Part 4 Outline

7 Fourier Transforms

Differential Equation of CT System

General form:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

N-th order differential equation

For example:

$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Consider the Fourier Transform of the derivative, $x'(t) = \frac{dx(t)}{dt}$ of a signal $x(t)$:

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathscr{F}}{\longleftrightarrow} j\omega\, X(j\omega)}$$

Further, the Fourier Transform of the kth derivative, $x^{(k)}(t)$ of a signal $x(t)$:

$$\boxed{\frac{d^k x(t)}{dt^k} \stackrel{\mathscr{F}}{\longleftrightarrow} (j\omega)^k\, X(j\omega)}$$

LCC Differential Equations: `O&W 4.7 pp.330-333`

Now solve the linear, constant coefficient differential equation:

$$\boxed{\sum_{k=0}^{K} a_k \, \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \, \frac{d^m x(t)}{dt^m}}$$

where we can interpret this as describing a system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fourier Transform both side of the differential equation to yield (see next slide)

$$\boxed{\sum_{k=0}^{K} a_k \, (j\omega)^k Y(j\omega) = \sum_{m=0}^{M} b_m \, (j\omega)^m X(j\omega)}$$

That is, take Fourier Transforms of both sides of

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

to yield

$$\mathscr{F}\left\{\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathscr{F}\left\{\sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}\right\}$$

$$\sum_{k=0}^{K} a_k \,\mathscr{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{m=0}^{M} b_m \,\mathscr{F}\left\{\frac{d^m x(t)}{dt^m}\right\}$$

$$\sum_{k=0}^{K} a_k \,(j\omega)^k Y(j\omega) = \sum_{m=0}^{M} b_m \,(j\omega)^m X(j\omega)$$

Fourier Transforms – Differential Equations

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$Y(j\omega) = \left(\frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k} \right) X(j\omega)$$

So

$$\boxed{H(j\omega) = \frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k}}$$

is the frequency response of the linear, constant coefficient differential equation system.

Called a "rational function of $j\omega$", ratio of polynomials in $j\omega$ or ω.

Filter Design: Can get different shaped $H(j\omega)$ by choosing different values for the a_k and b_m.

Example: An LTI system is described by the DE $\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=\frac{d}{dt}x(t)+4x(t)$, what is $h(t)$?

$$H(j\omega) = \frac{4+j\omega}{6+5j\omega+(j\omega)^2} = \frac{4+j\omega}{(j\omega+3)(j\omega+2)}$$

$$= \frac{2}{j\omega+2} - \frac{1}{j\omega+3}, \quad \text{TP13: } \frac{1}{a+j\omega} \longleftrightarrow e^{-at}u(t)$$

$$\therefore h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

Alternative Method: DE $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$, what is $h(t)$? Let $x(t) = e^{j\omega t}$

$$\frac{d}{dt}x(t) = j\omega e^{j\omega t}$$

$$y(t) = H(j\omega)e^{j\omega t}$$

$$\frac{d}{dt}y(t) = H(j\omega)j\omega e^{j\omega t}$$

$$\frac{d^2}{dt^2}y(t) = H(j\omega)(j\omega)^2 e^{j\omega t}$$

Alternative Method: DE $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$, what is $h(t)$?

$$(j\omega)^2 H(j\omega)\cancel{e^{j\omega t}} + 5j\omega H(j\omega)\cancel{e^{j\omega t}} + 6H(j\omega)\cancel{e^{j\omega t}}$$

$$= j\omega \cancel{e^{j\omega t}} + 4\cancel{e^{j\omega t}}$$

$$H(j\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = j\omega + 4$$

$$H(j\omega) = \frac{j\omega + 4}{(j\omega)^2 + 5j\omega + 6}$$

$$\vdots$$

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

Same answer