How hard is to untangle curves?

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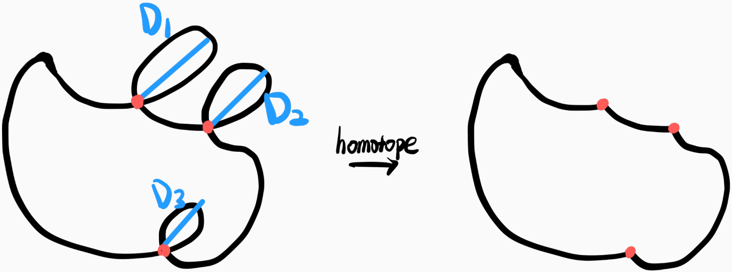
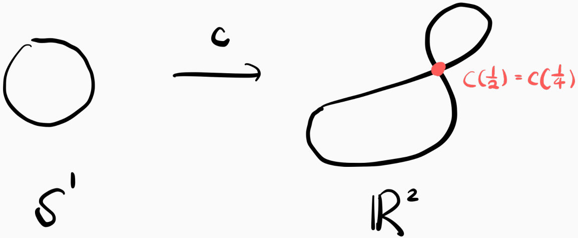
Motivation: To measure how hard it is to untangle curves



"Robustness of a curve $= D_1 + D_2 + D_3$"

We would like to quantify the **robustness** of the self intersection points, and see if they satisfy stability conditions.

Motivation: How to detect the self intersection points?



In this case, $(\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}) \in C^{-1}(\mathbb{D})$.

Given a plane curve $c : S^1 \to \mathbb{R}^2$, consider its diagonal map $C = (c, c) : S^1 \times S^1 \to \mathbb{R}^2 \times \mathbb{R}^2$ and the set of diagonal points $\mathbb{D} = \{(x, x) | x \in \mathbb{R}^2\} \subset \mathbb{R}^2 \times \mathbb{R}^2$. Then the **self-intersection points** of c can be encoded as the **preimage** $C^{-1}(\mathbb{D})$.

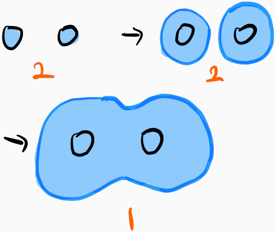
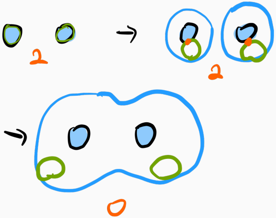
Now we would like to focus on the robustness of this preimage $C^{-1}(\mathbb{D})$!

Table of Contents

Background

Persistence vs Robustness

"How long it will survive?" VS "How much perturbation it will survive?"

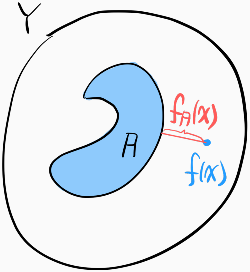
	Persistence	Robustness
Object of Interest	Space	Preimage of Maps
Parameter	**Level** in Sublevel Filtration	**Room** for Perturbation
Illustration		

Perturbations and Distance Functions

Definition

Given X a manifold and Y a Riemannian manifold, a map $h : X \to Y$ homotopic to f is a ρ-**perturbation** of $f : X \to Y$ if they differ by at most ρ with the sup norm: $\|h - f\|_\infty \leq \rho$.

For a submanifold $A \subset Y$, denote by $f_A : X \to \mathbb{R}$ the distance function $f_A(x) := \inf_{a \in A} \|f(x) - a\|$.



Distance Lemma

These induced maps have the following stability under sup norm of original maps:

Lemma 1 (Distance Lemma)

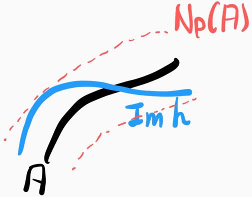
Let $f_A, g_A : X \to \mathbb{R}$ be the distance functions induced by $f, g : X \to Y$ with the submanifold $A \subset Y$. Then

$$\|f_A - g_A\|_\infty \leq \|f - g\|_\infty.$$

Induced Homomorphism

Suppose $||f - h||_\infty \leq \rho$. If $h(x) \in A$, then $f_A(x) \leq \rho$, so we have

$$h_A^{-1}(0) \subset f_A^{-1}([0, \rho]).$$



This inclusion induces a homomorphism between homology groups:

$$j_h : H_*(h_A^{-1}(0)) \longrightarrow H_*(f_A^{-1}([0, \rho])) =: F(\rho),$$

Finally, we denote by $f(r, s)$ the inclusion-induced homomorphism $f(r, s) : F(r) \longrightarrow F(s)$ for every $r \leq s$.

Well Group

Now we can define the well group of the growing preimage $f_A^{-1}([0, r])$ as follows:

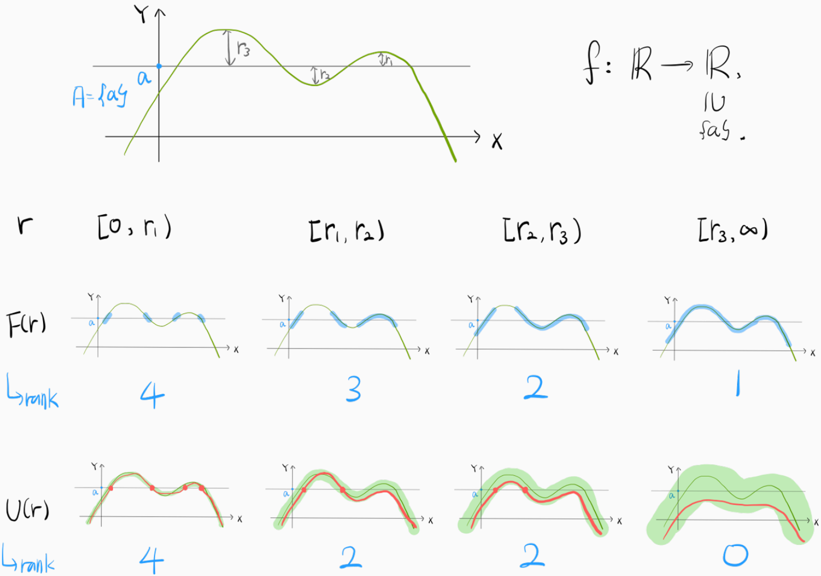
Definition

The **well group** of $f_A^{-1}([0, r])$ is the largest subgroup $U(r) \subset F(r)$ contained in

$$\bigcap_{h: X \to Y} \mathrm{Im} j_h \subset F(r+\delta)$$

where h ranges over all $(r+\delta)$-perturbations of f, and for sufficiently small $\delta > 0$.

Example of Well Group Computation



Issue: Not every perturbation of $C = (c, c)$ has the same form

Recall that given a curve $c : S^1 \to \mathbb{R}^2$, we would like to study **the preimage of the diagonal map** $C = (c, c) : S^1 \times S^1 \to \mathbb{R}^2 \times \mathbb{R}^2$. The definition of well group requires to consider **all** perturbations of $C = (c, c)$, most of which is just a random map $F : S^1 \times S^1 \to \mathbb{R}^2 \times \mathbb{R}^2$, **not** of the form $C' = (c', c')$ for some curve $c' : S^1 \to \mathbb{R}^2$.

Because we are untangling **curves** c instead of the digonal map C, this motivates us to should **restrict** our collection of perturbations to have a meaningful definition of well groups for c.

Shrinking Wellness Lemma

The following lemma serves as a **foundation** for the definition of robustness.

Lemma 2 (Shrinking Wellness Lemma)

For each $0 \leq r \leq s$*, we have*

$$U(s) \subset f(r,s)(U(r)).$$

Sketch of Proof.

This follows from the observation that the set of ρ-perturbations of f gets larger as ρ is bigger, making $\bigcap_h \mathrm{Im} j_h$ smaller. $\square$

Terminal Critical Values

Hence, the only way the well group can change is by **lowering its rank**. Thus in our setting, there are only finitely many $u_1 < u_2 < \ldots < u_l$, called **terminal critical values** at which the rank of $U(r)$ decreases. We extend this sequence by adding $u_0 = 0$ and $u_{l+1} = \infty$.

Denote by $U_i := U(u_i)$ and $F_i := F(u_i)$ for each $i = 0, \ldots, l+1$. We also write $f_{i,j} : F(i) \longrightarrow F(j)$ as the short hand for the inclusion induced homomorphism $f(u_i, u_j)$.

Zigzag Module – “Filtration for Well groups”

However, the well groups do not have a nice sublevel set filtrations with those $\{u_i\}$ as homology groups $F(i)'s$ do. Instead, they form a **zigzag module**:

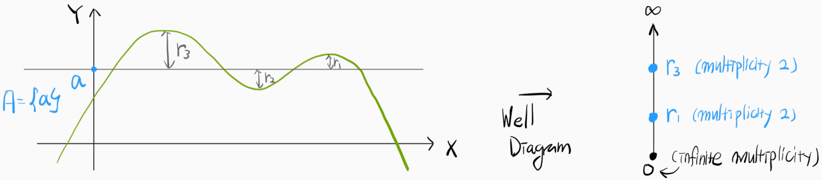
$$\dots \xrightarrow{a_{i-1}} Q_{i-1} \xleftarrow{b_{i-1}} U_i \xrightarrow{a_i} Q_i \xleftarrow{b_i} U_{i+1} \xrightarrow{a_{i+1}} Q_{i+1} \xleftarrow{b_{i+1}} \dots$$

In fact, having surjective a_i's and injective b_j's is equivalent to the fact that the birth only happens at U_0 from the general construction of zigzag module in [CS10].

From now on, to avoid the confusion with the birth and death in homology groups, we replace the terms *birth/death* in well group U_i with **get well/fall ill**.

Well Diagram

Since all classes get well at U_0, we can define the **well diagram** $\mathrm{Dgm}(U)$ as just the multiset of singletons u_i with multiplicities μ_i, together with the infinitely many copies of 0. Hence, the diagram itself is a multiset of values on the extended real $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.



Well diagram of previous example $f : \mathbb{R} \to \mathbb{R}$ with $A = \{a\}$.

Matching Lemma

Lemma 3 (Matching Lemma)

Let $0 \leq u_1 \leq u_2 \leq \ldots \leq u_M$ *and* $0 \leq v_1 \leq v_2 \leq \ldots \leq v_M$ *be the terminal critical values(possibly padding 0's to match the number* M *of values) associated with the maps* $f, g : X \to Y$ *with the same submanifold* $A \subset Y$. *Then the* **bottleneck distance** *between* $\mathrm{Dgm}(U)$ *and* $\mathrm{Dgm}(V)$ *can be calculated as:*

$$W_\infty(\mathrm{Dgm}(U), \mathrm{Dgm}(V)) = \max_{1 \leq i \leq M} |u_i - v_i|.$$

Sketch of Proof.

The proof relies on the purely combinatorial fact on the finitely many points on $\mathbb{R}$; It uses the fact that if $a \leq b$ and $c \leq d$, then $|a - c| + |b - d| \leq |a - d| + |b - c|$. $\square$

Robustness

Finally, we can define the robustness of each class in U_0 as follows:

Definition

for each homology class α in U_0, we can define the **robustness** $\varrho(\alpha)$ of α as the value u_i when α falls ill.

In other words, the robustness quantifies the stability of each homology class under perturbations of the map f. Observe that the robustness for well groups is a counterpart notion of the persistence for homology groups.

Bridge Lemma

Now we introduce the key lemma toward the stability of well diagrams. Note by Distance Lemma(Lemma 1), letting $\|f-g\|_\infty = \varepsilon$ gives $\|f_A - g_A\| \le \varepsilon$, so

$$g_A^{-1}([0,r]) \subset f_A^{-1}([0,r+\varepsilon])$$

This induces the **bridge map** $\mathcal{B}_r : G(r) \longrightarrow F(r+\varepsilon)$, where $G(r)$ is the homology group $H_*(g_A^{-1}([0,r]))$, which is dual to $F(r)$. Denoting by $V(r) \subset G(r)$ the well group associated to g, we have

Lemma 4 (Bridge Lemma)

$$U(r+\varepsilon) \subset \mathcal{B}_r(V(r)).$$

Sketch of Proof.

This follows from the fact that $(r+\varepsilon+\delta)$-perturbations of f include the $(r+\delta)$-perturbations of g by Distance Lemma. $\square$

Stability Theorem

Theorem 5 (Stability Theorem for Well Diagrams)

Let U, V be the well modules of the functions f_A, g_A defined by admissible, homotopic mappings $f, g : X \to Y$, where $A \subset Y$ is a submanifold. Then

$$W_\infty(\mathrm{Dgm}(U), \mathrm{Dgm}(V)) \leq \|f - g\|_\infty.$$

Sketch of Proof.

The proof uses the following two Lemmas:

- Matching Lemma
- Bridge Lemma

$\square$

Result 1: Generalization to restricted well groups

Restricted Well Group

Definition (Nested Collections of Perturbations)

Let $f : X \to Y$ and $A \subset Y$ be a submanifold. Let $\mathcal{F}_\rho$ be a nonempty subcollection of the ρ-perturbations of f. We say the collection $\{\mathcal{F}_\rho\}_{\rho \geq 0}$ is said to be **nested** if it satisfies

$$\mathcal{F}_\rho \subset \mathcal{F}_{\rho'} \qquad \text{for every } \rho \leq \rho'.$$

Definition (Restricted Well Group)

For a nested collections $\{\mathcal{F}_\rho\}_{\rho \geq 0}$ of perturbations of f, the $\{\mathcal{F}_\rho\}$-**restricted well group** of $f_A^{-1}([0, r])$ is defined to be the largest subgroup $\hat{U}(r) \subset F(r)$ such that contained in $\bigcap_{h \in \mathcal{F}_{r+\delta}} \mathrm{Im} j_h$, for a sufficiently small $\delta > 0$.

Shrinking Wellness Lemma for RWG

Lemma 6 (Shrinking Wellness Lemma for restricted well groups)

For each $0 \le r \le s$*, we have*

$$\hat{U}(s) \subset f(r,s)(\hat{U}(r)).$$

Proof.

By the nested property of parameterized collection of perturbations, the set $\mathcal{F}_\rho$ of restricted perturbations of f gets larger as ρ is bigger, making $\bigcap_{h \in \mathcal{F}_\rho} \mathrm{Im} j_h$ smaller. $\square$

Zigzag Modules and Robustness for RWG

We can construct zigzag modules for restricted well groups as before:

$$\ldots \xrightarrow{\hat{a}_{i-1}} \hat{Q}_{i-1} \xleftarrow{\hat{b}_{i-1}} \hat{U}_i \xrightarrow{\hat{a}_i} \hat{Q}_i \xleftarrow{\hat{b}_i} \hat{U}_{i+1} \xrightarrow{\hat{a}_{i+1}} \hat{Q}_{i+1} \xleftarrow{\hat{b}_{i+1}} \ldots$$

Thanks to the fact that Shrinking Wellness Lemma also holds for restricted well groups, $\hat{a}_i$'s are surjective, and $\hat{b}_j$'s are injective. **Thus, all classes get well at $\hat{U}_0$,** and from this we can construct the well diagram for $\hat{U}$ as the multiset of values u_i with multiplicities $\hat{\mu}$, denoting it as $\mathrm{Dgm}(\hat{U})$, together with infinitely many copies of 0. Similarly, for each homology class $[\alpha] \in \hat{U}(0)$, we define the **robustness** $\varrho(\alpha)$ of α to be the value u_i when α falls ill.

Toward Bridge Lemma

Since we will have different parameterized collections of perturbations for f and g, we need to ensure they are *compatible* each other to establish Bridge Lemma.

Definition (ε-compatibility)

Let $f, g : X \to Y$ and $A \subset Y$ be a submanifold. Let $\{\mathcal{F}_\rho\}_{\rho \geq 0}$ and $\{\mathcal{F}'_\rho\}_{\rho \geq 0}$ be parameterized collections of perturbations of f and g. Then $\{\mathcal{F}_\rho\}$ and $\{\mathcal{F}'_\rho\}$ are ε-**compatible** if

$$\mathcal{F}_\rho \subset \mathcal{F}'_{\rho+\varepsilon}, \quad \text{and} \quad \mathcal{F}'_\rho \subset \mathcal{F}_{\rho+\varepsilon}$$

for every $\rho \geq 0$. We say two restricted well modules are ε-**compatible** if their associated nested parameterized collections of perturbations are ε-compatible.

Bridge Lemma for RWG

With this additional condition, Bridge Lemma holds for restricted well modules as well.

Lemma 7 (Bridge Lemma for Restricted Well Module)

For ε-compatible restricted well modules $\hat{U}$ and $\hat{V}$, we have

$$\hat{U}(r+\varepsilon) \subset \mathcal{B}_r(\hat{V}(r)).$$

Proof.

The proof follows from the definition of ε-compatibility because for each $\delta > 0$, we have $\mathcal{F}_{r+\varepsilon+\delta} \supset \mathcal{F}'_{r+\delta}$. Hence, the intersection of the images of restricted $(r+\varepsilon+\delta)$-perturbations of f is contained in the intersection of the images of restricted $(r+\delta)$-perturbations of g, concluding the proof by definition of restricted well groups. $\square$

We remark that the **Matching Lemma** still holds for restricted well modules, as the proof relied only on the combinatorial facts of the points on $\mathbb{R}_{\geq 0}$.

Theorem A (Stability Theorem for Well Diagrams for Restricted Well Modules)

Let $\hat{U}, \hat{V}$ be the restricted well modules of the functions f_A, g_A defined by admissible, homotopic mappings $f, g : X \to Y$, where X, Y and $A \subset Y$ is a submanifold. Assume further $\hat{U}, \hat{V}$ are $\|f - g\|_\infty$-compatible. Then

$$W_\infty(\mathrm{Dgm}(\hat{U}), \mathrm{Dgm}(\hat{V})) \leq \|f - g\|_\infty$$

Result 2: Application – Stability theorem for robustness of self-intersection of curves/knots

Diagonal Perturbations

Now let us apply the aforementioned stability result (Theorem A) to the self-intersection points of curves/knots. To define the robustness of self intersection points, we construct our choice of parameterization for perturbations as follows:

Definition (Diagonal Perturbation)

Let $f : X \to Y$ be a map with $A \subset Y$ a submanifold. Consider a diagonal map $(f, f) : X \times X \to Y \times Y$. For each $\rho \geq 0$, define $\mathcal{F}_\rho$ as:

$$\mathcal{F}_\rho := \{(g, g) : X \times X \to Y \times Y| \quad \|f - g\|_\infty \leq \rho\},$$

and call $\mathcal{F}_\rho$ as the **diagonal ρ-perturbation** of (f, f).

Sanity Check for Diagonal Perturbations

Lemma 8 (Diagonal Perturbations $\Rightarrow$ Nested.)

The parameterized collection of diagonal perturbations are **nested**.

Lemma 9 (Diagonal Perturbations $\Rightarrow$ Compatible.)

Two parameterized collections of diagonal perturbations of f and g are $\|f - g\|_\infty$**-compatible.**

Robustness of self-intersection points of curves

Definition

Let $c : S^1 \to \mathbb{R}^2$ be a curve, with possibly finitely many self-intersection points. Define the **robustness** $\varrho(c)$ **of the curve** c as the sum of the robustness of *restricted* well group classes corresponding to the self-intersecting points, i.e.,

$$\varrho(c) := \sum_{[\alpha]\in\hat{U}(0)} \varrho(\alpha),$$

where $U(0) \subset H_0(C^{-1}(\mathbb{D}))$ is the initial restricted well group associated to the map C with the subset $\mathbb{D} \subset \mathbb{R}^2 \times \mathbb{R}^2$.

Note $C^{-1}(\mathbb{D})$ represents the self-intersecting points of c, together with infinitely many diagonal points in $S^1 \times S^1$.

Robustness of self-intersection points of knots

By replacing the 2d-curve c with the 3d-knot K, we can similarly define the robustness of self-intersection points for knots:

Definition

Let $K : S^1 \to \mathbb{R}^3$ be a knot, with possibly finitely many self-intersection points. Define the **robustness $\varrho(K)$ of the knot K** as the sum of the robustness of *restricted* well group classes corresponding to the self-intersecting points, i.e.,

$$\varrho(K) := \sum_{[\alpha] \in \hat{U}(0)} \varrho(\alpha),$$

where $\hat{U}(0) \subset H_0(\mathbb{K}^{-1}(\mathbb{D}))$ is the initial restricted well group associated to the map K with the subset $\mathbb{D} \subset \mathbb{R}^3 \times \mathbb{R}^3$.

Stability Result for Robustness of Self-Intersection points

Combining:

- Theorem A – Stability of restricted well diagrams
- Lemma 8,9 – Diagonal perturbations yield restricted well groups

Now the stability for the robustness of curves and knots follows.

Theorem B (Stability Results for Robustness of Self-intersection Points of Curves and Knots)

Denote by $I(X) < \infty$ the number of self-intersections of a curve(or knot) X. Then for two curves(or knots) X_1 and X_2 with possibly finitely many self- intersection points, the following inequality holds:

$$|\varrho(X_1) - \varrho(X_2)| \leq 2 \max \left(I(X_1), I(X_2) \right) \|X_1 - X_2\|_\infty.$$

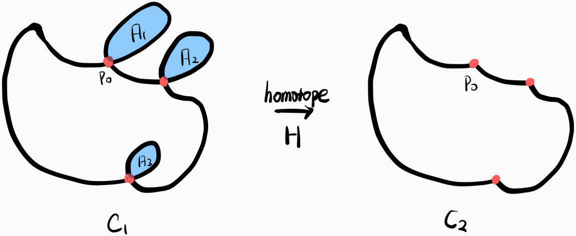
Future Directions

Minimum Homotopy Area

Definition

Given C_1, C_2 two curves based at p_0, we define the **minimum homotopy area** $\sigma(C_1, C_2)$ as the infimum of the homotopy area over all piecewise differentiable based homotopies between C_1 and C_2:

$$\sigma(C_1, C_2) = \inf_H \mathrm{Area}(H).$$



$$\sigma(C_1, C_2) = A_1 + A_2 + A_3.$$

Conjecture 1: Relation to Minimum Homotopy Area

Conjecture 1

Let $C : [0,1] \to \mathbb{R}^2$ be a piecewisely smooth curve, possibly with finitely many self-intersections. Then we have the following approximation:

$$\inf_{C' \simeq C} \sigma(C, C') = K\varrho(C)^2 + B$$

where C' ranges over the piecewise smooth **simple***(i.e. without self-intersection) curves homotopic to C, and $K, B \geq 0$ are constants.*

Though there is a polynomial-time algorithm for calculating minimum homotopy area of two "generic" curves [FKW17], **there has not been the stability results for minimum homotopy area between two curves.**

Conjecture 2: Compatibility between robustness of curves and that of knots

On the other hand, we conjecture the robustness $\varrho(K)$ of a 3d-knot goes well with its *projectivized approximation*. Namely,

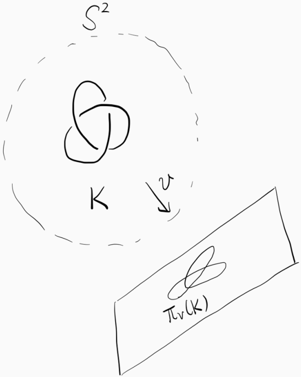
Conjecture 2

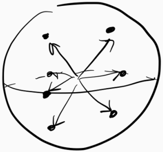
The robustness of knots is well approximated by its projectivized approximation. That is, given a 3d knot K, and $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-dense sampling $\mathcal{V}$ of unit vectors in the unit ball S^2, we have

$$\left| \sum_{v \in \mathcal{V}} \varrho(\pi_v(K)) A_v - \varrho(K) \right| < \varepsilon,$$

where $A_v := \inf_{\substack{w \neq v \\ w \in \mathcal{V}}} \|v - w\|^2.$

Illustration of Conjecture 2





$\mathcal{V}$ = Sample of vectors on S^2.

$$\boxed{\sum_{v\in\mathcal{V}} \varrho(\pi_v(K))A_v \quad \sim \quad \varrho(K)}$$

References

[CS10] Gunnar Carlsson and Vin de Silva (2010)
Zigzag persistence
Found. Comput. Math. 10, no. 4, 367405. MR 2657946

[FKW17] Brittany Terese Fasy, Selcuk Karakoc, and Carola Wenk (2017)
On minimum area homotopies of normal curves in the plane
arXiv preprint arXiv:1707.02251

[A10]Amit Patel (2010)
Reeb spaces and the robustness of preimages
Duke University

Any Questions?