

Worksheet 3

December 7, 2017

- Decide which of the three plots below corresponds to the parametrization

$$\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v) \rangle, \text{ for } (u, v) \in [0, 2\pi] \times [0, 2\pi]$$

set $v=0$, observe we find a circle of radius 3 on the xy plane.

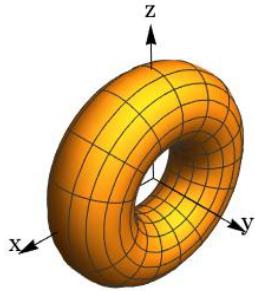


Figure 1: Plot 1

Figure 2: Plot 2

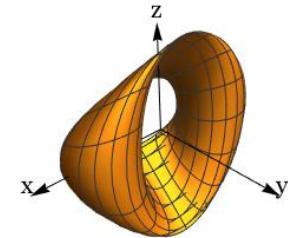
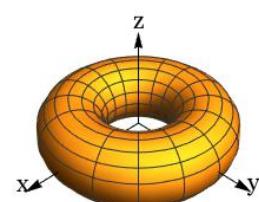


Figure 3: Plot 3

- Prove the first step towards showing Green's First Identity: Show that for any twice differentiable functions $u(x, y, z)$ and $v(x, y, z)$ we have

$$\operatorname{div}(u \nabla v) = \nabla u \cdot \nabla v + u \Delta v \quad (1)$$

Look at Lecture notes on Divergence Theorem.

3. Compute the line integral $\int_c \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle -y, x, 2 \rangle$ and c is the path that consists of the following line segments, as in Figure 5:

- A line segment from $(0, 0, 1)$ to $(-1, 1, 0)$.
- A line segment from $(-1, 1, 0)$ to $(1, 1, 0)$.
- A line segment from $(1, 1, 0)$ back to $(0, 0, 1)$.

Use Stokes' theorem: The three points lie on a plane that we have to find.

set $z = ax + by + c$ for the plane.

Plug in all three points:

$$1 = a \cdot 0 + b \cdot 0 + c \Rightarrow c = 1$$

$$0 = a(-1) + b \cdot 1 + 1 \quad | \quad \begin{cases} 2b + 2 = 0 \\ a + b + 1 = 0 \end{cases}$$

$$0 = a \cdot 1 + b \cdot 1 + 1 \quad | \quad \begin{cases} b = -1 \\ a = 0 \end{cases}$$

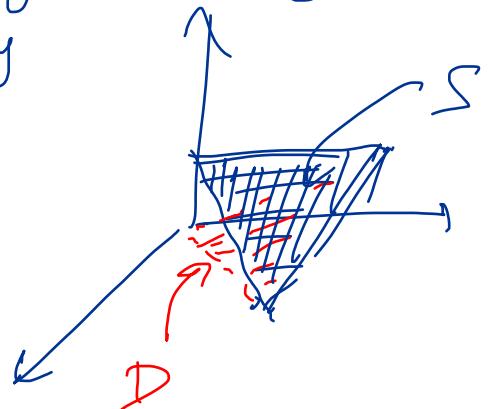
So $z = -y + 1$

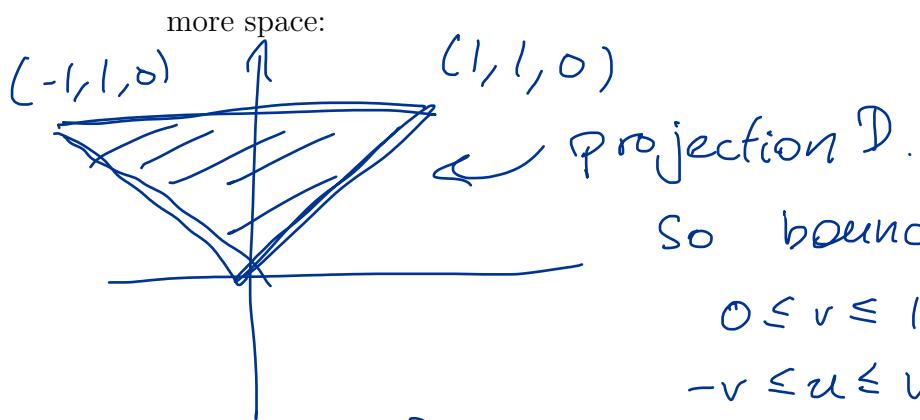
We can apply Stokes over the triangle S on the plane $z = -y + 1$ defined by the three points. Think of it as the graph of a function:

$$\vec{F}(u, v) = \langle u, v, -v+1 \rangle, (u, v) \in D.$$

To find D , project the triangle on xy plane
Project the three points to xy plane:

$$\begin{aligned} (0, 0, 1) &\rightarrow (0, 0, 0) \\ (1, 1, 0) &\rightarrow (1, 1, 0) \\ (-1, 1, 0) &\rightarrow (-1, 1, 0) \end{aligned}$$





so bounds for u, v :

$$0 \leq v \leq 1$$

$$-v \leq u \leq v$$

$$\vec{r}_u \times \vec{r}_v = \left\langle -\frac{\partial}{\partial u}(-v+1), -\frac{\partial}{\partial v}(-v+1), 1 \right\rangle = \langle 0, 1, 1 \rangle$$

This is upward orientation, we'll need downward b.c. of right hand rule.

$$\text{So we'll use } -\vec{r}_u \times \vec{r}_v = \langle 0, -1, -1 \rangle$$

$$\text{Find } \operatorname{curl} \vec{F} = \langle 0, 0, 2 \rangle$$

By Stokes' theorem:

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \operatorname{curl} \vec{F} \cdot d\vec{s} = \\ &= \int_0^1 \int_{-v}^v \langle 0, 0, 2 \rangle \cdot \langle 0, -1, -1 \rangle du dv \\ &= \int_0^1 -2 \cdot 2v dv = -2 v^2 \Big|_0^1 = -2 \end{aligned}$$

4. Let S be the surface that consists of the part of the cylinder $x^2 + y^2 = 1$ lying between the planes $z = 0$ and $z = -1$, together with the part of the sphere

$$x^2 + y^2 + (z + 1 + \sqrt{3})^2 = 4$$

that lies below the plane $z = -1$, and let S have orientation pointing away from the origin, as in picture 4.

- (a) Compute $\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle y + x, x + z, -z + y^2 \rangle$.

Hint: Modify the surface accordingly so you can use divergence theorem.

We can't use divergence theorem directly b/c. the surface isn't closed.
We'll attach a lid to it and make it closed. Call the lid S'

We need outward orientation

so $S \cup S'$ has outward orientation

The lid is a disk of radius 1 on xy plane, parametrize as

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 0 \rangle, r \in [0, 1]$$

$$\vec{r}_r = \langle \cos \theta, \sin \theta, 0 \rangle \quad \theta \in [0, 2\pi]$$

$$\vec{r}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \langle 0, 0, r \rangle \text{ upward pointing } \checkmark$$



If E is the interior of our closed surface,

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$$

more space:

$$\begin{aligned}\Rightarrow \iint_S \vec{F} \cdot d\vec{S} &= - \iint_{S'} \vec{F} \cdot d\vec{S} = \\ &= - \int_0^{2\pi} \int_0^1 \langle r \sin \theta + r \cos \theta, r \cos \theta, (r \sin \theta)^2 \rangle \cdot \langle 0, 0, r \rangle dr d\theta \\ &= - \int_0^{2\pi} \int_0^1 r^3 \sin^2 \theta dr d\theta = - \frac{1}{4} \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta \\ &= - \frac{1}{4} \cdot \frac{1}{2} \cdot 8\pi\end{aligned}$$

(b) *Find the surface area of S .

Here tricks won't help much, we need to do it by parametrizing the two parts of this surface.

For upper part, S_1 : $r(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$,

$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle \quad z \in [-1, 0]$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle \quad \theta \in [0, 2\pi]$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\text{So Area: } \int_{-1}^0 \int_0^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta + 0} d\theta dz \\ = 2\pi$$

(there are other ways to argue about this)

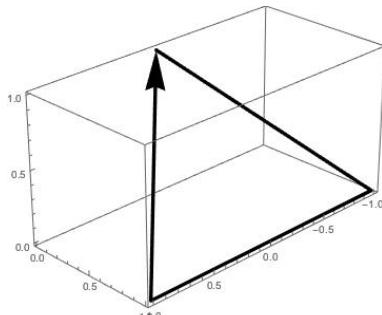
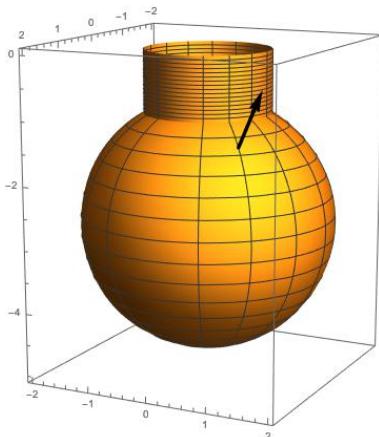


Figure 5: Problem 3

Figure 4: Problem 2

more space:

For the part of the sphere, write

$$\vec{r}(u,v) = \langle 2\sin u \cos v, 2\sin u \sin v, -1 - \sqrt{3} + 2\cos u \rangle$$

To find bounds: $v \in [0, 2\pi]$

For u , $z = -1 - \sqrt{3} + 2\cos u$ and $z = -1$

$$\Rightarrow -1 = -1 - \sqrt{3} + 2\cos u \Rightarrow$$

$$\Rightarrow \cos u = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{6}$$

So we take $u \in [\frac{\pi}{6}, \pi]$

Finally, as you can check

$$\vec{r}_u \times \vec{r}_v = \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\sin u \cos u \rangle$$

and $|\vec{r}_u \times \vec{r}_v| = 4\sin u$ so

$$\begin{aligned} \text{Area} &= \int_{\frac{\pi}{6}}^{\pi} \int_0^{2\pi} 4\sin u \, dv \, du = 8\pi \left(-\cos u \right) \Big|_{\frac{\pi}{6}}^{\pi} \\ &= 8\pi \left(1 - \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\text{So total area} = 2\pi + 8\pi \left(1 - \frac{\sqrt{3}}{2} \right)$$