Signal Processing

ENGN2228

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Second Semester

Lectures 13 and 14

Fourier Series – Where we are heading?

DT and CT LTI systems admit a simple description in terms of the impulse response. The output signal can be computed in terms of the convolution of the input and impulse response. In essence, not much else is needed.

Fourier Series, the Fourier Transform and other "frequency domain" descriptions provide an alternative (but equivalent) viewpoint with a number of key advantages:

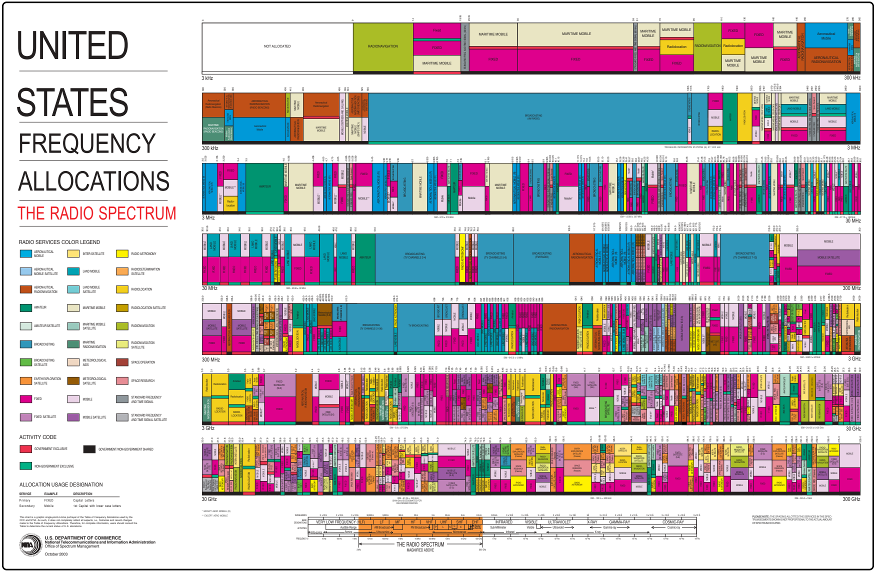
- Convolution is simplified using the Fourier representations. Convolution is "twisted and complicated" in the "time domain" but in the frequency domain it is "untwisted and simple".
- Studying the behavior of a LTI system in the frequency domain is actually natural and intuitive (after a few years).

Fourier Series – Where we are heading? (con't)

To this point we haven't said much about **design**. We want to build LTI systems that achieve certain design goals. This is best done in the frequency domain.

Design and resource allocation specifications almost always have a frequency domain formulation:

Fourier Series – Where we are heading? (con't)



Consider

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

which is a moving average of three consecutive terms. So $y[n]$ looks like a smooth version of $x[n]$.

Note that we use $1/3$ weights because if $x[n]$ is very smooth to start with then $y[n]$ should be equal or close to $x[n]$. So if $x[n] = 1$ for all n, that is, it is a constant, then $y[n] = 1/3 + 1/3 + 1/3 = 1$.

So

$$y[n] = \frac{1}{5}x[n-2] + \frac{1}{5}x[n-1] + \frac{1}{5}x[n] + \frac{1}{5}x[n+1] + \frac{1}{5}x[n+2]$$

works as a smoother as well. It should smooth more.

Ditto

$$y[n] = \frac{1}{9}x[n-2] + \frac{2}{9}x[n-1] + \frac{1}{3}x[n] + \frac{2}{9}x[n+1] + \frac{1}{9}x[n+2]$$

works as a smoother too. It relies more on the current value $x[n]$ in forming $y[n]$.

Which smoother works best? What are we trying to achieve?

Fourier Series – Where we are heading? (con't)

We are lacking tools to analyze which smoother is best. We are lacking tools for design. This is one main reason to look at the description of LTI systems in the Frequency or Fourier domain.

We can view the "slow" system

$$h_5[n] \triangleq \frac{1}{5}\delta[n-2] + \frac{1}{5}\delta[n-1] + \frac{1}{5}\delta[n] + \frac{1}{5}\delta[n+1] + \frac{1}{5}\delta[n+2]$$

as extracting the slow/smooth part of $x[n]$

$$x_s[n] \equiv x_5[n] \triangleq x[n] \star h_5[n]$$

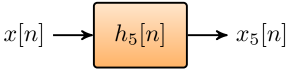


Fig: Smoother

This is called **filtering**. It is a **"low pass filter"**. It passes to the output the slowly varying parts of input $x[n]$ (mostly).

But what if I didn't want the slowly varying parts of input $x[n]$ but the opposite? Can we extract just the quickly varying parts (the parts blocked by $h_5[n]$)?

Fourier Series – Digression (con't)

Of course, the "fast" system is just the complement of the "slow" system

$$h_5^c[n] \triangleq \delta[n] - h_5[n] =$$
$$-\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]$$

and extracts the non-slow part of $x[n]$

$$x_s^c[n] \equiv x_5^c[n] \triangleq x[n] \star h_5^c[n]$$

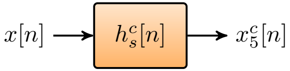


Fig: Complement of the Smoother

It is a **"high pass filter"**. It passes to the output the quickly varying parts of input $x[n]$ (mostly). Also evidently, $x[n]$ is perfectly split into the two parts $x[n] = x_5[n] + x_5^c[n]$.

Fourier Series – Digression (con't)

Finally, Goldilocks says she doesn't want the fast nor the slow but just the parts of $x[n]$ that are varying not too fast nor too slow. Can we figure this out? (Or should we just feed her to the bears?)

The complement of the five term moving average

$$h_5^c[n] \triangleq \delta[n] - h_5[n]$$
$$= -\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]$$

essentially blocks the very slow parts of $x[n]$ and lets through more than the complement of the three term moving average

$$h_3[n] \triangleq \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n+1]$$

so we can get the desired action via the **"band pass filter"**

$$h_5^c[n] \star h_3[n] = \text{whatever}$$

Fourier Series – Digression (con't)

$x[n]$ can be split into slow, medium (neither fast nor slow) and fast signals

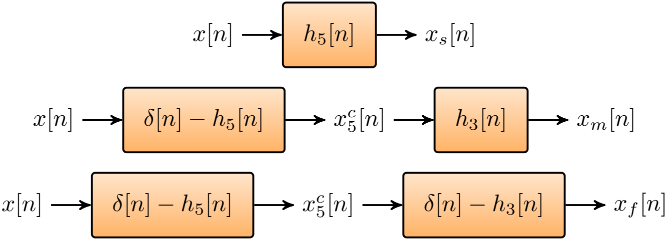


Fig: Slow, Medium and Fast Signal Processing

Here

$$x_m[n] + x_f[n] = x_5^c[n] \quad \text{and} \quad x_s[n] + x_5^c[n] = x[n]$$

$$x_s[n] + x_m[n] + x_f[n] = x[n]$$

Evidently, it seems useful to be able to characterize how a system treats fast, medium and slow inputs (then we can broadly regard it as low pass, high pass, band pass or some blurring of the three).

The most obvious thing to do is see how the system responds to **complex exponentials** (of different "frequencies"). This leads us to study this problem more completely.

Prior to studying the set of complex exponentials we can lay down the desirable characteristics of a set of basic signals to probe LTI systems:

- We need to be able to represent any **signal** (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or scanning in some sense).
- The response of an LTI **system** to any of these basic signals should be simple, useful and insightful.

To emphasize, it has to meet demands of both: i) signal representation and ii) systems characterization.

Fourier Series – Response to Complex Exponentials

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

$\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$

$u(t) \rightarrow \boxed{h(t)} \rightarrow s(t)$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

$\delta[n] \rightarrow \boxed{h[n]} \rightarrow h[n]$

$u[n] \rightarrow \boxed{h[n]} \rightarrow s[n]$

Previous focus was unit samples and impulses.

Alternative/new focus is "eigenfunctions of LTI systems" (what the?)

Start with CT LTI system convolution equation. Let the input be

$$x(t) = e^{st}$$

where $s \in \mathbb{C}$ is complex. Then, with $h(t)$ the impulse response,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau, \quad \text{with } x(t) = e^{st} \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \end{aligned}$$

(...yawn)

So $x(t) = e^{st}$ is a really good choice. Reflect on

$$y(t) = \underbrace{e^{st}}_{\text{the input}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\substack{\text{independent of } t \\ H(s)}}$$

The output equals the input apart from a complex multiplier

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \quad \in \mathbb{C}.$$

This is some sort of "transform" of the impulse response $h(t)$. What does this all mean?

Suppose I have a $N \times N$ square matrix $\mathbf{M}$ then if there is a N-vector $\boldsymbol{\mu}$ satisfying

$$\boldsymbol{M}\boldsymbol{\mu} = \lambda\boldsymbol{\mu}, \quad \lambda \in \mathbb{C}$$

then $\boldsymbol{\mu} \in \mathbb{C}^N$ is an **eigenvector** and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

We could draw

$$\boldsymbol{\mu} \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda\boldsymbol{\mu}$$

which reveals that the output vector/signal is equal to input vector/signal apart from a complex scale factor ("gain"). The action of matrix $\boldsymbol{M}$ is simple when the input is an eigenvector.

Fourier Series – Matrix Digression (con't)

More generally, there are a set of eigenvectors

$$\boldsymbol{M}\boldsymbol{\mu}_i = \lambda_i \boldsymbol{\mu}_i, \quad i = 1, 2, \ldots, N$$

and for each we could draw

$$\boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda_i \boldsymbol{\mu}_i$$

So different eigenvectors have different gains.

This is powerful because, take any $\boldsymbol{v} \in \mathbb{C}^N$ not generally an eigenvector and assume the set of N eigenvectors $\boldsymbol{\mu}_i$ span $\mathbb{C}^N$.

Fourier Series – Matrix Digression (con't)

Then

$$\boldsymbol{v} = \sum_{i=1}^{N} a_i \boldsymbol{\mu}_i$$

and we can draw

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i$$

These eigenvectors are a special set (for this matrix $\boldsymbol{M}$).

Fourier Series – Matrix Digression (con't)

Recall previous guidelines for a desirable signal "set"

- We need to can able to represent any **signal**/vector (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or spanning in some sense).
- The response of an LTI **system**/matrix to any of these basic signals should be simple, useful and insightful.

In fact it is more than a coincidence, since a system is an operator (maps functions to functions) and a matrix is a finite dimensional operator (maps vectors in $\mathbb{C}^N$ to $\mathbb{C}^N$).

In the back of our minds. . .

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i, \quad \lambda_i = \lambda_i(\boldsymbol{M})$$

Eigen-behavior of LTI Systems

$$\sum_{i=1}^{N} a_i e^{s_i t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i e^{s_i t}, \quad \lambda_i = H(s_i)$$

where

$$H(s_i) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s_i \tau}\, d\tau \in \mathbb{C}.$$

So being able to decompose signals into complex exponentials leads to a simple characterization of an LTI system.

Fourier Series – DT Case Eigenfunctions

What can we use to in the DT case for $x[n]$ analogous to CT $x(t) = e^{st}$?

Let the input be

$$x[n] = z^n$$

where $z \in \mathbb{C}$ is complex. Then, with $h[n]$ the impulse response,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k], \quad x[n] = z^n \\ &= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\ &= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}} \end{aligned}$$

Reflect

$$y[n] = \underbrace{z^n}_{\text{the input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{\text{independent of } n}$$

The output equals the input apart from a complex multiplier

$$\boxed{H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \in \mathbb{C}}$$

This is some sort of "transform" of the impulse response $h[k]$.

Fourier Stuff – Review Key Observation

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CT LTI system response to **arbitrary** input $x(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

versus

CT LTI system response to **specific, complex exponential** input $x(t) = e^{st}$, for some complex $s \in \mathbb{C}$,

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s) \in \mathbb{C}}$$

where $H(s)$ is just a complex number — call this the "complex gain".

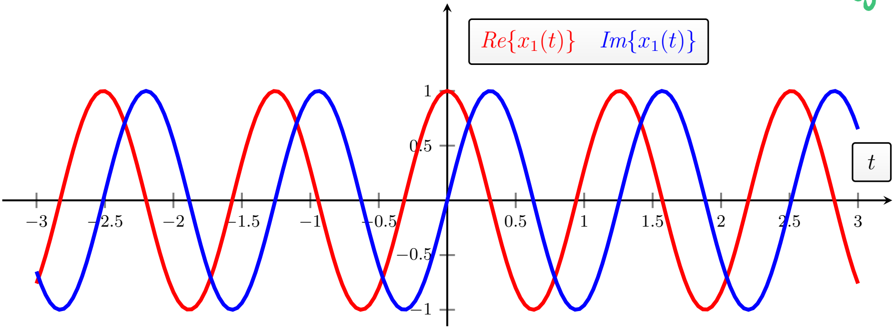
Fourier Stuff – Examples

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Example 1: Let

$$x_1(t) \triangleq e^{j\,5t}$$



be input to a LTI system with impulse response $h_1(t)$ and "complex gain"

$$H_1(j5) \triangleq 1 + j\sqrt{3}.$$

Find

$$y_1(t) = x_1(t) \star h_1(t), \quad \text{where } x_1(t) = e^{j\,5t}$$

For this input, we have

$$s = j5, \qquad \text{(recall } e^{st}\text{)}$$

and the output is

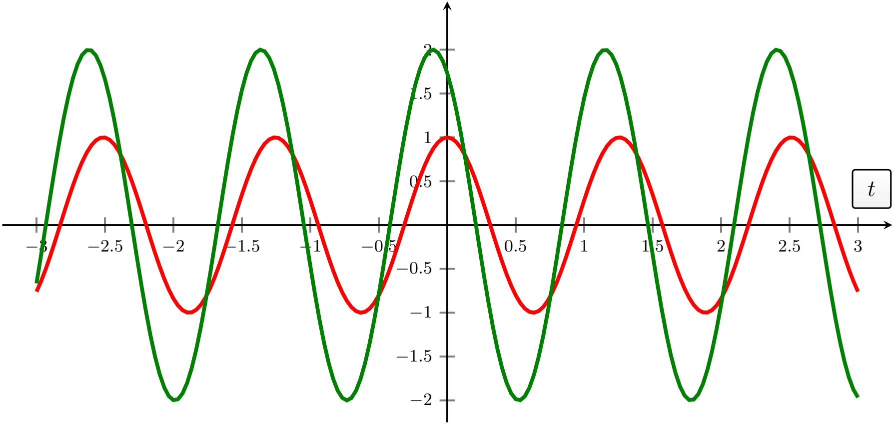
$s = j5$

$$\begin{aligned} y_1(t) &= e^{j\,5t} \star h_1(t) && \text{(convolution)} \\ &= e^{j\,5t} H_1(j5) && \text{(multiplication)} \\ &= e^{j\,5t}(1 + j\sqrt{3}) \\ &= 2\,e^{j\,5t} e^{j\,\pi/3} \\ &= 2\,e^{j\,(5t+\pi/3)} \end{aligned}$$

That is, the output is at the same frequency $\omega_1 = 5$ $(s = j\omega_1)$, is twice the amplitude of the input and is phase shifted by $\pi/3$.

Fourier Stuff – Examples

Plotting the real part of this output $y_1(t)$ (and input $x_1(t)$):



Example 2: Now let the input be only real, have a phase shift of $\pi/4$ and have magnitude 4:

$$x_1(t) \triangleq 4\cos(5t + \pi/4)$$

Let this be input to a LTI system with impulse response $h_1(t)$ which has gain

$$H_1(\pm j5) \triangleq 1 \pm j\sqrt{3}, \quad \text{(here } s = \pm j5\text{)},$$

(This notation means $H_1(j5) \triangleq 1 + j\sqrt{3}$ and $H_1(-j5) \triangleq 1 - j\sqrt{3}$ merged into one equation.) Find

$$y_1(t) = x_1(t) \star h_1(t)$$

Again we won't have to explicitly compute the convolution

$$y_1(t) = \int_{-\infty}^{\infty} 4\cos(5\tau + \pi/4)\, h_1(t-\tau)\, d\tau$$

and in fact we don't know $h_1(t)$ completely anyway.

For this input, we have

$$
\begin{aligned}
x_1(t) &= 4\cos(5t + \pi/4) \\
&= 4\frac{\left(e^{j\,(5t+\pi/4)} + e^{-j(5t+\pi/4)}\right)}{2} \\
&= (\overset{e^{j\pi/4}}{\underline{\sqrt{2} + \sqrt{2}j}})e^{j\,5t} + (\overset{e^{-j\pi/4}}{\underline{\sqrt{2} - \sqrt{2}j}})e^{-j5t}
\end{aligned}
$$

So we can use the principle of superposition with values $s = j5$ and $s = -j5$, that is, $s = \pm j5$,

$$
(\sqrt{2} \pm j\sqrt{2})e^{\pm \underbrace{j5}_{s}t} \longrightarrow \boxed{h_1(t)} \longrightarrow (\sqrt{2} \pm j\sqrt{2})H_1(\pm j5)e^{\pm j5t}
$$

For input

$$x_1(t) = 4\cos(5t + \pi/4)$$

the output is then

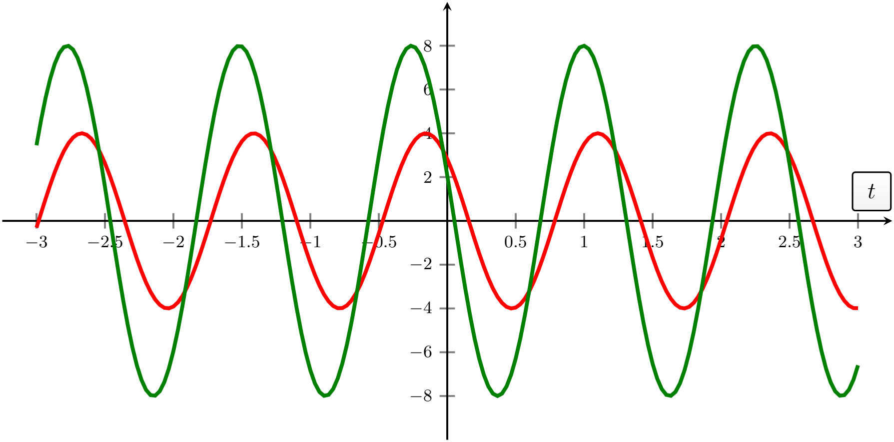
$$
\begin{aligned}
y_1(t) &= (\sqrt{2} + j\sqrt{2})\overbrace{(1 + j\sqrt{3})}^{H(j5)}e^{+j5t} + (\sqrt{2} - j\sqrt{2})\overbrace{(1 - j\sqrt{3})}^{H(-j5)}e^{-j5t} \\
&= (2e^{j\,\pi/4})2e^{j\,\pi/3}e^{+j5t} + (2e^{-j\pi/4})2e^{-j\pi/3}e^{-j5t} \\
&= 8\,\frac{\left(e^{j\,(5t+7\pi/12)} + e^{-j(5t+7\pi/12)}\right)}{2} \\
&= 8\,\cos(5t + 7\pi/12)
\end{aligned}
$$

Or equivalently

$$\int_{-\infty}^{\infty} 4\,\cos(5\tau + \pi/4)\,h_1(t-\tau)\,d\tau = 8\,\cos(5t + 7\pi/12)$$

Fourier Stuff – Examples

Plotting input $x_1(t)$ and output $y_1(t)$:



Fourier Stuff – Examples

$H(\pm j5) = 1 \pm j\sqrt{3}$

This calculation can be considerably simplified. Note

- $\left|H_1(\pm j5)\right| = 2$ so the gain (magnitude) is just 2.
- The phase of $H_1(\pm j5)$ is $\pm\pi/\cancel{4}3$ which is the **relative** phase shift of the output with respect to the input.
- Recall the system is time-invariant so the $\pi/4$ phase on the input is equivalent to a time shift and gets conveyed to the output.

Therefore,

$$x_1(t) = 4\cos(5t + \pi/4)$$

$$H_1(j5) = 2\, e^{j\,\pi/3} \quad \text{System}$$

$$y_1(t) = \underbrace{2 \times 4}_{8} \cos(5t + \underbrace{\pi/4 + \pi/3}_{7\pi/12})$$

Fourier Stuff – Examples

Example 3: Consider the series/cascade connection

$$x_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow y_1(t) \longrightarrow \boxed{h_2(t)} \longrightarrow y_2(t)$$

with LTI system $h_1(t)$ such that

$$H_1(j5) = 0.5\, e^{j\,\pi/4} \quad \text{and} \quad H_1(-j5) = 0.5\, e^{-j\pi/4}$$

and LTI system $h_2(t)$ such that

$$H_2(j5) = 0.3\, e^{j\,2\pi/3} \quad \text{and} \quad H_2(-j5) = 0.3\, e^{-j2\pi/3}$$

Compute

$$y_2(t) = x_1(t) \star h_1(t) \star h_2(t)$$

for input signal

$$x_1(t) \triangleq 3\cos(5t - \pi/7).$$

First test understanding:

- $y_1(t)$ has to look like
$$\alpha \cos(5t + \beta)$$
because the input, $x_1(t)$, is of that form.
- $y_2(t)$ has to look like
$$\gamma \cos(5t + \delta)$$
because the input, $x_2(t) = y_1(t)$ is of that form.
- We only have to find two real numbers (magnitude and phase) or, equivalently, one complex number.

Overall gain is

$$3 \times \underbrace{0.5 \times 0.3}_{0.15} = 0.45$$

Overall phase

$$-\pi/7 + \underbrace{\pi/4 + 2\pi/3}_{11\pi/12} = 65\pi/84$$

Therefore,

$$y_2(t) = 0.45 \cos(5t + 65\pi/84)$$

and we note that

$$H_1(j5)\, H_2(j5) = 0.15\, e^{j\, 11\pi/12}$$

and

$$H_1(-j5)\, H_2(-j5) = 0.15\, e^{-j\, 11\pi/12}.$$

Observation 1: This is revealing something very important. Cascading two LTI systems implies **convolving** their impulse responses. But for complex exponentials signals we only have to **multiply** complex gains. As we show much later this is the key property of frequency domain descriptions; series LTI systems lead to multiplications and parallel LTI systems lead to additions (superposition).

It seems like behavior of interconnections of LTI systems with complex exponential signals as inputs (and outputs) is relatively simple.

Observation 2: Houston we have a problem. OK so complex exponentials are simple to work with. What about more general signals?

Firstly we look at **periodic signals**. Fourier series, which we now consider, show that any periodic signal can be expressed into terms of appropriate linear combinations of complex exponential signals (which are themselves periodic). Then we can appeal to superposition to characterize the response of an LTI system to general (not necessarily complex exponential) periodic signals.

Eigen Behaviour of CT LTI Systems

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$\underbrace{H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{s=j\omega}$$

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

$$\underbrace{H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}$$

Eigen Behaviour of DT LTI Systems

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow H(z)z^n$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$z = e^{j\omega}$

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Definition of Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{j\omega_0 t}$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=0}^{N-1} a_k H(e^{j\omega_0 n}) e^{j\omega_0 n}$$

A big picture

- $e^{j\omega t}, e^{st}$ are eigen functions of CT LTI systems
- $e^{j\omega n}, z^n$ are eigen functions of DT LTI systems

CT

Study of CT LTI systems using $e^{j\omega t}$ → Fourier Series (FS) (periodic signals)

Study of CT LTI systems using $e^{j\omega t}$ → Fourier Transform (FT) (general signals)

Study of CT LTI systems using e^{st} → Laplace Transform (out of scope of this course)

DT

Study of DT LTI systems using $e^{j\omega n}$ → Deseret-Time Fourier Series (DTFS) (periodic signals)

Study of DT LTI systems using $e^{j\omega n}$ → Deseret-Time Fourier Transform (DTFT) (general signals)

Study of DT LTI systems using z^n → Z-transform (ENGN 4537)

Time and frequency domain properties for the four cases

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

week 8-11

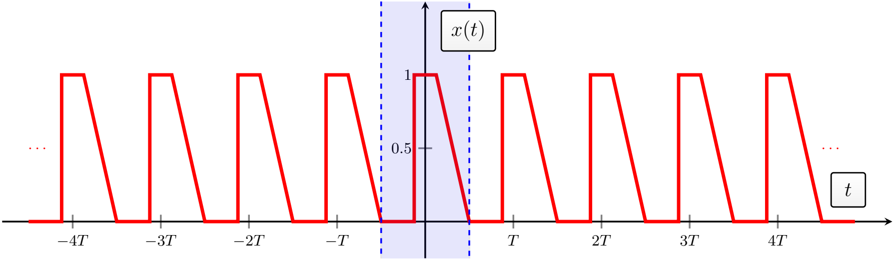
week 5-7

CT periodic signals satisfy

$$\boxed{x(t) = x(t+T), \quad \text{for all } t}$$

where T is the **fundamental period** (smallest positive T).

An example periodic waveform with fundamental period T is shown below.



Define the **fundamental frequency** associated with fundamental period T

$$\omega_0 = \frac{2\pi}{T}$$

Prototypical periodic signal is

$$e^{j\,\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

What other related signals (to this prototypical signal) are also periodic with period T?

It is reasonably clear that ($\iff$ means if and only if)

$$e^{j\,\omega t} \text{ is periodic with period } T \iff \omega = k\omega_0, \quad k \in \mathbb{Z}$$

To

- For each $|k| \neq 1$, the period T is not fundamental.
- For example, with $k = -3$, $e^{-j3\omega_0 t}$ has fundamental period $T/3$ but it is still periodic with period T. It is periodic with periods: $T/3$, $2T/3$, T, $4T/3$, etc.
- Further $k = 0$, which implies $\omega = 0$, is weird, it leads to just constant function.
- $k = -1$ leads to the conjugate of $k = 1$ and has fundamental period T also.

The infinite linear combination

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \, e^{jk2\pi t/T}$$

- is periodic with period T because all components are periodic with period T
- $k = 0$ term is the "DC" term ("direct current")
- $k = \pm 1$ terms are the first harmonic
- $k = \pm 2$ terms are the second harmonic
- The two kth terms are called the kth harmonic ($k > 0$) and have fundamental period kT.

In

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk2\pi t/T}}$$

the $\{a_k\}$ are the **Fourier Series coefficients**.

It's clear there are an infinite number of periodic functions that can be built this way but...

Question: Can any periodic function be expressed in such a way?

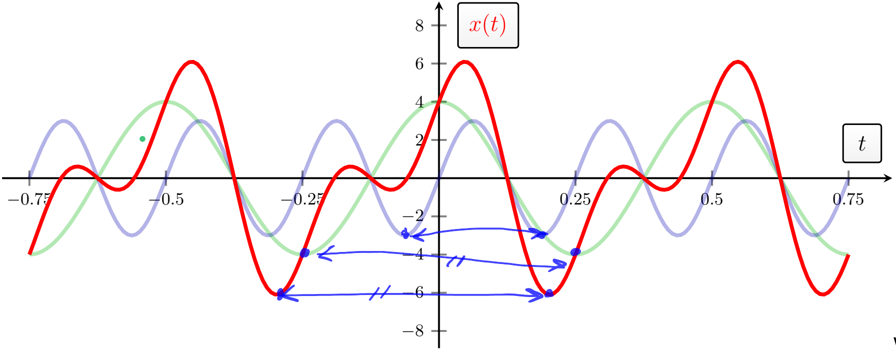
Answer: (Pretty well) yes.

Then we need to a way of determining the Fourier Series coefficients, that is:

$$\boxed{\text{Given } x(t) \text{ find } \{a_k\} \text{ for all } k \in \mathbb{Z}.}$$

(Simple) Example 1: If

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$



we find the fundamental frequency and fundamental period as

$$\omega_0 = 4\pi, \qquad T = \frac{2\pi}{\omega_0} = \frac{1}{2},$$

For

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

the Fourier Series coefficients are:

$$\begin{aligned} a_1 &= 2 \\ a_{-1} &= 2 \\ a_2 &= -3j/2 \\ a_{-2} &= 3j/2 \\ a_k &= 0, \quad \text{otherwise} \end{aligned}$$

meaning

$$x(t) = \underline{2}e^{j\,4\pi t} + \underline{2}e^{-j4\pi t} - \frac{3j}{2}e^{j\,8\pi t} + \frac{3j}{2}e^{-j8\pi t}$$

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For **real-valued** signals, other Fourier Series expressions are possible

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} \left(\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t) \right)$$

or

$$x(t) = \gamma_0 + \sum_{k=1}^{\infty} \left(\gamma_k \cos(k\omega_0 t + \theta_k) \right)$$

We'll stick to the complex exponential form. Then both positive and negative frequencies need to be used:

$$e^{jk\omega_0 t} \qquad e^{-jk\omega_0 t}$$

For (contrived)

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

we could read off (almost) the Fourier Series coefficients. But how do we compute them for a general function?

Given an arbitrary $x(t)$, how would we compute the coefficient a_{-37} in the expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \quad ?$$

Consider

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \int_T \overbrace{\left(\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \right)}^{x(t)} e^{-jn\omega_0 t}\, dt$$
$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_T e^{j\,(k-n)\omega_0 t}\, dt \right)$$

But

$$\int_T e^{j\,(k-n)\omega_0 t}\, dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$
$$= T\,\delta[k-n]$$

This is orthogonality. Here T is just a constant equal to the fundamental period.

Note we have written

$$\int_T$$

rather than

$$\int_0^T$$

because the integrand is periodic with period T. As such

$$\int_0^T \equiv \int_{-T/2}^{T/2} \equiv \int_{\epsilon}^{T+\epsilon}$$

(assuming the integrand is periodic with period T).

Hence

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \Big(\int_T e^{\,j\,(k-n)\omega_0 t}\, dt \Big)$$

$$= T \sum_{k=-\infty}^{\infty} a_k\, \delta[k-n] = a_n T$$

Fourier Stuff – Fourier Coefficients (cont'd)

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$