Part 5 Outline

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For DT LTI Systems the **Causality Property** can be written:

Theorem (Causal DT LTI System)

A DT LTI system is **causal** *if and only if its pulse response,* $h[n]$*, satisfies*

$$\boxed{h[n] = 0, \quad \text{for all } n < 0.}$$

- If $h[n] \neq 0$ for at least one $n = -n_0$ $(n_0 > 0)$ then the output at time n, $y[n]$, would contain term

$$h[-n_0]\, x[n + n_0],$$

for example, if $n_0 = 1$ and $h[-1] = 2$ then
$y[n] = \cdots + h[-1]\, x[n+1] + \cdots$, and hence would not be causal.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \ldots + h[-1]\, x[n+1] + \ldots$$

Part 5 Outline

DT System Properties – Stability Property

Stability: a bounded input $x[n]$ produces a bounded output $y[n]$.

Definition (DT LTI System Stability)

A DT LTI system is **stable**, with pulse respond $h[n]$, if and only if

$$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

~~is bounded if and only if the input is bounded.~~

DT System Properties
Stability Property

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |2^n| = \infty$$

- $h[n] \triangleq 2^n$ is not stable

system is non-causal $\Rightarrow$ system non-stable
because $h[n] \neq 0 \ \forall n < 0$

- $h[n] \triangleq 2^{-n}$ is not stable (consider $n \longrightarrow -\infty$)

$$\sum_{n=-\infty}^{\infty} |2^{-n}| = \infty \Rightarrow \text{non-stable}$$

- The following is stable:

$h[n] = 0 \ \forall n < 0$

$$h[n] \triangleq \begin{cases} 2^{-n} & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \text{causal}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |2^{-n}| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = \frac{1}{\frac{1}{2}} = 2 < \infty$$

$\Rightarrow$ system stable

DT System Properties – Review of System Properties

System properties:

- Time-invariant/ time-varying
- Memory/ memoryless
- Causal/ non-causal
  - Two ways to causality can be determined. One uses $h[n]$.
- Stable/ non-stable
  - Had to wait until defined $h[n]$ to introduce.
- Linear/ non-linear

Problem:

Determine whether or not each of the following signals are: i) **time-invariant**, ii) **linear**, iii) **casual**, iv) **stable**, and v) **memoryless**.

(a) $y[n] = x[n+3] - x[1-n]$

(b) $y[n] = \begin{cases} (-1)^n x[n], & x[n] \geq 0 \\ 2x[n], & x[n] < 0 \end{cases}$

(c) $y[n] = \sum_{k=n}^{\infty} x[k]$

Solution:

	TI	Linear	Causal	Stable	Memoryless
(a)	no	yes	no	yes	no
(b)	no	no	yes	yes	yes
(c)	yes	yes	no	no	no

Differential Equation of CT System

General form:

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)}$$

N-th order differential equation

For example:

$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

1st order

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

2nd

Difference Equation of DT System

General form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

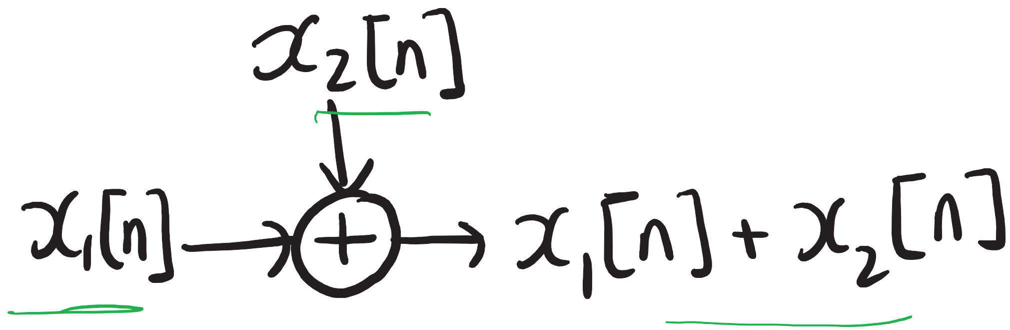
N-th order difference equation

Part 5 Outline

Difference Equation of DT System – Direct-Form I implementation

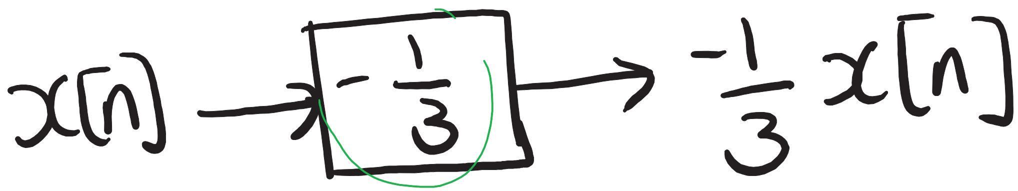
Block diagram representation of causal DT LTI systems:

- Summer (for two inputs)

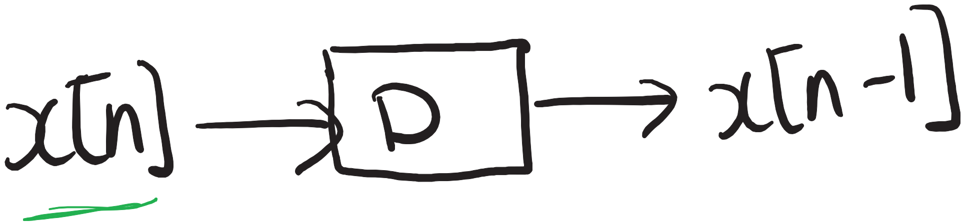


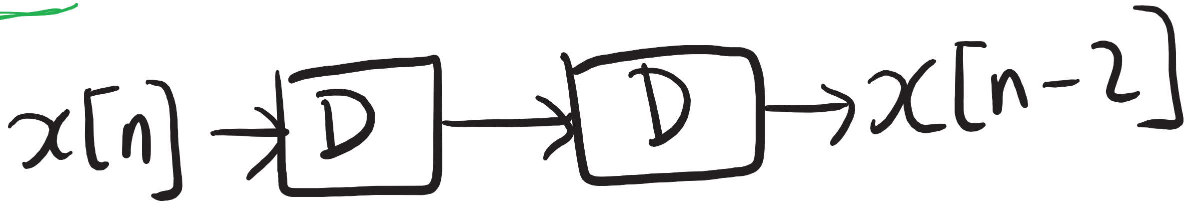
- Gain (can be negative or positive)

e.g.



- Delay



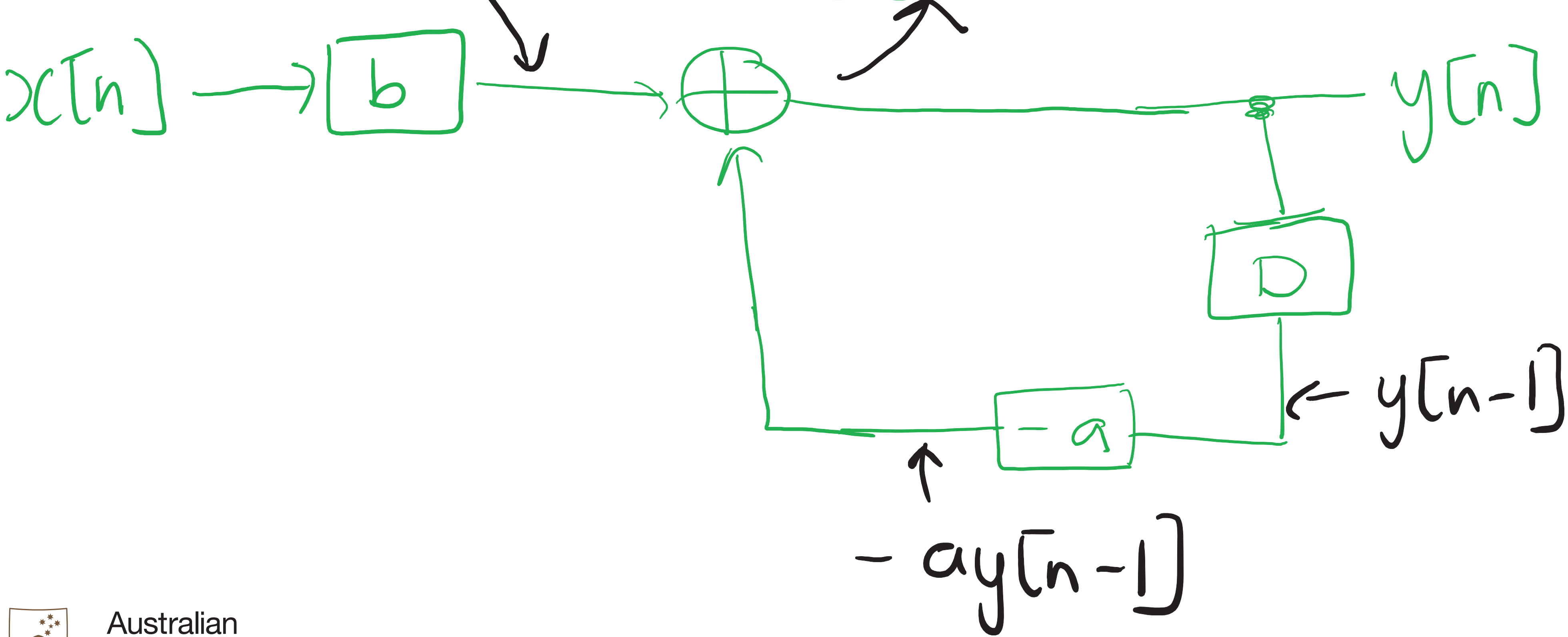


Difference Equation of DT System – Direct-Form I implementation

e.g. 1st order difference equation

$$y[n] + ay[n-1] = bx[n]$$

$$y[n] = bx[n] - ay[n-1]$$

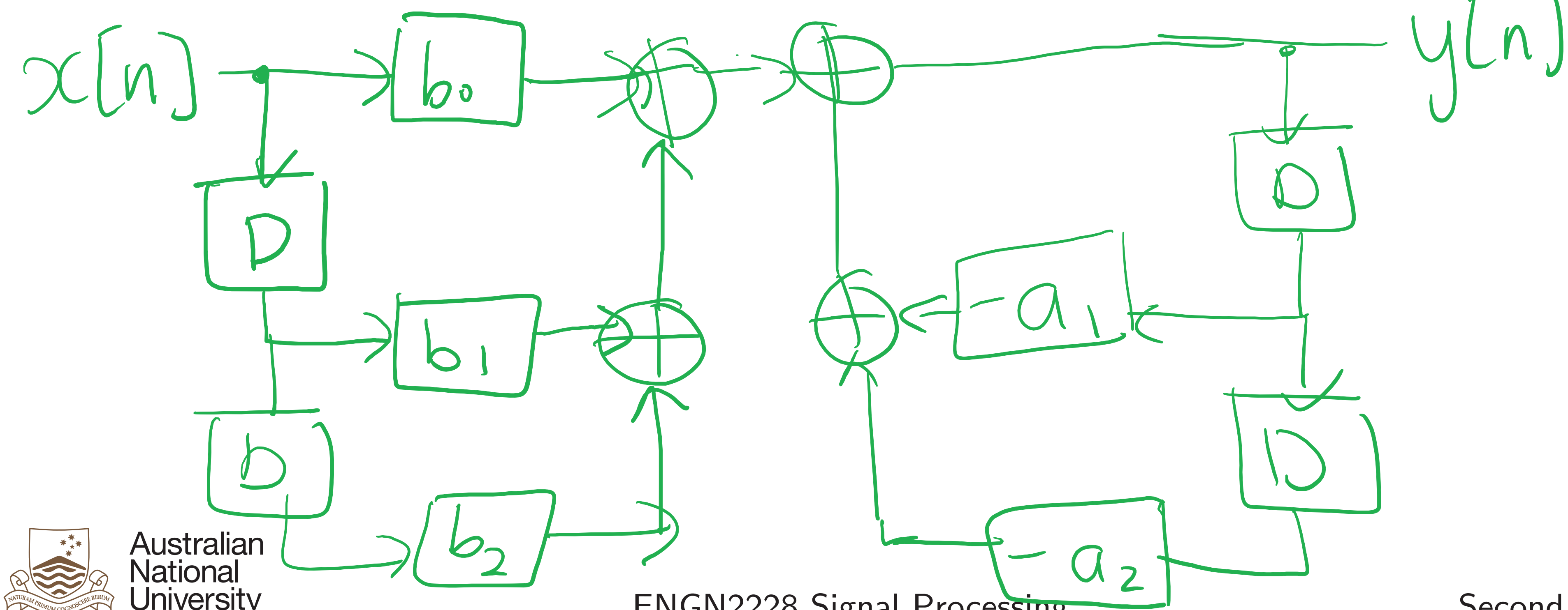


Difference Equation of DT System – Direct-Form I implementation

e.g. 2nd order difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

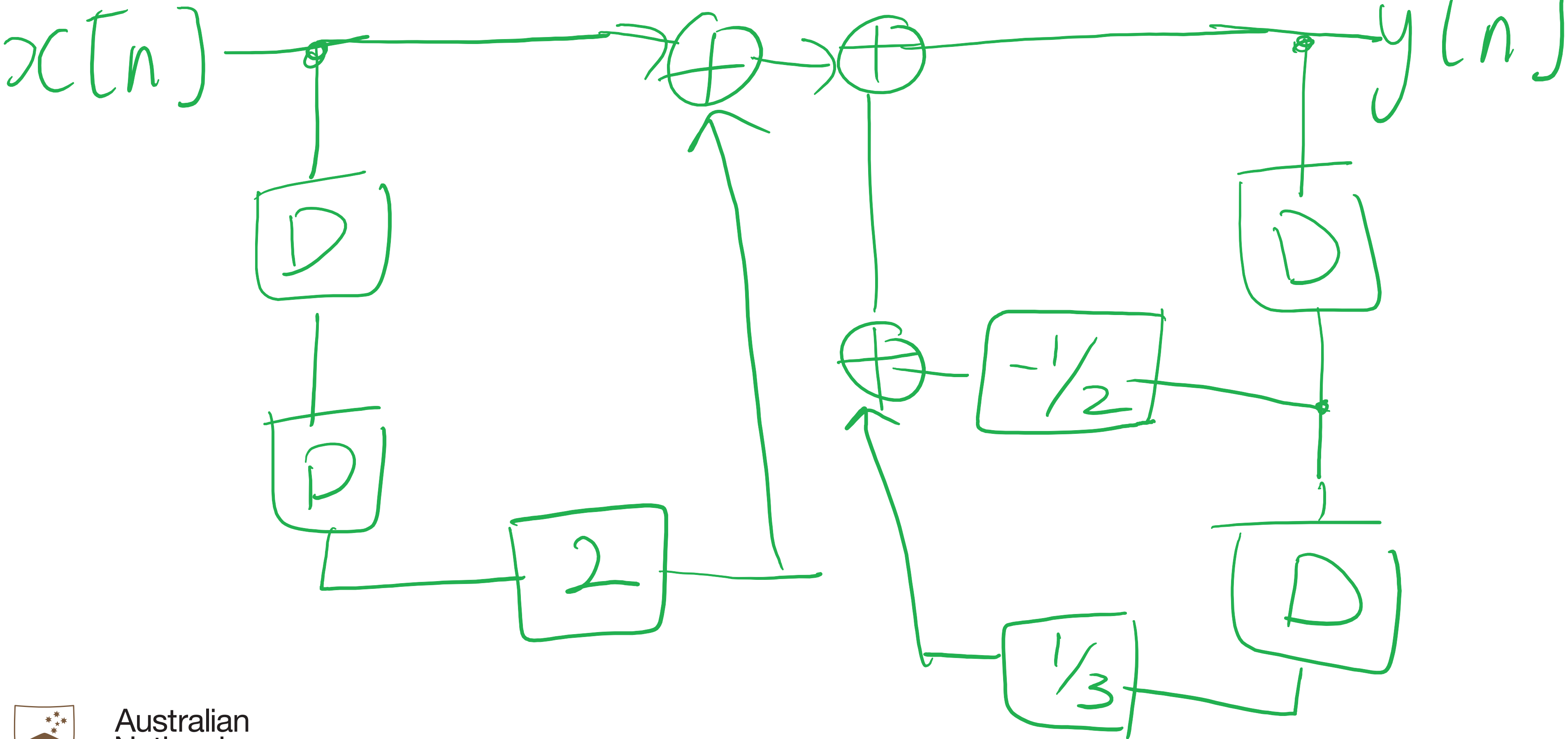
$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$



Difference Equation of DT System – Direct-Form I implementation

Try

$$y[n] = x[n] + 2x[n-2] - \frac{1}{2}y[n-1] + \frac{1}{3}y[n-2]$$



Part 5 Outline

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- Street Version
- Graphical Flip and Shift
- Other DT Convolution Methods
- Convolution with Impulses
- Commutative Property
- Distributive Property
- Associative Property

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- Causality Property
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- Review of System Properties

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23 **Finding the Impulse Response of a DT System**

$h[n]$

Finding the Impulse Response of a DT System

How to find $h[n]$?

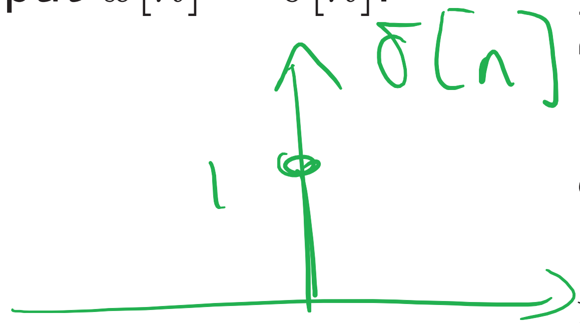
- Definition - $h[n]$ is the output of the system for an input $x[n] = \delta[n]$.

e.g.:

- $y[n] = x[n] + \frac{1}{2}x[n-1]$
- $h[n] = ?$
- Let $x[n] = \delta[n]$ then
- $h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$x[n] = \delta[n]$

$y[n] = h[n]$



$$y[n] = \frac{1}{3}y[n-1] + x[n-1]$$

Causal

DT LTI system initially at rest, find $h[n]$.

- Let $x[n] = \delta[n]$ then, $h[n] = \frac{1}{3}h[n-1] + \delta[n-1]$
- Have we solved for $h[n]$?

Finding the Impulse Response of a DT System – Example

$\frac{1}{3}h[n-1]$

$h[n] =$ [illegible] $+ \delta[n-1]$

- Use recursion to work out formula.
- System causal - therefore can use $h[n]$ causality property: as the system is initially at rest $h[n] = 0$ for $n < 0$.

$h[0] = \frac{1}{3}h[-1] + \delta[-1] = 0 + 0 = 0$

$h[1] = \frac{1}{3}h[0] + \delta[0] = 1$

$h[2] = \frac{1}{3}h[1] + \delta[1] = \frac{1}{3}$

$h[3] = \frac{1}{3}h[2] + \delta[2] = \frac{1}{9}$

Therefore $h[n] = \begin{cases} \left(\frac{1}{3}\right)^{(n-1)} & n > 0 \\ 0, & n \leq 0 \end{cases}$

$h[n] = \left(\frac{1}{3}\right)^{(n-1)} u[n-1]$

Finding the Impulse Response of a DT System – FIR vs IIR

- If a DT LTI system has a finite duration impulse response (i.e. $h[n]$ is nonzero only over a finite time interval), then the system is called a *Finite Impulse Response* (FIR) System.

- If a DT LTI system, with the condition of ~~instal~~ inital rest, will have an impulse response of infinite duration, then the system is called an *Infinite Impulse Response* (IIR) system.

- Important classification of systems - e.g. FIR and IIR filters.

$$y[n] = x[n+3] - x[1-n]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad , \quad h[n] = \underbrace{\delta[n+3]} - \underbrace{\delta[1-n]}$$

$$h[n] = 1 \quad \underline{n = -3}, \quad 1 \quad n = 1$$

$$\sum_{n=-\infty}^{\infty} |\underbrace{\delta[n+3]}_{\uparrow\, n=-3} - \underbrace{\delta[1-n]}_{\uparrow\, n=1}| = 1 + 1 = 2 < \infty$$

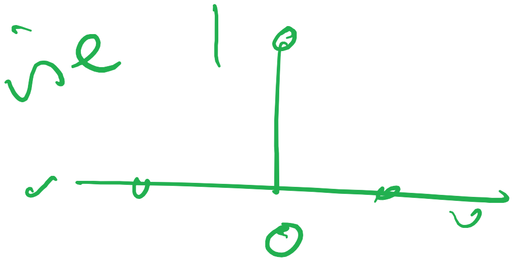
$\Rightarrow$ stable

non-causal as $h[-3] = 1$

$h[n] \neq 0 \quad n < 0$

$$y[n] = \begin{cases} \underline{(-1)^n x[n]} & , \ x[n] \geq 0 \\ \underline{2x[n]} & , \ x[n] < 0 \end{cases}$$

$$\delta[n] = 1, \ n=0 \ , \ 0 \text{ otherwise}$$



$$h[n] = \begin{cases} (-1)^n \delta[n] & , \ \delta[n] \geq 0 \\ \cancel{2\delta[n]} & \cancel{\delta[n] < 0} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |(-1)^n \delta[n]|$$

$$= 1 < \infty \Rightarrow \text{stable}$$

as $h[n] = 0, \ n < 0 \Rightarrow$ causal

$$y[n] = \sum_{k=n}^{\infty} x[k]$$

$$h[n] = \sum_{c=n}^{\infty} \delta[k] \qquad \delta[k] = \begin{cases} 1, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left| \sum_{k=n}^{\infty} \delta[k] \right|$$

if $n>0$, sum $=0$
if $n \leq 0$, sum $=1$

$$= \sum_{n=-\infty}^{0} 1 = \infty$$

$\Rightarrow$ unstable

non-causal as $h[n] \neq 0$ $n<0$