北京大学
PEKING UNIVERSITY

NOTE FOR SELF

Measure Theory

Author
Plote

Mentors
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Chapter 1
Riemann Integration

We begin with a few definitions needed before we can define the Riemann integral. Let R denote the complete ordered field of real numbers.(设 R 表示实数的完整有序字段)

1.1 Definition *partition*

Suppose $a, b \in \mathbb{R}$ with $a < b$. A partition of $[a, b]$ is a finite list of the form $x_0, x_1, ..., x_n$, where

$a = x_0 < x_1 < ... < x_n = b$.

We use a partition $x_0, x_1, ..., x_n$ of [a,b] to think of [a,b] as a union of closed subintervals, as follows:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup ...[x_{n-1}, x_n]$$

1.2 Definition *notation for infimum and supremum of a function*

If f is a real-valued function and A is a subset of the domain of f , then

$$\inf_A f = \inf\{f(x) : x \in A\} \text{ and} \sup_A f = \sup\{f(x) : x \in A\}$$

for example:

if function is $f(x) = x^2$, The domain is real numbers,Subset $A = [-2, 1]$

$$\{f(x) : x \in A\} = \{x^2 : x \in [-2, 1]\} = [0, 4]$$

$$\inf_A f = 0 \text{ and} \sup_A f = 4$$

1.3 Definition *lower and upper Riemann sums*

Suppose $f : [a, b] \to R$ is a bounded function and P is a partition $x_0, ..., x_n$ of $[a, b]$. The lower Riemann sum $L(f, P, [a, b])$ and the upper Riemann sum $U(f, P, [a, b])$ are defined by:

$$L(f, P, [a, b]) = \sup_A L(f, P, [a, b]) \text{ and } U(f, P, [a, b]) = \inf_A U(f, P, [a, b])$$

不难理解,lower Riemann sum 和 upper Riemann sum 就是近似 f 在 [a,b] 的面积,只不过区别就是一个从下限近似,一个从上限近似而已.

1.1 example *lower and upper Riemann sums*

Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$,Let P_n donote the partition $0, \frac{1}{n}, \frac{2}{n}, ..., 1$ of $[0, 1]$:

For the partition, we have $x_j - x_{j-1} = \frac{1}{n}$ for each $j = 1, ..., n$, Thus:

$$L(x^2, P_n, [0, 1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{(j-1)^2}{n^2} = \frac{2n^2 - 3n + 1}{6n^2}$$

and

$$U(x^2, P_n, [0, 1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{(j^2)^2}{n^2} = \frac{2n^2 + 3n + 1}{6n^2}$$

由此可得,分的足够多,面积足够的接近真实值,下黎曼和增大,上黎曼和减小

1.4 Definition *lower Riemann sums ≤ upper Riemann sums*

suppose:$f : [a, b] \to \mathbb{R}$ is bounded function and P, P’ are partitions of [a,b]. then:

$$L(f, P, [a, b]) \leq U(f, P', [a, b])$$

We have been working with lower and upper Riemann sums. Now we define the lower and upper Riemann integrals

1.5 Definition *lower and upper Riemann integrals*

suppose:$f : [a, b] \to \mathbb{R}$ is bounded function. The lower Riemann integral $L(f, [a, b])$ and the upper Riemann integral $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_{p} L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_{p} U(f, P, [a, b])$$

where the supremum and infimum above are taken over all partitions P of [a, b]

Same reason:

1.6 Definition *lower Riemann integral ≤ upper Riemann integral*

suppose:$f : [a, b] \to \mathbb{R}$ is bounded function. then:

$$L(f, [a, b]) \leq U(f, [a, b])$$