Signal Processing
ENGN2228

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Second Semester

Lectures 16 and 17

Fourier Series Properties – Notation

Previously we established:

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

We adopt the shorthand

$$x(t) \xleftrightarrow{\mathscr{F}} a_k$$

and use this extensively.

Dirichlet Conditions

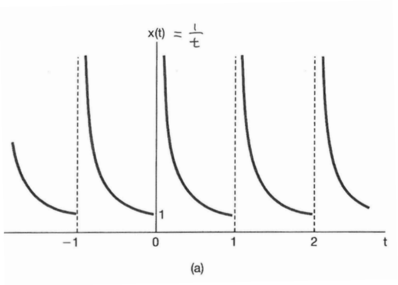
Fourier believed that any periodic signal could be expressed as a sum of sinusoids. However, this turned out not to be the case, although virtually all periodic signals arising in engineering do have a Fourier series representation. In particular, a periodic signal $x(t)$ has a Fourier series if it satisfies the following Dirichlet conditions:

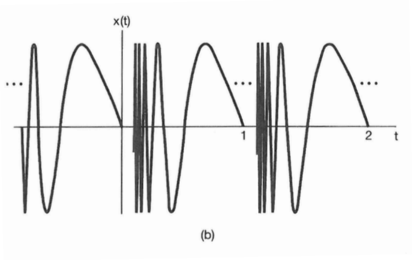
1. $x(t)$ is absolutely integrable over any period; that is,

$$\int_{T}^{a+T} |x(t)|\, dt < \infty \quad \text{for any } a$$

2. $x(t)$ has only a finite number of maxima and minima over any period.
3. $x(t)$ has only a finite number of discontinuities over any period.

Dirichlet Conditions (cont'd)





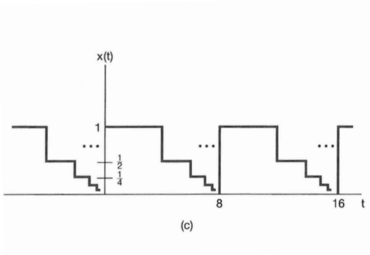


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

Fourier Series Properties – Motivation

What's coming:

- We're lazy, we don't want to do work if we can avoid it. We don't want to do something complicated if there is an easy way or a trick available.
- Computing Fourier Series is not hard but can be tedious.
- With a few Fourier Series we can synthesize others. A signal derived from an original signal via a simply transformation such as time-shift, scaling, compression, etc., should have a Fourier Series some how related to the original signal.
- There are many of these, see Table 3.1 in text.

Motivation – Motivation (cont'd)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Fourier Series Properties – Linearity

If

$$x(t) \xleftrightarrow{\mathscr{F}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\mathscr{F}} b_k$$

then

$$\boxed{\alpha\, x(t) + \beta\, y(t) \xleftrightarrow{\mathscr{F}} \alpha\, a_k + \beta\, b_k}$$

> *"Linear combinations of signals leads to identical linear combinations of the Fourier coefficients."*

For example, this can be used with parallel connection of signals.

$$x(t) \text{ real valued} \quad \Rightarrow \quad \boxed{a_{-k} = \overline{a_k} \equiv a_k^* \quad \text{(conjugate)}}$$

"Negative index Fourier coefficients are the complex conjugates of the positive index Fourier coefficients whenever the time domain signal is real valued (zero imaginary part)."

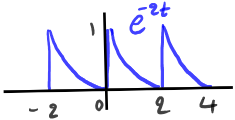
This implies for real valued signals

$$\mathit{Re}\{a_k\} \text{ is even,} \quad \mathit{Im}\{a_k\} \text{ is odd}$$

$$|a_k| \text{ is even,} \quad \angle a_k \text{ is odd}$$

Note that with any complex number written as $b\,e^{j\,\omega}$ with real $b > 0$ and angle ω, then $\angle b\,e^{j\,\omega} = \omega$, and $|b\,e^{j\,\omega}| = b$.

Example:



$$x(t) = e^{-2t} \quad 0 \leq t \leq 2$$

$$T = 2\,s, \quad \omega_0 = \frac{2\pi}{T} = \pi \text{ rad/s}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{1}{2}\int_0^2 e^{-2t}\,dt$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\Big|\, e^{-2t}\Big|_0^2$$

$$= \frac{1}{4}\left[1 - e^{-4}\right] = 0.245$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_0^2 e^{-2t}\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt$$

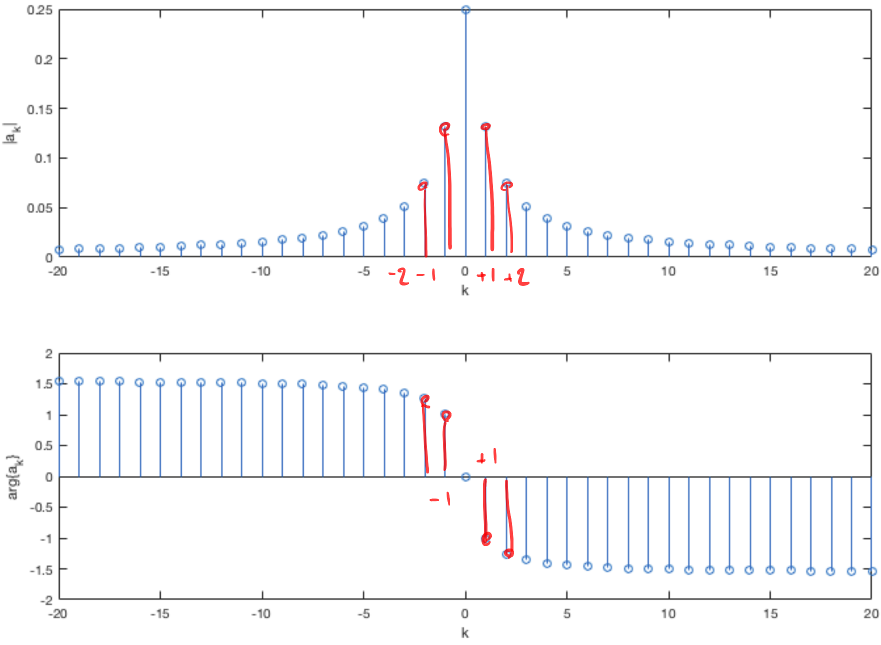
$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2+jk\pi}\right)\Big|\, e^{-(2+jk\pi)t}\Big|_0^2$$

$$= -\frac{1}{4+2jk\pi}\left[e^{-(4+j2\pi k)} - e^{0}\right]$$

$$= \frac{1}{4+2jk\pi}\left[1 - e^{-4}\underbrace{e^{-jk2\pi}}_{=1}\right]$$

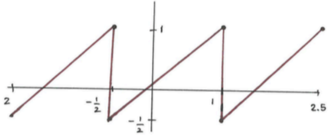
$$a_k = \frac{1 - e^{-4}}{4 + j2k\pi}$$

$\boxed{\text{From lecture 15}}$



|a_k|
0.25
0.2
0.15
0.1
0.05
0
-20
-15
-10
-5
0
5
10
15
20
k
-2 -1
+1 +2
arg{a_k}
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
+1
-1
k

Example Sawtooth Waveform



$$x(t) = t \qquad -\tfrac{1}{2} \le t \le 1$$

$$T = \tfrac{3}{2} \text{ sec}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{4\pi}{3} \text{ rad/s}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{2}{3}\int_{-\frac{1}{2}}^{1} t\,dt$$

$$= \frac{2}{3}\left|\frac{t^2}{2}\right|_{-\frac{1}{2}}^{1}$$

$$= \frac{2}{3}\left(\frac{1}{2} - \frac{1}{8}\right)$$

$$= \frac{1}{4}$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{2}{3}\int_{-\frac{1}{2}}^{1} t\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{2}{3}\left|\frac{e^{-jk\omega_0 t}(1 + jkt\omega_0)}{k^2\omega_0^2}\right|_{-\frac{1}{2}}^{1}$$

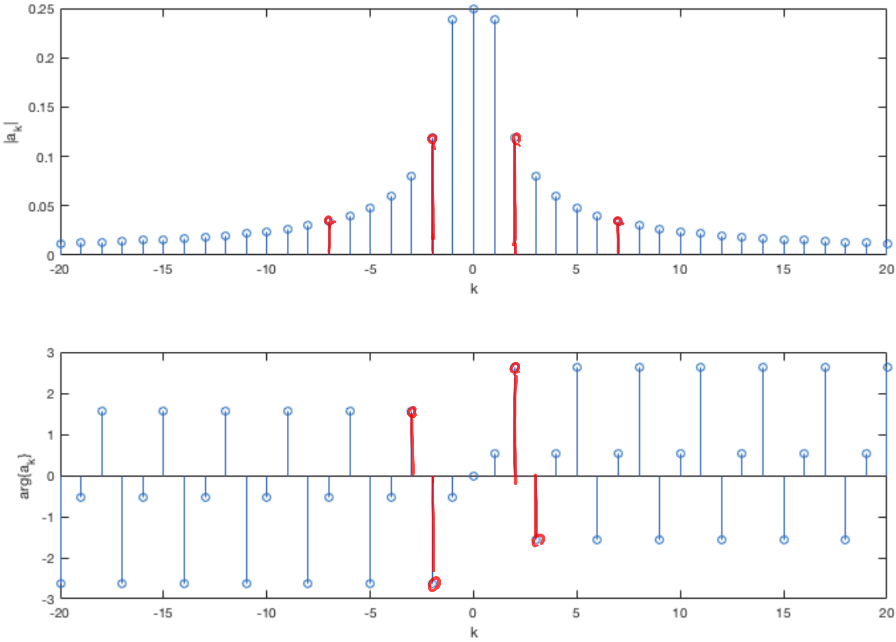
$$= \frac{2e^{-jk\omega_0}(1 + jk\omega_0)}{3k^2\omega_0^2} + \frac{j e^{\frac{jk\omega_0}{2}}(2j + k\omega_0)}{3k^2\omega_0^2}$$

$$= \frac{2}{3k^2\omega_0^2}e^{-jk\omega_0} - \frac{2}{3jk\omega_0}e^{-jk\omega_0} - \frac{2}{3k^2\omega_0^2}e^{\frac{jk\omega_0}{2}} - \frac{1}{3jk\omega_0}e^{\frac{jk\omega_0}{2}}$$

$$= \frac{2}{3k^2\omega_0^2}\left(e^{-jk\omega_0} - e^{\frac{jk\omega_0}{2}}\right) - \frac{2}{3jk\omega_0}\left(e^{-jk\omega_0} + \frac{1}{2}e^{\frac{jk\omega_0}{2}}\right)$$

Plot using Matlab

From lecture 15



0.25
0.2
0.15
0.1
0.05
0
|a_k|
-20
-15
-10
-5
0
5
10
15
20
k
3
2
1
0
-1
-2
-3
arg{a_k}
-20
-15
-10
-5
0
5
10
15
20
k

Given

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k$$

then

$$\boxed{x(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow} a_k\, e^{-jk\omega_0 t_0} \equiv a_k\, e^{-jk\,2\pi t_0/T}}$$

"Time shifting a signal introduces a linear phase shift $\propto t_0$ in the Fourier coefficients."

Fourier Series Properties – Parseval's Relation

It can be shown:

$$\boxed{\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2}$$

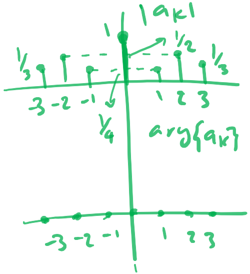
"Average ~~energy~~ power is the time domain is the same as the ~~energy~~ power in the frequency domain."

Example:

$$x(t) = 1 + \tfrac{1}{2}\cos 2\pi t + \cos 4\pi t + \tfrac{2}{3}\cos 6\pi t$$

$\omega_{o_1} = 2\pi \qquad \omega_{o_2} = 4\pi \qquad \omega_{o_3} = 6\pi$

$T_{o_1} = 1 \qquad T_{o_2} = \tfrac{1}{2} \qquad T_{o_3} = \tfrac{1}{3}$



Hence, $T_o = 1\,s$ so $\omega_o = 2\pi$ rad/s

We can write:

$$\underbrace{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t}}\quad \text{(I)}$$

$$x(t) = 1 + \tfrac{1}{2}\left[\frac{e^{j2\pi t} + e^{-j2\pi t}}{2}\right] + \left[\frac{e^{j4\pi t} + e^{-j4\pi t}}{2}\right] + \tfrac{2}{3}\left[\frac{e^{j6\pi t} + e^{-j6\pi t}}{2}\right]$$

$$\underbrace{= 1 + \tfrac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \tfrac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \tfrac{1}{3}\left[e^{j6\pi t} + e^{-j6\pi t}\right]}\quad \text{(II)}$$

I & II $\Rightarrow$ $a_0 = 1$ $\qquad a_2 = a_{-2} = \tfrac{1}{2}$ $\qquad a_k = 0$ other ks

$a_1 = a_{-1} = \tfrac{1}{4}$ $\qquad a_3 = a_{-3} = \tfrac{1}{3}$

$\boxed{\text{From lecture 15}}$

Average power?

$$x(t) = 1 + \tfrac{1}{2}\cos 2\pi t + \cos 4\pi t + \tfrac{2}{3}\cos 6\pi t$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$= \sum_{k=-3}^{3} |a_k|^2$$

$$= (1/3)^2 + (1/2)^2 + (1/4)^2 + (1)^2 + (1/4)^2 + (1/2)^2 + (1/3)^2$$

$$= \frac{133}{72} = 1.847$$

$$= \frac{1}{T}\int_T |x(t)|^2 dt$$

$$= \int_0^1 \left(1 + \tfrac{1}{2}\cos 2\pi t + \cos 4\pi t + \tfrac{2}{3}\cos 6\pi t\right) dt$$

$$= 1.84722$$

Given

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} b_k$$

then

$$\boxed{x(t)\, y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} c_k \triangleq \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell} = a_k \star b_k}$$

"The (pointwise) product of two periodic signals (of the same period, T) is the convolution of the Fourier coefficients."

For example, this can be used with serial/cascade connection of signals/systems.

Fourier Series Properties – Multiplication Property

With Fourier Series pairs (time-domain and frequency-domain) convolution in one domain is multiplication in the other.

At this point we have only considered the product in time-domain leading to convolution in frequency-domain.

$$
\begin{aligned}
x(t)\,y(t) &= \overbrace{\sum_{\ell} a_\ell\, e^{j\,\ell\omega_0 t}}^{x(t)}\, \overbrace{\sum_{m} b_m\, e^{j\,m\omega_0 t}}^{y(t)} \\
&= \sum_{\ell}\sum_{m} a_\ell b_m\, e^{j\,(\ell+m)\omega_0 t} \\
&= \sum_{k} \underbrace{\left(\sum_{\ell} a_\ell b_{k-\ell}\right)}_{c_k} e^{jk\omega_0 t} \qquad (\ell + m = k)
\end{aligned}
$$

From lecture 15

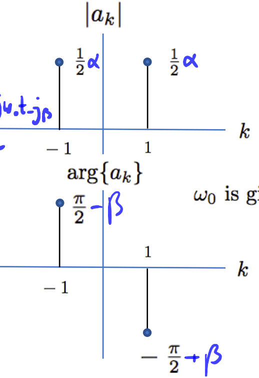
$$x(t) = \sin(\omega_0 t) \longrightarrow \alpha \sin(\omega_0 t + \beta)$$

$$= \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \longrightarrow \frac{\alpha}{2j}e^{j\omega_0 t + j\beta} - \frac{\alpha}{2j}e^{-j\omega_0 t - j\beta}$$

$$a_1 = \frac{1}{2j} = \frac{1}{2}e^{-j\frac{\pi}{2}} \quad (k=1) \longrightarrow \frac{\alpha}{2j}e^{j\beta}$$

$$a_{-1} = -\frac{1}{2j} = +\frac{1}{2}e^{j\frac{\pi}{2}} \quad (k=-1) \longrightarrow \frac{\alpha}{2j}e^{-j\beta}$$

$$a_k = 0 \ \text{ otherwise } \ (k \neq \pm 1)$$



ω_0 is given

$$\boxed{a_k = -\frac{1}{2j}\delta[k+1] + \frac{1}{2j}\delta[k-1]}$$

$$a_k = -\frac{\alpha}{2j}e^{-j\beta}\delta[k+1] + \frac{\alpha e^{j\beta}}{2j}\delta[k-1]$$

From lecture 15

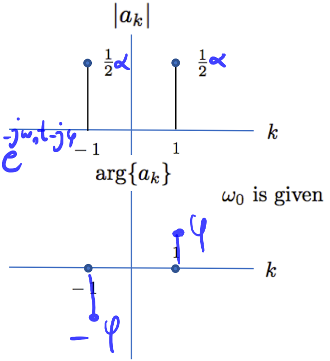
$$x(t) = \cos(\omega_0 t) \longrightarrow \alpha\cos(\omega_0 t + \varphi)$$

$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \longrightarrow \frac{\alpha}{2}e^{j\omega_0 t + j\varphi} + \alpha/2\, e^{-j\omega_0 t - j\varphi}$$

$$a_1 = \frac{1}{2} \; (k = 1) \longrightarrow \frac{\alpha}{2}e^{j\varphi}$$

$$a_{-1} = \frac{1}{2} \; (k = -1) \longrightarrow \alpha/2\, e^{-j\varphi}$$

$$a_k = 0 \text{ otherwise}$$



$$\boxed{a_k = \tfrac{1}{2}\delta[k+1] + \tfrac{1}{2}\delta[k-1]}$$

$$a_k = \frac{\alpha}{2}e^{-j\varphi}\delta[k+1] + \frac{\alpha}{2}e^{j\varphi}\delta[k-1]$$

Example: $x(t) = [1+\cos(2\pi t)][\sin(10\pi t+\pi/6)]$

Inspection Method

$$x(t) = \sin(10\pi t+\pi/6) + \cos(2\pi t)\sin(10\pi t+\pi/6)$$

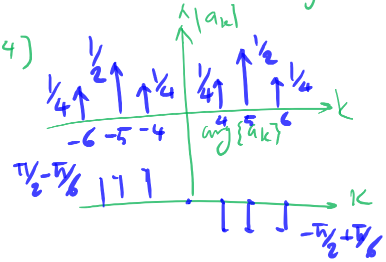
$$= \frac{e^{j\pi/6}}{2j}e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j}e^{-j2\pi t5} + \left(\frac{e^{j2\pi t}+e^{-j2\pi t}}{2}\right)\left(\frac{e^{j\pi/6}}{2j}e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j}e^{-j2\pi t5}\right)$$

$$= \frac{e^{j\pi/6}}{2j}e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j}e^{-j2\pi t5} + \frac{e^{j\pi/6}}{4j}e^{j2\pi t6} - \frac{e^{-j\pi/6}}{4j}e^{j2\pi t4} + \frac{e^{j\pi/6}}{4j}e^{j2\pi t4} - \frac{e^{-j\pi/6}}{4j}e^{-j2\pi t6}$$

$\omega_0 = 2\pi$

$$a_k = \frac{e^{j\pi/6}}{2j}\delta[k-5] - \frac{e^{-j\pi/6}}{2j}\delta[k+5] + \frac{e^{j\pi/6}}{4j}\delta[k-6] - \frac{e^{-j\pi/6}}{4j}\delta[k+6]$$

$$+ \frac{e^{j\pi/6}}{4j}\delta[k-4] - \frac{e^{-j\pi/6}}{4j}\delta[k+4]$$



$$x(t) = \underbrace{\left[1+\cos(2\pi t)\right]}_{x_1(t)}\underbrace{\left[\sin(10\pi t+\pi/6)\right]}_{x_2(t)}$$

$\omega_0 = 2\pi$

$$a_{k_1} = 1 + \tfrac{1}{2}\delta[k-1] + \tfrac{1}{2}\delta[k+1]$$

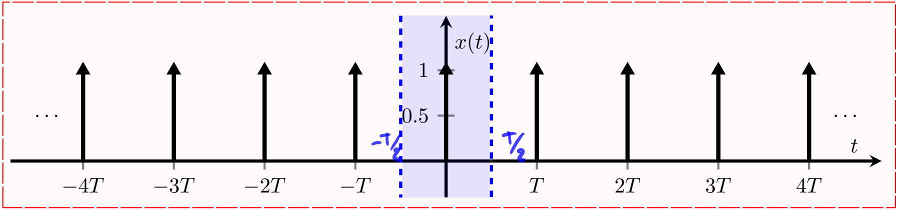
$$a_{k_2} = \frac{e^{j\pi/6}}{2j}\delta[k-5] - \frac{e^{-j\pi/6}}{2j}\delta[k+5]$$

$$a_{k_1} * a_{k_2} = \frac{e^{j\pi/6}}{2j}\delta[k-5] - \frac{e^{-j\pi/6}}{2j}\delta[k+5] + \frac{e^{j\pi/6}}{4j}\delta[k-6] - \frac{e^{-j\pi/6}}{4j}\delta[k+4]$$

$$+ \frac{e^{j\pi/6}}{2j}\delta[k-4] - \frac{e^{-j\pi/6}}{2j}\delta(k+6)$$

Periodic Impulse Train – Definition

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 0}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, dt$$

$$= \frac{1}{T} \quad \text{for all } k$$

Periodic Impulse Train – Definition (cont'd)

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k = \frac{1}{T},$$

that is,

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

has Fourier Series

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

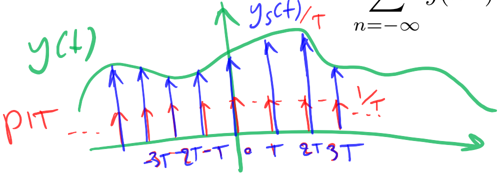
Periodic Impulse Train – Definition (cont'd)

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is known as the **sampling function**. As its name implies it is useful for sampling CT signals uniformly at time instants which are a multiple of T:

$$y_s(t) \triangleq y(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} y(nT)\, \delta(t - nT)$$



Periodic Impulse Train – Time Shifted PIT

Consider time shifting an **arbitrary** signal $x(t)$

$$x(t) \xleftrightarrow{\mathscr{F}} a_k$$

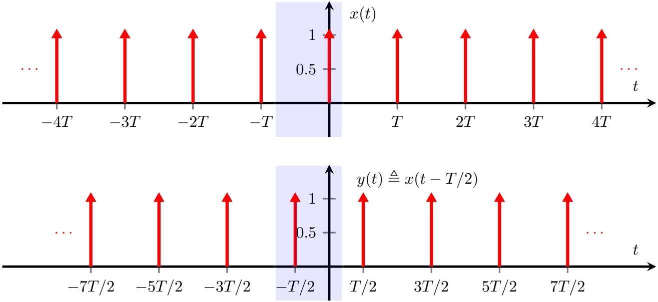
by the **specific** value of half the period, $t_0 = T/2$. By the "Time Shift" property, a linear phase shift is introduced relative to $x(t)$ in the Fourier coefficients

$$y(t) \triangleq x(t - T/2) \xleftrightarrow{\mathscr{F}} a_k e^{-jk\pi} \equiv (-1)^k a_k$$

from

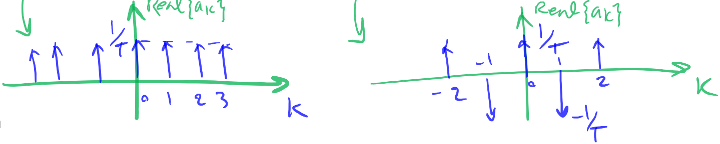
$$e^{-jk\omega_0 T/2} = e^{-jk\pi}, \qquad e^{-j\pi} = -1$$

Next we consider the case when $x(t)$ is the impulse train.



T/2

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{T} \quad \Rightarrow \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{(-1)^k}{T}$$



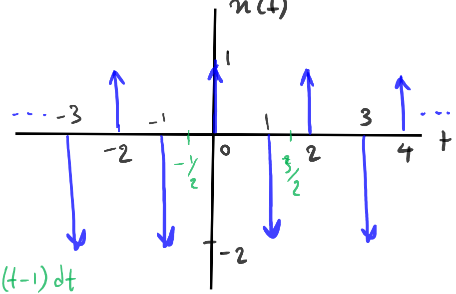
Periodic Impulse Train – Time Shifted PIT

- The Fourier coefficients for both the PIT and $T/2$ time shifted PIT are purely real. Note that in the time domain both the PIT and $T/2$ time shifted PIT are even functions.
- A more general delay, e.g., $t_0 = 0.3498T$, would lead to a time shifted PIT with complex Fourier coefficients.

If $|x(t)|$ is even and $\arg\{x(t)\}$ is odd $\longrightarrow$ a_k will be real values

If $|a_k|$ is even and $\arg\{a_k\}$ is odd $\longrightarrow$ $x(t)$ is a real function

Example:



$T = 2s$, $\omega_0 = \frac{2\pi}{T} = \pi$ rad/s

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

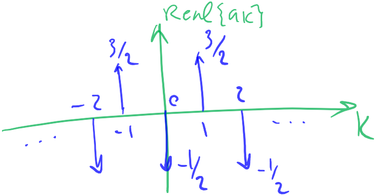
$$a_0 = \frac{1}{2}\int_{-1/2}^{3/2} x(t)\,dt = \frac{1}{2}\int_{-1/2}^{3/2} \delta(t) - 2\delta(t-1)\,dt$$

$$= \frac{1}{2}(1-2) = -\frac{1}{2}$$

$$a_k = \frac{1}{2}\int_{-1/2}^{3/2}\left[\delta(t) - 2\delta(t-1)\right]e^{-jk\omega_0 t}\,dt$$

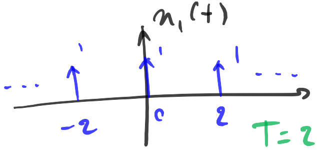
$$= \frac{1}{2}\int_{-1/2}^{3/2}\delta(t) - 2\delta(t-1)\,e^{-jk\omega_0}\,dt$$

$$= \frac{1}{2} - e^{-jk\omega_0} = \frac{1}{2} - \left(e^{-j\pi}\right)^k = \frac{1}{2} - (-1)^k$$

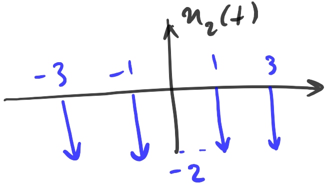


$\frac{1}{2j} = \frac{1}{2}j$

$\frac{1}{2}e^{-j\pi/2}$



$T = 2$



$p \rightarrow p|T$

$$x(t) = x_1(t) + x_2(t) = p(t) - 2p(t - T/2)$$

$$a_k = a_{k_p} - 2a_{k_p} e^{-jk\pi} = 1/T - \frac{2}{T} e^{-jk\pi} = 1/T - \frac{2}{T} (\underbrace{e^{-j\pi}}_{-1})^k$$

$$= 1/2 - (-1)^k$$

Periodic Convolution – Definition

By "Periodic Convolution" we mean convolution of periodic functions (of the same period T) in the time domain. But we need further qualifications.

Previously we found that multiplication in the time domain led to "discrete" convolution in the Fourier series (frequency) domain.

Doing the obvious

$$x(t) \star y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$$

has the hazard of often being infinite. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) \star y(t) = \infty$$

So we define a modified periodic convolution.

Define Periodic Convolution to an integral over any one period, e.g., $-T/2$ to $T/2$. (The ends will justify the means.)

$$z(t) \triangleq \int_T x(\tau)y(t-\tau)\,d\tau$$

Let

$$x(t) \xleftrightarrow{\mathscr{F}} a_k, \quad y(t) \xleftrightarrow{\mathscr{F}} b_k, \quad z(t) \xleftrightarrow{\mathscr{F}} c_k$$

Periodic Convolution – Key Result

Computing the Fourier coefficients of $z(t)$,

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T}\int_T \Big(\int_{-T/2}^{T/2} x(\tau) y(t-\tau)\, d\tau \Big)\, e^{-jk\omega_0 t}\, dt \\
&= \int_T \underbrace{\Big(\frac{1}{T}\int_T y(t-\tau)\, e^{-jk\omega_0 (t-\tau)} dt \Big)}_{b_k} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= b_k \int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= T\, a_k b_k
\end{aligned}
$$

Multiplication in frequency (Fourier series domain).

Periodic Convolution – Key Result (cont'd)

In summary, for signals sharing a common period T,

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k, \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} b_k, \quad z(t) \overset{\mathscr{F}}{\longleftrightarrow} c_k$$

related through Periodic Convolution

$$z(t) \triangleq \int_T x(\tau) y(t-\tau)\, d\tau$$

then

$$c_k = T\, a_k b_k$$

Alternatively

$$\Big(\frac{1}{T}\int_T x(\tau) y(t-\tau)\, d\tau\Big) \overset{\mathscr{F}}{\longleftrightarrow} a_k b_k$$

Time and frequency domain properties for the four cases

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties