Signal Processing
ENGN2228

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Second Semester

Part 1 Outline

What are Signals?

Definition (Signals)

Signals carry information in their variations. Mathematically signals are functions of one or more independent variables.

Examples of signals are:

- speech and audio signals, output from a microphone
- voltages, currents
- images, stock market prices, currency exchange rates
- temperature, weather data
- biological signals
- $\cdots$

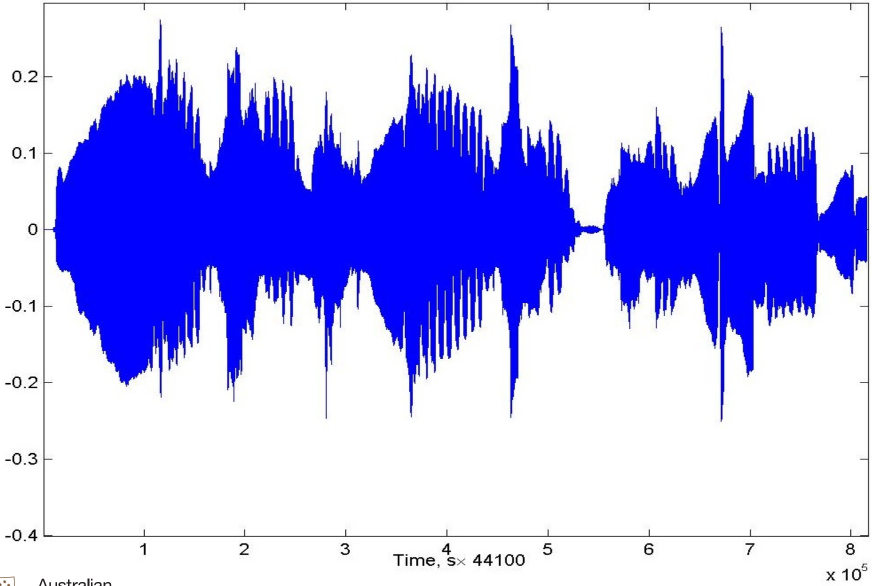
Signals are generally connected with physical quantities that vary with time or space or both.

Part 1 Outline

Examples of Signals – Currency (vs time)

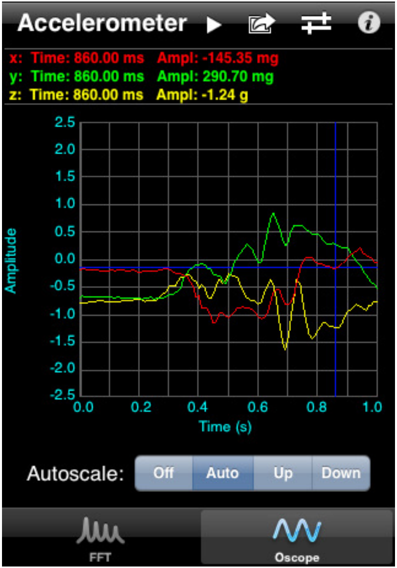
Examples of Signals – Share Price (vs time)

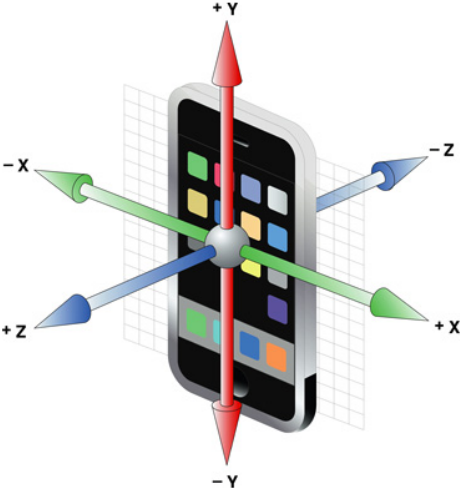
Examples of Signals – Flute Sound Waveform (vs time)



Examples of Signals – An Image (vs pixel space)

Examples of Signals – iPhone Accelerometers (vs time)





Part 1 Outline

Independent Variables (types)

Signals vary with respect to the "independent variables"

- **Continuous**
  - air temperature across Australia (latitude and longitude)
  - petrol price as a function of time

- **Discrete**
  - digital image pixels (xy), 3D medical image voxels (xyz)
  - petrol price for every day of the week

Independent Variables – (dimension)

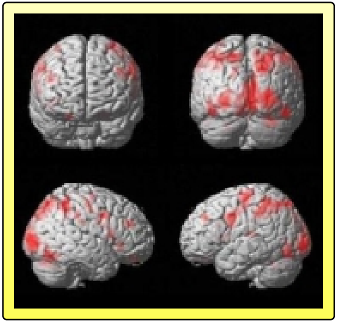
The independent variables can be one dimensional 1D, 2D, 3D, etc.

- A signal may vary with time (1D)
- An image varies with cartesian coordinates x and y in space (2D)
- The temperature can vary with position in a room horizontal x and y, and vertical z (3D)
- A movie is a 2D image that varies with time (3D)
- What dimension is a 3D movie?

Independent Variables – (dimension)

- fMRI – functional Magnetic Resonance Imaging – 3D volume of patient's brain is imaged every one or two seconds (4D, i.e., 3 space and 1 time dimensions)

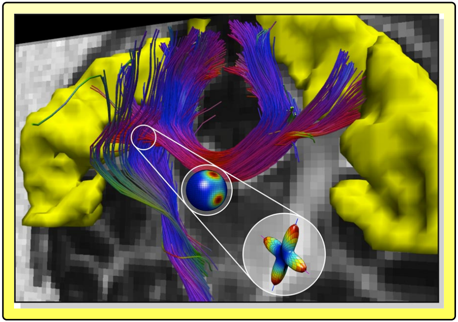
Oversized pencil sharpener



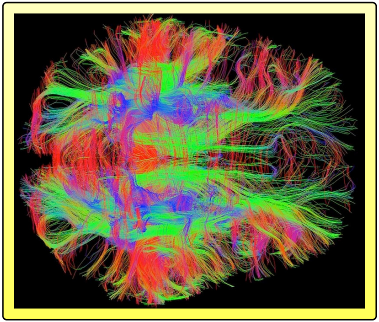
fMRI

Independent Variables – (dimension)

- dMRI – diffusion Magnetic Resonance Imaging – 3D volume of patient's brain is imaged and the diffusion of water molecules is measured in 3D within every voxel - 6D image.



6D dMRI image



Brain wiring

Independent Variables – (course focus on 1D)

- For this course we will focus on 1D
- That is, a single independent variable
- Most cases this is “time”

Part 1 Outline

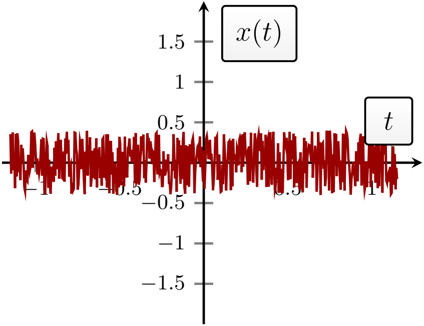
Definition (Continuous-Time, CT, Signals)

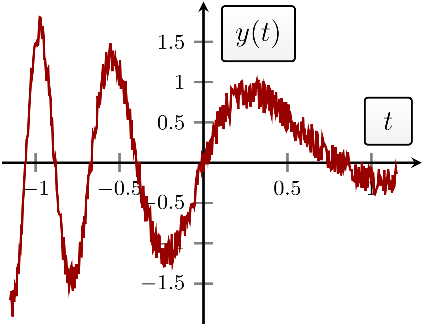
Continuous-Time Signals are signals whose independent variable is **continuous** and taken as time. That is, $x(t)$ with continuum t.

$$\boxed{x(t), \quad t \in \mathbb{R} \text{ (real numbers/time)}}$$

- Signals from the real physical world are generally CT, such as voltage, pressure, velocity, etc, as functions of time t
- Examples of real-valued CT signals (functions) are shown next: one noisy and the other some dying signal with additive noise.
- Followed by an example of a complex-valued CT signal.

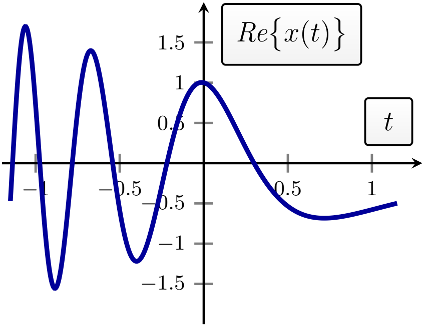
Continuous Time Signals – Examples (note (t))

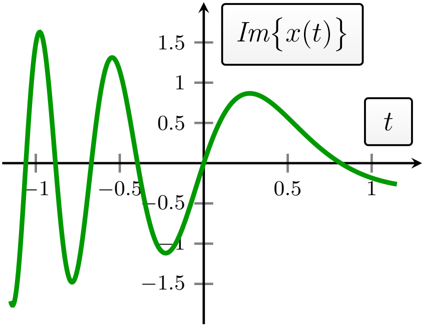




Continuous Time Signals – Complex Example

$$x(t) \triangleq \exp\big(2\pi j\, t\, \exp(-0.6\, t)\big)\, \exp(-0.5\, t) \in \mathbb{C}$$





Part 1 Outline

Discrete Time Signals

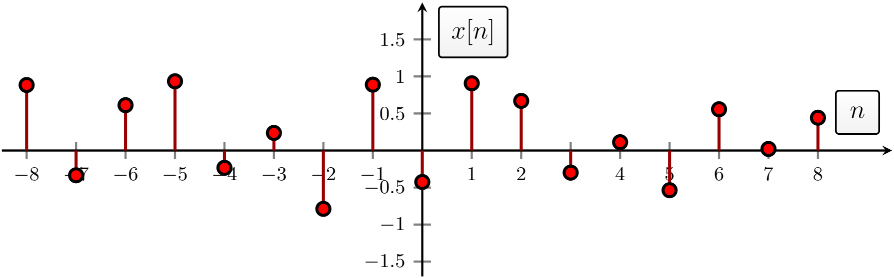
Definition (Discrete-Time (DT) Signals)

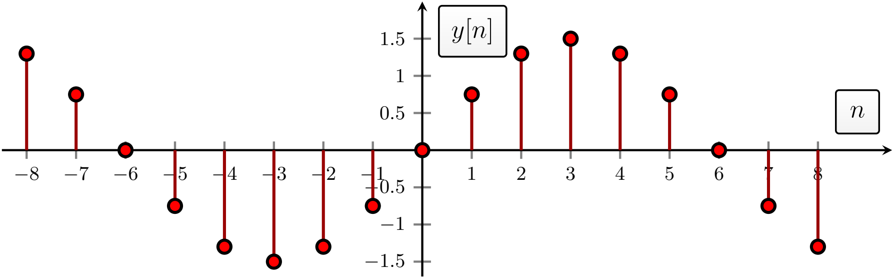
Discrete-Time Signals are signals whose independent variable takes on only a **discrete set of values** and are generally taken to be integer values. That is, $x[n]$ with discrete/integer n

$$\boxed{x[n], \quad n \in \mathbb{Z} \text{ (integers)}}$$

- Signals from the real physical world are generally not naturally DT, but most man-made and "sampled" signals are DT
- Examples of real-valued DT signals (discrete functions) are shown next slide:
  - $x[n]$ is somewhat random
  - $y[n]$ looks like a sampled sinusoid

Discrete Time Signals – Real-Valued Examples (note $[n]$)





Discrete Time Signals

Reminder:

Continuous-Time (CT) signals: $x(t)$ with t – continuous values[1]

Discrete-Time (DT) signals: $x[n]$ with n – integer values[1]

[1] Round brackets $\equiv$ continuous. Square brackets $\equiv$ discrete.

Discrete Time Signals

- Natural DT signals? Less common, e.g., DNA base sequence
- Most DT signals are man-made
- Images, digital music, stock market data, etc

DT signals are increasingly important because they are in a form that permit calculations, that is, “processing”, via computers (CPUs), graphics processing units (GPU’s) and digital signal processors (DSPs)

i) Intel CPU, ii) ATI RADEON GPU and iii) TI TMS320 DSP

Part 1 Outline

Periodic Signals (CT definition)

Definition (Periodic Continuous Time Signals)

CT Signal $x(t)$ is **periodic with period** $T > 0$ if

$$x(t) = x(t + T)$$

for all real $t \in \mathbb{R}$ (or $\forall t \in \mathbb{R}$).

- If the signal is given by

$$x(t) = \sin(t),$$

  then $x(t) = x(t + 2\pi)$, $\forall t$, and $x(t) = x(t + 4\pi)$, $\forall t$, etc.
- So, $x(t) = \sin(t)$ is periodic with period $T = 4\pi$. But, of course, $T = 4\pi$ is not the smallest period. This motivates the following definition:

Periodic Signals (CT definition)

Definition (Fundamental Period T_0)

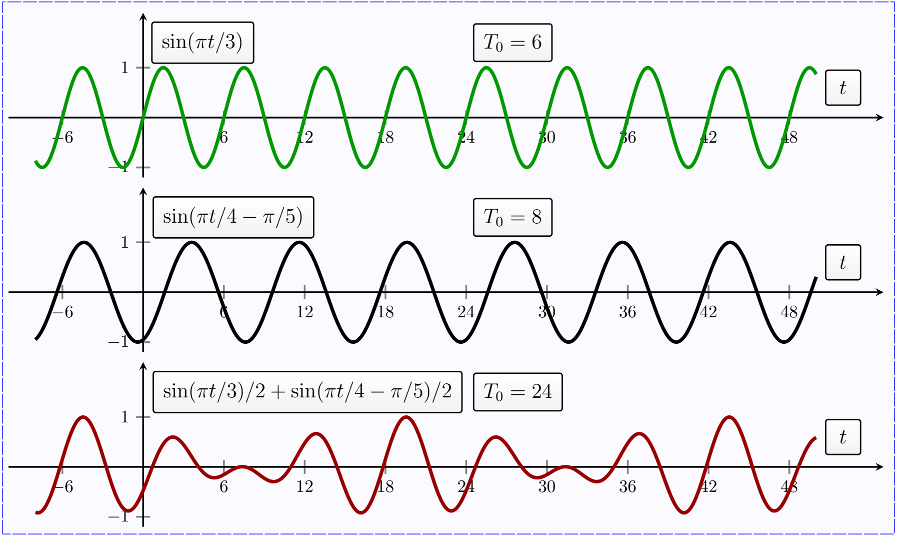
The fundamental period, $T_0 > 0$, is the **smallest positive period** T for which $x(t)$ is periodic.

- If $x(t) = \sin(t)$, then $T_0 = 2\pi$.
- If $x(t) = A\sin(\omega_0 t + \theta)$ or $x(t) = A\cos(\omega_0 t + \theta)$, then

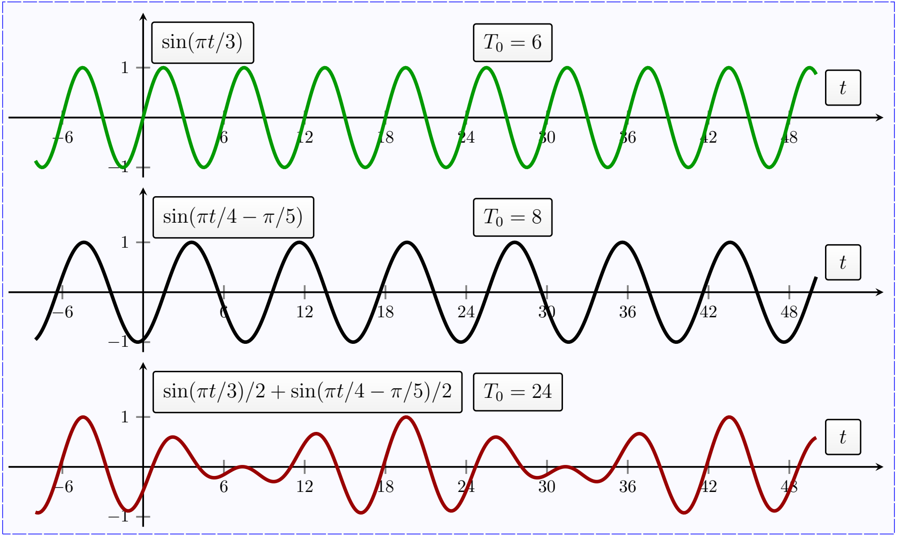
$$\boxed{T_0 = \frac{2\pi}{\omega_0}}$$

- Also $x(t) = \exp(jt)$ has fundamental period $T_0 = 2\pi$. So complex signals can be periodic too. Use same formula calculate T_0
- $x(t) = \sin(4t)$ has fundamental period $T_0 = \pi/2$.
- $x(t) = \sin(3t)$ has fundamental period $T_0 = 2\pi/3$.
- $x(t) = \sin(t) + \sin(4t)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(3t) + \sin(4t)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(3t) + \sin(4t + \pi/7)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(\pi\, t/3)/2 + \sin(\pi\, t/4 - \pi/5)/2$ has fundamental period $T_0 = 24$.

Periodic Signals – Example



Examples of Signals – Example



Periodic Signals (DT definition)

Definition (Periodic Discrete Time Signals)

DT Signal $x[n]$ is **periodic with integer period** $N > 0$ if

$$x[n] = x[n+N]$$

for all integer $n \in \mathbb{Z}$.

Definition (Fundamental Period N_0)

The fundamental period, $N_0 > 0$, is the **smallest positive integer** N for which $x[n]$ is periodic.

- For DT sinusoidal and exponential signals:

$$\boxed{N = \frac{2\pi}{\omega_0} m}$$

where $m = 1, 2, \ldots$ such that N is an integer.

- $x[n] = \cos[2\pi n]$, $\omega_0 = 2\pi$ therefore $N = \frac{2\pi}{2\pi} m$. For $m = 1$, $N_0 = 1$.
- $x[n] = 5 \sin[\frac{6\pi n}{35}]$, $\omega_0 = \frac{6\pi}{35}$ therefore $N = \frac{35}{3} m$. For $m = 3$, $N_0 = 35$.
- $x[n] = \cos\left[\frac{\pi}{2} n\right] + \cos\left[\frac{\pi}{4} n\right]$, $N_0 = 8$ as lowest common multiple.

Periodic Signals[(DT definition)]1.2.2page 12

- Are CT sinusoids always periodic? YES
- Are DT sinusoids always periodic? NO - depends on the sampling rate
  - ω_0 must be a rational multiple of 2π.
- $x[n] = \cos[\frac{n}{6}]$, $\omega_0 = \frac{1}{6}$, $N = \frac{2\pi}{\frac{1}{6}} = 12m\pi$ so non-periodic.

Part 1 Outline

Signal Energy and Power (background)

Definition (Power vs Energy)

Power (Watts) is energy (Joules) transferred per unit of time (seconds).

- Power is the rate at which energy is delivered.
- Understand the difference between energy and power.

Signal Energy and Power (physical motivation)

For a resistor R continuous time **instantaneous power** in a circuit is the product of voltage and current

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t) = R\,i^2(t)$$

The **total energy** dissipated from time t_1 to time t_2 is

$$\boxed{\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt}$$

and the **average power** over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

such that the energy delivered over the time interval $t_2 - t_1$ at the average power equals the total energy. It is the mean rate that energy is delivered.

Signal Energy and Power (signal abstraction)

Definition (Total energy of a continuous time signal $x(t)$)

Total energy of a continuous time signal $x(t)$ between real times instants t_1 and t_2 is

$$\boxed{\int_{t_1}^{t_2} \left|x(t)\right|^2 dt}$$

Definition (Total energy of a discrete time signal $x[n]$)

Total energy of a discrete time signal $x[n]$ between integer time instants n_1 and n_2 is

$$\boxed{\sum_{n=n_1}^{n_2} \left|x[n]\right|^2}$$

- Signals can be real or complex valued; $|\cdot|$ means "absolute" value for real and "magnitude" for complex
- Don't worry about the physical interpretation; this is just a definition

Signal Energy and Power (Infinite Time Interval Energy)

Definition (Infinite time interval total energy of a continuous time signal $x(t)$)

$$E_\infty = \lim_{T \longrightarrow \infty} \int_{-T}^{T} \left| x(t) \right|^2 dt$$

Definition (Infinite time interval total energy of a discrete time signal $x[n]$)

$$E_\infty = \lim_{N \longrightarrow \infty} \sum_{n=-N}^{N} \left| x[n] \right|^2$$

- Total energy may be finite or infinite.

Signal Energy and Power (Infinite Time Interval Time-Average Power)

Definition (Infinite time interval time-average power of a continuous time signal $x(t)$)

$$\boxed{P_\infty = \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left| x(t) \right|^2 dt}$$

Definition (Infinite time interval time-average power of a continuous time periodic signal)

$$\boxed{P_\infty = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| x(t) \right|^2 dt}$$

Signal Energy and Power (Infinite Time Interval Time-Average Power)

Definition (Infinite time interval time-average power of a discrete time signal $x[n]$)

$$P_\infty = \lim_{N \longrightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2$$

Definition (Infinite time interval time-average power of a discrete time periodic signal)

$$P_\infty = \frac{1}{N} \sum_{n=0}^{N-1} \left|x[n]\right|^2$$

Signal Energy and Power (Examples)

- $$x[n] = \begin{cases} \cos(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

  Using $E_\infty = \lim_{N \longrightarrow \infty} \sum_{n=-N}^{N} \left|x[n]\right|^2$:

  $E_\infty = 9$

- $$x(t) = 5\cos(\pi t) + \sin(5\pi t)$$

  Using $P_\infty = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|x(t)\right|^2 dt$ get $P_\infty = 13$

Signal Energy and Power (various cases)

- P_∞ may be finite or infinite. Natural signals are expected to be finite power.
- $E_\infty < \infty$ implies $P_\infty = 0$. Finite energy signals have zero average power over the infinite interval.
- $P_\infty > 0$ implies $E_\infty = \infty$. Finite average power signals end up delivering infinite energy over the infinite interval.
- Both $P_\infty = \infty$ and $E_\infty = \infty$ also mathematically possible, but not of much engineering interest.

Part 1 Outline

Odd and Even Signals even signals

Definition (Even Continuous Time Signals)

CT Signal $x(t)$ is **even** if

$$\boxed{x(-t) = x(t), \quad \text{for all } t}$$

i.e. it is symmetric about the y-axis.

We find the even component of a signal $x(t)$ as:

$$\begin{aligned} x_e(t) &= \frac{1}{2}\big(x(t) + x(-t)\big) \\ &= \frac{1}{2}\big(x(-t) + x(t)\big) \\ &= x_e(-t) \end{aligned}$$

Odd and Even Signals (odd signals)

Definition (Odd Continuous Time Signals)

CT Signal $x(t)$ is **odd** if

$$\boxed{x(-t) = -x(t), \qquad \text{for all } t}$$

i.e. anti-symmetric about the y-axis.

We find the odd component of a signal $x(t)$ as:

$$\begin{aligned} x_o(t) &= \frac{1}{2}\big(x(t) - x(-t)\big) \\ &= -\frac{1}{2}\big(x(-t) - x(t)\big) \\ &= -x_o(-t) \end{aligned}$$

Odd and Even Signals

For any signal $x(t)$, then we can write

$$\boxed{x(t) = \underbrace{\frac{1}{2}\big(x(t) + x(-t)\big)}_{x_e(t)} + \underbrace{\frac{1}{2}\big(x(t) - x(-t)\big)}_{x_o(t)}}$$

which, in a way, seems completely daft since we introduce a time reversed signal, $x(-t)$ which is cancelled.

Other notation `(see O&W p.14)`

$$\mathit{Ev}\{x(t)\} \triangleq x_e(t) = \frac{1}{2}\big(x(t) + x(-t)\big)$$
$$\mathit{Od}\{x(t)\} \triangleq x_o(t) = \frac{1}{2}\big(x(t) - x(-t)\big)$$

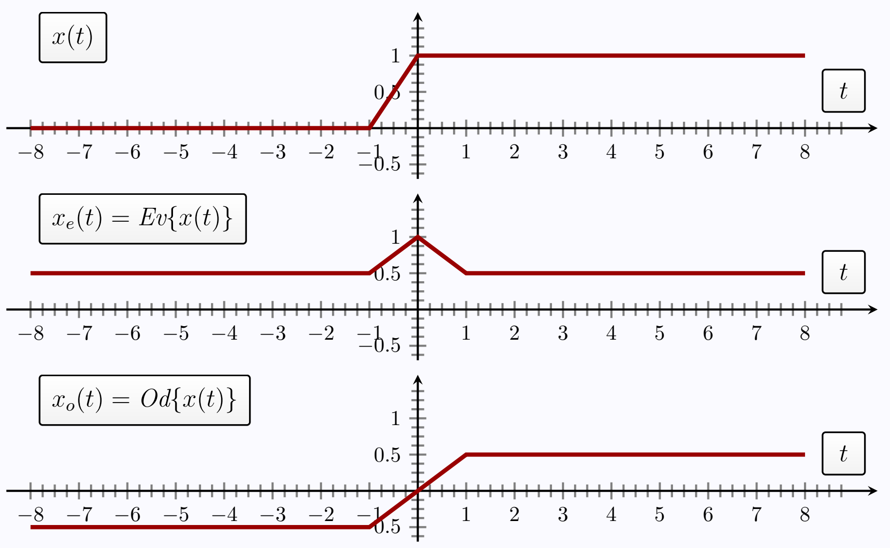
Odd and Even Signals (CT Signal Decomposition)

$$\boxed{x(t) = \underbrace{\frac{1}{2}\big(x(t) + x(-t)\big)}_{x_e(t)} + \underbrace{\frac{1}{2}\big(x(t) - x(-t)\big)}_{x_o(t)}}$$

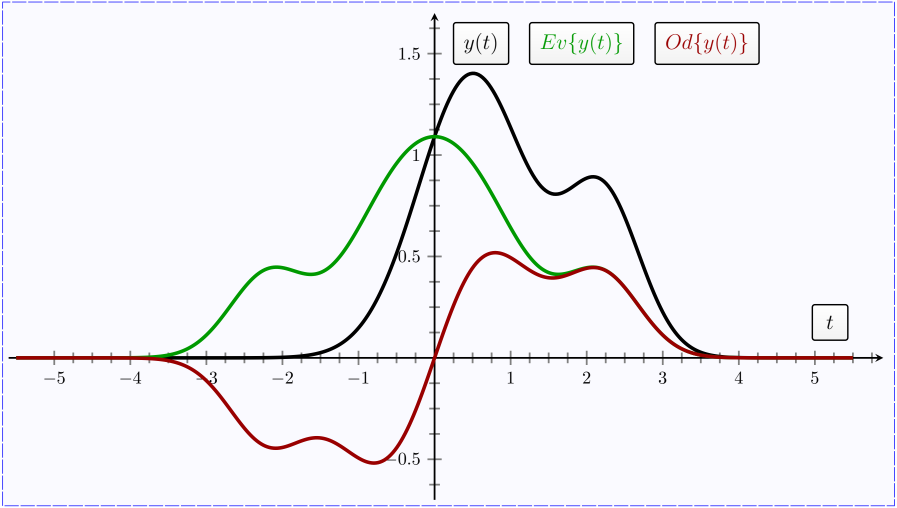
means we can always decompose any CT signal into the sum of an odd CT signal and an even CT signal.

Even signals are **symmetric** and odd signal are **anti-symmetric**. Both properties play an important role in simplifying signal processing.

Odd and Even Signals (CT Example)



Odd and Even Signals (CT 2nd Example)



Odd and Even Signals (CT 3rd Example)

$$x(t) = 3t^3 - 2t^2 + 5t - 7$$

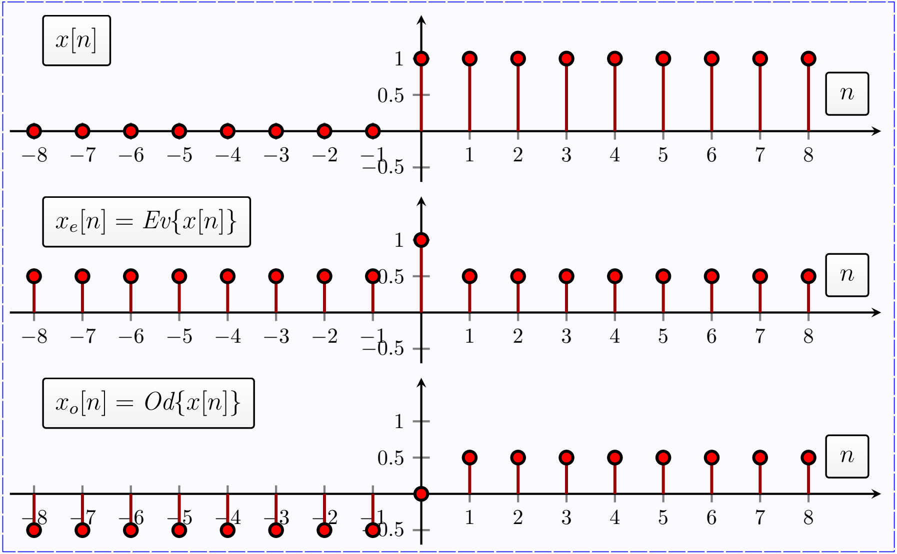
Odd and Even Signals (DT Signal Decomposition)

For DT signals

$$\boxed{x[n] = \underbrace{\frac{1}{2}\big(x[n] + x[-n]\big)}_{x_e[n]} + \underbrace{\frac{1}{2}\big(x[n] - x[-n]\big)}_{x_o[n]}}$$

means we can always decompose any DT signal into the sum of an odd DT signal and an even CT signal.

Odd and Even Signals (DT Example)



Odd and Even Signals

Useful property, e.g.:

- Diffusion magnetic resonance imaging (dMRI) signal is antipodally symmetric (even in 3D)
- This property can be used to reduce the number of samples required
- Important - number of measurements governs scan time

8 **Complex Number Revision**

9 CT Exponential Signals

10 DT Exponential Signals

11 Periodicity of DT Exponential Signals

12 Signal Transformations

Complex Number Revision

Engineers use j to denote the imaginary unit, mathematicians use i

$$j = \sqrt{-1}$$

$$j^2 = -1$$

$$\frac{1}{j} = -j$$

Three identities with complex exponentials $e^{j\theta}$ used heavily in this course:
Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Number Revision

Some examples:

$$
\begin{aligned}
x[n] &= e^{j2\pi n} \\
&= (e^{j2\pi})^n \\
&= [cos(2\pi) + j\sin(2\pi)]^n \\
&= 1^n \\
&= 1
\end{aligned}
$$

Is this periodic?

$$
\begin{aligned}
x[n] &= e^{j\pi n} \\
&= (e^{j})^n \\
&= [cos(\pi) + j\sin(\pi)]^n \\
&= (-1)^n
\end{aligned}
$$

What is the period?

Complex Number Revision– Rectangular and Polar Form

- Rectangular form $x + jy$
- Polar form $re^{j\theta}$, $-\pi < \theta \le \pi$
- Convention in signal processing is to express angle θ in radians.
- Can use functions on your calculator to convert. Make sure it is in radians mode.
- Or use trig identities.

Complex Number Revision– Rectangular and Polar Form

Rectangular to polar:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar to rectangular:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\sqrt{3} + j^3)(1 - j)$$

now $j^3 = j \times j^2 = -j$ so

$$\begin{aligned}(\sqrt{3} + j^3)(1 - j) &= (\sqrt{3} - j)(1 - j) \\ &= \sqrt{3} - j\sqrt{3} - j + j^2 \\ &= (\sqrt{3} - 1) - j(\sqrt{3} + 1) \\ &= 2.828e^{-j1.308} = 2\sqrt{2}e^{-j\frac{5\pi}{12}}\end{aligned}$$

Either version of answer is fine.

Complex Number Revision– Complex Conjugate

In rectangular form:

$$z = x + jy$$

$$z^* = x - jy$$

e.g. $z_1 = 4 - j3$, $z_1^* =$

In polar form:

$$re^{j\theta}$$

$$re^{-j\theta}$$

$z_1 = 5e^{j0.5\pi}$, $z_1^* =$

Part 2 Outline

CT Exponential Signals

- A fundamental signal class is the complex exponential signals

$$x(t) = Ce^{\alpha t}$$

  with C and α complex numbers ($C, \alpha \in \mathbb{C}$).

- This is a compact way to represent: pure sinusoids, real exponentials, exponentially growing or decaying sinusoids, etc. They are basic building blocks from which we can construct many signals of interest.

- With $C = |C|e^{j\theta}$ and $\alpha = r + j\omega_0$, we can write

$$x(t) = |C|e^{rt}e^{j(\omega_0 t + \theta)}$$

  Parameter $|C|$ is the magnitude, r is the exponential growth, ω_0 is the fundamental frequency of the oscillation in rad/sec and θ is the phase offset.

- We should be comfortable with $j \triangleq \sqrt{-1}$ such that $j^2 = -1$.

CT Exponential Signals

- Note, Euler's relation `O&W p.71`

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

which has fundamental period $T_0 = 2\pi/|\omega_0|$.

- Signal $x(t) = e^{j\omega_0 t}$ has finite, in fact unity, average power

$$\begin{aligned} P_\infty &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left|x(t)\right|^2 dt \\ &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt \\ &= 1 \end{aligned}$$

and (necessarily) infinite total energy, $E_\infty = \infty$.

CT Exponential Signals

- Many identities can be easily derived such as

$$
\begin{aligned}
A\cos(\omega_0 t + \phi) &= \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t} \\
&= A\,\mathit{Re}\{e^{j(\omega_0 t + \phi)}\}
\end{aligned}
$$

$$
A\sin(\omega_0 t + \phi) = A\,\mathit{Im}\{e^{j(\omega_0 t + \phi)}\}
$$

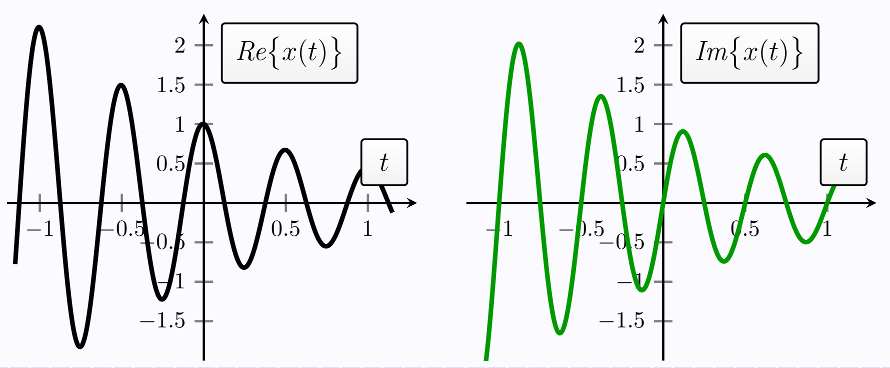
  See `O&W 1.3.1 p.17`

- Get comfortable and competent with such calculations.

CT Exponential Signals – Example

With $C = 1$ and $\alpha = -0.5 + j4\pi$ in $x(t) = Ce^{\alpha t}$ we have

$$\boxed{x(t) \triangleq \exp\left(4\pi j\, t\right) \exp(-0.5\, t) \in \mathbb{C}}$$



CT Exponential Signals – Periodicity

$$x(t) = e^{j\frac{2\pi}{3}t}$$

- Is $x(t)$ periodic?
- Expanding $x(t) = \cos(\frac{2\pi}{3}t) + \sin(\frac{2\pi}{3}t)$.
- Real and imaginary parts periodic.
- Hence $x(t)$ periodic.
- Comparing with $e^{j\omega_0 t}$, $\omega_0 = \frac{2\pi}{3}$ so $T = 3$.

Part 2 Outline

DT Exponential Signals

- A fundamental signal class is the class of DT complex exponential signals

$$x[n] = C\alpha^n, \quad n \in \mathbb{Z}$$

  with C and α complex numbers ($C, \alpha \in \mathbb{C}$).

- Generally DT signals emulate the properties of CT signals or vice versa.
- Slightly weird things can emerge for DT signals though. You should rely on mathematical analysis to resolve any confusion rather than trying to memorize any details. However, you should make a mental note to be wary.
- First we look at the case where $|\alpha| = 1$ which we can write in the form

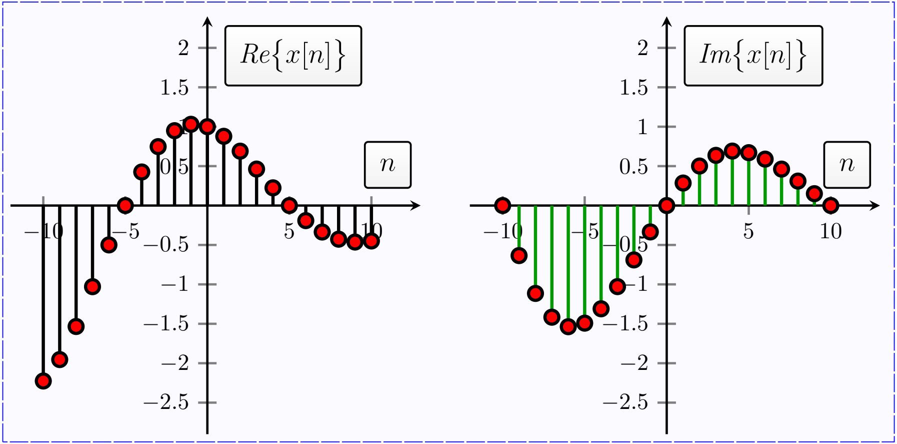
$$x[n] = e^{j\omega_0 n}, \quad n \in \mathbb{Z},$$

  that is, $C = 1$ and $\alpha = e^{j\omega_0}$ for some $0 \leq \omega_0 < 2\pi$.

DT Exponential Signals – Example

With $C = 1$ and $\alpha = -0.08 + j0.1\pi$ in $x[n] = Ce^{\alpha n}$ we have

$$\boxed{x[n] \triangleq \exp\left(0.1\pi j\, n\right) \exp(-0.08\, n) \in \mathbb{C}}$$



Part 2 Outline

Periodicity of DT Exponential Signals

- Note that CT signal $x(t) = e^{j\omega_0 t}$ is **periodic** (with fundamental period $T_0 = 2\pi/|\omega_0|$) for all values of $\omega_0 \neq 0$). This is not quite true of the analogous DT exponential signal.
- For
$$x[n] = e^{j\omega_0 n}, \quad n \in \mathbb{Z}$$
to be periodic we need $x[n] = x[n+N]$ for some period N (which may or may not be the fundamental period N_0).
- The issue is that N may not be, and in general is not likely to be, synchronized with the period implicit in ω_0 which is $2\pi/|\omega_0|$ and is generally not an integer.
- Sometimes $x[n] = e^{j\omega_0 n}$ is periodic, e.g., if $2\pi/|\omega_0| = N$ is an integer then $x[n] = e^{j\omega_0 n}$ has period N. (This is not quite the only case for periodicity, see next slide.)
- Generally $x[n] = e^{j\omega_0 n}$ is not periodic.

Periodicity of DT Exponential Signals

- Other possibilities for periodicity to arise are when $2\pi k/|\omega_0| = N$ for integer $k \in \mathbb{Z}$ then $x[n] = e^{j\omega_0 n}$ has (integer) period kN.
- Plonking $\omega_0 = 2\pi k/N$ into $x[n] = e^{j\omega_0 n}$ leads to the harmonic set $\phi_k[n] \triangleq e^{jk(2\pi/N)n}$ where $k \in \mathbb{Z}$ (integer values). However, only N of these are distinct

$$\phi_k[n] \triangleq e^{jk(2\pi/N)n}, \quad k = 0, 1, \ldots, N-1 \tag{1}$$

  For example, $k = 3$ and $k = N + 3$ are not distinct because $\phi_k[n] = \phi_{k+N}[n]$ for all n. This is related to "aliasing" that we will meet later.
- As bland as (1) looks, in fact this set is one of the most important sets of signals and plays a fundamental role in most signal processing systems, most modern communications systems, digital television, most scientific processing, etc.

Periodicity of DT Exponential Signals

$$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$$

- $\omega_{01} =$
- Therefore $N_1 = m\frac{2\pi}{\omega_{01}} =$
- $\omega_{02} =$
- Therefore $N_2 = m\frac{2\pi}{\omega_{02}} =$
- Fundamental period, $N_0 = 24$ samples

Signal Transformations–dependent variable

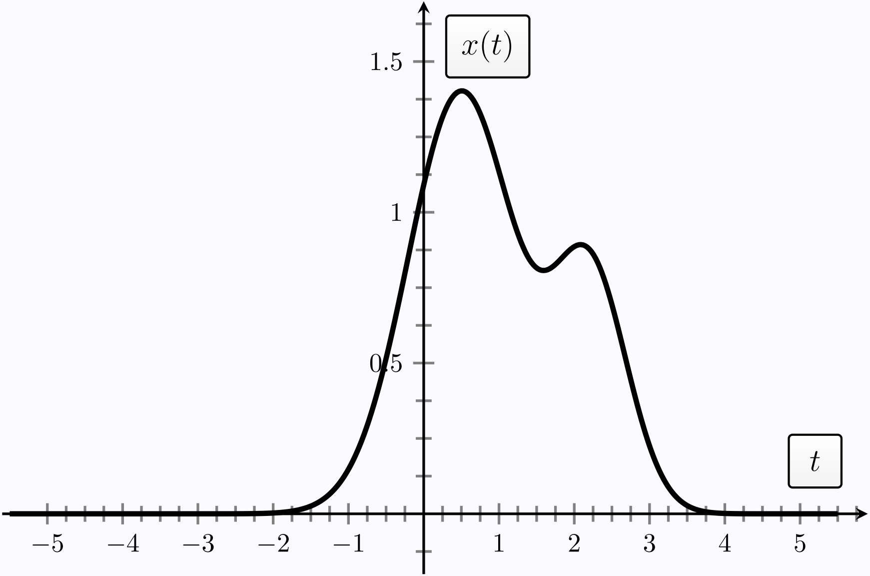
Already familiar with transformations of the signal such as:

Operation	CT	DT
Amplitude scaling (amplifier)	$y(t) = Cx(t)$	$y[n] = Cx[n]$
Addition (summing amplifier)	$y(t) = x_1(t) + x_2(t)$	$y[n] = x_1[n] + x_2[n]$
Multiplication	$y(t) = x_1(t)x_2(t)$	$y[n] = x_1[n]x_2[n]$

Signal Transformations (independent var.)

- A signal $x(t)$ at some point transferred to another point (e.g., via communication) would generally suffer a time delay, $x(t-\Delta)$. This and other simple scenarios defines a simple but important class of **signal transformations** which are easily characterized.
- These signal transformations can be understood in terms of “affine” transformations on the independent variable
- For time as the independent variable, this just means scaling (compressing or expanding), reversing and shifting (delaying or advancing) time
- Other signal transformations not of this form are possible and treated later.

Signal Transformations (CT reference signal)



Signal Transformations (time shift)

Time Shift

Independent variable change

$$t \longrightarrow t + \beta$$

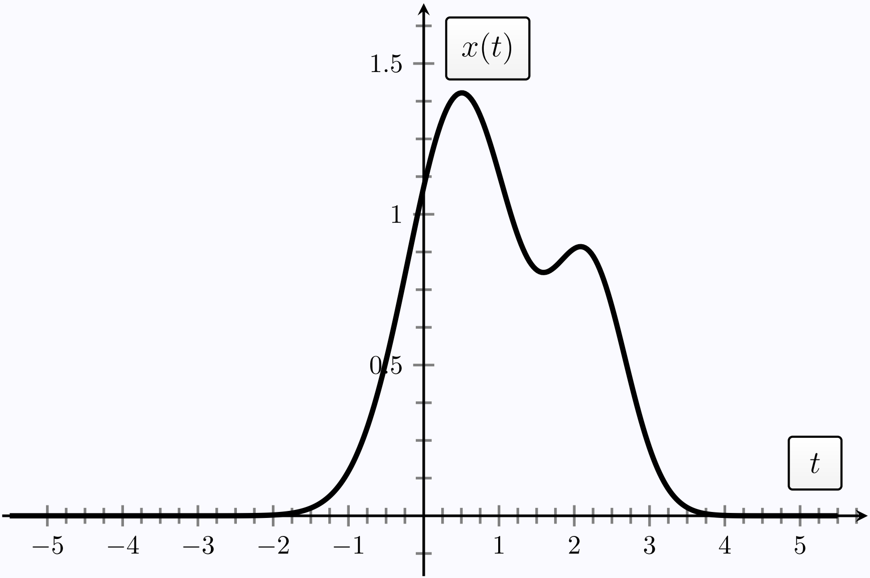
induces the signal change

$$x(t) \longrightarrow y(t) \triangleq x(t + \beta),$$

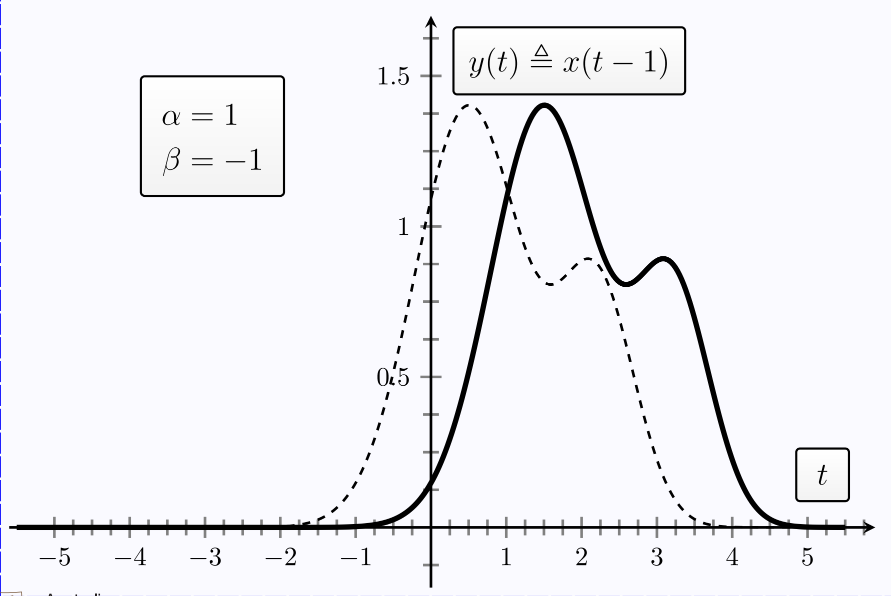
where $\beta \in \mathbb{R}$. If $\beta < 0$ then signal is delayed (shifts to the right), if $\beta = 0$ then no change, and if $\beta > 0$ then signal is advanced (shifts to the left).

(There is another parameter, α, which we'll meet shortly but for a time shift it is $\alpha = 1$.)

Signal Transformations (CT time shift)



Signal Transformations (CT time shift)



Time Scaling

Independent variable change

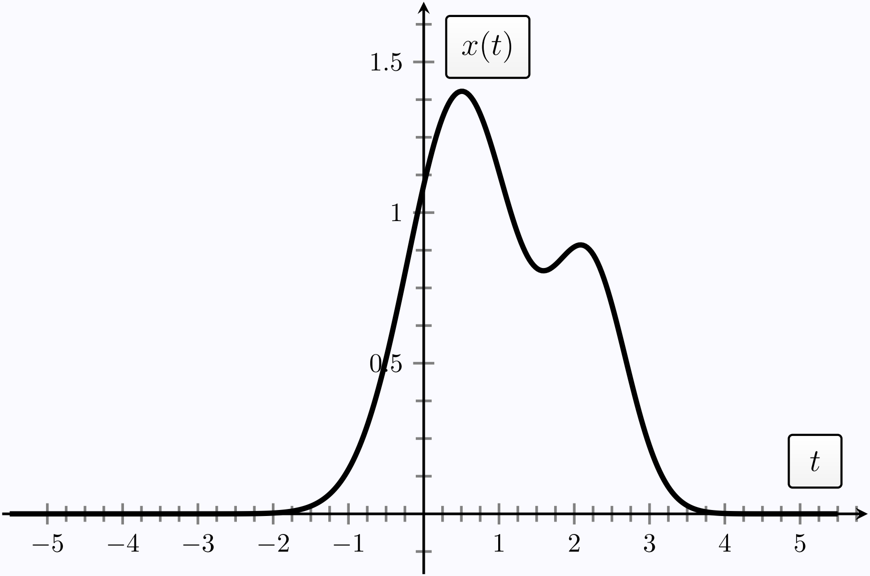
$$t \longrightarrow \alpha\, t$$

induces the signal change

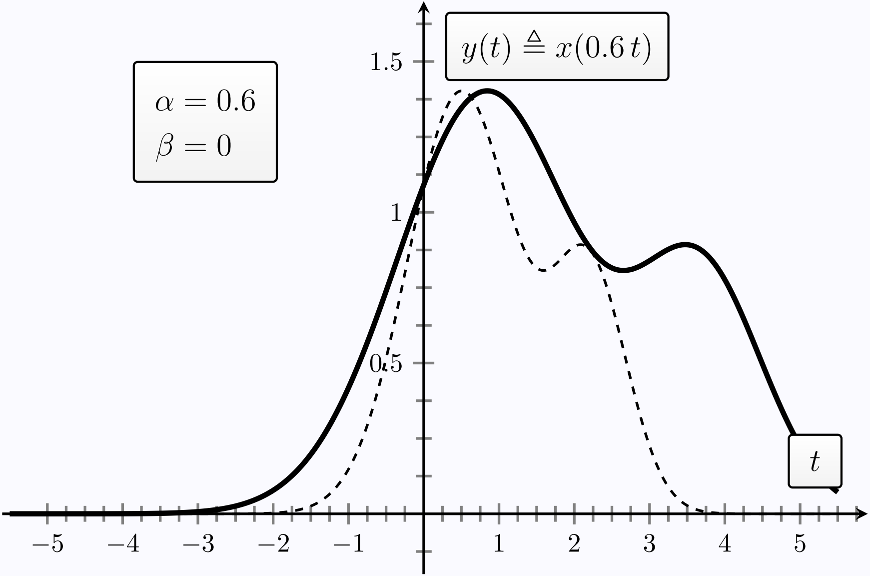
$$x(t) \longrightarrow y(t) \triangleq x(\alpha\, t),$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq 0$.
If $\alpha > 1$ then the signal is compressed, if $\alpha < 1$ then the signal is expanded.

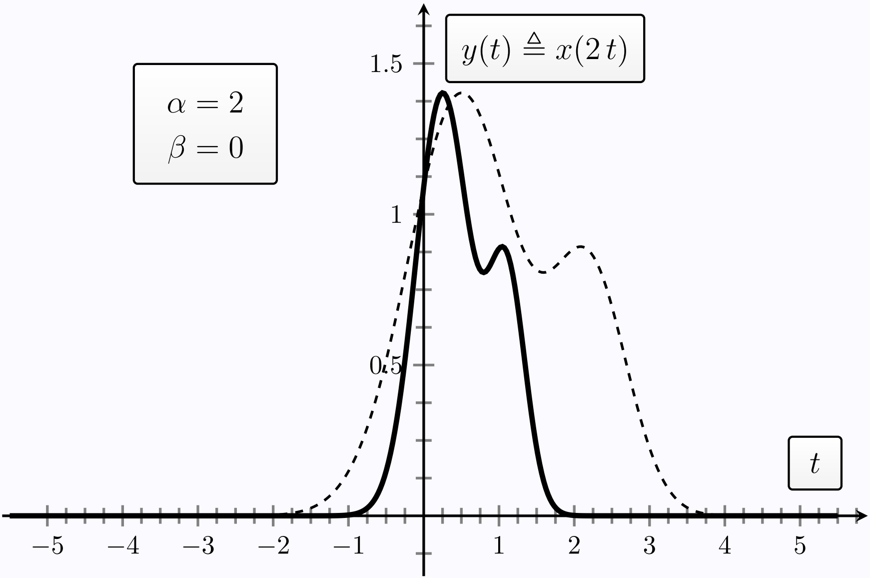
Signal Transformations (CT time scaling)



Signal Transformations (CT time scaling - expand)



Signal Transformations (CT time scaling - compress)



Time Reverse

Independent variable change

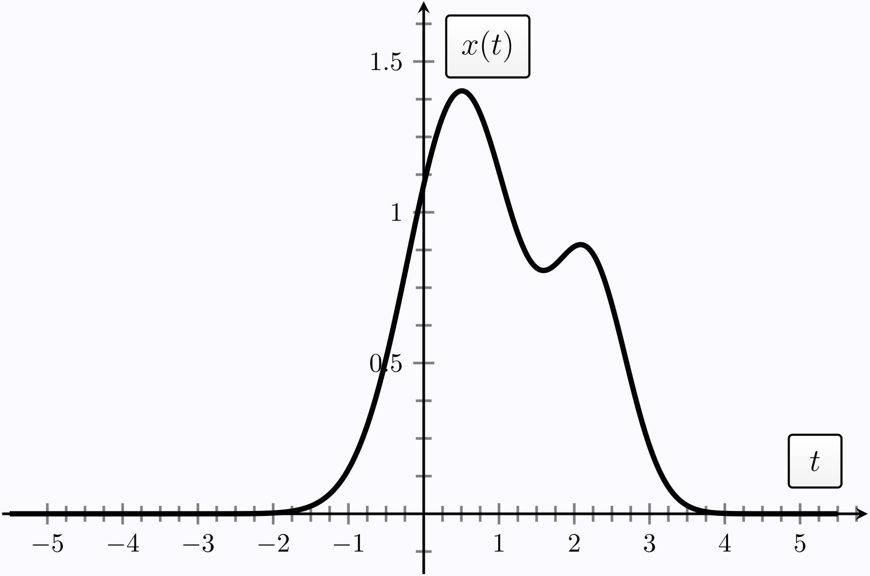
$$t \longrightarrow -t$$

induces the signal change

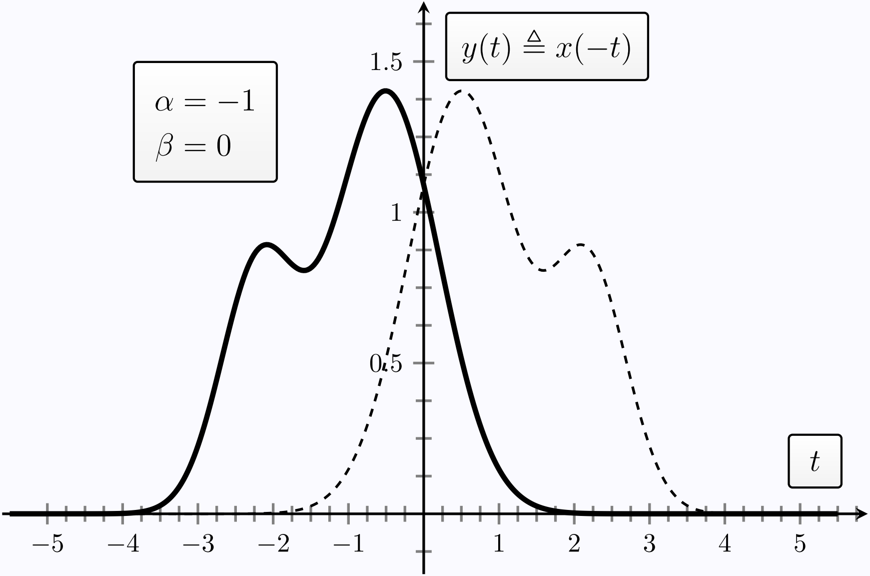
$$x(t) \longrightarrow y(t) \triangleq x(-t)$$

Where $\alpha = -1$.

Signal Transformations (CT time reversal)



Signal Transformations (CT time reversal)



Affine Transformation[2]

Independent variable change

$$t \;\longrightarrow\; \alpha\, t + \beta$$

induces the signal change

$$x(t) \;\longrightarrow\; y(t) \triangleq x(\alpha\, t + \beta)$$

where

$\alpha \in \mathbb{R}$ **time scales** expands whenever $|\alpha| < 1$ and
compresses whenever $|\alpha| > 1$ and/or
time-reverses whenever $\alpha < 0$

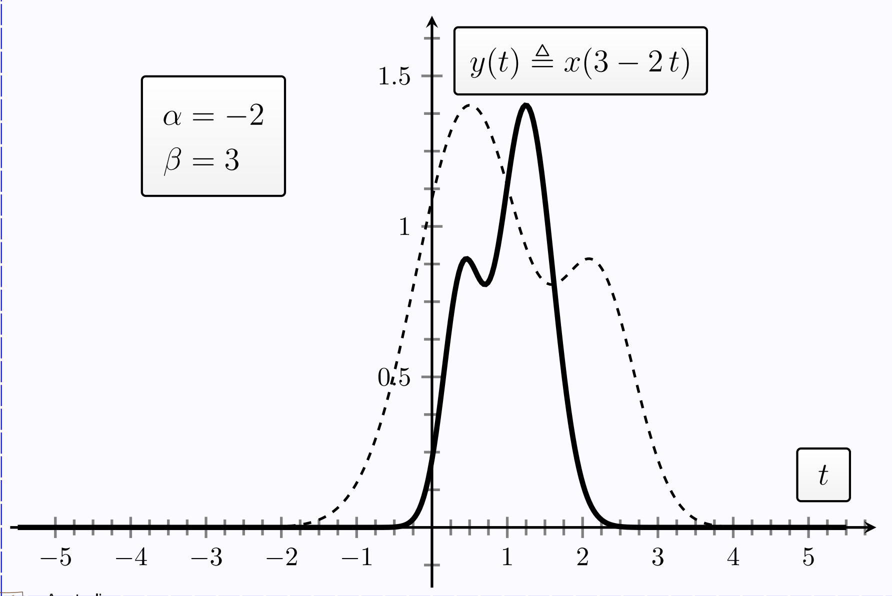
$\beta \in \mathbb{R}$ **time shifts** forward in time whenever $\beta < 0$ and
backward in time whenever $\beta > 0$

[2]affine $\equiv$ linear + constant

Signal Transformations (CT affine transformation)

- Single framework captures three transformations in one.
- Apply transformations in this order:
  - Apply time shift.
  - Then time scaling.
  - Finally time reversal.
- Check answer by substituting in a range of values of the independent variable.

Signal Transformations (CT affine time change)



Signal Transformations (independent variable)

- To this point we have only considered transformations

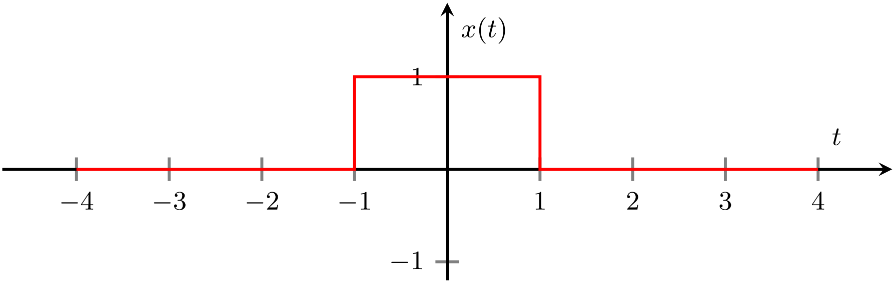
$$x(t) \longrightarrow y(t) \triangleq x(\alpha\, t + \beta)$$

which affinely transform the independent variable.

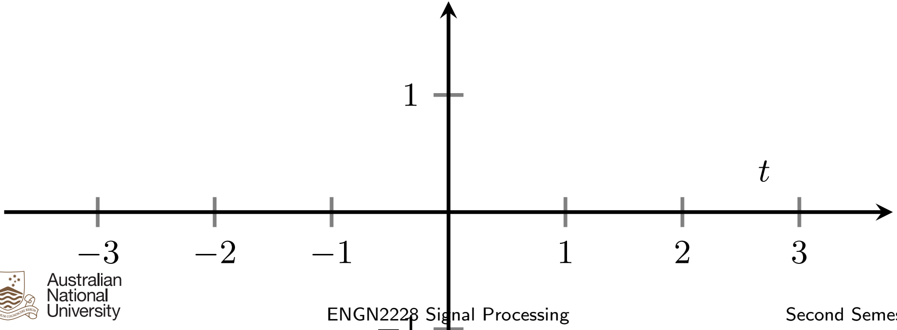
- But this **excludes** other simple signal transformations such as

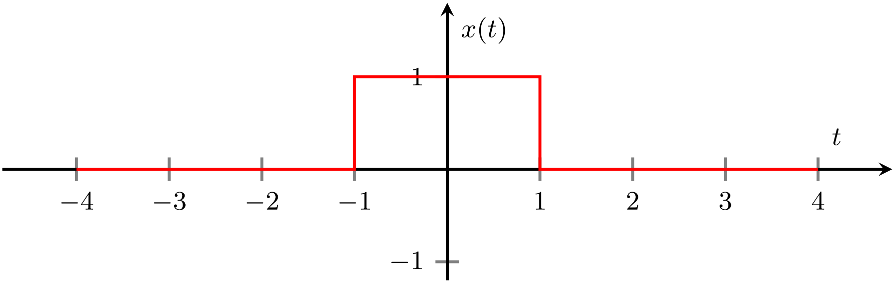
$$\begin{aligned}
x(t) &\longrightarrow y(t) \triangleq 3\, x(t) \\
x(t) &\longrightarrow y(t) \triangleq x(t) + 2.7 \\
x(t) &\longrightarrow y(t) \triangleq x(t^2) \\
x(t) &\longrightarrow y(t) \triangleq \big(x(t)\big)^5 \\
x(t) &\longrightarrow y(t) \triangleq 23
\end{aligned}$$

Signal Transformations (CT examples)

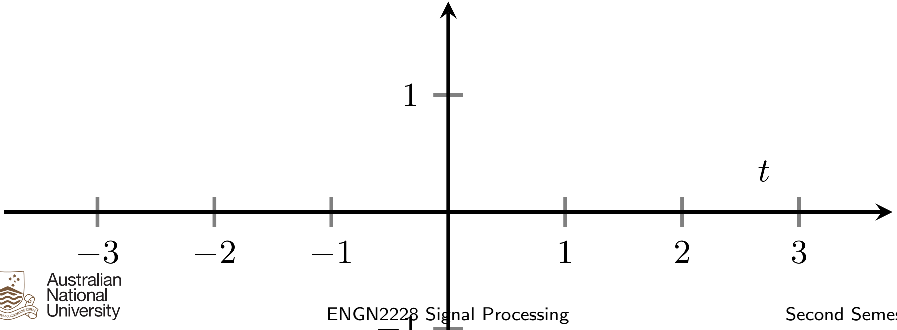


$y(t) = x(3t)$:

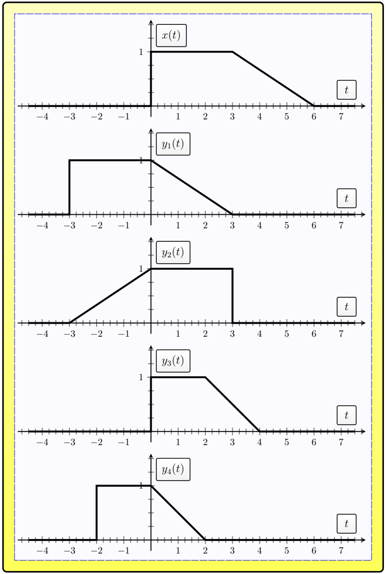




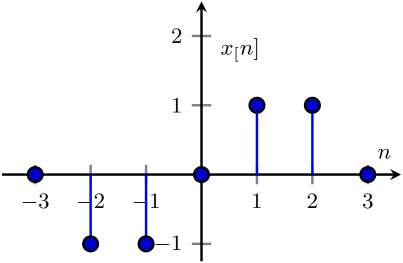
$y(t) = x(3t + 2)$:



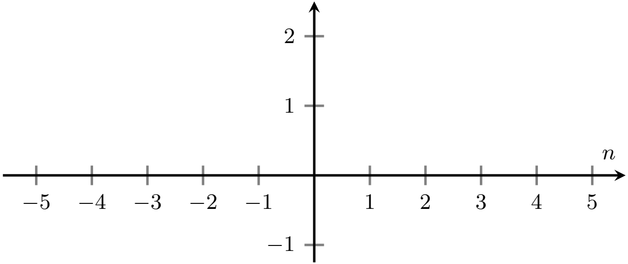
Signal Transformations (CT examples)



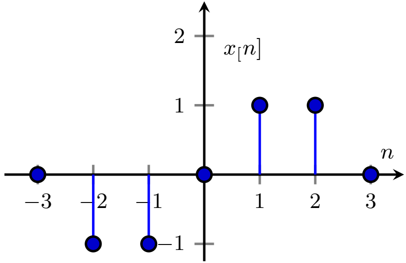
- $x(t)$
- $y_1(t) = x(t+3)$, that is, $\alpha = 1 \quad \beta = 3$
- $y_2(t) = x(-t+3)$, that is, $\alpha = -1 \quad \beta = 3$
- $y_3(t) = x(1.5\,t)$, that is, $\alpha = 1.5 \quad \beta = 0$
- $y_4(t) = x(1.5\,t+3)$, that is, $\alpha = 1.5 \quad \beta = 3$



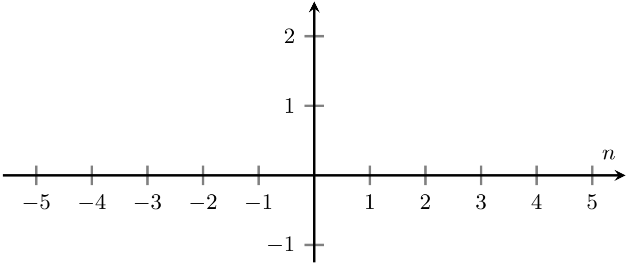
Draw $y[n] = x[n+3]$:



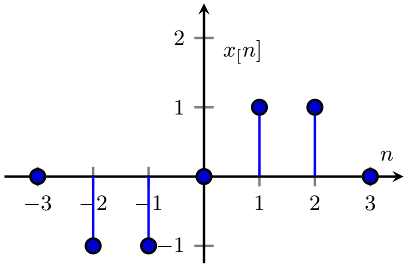
Signal Transformations (DT examples)



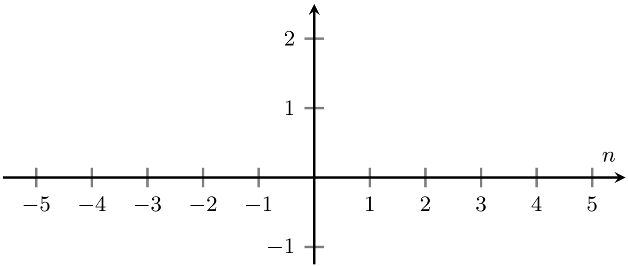
Draw $y[n] = x[2n+3]$:



Signal Transformations (DT examples)



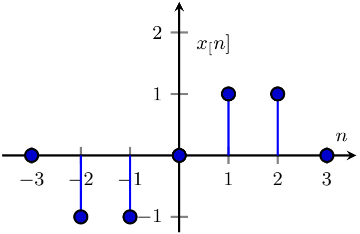
Draw $y[n] = x[-n]$:

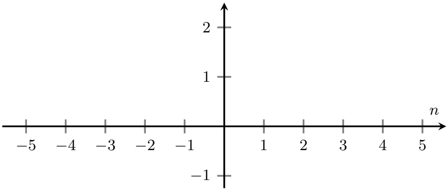


Signal Transformations (DT examples)

What about $x\left[\frac{n}{2}\right]$?

- Original samples remain.
- Then introduce more values by interpolation, zero-padding etc.
- Won't get questions on this as ambiguous.





Part 3 Outline

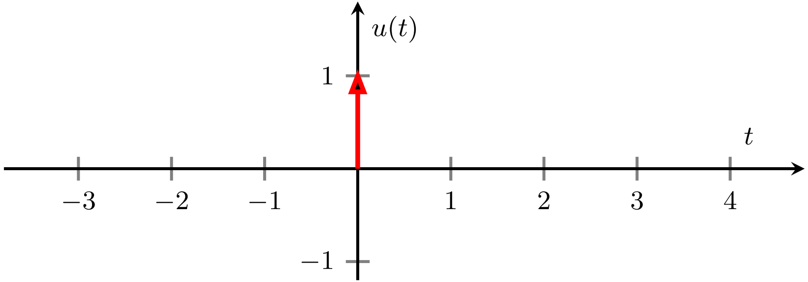
13. **Special Test Signals**
14. CT and DT Systems
15. Interconnections of Systems
16. System Examples
    - Electrical
    - Mechanical
    - Thermal
    - Edge Detector
17. System Properties
    - Causality
    - Memory
    - Time-Invariance
    - Linear & Nonlinear
    - Audio Example

Special Test Signals - Impulse Function

Define the CT **impulse/delta function** (this is a signal) `O&W 1.4 pages 30-32`

$$\delta(t) \triangleq \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

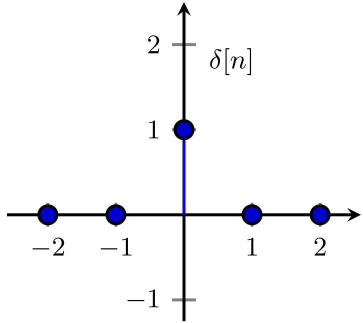
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



Special Test Signals - Impulse Function

Define **unit impulse/ sample** (this is a signal) `O&W 1.4 pages 30-32`

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

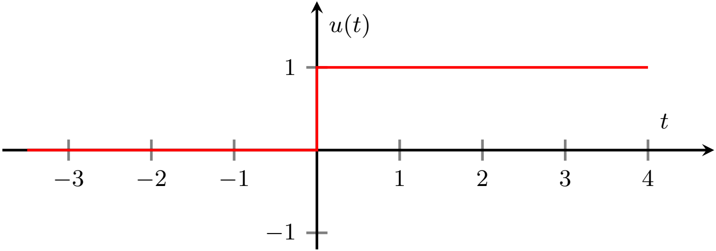


Special Test Signals - Unit Step

- Define CT **unit step** (this is a signal) `O&W 1.4.2 pages 32-38`

$$u(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$
$$= \sum_{k=-\infty}^{n} \delta[k]$$

Can think of as a switch closing at $t = 0$.



Special Test Signals - Unit Step

- Define **unit step** (this is a signal) `O&W 1.4.2 pages 32-38`

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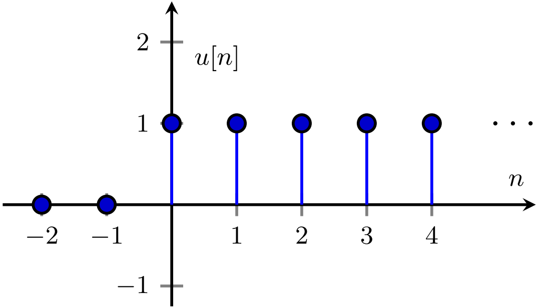
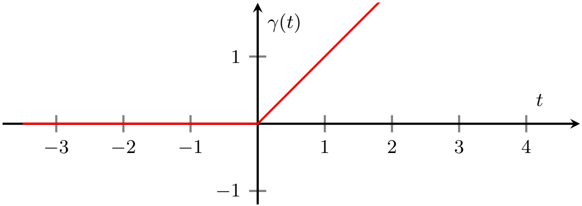
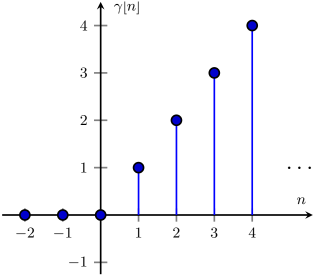


Special Test Signals - Ramp Function

Is the integral of the step function





Special Test Signals - Examples

- $x[n] = \delta[n-1] + \delta[n+2]$

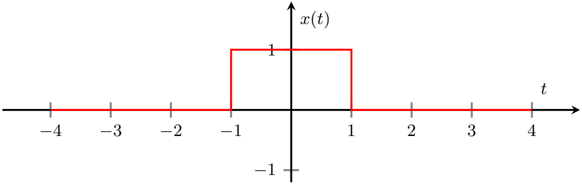
- $-\delta[n+1]$

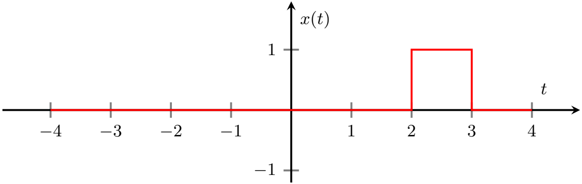
- $x[n] = u[n+1]$

- $x[n] = -u[n-1]$

Special Test Signals - Signal Representation

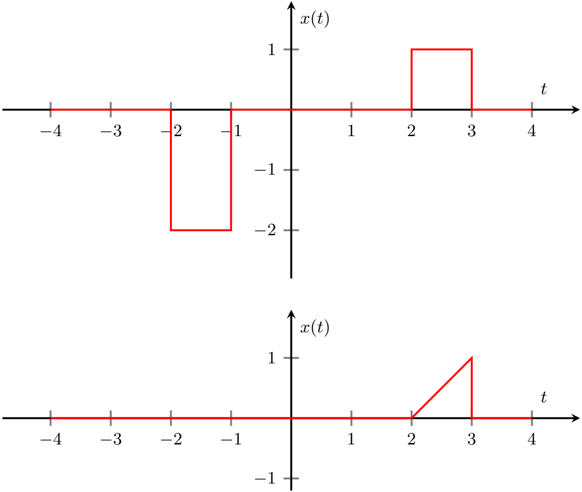
Decomposing functions into sum of shifted step functions e.g.:





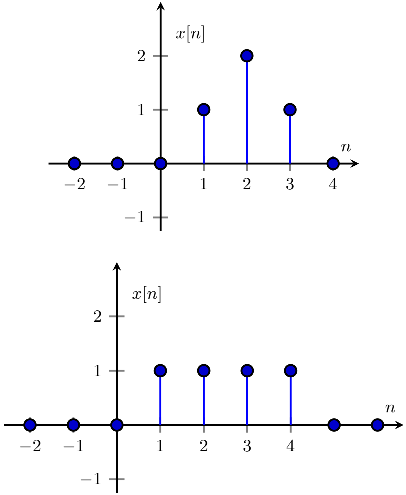
Special Test Signals - Signal Representation

Decomposing functions into sum of scaled unit step functions e.g.:



Special Test Signals - Signal Representation

Decomposing functions into sum of shifted impulse functions e.g.:



Special Test Signals - Signal Representation

Some important relations:

- $$\delta[n] = u[n] - u[n-1]$$

- $$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \ldots = \sum_{k=0}^{\infty} \delta[n-k]$$

- setting $m = n - k$ (p.31 textbook):

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Part 3 Outline

CT and DT Systems

Signals & Systems
section 1.5
pages 38-41

- A **system is a box** with an input signal $x(t)$ and an output signal $y(t)$

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$$

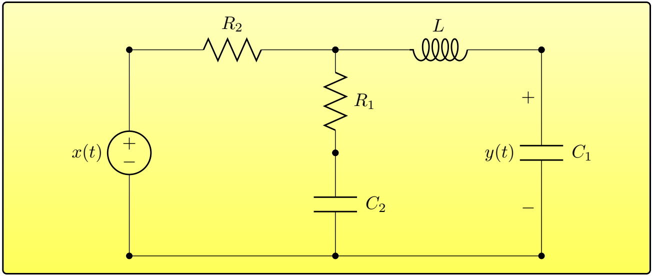
- In the discrete case with an input signal $x[n]$ and an output signal $y[n]$

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Mathematicians and physicists would say a system is an operator
- Signals are functions (containing useful information) and systems are things that transform one type of signal (function) to another signal (function)

CT and DT Systems (examples)

- An RLC circuit can be regarded as a system



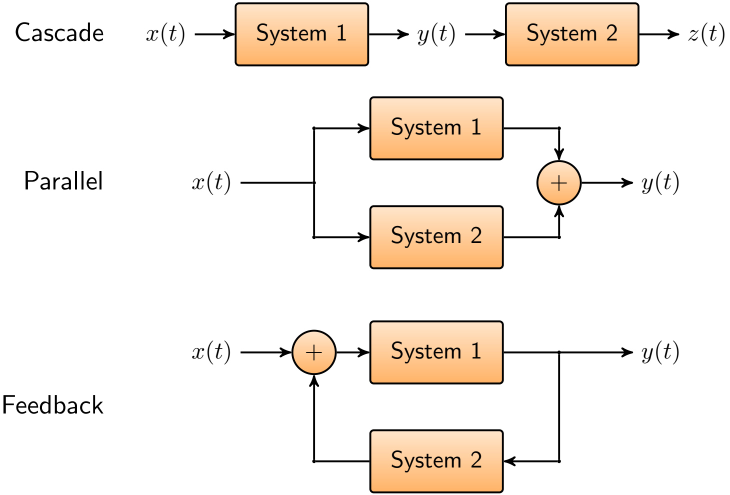
where the $x(t)$ is a voltage source and $y(t)$ is the voltage across the capacitor.

- There are a plethora of systems derived from the RLC circuit. The input $x(t)$ can be any voltage or current and output $y(t)$ can be any voltage or current.

CT and DT Systems

- Dynamics of a car in response to steering.
- An algorithm for predicting the BHP stock price.
- Medical image feature enhancement processing algorithms.
- Basically anything that has an output that responds to an input, e.g., a horse
- This course we focus on systems with one input and one output.
- This course we focus on systems that are linear (whatever that means).

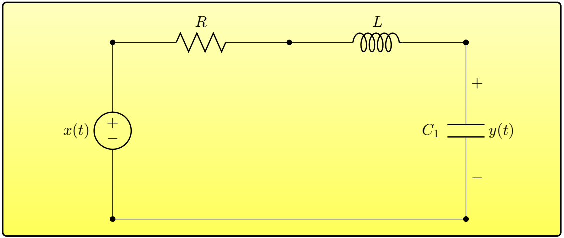
- More complex systems are the interconnection of simpler or component subsystems.



Interconnections of Systems

- Understanding, designing and analysing complex interconnections of systems, in cascade (series), parallel and feedback is a core element of modern engineering.
- Complex control systems for a modern aircraft with high order dynamical flight models.
- Almost bewilderingly complex mobile communications systems (that work).

System Examples – Electrical



$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

System Examples – Electrical (cont'd)

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{2}$$

This is a system? (Yes)

- Signal $x(t)$ is the input. Signal $y(t)$ is the output and responds to, or depends on, this input.
- The coefficients LC and RC are constants and don't depend on time or $x(t)$ or $y(t)$.

CT systems can be described by differential equations.

System Examples – Electrical (cont'd)

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

- [Addition] The system in (2) is a second order differential equation. It is **linear**. If $y_1(t)$ is the result of setting $x(t) = x_1(t)$ and $y_2(t)$ is the result of setting $x(t) = x_2(t)$ then $y(t) = y_1(t) + y_2(t)$ is the result of setting $x(t) = x_1(t) + x_2(t)$
- [Scaling] If $y_1(t)$ is the result of setting $x(t) = x_1(t)$ then $3y_1(t)$ is the result of setting $x(t) = 3x_1(t)$

System Examples – Electrical (cont'd)

- In combination (that is, added and scaling together), for input

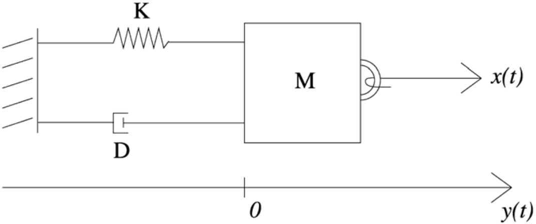
$$x(t) = \alpha_1 \, x_1(t) + \alpha_2 \, x_2(t)$$

the output is

$$y(t) = \alpha_1 \, y_1(t) + \alpha_2 \, y_2(t)$$

where α_1 and α_2 are (complex) scalars.
- This is linearity / superposition.

Part 3 Outline



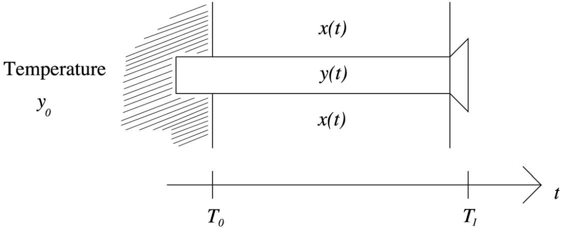
- Force Balance
$$M\frac{d^2y(t)}{dt^2} + D\frac{dy(t)}{dt} + Ky(t) = x(t)$$
where M is mass, K is string constant, D is damping and $x(t)$ is the applied force.
- The coefficients M, D and K are RC constants and don't depend on time or $x(t)$ or $y(t)$.

Observations:

- this can be viewed as a mechanical analogue of the previous electrical example
- different physical systems / analogues may have identical or very similar mathematical descriptions
- generally you have a strong or familiar domain, say electrical, from which you can interpret other systems (e.g., resistance interpretation of damping)

Part 3 Outline

System Examples – Thermal



- t – distance along the cooling fin
- $y(t)$ – fin temperature as a function of distance
- $x(t)$ – surrounding temperature along fin

System Examples – Thermal (cont'd)

$$\frac{d^2y(t)}{dt^2} = k\big(y(t) - x(t)\big)$$
$$y(T_0) = y_0$$
$$\frac{dy}{dt}(T_1) = 0$$

- Here the independent variable, t, is space (not time). (OK so the notation is not the greatest.)
- Here we have boundary conditions rather than initial conditions.

Part 3 Outline

System Examples – Edge Detector

- A rough edge detector acting on a DT signal (sequence)

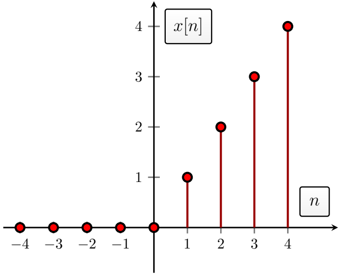
$$\begin{aligned} y[n] &= x[n+1] - 2x[n] + x[n-1] \\ &= \Big(x[n+1] - x[n]\Big) - \Big(x[n] - x[n-1]\Big) \end{aligned}$$

  which is a second difference. This emulates the second derivative for CT signals "change of slope".

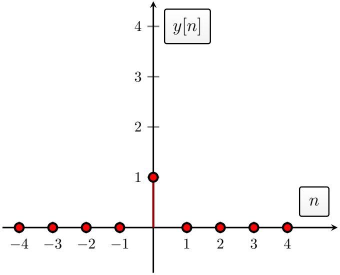
- This is a system, $x[n]$ is the input DT signal and $y[n]$ is output detector.
- If $x[n] = n$ (a linear ramp) then $y[n] = 0$ for all n.

System Examples – Edge Detector (cont'd)

- If $x[n] = n\,u[n]$ $(= n \times u[n])$, where $u[n]$ is a step then $y[n] = 0$ for all n except $n = 0$ where $y[0] = 1$.



$x[n]$



$y[n]$

- Here $y[n]$ is a bit weird because how did it know at time $n = 0$ that the input was changing at time $n = 1$? This system is not "causal" which we treat shortly.

$$y[n] = x[n+1] - 2x[n] + x[n-1]$$

System Examples – Observations

Observations:

- Differential equations and difference equations form an important class of systems.
- A system is not fully characterized by just the "dynamical" equations but also the initial conditions (or boundary conditions).

Part 3 Outline

System Properties

Why study system properties?

- important practical / physical implications
- system Properties imply structure that we can exploit to analyse and understand systems more deeply

Part 3 Outline

Definition (Causality)

A system is **causal** if the output at any time depends on values of the input at only the present and past times.

- All real time-based physical systems are causal. Time flows in one direction. Effect occurs after cause.
- Non-causal systems are the play thing of science fiction. (Don't murder any of your ancestors.)
- Causality relates to time. For other independent variables, like space, there need not be such a constraint. We can approach a point in space from any direction in general without pondering the consequences of strangling an unsuspecting ancestor.

System Properties – Causality

Terminology: causal, non-causal, anti-causal and strictly causal

- "Non-causal" means there is some output that anticipates the input for some input. For other input-output combinations the system may appear causal. (The set of numbers $\{0, -3, 7, 3, 4, 2\}$ is not positive, since at least one and not all elements are negative.)
- "Strictly causal" means the output depends on the past but not the present nor future. For example, $y[n]$ can be a function of $x[n-1]$, $x[n-2]$, $\ldots$ but not a function of $x[n]$ nor $x[n+1]$, $x[n+2]$, $\ldots$
- "Anti-causal" systems always violate causality (output depends only on the future of the input). They are a type of time reversal of a strictly causal system.

System Properties – Causality

Examples: Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by
$$y(t) = \big(x(t-1)\big)^2$$
is causal, e.g., $y(10)$ depends on $x(9)$, $y(t)$ depends strictly on past $x(t)$.
- The CT system $x(t) \longrightarrow y(t)$ described by
$$y(t) = x(t+1)$$
is non-causal, e.g., $y(13) = x(14)$, $y(t)$ depends on strictly future $x(t)$.
- Note a CT system is non-causal even if it is only non-causal at one time instant.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = x[-n]$$

is non-causal, e.g., $y[-5] = x[5]$ (but not anti-causal, as $y[5] = x[-5]$). $y[n]$ is the time-reversal of $x[n]$.

Examples (cont'd): Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = x(-t)$$

is non-causal. That is, the system that time reverses an input signal is a non-causal system.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \left(\frac{1}{2}\right)^{n+1} \big(x[n-1]\big)^3$$

is causal. The weighting $(1/2)^{n+1}$ is decaying with time n increasing but this is independent of signal $x[n]$.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is ...

System Properties – Memory

Definition (Memory)

A system is said to be memoryless if its output for each value of t or n at a given time is dependent on input at only the same time.

For example:

- $v(t) = Ri(t)$ - memoryless (resistor is memoryless)
- $y[n] = (2x[n] - x^2[n])$ - memoryless
- $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$ - memory (capacitor has memory)
- $y[n] = x[n] + y[n-1]$ - memory
- $y[n] = \frac{1}{3}\big(x[n+1] + x[n] + x[n-1]\big)$ - memory

A system is said to possess memory if its output signal depends on past of future values of the input signal.
All memoryless systems are causal, vice versa is not true.

System Properties – Time-Invariance (cont'd)

Definition (CT System Time-Invariance)

A CT system is **time-invariant** if

$$x(t) \longrightarrow y(t)$$

then

$$x(t - t_0) \longrightarrow y(t - t_0)$$

for all $t_0 \in \mathbb{R}$.

- Time-Invariance means "doesn't change with time". It is a property of a system and not of the signals input and output (which are obviously functions of time). It means that if a caveman put a signal through a TI system then the output would be the same as the same signal today.
- Only a system can be time-invariant. It is senseless to say a signal is time-invariant.

Definition (DT System Time-Invariance)

A DT system is **time-invariant** if

$$x[n] \longrightarrow y[n]$$

then

$$x[n-n_0] \longrightarrow y[n-n_0]$$

for all $n_0 \in \mathbb{Z}$.

System Properties – Time-Invariance (cont'd)

Examples:

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = \big(x(t+1)\big)^2$$

  is time-invariant (TI).

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \left(\frac{1}{2}\right)^{n+1} \big(x[n-1]\big)^3$$

  is not time-invariant.

Not time-invariant is preferably called time-varying (don't use the expression "time variant").

Examples:

$$y(t) = \big(x(t+1)\big)^2$$

Examples:

$$y[n] = \left(\frac{1}{2}\right)^{n+1} \big(x[n-1]\big)^3$$

System Properties – Linear & Nonlinear

- Many, some say most, systems are **nonlinear**. For example, diodes, car dynamics, etc.
- In this course we focus of **linear** systems.
- Don't confuse nonlinear with time-varying linear, e.g. $2x + 3$ is a linear equation but system non-linear.
- Linear models are a very important class of models because:
  - they are mathematically tractable
  - they can model small signal variations in nonlinear systems
  - they model accurately circuit elements such as resistors, capacitors, etc.
  - they can provide insights into the behaviour of more complex nonlinear systems

System Properties – Linear & Nonlinear (cont'd)

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t)$$

then

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for complex scalars α_1 and α_2.

Definition (Nonlinear System)

A **nonlinear** system is a system which is not linear.

System Properties – Linear & Nonlinear (cont'd)

An equivalent definition:

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_k(t) \longrightarrow y_k(t)$$

then

$$\sum_k \alpha_k x_k(t) \longrightarrow \sum_k \alpha_k y_k(t)$$

for complex scalars α_k.

Definition (Linear System)

A DT system is **linear** if superposition holds. If

$$x_1[n] \longrightarrow y_1[n] \text{ and } x_2[n] \longrightarrow y_2[n]$$

then

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

for complex scalars α_1 and α_2.

Definition (Linear System)

A DT system is **linear** if superposition holds. If

$$x_k[n] \longrightarrow y_k[n]$$

then

$$\sum_k \alpha_k x_k[n] \longrightarrow \sum_k \alpha_k y_k[n]$$

for complex scalars α_k.

- For linear systems, zero input gives zero output.

System Properties – Examples

causal/non-causal, linear/nonlinear, time-invariant/time varying:

$y(t) = \big(x(t)\big)^2 = x^2(t)$ is a square law and as a system is:

- Time-invariant, do proof
- Causal and memoryless (current output depends only on current input)
- Nonlinear (it is quadratic), do proof

causal/non-causal, linear/nonlinear, time-invariant/time varying:
$y(t) = x(2t)$ is a compression in time and as a system is:

- Non-causal, since for $t > 0$ we have $2t > t$, for example, at time $t = 3$ we have $y(3) = x(6)$ which is a time advance of 3. Note for $t < 0$ we have $2t < t$, for example, at time $t = -3$ we have $y(-3) = x(-6)$ which is a delay of 3 (that is, it acts causally at time $t = -3$).
- Linear, do proof.
- Time-varying, do proof.

System Properties – Examples

causal/non-causal, linear/nonlinear, time-invariant/time varying:
$y[n] = x[n+1] - x[n-1]$ as a system is:

- Non-causal because uses future input $x[n+1]$.
- Linear, do proof.
- Time-invariant, do proof.

System Properties – Examples

System	Linear	Time-Invariant	Causal	Memoryless
$y[n] = 2x[n]$				
$y[n] = 2x[n] + 3$				
$y[n] = x[-n]$				
$y(t) = tx(t)$				
$y(t) = \cos(3t)x(t)$				
$y(t) = \sin\big(x(t)\big)$				
$y(t) = t^2 x(t-1)$				

System Properties – Linear & Nonlinear

Are all these combinations possible?

- Linear, time-invariant and causal?
- Linear, time-invariant and non-causal?
- Linear, time-varying and causal?
- Linear, time-varying and non-causal?
- Nonlinear, time-invariant and causal?
- Nonlinear, time-invariant and non-causal?
- Nonlinear, time-varying and causal?
- Nonlinear, time-varying and non-causal?

Yes, all combinations are possible.

Homework Problem: generate system examples for each of the 8 cases above.

Part 3 Outline

System Properties – Audio Example

Enough of the maths, let's look at whether we can do something useful/real with what we have learnt so far.

- The DT system $x[n] \longrightarrow y[n]$ described by

$$\boxed{y[n] = 0.6\,x[n] + 0.4\,x[n-3000]}$$

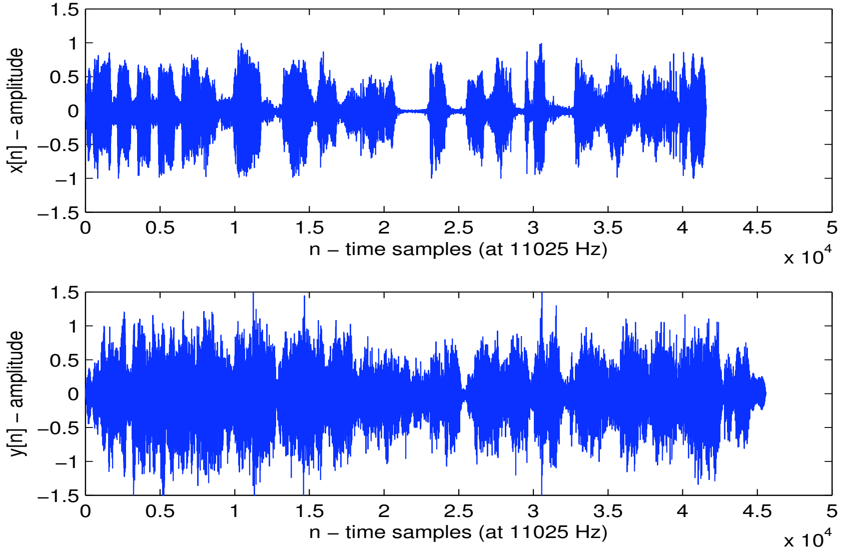
  is: a) linear, b) causal and c) time-invariant.

- Take the input $x[n]$ to be a "sound bite" with samples

$$\begin{aligned} x[0] &= -0.0781 \\ x[1] &= -0.0547 \\ x[2] &= -0.0469 \\ x[3] &= -0.0312 \\ x[4] &= 0.0000 \\ &\vdots \\ x[41580] &= 0.0859 \end{aligned}$$

- Components of sequence $x[n]$ are samples of an audio waveform taken at sampling rate, in this case, of $11,025$ Hz. $x[0]$ is the sample at time 0 seconds, $x[1]$ is the sample at time $1/11,025$ seconds, ... The total real time length is 3.77 seconds (41581 samples).
- Sample values are normalized to range $[-1,+1]$, that is, maximum is $+1$ and minimum is -1 and no signal (quiet) is 0.
- The DT output signal $y[n]$ can be regarded as an output $11,025$ Hz audio clip. What does it sound like?

System Properties – Audio Example (cont'd)



System Properties – Audio Example (cont'd)

- Output signal $y[n]$ is longer. Why?
- Output signal $y[n]$ has range outside $[-1, +1]$. Why?
- Direct path 0.6 plus 0.4 echo at $3000/11,025 \approx 0.2731$ seconds or 93 metres longer (speed sound 340m/s).
- If sounds are played back at $8,000$ Hz rate does the pitch go up or down? Does the sound play longer or shorter?
- If the original $11,025$ Hz sound bite is Daffy Duck which cartoon character does the same sound bite sound like when played back at $8,000$ Hz? **Yosemite Sam**

System Properties – Audio Example (cont'd)

```
d=wavread( 'daffyin.wav' );
fs=11025;
f8=8000;
wavwrite( d, f8, 'daffyin8.wav' );
sound( d, fs );
r(1)=0.6;
r(3001)=0.4;
dd = conv( d, r );
sound( dd, fs );
wavwrite( dd, fs, 'daffyout.wav' );
wavwrite( dd, f8, 'daffyout8.wav' );
subplot(2,1,1);
plot(d)
xlabel('n - time samples (at 11025 Hz)')
ylabel('x[n] - amplitude')
axis([0 50000 -1.5 1.5])
subplot(2,1,2);
plot(dd)
xlabel('n - time samples (at 11025 Hz)')
ylabel('y[n] - amplitude')
axis([0 50000 -1.5 1.5])
```

18 **Summary to this Point**

19 DT LTI Systems

- Using DT LTI Property
- Superposition + Time-Invariance
- Signal Representation
- Unit Pulse Response
- Examples
- More General TV Case
- Commutative Property

Summary to this Point

- Signals carry some desired information
- Signals can be CT or DT, that is, their domain is time $t \in \mathbb{R}$ for CT or time index $n \in \mathbb{Z}$ for DT.
- Properties of CT signals and DT signals are similar (and can be treated together).
- The independent variable is usually time (1D), but may be space (1D, 2D or 3D) or time (1D) or combinations, or some other parameter(s) altogether.
- Signals can be classified under various labels such as: periodic, complex exponential, etc.
- Signals are inputs and outputs of systems (systems model how signals are generated or processed or both).

Summary to this Point – (cont'd)

- Systems are usually described through differential equations for CT and difference equations for DT + initial conditions or boundary conditions.
- Systems descriptions for diverse physical situation (e.g., electrical, thermal, mechanical, financial, ...) may be very similar or identical.
- Systems can be linear or nonlinear, causal or non-causal, time-invariant or time-varying. This classification is not exhaustive as there are additional "interesting" properties.
- For causality, it is implicit the independent variable is time. Systems which model physical systems are causal by nature. But non-causal system descriptions are still useful.
- Systems which are linear and time-invariant (LTI) are special (easier, powerful theory and very useful). Now look at these in more detail.

Part 4 Outline

18 Summary to this Point

19 **DT LTI Systems**

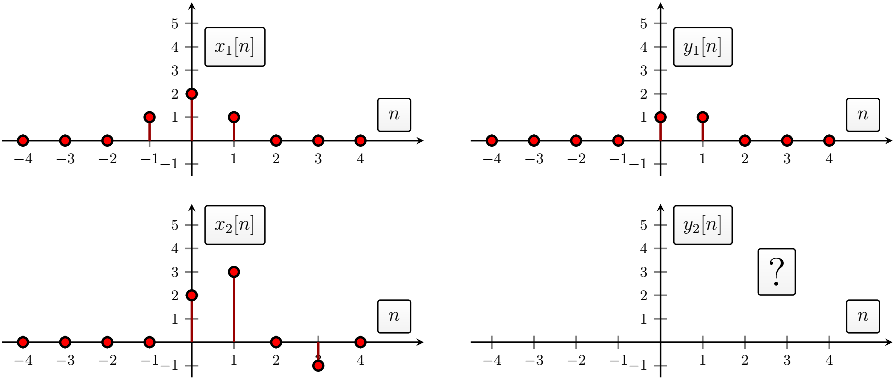
- Using DT LTI Property
- Superposition + Time-Invariance
- Signal Representation
- Unit Pulse Response
- Examples
- More General TV Case
- Commutative Property

LTI

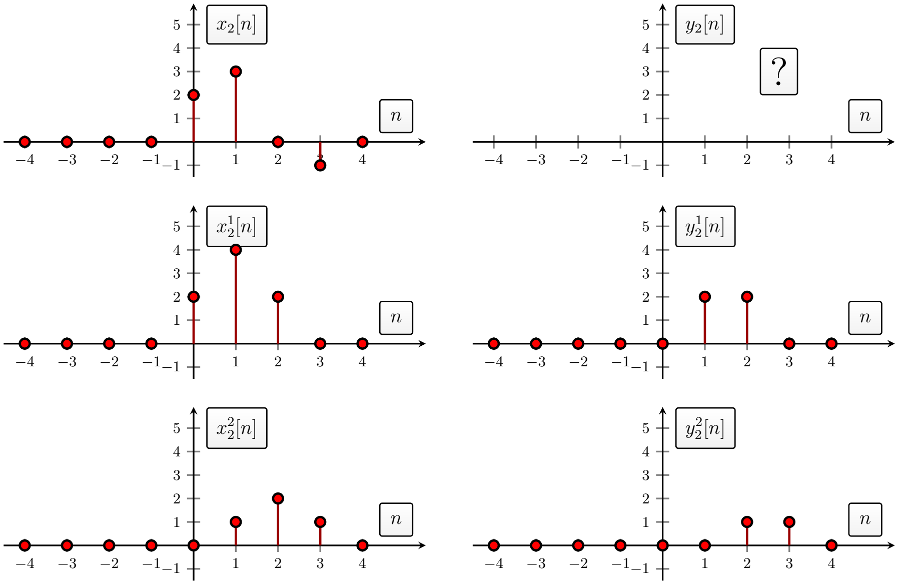
- Linear Time-Invariant (LTI) Systems
- That is systems which have the properties of linearity and time-invariance
- Practically important - model many physical processes
- Extensive analysis tools for LTI systems - e.g. convolution
- Response of an LTI system can be understand from the response to a special type of input signal
- LTI systems can be characterized by a “signal” called a unit sample (DT) or an impulse (CT)

DT LTI Systems – Using DT LTI Property

A system is linear and time-invariant. It is unknown except that output $y_1[n]$ occurs for input $x_1[n]$, that is, $x_1[n] \longrightarrow y_1[n]$. Can we work out the output $y_2[n]$ of this system when the input is $x_2[n]$ ($x_2[n] \longrightarrow y_2[n]$)?



DT LTI Systems – Using DT LTI Property



DT LTI Systems – Using DT LTI Property

Note that $x_2^1[n]$ and $x_2^2[n]$ are shifted and scaled versions of $x_1[n]$, and further that

$$x_2[n] = x_2^1[n] - x_2^2[n]$$

So by linearity and time-invariance

$$y_2[n] = y_2^1[n] - y_2^2[n]$$

DT LTI Systems – Superposition + Time-Invariance

- By superposition and time-invariance

$$x_k[n] \longrightarrow y_k[n]$$

$$x[n] = \sum_k a_k x_k[n] \longrightarrow y[n] = \sum_k a_k y_k[n]$$

  $x_k[n]$ is an indexed set of candidate building blocks.

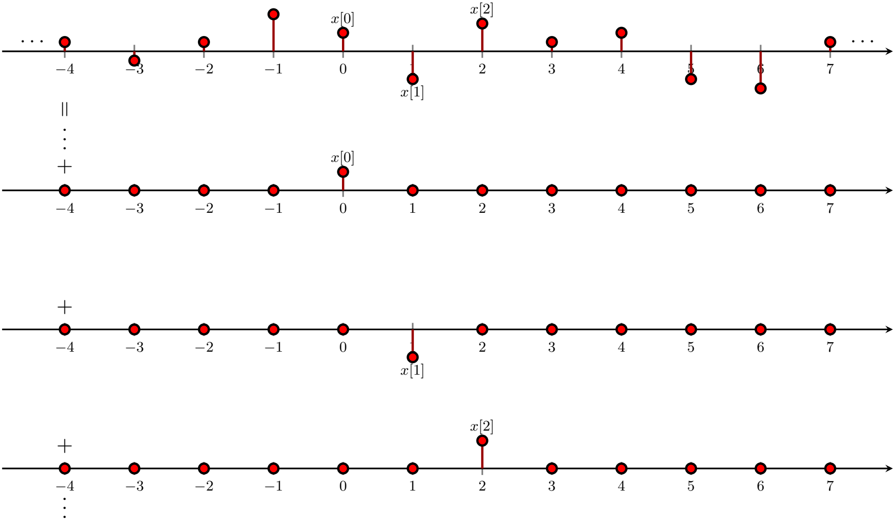
- Seek basic building blocks to represent any signal, that is, linear combinations of these building blocks.
- The response of LTI systems to these basic building blocks, if properly chosen, is elegant and powerful.
- In DT the natural choice of building block signals are **time shifted unit samples**.

DT LTI Systems – Signal Representation

Finding the output of a linear system:

- **Any signal can be represented as a combination of scaled and shifted DT unit impulses.**
- Hence, find the response of a system to a unit impulse.
- By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.

DT LTI Systems – Signal Representation (cont'd)



- Unit impluse/sample (this is a signal)

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- Expand $x[n]$ in terms of shifted unit samples

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + \\ x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

"Sifting Property" of the unit sample $\delta[n]$

DT LTI Systems – Unit Pulse Response

Finding the output of a linear system:

- Any signal can be represented as a combination of scaled and shifted DT unit impulses.
- **Hence, find the response of a system to a unit impulse.**
- By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.

DT LTI Systems – Unit Pulse Response

$$\delta[n] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow h[n-k]$$

DT LTI Systems – Unit Pulse Response

Finding the output of a linear system:

- Any signal can be represented as a combination of scaled and shifted DT unit impulses.
- Hence, find the response of a system to a unit impulse.
- **By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.**

DT LTI Systems – Unit Pulse Response

$$x[n] \longrightarrow \boxed{\text{DT LTI System } h[n]} \longrightarrow y[n]$$

$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

- For an LTI system, define **unit sample response** $h[n]$

$$\delta[n] \longrightarrow h[n],$$

that is, $h[n]$ is the response of the LTI system to a kick at $k = 0$.

- But from time-invariance

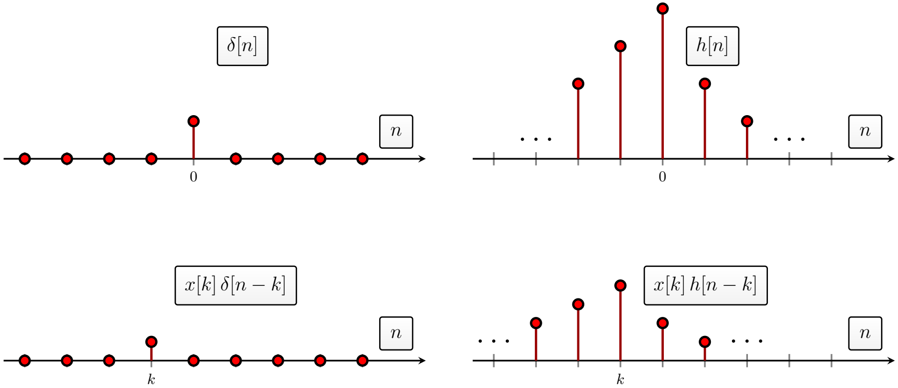
$$\delta[n-k] \longrightarrow h[n-k],$$

that is, kick at k is the same as a kick at 0 shifted by k. Hence, $h[n]$ completely characterizes a LTI System !!!

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

DT LTI Systems – Unit Pulse Response

Interpretation:



DT LTI Systems – Unit Pulse Response

- Convolution sum, here $\star$ denotes a binary operation not multiplication

$$y[n] = x[n] \star h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- In MATLAB: `y = conv( x, h );`
- Weird but (follows from definition)

$$y[n] = x[n] \star h[n] = h[n] \star x[n]$$

- LTI System is characterized/parametrized by $h[n]$ which looks like a signal—this signal is the output of the LTI system to a unit sample input.

DT LTI Systems – Examples

Simple Reverb System: (revisit)

$$y[n] = 0.6\,x[n] + 0.4\,x[n - 3000]$$

- The unit sample response is

$$h[n] = 0.6\,\delta[n] + 0.4\,\delta[n - 3000]$$

- If $x[n]$ is the daffy audio signal then

$$y[n] = h[n] \star x[n]$$

  is the reverberant daffy output.

- To emulate the reverberation in the Opera House, fire a pistol $\delta[n]$ and record $h[n]$.

An accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is causal LTI.

- Unit sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

  where $u[n]$ is the unit step.

- In convolution form

$$x[n] \star u[n] = \sum_{k=-\infty}^{n} x[k]$$

Part 4 Outline

18 Summary to this Point

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DT LTI Systems – More General TV Case

- Response (output $y[n]$) of a **time-varying (TV) linear system** to $x[n]$ can be thought of in terms of superposition of unit sample responses.
- So we want to figure out the response in $y[n]$ to the $\delta[n-k]$ components in $x[n]$.
- Define $h_k[n]$ as the response to $\delta[n-k]$:

$$\delta[n-k] \longrightarrow h_k[n]$$

- By superposition, this linear (generally time-varying) system is given by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

- We need to know the response of a time-varying linear system to an **infinite** number of shifted unit samples to fully characterize it.

DT LTI Systems – More General TV Case

- If I give a linear system a kick then it has a response.
- But the response may be different depending on when I kick it. It can still be linear.
- If the response is different depending on when I kick it it is called **time-varying (TV)**. This is why we wrote

$$h_k[n]$$

which is the result of a kick at time k, $\delta[n-k]$.

DT LTI Systems – More General TV Case

We need all of

$$
\begin{array}{cccccccc}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h_{-2}[-3] & h_{-2}[-2] & h_{-2}[-1] & h_{-2}[0] & h_{-2}[1] & h_{-2}[2] & \cdots \\
\cdots & h_{-1}[-3] & h_{-1}[-2] & h_{-1}[-1] & h_{-1}[0] & h_{-1}[1] & h_{-1}[2] & \cdots \\
\cdots & h_{0}[-3] & h_{0}[-2] & h_{0}[-1] & h_{0}[0] & h_{0}[1] & h_{0}[2] & \cdots \\
\cdots & h_{1}[-3] & h_{1}[-2] & h_{1}[-1] & h_{1}[0] & h_{1}[1] & h_{1}[2] & \cdots \\
\cdots & h_{2}[-3] & h_{2}[-2] & h_{2}[-1] & h_{2}[0] & h_{2}[1] & h_{2}[2] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

to be able to compute the response of a **time-varying** linear system (which is a pain). This is not LTI - LTI is a special case.

DT LTI Systems – More General TV Case

As a special case of a TV Linear System we have $h_k[n] = h[n-k]$, then:

$$\begin{array}{cccccccc}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h[-1] & h[0] & h[1] & h[2] & h[3] & h[4] & \cdots \\
\cdots & h[-2] & h[-1] & h[0] & h[1] & h[2] & h[3] & \cdots \\
\cdots & h[-3] & h[-2] & h[-1] & h[0] & h[1] & h[2] & \cdots \\
\cdots & h[-4] & h[-3] & h[-2] & h[-1] & h[0] & h[1] & \cdots \\
\cdots & h[-5] & h[-4] & h[-3] & h[-2] & h[-1] & h[0] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We just need the unit sample response, all the $h[n]$.

Part 4 Outline

18 Summary to this Point

19 **DT LTI Systems**

- Using DT LTI Property
- Superposition + Time-Invariance
- Signal Representation
- Unit Pulse Response
- Examples
- More General TV Case
- Commutative Property

DT LTI Systems – Commutative Property

From the definition of convolution it can be shown (we do this later in Part 5)

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$

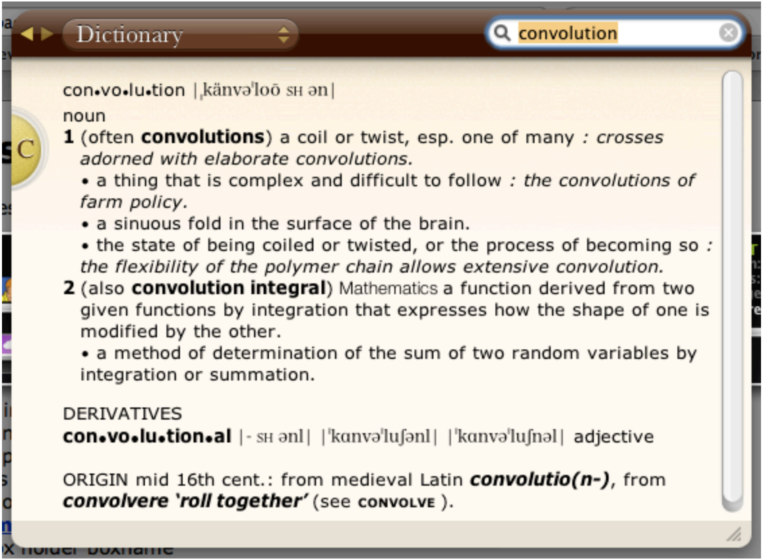
- Can be used to reinterpret LTI systems. $x[n]$ as input signal to a system with unit sample response $h[n]$ is the same as $h[n]$ as input signal to a system with unit sample response $x[n]$.
- Can be used as a tool or trick:

$$s[n] = u[n] \star h[n] = h[n] \star u[n]$$

the second is the "accumulator" system. Hence

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

DT Convolution – Street Version



Dictionary

convolution

con•vo•lu•tion |ˌkänvəˈlo͞o sh ən|

noun

1 (often **convolutions**) a coil or twist, esp. one of many *: crosses adorned with elaborate convolutions.*

• a thing that is complex and difficult to follow *: the convolutions of farm policy.*

• a sinuous fold in the surface of the brain.

• the state of being coiled or twisted, or the process of becoming so *: the flexibility of the polymer chain allows extensive convolution.*

2 (also **convolution integral**) Mathematics a function derived from two given functions by integration that expresses how the shape of one is modified by the other.

• a method of determination of the sum of two random variables by integration or summation.

DERIVATIVES

con•vo•lu•tion•al |- sh ənl| |ˈkɑnvəˈluʃənl| |ˈkɑnvəˈluʃnəl| adjective

ORIGIN mid 16th cent.: from medieval Latin ***convolutio(n-)***, from ***convolvere 'roll together'*** (see CONVOLVE).

Part 5 Outline

20 **DT Convolution**
- Street Version
- Graphical Flip and Shift
- Other DT Convolution Methods
- Convolution with Impulses
- Commutative Property
- Distributive Property
- Associative Property

21 **DT System Properties**
- Causality Property
- Stability Property
- Review of System Properties

22 **Difference Equation of DT System**
- Direct-Form I implementation

23 **Finding the Impulse Response of a DT System**

DT Convolution – Graphical Flip and Shift

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Looking at the equation for convolution we can arrive at the following procedure:

1. Choose one signal to be $x[n]$, the other is then $h[n]$
2. Draw them on the k-axis
3. Flip $h[k]$ around $k = 0$ (the y-axis)
4. Shift the flipped version of h to the right by n
5. Multiply $x[k]$ by flipped/shifted version of $h[k]$ and sum over all values of k
6. The summation only gives you $y[n]$ for one value of n

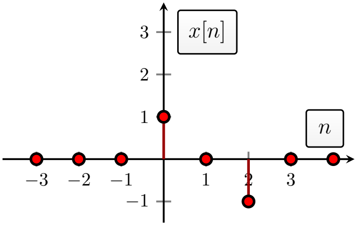
DT Convolution – Graphical Flip and Shift

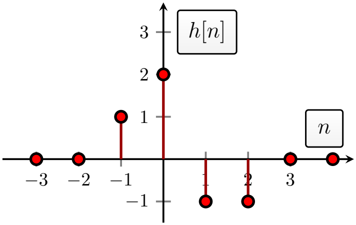
Consider the convolution of

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < 0, \text{ or } n > 2) \end{cases}$$

and (non-causal) pulse response

$$h[n] = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ -1 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < -1, \text{ or } n > 2) \end{cases}$$

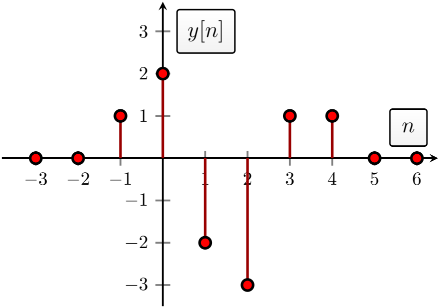




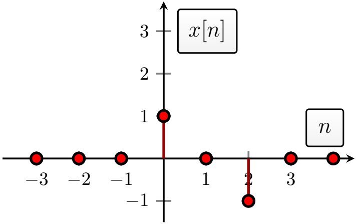
DT Convolution – Graphical Flip and Shift (cont'd)

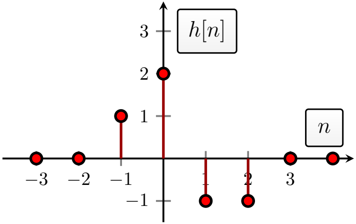
The convolution is

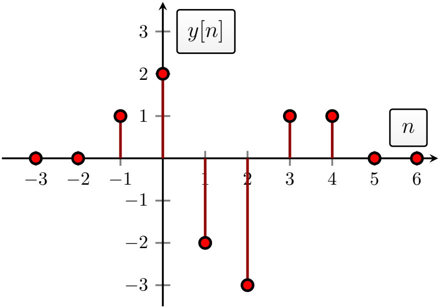
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \begin{cases} 1 & n=-1 \\ 2 & n=0 \\ -1-1=-2 & n=1 \\ -1-2=-3 & n=2 \\ 1 & n=3 \\ 1 & n=4 \\ 0 & \text{otherwise } (n<-1, \text{ or } n>4) \end{cases}$$



DT Convolution – Graphical Flip and Shift (cont'd)



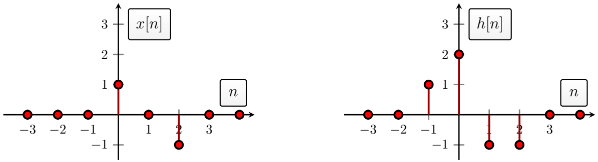


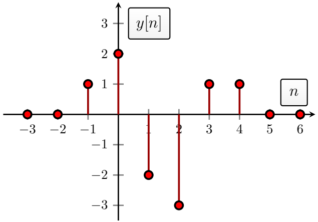


DT Convolution – Graphical Flip and Shift (cont'd)

Checks:

- length($y[n]$) $=$ length($x[n]$) $+$ length($h[n]$) -1
- $y[n]$ will begin at start($x[n]$)+start($h[n]$)
- $y[n]$ will end at end($x[n]$) $+$ end($h[n]$)





Part 5 Outline

Other DT Convolution Methods – Matlab

The calculation of DT convolution is a means to combine two vectors/sequences

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & -1 & -1 \end{bmatrix}$$

to generate a new vector/sequence

$$\begin{bmatrix} 1 & 2 & -2 & -3 & 1 & 1 \end{bmatrix}$$

and indeed this is what MATLAB command

```
y = conv( x, h );
```

does. Where else does this exact type of calculation appear?

Consider the polynomial multiplication:

$$(1x^2 - 1)(x^3 + 2x^2 - x - 1) = x^5 + 2x^4 - 2x^3 - 3x^2 + x + 1$$

- $x[n]$ and $h[n]$ provide the LHS polynomials' coefficients.
- $y[n]$ provides the RHS polynomial's coefficients.
- So DT convolution appears to be related to polynomial multiplication.
- Here the non-causal shift in $h[n]$ has not been factored in (but can be).
- x is the polynomial indeterminate, it can be thought of as a unit time shift.

Other DT Convolution Methods – Matrix Version

Matrix version of convolution

$$\begin{array}{rrr} -1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

- Here $h[n]$ is implemented as a 6×3 Toeplitz matrix. The rows are formed by reversing $h[n]$ and shifting.
- The orange portions above are just an aid to understand and not part of the matrix.

Part 5 Outline

20 **DT Convolution**
- Street Version
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- Distributive Property
- Associative Property

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23 **Finding the Impulse Response of a DT System**

DT Convolution – Convolution with Impulses

Convolution of a signal with a shifted impulse shifts the signal to the location of the impulse

- $u[n] * \delta[n-1] = u[n-1]$

- $\delta[n] * \delta[n-4] = \delta[n-4]$

- $\delta[n-1] * \delta[n-4] = \delta[n-5]$

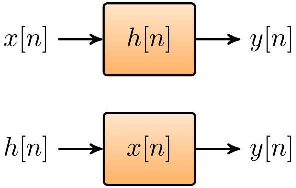
- $(\delta[n] + 2\delta[n-1] - \delta[n-2]) * 2\delta[n-4] = 2\delta[n-4] + 4\delta[n-5] - 2\delta[n-6]$

Part 5 Outline

DT Convolution – Commutative Property

Previously we noted the **Commutative Property**:

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$



DT Convolution – Commutative Property

- This follows from (change variables to $\ell = n - k$)

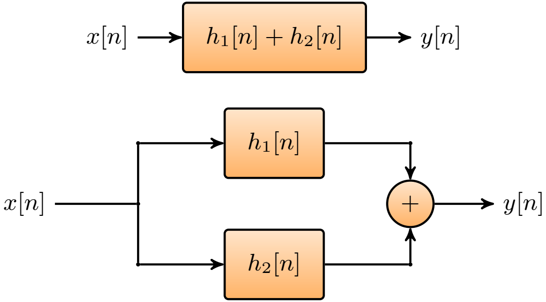
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$
$$= \sum_{\ell=-\infty}^{\infty} x[n-\ell]\, h[\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, x[n-\ell]$$

- Alternatively, if we have two polynomials say $p(x)$ and $q(x)$ then $p(x)q(x) = q(x)p(x)$. Polynomial multiplication is commutative. Convolution is commutative.

DT Convolution – Distributive Property

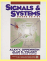
Consider an input signal $x[n]$ and two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **parallel**, then we have the **Distributive Property**:

$$\boxed{x[n] \star \big(h_1[n] + h_2[n]\big) = x[n] \star h_1[n] + x[n] \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in parallel into a single equivalent DT LTI system (by **adding** the pulse responses).

DT Convolution – Associative Property



Signals & Systems
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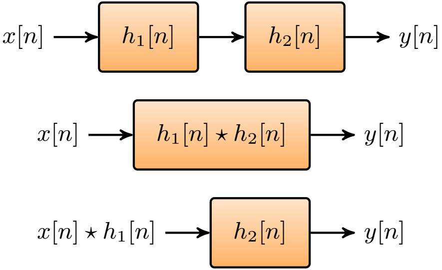
Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

$$\boxed{x[n] \star \big(h_1[n] \star h_2[n]\big) = \big(x[n] \star h_1[n]\big) \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).

Part 5 Outline

Part 5 Outline

For DT LTI Systems the **Causality Property** can be written:

Theorem (Causal DT LTI System)

A DT LTI system is **causal** *if and only if its pulse response, $h[n]$, satisfies*

$$h[n] = 0, \quad \textit{for all } n < 0.$$

- If $h[n] \neq 0$ for at least one $n = -n_0$ $(n_0 > 0)$ then the output at time n, $y[n]$, would contain term

$$h[-n_0]\, x[n + n_0],$$

  for example, if $n_0 = 1$ and $h[-1] = 2$ then
  $y[n] = \cdots + h[-1]\, x[n+1] + \cdots$, and hence would not be causal.

Part 5 Outline

DT System Properties – Stability Property

Stability: a bounded input $x[n]$ produces a bounded output $y[n]$.

Definition (DT LTI System Stability)

A DT LTI system is **stable**, with pulse respond $h[n]$, if and only if

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

is bounded if and only if the input is bounded.

DT System Properties
Stability Property

- $h[n] \triangleq 2^n$ is not stable
- $h[n] \triangleq 2^{-n}$ is not stable (consider $n \longrightarrow -\infty$)
- The following is stable:

$$h[n] \triangleq \begin{cases} 2^{-n} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Part 5 Outline

DT System Properties – Review of System Properties

System properties:

- Time-invariant/ time-varying
- Memory/ memoryless
- Causual/ non-causal
  - Two ways to causality can be determined. One uses $h[n]$.
- Stable/ non-stable
  - Had to wait until defined $h[n]$ to introduce.
- Linear/ non-linear

Problem:

Determine whether or not each of the following signals are: i) **time-invariant**, ii) **linear**, iii) **casual**, iv) **stable**, and v) **memoryless**.

(a) $y[n] = x[n+3] - x[1-n]$

(b) $y[n] = \begin{cases} (-1)^n x[n], & x[n] \geq 0 \\ 2x[n], & x[n] < 0 \end{cases}$

(c) $y[n] = \sum_{k=n}^{\infty} x[k]$

Solution:

	TI	Linear	Causal	Stable	Memoryless
(a)	no	yes	no	yes	no
(b)	no	no	yes	yes	yes
(c)	yes	yes	no	no	no

Part 5 Outline

Differential Equation of CT System

General form:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

N-th order differential equation

For example:

$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Difference Equation of DT System

General form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

N-th order difference equation

Difference Equation of DT System – Direct-Form I implementation

Block diagram representation of causal DT LTI systems:

- Summer (for two inputs)

- Gain (can be negative or positive)

- Delay

Difference Equation of DT System – Direct-Form I implementation

e.g. 1st order difference equation

$$y[n] + ay[n-1] = bx[n]$$

Difference Equation of DT System – Direct-Form I implementation

e.g. 2nd order difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

Difference Equation of DT System – Direct-Form I implementation

Try

$$y[n] = x[n] + 2x[n-2] - \frac{1}{2}y[n-1] + \frac{1}{3}y[n-2]$$

Part 5 Outline

Finding the Impulse Response of a DT System

How to find $h[n]$?

- Definition - $h[n]$ is the output of the system for an input $x[n] = \delta[n]$.

e.g.:

- $y[n] = x[n] + \frac{1}{2}x[n-1]$
- $h[n] = ?$
- Let $x[n] = \delta[n]$ then
- $h[n] =$

Finding the Impulse Response of a DT System – Example

$$y[n] = \frac{1}{3}y[n-1] + x[n-1]$$

DT LTI system initially at rest, find $h[n]$.

- Let $x[n] = \delta[n]$ then, $h[n] =$
- Have we solved for $h[n]$?

Finding the Impulse Response of a DT System – Example

$h[n] = \frac{1}{3}h[n] + \delta[n-1]$

- Use recursion to work out formula.
- System causal - therefore can use $h[n]$ causality property: as the system is initially at rest $h[n] = 0$ for $n < 0$.

$h[0] =$

$h[1] =$

$h[2] =$

$h[3] =$

Therefore $h[n] =$

Finding the Impulse Response of a DT System – FIR vs IIR

- If a DT LTI system has a finite duration impulse response (i.e. $h[n]$ is nonzero only over a finite time interval), then the system is called a *Finite Impulse Response* (FIR) System.

- If a DT LTI system, with the condition of instal rest, will have an impulse response of infinite duration, then the system is called an *Infinite Impulse Response* (IIR) system.

- Important classification of systems - e.g. FIR and IIR filters.