. If $y=0$ then $x=-1$ or $x=2$ so $(-1,0)$, $(2,0)$. If $x=2 \Rightarrow$ any y works. so $(2,y)$ for all y.

(b) Compute its linearization at the critical point $(-1, 0)$.

$$J = \begin{bmatrix} -y & 2-x \\ +4(1+x)-4(2-x) & 0 \end{bmatrix}, \text{ at } (-1,0): \quad J = \begin{bmatrix} 0 & 3 \\ -12 & 0 \end{bmatrix}$$

$$u' = \begin{bmatrix} 0 & 3 \\ -12 & 0 \end{bmatrix}$$

(c) What is the type and stability of the phase plane portrait of the *linearized* system you found in Part (b) (spiral sink, spiral source, saddle etc)?

Eigenvalues: $\begin{vmatrix} -\lambda & 3 \\ -12 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 36 = 0 \Rightarrow \lambda = \pm 6i$

so stable center

(d) What can we deduce about the type and stabilty of the critical point $(-1, 0)$ of the nonlinear system? If there are more than one possibilities, list them all.

stable center or a. stable spiral sink or unstable spiral source.

(e) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{dy}{dt} / \dfrac{dx}{dt}$ to obtain the trajectories of the nonlinear system in implicit form[1]. What do you conclude about the stability and type of the critical point $(-1, 0)$ of the nonlinear system (7)? Justify your answer.

$$\frac{dy}{dx} = \frac{-4(1+x)(2-x)}{y(2-x)} = \frac{-4(1+x)}{y}$$

$$\Rightarrow \quad y\,dy = -4(1+x)\,dx \Rightarrow y^2 = -4x - 2x^2 + C$$

$$\Rightarrow y^2 + 4x^2 + 4x = C \Rightarrow y^2 + 2x^2 + 4x + 2 = \tilde{C}$$

$$\Rightarrow y^2 + 2(x+1)^2 = \tilde{C}$$

ellipses centered at $(-1,0)$

So stable center.

[1]This means in the form $F(x, y) = C$, as opposed to $y = f(x)$, for example.