Math 324 A - Winter 2018
Final Exam
Tuesday, March 13, 2018

Name: ______________________________

UW email address: ______________________________

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THIS EXAM IS DOUBLE SIDED

- There are 8 problems spanning 8 pages (your last nonempty page should be numbered as 8). Please make sure your exam contains all these questions.
- You are allowed to use a TI-30x IIS Calculator and one double sided hand-written 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- Scratch paper is available. Please do not use your own.
- You have 110 minutes to complete the exam. Budget your time wisely.
Do not spend too much time on an individual problem, unless you are done with all the rest.

GOOD LUCK!

1. (8 pts.)You do not need to justify your answers.

   (i) You are given the following plots of vector fields:

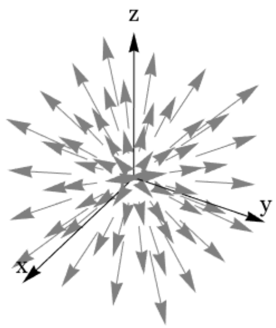


Figure 1: Plot A

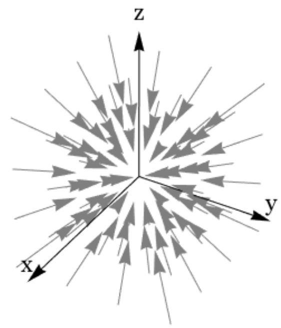


Figure 2: Plot B

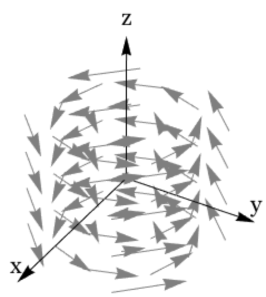


Figure 3: Plot C

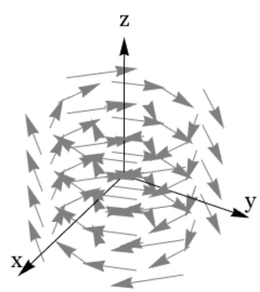


Figure 4: Plot D

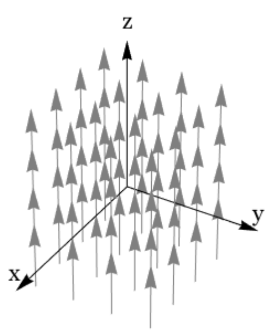


Figure 5: Plot E

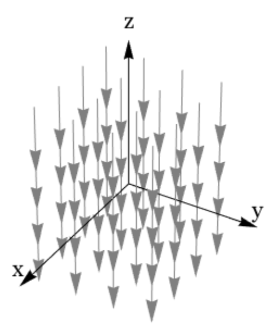


Figure 6: Plot F

   (a) It is given that only one of them has everywhere positive divergence. Which one? A

   (b) It is given that only one of them has everywhere downward pointing curl. Which one? D

(ii) Mark the following sentence as **true** or **false**.

Let c be the unit circle in $\mathbb{R}^2$ parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x, y)$ we have

$$\int_{-c} f(x,y)dx = -\int_c f(x,y)dx.$$

True **False**

(iii) Mark the following sentence as **true** or **false**.

Let S denote the unit sphere in $\mathbb{R}^3$ with positive (outward) orientation and $\tilde{S}$ the unit sphere with negative (inward) orientation. Then, for any continuous scalar function $f(x, y, z)$,

$$\iint_S f(x,y,z)dS = -\iint_{\tilde{S}} f(x,y,z)dS.$$

True **False**

(iv) Mark the following sentence as **true** or **false**.

Let S denote the upper hemisphere of the unit sphere centered at the origin in $\mathbb{R}^3$ (the one that satisfies $z \geq 0$), with **upward** orientation, and $\tilde{S}$ the unit disk on the plane $z = 0$, centered at the origin, again with **upward** orientation. Then, for any vector field $\vec{F}(x, y, z)$ with differentiable coefficients

$$\iint_S \operatorname{curl} \vec{F}(x,y,z) \cdot d\vec{S} = \iint_{\tilde{S}} \operatorname{curl} \vec{F}(x,y,z) \cdot d\vec{S}.$$

True **False**

2. (5 pts.) It is given that the vector field $\vec{F}(x, y) = \langle \underbrace{\sin(y) + 3x^2}_{P}, \underbrace{x\cos(y) + 2y}_{Q} \rangle$ is conservative on $\mathbb{R}^2$. Compute a potential function for it.

$$P = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = x\sin y + x^3 + g(y)$$

$$\frac{\partial f}{\partial y} = Q \Rightarrow x\cos y + g'(y) = x\cos y + 2y$$

$$\Rightarrow g(y) = y^2 + c$$

So $f(x,y) = x\sin y + x^3 + y^2 + c$

3. (12 pts.) **Answers without supporting work will not receive credit**. Let

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$$

be the sphere of radius R centered at the origin (For all the questions below, R will appear in the final result). Each part can be answered regardless of whether you have answered the other parts.

3

(i) Compute the **volume** enclosed by S (that is, the volume of the ball of radius R).

Spherical coords

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= \frac{\rho^3}{3}\Big|_0^R \left(-\cos\varphi\right)\Big|_0^{\pi} \cdot 2\pi = 4\pi\frac{R^3}{3}$$

5

(ii) Compute the **surface area** of S.

$$\vec{r}(\varphi,\theta) = \langle R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi\rangle \quad , \varphi \in [0,\pi]$$
$$\theta \in [0, 2\pi]$$

$$\vec{r}_\varphi(\varphi,\theta) = \langle R\cos\varphi\cos\theta, R\cos\varphi\sin\theta, -R\sin\varphi\rangle$$

$$\vec{r}_\theta(\varphi,\theta) = \langle -R\sin\varphi\sin\theta, R\sin\varphi\cos\theta, 0\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta(\varphi,\theta) = \langle R^2\sin^2\varphi\cos\theta, R^2\sin^2\varphi\sin\theta, R^2\sin\varphi\cos\varphi\cos^2\theta + R^2\cos\varphi\sin\varphi\sin^2\theta\rangle$$

$$\Rightarrow |\vec{r}_\varphi \times \vec{r}_\theta| = \left(R^4\sin^4\varphi\cos^2\theta + R^4\sin^4\varphi\sin^2\theta + R^4\cos^2\varphi\sin^2\varphi\right)^{\frac{1}{2}}$$

$$= R^2\sqrt{\sin^2\varphi} = R^2\sin\varphi$$

$$A = \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi \, d\varphi \, d\theta = R^2 \cdot 2\pi \cdot 2 = 4\pi R^2$$

(iii) Find an equation describing the **tangent plane** to S at the point $(\frac{1}{2}R, \frac{1}{2}R, \frac{\sqrt{2}}{2}R)$.

4

Use ∇f, $f(x,y,z) = x^2 + y^2 + z^2$

$\nabla f = \langle 2x, 2y, 2z\rangle$. Plug in $\langle \frac{1}{2}R, \frac{1}{2}R, \frac{\sqrt{2}}{2}R\rangle$

$$\langle x - \tfrac{1}{2}R, y - \tfrac{1}{2}R, z - \tfrac{\sqrt{2}}{2}R\rangle \cdot \langle R, R, \sqrt{2}R\rangle = 0.$$

4. (15 pts.) The two parts can be answered independently

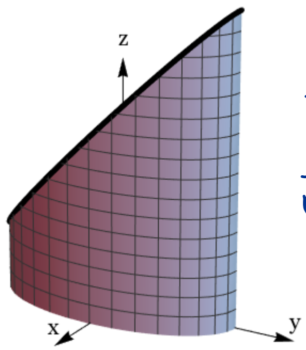
Let S be the surface consisting of the part of the generalized cylinder $x^2 + \frac{y^2}{4} = 1$, between the planes $z = 0$ and $z = y + 3$, that also satisfies $x > 0$. We give S orientation towards the positive x axis (this means that the x coordinate of the unit normal vector field has to be always positive).

9

(i) Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle y, x, z \rangle$.

$\vec{r}(u,v) = \langle \cos u, 2\sin u, v \rangle$ $\quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$0 \le v \le 2\sin u + 3$



$\vec{r}_u = \langle -\sin u, 2\cos u, 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2\cos u, \sin u, 0 \rangle$ correct orientation

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\sin u + 3} \langle 2\sin u, \cos u, v \rangle \cdot \langle 2\cos u, \sin u, 0 \rangle \, dv\, du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin u + 3)\, 5 \sin u \cos u \, du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 10 \sin^2 u \cos u + 15 \sin u \cos u \, du$$

$$= \frac{10}{3} \sin^3 u + \frac{15}{2} \sin^2 u \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{10}{3} \cdot 2 = \frac{20}{3}$$

6

(ii) Compute the line integral $\int_c \vec{G} \cdot d\vec{r}$, where c is the intersection of the cylinder $x^2 + \frac{y^2}{4} = 1$ and the plane $z = y + 3$ with $x \ge 0$ (black in the picture), transversed in direction from $(0, 2, 5)$ to $(0, -2, 1)$ and $\vec{G}(x, y, z) = \langle 0, 0, z \rangle$

$-c(t) = \langle \cos t, 2\sin t, 2\sin t + 3 \rangle$ $\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

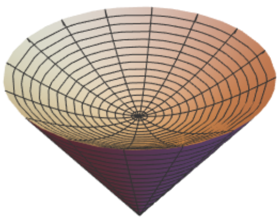
So

$$\int_c \vec{G} \cdot d\vec{r} = -\int_{-c} \vec{G} \cdot d\vec{r} = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle 0, 0, 2\sin t + 3 \rangle \cdot \langle -\sin t, 2\cos t, 2\cos t \rangle \, dt$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \sin t \cos t + 6 \cos t \, dt = -\left(2\sin^2 t + 6 \sin t\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= -12$$

5. (12 pts.) Let E be the solid bounded below by the half cone $z = \sqrt{2}\sqrt{x^2+y^2}$ and bounded above by a sheet of a two sheeted hyperboloid, given by $z = \sqrt{1+x^2+y^2}$. Let S be the boundary surface of E, with **inward** orientation. Compute the flux of the vector field $\vec{F}(x,y,z) = \langle x, -zx, x^2+y^2\rangle$ across S. That is, find $\iint_S \vec{F}\cdot d\vec{S}$.



Use divergence Theorem. Inward or.

$$\iint_S \vec{F}\cdot d\vec{S} = - \iiint_E \operatorname{div}\vec{F}\, dV$$

$$= - \iiint_E (1+0+0)\, dV = -\operatorname{Vol}(E)$$

In cylindrical coords:

V: $\sqrt{2}\, r \le z \le \sqrt{1+r^2}$

$0 \le \theta \le 2\pi$

$\sqrt{2}\, r = \sqrt{1+r^2} \Rightarrow 2r^2 = 1+r^2 \Rightarrow r^2 = 1 \Rightarrow r = \pm 1 \Rightarrow r = 1$

$0 \le r \le 1$

So

$$\iint_S \vec{F}\cdot d\vec{S} = - \int_0^{2\pi}\int_0^1\int_{\sqrt{2}r}^{\sqrt{1+r^2}} r\, dz\, dr\, d\theta$$

$$= -2\pi \int_0^1 r\sqrt{1+r^2} - \sqrt{2}\, r^2\, dr$$

$$= -\pi \int_0^1 2r\sqrt{1+r^2} + 2\pi \int_0^1 \sqrt{2}\, r^2\, dr$$

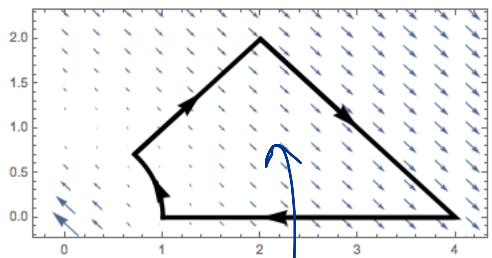
$$= -\pi \int_1^2 \sqrt{u}\, du + 2\pi\sqrt{2}\, \frac{r^3}{3}\Big|_0^1 = -\pi\, \frac{2}{3} u^{\frac{3}{2}}\Big|_1^2 + \frac{2\pi}{3}\sqrt{2}$$

$$= -\frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right) + \frac{2\pi}{3}\sqrt{2}$$

6. (12 pts.) Find the work of a force field $\vec{F}(x,y) = \langle \overbrace{\frac{1}{8}\ln(x^2+y^2)}^{P}, \overbrace{-\frac{1}{8}\ln(x^2+y^2)}^{Q} \rangle$, produced when an object is moving along a closed path on the plane consisting of the following curves:

- A segment of the line $y = x$ from $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ to $(2,2)$.
- A segment of the line $y = 4 - x$ from (2,2) to (4,0).
- A segment of the line $y = 0$ from (4,0) to (1,0).
- An arc of the circle $x^2 + y^2 = 1$ from (1,0) to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Hint: Use Green's Theorem.



D

Use Green's Theorem. Negative or.

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= -\iint_D -\frac{1}{8}\frac{1}{x^2+y^2}\cdot 2x - \frac{1}{8(x^2+y^2)}2y\, dA$$

$$= +\iint_D \frac{1}{4}\frac{x+y}{x^2+y^2}\, dA$$

In polar: $D: \ 0 \le \theta \le \frac{\pi}{4}$. $\quad y = 4-x \Rightarrow r\sin\theta = 4 - r\cos\theta$

$0 \le r \le \frac{4}{\sin\theta + \cos\theta}$ $\qquad r = \frac{4}{\sin\theta+\cos\theta}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{4}} \int_1^{\frac{4}{\sin\theta+\cos\theta}} \frac{1}{4}\frac{r\sin\theta + r\cos\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_1^{\frac{4}{\sin\theta+\cos\theta}} \frac{1}{4}(\sin\theta + \cos\theta)\, dr\, d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 1 - \frac{\sin\theta + \cos\theta}{4}\, d\theta =$$

$$= \frac{\pi}{4} - \left(\frac{-\cos\theta + \sin\theta}{4}\right)\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{4}$$

7. (12 pts.)Let c be the curve with parametrization

$$c(t) = (\cos(-t), \sin(-t), \cos(-2t)), t \in [0, 2\pi],$$

which lives on the surface

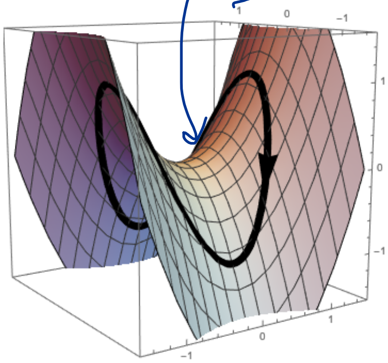
$$z = x^2 - y^2,$$

as in the picture. Compute $\int_c \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x, y, z) = \langle \sin(x) + y, \cos(y), \cancel{x^2} + z^2 \rangle.$$

Hint: Use Stokes' Theorem

S



With Stokes:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

Need downward orientation for surface.

Param. surf. $\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$, $(u,v) \in$ unit disk bec. curve is over unit circle.

$\vec{r}_u \times \vec{r}_v = \langle -2u, 2v, 1 \rangle$ ← upward or., use $-\vec{r}_u \times \vec{r}_v$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \sin x + y & \cos y & z^2 \end{vmatrix} = \langle 0, 0, -1 \rangle$$

So

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{u^2+v^2 \le 1} \langle 0, 0, -1 \rangle \cdot (-\langle -2u, 2v, 1 \rangle) \, dA$$

$$= \int_0^1 \int_0^{2\pi} r \, d\theta \, dr = \pi$$

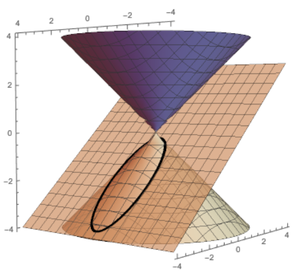
8. (12 pts.) Give a parametrization for the black curve in the picture, which is given as the intersection of the cone

$$z^2 = x^2 + y^2$$

and the plane

$$z = \frac{\sqrt{3}}{2}x - \frac{1}{2}.$$

The curve can have any orientation you prefer, but **you must provide the domain of your parametrization**. That is, you are expected to find an expression of the form $c(t) = \langle x(t), y(t), z(t)\rangle$, $t \in [a, b]$ for the curve.



$$z(t) = \frac{\sqrt{3}}{2}x(t) - \frac{1}{2}$$

$$z^2 = x^2 + y^2$$

$$\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\right)^2 = x^2 + y^2 \Rightarrow \frac{3}{4}x^2 - \frac{1}{2}\sqrt{3}x + \frac{1}{4} = x^2 + y^2$$

$$\Rightarrow \frac{1}{4}x^2 + \frac{1}{2}\sqrt{3}x + y^2 = \frac{1}{4}$$

$$\Rightarrow x^2 + 2\sqrt{3}x + 4y^2 = 1$$

$$\Rightarrow (x + \sqrt{3})^2 + 4y^2 = 4$$

$$\Rightarrow \left(\frac{x + \sqrt{3}}{2}\right)^2 + y^2 = 1$$

Set $\frac{x + \sqrt{3}}{2} = \cos t \quad y = \sin t$

$x = -\sqrt{3} + 2\cos t, \quad y = \sin t,$

$z = \frac{\sqrt{3}}{2}(-\sqrt{3} + 2\cos t) - \frac{1}{2}$

$t \in [0, 2\pi]$.

Now that you're done, go back and make sure that you didn't miss any page with a problem!