Example 1. *Find the surface area of the sphere centered at the origin with radius r.*

Solution. As a remark, the fact that the sphere is centered at the origin doesn't matter, as its surface area doesn't depend on the center! The equation of the sphere is

$$x^2 + y^2 + z^2 = R^2.$$

Note that we've seen how to calculate the surface areas of graphs of functions, but the sphere is **not** the graph of a function (why?). However, both its upper and lower hemisphere **are** graphs of functions and, in fact, they have the same surface area. So we'll consider the function

$$z = f(x, y) = \sqrt{R^2 - x^2 - y^2} \tag{1}$$

representing the upper hemisphere. To find the domain of integration, we intersect with the xy plane and find, by (1),

$$z = 0 \implies x^2 + y^2 = R^2.$$

So, the domain of integration has to be the disk

$$D = \{(x, y) : x^2 + y^2 \leq R^2\}.$$

Then we compute partial derivatives and find

$$f_x(x, y) = \frac{-2x}{2\sqrt{R^2 - x^2 - y^2}}$$

and

$$f_y(x, y) = \frac{-2y}{2\sqrt{R^2 - x^2 - y^2}}.$$

Therefore,

$$\begin{aligned}
S =& \iint_D \sqrt{(f_x(x, y))^2 + (f_y(x, y))^2 + 1} dA \\
=& \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}} dA \\
=& \int_0^{2\pi} \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} dr d\theta \\
=& \int_0^{2\pi} \int_R^0 -\frac{R}{2} u^{-\frac{1}{2}} du d\theta \\
=& 2\pi R^2.
\end{aligned}$$

Recall that all this is only for the upper hemisphere, so to find the total surface area we multiply $\times 2$ and we find

$$Area = 4\pi R^2.$$

$\square$