CS6170 FINAL PROJECT REPORT

SANGHOON KWAK(U1262238)

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1. Team Members

Sanghoon Kwak (u1262238)

2. Introduction

Given that conventional knot invariants are expensive for computation, we would like to establish computationally efficient topological descriptor for knots. In this regard, we use already established notion of robustness of space introduced by Amit Patel [3] to define **the robustness of self intersection points** of knots. However, the robustness of self-intersection points is not well defined using the original definition by Patel as the definition of perturbations of functions is too *rigid*. Motivated from this observation, we tweak the definition of perturbations and generalize the definition of well modules, and establish the same stability result by Patel for generalized well modules. From this new definition and results, we could derive the well-defined notions of robustness of self-intersection points of curves/knots, and prove their stability result.

Thereafter, we would like to find the potential relationship of the robustness of self intersection points to the minimum homotopy area of curves, and see how the robustness of self intersection points of 3d knots is well approximated by the average of that of projected 2d knots.

In Section 3, we summarize Patel's paper [3] to introduce the definition of robustness and the collection of essential notions toward the established stability result of robustness. In Subsection 4.1, we start by generalizing the notion of well groups by restricting the class of perturbations, and establish the same set-up as for original stability result, culminating to prove the stability result for the generalized well groups. In Subsection 4.2, we define the robustness of curves/knots, and we will apply the generalized stability theorem in Subsection 4.1 to the self-intersections of curves/knots and establish the related stability result. Finally, in Section 5, we provide two conjectures inspired from our two main results in Section 4.

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3. Background and Related Work

To define what robustness is, we need to define some preliminary notions.

Definition 1. Given X a manifold and Y a Riemannian manifold, a map $h : X \to Y$ homotopic to f is a **ρ-perturbation** of $f : X \to Y$ if they differ by at most ρ with the sup norm: $\|h - f\|_\infty \leq \rho$.

For a submanifold $A \subset Y$, denote by $f_A : X \to \mathbb{R}$ the distance function $f_A(x) := \inf_{a \in A} \|f(x) - a\|$. These induced maps have the following stability under sup norm of original maps:

Lemma 2 (Distance Lemma)**.** *Let $f_A, g_A : X \to \mathbb{R}$ be the distance functions induced by $f, g : X \to Y$ with the submanifold $A \subset Y$. Then*

$$\|f_A - g_A\|_\infty \leq \|f - g\|_\infty.$$

Sketch of Proof. This follows from the definition of the infimum and triangle inequality with respect to the metric on Y. $\square$

Since the image of ρ-perturbation h of f must be contained in the ρ-neighborhood of A, it follows that

$$h_A^{-1}(0) \subset f_A^{-1}([0, \rho]).$$

This inclusion induces a homomorphism between homology groups:

$$j_h : H_*(h_A^{-1}(0)) \longrightarrow H_*(f_A^{-1}([0, \rho])) =: F(\rho),$$

where we denoted by $F(\rho)$ the latter homology group for brevity. Finally, we denote by $f(r, s)$ the inclusion-induced homomorphism $f(r, s) : F(r) \longrightarrow F(s)$ for every $r \leq s$. Now we can define the well group of the growing preimage $f_A^{-1}([0, r])$ as follows:

Definition 3. The **well group** of $f_A^{-1}([0, r])$ is the largest subgroup $U(r) \subset F(r)$ such that the image of the composition $U(r) \hookrightarrow F(r) \overset{f(r,r+\delta)}{\longrightarrow} F(r + \delta)$ is contained in

$$\bigcap_{h:X\to Y} \mathrm{Im} j_h \subset F(r + \delta)$$

where h ranges over all $(r + \delta)$-perturbations of f, and for sufficiently small $\delta > 0$.

With this definition, we have the following important property on well groups.

Lemma 4 (Shrinking Wellness Lemma)**.** *For each $0 \leq r \leq s$, we have*

$$U(s) \subset f(r, s)(U(r)).$$

Sketch of Proof. This follows from the observation that the set of ρ-perturbations of f gets larger as ρ is bigger, making $\bigcap_h \mathrm{Im} j_h$ smaller. $\square$

Since the homomorphism cannot increase the rank of the group, it follows that the only way the well group can change is by lowering its rank. Hence if we assume that $f : X \to Y$ is *admissible*, namely $F(0) \supset U(0)$ being of finite rank, there are only finitely many $u_1 < u_2 < \ldots < u_l$, called **terminal critical values** at which the rank of $U(r)$ decreases. We extend this sequence by adding $u_0 = 0$ and $u_{l+1} = \infty$. Denote by $U_i := U(u_i)$ and $F_i := F(u_i)$ for each $i = 0, \ldots, l + 1$. We also write $f_{i,j} : F(i) \longrightarrow F(j)$ as the short hand for the inclusion induced homomorphism $f(u_i, u_j)$.

However, the well groups do not have a nice sublevel set filtrations with those $\{u_i\}$ as homological groups do. Instead, they form a **zigzag module**, constructed as follows. Let $K_i = U_i \cap \ker f_{i,i+1}$ and consider the quotient group $Q_i = U_i/K_i$. Then define the forward map $a_i : U_i \to Q_i$ as the natural quotient map, and then it is obviously surjective. Define the backward map $b_i : U_{i+1} \to Q_i$ as $\eta \mapsto \xi + K_i$, where $\xi \in U_i$ satisfies $f_{i,i+1}(\xi) = \eta$. Here we remark that b_i is well-defined by Shrink Wellness Lemma (Lemma 4) because $U_{i+1} \subset f_{i,i+1}(U_i)$. Also, by the First Isomorphism

Theorem $Q_i \overset{\phi}{\cong} \mathrm{Im}(f_{i,i+1}|_{U_i}) \supset U_{i+1}$ and b_i is the restriction of above isomorphism ϕ^{-1} to U_{i+1}, so it is injective. The following is a schematic diagram for zigzag module for U_i.

$$\ldots \xrightarrow{a_{i-1}} Q_{i-1} \xleftarrow{b_{i-1}} U_i \xrightarrow{a_i} Q_i \xleftarrow{b_i} U_{i+1} \xrightarrow{a_{i+1}} Q_{i+1} \xleftarrow{b_{i+1}} \ldots$$

In fact, having surjective a_i's and injective b_j's is equivalent to the fact that the birth only happens at U_0 from the general construction of zigzag module in [1].

From now on, to avoid the confusion with the birth and death in homology groups, we replace the terms *birth/death* in well group U_i with **get well/fall ill**. Now note that the drop in rank from U_{i-1} to U_i is

$$\mu_i := \mathrm{rank}(\ker a_{i-1}) + \mathrm{rank}(\mathrm{coker}\, b_{i-1}).$$

Since all classes get well at U_0, we can define the **well diagram** $\mathrm{Dgm}(U)$ as just the multiset of singletons u_i with multiplicities μ_i, together with the infinitely many copies of 0. Hence, the diagram itself is a multiset of values on the extended real $\overline{R} = \mathbb{R} \cup \{+\infty\}$. The following lemma suggests that the bottleneck distance between two well diagrams can be found by performing the inversion-free matching.

Lemma 5 (Matching Lemma)**.** *Let*

$$0 \le u_1 \le u_2 \le \ldots \le u_M$$
$$0 \le v_1 \le v_2 \le \ldots \le v_M$$

be the terminal critical values(possibly padding 0's to match the number M of values) associated with the maps $f, g : X \to Y$ with the same submanifold $A \subset Y$. Then the bottleneck distance between $\mathrm{Dgm}(U)$ *and* $\mathrm{Dgm}(V)$ *can be calculated as:*

$$W_\infty(\mathrm{Dgm}(U), \mathrm{Dgm}(V)) = \max_{1 \le i \le M} |u_i - v_i|.$$

Sketch of Proof. The proof relies on the purely combinatorial fact on the finitely many points on $\mathbb{R}$; It uses the fact that if $a \le b$ and $c \le d$, then $|a-c| + |b-d| \le |a-d| + |b-c|$. □

Finally, we can define the robustness of each class in U_0 as follows:

Definition 6. for each homology class α in U_0, we can define the **robustness** $\varrho(\alpha)$ of α as the value u_i when α falls ill.

In other words, the robustness quantifies the stability of each homology class under perturbations of the map f. Observe that the robustness for well groups is a counterpart notion of the persistence for homology groups.

The property of the forward/backward maps being surjective/injective respectively enables us to have the **compatible bases** of U_0. By compatibility, it means that the bases of U_0 induced from the following decomposition

$$U_0 \cong \ker a_0 \oplus \ldots \oplus \ker a_{l+1} \oplus \mathrm{coker}\, b_l \oplus \ldots \oplus \mathrm{coker}\, b_0$$

automatically *determines* the bases of the following **left filtration** of the zigzag module for U_i's:

$$0 \longrightarrow A_0 \longrightarrow \ldots \longrightarrow A_{l+1} = B_{l+1} \longrightarrow \ldots \longrightarrow B_0 = U_0,$$

where

$$A_i := u_{0,i}^{-1}(K_i) \qquad B_i = u_{0,i}^{-1}(U_i),$$

and $u_{0,i} := f_{0,i}|_{U_0}$. This is because $A_i/A_{i-1} \cong \ker a_i$ and $B_i/b_{i+1} \cong \mathrm{coker}\, b_i$ for all i, so the bases for the $\ker a_i$ and $\mathrm{coker}\, b_i$'s determine the bases of each of A_i and B_i's.

Now we introduce the key lemma toward the stability of well diagrams. Note by Distance Lemma(Lemma 2), letting $\|f - g\|_\infty = \varepsilon$ gives $\|f_A - g_A\| \le \varepsilon$, so

$$g_A^{-1}([0,r]) \subset f_A^{-1}([0, r+\varepsilon])$$

This induces the **bridge map** $\mathcal{B}_r : G(r) \longrightarrow F(r+\varepsilon)$, where $G(r)$ is the homology group $H_*(g_A^{-1}([0,r]))$, which is dual to $F(r)$. Denoting by $V(r) \subset G(r)$ the well group associated to g, we have

Lemma 7 (Bridge Lemma)**.**

$$U(r+\varepsilon) \subset \mathcal{B}_r(V(r))$$

Sketch of Proof. This follows from the fact that $(r+\varepsilon+\delta)$-perturbations of f include the $(r+\delta)$-perturbations of g by Distance Lemma. ☐

Similar to Shrink Wellness Lemma (Lemma 4), Bridge Lemma allows us to construct a new zigzag module out of two well groups U, V:

$$V(r) \longrightarrow Q(r) \longleftarrow U(r+\varepsilon),$$

where $Q(r) := V(r)/(V(r) \cap \ker \mathcal{B}_r)$. Here again the forward maps are surjective and the backward maps are injective. Parallel to earlier discussions, we can construct the left filtration out of this zigzag module, and their compatible basis.

Specifically, in the proof of stability theorem we set $r = 0$ and construct a new left filtration W by landing the bridge from $G(0)$ to $F(\varepsilon)$. Then it follows that *the two quotient groups $U(\varepsilon)$ and $W(\varepsilon)$ are decomposed by the kernels and cokernels in parallel, so that their bases are compatible.*

Now we give the main theorem of [3].

Theorem 8 (Stability Theorem for Well Diagrams)**.** *Let U, V be the well modules of the functions f_A, g_A defined by admissible, homotopic mappings $f, g : X \to Y$, where $A \subset Y$ is a submanifold. Then*

$$W_\infty(\mathrm{Dgm}(U), \mathrm{Dgm}(V)) \leq \|f-g\|_\infty.$$

Sketch of Proof. The proof consists of the following steps:

- Let $\varepsilon := \|f-g\|_\infty$. Toward Matching Lemma (Lemma 5), we match the points $u \leq \varepsilon$ in $\mathrm{Dgm}(U)$ with copies of 0 in $\mathrm{Dgm}(V)$.
- For the rest of the points in $\mathrm{Dgm}(U)$, we will match them using the parallel module W to U constructed as above. To be precise, for each $\alpha \in U(0)$, we can pick $\beta \in W(0) = V(0)$ so that the images of α, β in $U(\varepsilon) = W(\varepsilon)$ coincide using the parallel bases.
- Setting the robustness of α to be r, then $r > \varepsilon$ by our first step. Then construct another auxiliary well module X from U, V by landing two bridges to argue that β falls ill within the interval $[r-\varepsilon, r+\varepsilon]$. Here we need to use Bridge Lemma (Lemma 7) twice.
- Then the matching (α, β) gives radii difference $\leq \varepsilon$, so by Matching Lemma (Lemma 5), this gives

$$W_\infty(\mathrm{Dgm}(U), \mathrm{Dgm}(V)) \leq \varepsilon = \|f-g\|_\infty.$$ ☐

4. Results

4.1. Result (1): Generalization to *Restricted* well groups.

We first generalize the definition of well group by allowing restriction on the choice of perturbations.

Definition 9 (Restricted Well Group)**.** Let $f : X \to Y$ and $A \subset Y$ be a submanifold. Let $\mathcal{F}_\rho$ be a nonempty subcollection of the ρ-perturbations of f. We say the parameterized collection of perturbations $\{\mathcal{F}_\rho\}_{\rho \geq 0}$ is said to be **nested** if it satisfies the following nested property:

$$\mathcal{F}_\rho \subset \mathcal{F}_{\rho'} \qquad \text{for every } \rho \leq \rho'.$$

For a nested collections $\{\mathcal{F}_\rho\}_{\rho\geq 0}$ of perturbations of f, the $\{\mathcal{F}_\rho\}$**-restricted well group** of $f_A^{-1}([0,r])$ is defined to be the largest subgroup $\hat{U}(r) \subset F(r)$ such that the image of the composition of $\hat{U}(r) \hookrightarrow F(r) \overset{f(r,r+\delta)}{\longrightarrow} F(r+\delta)$ is contained in

$$\bigcap_{h\in\mathcal{F}_{r+\delta}} \mathrm{Im} j_h,$$

for a sufficiently small $\delta > 0$.

We remark here that since there is a fewer collection of perturbations, the restricted well groups are *larger* than the usual well groups: $U(r) \subset \hat{U}(r)$.

From now on, we will see the restricted well groups still have all the constructions and results of the regular well groups. Recall $f(r,s) : H(F(r)) \to H(F(s))$ is the homomorphism induced by inclusion $F(r) \subset F(s)$ for $r \leq s$.

Lemma 10 (Shrinking Wellness Lemma for restricted well groups)**.** *For each* $0 \leq r \leq s$*, we have*

$$\hat{U}(s) \subset f(r,s)(\hat{U}(r)).$$

Proof. By the nested property of parameterized collection of perturbations, the observation for Lemma 4 still holds: The set $\mathcal{F}_\rho$ of restricted perturbations of f gets larger as ρ is bigger, making $\bigcap_{h\in\mathcal{F}_\rho} \mathrm{Im} j_h$ smaller. $\square$

We can construct zigzag modules for restricted well groups as before:

$$\ldots \xrightarrow{\hat{a}_{i-1}} \hat{Q}_{i-1} \xleftarrow{\hat{b}_{i-1}} \hat{U}_i \xrightarrow{\hat{a}_i} \hat{Q}_i \xleftarrow{\hat{b}_i} \hat{U}_{i+1} \xrightarrow{\hat{a}_{i+1}} \hat{Q}_{i+1} \xleftarrow{\hat{b}_{i+1}} \ldots$$

where we define $\hat{Q}_i$ to be the quotient group $\hat{U}_i/\hat{K}_i$, and $\hat{K}_i := \hat{U}_i \cap \ker f_{i,i+1}$. As before, define the forward maps $\hat{a}_i$ as quotient map and the backward maps $\hat{b}_j$ as the restricted map of the inverse of the map given by First Isomorphism Theorem. Thanks to the fact that Shrinking Wellness Lemma also holds for restricted well groups, $\hat{a}_i$'s are surjective, and $\hat{b}_j$'s are injective. Thus, all classes *get well* at $\hat{U}_0$. Still the drop in rank from $\hat{U}_{i-1}$ to $\hat{U}_i$ is

$$\hat{\mu}_i := \mathrm{rank}(\ker \hat{a}_{i-1}) + \mathrm{rank}(\mathrm{coker}\, \hat{b}_{i-1}),$$

and from this we can construct the well diagram for $\hat{U}$ as the multiset of values u_i with multiplicities $\hat{\mu}$, denoting it as $\mathrm{Dgm}(\hat{U})$, together with infinitely many copies of 0. Similarly, for each homology class $[\alpha] \in \hat{U}(0)$, we define the **robustness** $\varrho(\alpha)$ of α to be the value u_i when α falls ill.

We remark that the **Matching Lemma** still holds for restricted well modules, as the proof relied only on the combinatorial facts of the points on $\mathbb{R}_{\geq 0}$. Also, by Shrinking Wellness Lemma, we have **compatible left filtration basis** for $\hat{U}_0$ as before.

Finally, let us establish the Bridge Lemma for restricted well modules. Let $f, g : X \to Y$ be the maps with the same submanifold $A \subset Y$. First, by Distance Lemma (Lemma 2) we still have the same **bridge map** $\mathcal{B}_r : G(r) \to F(r+\varepsilon)$ as before. Since we will have different parameterized collections of perturbations for f and g, we need to ensure they are *compatible* each other to establish Bridge Lemma.

Definition 11 (ε-compatibility)**.** Let $f, g : X \to Y$ and $A \subset Y$ be a submanifold. Let $\{\mathcal{F}_\rho\}_{\rho\geq 0}$ and $\{\mathcal{F}'_\rho\}_{\rho\geq 0}$ be parameterized collections of perturbations of f and g, respectively. Then $\{\mathcal{F}_\rho\}_{\rho\geq 0}$ and $\{\mathcal{F}'_\rho\}_{\rho\geq 0}$ are said to be ε**-compatible** if

$$\mathcal{F}_\rho \subset \mathcal{F}'_{\rho+\varepsilon}, \quad \text{and} \quad \mathcal{F}'_\rho \subset \mathcal{F}_{\rho+\varepsilon}$$

for every $\rho \geq 0$. We say two restricted well modules are ε**-compatible** if their associated nested parameterized collections of perturbations are ε-compatible.

With this additional condition, Bridge Lemma holds for restricted well modules as well.

Lemma 12 (Bridge Lemma for Restricted Well Module)**.** *For ε-compatible restricted well modules $\hat{U}$ and $\hat{V}$, we have*

$$\hat{U}(r+\varepsilon) \subset \mathcal{B}_r(\hat{V}(r)).$$

Proof. The proof follows from the definition of ε-compatibility because for each $\delta > 0$, we have $\mathcal{F}_{r+\varepsilon+\delta} \supset \mathcal{F}'_{r+\delta}$. Hence, the intersection of the images of restricted $(r+\varepsilon+\delta)$-perturbations of f is contained in the intersection of the images of restricted $(r+\delta)$-perturbations of g, concluding the proof by definition of restricted well groups. $\square$

Using Bridge Lemma, we can obtain new modules with parallel compatible bases as before, by landing bridges between two different well modules. Given that all the settings for restricted well modules have been established, now we can state our version of stability theorem:

Theorem A (Stability Theorem for Well Diagrams for Restricted Well Modules)**.** *Let $\hat{U}, \hat{V}$ be the restricted well modules of the functions f_A, g_A defined by admissible, homotopic mappings $f, g : X \to Y$, where X, Y and $A \subset Y$ is a submanifold. Assume further $\hat{U}, \hat{V}$ are $\|f-g\|_\infty$-compatible. Then*

$$W_\infty(\mathrm{Dgm}(\hat{U}), \mathrm{Dgm}(\hat{V})) \leq \|f-g\|_\infty$$

Proof. The proof is the verbatim for the proof of Theorem 8, because it the proof consists of:

- Matching Lemma
- Bridge Lemma and Existence of Compatible Bases for Left Filtration: For construction of new modules $\hat{X}, \hat{W}$ out of $\hat{U}, \hat{V}$.
- Bridge Lemma: For getting the ill radii interval for β as in the proof of Theorem 8.

all of which we have been already established for restricted well modules. $\square$

4.2. **Result (2): Application – Stability theorem for robustness of self-intersection of curves/knots.** Now let us apply the aforementioned stability result (Theorem A) to the self-intersection points of curves/knots. To define the robustness of self intersection points, we construct our choice of parameterization for perturbations as follows:

Definition 13. Let $f : X \to Y$ be a map with $A \subset Y$ a submanifold. Consider a diagonal map $(f, f) : X \times X \to Y \times Y$. For each $\rho \geq 0$, define $\mathcal{F}_\rho$ as:

$$\mathcal{F}_\rho := \{(g, g) : X \times X \to Y \times Y | \quad \|f-g\|_\infty \leq \rho\},$$

and call $\mathcal{F}_\rho$ as the **diagonal ρ-perturbation** of (f, f).

Observe that $(f, f) \in \mathcal{F}_\rho$ for every $\rho \geq 0$, so it is nonempty. Also, it is obvious by construction $F_\rho \subset F_{\rho'}$ for $\rho \leq \rho'$. Hence, we can conclude that

Lemma 14. *The parameterized collection of diagonal perturbations are **nested**.*

Also, we have compatibility for multiple parameterized collections of diagonal perturbations:

Lemma 15. *Two parameterized collections of diagonal perturbations of f and g are $\|f-g\|_\infty$-compatible.*

Proof. Say $\mathcal{F}$ and $\mathcal{F}'$ are the two parameterized collections of f and g respectively. Let $\varepsilon := \|f-g\|_\infty$. It suffices to show $\mathcal{F}_\rho \subset \mathcal{F}'_{\rho+\varepsilon}$ are ε-compatible. Pick $(h, h) \in F_\rho$. Then by triangle inequality,

$$\|h-g\|_\infty \leq \|h-f\|_\infty + \|f-g\|_\infty \leq \rho + \varepsilon,$$

so $h \in \mathcal{F}'_{\rho+\varepsilon}$. $\square$

Now we can define the robustness of self-intersection points of curves and knots with the restricted well groups associated with the diagonal perturbations as follows.

Definition 16 (Definition of robustness of self-intersection points of 2d curves)**.** Let $c : S^1 \to \mathbb{R}^2$ be a curve, with possibly finitely many self-intersection points. Consider the diagonal map $C = (c, c) : S^1 \times S^1 \to \mathbb{R}^2 \times \mathbb{R}^2$, and the set of diagonal points $\mathbb{D} = \{(p, p)\}_{p \in \mathbb{R}^2} \subset \mathbb{R}^2 \times \mathbb{R}^2$. Define the **robustness** $\varrho(c)$ **of the curve** c as the sum of the robustness of *restricted* well group classes corresponding to the self-intersecting points, i.e.,

$$\varrho(c) := \sum_{[\alpha] \in \hat{U}(0)} \varrho(\alpha),$$

where $U(0) \subset H_0(C^{-1}(\mathbb{D}))$ is the initial restricted well group associated to the map C with the subset $\mathbb{D} \subset \mathbb{R}^2 \times \mathbb{R}^2$.

Since c is assumed to have at most finitely many self-intersecting points, $\hat{U}(0) \subset F(0)$ has finite rank. Hence the above sum for $\varrho(c)$ is indeed a finite sum, so $\varrho(c)$ is well-defined. Note $C^{-1}(\mathbb{D})$ represents the self-intersecting points of c, together with infinitely many diagonal points in $S^1 \times S^1$.

By replacing the 2d-curve c with the 3d-knot K, we can similarly define the robustness of self-intersection points for knots:

Definition 17 (Definition of robustness of self-intersection points of 3d knots)**.** Let $K : S^1 \to \mathbb{R}^3$ be a knot, with possibly finitely many self-intersection points. Consider the diagonal map $\mathbb{K} = (K, K) : S^1 \times S^1 \to \mathbb{R}^3 \times \mathbb{R}^3$, and the diagonal points $\mathbb{D} = \{(p, p)\}_{p \in \mathbb{R}^3} \subset \mathbb{R}^3 \times \mathbb{R}^3$. Define the **robustness** $\varrho(K)$ **of the knot** K as the sum of the robustness of *restricted* well group classes corresponding to the self-intersecting points, i.e.,

$$\varrho(K) := \sum_{[\alpha] \in \hat{U}(0)} \varrho(\alpha),$$

where $\hat{U}(0) \subset H_0(\mathbb{K}^{-1}(\mathbb{D}))$ is the initial restricted well group associated to the map K with the subset $\mathbb{D} \subset \mathbb{R}^3 \times \mathbb{R}^3$.

Now the stability for the robustness of curves and knots follows from the stability result for the well diagrams of restricted well modules(Theorem A), together with the compatibility result for parameterized collections of diagonal perturbations(Lemma 15).

Theorem B (Stability Results for Robustness of Self-intersection Points of Curves and Knots)**.** *Denote by $I(X) < \infty$ the number of self-intersections of a curve(or knot) X. Then for two curves(or knots) X_1 and X_2 with possibly finitely many self- intersection points, the following inequality holds:*

$$|\varrho(X_1) - \varrho(X_2)| \leq 2 \max\left(I(X_1), I(X_2)\right) \|X_1 - X_2\|_\infty.$$

Proof. It suffices to prove for the curves only because the proof for knots is verbatim by replacing the curves with the knots. Let $c_1, c_2 : S^1 \to \mathbb{R}^2$ be two curves, possibly with finitely many intersection points, and $\hat{U}$ and $\hat{V}$ be the restricted well modules associated to parameterizations of diagonal perturbations of (c_1, c_1) and (c_2, c_2) respectively, with the same diagonal set $\mathbb{D} \subset \mathbb{R}^2 \times \mathbb{R}^2$. Arrange the robustness $\varrho(\alpha)$ of each class $[\alpha] \in \hat{U}(0)$ and $\hat{V}(0)$ as:

$$\begin{gathered} 0 \leq u_1 \leq u_2 \leq \ldots \leq u_M \\ 0 \leq v_1 \leq v_2 \leq \ldots \leq v_M. \end{gathered}$$

where we add zeros to either $\{u_i\}$ or $\{v_i\}$ make sure we have two sequences of the same length, while removing zeros from both sequences until one of them has no zeros in the sequence. Since each nonzero robustness value corresponds to a pair $(a, b) \in S^1 \times S^1$ where $a \neq b$ but have the same image in the curve, (b, a) is another such pair with the the same robustness. Hence, there are two nonzero (same) robustness values for each self-intersection point of a curve. Therefore, it follows that $M = 2 \max(I(c_1), I(c_2))$.

Then by definition of robustness for curves and definition of bottleneck distance,

$$|\varrho(c_1) - \varrho(c_2)| = \left|\sum_{i=1}^{M} u_i - \sum_{i=1}^{M} v_i\right| \leq M \max_{i=1,\ldots,M} |u_i - v_i| = M \cdot W_\infty(\mathrm{Dgm}(\hat{U}), \mathrm{Dgm}(\hat{V})),$$

where we used triangle inequality, and the last equality follows from Matching Lemma. By Lemma 15, $\hat{U}$ and $\hat{V}$ are $\|c_1 - c_2\|_\infty$-compatible. Therefore, by Theorem A, we conclude

$$\begin{aligned} |\varrho(c_1) - \varrho(c_2)| &\leq M W_\infty(\mathrm{Dgm}(\hat{U}), \mathrm{Dgm}(\hat{V})) \leq M\|c_1 - c_2\|_\infty \\ &= 2\max(I(c_1), I(c_2))\|c_1 - c_2\|_\infty \end{aligned}$$

□

5. Insights: Two Conjectures

We would like to see how robustness of 2d curves is related to the similar notion *minimum homotopy area*, which is defined as follows:

Definition 18 ([2])**.** Given C_1, C_2 two curves based at p_0, and a homotopy H between C_1 and C_2, let $E_H : \mathbb{R}^2 \to \mathbb{Z}$ be defined as

$$E_H(x) := \left|\pi_0(H^{-1}(x))\right|,$$

namely, assign $x \in \mathbb{R}^2$ to the number of intermediate curves $H|_{[0,1]\times t}$ sweeps over x. Then define the homotopy area $\mathrm{Area}(H)$ of the homotopy H as the integral:

$$\mathrm{Area}(H) := \int_{\mathbb{R}^2} E_H(x)dx.$$

Finally, we define the **minimum homotopy area** $\sigma(C_1, C_2)$ as the infimum of the homotopy area over all piecewise differentiable based homotopies between C_1 and C_2:

$$\sigma(C_1, C_2) = \inf_H \mathrm{Area}(H).$$

Though there is a polynomial-time algorithm for calculating minimum homotopy area of two "generic" curves [2], *there has not been the stability results for minimum homotopy area between two curves to best of our knowledge.*

We give the following conjecture:

Conjecture 1. Let $C : [0,1] \to \mathbb{R}^2$ be a piecewisely smooth curve, possibly with finitely many self-intersections. Then we have the following approximation:

$$\inf_{C' \simeq C} \sigma(C, C') = A\varrho(C)^2 + B$$

where C' ranges over the piecewise smooth *simple*(i.e. without self-intersection) curves homotopic to C, and $A, B \geq 0$ are some constants. Furthermore, this infimum is achieved, namely there exists a piecewisely smooth *simple* curve C_0 homotopic to C, where

$$\sigma(C, C_0) = \inf_{C' \simeq C} \sigma(C, C').$$

If this conjecture were true, then we establish the stability result for minimum homotopy area as a direct corollary.

Switching a gear little bit, we want to see how the robustness $\varrho(K)$ of a 3d-knot is compatible with its *projectivized approximation.*

Conjecture 2. The robustness of knots is well approximated by its projectivized approximation. That is, given a 3d knot K, and $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-dense sampling $\mathcal{V}$ of unit vectors in the unit ball S^2, we have

$$\left|\sum_{v\in\mathcal{V}} \varrho(\pi_v(K))A_v - \varrho(K)\right| < \varepsilon,$$

where $A_v := \inf_{\substack{w\neq v\\ w\in\mathcal{V}}} \|v-w\|^2$.

6. Conclusions and Discussions

- **What is an overview of your project?** We established stability results for curves and knots generalizing the notion of robustness.
- **What are your project objectives?** Modify the classical definition of robustness to fit into the situation of self-intersecting points of curves/knots. Then prove its stability result with the new definition, and adapt it to prove the stability results for robustness of self-intersection points of curves/knots.
- **What questions does your project address?** Is there a way to define the well group in a new way to address the robustness of self-intersecting points of curves as intersections of $\mathrm{Im} j_h$ where h is of the form diagonal maps $C = (c, c)$? Is there a way to quantify the robustness of S^1-immersion in any dimension of Euclidean space in general?
- **What are the key insights based on your results?** Without having the full set of perturbations, we could establish the stability result for the well groups, with some restriction of hierarchy between the collections of perturbations.
- **What are future directions?** One direction is to establish stability result for minimum homotopy area for curves using our result on robustness. The other direction is to see if the robustness for knots can be approximated by its projectivized average over sampled S^2-projections.

References

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