Worksheet 3

December 7, 2017

1. Decide which of the three plots below corresponds to the parametrization

$$\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v) \rangle, \text{ for } (u, v) \in [0, 2\pi] \times [0, 2\pi]$$

set v=0, observe we find a
cirle of radius 3 on the xy plane.

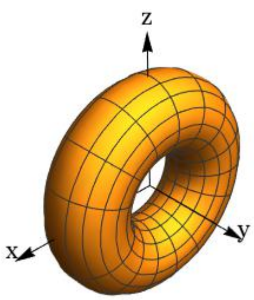


Figure 1: Plot 1

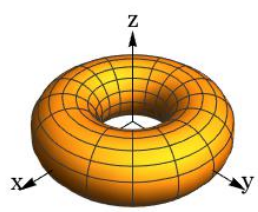


Figure 2: Plot 2

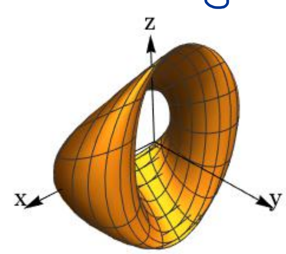


Figure 3: Plot 3

2. Prove the first step towards showing Green's First Identity: Show that for any twice differentiable functions $u(x, y, z)$ and $v(x, y, z)$ we have

$$\operatorname{div}(u\nabla v) = \nabla u \cdot \nabla v + u\Delta v \tag{1}$$

Look at Lecture notes on Divergence
Theorem.

3. Compute the line integral $\int_c \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle -y, x, 2 \rangle$ and c is the path that consists of the following line segments, as in Figure 5:

- A line segment from $(0,0,1)$ to $(-1,1,0)$.
- A line segment from $(-1,1,0)$ to $(1,1,0)$.
- A line segment from $(1,1,0)$ back to $(0,0,1)$.

Use Stokes' theorem: The three points lie on a plane that we have to find.

set $z = ax + by + c$ for the plane.

Plug in all three points:

$$1 = a \cdot 0 + b \cdot 0 + c \Rightarrow c = 1$$

$$\left.\begin{array}{l} 0 = a(-1) + b \cdot 1 + 1 \\ 0 = a \cdot 1 + b \cdot 1 + 1 \end{array}\right| \Rightarrow \begin{cases} 2b + 2 = 0 \\ a + b + 1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} b = -1 \\ a = 0 \end{cases}$$

So $z = -y + 1$

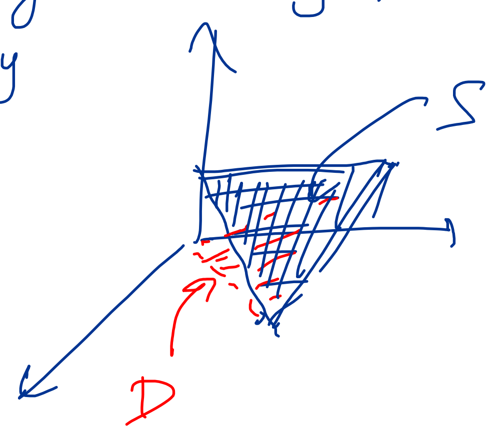
We can apply Stokes over the triangle S on the plane $z = -y + 1$ defined by the three points. Think of it as the graph of a function:

$$\vec{r}(u,v) = \langle u, v, -v+1 \rangle, \quad (u,v) \in D.$$

To find D, project the triangle on xy plane

Project the three points to xy plane:

$$(0,0,1) \longrightarrow (0,0,0)$$
$$(1,1,0) \longrightarrow (1,1,0)$$
$$(-1,1,0) \longrightarrow (-1,1,0)$$



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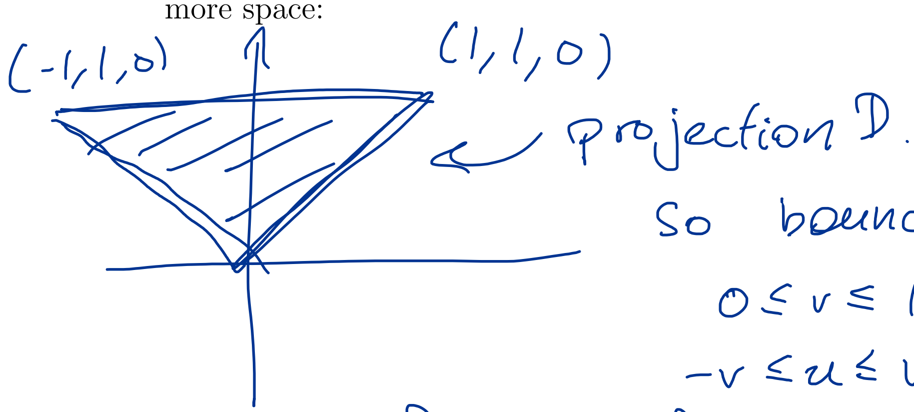

So bounds for u,v:

$0 \le v \le 1$

$-v \le u \le v$

$\vec{r}_u \times \vec{r}_v = \langle -\frac{\partial}{\partial u}(-v+1), -\frac{\partial}{\partial v}(-v+1), 1 \rangle = \langle 0, 1, 1 \rangle$

This is upward orientation, we'll need downward bec. of right hand rule.

So we'll use $-\vec{r}_u \times \vec{r}_v = \langle 0, -1, -1 \rangle$

Find $\operatorname{curl} \vec{F} = \langle 0, 0, 2 \rangle$

By Stokes' theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} =$$

$$= \int_0^1 \int_{-v}^{v} \langle 0,0,2 \rangle \cdot \langle 0,-1,-1 \rangle \, du\, dv$$

$$= \int_0^1 -2 \cdot 2v \, dv = -2v^2 \Big|_0^1 = -2$$

4. Let S be the surface that consists of the part of the cylinder $x^2 + y^2 = 1$ lying between the planes $z = 0$ and $z = -1$, together with the part of the sphere

$$x^2 + y^2 + (z + 1 + \sqrt{3})^2 = 4$$

that lies below the plane $z = -1$, and let S have orientation pointing away from the origin, as in picture 4.

(a) Compute $\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle y + x, x + z, -z + y^2 \rangle$.
Hint: Modify the surface accordingly so you can use divergence theorem.

We can't use divergence theorem directly bec. the surface isn't closed. We'll attach a lid to it and make it closed. Call the lid S'

We need outward orientation so $S \cup S'$ has outward orientation

The lid is a disk of radius 1 on xy plane, parametrize as

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle$, $r \in [0,1]$, $\theta \in [0, 2\pi]$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 0 \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r \rangle$$ upward pointing ✓

If E is the interior of our closed surface,

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{=0} \, dV$$

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$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = -\iint_{S'} \vec{F}\cdot d\vec{S} =$$

$$= -\int_0^{2\pi}\int_0^1 \langle r\sin\theta + r\cos\theta, r\cos\theta, (r\sin\theta)^2\rangle \cdot \langle 0,0,r\rangle\, dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^1 r^3\sin^2\theta\, dr\, d\theta = -\frac{1}{4}\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$= -\frac{1}{4}\,\frac{1}{2}\cdot 2\pi$$

(b) *Find the surface area of S.

Here tricks won't help much, we need to do it by parametrizing the two parts of this surface.

For upper part, S_1: $r(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$,

$z \in [-1, 0]$

$\theta \in [0, 2\pi]$

$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$

$\vec{r}_z = \langle 0, 0, 1 \rangle$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$

So Area: $$\int_{-1}^{0} \int_{0}^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta + 0}\, d\theta\, dz$$

$$= 2\pi$$

(there are other ways to argue about this)

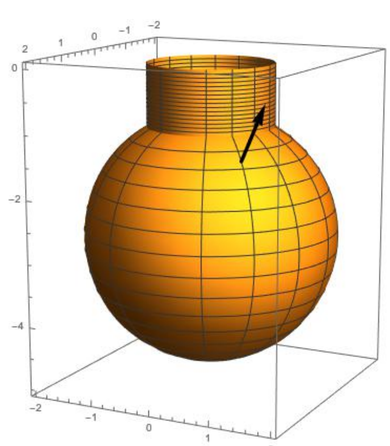

Figure 4: Problem 2

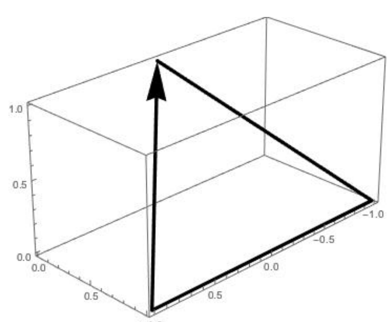

Figure 5: Problem 3

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For the part of the sphere, write

$\vec{r}(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, -1-\sqrt{3}+2\cos u \rangle$

To find bounds: $v \in [0, 2\pi]$

For u, $z = -1-\sqrt{3}+2\cos u$ and $z = -1$

$\Rightarrow -1 = -1-\sqrt{3}+2\cos u \Rightarrow$

$\Rightarrow \cos u = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{6}$

So we take $u \in [\frac{\pi}{6}, \pi]$

Finally, as you can check

$\vec{r}_u \times \vec{r}_v = \langle 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u \rangle$

and $|\vec{r}_u \times \vec{r}_v| = 4\sin u$ so

$$\text{Area} = \int_{\frac{\pi}{6}}^{\pi}\int_0^{2\pi} 4\sin u \, dv\, du = 8\pi \left(-\cos u\right)\Big|_{\frac{\pi}{6}}^{\pi}$$

$$= 8\pi\left(1 - \frac{\sqrt{3}}{2}\right)$$

So total area $= 2\pi + 8\pi\left(1 - \frac{\sqrt{3}}{2}\right)$