Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 3 – SOLUTIONS

Continuous-time Convolution

Problem Set 3-1

Use the graphical flip/shift method, showing intermediate working to perform the following CT convolutions to find $y(t)$. NOTE: solution can be checked using Matlab.

(a) $x(t) = (t-1)\{u(t-1) - u(t-3)\}$ and $h(t) = u(t+1) - 2u(t-2)$

Solution:

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ -t^2 + 6t - 7 & 3 \le t < 5 \\ -2 & t \ge 5 \end{cases}$$

(b) $x(t) = u(t) - u(t-2)$ and $h(t) = e^{-t}u(t)$

Solution:

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & 0 \le t < 2 \\ (e^2 - 1)e^{-t} & t \ge 2 \end{cases}$$

(c) $x(t) = u(t) - 2u(t-2) + u(t-5)$ and $h(t) = e^{2t}u(1-t)$. Hint: convolution is commutative

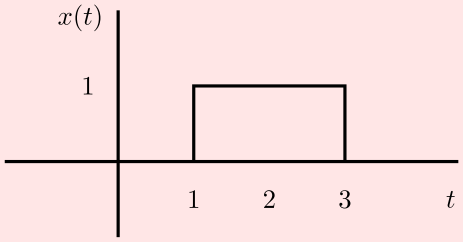
Solution:

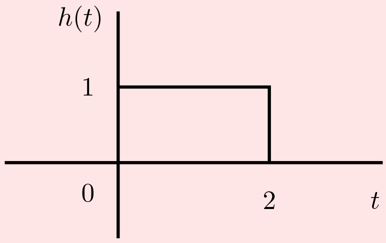
$$y(t) = \begin{cases} \frac{1}{2}[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}] & t \le 1 \\ \frac{1}{2}[e^2 + e^{2(t-5)} - 2e^{2(t-2)}] & 1 \le t \le 3 \\ \frac{1}{2}[e^{2(t-5)} - e^2] & 3 \le t \le 6 \\ 0 & 6 < t \end{cases}$$

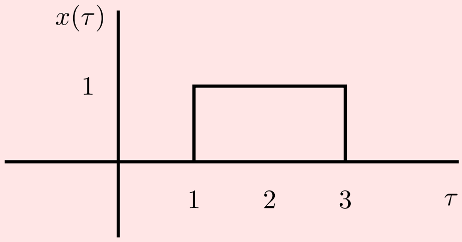
Problem Set 3-2

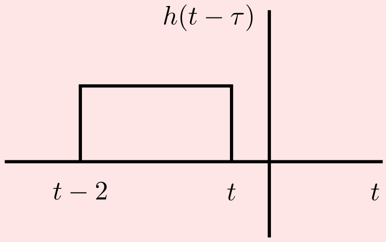
Compute the convolution $y(t) = x(t) * h(t)$ when $x(t) = u(t-1) - u(t-3)$ and $h(t) = u(t) - u(t-2)$.

Solution: The signals are shown graphically as follows :









For $t < 1$, there is no over-lap, hence $y(t) = 0$.
For $1 \le t \le 3$,

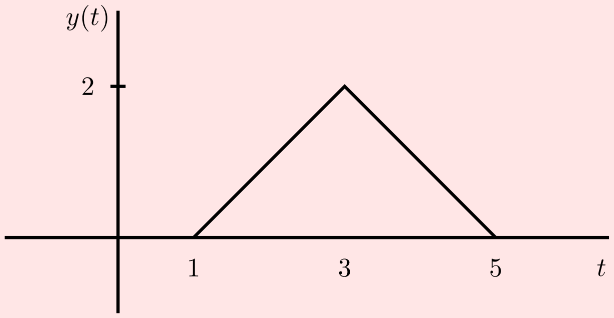
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{1}^{t} (1)(1)d\tau \\ &= t-1 \end{aligned}$$

For $3 \le t < 5$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{t-2}^{3} (1)(1)d\tau \\ &= 3-(t-2) = 5-t \end{aligned}$$

For $t \ge 5$, there is no over-lap, hence $y(t) = 0$.

The plot for $y(t)$ looks like the following :



Problem Set 3-3

Consider the following convolution

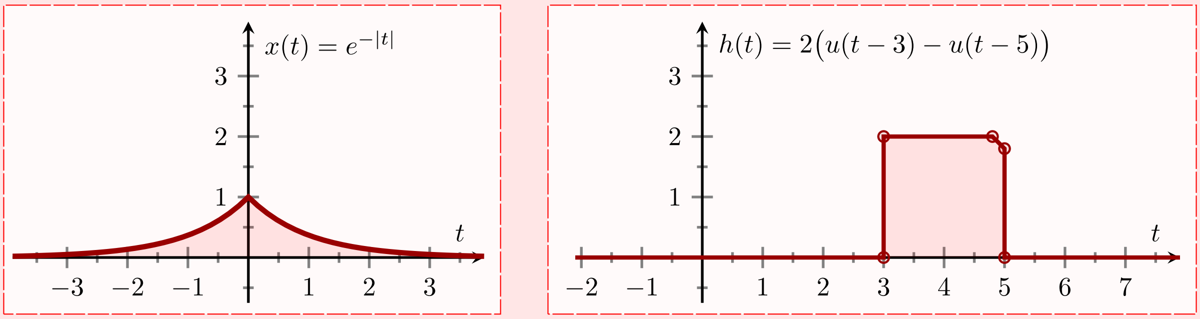
$$y(t) = x(t) \star h(t)$$

where

$$x(t) = e^{-|t|} \quad \text{and} \quad h(t) = 2\,u(t-3) - 2\,u(t-5).$$

(a) Draw $x(t)$ and $h(t)$.

Solution:

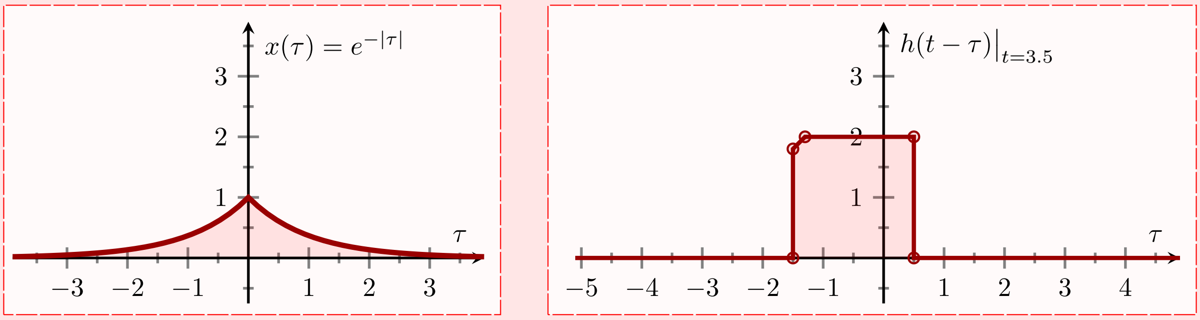


Note that $h(t)$ is drawn with a bite taken out of it, which helps interpret the convolution later. □

(b) Evaluate the integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Solution: Notice $h(t)$ has been flipped and shifted, and $h(t-\tau)$ is drawn for value $t = 3.5$. For example at $\tau = -1.5$, then $h(3.5 - (-1.5)) = h(5)$, which is indicated with the "bitten edge".



So as $h(t-\tau)$ slides around (as we vary t) then the overlap with $x(\tau)$ will consist of 3 cases:

(i) $t < 3$: here $h(t-\tau)$ overlaps with the left part of $x(\tau)$ where τ is negative

$$y(t) = 2\int_{t-5}^{t-3} e^{\tau}\, d\tau = 2\, e^{\tau}\Big|_{t-5}^{t-3} = 2(e^{t-3} - e^{t-5})$$

(ii) $3 \le t \le 5$: here $h(t-\tau)$ overlaps with the right part of $x(\tau)$ where τ is positive

$$y(t) = \int_{t-5}^{0} 2e^{\tau}\, d\tau + \int_{0}^{t-3} 2e^{-\tau}\, d\tau = \cdots = 4 - 2\,e^{t-5} - 2\,e^{3-t}$$

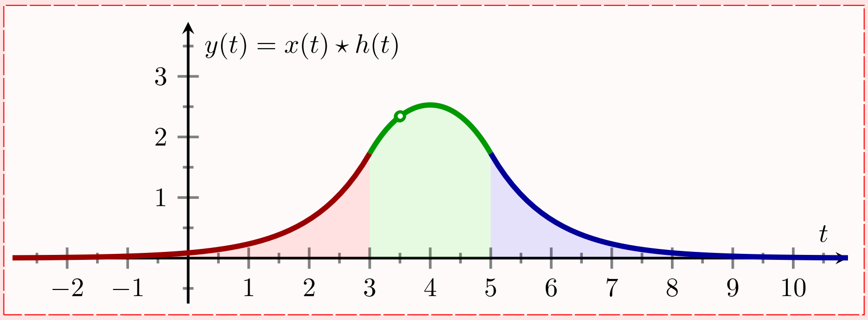
(iii) $t > 5$: here $h(t-\tau)$ overlaps across the peak of $x(\tau)$

$$y(t) = 2\int_{t-5}^{t-3} e^{-\tau}\, d\tau = -2\,e^{-\tau}\Big|_{t-5}^{t-3} = 2(e^{5-t} - e^{3-t})$$

The output $y(t)$ consists of these three parts and is plotted next. □

(c) Draw $y(t)$.

Solution: Combining the three segments we get the final answer



Hopefully you are convinced that computing the convolution from the integral expression is a pain and error prone. There are other ways of computing this convolution. Also on the figure the point for $t = 3.5$ is marked and this corresponds to the integral of the product of the two functions shown in part (b). □

Problem Set 3-4

In 3-3, we note that the centre of $h(t)$ is at $t = 4$. Review and understand the following:

$$\begin{aligned} h(t) &= 2\big(u(t-3) - u(t-5)\big) \\ &= 2\big(u(t+1) - u(t-1)\big) \star \delta(t-4) \end{aligned}$$

Therefore

$$\begin{aligned} y(t) &= x(t) \star h(t) \\ &= \Big(x(t) \star 2\big(u(t+1) - u(t-1)\big)\Big) \star \delta(t-4). \end{aligned}$$

So we can adjust for the delay of 4 at the end of a convolution that uses the even function

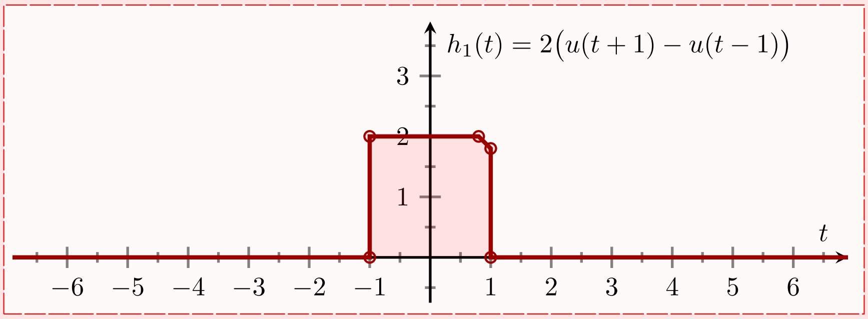
$$h_1(t) = 2\big(u(t+1) - u(t-1)\big)$$

instead of the original $h(t)$.

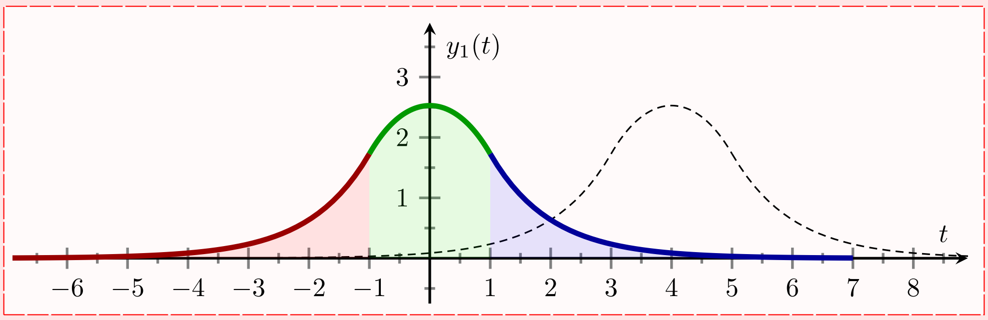
(a) Draw

$$y_1(t) = x(t) \star h_1(t) = x(t) \star 2\big(u(t+1) - u(t-1)\big).$$

Solution: First we show $h_1(t)$:



and then $y_1(t)$ is given by



Here $y(t)$ is shown dashed. □

(b) Confirm $y_1(t)$'s relationship with $y(t)$.

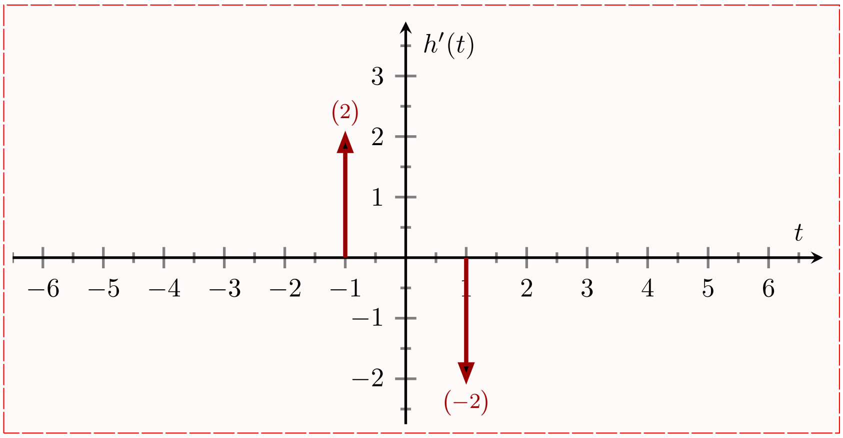
Solution: Indeed $y(t) = y_1(t-4)$. $\square$

Problem Set 3-5

(Hard) The derivative of $h(t)$ is given by

$$h'(t) = 2\,\delta(t+1) - 2\,\delta(t-1)$$

and is drawn below



In the lecture notes it was shown how to compute the convolution using the $u_n(t)$ functions (where $\delta(t) = u_0(t)$, $u(t) = u_{-1}(t)$, etc.).

(a) Confirm $x(t) \star h(t) = x(t) \star u_1(t) \star u_{-1}(t) \star h(t)$.

Solution: Convolving with $\delta(t)$ does nothing, and $u_1(t) \star u_{-1}(t) = \delta(t)$. $\square$

(b) Use this expression to evaluate the convolution.

Solution: We have

$$\begin{aligned} x(t) \star h(t) = x(t) \star u_1(t) \star u_{-1}(t) \star h(t) &= x(t) \star u_1(t) \star h(t) \star u_{-1}(t) \\ &= x(t) \star h'(t) \star u_{-1}(t) \\ &= \int_{-\infty}^{t} x(\tau) \star h'(\tau)\, d\tau \end{aligned}$$

but

$$\begin{aligned} x(\tau) \star h'(\tau) &= x(\tau) \star \big(2\,\delta(t+1) - 2\,\delta(t-1)\big) \\ &= 2\,e^{-|t+1|} - 2\,e^{-|t-1|} \end{aligned}$$

and so

$$y(t) = 2\int_{-\infty}^{t} e^{-|\tau+1|}\, d\tau - 2\int_{-\infty}^{t} e^{-|\tau-1|}\, d\tau.$$

And give this to zombies to compute. There are better things in life to do. $\square$

Problem Set 3-6

Let $h(t)$ be the triangular pulse shown in Fig. 1 and $x(t)$ be the impulse train depicted in Fig. 2. That is,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Determine and sketch $y(t) = x(t) \star h(t)$ for the following values of T:

(a) $T = 2$

(b) $T = 1.5$

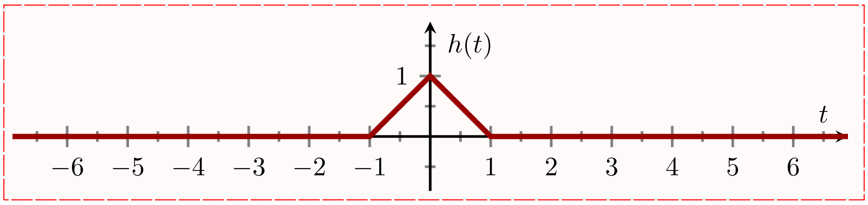


Figure 1: Triangular pulse $h(t)$.

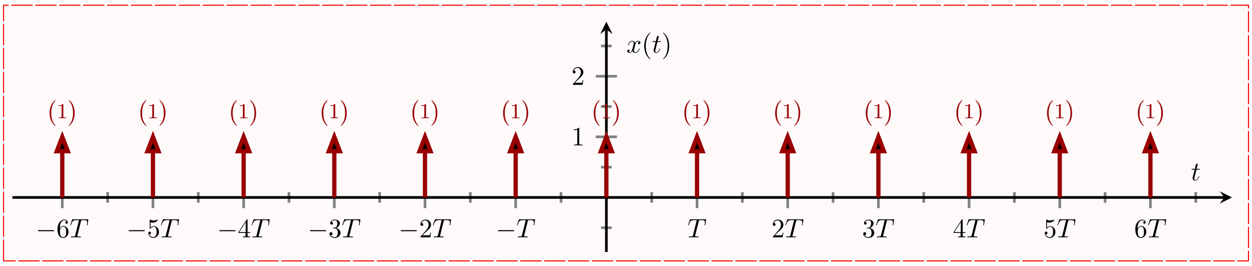


Figure 2: Impulse train $x(t)$ with spacing T.

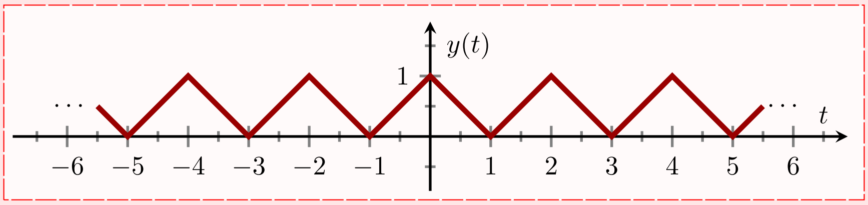
Solution: Using the following time-sifting property of impulse function

$$\int_{-\infty}^{\infty} \delta(t-T)h(t)\,dt = h(T),$$

we can write the output $y(t)$ as

$$y(t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-\tau)\delta(\tau - kT)\,d\tau = \sum_{k=-\infty}^{\infty} h(t-kT).$$

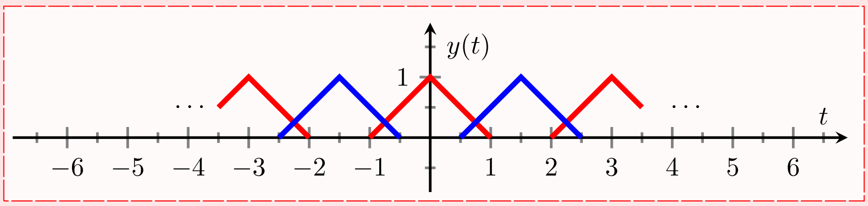
(a) For $T = 2$, $y(t)$ is sketched below:



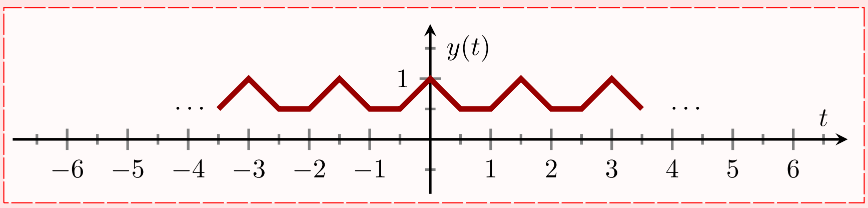
Since the triangular pulse has a width of $T = 2$, the neighbouring pulses do not overlap with each other.

(b) For $T = 1.5$, the centre of the neighbouring triangular pulses is $T = 1.5$ apart, resulting in an overlap between the triangular pulses.

We first sketch $y(t)$ where we show the neighbouring triangular pulses in different colours:



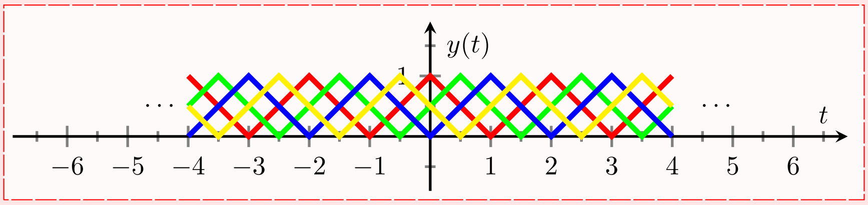
The output $y(t)$ is a sum of red and blue pulses. If we were mathematicians, we should find equation of the lines to find the sum of two pulses in the overlap interval. But we are engineers, we can sum the two pulses graphically to find the output $y(t)$:



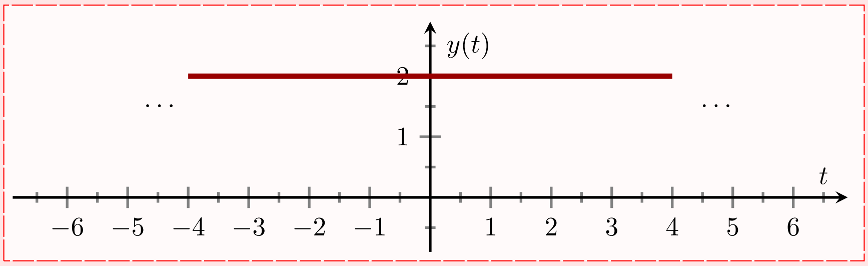
□

(b, old version of the assignment when $T = 0.5$) For $T = 0.5$, the centre of the neighbouring triangular pulses is $T = 0.5$ apart, due to which there is an overlap between the pulses.

We first sketch $y(t)$ where we show the neighbouring triangular pulses in different colours:



The output $y(t)$ is a sum of four different colour pulses. It can be noted (graphically) that the sum of four different pulses at any time t is $y(t) = 2$, which is sketched below:



□

Problem Set 3-7

A continuous-time LTI system has impulse response given by the shifted unit step

$$h(t) = u(t - t_0) \equiv \begin{cases} 1 & t \geq t_0 \\ 0 & \text{otherwise,} \end{cases}$$

where t_0 is a fixed time shift. If $y(t) = x(t) \star h(t)$, show that

$$y(t) = \int_{-\infty}^{t} x(\tau - t_0)\, d\tau.$$

$\big($*Hint:* Note the similarity to $x(t) \star h(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau.\big)$

Solution: Before carrying out the convolution operation, we manipulate the system's impulse response $h(t)$ by time-shifting and folding so that it becomes

$$h(t - \tau) = u(t - \tau - t_0) \equiv \begin{cases} 1 & \tau \leq t - t_0 \\ 0 & \tau > t_0 \end{cases}$$

Next using the hint which gives the integral expression for the convolution $x(t) \star h(t)$ we can write

$$y(t) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau - t_0)\, d\tau = \int_{-\infty}^{t - t_0} x(\tau)\, d\tau.$$

With the change of variable $\tau = z - t_0 \implies d\tau = dz$ the integral in dz becomes

$$y(t) = \int_{-\infty}^{t} x(z - t_0)\, dz,$$

which is equivalent to desired expression for the output $y(t)$. □

Continuous-time Impulse Response

Problem Set 3-8

Consider a continuous-time LTI system with input signal $x(t)$ and output signal $y(t)$ given by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau.$$

Find the impulse response of the system, $h(t)$ where $y(t) = x(t) \star h(t)$. You should express your answer in terms of the unit step function nothing that it can be written

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

$\big($*Hint:* Remember that $h(t)$ is given by $y(t)$ when $x(t) = \delta(t)$.$\big)$

Solution: There are a number of way to solve this problem. One slick way is to note that

$$x(t)\, \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$$

for any time offset t_0. In this solution we will use the above with $t_0 = 2$.

It then follows that

$$\begin{aligned} h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau - 2)\, d\tau &= \int_{-\infty}^{t} e^{-(t-2)} \delta(\tau - 2)\, d\tau \\ &= e^{-(t-2)} \int_{-\infty}^{t} \delta(\tau - 2)\, d\tau = e^{-(t-2)} u(t-2) \qquad \text{using unit step} \end{aligned}$$

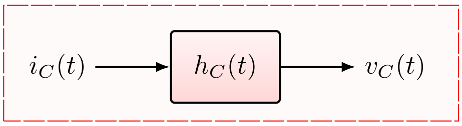
is the impulse response of the system. □

Problem Set 3-9

(a) The voltage and current for a capacitor C are related through

$$i_C(t) = C \frac{dv_C(t)}{dt}.$$

Consider the LTI system with input $i_C(t)$ and output $v_C(t)$. What is the impulse response $h_C(t)$ of the system? (You can express the result using the $u_k(t)$ functions defined in Part 7 of the lecture notes, or Section 2.5 of the text.)

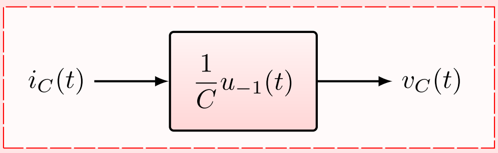


Solution: We write the output $v_C(t)$ in terms of input $i_C(t)$ as follows:

$$v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i_C(t)\, dt$$

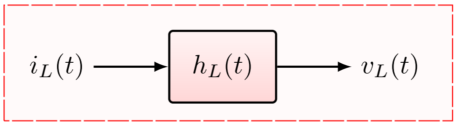
Then the impulse response is given by

$$\boxed{h_C(t) = \frac{1}{C} u_{-1}(t) \equiv \frac{1}{C} u(t),}$$



which is a scaled unit step. □

(b) The inductor L can be thought of as the dual of the capacitor C where the transformation can be achieved by $L \leftrightarrow C$, $v_L(t) \leftrightarrow i_C(t)$ and $i_L(t) \leftrightarrow v_C(t)$. What is the impulse response $h_L(t)$ for the LTI system with input $i_L(t)$ and output $v_L(t)$

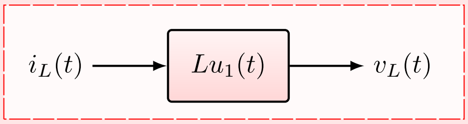


Solution: We write the output $v_L(t)$ in terms of input $i_L(t)$ as follows:

$$v_L(t) = L\frac{i_L(t)}{dt}.$$

Then the impulse response is given by

$$\boxed{h_L(t) = Lu_1(t),}$$



which is a scaled unit doublet (derivative of $\delta(t)$). □

Continuous-time System Properties

Problem Set 3-10

If a system is memoryless is it causal?

Solution: Yes. ☐

Problem Set 3-11

If a system is non-causal can it be memoryless?

Solution: No, by the contrapositive of the previous question. That is, saying "memoryless implies causal" (which is true) is the same as saying "not causal implies not memoryless". If you are not familiar with the contrapositive then: "rain implies clouds", therefore "no clouds implies no rain". ☐

Problem Set 3-12

Determine whether each of the following systems, where $x(t)$ or $x[n]$ is the input signal and $y(t)$ or $y[n]$ is the output signal, are: i) linear, ii) time-invariant, and iii) causal and iv) Memoryless.

System	Linear	Time-Invariant	Causal	Memoryless
$y(t) = x(t-1)$				
$y[n] = x[1-n]$				
$y(t) = 2x(t) + 3$				
$y(t) = x(5t)$				
$y(t) = x(t/5)$				
$y(t) = \text{Real}\{x(t)\}$				
$y[n] = \sum_{k=0}^{\infty} x[k]$				
$y[n] = \sum_{k=-10}^{n-3} x[k]$				
$y(t) = \sin\left(2\pi x(t/5)\right)$				
$y[n] = \cos(2\pi n)x[n]$				
$y[n] = \cos(\pi n)x[n]$				
$y[n] = \sum_{k=-10}^{5} x[k]$				

Solution:

System	Linear	Time-Invariant	Causal	Memoryless
A: $y(t) = x(t-1)$	✓	✓	✓	X
B: $y[n] = x[1-n]$	✓	X	X	X
C: $y(t) = 2x(t) + 3$	X	✓	✓	✓
D: $y(t) = x(5t)$	✓	X	X	X
E: $y(t) = x(t/5)$	✓	X	X	X
F: $y(t) = \mathrm{Real}\{x(t)\}$	X	✓	✓	✓
G: $y[n] = \sum_{k=0}^{\infty} x[k]$	✓	X	X	X
H: $y[n] = \sum_{k=-10}^{n-3} x[k]$	✓	X	✓	X
I: $y(t) = \sin\big(2\pi x(t/5)\big)$	X	X	X	X
J: $y[n] = \cos(2\pi n)x[n]$	✓	✓	✓	✓
K: $y[n] = \cos(\pi n)x[n]$	✓	X	✓	✓
L: $y[n] = \sum_{k=-10}^{5} x[k]$	✓	X	X	X

For the more challenging and trickier ones.

- **C:** put zero function in and the constant value of 3 comes out. Then scale this input by 7, say, and the output is still 3 and not 21 (which it would be if it were linear).
- **D:** $y(t)$ at $t = 1$ needs $x(t)$ at (future) $t = 5$.
- **E:** $y(t)$ at $t = -1$ needs $x(t)$ at (future) $t = -1/5$.
- **F:** Very tricky. Multiply the input by a complex scalar (e.g., $j = \sqrt{-1}$) generally doesn't multiply the output by the same complex scalar.
- **G:** If input is $x_1[n] = \delta[n+1]$ then $y_1[n] = 0$; then shifted input $x_2[n] = x[n-1] = \delta[n]$ gives $y_2[n] = 1 \neq y_1[n] = 0$.
- **H:** Linear and causal. It is time-varying because then number of inputs it combines varies with n.
- **J:** A trick because $\cos(2\pi n) = 1$ (for integer n).
- **K:** $\cos(2\pi n) = (-1)^n$
- **L:** If input is $x_1[n] = \delta[n+11]$ then $y_1[n] = 0$; then shifted input $x_2[n] = x[n-1] = \delta[n+10]$ gives $y_2[n] = 1 \neq y_1[n] = 0$.

□