Signal Processing
ENGN2228

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Second Semester

Lecture 15

Fourier Stuff – Fourier Coefficients (cont'd)

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

Example: $x(t) = 2\sin(2\pi t)$

$\omega_0 = 2\pi$

$T = \frac{2\pi}{\omega_0} = 1\,s$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t} \qquad \text{I}$$

$$x(t) = 2\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right]$$

$$= \frac{1}{j} e^{j2\pi t} - \frac{1}{j} e^{-j2\pi t}$$

$$= -j e^{j2\pi t} + j e^{-j2\pi t}$$

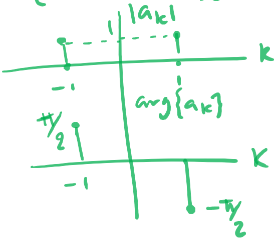
$$= \underbrace{e^{-j\pi/2}}_{a_1} e^{j2\pi t} + \underbrace{e^{j\pi/2}}_{a_{-1}} e^{-j2\pi t} \qquad \text{II}$$

Comparing I & II

for $k=1$, $a_1 = e^{-j\pi/2} = -j$

for $k=-1$, $a_{-1} = e^{j\pi/2} = j$

$$a_k = \begin{cases} j & k=-1 \\ -j & k=1 \\ 0 & \text{otherwise} \end{cases}$$



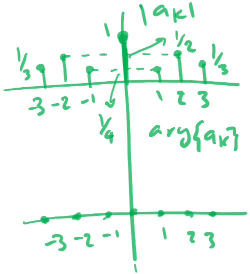
Example:

$$x(t) = 1 + \tfrac{1}{2}\cos 2\pi t + \cos 4\pi t + \tfrac{2}{3}\cos 6\pi t$$

$\omega_{0_1} = 2\pi$ $\quad \omega_{0_2} = 4\pi$ $\quad \omega_{0_3} = 6\pi$

$T_{0_1} = 1$ $\quad T_{0_2} = \tfrac{1}{2}$ $\quad T_{0_3} = \tfrac{1}{3}$

Hence, $T_0 = 1\,s$ so $\omega_0 = 2\pi$ rad/s



We can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t} \qquad \text{I}$$

$$x(t) = 1 + \tfrac{1}{2}\left[\frac{e^{j2\pi t} + e^{-j2\pi t}}{2}\right] + \left[\frac{e^{j4\pi t} + e^{-j4\pi t}}{2}\right] + \tfrac{2}{3}\left[\frac{e^{j6\pi t} + e^{-j6\pi t}}{2}\right]$$

$$= 1 + \tfrac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \tfrac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \tfrac{1}{3}\left[e^{j6\pi t} + e^{-j6\pi t}\right] \qquad \text{II}$$

I & II $\Rightarrow$ $a_0 = 1$ $\quad a_2 = a_{-2} = \tfrac{1}{2}$ $\quad a_k = 0$ other ks

$a_1 = a_{-1} = \tfrac{1}{4}$ $\quad a_3 = a_{-3} = \tfrac{1}{3}$

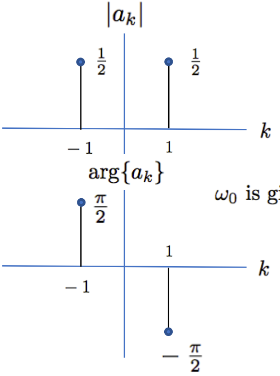
Fourier series of sine function

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$a_1 = \frac{1}{2j} = \frac{1}{2}e^{-j\frac{\pi}{2}} (k = 1)$$

$$a_{-1} = -\frac{1}{2j} = -\frac{1}{2}e^{j\frac{\pi}{2}} (k = -1)$$

$$a_k = 0 \text{ otherwise } (k \neq \pm 1)$$



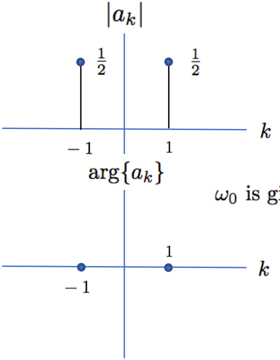
ω_0 is given

$$\boxed{a_k = -\frac{1}{2j}\delta[k+1] + \frac{1}{2j}\delta[k-1]}$$

Fourier series of cosine function

$$x(t) = \cos(\omega_0 t)$$
$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$a_1 = \frac{1}{2} \; (k = 1)$$
$$a_{-1} = \frac{1}{2} \; (k = -1)$$
$$a_k = 0 \text{ otherwise}$$

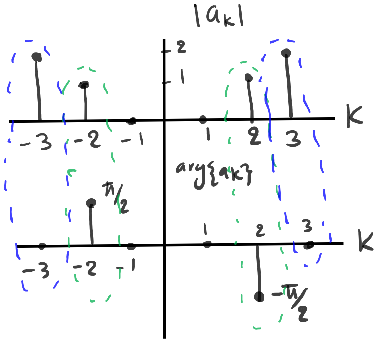


ω_0 is given

$$\boxed{a_k = \tfrac{1}{2}\delta[k+1] + \tfrac{1}{2}\delta[k-1]}$$

Example:

$\omega_0 = \pi$ rad/s

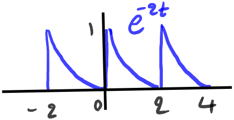


$$x(t) = \alpha_1 \sin(\beta_1 \omega_0 t) + \alpha_2 \cos(\beta_2 \omega_0 t)$$

$\alpha_1 = 2 \quad \beta_1 = 2 \quad \alpha_2 = 4 \quad \beta_2 = 3, \; \omega_0 = \pi$

$$x(t) = 2\sin(2\pi t) + 4\cos(3\pi t)$$

Example:



$x(t) = e^{-2t} \quad 0 \leq t \leq 2$

$T = 2\,s, \quad \omega_0 = \frac{2\pi}{T} = \pi \text{ rad/s}$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{1}{2}\int_0^2 e^{-2t}\,dt$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\left| e^{-2t}\right|_0^2$$

$$= \frac{1}{4}\left[1 - e^{-4}\right] = 0.245$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_0^2 e^{-2t}\,e^{-jk\omega_0 t}\,dt$$

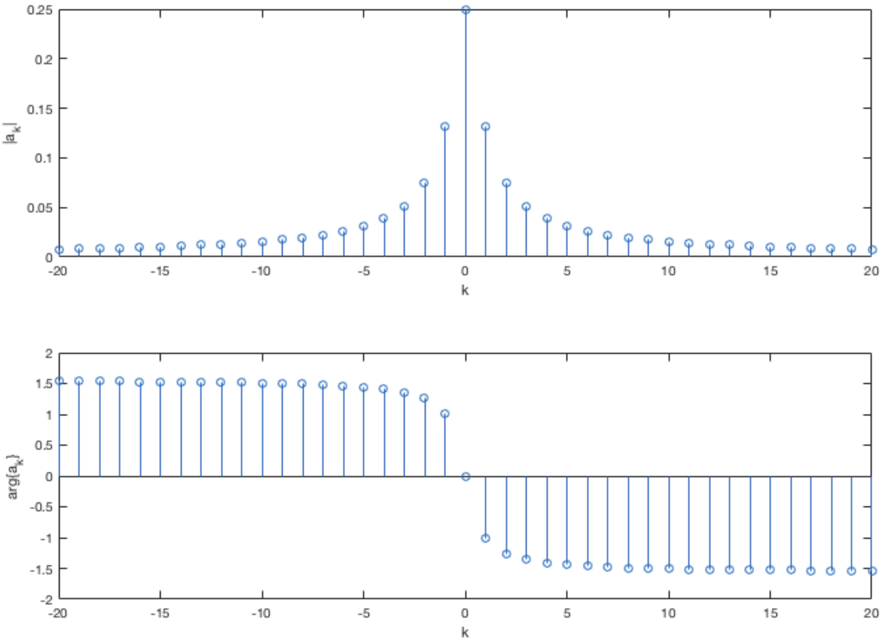
$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt$$

$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2+jk\pi}\right)\left| e^{-(2+jk\pi)t}\right|_0^2$$

$$= -\frac{1}{4+2jk\pi}\left[e^{-(4+j2\pi k)} - e^{0}\right]$$

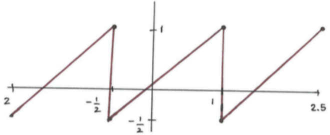
$$= \frac{1}{4+2jk\pi}\left[1 - e^{-4}\underbrace{e^{-jk2\pi}}_{=1}\right]$$

$$a_k = \frac{1-e^{-4}}{4+j2k\pi}$$



0.25
0.2
0.15
0.1
0.05
0
$|a_k|$
-20
-15
-10
-5
0
5
10
15
20
k
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
$\arg(a_k)$
-20
-15
-10
-5
0
5
10
15
20
k

Example Sawtooth Waveform



$$x(t) = t \qquad -\tfrac{1}{2} \le t \le 1$$

$$T = \tfrac{3}{2} \text{ sec}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{4\pi}{3} \text{ rad/s}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{2}{3}\int_{-\frac{1}{2}}^{1} t\,dt$$

$$= \frac{2}{3}\left|\frac{t^2}{2}\right|_{-\frac{1}{2}}^{1}$$

$$= \frac{2}{3}\left(\frac{1}{2} - \frac{1}{8}\right)$$

$$= \frac{1}{4}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{2}{3}\int_{-\frac{1}{2}}^{1} t\, e^{-jk\omega_0 t}\,dt$$

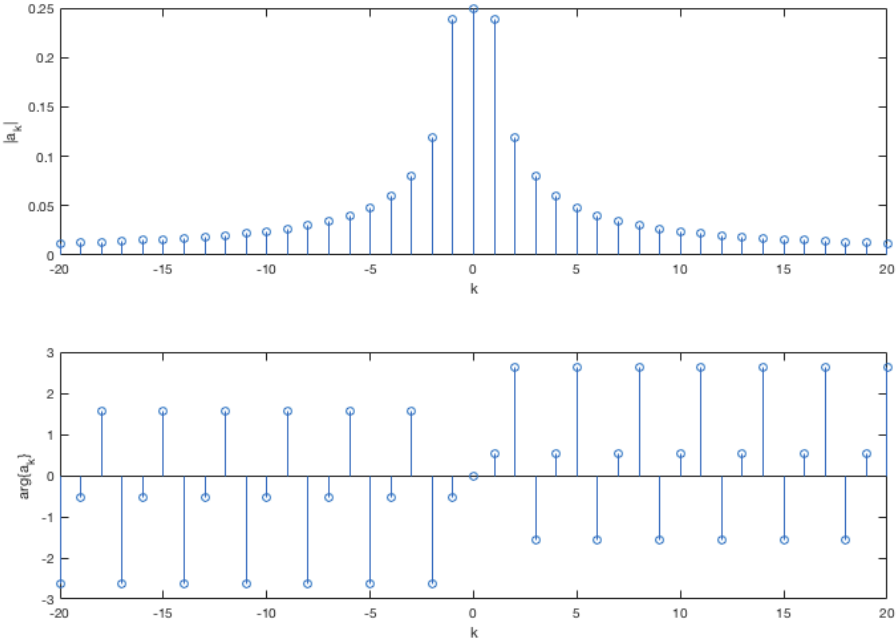
$$= \frac{2}{3}\left|\frac{e^{-jk\omega_0 t}(1 + jkt\omega_0)}{k^2\omega_0^2}\right|_{-\frac{1}{2}}^{1}$$

$$= \frac{2e^{-jk\omega_0}(1 + jk\omega_0)}{3k^2\omega_0^2} + \frac{j e^{\frac{jk\omega_0}{2}}(2j + k\omega_0)}{3k^2\omega_0^2}$$

$$= \frac{2}{3k^2\omega_0^2}e^{-jk\omega_0} - \frac{2}{3jk\omega_0}e^{-jk\omega_0} - \frac{2}{3k^2\omega_0^2}e^{\frac{jk\omega_0}{2}} - \frac{1}{3jk\omega_0}e^{\frac{jk\omega_0}{2}}$$

$$= \frac{2}{3k^2\omega_0^2}\left(e^{-jk\omega_0} - e^{\frac{jk\omega_0}{2}}\right) - \frac{2}{3jk\omega_0}\left(e^{-jk\omega_0} + \frac{1}{2}e^{\frac{jk\omega_0}{2}}\right)$$

Plot using Matlab


0.25
0.2
0.15
0.1
0.05
0
|a_k|
-20
-15
-10
-5
0
5
10
15
20
k
3
2
1
0
-1
-2
-3
arg{a_k}
-20
-15
-10
-5
0
5
10
15
20
k

Finding Fourier Series by inspection vs analysis

$$x(t) = 2\sin(2\pi t) \quad \omega_0 = 2\pi \quad T = \tfrac{2\pi}{\omega_0} = 1 \text{ sec}$$

Analysis Eq.

$$\begin{aligned} a_0 &= \frac{1}{T}\int_T x(t)dt \\ &= \frac{1}{T}\int_0^1 2\sin(2\pi t)dt \\ &= 2\int_0^1 \sin(2\pi t)dt \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^1 2\sin(2\pi t)e^{-jk\omega_0 t}dt \\ &= \begin{cases} 0 & k \neq \pm 1 \\ j & k = -1 \\ -j & k = 1 \end{cases} \end{aligned}$$

$=$

Inspection method

$$\begin{aligned} x(t) &= 2\sin(2\pi t) \\ &= 2\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right] \\ &= \frac{1}{j}\left[e^{j2\pi t} - e^{-j2\pi t}\right] \\ &= -je^{j2\pi t} + je^{-j2\pi t} \quad \text{(I)} \end{aligned}$$

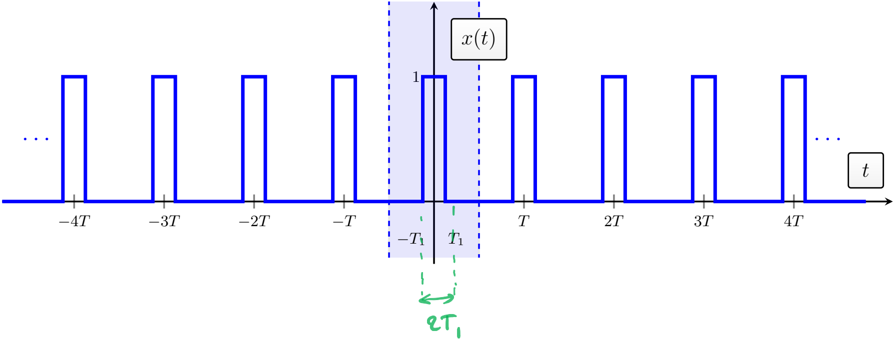
$$x(t) = \textstyle\sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t} \quad \text{(II)}$$

Comparing (I) and (II)

$$a_k = \begin{cases} -j & k = 1 \\ j & k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \le T/2$$



Calculating FS coefficients for a periodic Rectangular Wave

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt \\
&= \frac{1}{T}\int_{-T_1}^{T_1} (1)dt \\
&= \frac{1}{T}[t]_{-T_1}^{T_1} \\
&= \frac{2T_1}{T}.
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_{-T_1}^{T_1} (1)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\left(-\frac{1}{jk\omega_0}\right)\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} \\
&= -\frac{1}{jkT\omega_0}\left\{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right\} \\
&= \frac{2}{kT\omega_0}\left\{\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right\} \\
&= \frac{2}{kT\omega_0}\sin(k\omega_0 T_1) \\
&= \frac{\sin(k\omega_0 T_1)}{k\pi}.
\end{aligned}
$$

Calculating FS coefficients for a periodic Rectangular Wave (cont'd)

$T_1 = T/8$

$$a_0 = \frac{2T_1}{T} = \frac{2}{T}\frac{T}{8} = \frac{1}{4} = 0.25$$

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin\left(k\frac{2\pi}{T}\frac{T}{8}\right)}{k\pi} = \frac{\sin\left(k\frac{\pi}{4}\right)}{k\pi}$$

$$a_1 = \frac{\sin\left(\frac{\pi}{4}\right)}{\pi} = \frac{1}{\sqrt{2}\pi} = 0.225079$$

$$a_{-1} = \frac{\sin\left(-\frac{\pi}{4}\right)}{-\pi} = \frac{1}{\sqrt{2}\pi} = 0.225079$$

$$a_2 = \frac{\sin\left(\frac{\pi}{2}\right)}{2\pi} = \frac{1}{2\pi} = 0.15915$$

$$a_{-2} = \frac{\sin\left(-\frac{\pi}{2}\right)}{-2\pi} = \frac{1}{2\pi} = 0.15915$$

$$a_3 = \frac{\sin\left(\frac{3\pi}{4}\right)}{3\pi} = \frac{1}{3\sqrt{2}\pi} = 0.075$$

$$a_{-3} = \frac{\sin\left(-\frac{3\pi}{4}\right)}{-3\pi} = \frac{1}{3\sqrt{2}\pi} = 0.075$$

$$a_4 = \frac{\sin(\pi)}{4\pi} = 0$$

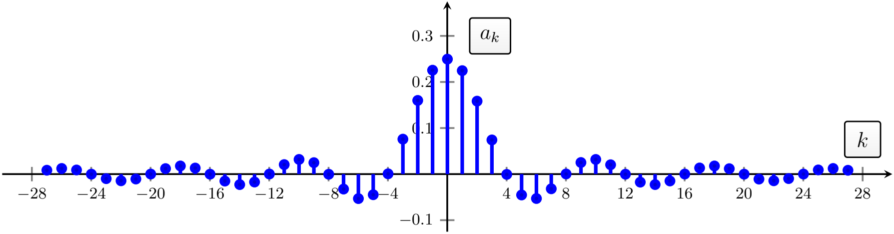
$$a_{-4} = \frac{\sin(-\pi)}{-4\pi} = 0$$

$$a_5 = \frac{\sin\left(\frac{5\pi}{4}\right)}{5\pi} = \frac{-\sqrt{2}}{10\pi} = -0.045$$

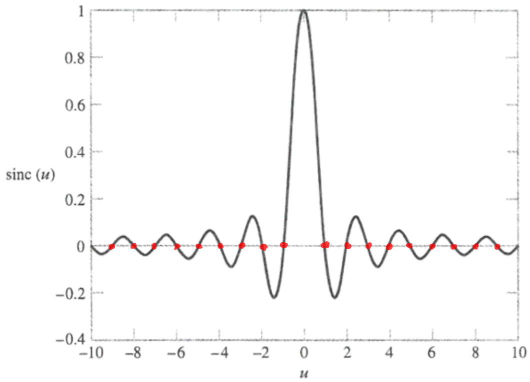
$$a_{-5} = \frac{\sin\left(-\frac{5\pi}{4}\right)}{-5\pi} = \frac{-\sqrt{2}}{10\pi} = -0.045$$

and so on ...

With $T_1 = T/8$ the periodic rectangular wave has Fourier coefficients, $\{a_k\}$, as follows



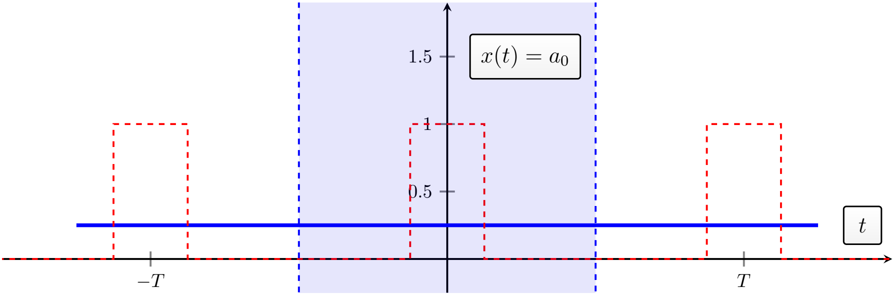
Sinc function

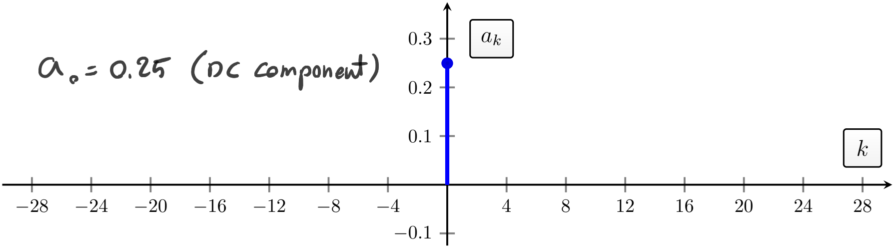


Sinc function $\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$.

$$\boxed{\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}}$$

Fourier Stuff – Periodic Rectangular Wave

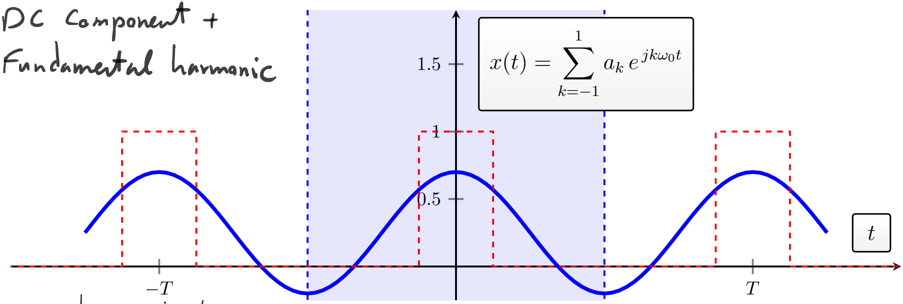




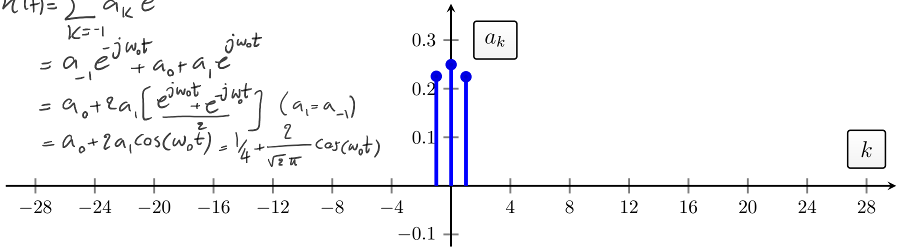
$a_0 = 0.25$ (DC component)

Fourier Stuff – Periodic Rectangular Wave

DC Component + Fundamental harmonic

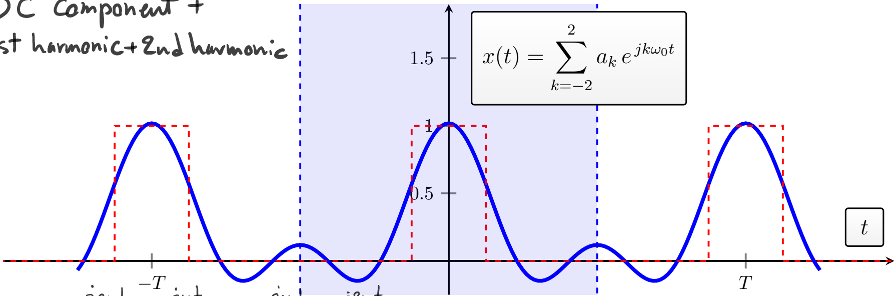


$$x(t) = \sum_{k=-1}^{1} a_k e^{jk\omega_0 t}$$
$$= a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t}$$
$$= a_0 + 2a_1 \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right] \quad (a_1 = a_{-1})$$
$$= a_0 + 2a_1 \cos(\omega_0 t) = \frac{1}{4} + \frac{2}{\sqrt{2}\pi} \cos(\omega_0 t)$$



Convenor: R. A. Kennedy

DC component +
1st harmonic + 2nd harmonic

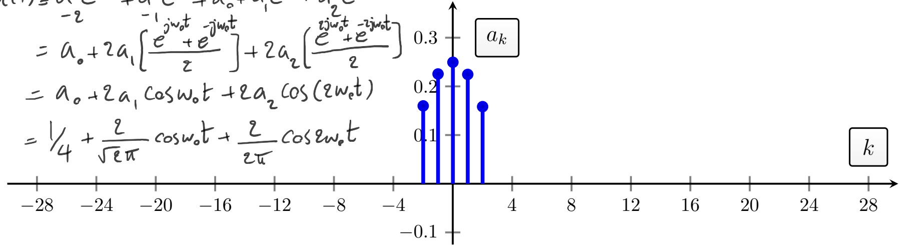


$$x(t) = a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}$$

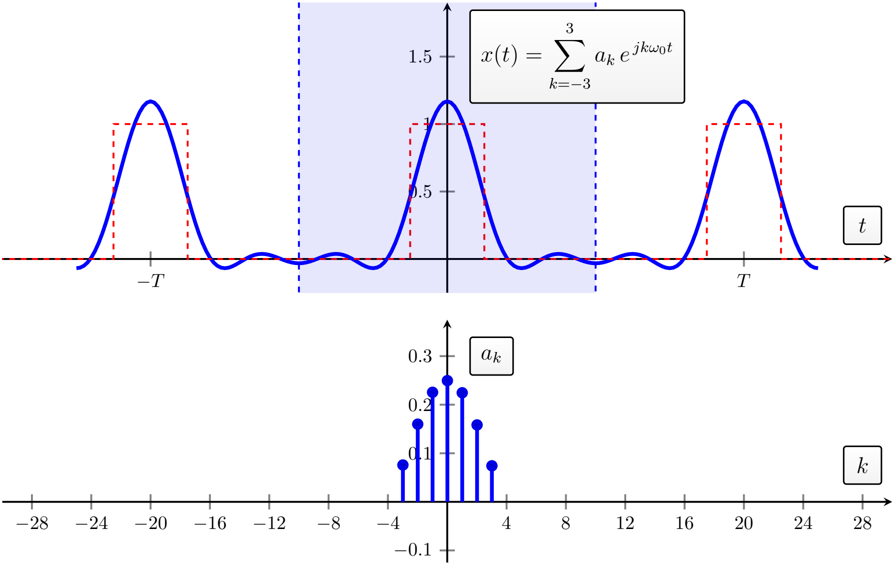
$$= a_0 + 2a_1 \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right] + 2a_2 \left[\frac{e^{2j\omega_0 t} + e^{-2j\omega_0 t}}{2} \right]$$

$$= a_0 + 2a_1 \cos \omega_0 t + 2a_2 \cos(2\omega_0 t)$$

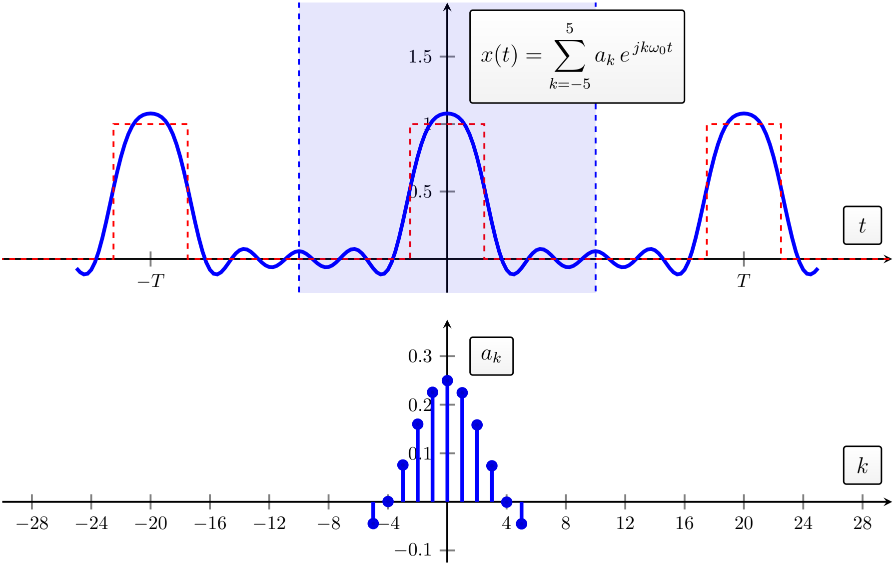
$$= \frac{1}{4} + \frac{2}{\sqrt{2}\pi} \cos \omega_0 t + \frac{2}{2\pi} \cos 2\omega_0 t$$



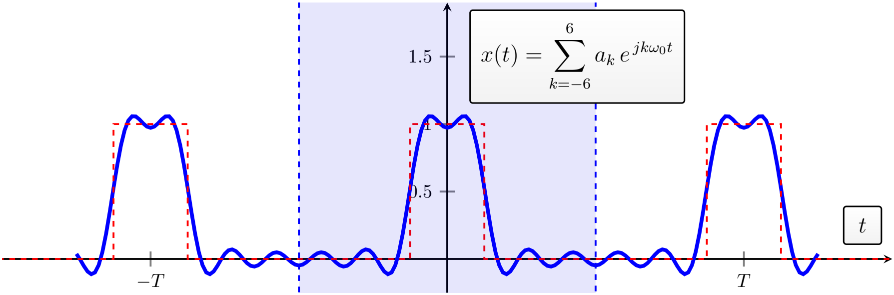
Fourier Stuff – Periodic Rectangular Wave

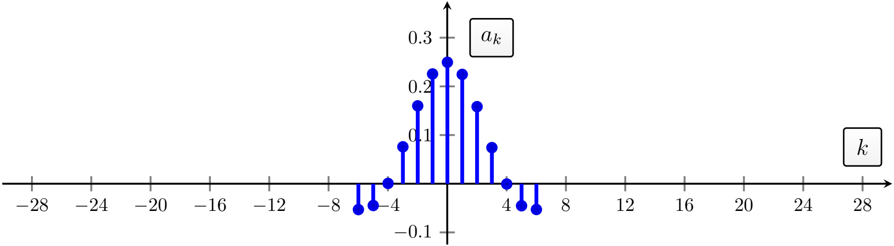


Fourier Stuff – Periodic Rectangular Wave

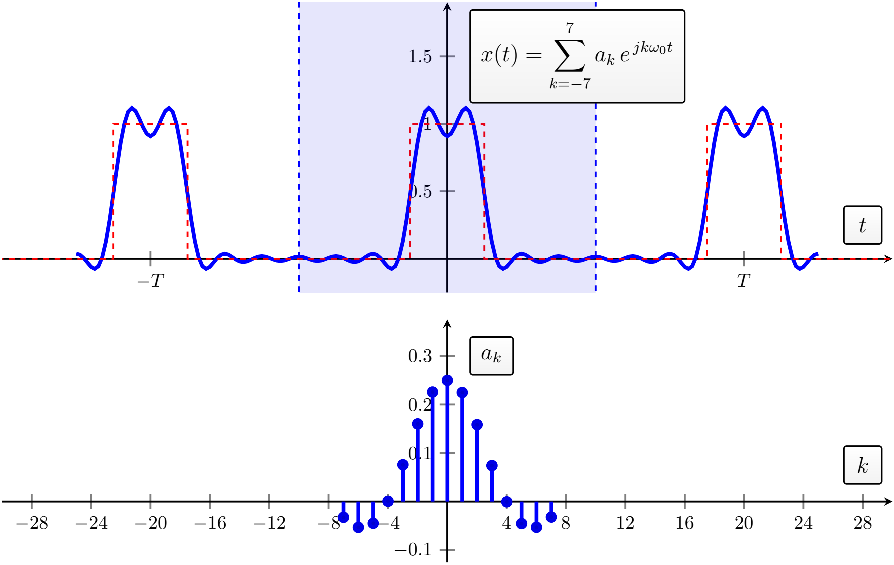


Fourier Stuff – Periodic Rectangular Wave

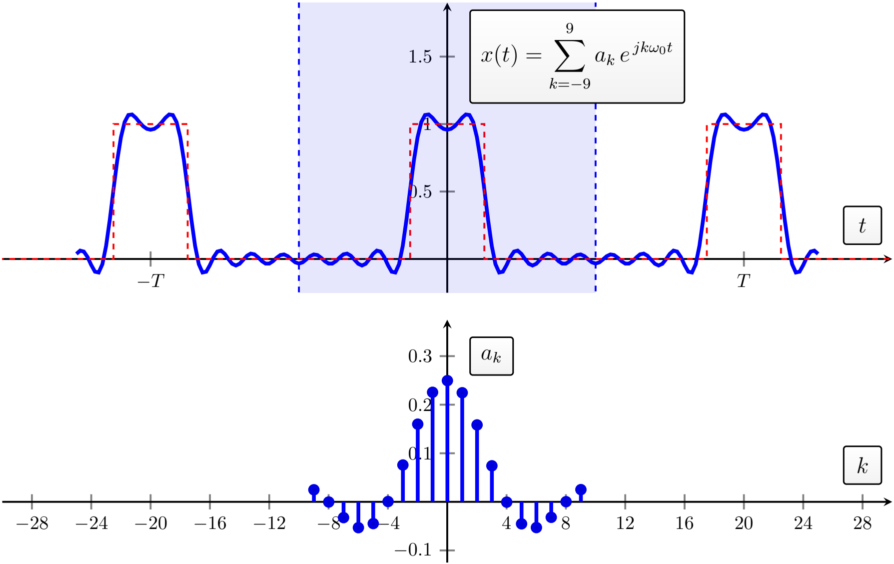




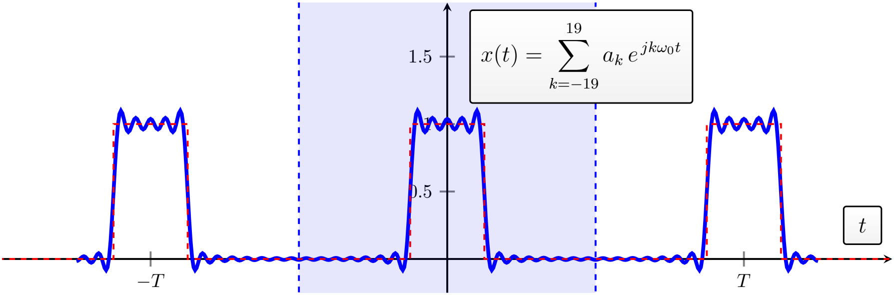
Fourier Stuff – Periodic Rectangular Wave

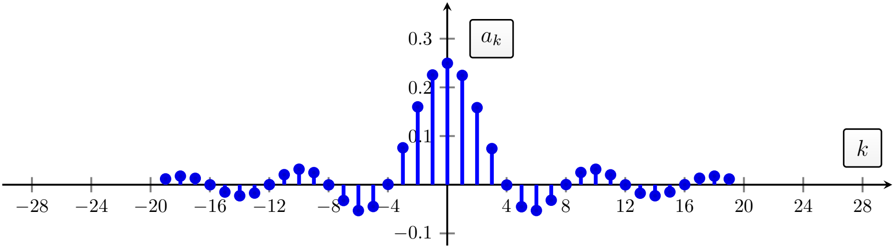


Fourier Stuff – Periodic Rectangular Wave

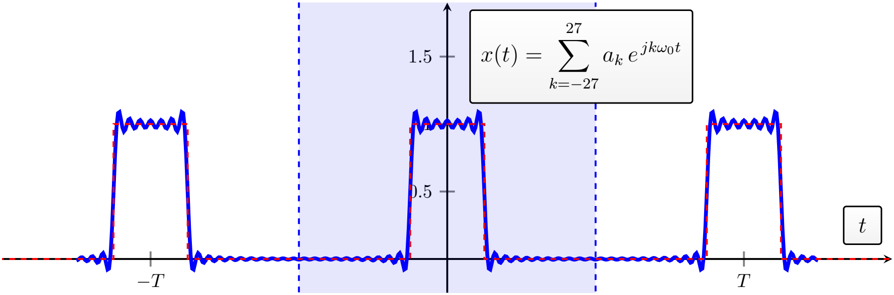


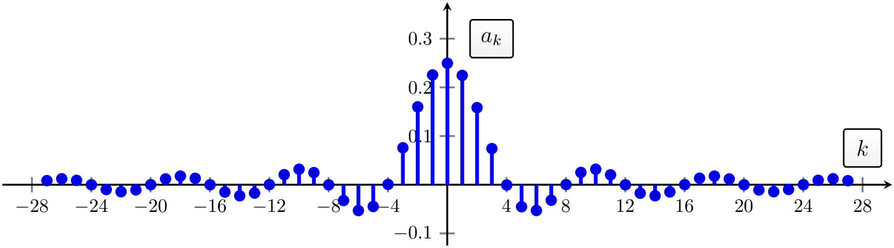
Fourier Stuff – Periodic Rectangular Wave





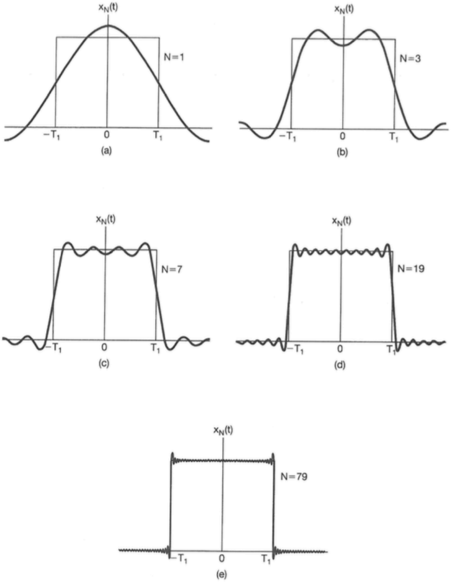
Fourier Stuff – Periodic Rectangular Wave





Convergence of the Fourier Series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$



Time and frequency domain properties for the four cases

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties