Signal Processing
ENGN2228

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Second Semester

Lecture 33

Part 0 Slide 1/many Convenor: R. A. Kennedy

Sampling – Background

CT signals (and systems) describe the physical world. We are interested in converting them to DT signals. Why?

- why not?
- we what to process them with a computer or digital signal processor (DSP)
- it may reduce the complexity of the signal
- we can use MATLAB to play with them
- they are amenable for storage (hard disk)

We have the results from earlier lectures to fully understand the process of converting a CT signal to a meaningful DT signal through "sampling". What is sampling?

Sampling – Background

Sampling is taking snapshots of some signal $x(t)$ every T seconds. Here T is the sampling period. For example, the standard CD audio sampling rate is 44.1 kHz which means $T = 1/44100 = 0.00002267\cdots$ seconds.

44.1 kHz is a weird number. It was chosen to: 1) be high enough and 2) to make it hard to convert from 48 kHz professional audio by designers who probably would have failed this course. And then there is 96 kHz audio which is a load of nonsense to make it appeal to the same fools who think oxygen free speaker cable makes audio sound better.

Sampling – Background

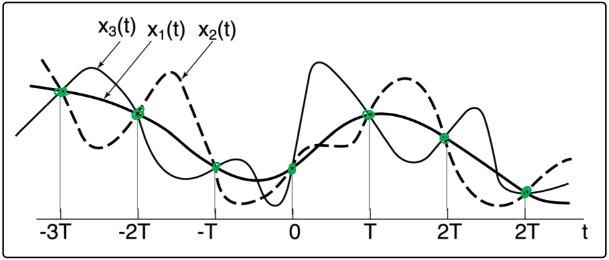
With T the sampling period, one way to create a DT signal, $x[n]$, from a given CT signal, $x(t)$, is via

$$x[n] = x(nT), \quad n \in \mathbb{Z}$$

noting $x(nT)$ means $x(t)$ evaluated or sampled at time $t = nT$.

Sampling – Background

We note that lots of CT signals have the same samples



and, therefore, information is thrown away when we sample. We lose the behavior of the signal between the samples. Or do we? Do we *always* lose information? Under what conditions can we recover or reconstruct the original CT signal $x(t)$ from its samples $x[n] = x(nT)$?

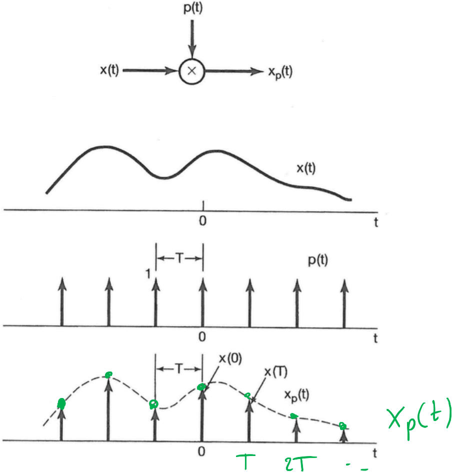
Sampling – Background

This is a non-trivial and somewhat remarkable property. If true then we can do digital signal processing (signal processing of sampled signals) without compromising the signal we are interested in.

What we expect is:

- The signals might need to have some special property (like smoothness).
- The sampling period T is short enough, or equivalently the sampling rate $1/T$ is high enough, relative to the variations we see in the signal.
- The complete CT signal $x(t)$ must be mathematically explicitly expressible in terms of its samples $x[n] = x(nT)$.

Sampling – Impulse Sampling:



Sampling – Impulse Sampling

We need a convenient mathematical description of the sampling process. On the way through the course the signals and operations required to do this have been introduced.

Multiply the signal $x(t)$ by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

which yields

$$x_p(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT)$$

By "sifting" property of the delta function:

$$\boxed{x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)}$$

$x_p(t)$ is a continuous time sampled signal. We only need to know $x(t)$ at $t = nT$ for integer $n \in \mathbb{Z}$, these are the samples.

For all other values of t (namely all t taking reals values not equal to the integers), the information in $x(t)$ is ignored or discarded. That is, the vast majority of $x(t)$ is discarded.

So surely hoping that $x_p(t)$ still contains all the information in $x(t)$ seems delusional. But the remarkable thing is, under the right conditions, we can figure out what $x(t)$ is from the continuous time sampled signal $x_p(t)$. Let's understand this further.

Time domain operation

$$x_p(t) = x(t)\, p(t)$$

implies, by the multiplication property, in the frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \star P(j\omega)$$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega), \quad x_p(t) \overset{\mathscr{F}}{\longleftrightarrow} X_p(j\omega), \quad p(t) \overset{\mathscr{F}}{\longleftrightarrow} P(j\omega)$$

where

$$\boxed{P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,\omega_s)}$$

and $\omega_s \triangleq 2\pi/T$ is the sampling frequency (in rad/sec).

Sampling – Frequency Domain Analysis

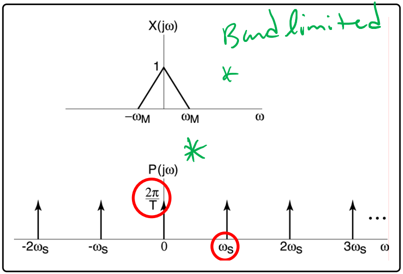
So we have

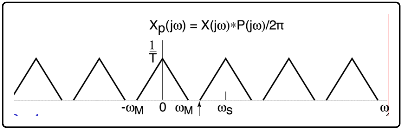
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega) \star \delta(\omega - k\,\omega_s)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\,\omega_s)\big)$$

Recall $x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)$ and $x_p(t) \xleftrightarrow{\mathscr{F}} X_p(j\omega)$.

So the frequency content of $x_p(t)$ is a periodic copy, with the frequency shifts given by $k\,\omega_s$ for integer $k \in \mathbb{Z}$, of the frequency content of $x(t)$ (with some scaling given by $1/T$).

Sampling – Illustration





More than an illustration but an important special case.

Here, have a band-limited signal such that $X(j\omega) = 0$ for $|\omega| > \omega_M$ and illustrated for

$$\omega_s - \omega_M > \omega_M$$

or

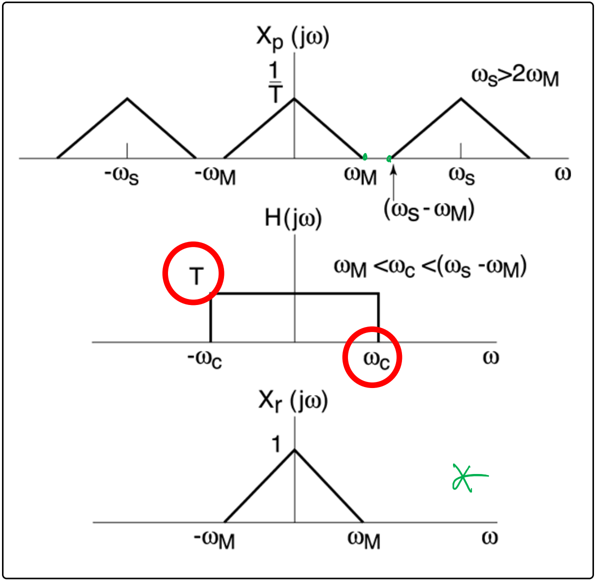
$$\omega_s > 2\,\omega_M$$

If the frequency content of the signal is appropriately low pass then there is no overlap in the periodic repetition of the spectrum. Hence we can use an ideal low pass filter to *perfectly* recover the signal.

A suitable bandwidth of the low pass filter is ω_c where

$$\omega_M < \omega_c < (\omega_s - \omega_M)$$

This condition can be "graphically" determined.


$X_p(j\omega)$
$\frac{1}{T}$
$\omega_S > 2\omega_M$
$-\omega_S$
$-\omega_M$
ω_M
ω_S
ω
$(\omega_S - \omega_M)$
$H(j\omega)$
T
$\omega_M < \omega_c < (\omega_S - \omega_M)$
$-\omega_c$
ω_c
ω
$X_r(j\omega)$
1
$-\omega_M$
ω_M
ω

Theorem (Sampling Theorem)

Suppose $x(t)$ is band-limited so that

$$X(j\omega) = 0 \quad \textit{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$ if

$$\omega_s \triangleq \frac{2\pi}{T} > 2\,\omega_M \equiv \text{“\textit{Nyquist Rate}”}$$

The theorem provides sufficient conditions. You could generate a whole bunch of similar results for bandpass signals or signals with nicely spaced frequency holes, etc.

Example

Generally the top end of human hearing is given as 20 kHz (which declines with age and the number of night-clubs you frequent). The Nyquist rate or frequency is twice the highest frequency content, so the sample rate in Hz needs to be at least 40 kHz (or $2\,\omega_M = 80,000\pi$ in radians/sec).

(Continued)

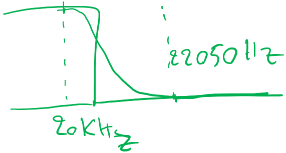
Sampling – Sampling Theorem

Note that the CD audio rate is at least 40 kHz since it is 44.1 kHz. Why is it not 40 kHz?

- If it were 40 kHz then we would need an ideal low pass filter with cut-off at 20 kHz exactly. An ideal low pass filter has a long sinc shaped impulse response which is not nice (expensive and complicated to implement).
- Since there is a gap between 40 kHz and 44.1 kHz, the reconstruction filter needs only to satisfy

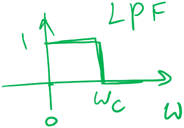
$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 2\pi \times 20{,}000 \\ 0 & \text{for } |\omega| > 2\pi \times 22{,}050 \\ \text{don't care} & \text{otherwise} \end{cases}$$

So we can design a cheaper filter.

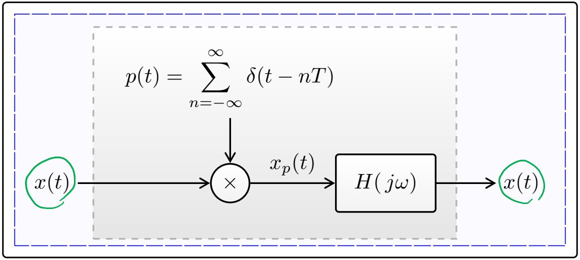


Sampling – Sampling is Time-Varying

Multiplying a signal $x(t)$ by a time-varying (non-constant) function $p(t)$ implies sampling is a time-varying operation. However, sometimes combinations of such sampling and filtering reduces to something simple. For



$$\boxed{x(t) \text{ s.t. } X(j\omega) = 0 \text{ for } |\omega| > 2\pi/T}$$



with $H(j\omega)$ an ideal low pass filter. Overall this acts like the identity.

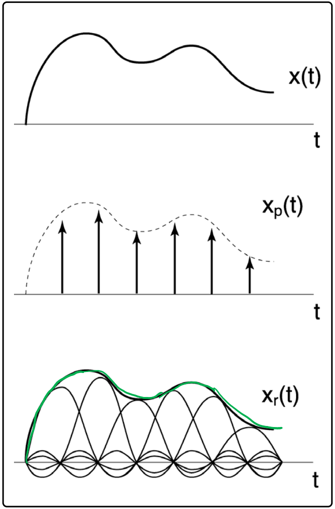
The Sampling Theorem is obvious in the frequency domain but it is useful to see what the equivalent time domain operation is (for implementation). Recall ω_c is the cut-off of the filter which we take to be an ideal LPF.

$$
\begin{aligned}
x_r(t) &= x_p(t) \star h(t), \qquad h(t) \triangleq \frac{T \sin \omega_c t}{\pi t} \quad \text{LPF in time} \\
&= \Big(\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \Big) \star h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)
\end{aligned}
$$

That is,

$$
\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \frac{T\, \sin\big(\omega_c (t - nT)\big)}{\pi (t - nT)}}
$$

Sampling – Time Domain Reconstruction



Sampling – Interpolation Methods

Note that in going from $x(nT)$ to $x(t)$ we are filling in the values in the time gaps, which is just interpolation. So the Sampling Theorem reconstruction can be viewed as a special/optimal interpolation. We can then enumerate a few different types of interpolation

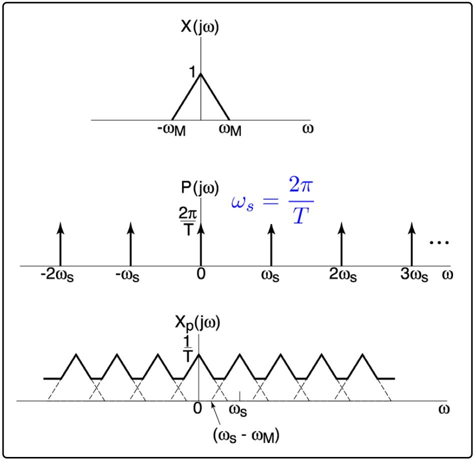
- Band-limited Interpolation — primarily the sinc style interpolation
- Zero-Order Hold — reconstructs a piecewise constant signal through the sample points
- First-Order Hold — reconstructs a piecewise linear interpolation joining the sample points (non-causal)

Sampling – Undersampling and Aliasing

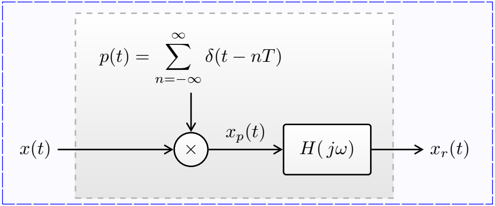
For the band-limited signal case, when the sampling rate is insufficient, that is,

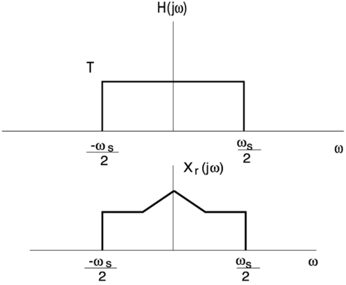
$$\omega_s \leq 2\omega_M$$

there will be frequency overlapping.



Sampling – Undersampling and Aliasing





Higher frequencies of $x(t)$ are aliased to lower frequencies.

Example

With

$$x(t) = \cos(\omega_0 t), \quad \text{where } \omega_M = \omega_0.$$

If $\omega_s > 2\omega_0$ then there is perfect reconstruction

$$x_r(t) = \cos(\omega_0 t)$$

If $\omega_s < 2\omega_0$ then there is aliasing

$$x_r(t) = \cos\big((\omega_s - \omega_0)t\big)$$

Undersampling example:

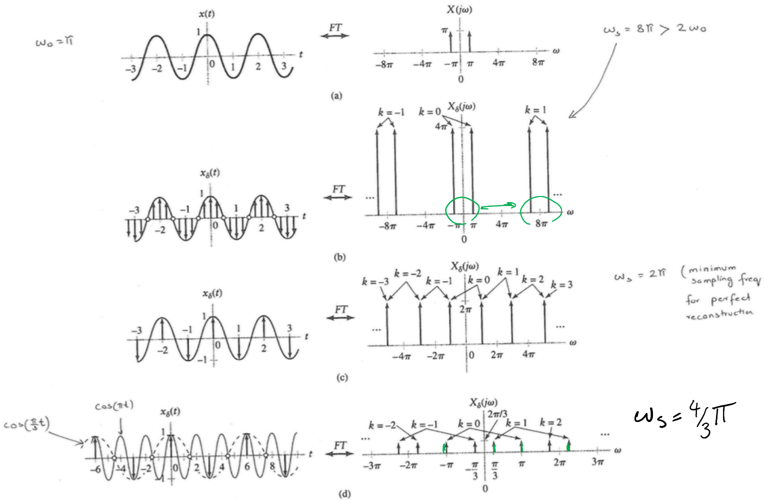


FIGURE 4.24 The effect of sampling a sinusoid at different rates (Example 4.9). (a) Original signal and FT. (b) Original signal, impulse-sampled representation and FT for $T_s = 1/4$. (c) Original signal, impulse-sampled representation and FT for $T_s = 1$. (d) Original signal, impulse-sampled representation and FT for $T_s = 3/2$. A cosine of frequency $\pi/3$ is shown as the dashed line.

Sampling – Practical Sampling

Sampling with impulses is an idealization. An infinitely narrow time portion of a signal would have zero energy in the limit (which we can't amplify by infinity like we do mathematically).

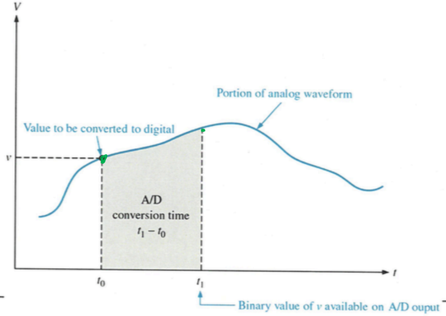
We could add or integrate the signal around the sampling time instant to gather enough energy to get a meaningful reading. This is a distortion but not a great one. This leads to the concept of a "Zero-Order Hold" which has a convenient mathematical or system model (shown next). It is important to have such a model since it can reveal how close to ideal impulse like sampling we can get.

Reconstruction of Practical Sampling:

ADC Parameters

- **Conversion Time:** The conversion of a value on an analog waveform into a digital quantity is not an instantaneous event and takes a certain amount of time.

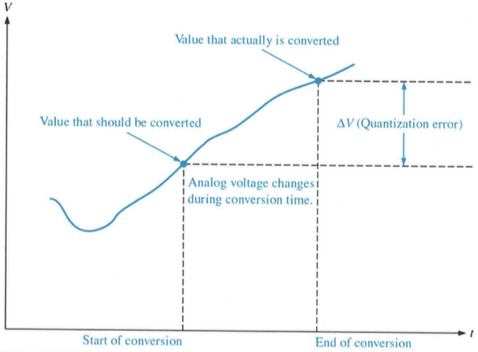
- It can range from microseconds to milliseconds.



Reconstruction of Practical Sampling:

ADC Parameters

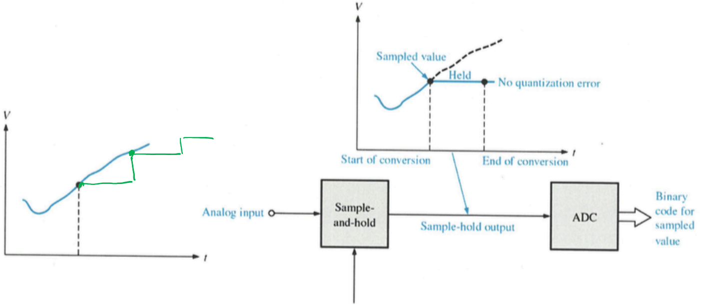
- **Quantization Error:** The change in the value of the analog signal during the conversion time produces an error called quantization error.



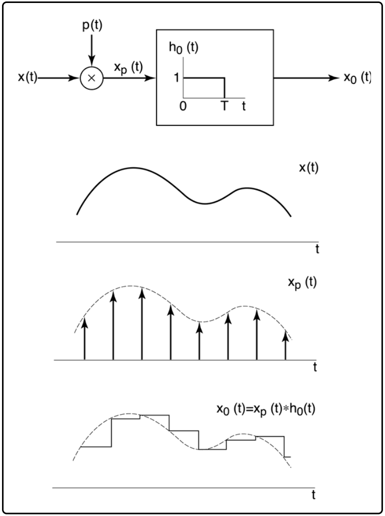
Reconstruction of Practical Sampling:

ADC Parameters

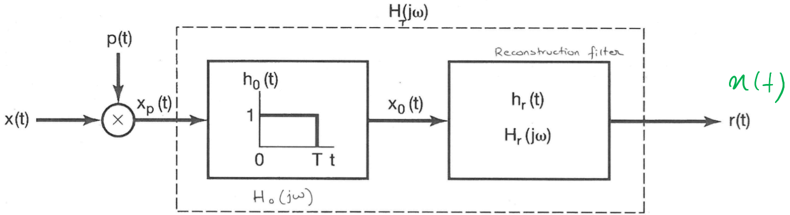
- A **sample-and-hold amplifier** (e.g. AD585) is used to avoid quantization error. It quickly samples the analog input and then holds the sampled value constant for the duration of the conversion time.

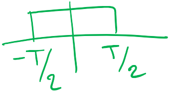


For details, SEE: Handout pg 712

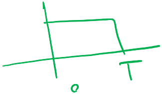

p(t)
x(t)
×
xp (t)
h0 (t)
1
0
T
t
x0 (t)
x(t)
t
xp (t)
t
x0 (t)=xp (t)∗h0(t)
t

Reconstruction of Practical Sampling:





$$\longleftrightarrow \quad \frac{2\sin(\omega T/2)}{\omega}$$



$$\longleftrightarrow \quad e^{-j\omega T/2}\,\frac{2\sin(\omega T/2)}{\omega} = H_0(j\omega)$$

$$H_T(j\omega) = H_0(j\omega)\,H_r(j\omega)$$

$$H_r(j\omega) = \frac{H_T(j\omega)}{H_0(j\omega)} = \frac{H_T(j\omega)\,e^{j\omega T/2}}{\frac{2\sin(\omega T/2)}{\omega}}$$

Reconstruction of Practical Sampling:

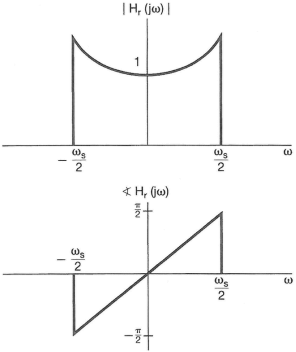
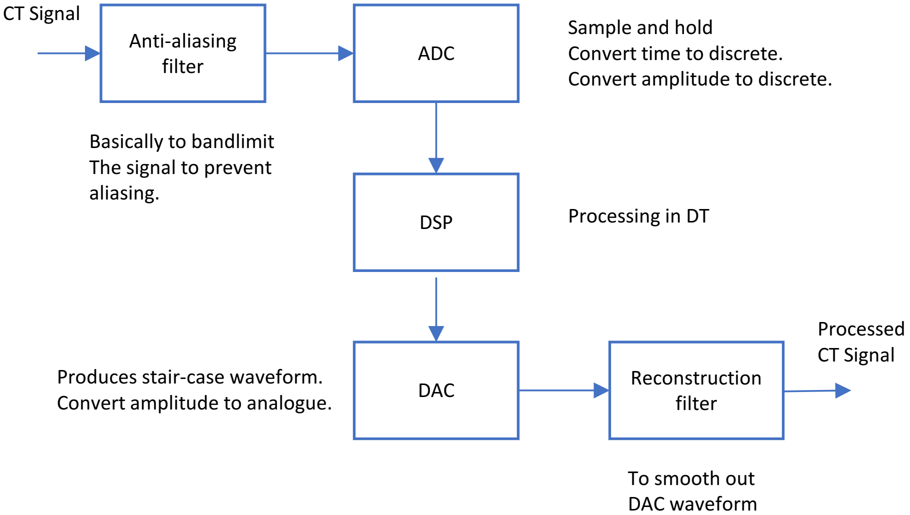


Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

Overview of Digital Signal Processing:

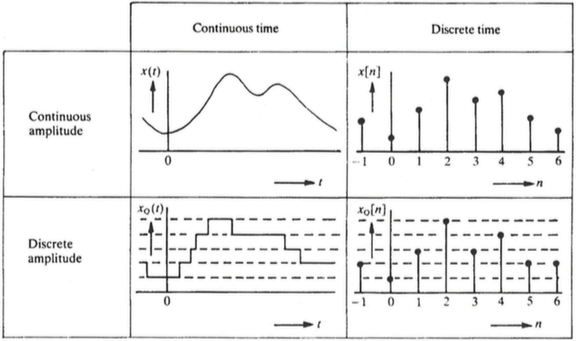


Reconstruction of Practical Sampling:

Analog vs. Digital Signals

- Signals can be either continuous or discrete in both time and amplitude.

Analog Signal



Sampled digital signal

DAC Output

ADC Output (Digital Signal)

CT Processing via DT – Digital Processing

The Importance:

We can convert band-limited CT signals to DT signals **without any loss of information**. We want to measure a signal and process it. This can be done in the DT domain making it amenable to digital processing, that is, processing in a computer rather than building a circuit or other approach.

One can think of time being sampled or being made discrete. One also samples or **discretizes the values** (like the voltage levels) and this is really what we mean by "digital" — discrete in time **and** discrete in amplitude.

CT Processing via DT – Digital Processing

Digital Processing Means:

- MATLAB
- "PC" with PCI (Multi-Channel) 8/12/16-bit A/D
- DSP such as Texas Instruments TMS320 series
- FPGA – Field Programmable Gate Array
- ASIC – Application-specific integrated circuit