Signal Processing

ENGN2228

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Second Semester

Review

Partial Fraction Expansion stuff:

$$X\left(e^{j\omega}\right) = \frac{5 - 2e^{-j\omega}}{1 + 5e^{-j\omega} + 8e^{-j2\omega}}$$

$$\alpha = \frac{-5+\sqrt{25-32}}{16} = \frac{-5+\sqrt{-7}}{16} = \frac{-5+j\sqrt{7}}{16}$$

$$\beta = \frac{-5-\sqrt{25-32}}{16} = \frac{-5-\sqrt{-7}}{16} = \frac{-5-j\sqrt{7}}{16}$$

let $v = e^{-j\omega}$ $\quad X(v) = \dfrac{5-2v}{1+5v+8v^2}$

$$X(v) = \frac{F(v)}{(v-\alpha)(v-\beta)} \qquad \alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$= \frac{F(v)}{c+bv+av^2} \qquad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

partial Fraction Expansion

$$X(v) = \frac{F(v)}{(v-\alpha)(v-\beta)} = \frac{F(v)/\alpha\beta}{(1-v/\beta)(1-v/\alpha)} = \frac{A}{(1-v/\beta)} + \frac{B}{(1-v/\alpha)}$$

$$x[n] = \left[A\left(1/\alpha\right)^n + B\left(1/\beta\right)^n\right]u[n]$$

Time vs. Frequency domain:

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

Review problems:

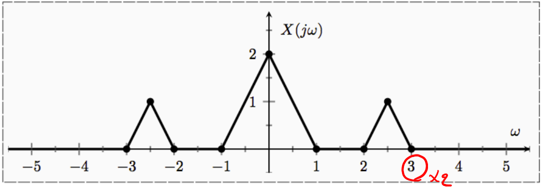


Figure 9: The Fourier transform $X(j\omega)$ of some CT signal $x(t)$.

(a) **[4 marks]** Consider a sampled version of the CT signal $x(t)$, with Fourier transform $X(j\omega)$, shown in Figure 9, given by the CT signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

where T is the sampling interval in time and the sampling rate is given by $\omega_s = 2\pi/T$. Draw/plot $X_p(j\omega)$, the Fourier transform of $x_p(t)$, for the range $-10 \leq \omega \leq 10$ for the following values of the sampling rate $\omega_s = 2\pi/T$

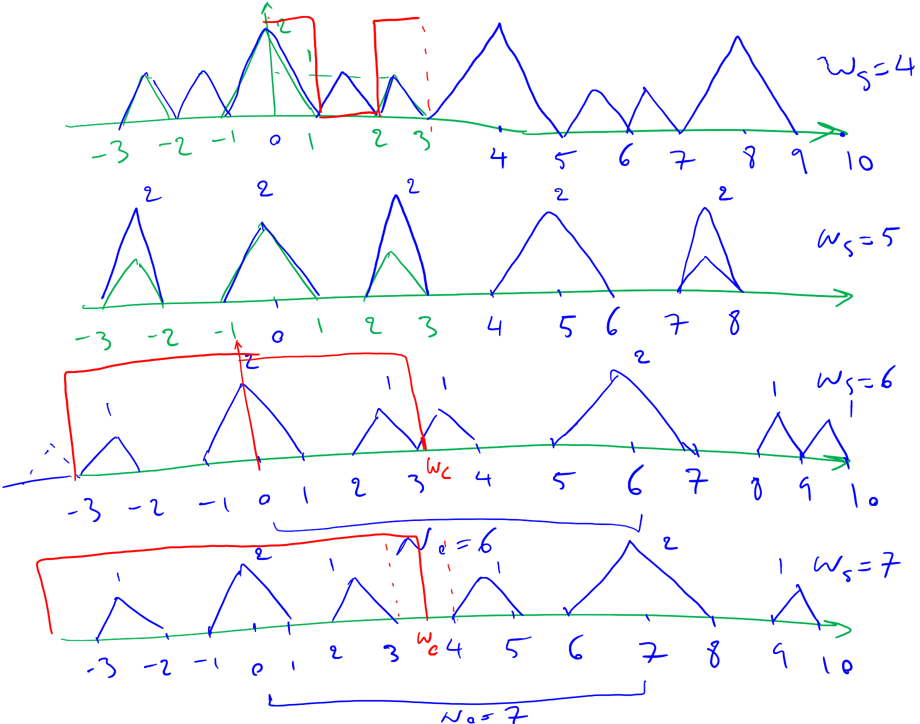
i) $\omega_s = 4$ rad/sec

ii) $\omega_s = 5$ rad/sec

iii) $\omega_s = 6$ rad/sec

iv) $\omega_s = 7$ rad/sec

corresponding to $T = \pi/2,\ 2\pi/5,\ \pi/3,\ 2\pi/7$ sec.



$w_s = 4$
$w_s = 5$
$w_s = 6$
$w_s = 7$
w_c
$w_c = 6$
$w_c = 7$

Review problems:

(b) [**6 marks**] Consider recovery of $x(t)$ from $x_p(t)$ using only an ideal low-pass filter whose cutoff or bandwidth is given by ω_c rad/sec. For each of the 4 values of the sampling rate from part (a) state whether recovery using only an ideal pass filter is possible (yes/no) and, if it is possible, what is the least bandwidth ω_c (in rad/sec), and greatest bandwidth ω_c (in rad/sec), that can be used to perfectly recover $x(t)$ from $x_p(t)$.

You can answer this question by copying and filling in the missing entries of the table below.

Sampling Rate ω_s	Recovery Possible?	Least ω_c	Greatest ω_c
4 rad/sec	✗	–	–
5 rad/sec	✗	–	–
6 rad/sec	✓	3	3
7 rad/sec	✓	3	4

In the case that recovery is not possible, then there are no values for least and greatest ω_c and you should indicate "n/a" for not applicable.

Review problems:

(c) [**1 mark**] Given recovery of $x(t)$ from $x_p(t)$ using only an ideal pass filter, what is the Nyquist sampling rate in rad/sec?

6 rad/sec $= \omega_N$

(d) [**4 marks**] This part considers recovery of $x(t)$ from $x_p(t)$ but not restricted to using an ideal pass filter. In any of the four cases where recovery of $x(t)$ from $x_p(t)$ using only an ideal pass filter fails, indicate whether the signal can still be recovered and how (what combination of LTI filters could be used). Justify your answer.

Inverse band-limit filter for the first case.

Review problems:

(a) **[2 marks]** The analysis and synthesis equations for a DT signal $x[n]$ and its Fourier transform $X(e^{j\omega})$ are given below:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \qquad \text{(Synthesis Equation)}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \qquad \text{(Analysis Equation)}$$

and can be represented as $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$. Express in a few sentences your understanding of these two equations.

(b) **[3 marks]** Using the identities in part (a), or otherwise, show:

$$x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega}) \qquad (2)$$

(c) **[2 marks]** Describe in words the physical meaning of the property (2), that is, what it means in the time-domain and what it means in the frequency-domain.

$$x[n-n_0] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega(n-n_0)}\, d\omega = \frac{1}{2\pi}\int_{2\pi} \underbrace{e^{-j\omega n_0} X(e^{j\omega})}_{X'} e^{j\omega n}\, d\omega$$

$$x[n-n_0] \longleftrightarrow X' = e^{-j\omega n_0} X(e^{j\omega})$$

Review problems:

(d) **[3 marks]** Consider the causal LTI system characterized by the difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 3\,x[n] \tag{3}$$

relating its input $x[n]$ to its output $y[n]$. Using the property (2), or otherwise, determine an expression for the frequency response:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

where $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$ and $y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega})$.

$$Y(e^{j\omega}) - \tfrac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \tfrac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 3X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \tfrac{3}{4}e^{-j\omega} + \tfrac{1}{8}e^{-j2\omega}\right] = 3X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}} = H(e^{j\omega}) = \frac{3}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

Review problems:

(e) **[1 mark]** What is the DC gain ($\omega = 0$) of the system in (3)?

$\omega = 0$ $\quad H(e^{j\omega}) \Rightarrow H(e^{j(0)}) = H(1) = \frac{3}{\frac{1}{2} \times \frac{3}{4}} = 8$

(f) **[1 mark]** What is the gain at the maximum frequency ($\omega = \pi$) of the system in (3)?

$\omega = \pi$ $\quad H(e^{j\pi}) = H(-1) = 8/5$

(g) **[3 marks]** Use a partial-fraction expansion to show the (unit) impulse response is

$$h[n] = 6\left(\frac{1}{2}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n]$$

The following DT Fourier transform pair should be useful

$$\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}}, \quad |\alpha| < 1.$$

$$H(e^{j\omega}) = \frac{3}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} = \frac{A}{(1 - \frac{1}{2}e^{-j\omega})} + \frac{B}{(1 - \frac{1}{4}e^{-j\omega})}$$

$A = 6$

$B = -3$

$6\left(\frac{1}{2}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n] = h[n]$

Review problems:

What is the DT convolution, $y[n] = x[n] \star h[n]$, of the two signals

$$x[n] = \delta[n] + \delta[n-2] \quad \text{and} \quad h[n] = 2\,\delta[n-3]$$

a. $y[n] = \delta[n] + \delta[n-2] + 2\delta[n-3]$

b. $y[n] = 2\,\delta[n-3] + 2\,\delta[n-5]$

c. $y[n] = 2\,\delta[n+3] + 2\,\delta[n+1]$

d. $y[n] = 2\,\delta[n] + 2\,\delta[n-2]$

e. None of the above.

$$x[n] * h[n] = \left[\delta[n] + \delta[n-2]\right] * 2\,\delta[n-3]$$

$$= 2\,\delta[n-3] + 2\delta[n-2-3] = 2\,\delta[n-3] + 2\delta[n-5]$$

Review problems:

For the DT signal $x[n] = \sin(n/16)$ which of the following is true.

a. It has only one (non-zero) term in its Fourier series.

b. It has two (non-zero) terms in its Fourier series.

c. It has an infinite number of terms in its Fourier series.

d. It has 16 terms in its Fourier series.

e. It does not have a Fourier series.

$\omega = 1/16 \qquad \frac{\omega}{2\pi} m = \frac{1/16}{2\pi} m = \frac{1}{32\pi} m$

Review problems:

For $x[n]$ shown in Figure 1(a), which of the following is correct?

a. $x[n] = -\delta[n+1]$

b. $x[n] = -\delta[n-1]$

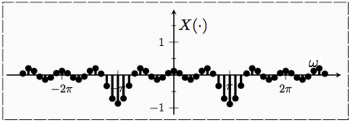
c. $x[n] = -1$

d. $x[n] = +1$

e. $x[n+1] = \delta[n]$

Review problems:

Based on the structure of the frequency domain representation $X(\cdot)$ (period 2π) shown to the right which of the following is the best description of the time-domain signal?



a. The time-domain signal is DT and periodic in time.

b. The time-domain signal is CT and periodic in time.

c. The time-domain signal is DT and not periodic in time.

d. The time-domain signal is CT and not periodic in time.

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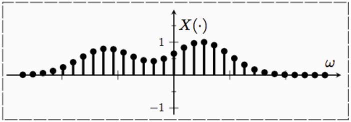

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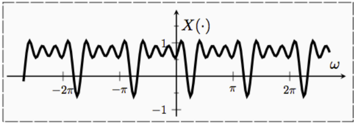

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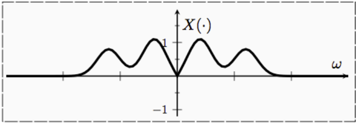

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b. The time-domain signal is CT and periodic in time.

c. The time-domain signal is DT and not periodic in time.

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Review problems:

[Difficult] A DT LTI system with input

$$x[n] = (0.5)^n u[n]$$

has the following output

$$y[n] = \delta[n] - 0.5\,\delta[n-1].$$

What is the impulse response, $h[n]$, of the system, that is, the response when $x[n] = \delta[n]$?
(Hint: try to express $\delta[n]$ in terms of $x[n]$ and its scaled shifted version.)

a. $h[n] = \delta[n] + 0.5\,\delta[n-1] + 0.25\,\delta[n-2]$

b. $h[n] = \delta[n] - \delta[n-1] + 0.25\,\delta[n-2]$

c. $h[n] = \delta[n] - \delta[n-1]$

d. $h[n] = \delta[n]$

e. There is not enough information to determine the impulse response.

$\delta[n] = x[n] - 0.5\,x[n-1]$

$h[n] = y[n] - 0.5\,y[n-1] = \delta[n] - 0.5\,\delta[n-1] - 0.5\,\delta[n-1]$

$+ 0.25\,\delta[n-2]$

Review problems:

A CT LTI system with input $x(t) = 4\,\delta(t-2)$ generates an output $y(t)$. What is the impulse response of the system, that is, what is the output $h(t)$ when the input is $x(t) = \delta(t)$?

a. The impulse response is $h(t) = 0.25\,y(t+2)$

b. The impulse response is $h(t) = 4\,y(t+2)$

c. The impulse response is $h(t) = 0.25\,y(t-2)$

d. The impulse response is $h(t) = 4\,y(t-2)$

e. The impulse response is $h(t) = 4\,\delta(t-2)$

$x(t) = \delta[t] \longrightarrow h[t)$

$x(t) = 4\,\delta(t-2) \longrightarrow y(t)$

$\delta(t) = \dfrac{x(t+2)}{4} \longrightarrow h(t) = \dfrac{y(t+2)}{4}$

Review problems:

What is the frequency of the positive 3rd harmonic ($k = 3$) of the DT periodic signal

$$x[n] = (-1)^{3n} \cos(\pi n/3)?$$

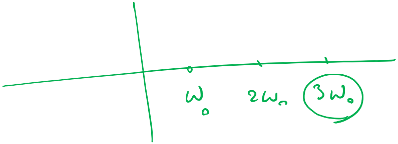
a. $\pi/2$ rad/sec
b. π rad/sec
c. 2π rad/sec
d. 6 rad/sec
e. 18 rad/sec

$\omega_0 = \pi/3$

$\frac{\omega/M}{2\pi} = \frac{\pi/3\,M}{2\pi} = 1/6\,M \Rightarrow M = 6$

$N_0 = 6$ Samples

$\omega_0 = \pi/3 \quad 3\omega_0 = \pi$

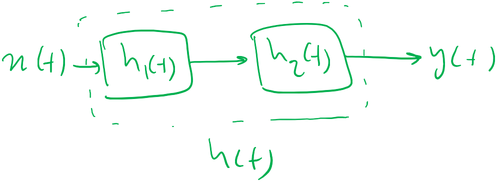


Consider the series/cascade connection of the two CT LTI systems with impulse responses $h_1(t)$ and $h_2(t)$, input $x(t)$ and output $y(t)$, as shown in Figure 6:

Suppose $h_1(t) = \delta(t - 2.3)$ and $h_2(t) = \delta(t - 1.5)$ then which of the following is true?

- a. $y(t) = x(t - 0.8)$
- b. $y(t) = x(t + 0.8)$
- c. $y(t) = x(t - 1.5)$
- d. $y(t) = x(t - 3.8)$
- e. $y(t) = x(t + 3.8)$

$$h(t) = h_1(t) * h_2(t) = \delta(t-2.3) * \delta(t-1.5) = \delta(t \underbrace{-1.5-2.3}_{-3.8})$$



$$= \delta(t-3.8)$$

Review problems:

A DT LTI system with input $x[n]$ and output $y[n]$ is given by

$$y[n] = x[n] + \alpha\, x[n-1] + \beta\, x[n-2]$$

for some constants α and β. What values of α and β yield a high pass filter with DT frequency response at $\omega = 0$ of zero (DC gain of zero), and frequency response at $\omega = \pi$ of 1 (gain at the maximum frequency of unity)?

a. $\alpha = 1$ and $\beta = 0$

b. $\alpha = -1$ and $\beta = 0$

c. $\alpha = 0$ and $\beta = -1$

d. $\alpha = -1/2$ and $\beta = -1/2$

e. $\alpha = 2$ and $\beta = -1$

$$Y(e^{jw}) = X(e^{jw}) + \alpha e^{-jw} X(e^{jw}) + \beta e^{-j2w} X(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = 1 + \alpha e^{-jw} + \beta e^{-2jw}$$

$$\left.\begin{aligned} w=0 &\longrightarrow H(0) = 1+\alpha+\beta = 0 \\ w=\pi &\longrightarrow H(-1) = 1-\alpha+\beta = 1 \end{aligned}\right\} \longrightarrow \begin{aligned} \alpha &= -1/2 \\ \beta &= -1/2 \end{aligned}$$

Review problems:

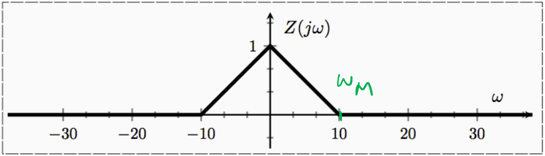


Figure 7: A frequency domain spectrum, $Z(j\omega)$, which is purely real. The units of ω are radians/sec

For the CT signal $z(t)$ with $Z(j\omega)$ shown in Figure 7, what is the minimum sampling frequency, ω_s so that $z(t)$ can be recovered from its samples using an ideal low-pass filter?

a. $\omega_s = 10$ radians/sec
b. $\omega_s = 20$ radians/sec
c. $\omega_s = 40$ radians/sec
d. $\omega_s = 60$ radians/sec
e. $\omega_s = 80$ radians/sec

Nyquist freq $= 2 \times \omega_M$

$\omega_s = 2 \times 10 = 20$ rad/sec