Math 324 C - Winter 2017

Final Exam v.B

Wednesday, March 15, 2017

Name: ______________________________

Student ID Number: ______________________________

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- There are 7 problems spanning 7 pages (your last page should be numbered as 7). Please make sure your exam contains all these questions.
- You are allowed to use a scientific calculator (**no graphing calculators**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems (unless explicitly instructed otherwise). The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the back of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 110 minutes to complete the exam. Budget your time wisely.
  Do not spend too much time on an individual problem, unless you are done with all the rest.
- You are not allowed to discuss this exam with other people until 5.00 pm today.

GOOD LUCK!

1. (8 pts.) **You do not need to explain your answers for this problem.**

(a) Mark the following sentence as **true** or **false**. Let c be the unit circle in $\mathbb{R}^2$ parametrized counterclockwise, so that $-c$ is the unit circle parametrized clockwise. Then for every scalar valued continuous function $f(x,y)$ we have

$$\int_{-c} f(x,y)dx = -\int_{c} f(x,y)dx.$$

True **False**

(b) Mark the following sentence as **true** or **false**. Let S denote the unit ball in $\mathbb{R}^3$ with positive(outward) orientation and $\tilde{S}$ the unit ball with negative (inward) orientation. Then, for any vector field $\vec{F}(x,y,z)$ with continuous coefficients

$$\int_{S} \vec{F}(x,y,z)\cdot d\vec{S} = -\int_{\tilde{S}} \vec{F}(x,y,z)\cdot d\vec{S}.$$

True **False**

(c) Mark the following sentence as **true** or **false**. Let S denote the upper hemisphere of the unit ball centered at the origin in $\mathbb{R}^3$ (the one that satisfies $z \geq 0$), with **upward** orientation, and $\tilde{S}$ the lower hemisphere of the unit ball centered at the origin (the one that satisfies $z \leq 0$), again with **upward** orientation. Then, for any vector field $\vec{F}(x,y,z)$ with differentiable coefficients

$$\iint_{S} \operatorname{curl}\vec{F}(x,y,z)\cdot d\vec{S} = \iint_{\tilde{S}} \operatorname{curl}\vec{F}(x,y,z)\cdot d\vec{S}.$$

True **False**

2. (6 pts.) Show the following version of the product rule: Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field, where P, Q are differentiable scalar valued functions, and let $g(x,y)$ be a differentiable scalar valued function. Then

$$\operatorname{div}(g\vec{F}) = g\operatorname{div}(\vec{F}) + (\nabla g)\cdot\vec{F}.$$

Make sure that each step follows clearly from the previous one, otherwise you may not receive full credit.

$$\operatorname{div}(g\vec{F}) = \operatorname{div}(g<P, Q>) =$$

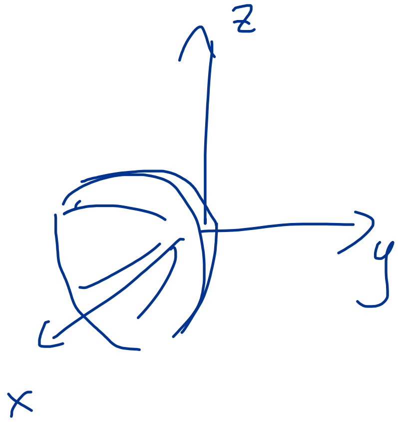
$$= \operatorname{div}(<gP, gQ>) = \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gQ)$$

$$= \frac{\partial g}{\partial x}P + g\frac{\partial P}{\partial x} + \frac{\partial g}{\partial y}Q + g\frac{\partial Q}{\partial y}$$

$$= \nabla g \cdot <P, Q> + g\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$$

$$= g\operatorname{div}\vec{F} + \nabla g\cdot\vec{F}$$

3. Find the mass of a thin piece of aluminum foil occupying the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \le 4$, assuming that its density at the point (x, y, z) is



$$\rho(x, y, z) = \sqrt{\frac{x}{4x+1}}.$$

Parametrize:

$\vec{r}(u,v) = \langle u^2+v^2, u, v \rangle$, $(u,v) \in D$

To find D, project paroboloid on yz plane:

$\left.\begin{array}{l} x = y^2+z^2 \\ x = 4 \end{array}\right\} \Rightarrow \quad y^2+z^2 = 4 \Rightarrow$ projection is the disk of radius 2 on yz plane, so:

$$D = \{(u,v) : u^2+v^2 \le 4\}$$

$$\vec{r}_u = \langle 2u, 1, 0 \rangle, \quad \vec{r}_v = \langle 2v, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j}(-2u) + \vec{k}(-2v)$$

$$m = \iint_S \rho(x,y,z)\,dS = \iint_D \frac{\sqrt{u^2+v^2}}{\sqrt{4(u^2+v^2)+1}} \sqrt{1+4u^2+4v^2}\,dA$$

polar

$$= \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta = 2\pi \left.\frac{r^3}{3}\right|_0^2 = \frac{8 \cdot 2\pi}{3}$$

4. (10 pts.) Let S be the onion-like surface obtained from the revolution of the graph of the function $z = \sin(y) + 1$, $-\frac{\pi}{2} \le y \le \pi$, around the y-axis (look at the picture).

Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \cancel{\langle y, y^2, x+z \rangle}$. → unit normal away from origin

$\langle 0,1,0 \rangle$

Parametrize the surface of revolution:

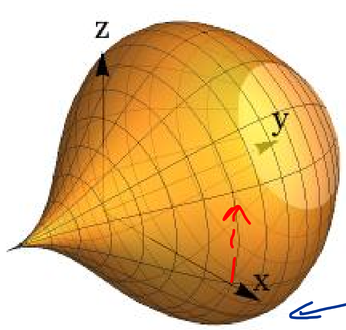
$$r(u,v) = \langle (1+\sin v)\cos u,\ v,\ (1+\sin v)\sin u \rangle \qquad u \in [0, 2\pi],\ v \in \left[-\frac{\pi}{2}, \pi\right]$$

$$\vec{r}_u = \langle -(1+\sin v)\sin u,\ 0,\ (1+\sin v)\cos u \rangle$$

$$\vec{r}_v = \langle \cos v \cos u,\ 1,\ \cos v \sin u \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(1+\sin v)\sin u & 0 & (1+\sin v)\cos u \\ \cos v \cos u & 1 & \cos v \sin u \end{vmatrix} =$$

$$= \vec{i}\,(-\cos u(1+\sin v)) + \vec{j}\,\big((1+\sin v)\cos v \sin^2 u + (1+\sin v)\cos v \cos^2 u\big) + \vec{k}\,(-(1+\sin v))\sin u)$$



$$= \langle -\cos u(1+\sin v),\ (1+\sin v)\cos v,\ -(1+\sin v)\sin u \rangle$$

Plug in $u=0$, $v=0$,

$\vec{r}(0,0) = \langle 1,0,0 \rangle$, $\vec{r}_u \times \vec{r}_v(0,0) = \langle -1, 1, 0 \rangle$

it doesn't work, it's pointing inside

So:

$$\iint_S \vec{F} \cdot d\vec{S} = -\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} \vec{F} \cdot \vec{r}_u \times \vec{r}_v(u,v)\, dv\, du = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\pi} -(1+\sin v)\cos v\, dv\, du$$

$$= -2\pi \int_{-\frac{\pi}{2}}^{\pi} \cos v + \frac{1}{2}\sin 2v\, dv = -2\pi \cdot \frac{1}{2}$$

5. (10 pts) (The two parts are not related)

(a) Find the tangent plane to the surface described implicitly by $z^3 = x^2 - y^4 + zxy$ at $(1, 1, 1)$

Level set of $F(x,y,z) = z^3 - x^2 + y^4 - zxy$

$\nabla F = \langle -2x - zy,\ 4y^3 - xz,\ 3z^2 - xy \rangle$

$\Rightarrow \nabla F(1,1,1) = \langle -3, 3, 2 \rangle$

Therefore:

$(\langle x, y, z\rangle - \langle 1,1,1\rangle) \cdot \langle -3, 3, 2\rangle = 0$

$\Leftrightarrow -3(x-1) + 3(y-1) + 2(z-1) = 0$

(b) (10 pts.) Let E be the solid **in the first octant** bounded by the coordinate planes, the cylinder $x^2 + z^2 = 1$ and the plane $y = 3 - x$, as in the picture. For a function $f(x, y, z)$, set up an integral $\iiint_E f(x, y, z)dV$ in the following way:

$$\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{3-x} f(x, y, z)\, dydzdx$$

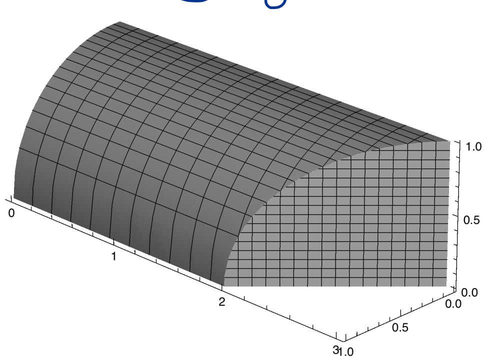
$x^2 + z^2 = 0$ $\qquad 0 \le y \le 3 - x$

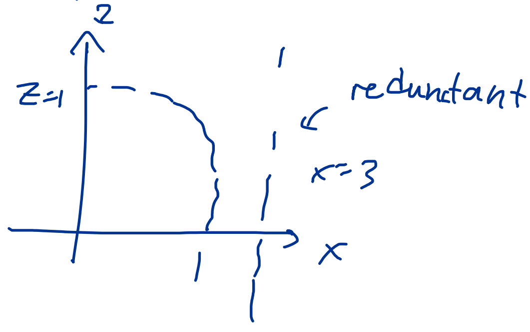
(*) $x = 0$, $y = 0$, $z = 0$

Proj. on xz plane:

$x^2 + z^2 = 0$

(**) $y = 3 - x$ $\qquad$ (*), (**) $\Rightarrow 3 - x = 0 \Rightarrow x = 3$





6. (9 pts) Let E be the solid of the picture below, bounded below by the paraboloid $z = 4x^2 + 4y^2$ and bounded above by the cone $z = 8 - 4\sqrt{x^2 + y^2}$.

(a) Compute the volume of E.

Use cylindrical coords:

cone: $z = 8 - 4\sqrt{x^2+y^2} \Rightarrow z = 8 - 4r$

paraboloid: $z = 4x^2 + 4y^2 \Rightarrow z = 4r^2$

Find projection of their intersection on xy plane: $\begin{cases} z = 8-4r \\ z = 4r^2 \end{cases} \Rightarrow 4r^2 + 4r - 8 = 0$

$\Rightarrow r = 1$ or $r = -2$

$\Rightarrow r = 1$

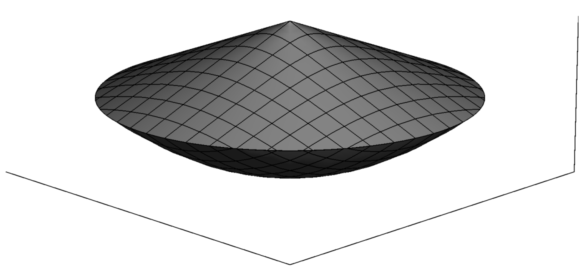
$$\text{volume} = \int_0^{2\pi} \int_0^1 \int_{4r^2}^{8-4r} 1 \cdot r \, dz \, dr \, d\theta = 2\pi \int_0^1 r(8 - 4r - 4r^2) dr$$

$$= 2\pi \int_0^1 8r - 4r^2 - 4r^3 dr = 2\pi \left[4r^2 - \frac{4r^3}{3} - r^4\right]_0^1 = 2\pi\left(4 - \frac{4}{3} - 1\right) = \frac{10\pi}{3}$$

(b) If $\vec{F} = \langle y+x, y, 4z \rangle$ and S is the boundary of E with **inward orientation**, compute $\iint_S \vec{F} \cdot d\vec{S}$. (Hint: Use the divergence theorem).

By divergence Theorem, since orientation is inward,

$$\iint_S \vec{F} \cdot d\vec{S} = - \iiint_E \operatorname{div} \vec{F} \, dV = - \iiint_E 1 + 1 + 4 \, dV = -6 \cdot \frac{10\pi}{3} = -20\pi$$



7. (10 pts.) Let S be the unit sphere centered at the origin. Let c be the path consisting of the following curves, as in the picture at the bottom of the page:

- An arc of the intersection of S with the plane $y = x$, from (0,0,1) to $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $z = -\dfrac{1}{2}$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ (the one satisfying $x \geq 0$).
- An arc of the intersection of S with the plane $y = -x$, from $(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$ to (0,0,1) (the one satisfying $x \geq 0$).

Let $\vec{F}(x, y, z) = \langle -yx, x^2, z\rangle$. Compute $\int_c \vec{F} \cdot d\vec{r}$ (you may do it directly, or use one of the theorems of chapter 16; if you do so, clearly state which theorem you are using).

Easier with Stokes: c is the boundary of a surface S' on the sphere. Parametrize sphere:

$$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle,$$

$$\vec{r}_u \times \vec{r}_v(u,v) = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u\rangle$$

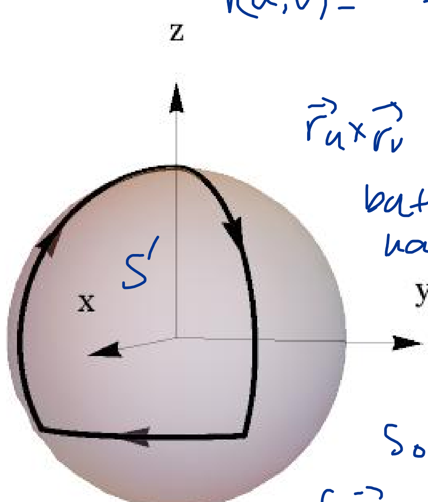
We need correct bounds. $y = x \Rightarrow \cos v = \sin v \Rightarrow v = \frac{\pi}{4}$

$y = -x \Rightarrow \cos v = -\sin v \Rightarrow v = -\frac{\pi}{4}$

$z = -\frac{1}{2} \Rightarrow \cos u = -\frac{1}{2} \Rightarrow u = \frac{2\pi}{3}$

So: S' can be parametrized as

$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle$, $u \in [0, \frac{2\pi}{3}]$, $v \in [-\frac{\pi}{4}, \frac{\pi}{4}]$



$\vec{r}_u \times \vec{r}_v$ gives <u>outward</u> orientation but we need inward bec. of right hand rule.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yx & x^2 & z \end{vmatrix} = \langle 0, 0, 3x\rangle$$

So by Stokes' thm,

$$\int_c \vec{F} \cdot d\vec{r} = \iint_{S'} \text{curl } \vec{F} \cdot d\vec{S} =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} \langle 0, 0, 3\sin u \cos v\rangle \cdot (-\langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u\rangle)\, du\, dv$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{2\pi}{3}} -3\sin^2 u \cos u \cos v\, du\, dv = \left[\sin^3 u\right]_0^{\frac{2\pi}{3}} \left[-\sin v\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\sqrt{3}}{2}\right)^3 (-\sqrt{2})$$