Fourier Transforms – Up to this Point

can also handle periodic

Time-domain properties	Periodic	Non-periodic	
Continuous	Fourier series (FS)	Fourier transform (FT)	Non-periodic
Discrete	Discrete-time Fourier series (DTFS)	Discrete-time Fourier transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

Time-domain Property	Frequency-domain Property
continuous	non-periodic
discrete	periodic
periodic	discrete
non-periodic	continuous

7. **Fourier Transforms**
   - Periodic Signals
   - Up to this Point
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations

Fourier Transforms – Properties

- Given in exam
- Proof of properties not accessible

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transforms – Properties

With $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega)$ then from simple manipulations of the Fourier Transform integral:

Linearity/Superposition:

$$\boxed{a\,x(t) + b\,y(t) \overset{\mathscr{F}}{\longleftrightarrow} a\,X(j\omega) + b\,Y(j\omega)}$$

Time Shift:

$$\boxed{x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0}\,X(j\omega)}$$

Time Scale:

$$\boxed{x(at) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}\,X(j\omega/a)}$$

The differentiation property: if

$$\boxed{x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)}$$

then

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathscr{F}} j\omega\, X(j\omega)}$$

That is, the operation of differentiation in the time domain is replaced by multiplication by $j\omega$ in the frequency domain.
The integration property:

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathscr{F}} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)}$$

The impulse term on the RHS of the equation reflects the DC or average value that can result from integration.

CT unit step $u(t)$ can be written as:

$\boxed{u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau}$ running integral of the unit impulse.

$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$ superposition of an infinite number of delayed impulses.

Use the first relationship and the integration property to work out the FT of $u(t)$:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

$$x(\tau) = \delta(\tau), \quad \delta(\tau) \xleftrightarrow{F} 1$$

$$u(t) \xleftrightarrow{F} \frac{1}{j\omega} \times 1 + \pi\delta(\omega)$$

This is Transform Pair 11.

Recap:

$$u(t) \xleftrightarrow{\mathscr{F}} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\mathscr{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\mathscr{F}^{-1}\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} = u(t)$$

- Frequency domain function $\frac{1}{j\omega} + \pi\delta(\omega)$ is the *Fourier Transform* of $u(t)$
- Time domain function $u(t)$ is the *Inverse Fourier Transform* of $\frac{1}{j\omega} + \pi\delta(\omega)$

Fourier Transforms – Properties

- Have now gone through all the transform pairs in the table.
- Can use these transform pairs and the properties of the FT to calculate the FT of many signals.
- Much easier than applying the analysis equation directly.
- Need to be able to select which transform pair to use and which properties to apply.

Fourier Transforms – Examples

Example 1: $x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$

TP13: $e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$\therefore$ $e^{-t}u(t) \longleftrightarrow \dfrac{1}{1+j\omega}$

$e^{-2t}u(t) \longleftrightarrow \dfrac{1}{2+j\omega}$

linearity property: $ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega)$

$\therefore$ $2e^{-t}u(t) - 3e^{-2t}u(t) \longleftrightarrow \dfrac{2}{1+j\omega} - \dfrac{3}{2+j\omega}$

Fourier Transforms – Examples

Example 2: $x(t) = e^{-2t}u(t-3)$

TP13: $e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

so $\underbrace{e^{-2t}u(t)}_{z(t)} \longleftrightarrow \underbrace{\dfrac{1}{2+j\omega}}_{z(j\omega)}$

$z(t-3) \longleftrightarrow e^{-j\omega 3} z(j\omega)$

$\parallel$

$e^{-2(t-3)}u(t-3) \longleftrightarrow \dfrac{e^{-j\omega 3}}{2+j\omega}$

$e^{-2t}e^{6}u(t-3)$

$e^{-2t}u(t-3) \longleftrightarrow \dfrac{e^{-j\omega 3}}{(2+j\omega)e^{6}}$

In the CT non-periodic (in general) signal case:

Definition (Parseval's Relation)

The total energy in time domain equals total energy in frequency domain:

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega}$$

The term

$$\boxed{\frac{1}{2\pi} |X(j\omega)|^2}$$

is called the spectral density (energy per unit frequency).

Part 4 Outline

7 **Fourier Transforms**
- Periodic Signals
- Up to this Point
- Properties
- Convolution
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- Partial Fractions
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- Multiplication Property
- Differential Equations

Fourier Transforms – Convolution

Convolution:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\boxed{y(t) = h(t) \star x(t) \xleftrightarrow{\mathscr{F}} Y(j\omega) = H(j\omega)\, X(j\omega)}$$

where

$$h(t) \xleftrightarrow{\mathscr{F}} H(j\omega)$$

This follows from the eigenfunction property of the $e^{j\omega t}$ which is central to the definition of the Fourier Transform (see over).

Terminology, $H(j\omega)$ is the **frequency response**.

Fourier Transforms – Convolution

Synthesis equation / inv. F.T.

$$x(t) = \mathscr{F}^{-1}\{X(j\omega)\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Recall the eigenfunction property:

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) H(j\omega)\, e^{j\omega t}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(j\omega)\, X(j\omega)}_{Y(j\omega)}\, e^{j\omega t}\, d\omega$$
$$= \mathscr{F}^{-1}\{H(j\omega)\, X(j\omega)\} \equiv \mathscr{F}^{-1}\{Y(j\omega)\}$$

Hence

$$\boxed{y(t) = h(t) \star x(t) \iff Y(j\omega) = H(j\omega)\, X(j\omega)}$$

Part 4 Outline

7 **Fourier Transforms**

Fourier Transforms – Transform Method

Convolution via transform techniques:

- Do Fourier Transform to convert problem to Fourier/frequency domain.

  $x(t) \longleftrightarrow X(j\omega)$, $h(t) \longleftrightarrow H(j\omega)$

- Do convolution via multiplication of Fourier Transforms.

  $Y(j\omega) = H(j\omega)X(j\omega)$

- Do algebraic manipulations, e.g., partial fractions, to get into form where know what inverse Fourier Tranform is.

- Do Inverse Fourier Transform to get back time domain signal.

  $y(t)$

Part 4 Outline

7 **Fourier Transforms**

- Periodic Signals
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- Filter Cascade
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Fourier Transforms – Partial Fractions

Example 1: Consider TP13: $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$

$$h(t) = e^{-t}u(t) \quad \text{and} \quad x(t) = e^{-2t}u(t)$$

What is $y(t) = h(t) \star x(t)$?

Obviously could grind through a convolution integral. But generally going via the Fourier Transform is quicker and easier.

$X(j\omega) = \frac{1}{2+j\omega}$

$H(j\omega) = \frac{1}{1+j\omega}$

Fourier Transforms – Partial Fractions

$$Y(j\omega) = H(j\omega)\,X(j\omega)$$
$$= \frac{1}{(1+j\omega)}\,\frac{1}{(2+j\omega)} = \frac{1}{(1+j\omega)(2+j\omega)}$$

Do we know what the Inverse Fourier Transform of this is? Not really. However, we can do a "partial fraction expansion" (trick):

$$Y(jw) = \frac{A}{1+jw} + \frac{B}{2+jw}, \quad \text{Let } v = jw$$

$$Y(v) = \frac{A}{1+v} + \frac{B}{2+v} = \frac{1}{(1+v)(2+v)}$$

$$A = \left[(1+v)Y(v)\right]_{v=-1} = \frac{\cancel{(1+v)}}{\cancel{(1+v)}\,(2+v)}\Bigg|_{v=-1} = \frac{1}{1} = 1$$

$$B = \left[(2+v)Y(v)\right]_{v=-2} = \frac{\cancel{(2+v)}}{(1+v)\cancel{(2+v)}}\Bigg|_{v=-2} = \frac{1}{-1} = -1$$

$$\therefore\ Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}\ ,\ \frac{1}{a+j\omega} \longleftrightarrow e^{-at}u(t)$$

$$\Rightarrow y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

Example 2: Consider

$h(t) \leftrightarrow H(j\omega)$

$$H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)^2(j\omega+3)}$$

$$= \frac{A}{j\omega+1} + \frac{B}{(j\omega+1)^2} + \frac{C}{j\omega+3}$$

What is $h(t)$?

Let $v = j\omega$

$$H(v) = \frac{(v+2)}{(v+1)^2(v+3)} = \frac{A}{v+1} + \frac{B}{(v+1)^2} + \frac{C}{v+3}$$

using method of residuals

$$C = \left[(v+3)H(v)\right]_{v=-3} = \left.\frac{\cancel{(v+3)}(v+2)}{(v+1)^2\cancel{(v+3)}}\right|_{v=-3} = \frac{-1}{4}$$

$$B = \left[(v+1)^2 H(v)\right]_{v=-1} = \left.\frac{\cancel{(v+1)^2}(v+2)}{\cancel{(v+1)^2}(v+3)}\right|_{v=-1} = \frac{1}{2}$$

$$A = \frac{1}{(2-1)!}\frac{d}{dv}\left[(v+1)^2 H(v)\right]_{v=-1} = \frac{d}{dv}\left[\frac{\cancel{(v+1)^2}\, v+2)}{\cancel{(v+1)^2}(v+3)}\right]_{v=-1}$$

$$= \frac{\cancel{v}+3-(\cancel{v}+2)}{(v+3)^2}\Bigg|_{v=-1} = \frac{1}{4}$$

$$\therefore H(jw) = \frac{\frac{1}{4}}{jw+1} + \frac{\frac{1}{2}}{(jw+1)^2} - \frac{1/4}{jw+3}$$

We know $\frac{1}{a+jw} \longleftrightarrow e^{-at}u(t)$

$$\frac{1}{(a+jw)^2} \longleftrightarrow te^{-at}u(t)$$

$$\therefore h(t) = 1/4\, e^{-t}u(t) + 1/2\, te^{-t}u(t) - 1/4\, e^{-3t}u(t)$$

$y(t) = x(t) * h(t)$

Example 3: Consider

$$x(t) = e^{-2t}u(t) \quad \text{and} \quad y(t) = e^{-t}u(t)$$

What is $h(t)$? Deconvolution example - solve in the frequency domain.

$X(j\omega) = \dfrac{1}{2+j\omega}$

$e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$Y(j\omega) = \dfrac{1}{1+j\omega}$

Fourier Transforms – Partial Fractions

$$Y(j\omega) = H(j\omega)\,X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{1}{1+j\omega}}{\frac{1}{2+j\omega}} = \frac{2+j\omega}{1+j\omega}$$

don't need to use partial fractions as num>den

$$= \frac{1+1+j\omega}{1+j\omega} = 1 + \frac{1}{1+j\omega}$$

$$h(t) = \delta(t) + e^{-t}u(t)$$

Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

What is $y(t)$? If tried to solve in the time domain:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \frac{\sin(\omega_i \tau)}{\pi \tau} \frac{\sin(\omega_c (t-\tau))}{\pi (t-\tau)} d\tau$$

Fourier Transforms – Partial Fractions

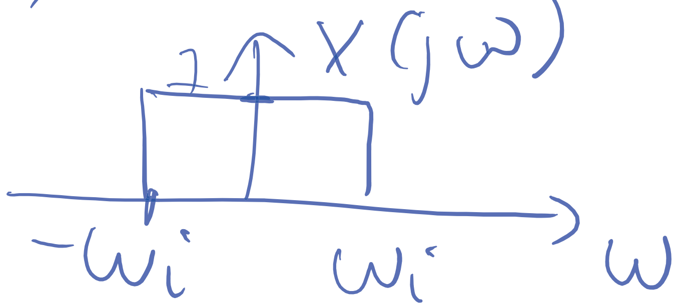
Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

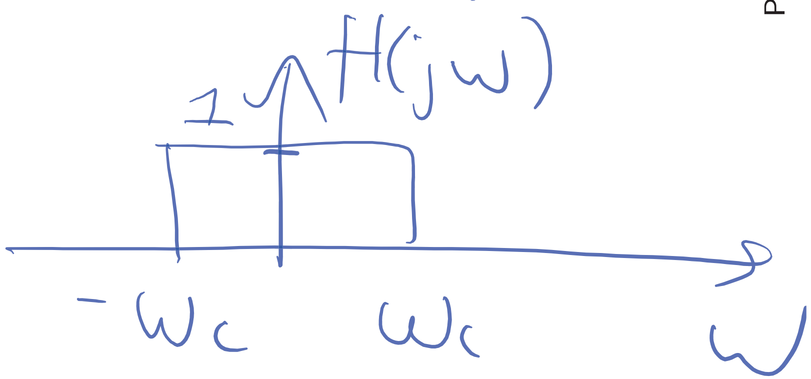
What is $y(t)$? Solving in the frequency domain:

TP9: $\dfrac{\sin Wt}{\pi t} \longleftrightarrow X(j\omega) = \begin{cases} 1 & , |\omega| < W \\ 0 & , |\omega| > W \end{cases}$

$$X(j\omega) = \begin{cases} 1 & , |\omega| < \omega_i \\ 0 & , |\omega| > \omega_i \end{cases}$$



$$H(j\omega) = \begin{cases} 1 & , |\omega| < \omega_c \\ 0 & , |\omega| > \omega_c \end{cases}$$



Fourier Transforms – Partial Fractions

Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

What is $y(t)$? Solving in the frequency domain:

$$Y(j\omega) = H(j\omega)X(j\omega) = \begin{cases} 1, & |\omega| < \omega_0 = \min\{\omega_c, \omega_i\} \\ 0, & \text{elsewhere} \end{cases}$$

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} = h(t) & \text{if } \omega_c \le \omega_i \\[2ex] \dfrac{\sin \omega_i t}{\pi t} = x(t) & \text{if } \omega_i \le \omega_c \end{cases}$$

Part 4 Outline

7 Fourier Transforms

- Periodic Signals
- Up to this Point
- Properties
- Convolution
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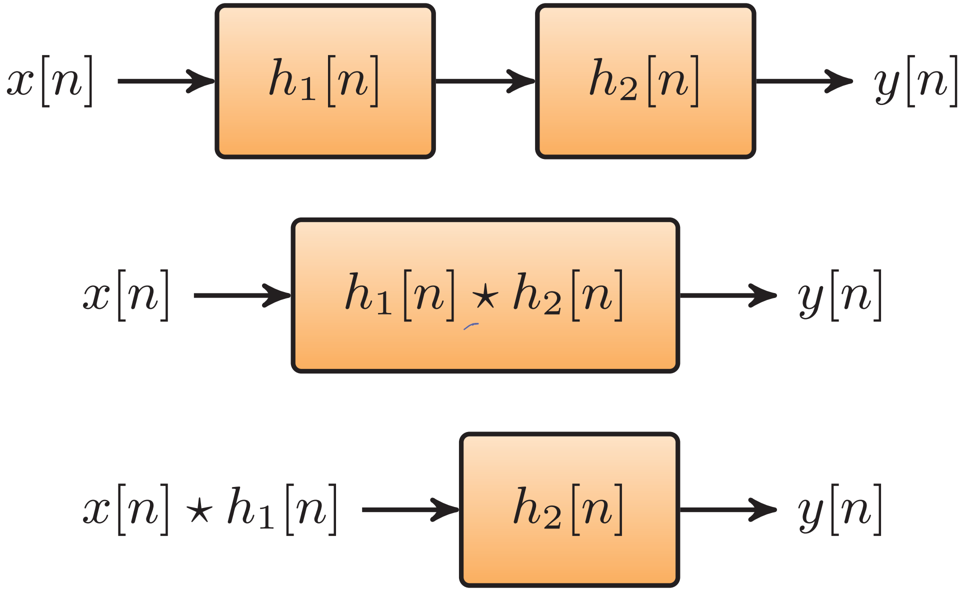
- Filter Cascade
- Multiplication Property
- Differential Equations

Fourier Transforms – Filter Cascade

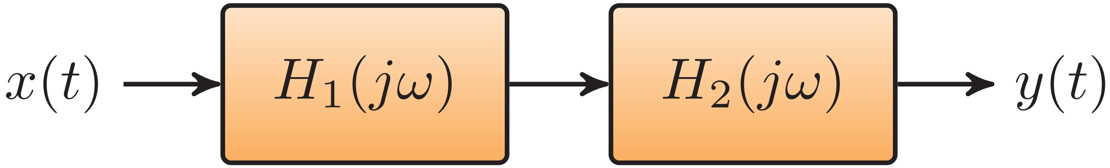
Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

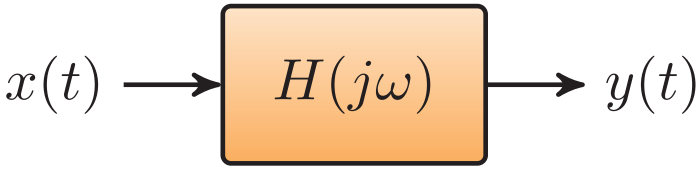
$$\boxed{x[n] \star \big(h_1[n] \star h_2[n]\big) = \big(x[n] \star h_1[n]\big) \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).

Example 1: Cascading filters





Then the convolution theorem gives

$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

Note that if $H_1(j\omega) = H_2(j\omega)$ then $H_1^2(j\omega) \equiv \big(H_1(j\omega)\big)^2$ tends to have a sharper frequency selectivity/cutoff, for example, as some frequency, attenuation by 10% becomes attenuation by almost 20%, $0.9^2 = 0.81$.

Fourier Transforms – Filter Cascade

Example 2: Cascading ideal LPF filters. What is

$$\underbrace{\frac{\sin(4\pi t)}{\pi t}}_{x(t)} \star \underbrace{\frac{\sin(8\pi t)}{\pi t}}_{h(t)} = ? \qquad \frac{\sin(4\pi t)}{\pi t}$$

Think about $x(t)$ as the input into LTI system with impulse response $h(t)$.

Note that: $x(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 4\pi$; and $h(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 8\pi$

Answer:

$$\frac{\sin(4\pi t)}{\pi t} \star \frac{\sin(8\pi t)}{\pi t} = \frac{\sin(4\pi t)}{\pi t}$$

That's weird. Imagine the pain to calculate:

$$\int_{-\infty}^{\infty} \frac{\sin(4\pi\tau)}{\pi\tau} \frac{\sin(8\pi(t-\tau))}{\pi(t-\tau)}\, d\tau$$

But the result is clear if we consider the frequency domain.

Useful result: If we cascade (ideal) LPFs the effect is the same as just applying the (ideal) LPF of the least bandwidth (least cut-off ω_c).

Fourier Transforms – Filter Cascade

$$\frac{\sin(3.4\pi t)}{\pi t} \star \frac{\sin(2.7\pi t)}{\pi t} \star \frac{\sin(4.1\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$

$$\frac{\sin(3.7\pi t)}{\pi t} \star \frac{\sin(5.7\pi t)}{\pi t} \star \frac{\sin(5.1\pi t)}{\pi t} \star \frac{\sin(4.0\pi t)}{\pi t} \star$$

$$\frac{\sin(3.3\pi t)}{\pi t} \star \frac{\sin(2.8\pi t)}{\pi t} \star \frac{\sin(4.9\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$

$$\frac{\sin(7.4\pi t)}{\pi t} \star \frac{\sin(8.7\pi t)}{\pi t} \star \frac{\sin(5.3\pi t)}{\pi t} \star \frac{\sin(6.1\pi t)}{\pi t} \star$$

$$\frac{\sin(5.2\pi t)}{\pi t} \star \frac{\sin(8.8\pi t)}{\pi t} \star \frac{\sin(9.1\pi t)}{\pi t} \star \frac{\sin(8.2\pi t)}{\pi t} \star$$

$$\frac{\sin(4.5\pi t)}{\pi t} \star \frac{\sin(3.5\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} = \frac{\sin(2.7\pi t)}{\pi t}$$

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Fourier Transforms – Multiplication Property

Convolution theory states

$$\boxed{x(t) \star y(t) \xleftrightarrow{\mathscr{F}} X(j\omega)\, Y(j\omega)}$$

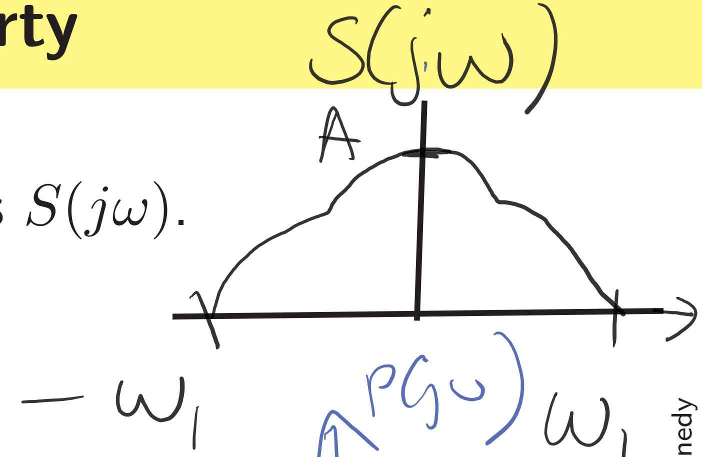
then by "symmetry" (and a bit of book-keeping)

$$\boxed{x(t) \cdot y(t) \xleftrightarrow{\mathscr{F}} \frac{1}{2\pi} X(j\omega) \star Y(j\omega)}$$

where, convolution in frequency,

$$\frac{1}{2\pi} X(j\omega) \star Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\zeta)\, Y\big(j(\omega - \zeta)\big)\, d\zeta$$

Example: Suppose $s(t)$ is a signal whose spectrum is $S(j\omega)$. Sketch the spectrum of $r(t) = s(t) \times \cos(\omega_0 t)$:



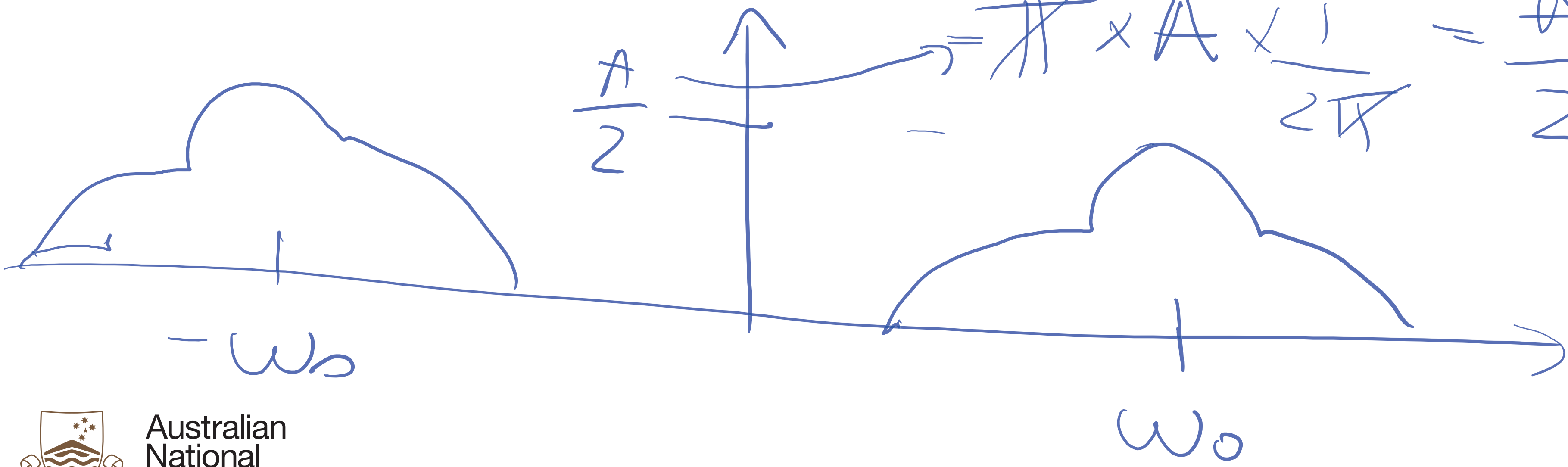
$R(j\omega)$?

$p(t) = \cos(\omega_0 t)$

$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega - \omega_0)$

$s(t) \times p(t) \longleftrightarrow \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$

$= \cancel{\pi} \times A \times \frac{1}{2\cancel{\pi}} = \frac{A}{2}$



Real-world Example: Amplitude modulation (AM):

> *AM or Amplitude Modulation is a method of radio broadcasting where the frequency is modulated or varied by its changing amplitude. Radio frequencies for AM broadcasts are expressed in kilohertz (kHz).*

ABC Canberra 666 means the centre frequency is 666 kHz. To express in terms of ω (radians per second):

$$\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots \quad \text{radians per second}$$

What is the Fourier Transform theory behind 666 Canberra?

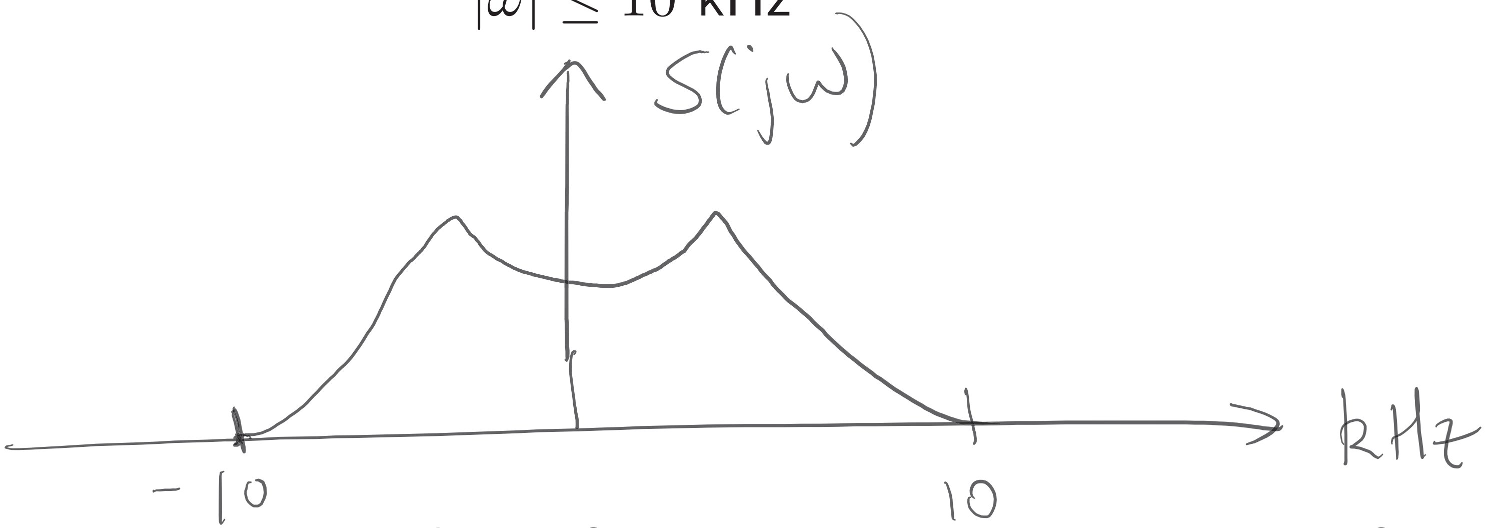
Fourier Transforms – Multiplication Property

First we take an audio signal, $s(t)$. The Fourier Transform yields:

$$s(t) \stackrel{\mathscr{F}}{\longleftrightarrow} S(j\omega)$$

Since $s(t)$ is audio its frequencies are limited to what people can hear. The human range, for teenagers and younger is roughly 1 Hz to 20 kHz. For various reasons AM audio is further limited to 10 kHz, so $s(t)$ is low pass and occupies the frequency range

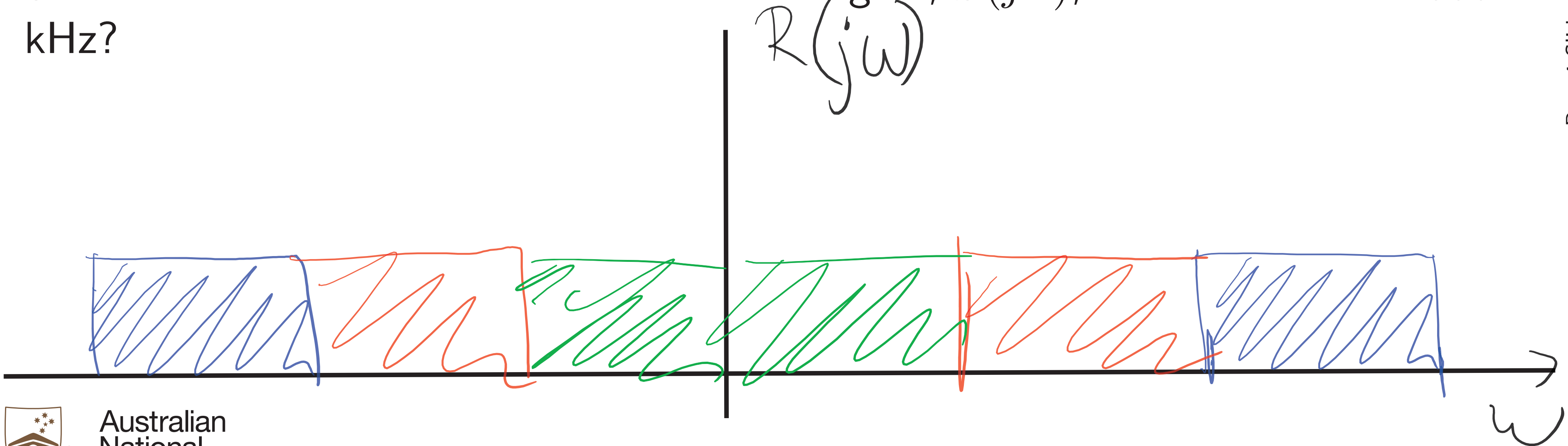
$$|\omega| \leq 10 \text{ kHz}$$



Imagine, we could transmit such a signal directly over radio (called baseband). There would be a number of problems: interference from other baseband transmitters, ridiculously huge antennas, etc.

So ABC Canberra 666 really means $666,000 \pm 10,000$ Hz. Different stations are centred on different frequencies so that they don't interfere.

So how do we translate a baseband audio signal, $S(j\omega)$, to be centred on 666 kHz?



Fourier Transforms – Multiplication Property

Use time-domain multiplication `O&W 4.5 pp.323-324`

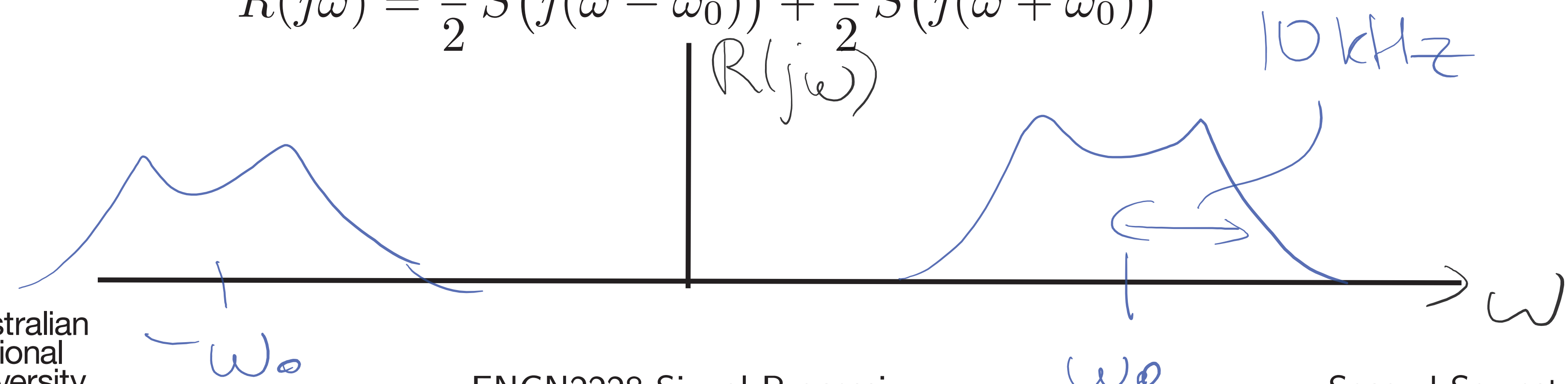
$$r(t) = s(t) \cdot p(t) \xleftrightarrow{\mathscr{F}} R(j\omega) = \frac{1}{2\pi} S(j\omega) \star P(j\omega)$$

with "modulator"

$$p(t) = \cos(\omega_0 t) \xleftrightarrow{\mathscr{F}} P(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

where $\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots$. This implies for any audio signal $s(t)$ that

$$R(j\omega) = \frac{1}{2} S(j(\omega - \omega_0)) + \frac{1}{2} S(j(\omega + \omega_0))$$



Fourier Transforms – Multiplication Property

The frequency range for the AM modulated signal $r(t)$ is non-zero in

$$\big||\omega| - 4,184,601.41\ldots\big| \leq 10 \text{ kHz}$$

which is in RF.

At the receiver, you can multiply again by $\cos(\omega_0 t)$ to move the signal back to baseband and low pass filter to move components that go to $2\omega_0$ `O&W 4.5 pp.323-324`

$$\cos^2(\omega_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)$$

This being the combination of the modulator (transmitter) and demodulator (receiver).