Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

ASSIGNMENT 2 – SOLUTIONS

Due Date: Friday, 27 October 2017, 11:55 PM (Week 12)

Late Submission Policy: Submit by the due date and time. Late assessment is *not accepted* for this course. That is, late submissions will get 0 marks.

This policy is to support the majority of students who complete and submit on time. This hard deadline enables the quick release of the assignment solutions.

Wattle submission: Upload your report as PDF format as a *multipage, single file* in Wattle. Use a scanner or scan-type smartphone app to create the PDF from your handwritten solutions.

Assignment format: 7 problems, for a total of 100 marks.

Value: 8% of total course assessment.

Solution: Will be posted on Wattle by Saturday, 28 October 2017. Marked assignments will be returned back in Wattle within 10–14 days.

Relationship to textbook: This assignment is related to Chapters 3–5 in the textbook, and Problem Sets 4 and 5. It is intended to aid you in your preparation for the final exam.

Declaration by the Student: All assessment task submissions, regardless of mode of submission, require agreement to the following declaration by the student:
"I declare that this work:

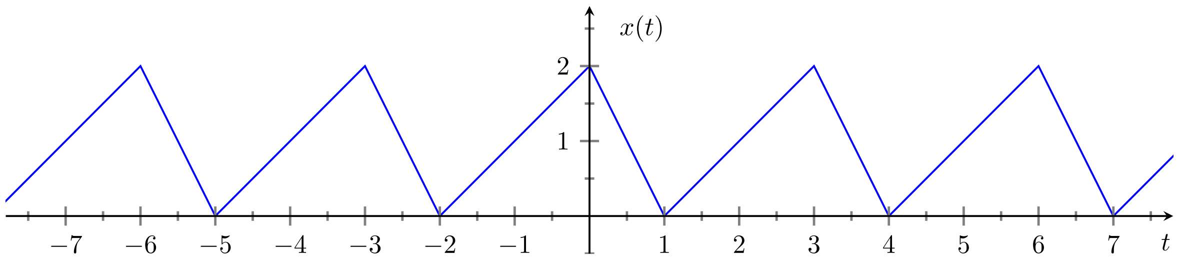
- upholds the principles of academic integrity, as defined in the University Academic Misconduct Rules;
- is original, except where collaboration (for example group work) has been authorised in writing by the course convener in the course outline and/or Wattle site;
- is produced for the purposes of this assessment task and has not been submitted for assessment in any other context, except where authorised in writing by the course convener;
- gives appropriate acknowledgement of the ideas, scholarship and intellectual property of others insofar as these have been used;
- in no part involves copying, cheating, collusion, fabrication, plagiarism or recycling."

Release Date: Monday, 31 July 2017 (Monday Week 2)

In the following: $\delta[n]$ and $u[n]$ represent the Dirac and unit step functions for discrete-time (DT). Similarly $\delta(t)$ and $u(t)$ for continuous-time (CT). Convolution of signals is written $x[n] \star h[n]$ or $x(t) \star h(t)$. Please indicate any identities or formulas used in the simplification of the results. You must show all steps taken to arrive at your answers.

Problem 1 7 marks

Determine the Fourier Series representation of the periodic CT signal in the following figure:



Solution: From inspecting the graph, $T_0 = 3$ so $\omega_0 = \frac{2\pi}{3}$.

$$x(t) = \begin{cases} 2+t, & -2 < t < 0 \\ 2-2t, & 0 < t < 1 \end{cases}$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{3}\Big[\int_{-2}^{0} (t+2) e^{-jk\frac{2\pi}{3}t}\, dt + \int_0^1 (2-2t) e^{-jk\frac{2\pi}{3}t}\, dt\Big] \end{aligned}$$

When $k = 0$

$$a_0 = \frac{1}{3}\Big[\int_{-2}^{0} (t+2)\, dt + \int_0^1 (2-2t)\, dt\Big] = 1$$

when $k \neq 0$, let $g(t) = \frac{d}{dt}x(t)$ and let b_k be the Fourier coefficients for $g(t)$ then

$$\begin{aligned} b_k &= \frac{1}{T}\int_T g(t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{3}\Big[\int_{-2}^{0} (1) e^{-jk\omega_0 t}\, dt + \int_0^1 (-2) e^{-jk\omega_0 t}\, dt\Big] \\ &= \frac{-1}{3jk\omega_0}(1 - e^{-j2k\omega_0} - 2e^{-jk\omega_0} + 2) \\ &= \frac{1}{3jk\omega_0}(e^{-j2k\omega_0} + 2e^{-jk\omega_0} - 3) \end{aligned}$$

now $a_k = \frac{1}{jk\omega_0} b_k$ using the integration property of the Fourier transform because x(t) is the integral of g(t). Therefore,

$$a_k = \frac{1}{jk\omega_0}\Big[\frac{1}{3jk\omega_0}(e^{-j2k\omega_0} + 2e^{-jk\omega_0} - 3)\Big] = \frac{3}{k^2 4\pi^2}(3 - 2e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{4\pi}{3}})$$

□

Problem 2   5 marks

The Fourier Series coefficients of a continuous time signal with period 4 is specified as

$$a_k = \begin{cases} j\,k, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$$

Determine the signal $x(t)$:

Solution: The only Fourier series coefficients are $a_1 = j$, $a_{-1} = -j$, $a_2 = 2j$ and $a_{-2} = -2j$. Since

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}$$

Then

$$\begin{aligned} x(t) &= j[e^{j\omega_0 t} - e^{-j\omega_0 t}] + 2j[e^{j2\omega_0 t} - e^{-j2\omega_0 t}] \\ &= (2j)(j)\sin(\omega_0 t) + (2j)(2j)\sin(2\omega_0 t) \\ &= -2\sin(\frac{\pi}{2}t) - 4\sin(\pi t) \end{aligned}$$

□

Problem 3

For each of the following pulse responses shown in the figures:

(i) give an expression for the pulse response $h[n]$,

(ii) the frequency response

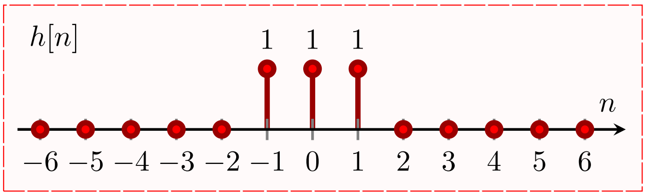
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega},$$

(iii) say whether $H(e^{j\omega})$ as a function of ω is even or odd

(iv) discuss the phase of $H(e^{j\omega})$ and by looking at the slope say what the (group) delay is

(v) and sketch/plot the magnitude $\left|H(e^{j\omega})\right|$

(a) **[2 marks]**



Real and even.

Solution:

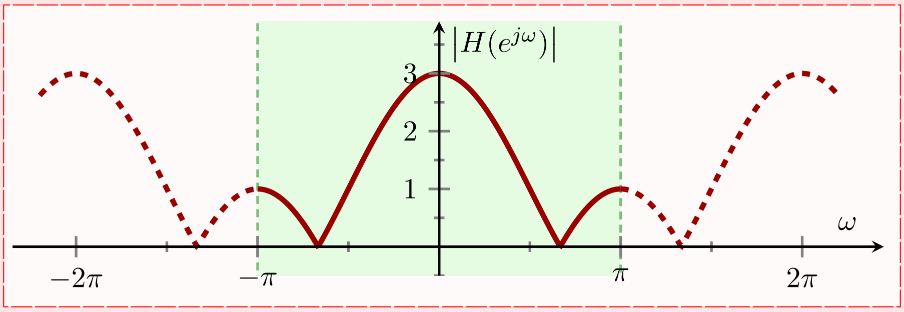
(i) $h[n] = h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1] = \delta[n+1] + \delta[n] + \delta[n-1]$

(ii) $H(e^{j\omega}) = h[-1]e^{j\omega} + h[0] + h[1]e^{-j\omega} = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos\omega$

(iii) $H(e^{j\omega})$ is even

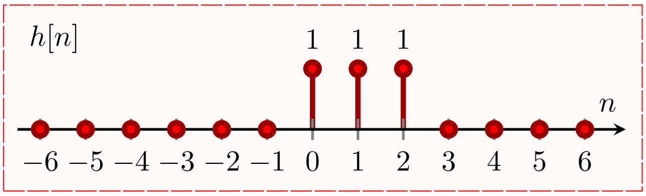
(iv) phase is flat and equals 0, the delay being the slope is 0

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos\omega|$ and is plotted below:



□

(b) **[2 marks]**



Real and causal.

Solution:

(i) $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

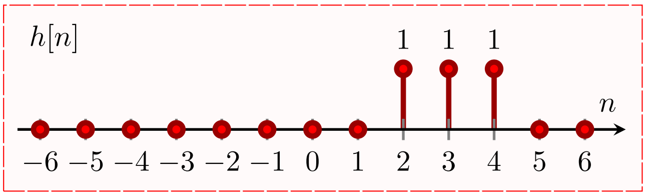
(ii) $H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(1 + 2\cos\omega)$

(iii) $H(e^{j\omega})$ is complex-valued but $\left|H(e^{j\omega})\right|$ is even

(iv) phase is linear and equals $-\omega$, the delay being the negative slope is 1

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos\omega|$ as previously plotted □

(c) **[2 marks]**



Real with larger delay.

Solution:

(i) $h[n] = \delta[n-2] + \delta[n-3] + \delta[n-4]$

(ii) $H(e^{j\omega}) = e^{-3j\omega}(1 + 2\cos\omega)$

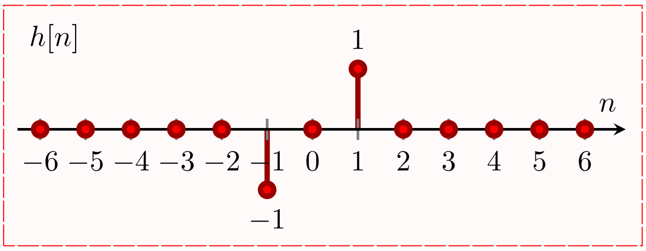
(iii) $H(e^{j\omega})$ is complex-valued but $\left|H(e^{j\omega})\right|$ is even

(iv) phase is linear and equals -3ω, the delay being the negative slope is 3

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos\omega|$ as previously plotted

□

(d) **[2 marks]**



Real and odd.

Solution:

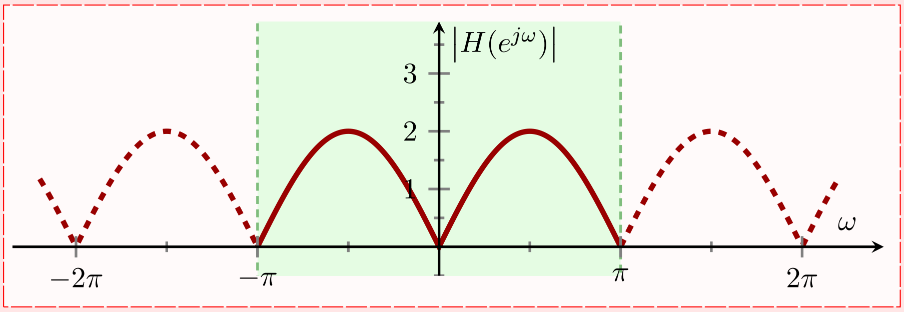
(i) $h[n] = h[-1]\delta[n+1] + h[1]\delta[n-1] = -\delta[n+1] + \delta[n-1]$

(ii) $H(e^{j\omega}) = h[-1]e^{j\omega} + h[1]e^{-j\omega} = -e^{j\omega} + e^{-j\omega} = -2j\left(\dfrac{e^{j\omega} - e^{-j\omega}}{2j}\right) = 2\,e^{j3\pi/2}\sin\omega$

(iii) $H(e^{j\omega})$ is odd since $\sin\omega$ is odd, and $\left|H(e^{j\omega})\right|$ is even

(iv) phase is constant at $3\pi/2$ and the delay (slope with respect to ω) is zero

(v) $\left|H(e^{j\omega})\right| = |2\sin\omega|$ and is plotted below:



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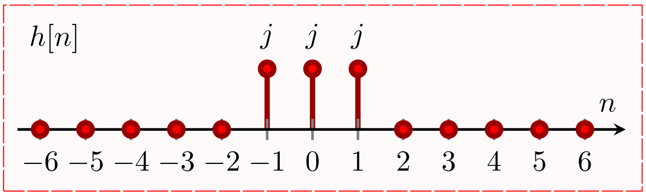
(e) **[2 marks]**



Imaginary and even.

Solution:

(i) $h[n] = h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1] = j\delta[n+1] + j\delta[n] + j\delta[n-1]$

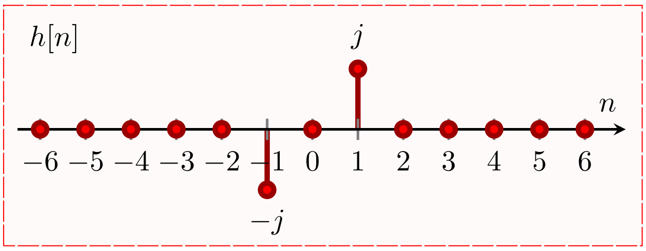
(ii) $H(e^{j\omega}) = h[-1]e^{j\omega} + h[0] + h[1]e^{-j\omega} = j\big(e^{j\omega} + 1 + e^{-j\omega}\big) = e^{j\pi/2}(1 + 2\cos\omega)$

(iii) $H(e^{j\omega})$ is even

(iv) phase is flat and equals $\pi/2$, the delay being the slope is 0

(v) $\big|H(e^{j\omega})\big| = |1 + 2\cos\omega|$ as previously plotted $\square$

(f) **[2 marks]**



Imaginary and odd.

Solution:

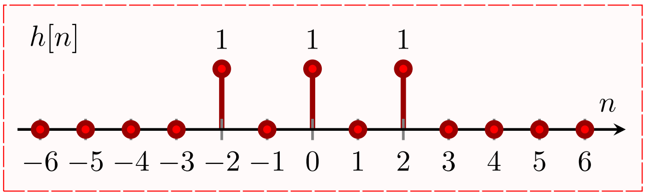
(i) $h[n] = h[-1]\delta[n+1] + h[1]\delta[n-1] = -j\delta[n+1] + j\delta[n-1]$

(ii) $H(e^{j\omega}) = h[-1]e^{j\omega} + h[1]e^{-j\omega} = -je^{j\omega} + je^{-j\omega} = 2\left(\dfrac{e^{j\omega} - e^{-j\omega}}{2j}\right) = 2\sin\omega$ is purely real-valued. Notice that $h[-1] = \overline{h[1]}$.

(iii) $H(e^{j\omega})$ is odd since $\sin\omega$ is odd, and $\big|H(e^{j\omega})\big|$ is even

(iv) phase is constant at 0 and the delay is zero

(v) $\big|H(e^{j\omega})\big| = |2\sin\omega|$ as previously plotted $\square$

(g) [2 marks] 

Real, even & spaced out.

Solution:

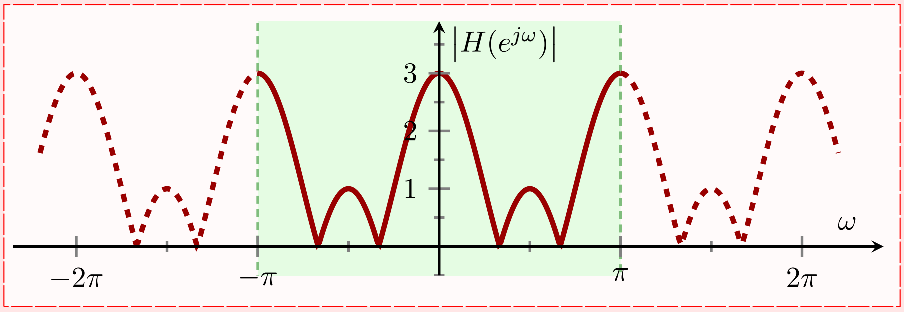
(i) $h[n] = h[-2]\delta[n+2] + h[0]\delta[n] + h[2]\delta[n-2] = \delta[n+2] + \delta[n] + \delta[n-2]$

(ii) $H(e^{j\omega}) = h[-2]e^{j\omega} + h[0] + h[2]e^{-j\omega} = e^{j2\omega} + 1 + e^{-j2\omega} = 1 + 2\cos 2\omega$

(iii) $H(e^{j\omega})$ is even and purely real-valued

(iv) phase is flat and equals 0, the delay being the slope is 0

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos 2\omega|$ and is plotted below:



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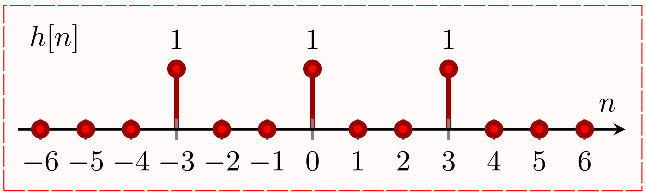
(h) [2 marks] 

Real, even & spaced out.

Solution:

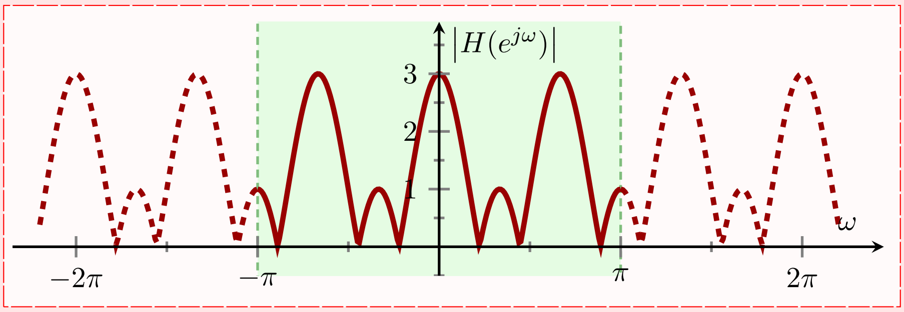
(i) $h[n] = h[-3]\delta[n+3] + h[0]\delta[n] + h[3]\delta[n-3] = \delta[n+3] + \delta[n] + \delta[n-3]$

(ii) $H(e^{j\omega}) = h[-3]e^{j\omega} + h[0] + h[3]e^{-j\omega} = e^{j3\omega} + 1 + e^{-j3\omega} = 1 + 2\cos 3\omega$

(iii) $H(e^{j\omega})$ is even and purely real-valued

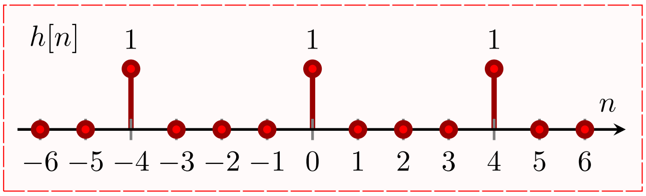
(iv) phase is flat and equals 0, the delay being the slope is 0

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos 3\omega|$ and is plotted below:



□

(i) **[2 marks]**



Real, even & spaced out.

Solution:

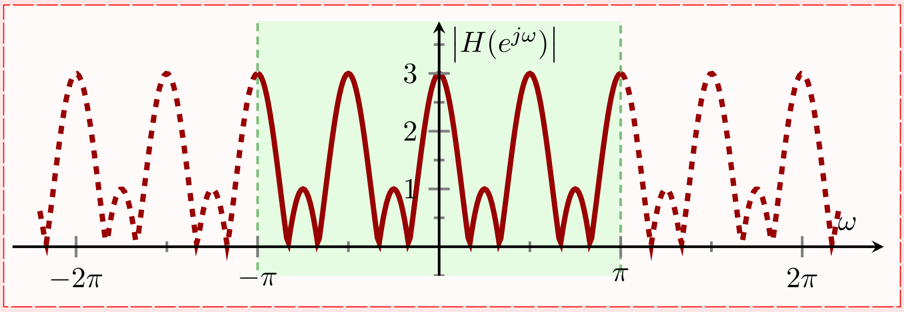
(i) $h[n] = h[-4]\delta[n+4] + h[0]\delta[n] + h[4]\delta[n-4] = \delta[n+4] + \delta[n] + \delta[n-4]$

(ii) $H(e^{j\omega}) = h[-4]e^{j\omega} + h[0] + h[4]e^{-j\omega} = e^{j4\omega} + 1 + e^{-j4\omega} = 1 + 2\cos 4\omega$

(iii) $H(e^{j\omega})$ is even and purely real-valued

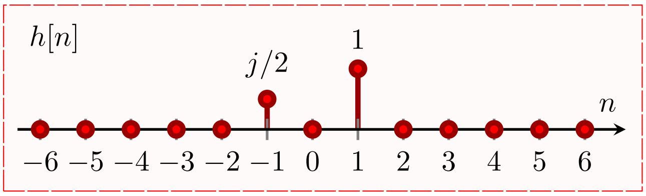
(iv) phase is flat and equals 0, the delay being the slope is 0

(v) $\left|H(e^{j\omega})\right| = |1 + 2\cos 4\omega|$ and is plotted below:



□

(j) **[2 marks]**



Mixed real and imaginary.

Solution:

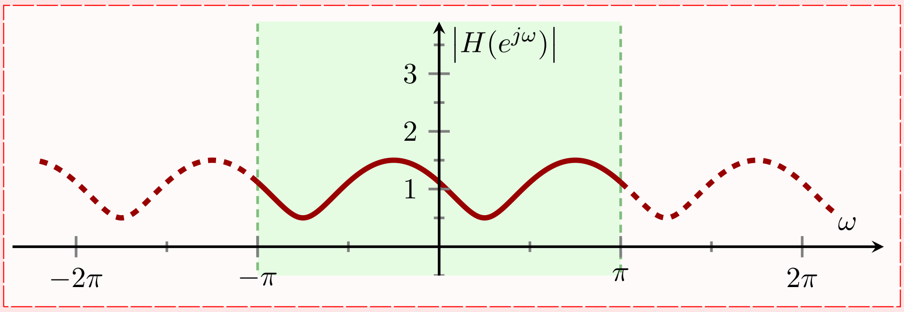
(i) $h[n] = h[-1]\delta[n+1] + h[1]\delta[n-1] = (j/2)\delta[n+1] + \delta[n-1]$

(ii) $H(e^{j\omega}) = (j/2)e^{j\omega} + e^{-j\omega}$ then use Euler

(iii) $H(e^{j\omega})$ is neither even nor odd and mixed real- and complex-valued

(iv) phase is all over the shop

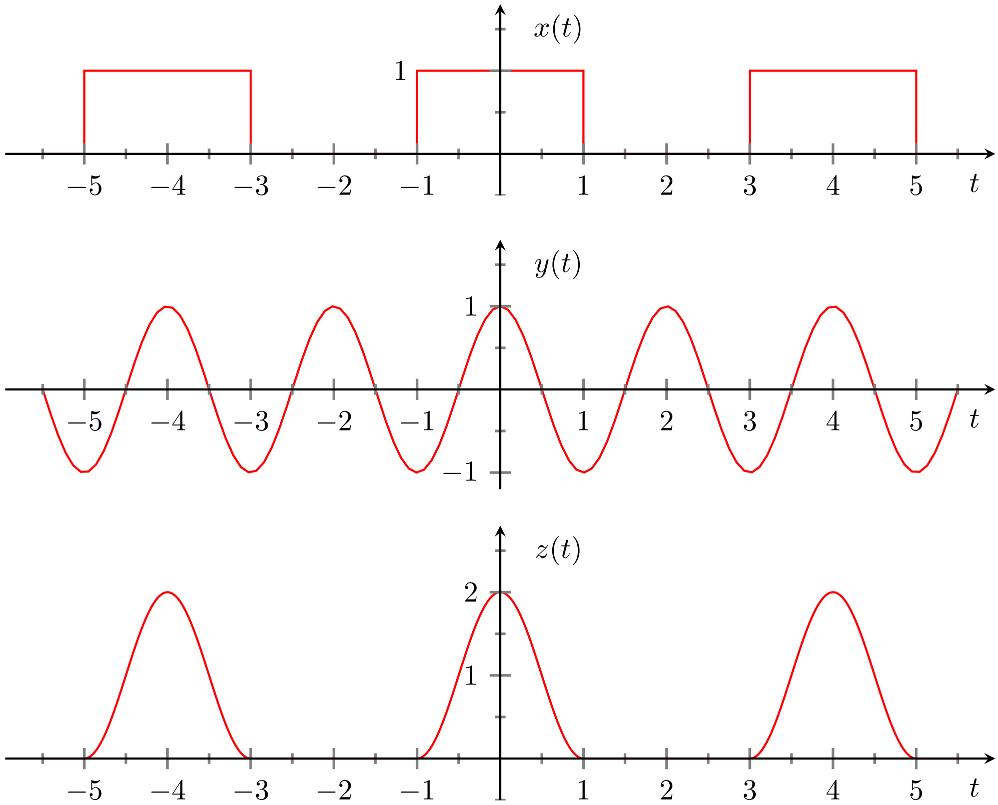
(v) $\left|H(e^{j\omega})\right| = \left((\cos\omega + (1/2)\sin\omega)^2 + ((1/2)\cos\omega - \sin\omega)^2\right)^{1/2}$ and is plotted below:



□

Problem 4

Consider the following CT periodic signals $x(t)$, $y(t)$ and $z(t)$



(a) [**5 marks**] Determine the fundamental frequency, ω_0, period T_0 and Fourier series coefficients, a_k, for $x(t)$ as shown (the central portion is given by $x(t) = \chi_{[-1,1]}(t)$).

Solution: The fundamental period is $T_0 = 4$ and, therefore, the fundamental frequency is $\omega_0 = 2\pi/4 = \pi/2$ rad/s.

Fourier series coefficients are given by

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt.$$

Therefore

$$a_0 = \frac{1}{4}\int_{-2}^{2} e^{-jk\frac{\pi}{2}t}\, dt = \frac{1}{4}\int_{-1}^{1} dt = \frac{1}{2}$$

and

$$a_k = \frac{1}{4}\int_{-1}^{1} e^{-jk\omega_0 t}\, dt = \frac{1}{4}\frac{e^{-jk\omega_0 t}}{-ik\omega_0}\Big|_{-1}^{1} = \frac{1}{2}\frac{\sin k\omega_0}{k\omega_0} = \frac{1}{2}\mathrm{sinc}(k\frac{\pi}{2})$$

□

(b) [**7 marks**] Determine the fundamental frequency, period and Fourier series coefficients, b_k, for $y(t) = \cos(\pi t)$.

Solution: The fundamental period is $T_0 = 2$ and, therefore, the fundamental frequency is

$$\omega_0 = 2\pi/2 = \pi \quad \text{rad/s}.$$

Fourier series coefficients are given by

$$b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt.$$

Therefore

$$\begin{aligned}
b_k &= \frac{1}{2}\int_{-1}^{1}\Big[\frac{e^{j\pi t}+e^{-j\pi t}}{2}\Big]e^{-jk\pi t}\, dt \\
&= \frac{1}{2}\Big[\frac{1}{2}\int_{-1}^{1} e^{j\pi(1-k)t}\, dt + \frac{1}{2}\int_{-1}^{1} e^{-j\pi(1+k)t}\, dt\Big] \\
&= \frac{1}{2}\Big[\frac{e^{j\pi(1-k)t}}{j\pi(1-k)} + \frac{e^{-j\pi(1+k)t}}{-j\pi(1+k)}\Big]\Big|_{-1}^{1} \\
&= \frac{1}{2}\Big[\frac{e^{j\pi(1-k)} - e^{-j\pi(1-k)}}{j\pi(1-k)} + \frac{e^{-j\pi(1+k)} - e^{j\pi(1+k)}}{-j\pi(1+k)}\Big] \\
&= \frac{1}{2}\Big[\frac{\sin\pi(1-k)}{\pi(1-k)} + \frac{\sin\pi(1+k)}{\pi(1+k)}\Big] \\
&= \frac{1}{2}\Big[\text{sinc}[\pi(1-k)] + \text{sinc}[\pi(1+k)]\Big]
\end{aligned}$$

Since $\text{sinc}(x) = 0$ for $x = \pm n\pi$, $n \in$ Natural numbers $n = 1, 2, 3\ldots$. Hence,

$$\text{sinc}[\pi(1-k)] = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}[\pi(1+k)] = \begin{cases} 1 & k = -1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$b_k = \begin{cases} \frac{1}{2} & k = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

i.e.,

$$b_k = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1].$$

□

(c) [**2 marks**] Determine the fundamental frequency and fundamental period for

$$z(t) = x(t)\big(1 + \cos(\pi t)\big).$$

Solution: The fundamental period is $T_0 = 4$ and, therefore, the fundamental frequency is

$$\omega_0 = 2\pi/4 = \pi/2 \quad \text{rad/s}.$$

□

(d) **[6 marks]** Using the results of parts (a) and (b) determine the Fourier coefficients c_k for $z(t)$. (It is recommended that you use, with care, the Fourier Series Multiplication Property.)

Solution: Before we use the Fourier Series Multiplication Property we need a common period. So we need to re-express the Fourier Series in Part (b) with fundamental period $T_0 = 4$. $b_k = \frac{1}{2}\delta[k-2] + \frac{1}{2}\delta[k+2]$ for $T_0 = 4$.

$$z(t) = x(t)[1 + \cos \pi t] = x(t) + x(t)y(t)$$

Using multiplication property Fourier Transform

$$\begin{aligned}
c_k &= a_k + \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k + \sum_{l=-\infty}^{\infty} a_{k-l} b_l \\
&= a_k + \sum_{l=-\infty}^{\infty} a_{k-l} \Big[\frac{1}{2}\delta[l-2] + \frac{1}{2}\delta[l+2]\Big] \\
&= a_k + \frac{1}{2}[a_{k-2} + a_{k+2}] \\
&= \frac{1}{4}\Big[2\,\mathrm{sinc}(k\frac{\pi}{2}) + \mathrm{sinc}((k-2)\frac{\pi}{2}) + \mathrm{sinc}((k+2)\frac{\pi}{2})\Big]
\end{aligned}$$

□

Problem 5

Determine the following convolutions using the approach specified:

(a) **[6 marks]** Use discrete time convolution sum to determine the convolution $y[n] = x[n] \star h[n]$ where $x[n] = \left(\frac{3}{4}\right)^n u[n]$ and $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

You may use any of the following sums in your calculations:

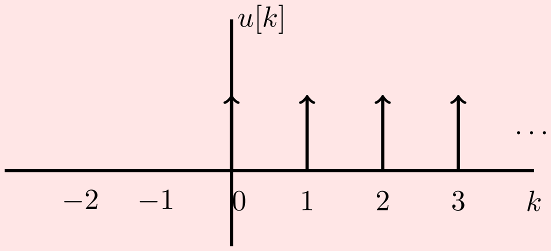
$$\sum_{k=0}^{n} \alpha^k \beta^{n-k} = \frac{\alpha^{1+n} - \beta^{1+n}}{\alpha - \beta}$$

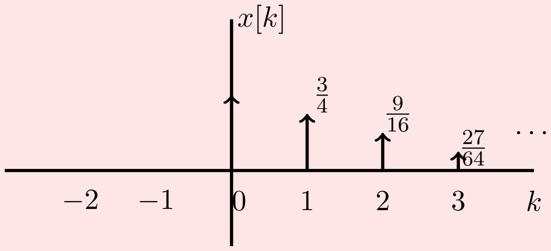
$$\sum_{k=0}^{n} \alpha^k \beta^k = \frac{\alpha^{1+n}\beta^{1+n} - 1}{\alpha\beta - 1}$$

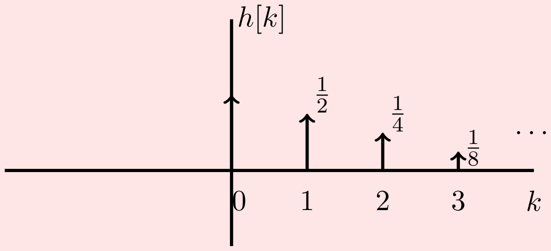
$$\sum_{k=0}^{\infty} \alpha^k \beta^{n-k} = \frac{\beta^{n+1}}{\beta - \alpha}$$

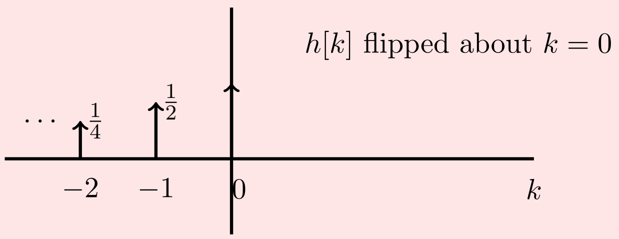
$$\sum_{k=0}^{\infty} \alpha^k \beta^k = \frac{1}{1 - \alpha\beta}$$

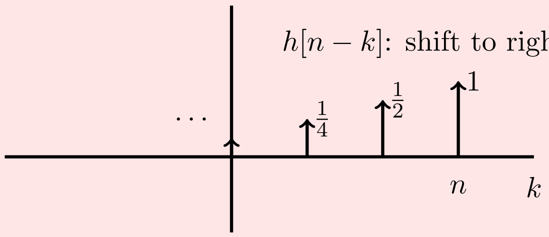
Solution: The signals are shown graphically as follows :











For $n < 0$, there is no overlap. Hence, $y[n] = 0$.
For $n \geq 0$, we have

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^{n} \left(\frac{1}{2}\right)^{n-k} \\
&= \frac{\left(\frac{3}{4}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+1}}{\left(\frac{3}{4}\right) - \left(\frac{1}{2}\right)} \\
&= 4\left(\left(\frac{3}{4}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+1}\right) \\
&= 3\left(\frac{3}{4}\right)^{n} - 2\left(\frac{1}{2}\right)^{n}
\end{aligned}
$$

Hence,

$$
y[n] = \left(3\left(\frac{3}{4}\right)^{n} - 2\left(\frac{1}{2}\right)^{n}\right) u[n].
$$

The plot of $y[n]$ is shown below. □

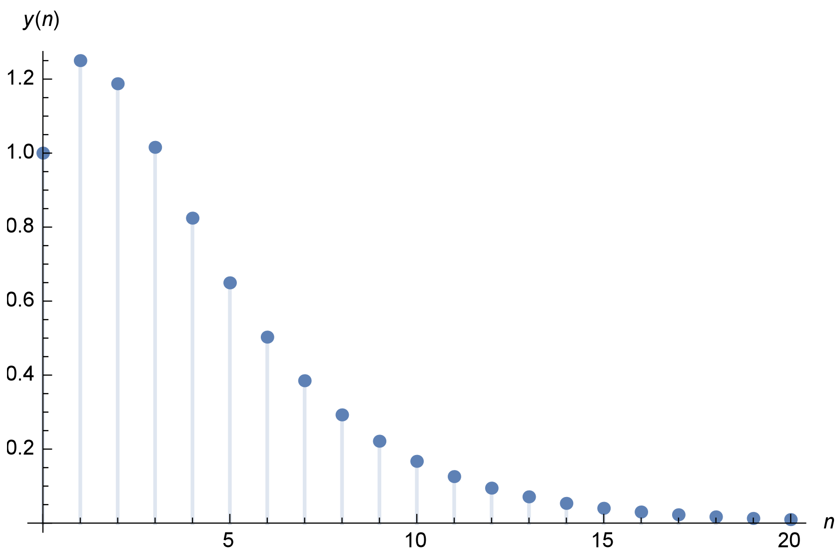


Figure 1: The convolution output $y[n]$ for parts (a) and (b).

(b) **[6 marks]** Use Fourier transforms to determine the convolution $y[n] = x[n] \star h[n]$ where $x[n] = \left(\frac{3}{4}\right)^n u[n]$ and $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Show that your answer is the same as in part (a) above.

Solution: We have

$$\begin{aligned} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) &= \left(\frac{1}{1-\frac{3}{4}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) \\ &= \frac{-2}{1-\frac{1}{2}e^{-j\omega}} + \frac{3}{1-\frac{3}{4}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we get

$$\begin{aligned} y(t) &= 3\left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n] \\ &= \left(3\left(\frac{3}{4}\right)^n - 2\left(\frac{1}{2}\right)^n\right) u[n] \end{aligned}$$

□

(c) **[4 marks]** Use Fourier transforms to determine the convolution $y[n] = x[n] \star h[n]$ where $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$ and $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

Solution: We have

$$\begin{aligned} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) &= \left(\frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) \\ &= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2} \end{aligned}$$

Taking the inverse Fourier transform, we get

$$y(t) = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] - (n+1)\left(\frac{1}{4}\right)^n u[n]$$

□

(d) **[5 marks]** Use Fourier transforms to determine the convolution $y(t) = x(t) \star h(t)$ where $x(t) = te^{-2t}u(t)$ and $h(t) = e^{-4t}u(t)$.

Solution: We have

$$\begin{aligned} Y(j\omega) = X(j\omega)H(j\omega) &= \left(\frac{1}{(2+j\omega)^2}\right)\left(\frac{1}{4+j\omega}\right) \\ &= \frac{1/4}{4+j\omega} - \frac{1/4}{2+j\omega} + \frac{1/2}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t)$$

□

(e) **[3 marks]** Use Fourier transforms to determine the convolution $y(t) = x(t) \star h(t)$ where $x(t) = te^{-2t}u(t)$ and $h(t) = te^{-4t}u(t)$.

Solution: We have

$$\begin{aligned} Y(j\omega) = X(j\omega)H(j\omega) &= \left(\frac{1}{(2+j\omega)^2}\right)\left(\frac{1}{(4+j\omega)^2}\right) \\ &= \frac{1/4}{(2+j\omega)^2} - \frac{1/4}{2+j\omega} + \frac{1/4}{4+j\omega} + \frac{1/4}{(4+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t) + +\frac{1}{4}te^{-4t}u(t)$$

□

Problem 6

A casual and stable LTI system has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

and frequency domain description:

$$Y(j\omega) = H(j\omega)\,X(j\omega)$$

(a) [**4 marks**] Using the property:

$$\frac{d^k z(t)}{dt^k} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^k\, Z(j\omega), \quad k = 0, 1, 2, \ldots$$

where

$$z(t) \overset{\mathcal{F}}{\longleftrightarrow} Z(j\omega),$$

find the **differential equation** relating the input $x(t)$ (and its derivatives) and output $y(t)$ (and its derivatives) of the system with frequency response $H(j\omega)$.

Solution:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

That is,

$$\big(6 + (j\omega)^2 + 5j\omega\big)Y(j\omega) = \big(j\omega + 4\big)X(j\omega)$$

thus

$$(j\omega)^2 Y(j\omega) + 5j\omega\, Y(j\omega) + 6\, Y(j\omega) = j\omega\, X(j\omega) + 4\, X(j\omega)$$

and using the derivative property gives

$$\boxed{\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\, y(t) = \frac{dx(t)}{dt} + 4\, x(t).}$$

Going in the opposite direction is equally straightforward. Fourier technique is a way of transforming linear differential equations to rational polynomials equations and finding algebraic methods to solve them. But more importantly the frequency response directly informs you of the operation characteristics of the differential equations acting as a system or filter. □

(b) [**4 marks**] Find the **partial fraction expansion** of $H(j\omega)$.

Solution: We try to find constants A and B such that

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \frac{A}{j\omega + 2} + \frac{B}{j\omega + 3}$$

because if we can then the components on the right-hand side are something we recognize and can handle. Hence

$$\frac{A}{j\omega + 2} + \frac{B}{j\omega + 3} = \frac{(3A + 2B) + j\omega(A + B)}{6 - \omega^2 + 5j\omega},$$

which implies

$$3A + 2B = 4$$
$$A + B = 1.$$

This has unique solution $A = 2$ and $B = -1$ hence the partial fraction expansion is

$$\boxed{H(j\omega) = \frac{2}{j\omega + 2} - \frac{1}{j\omega + 3}}.$$

Note: students can and should use the residue method to find partial fractions. But if you want to use the linear algebra method which is illustrated above, this is fine as well. □

(c) [**4 marks**] Using the partial fraction expansion and the property

$$e^{-at}\,u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega},$$

where $u(t)$ is the unit step function, determine the **impulse response** $h(t)$ corresponding to frequency response $H(j\omega)$.

Solution: We have

$$e^{-2t}\,u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2+j\omega} \quad \text{and} \quad e^{-3t}\,u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{3+j\omega}$$

and so

$$2\,e^{-2t}\,u(t) - e^{-3t}\,u(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega+2} - \frac{1}{j\omega+3}$$

That is,

$$\boxed{h(t) = \left(2\,e^{-2t} - e^{-3t}\right)u(t).}$$

□

Problem 7

(a) **[3 marks]** Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ for the system whose input is $x[n]$ and output is $y[n]$:

$$y[n] = x[n] - 0.5\,x[n-1]$$

Solution: The main identity we need is

$$z[n-k] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega k}\,Z(e^{j\omega}), \quad k = 0, 1, 2, \ldots$$

where

$$z[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Z(e^{j\omega}),$$

Take FT of both sides yields

$$Y(e^{j\omega}) = X(e^{j\omega}) - 0.5\,e^{-j\omega} X(e^{j\omega})$$

then

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - 0.5\,e^{-j\omega}.$$

For the impulse response, by definition $h[n]$ is the output $y[n]$ when the input $x[n] = \delta[n]$; so substitute this into the LDE

$$h[n] = \delta[n] - 0.5\,\delta[n-1]$$

which is the desired impulse response. □

(b) **[3 marks]** Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ for the system whose input is $x[n]$ and output is $y[n]$:

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

Solution: As we saw a delay of 1 time unit introduces a factor $e^{-j\omega}$ into the FT. Similarly a delay of 2 time units introduces a factor $e^{-j2\omega}$ into the FT. So

$$Y(e^{j\omega})\big(1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}\big) = X(e^{j\omega})$$

and, therefore,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}.$$

which is the desired frequency response. Using partial fractions

$$H(e^{j\omega}) = \frac{-6}{(3+e^{-j\omega})(-2+e^{-j\omega})} = \frac{A}{3+e^{-j\omega}} + \frac{B}{-2+e^{-j\omega}} = \frac{\frac{2}{5}}{1-\big(\frac{-1}{3}\big)e^{-j\omega}} + \frac{\frac{3}{5}}{1-\big(\frac{1}{2}\big)e^{-j\omega}}.$$

The impulse response is,

$$h[n] = \frac{2}{5}\Big(-\frac{1}{3}\Big)^n u[n] + \frac{3}{5}\Big(\frac{1}{2}\Big)^n u[n].$$

□

(c) **[3 marks]** Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ for the system whose input is $x[n]$ and output is $y[n]$:

$$y[n] - 0.3\,y[n-1] + 0.1\,y[n-2] = 3x[n] - x[n-1]$$

Solution:

$$Y(e^{j\omega})\big(1 - 0.3e^{-j\omega} + 0.1e^{-j2\omega}\big) = X(e^{j\omega})\big(3 - e^{-j\omega}\big)$$

and, therefore,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3 - e^{-j\omega}}{1 - 0.3e^{-j\omega} + 0.1e^{-j2\omega}}.$$

which is the desired frequency response.

Using partial fractions,

$$H(e^{j\omega}) = \frac{A}{1 - ae^{-j\omega}} + \frac{B}{1 - be^{-j\omega}},$$

where $A = \frac{93 - 11\sqrt{31}j}{62}$, $B = \frac{3}{2} + \frac{11\sqrt{31}j}{62}$, $a = \frac{3}{20} - \frac{\sqrt{31}j}{20}$ and $b = \frac{3}{20} + \frac{\sqrt{31}j}{20}$.

The impulse response is,

$$h[n] = \big(Aa^n + Bb^n\big)u[n]$$

□

(d) **[3 marks]** The filter described by

$$y[n] = x[n] + \frac{1}{3}\,x[n-1]$$

is connected in parallel with another filter with impulse response $g[n]$ such that the resulting parallel interconnection has the frequency response

$$H(e^{j\omega}) = \frac{-\frac{1}{9}e^{-j2\omega}}{1 - \frac{1}{3}e^{-j\omega}}.$$

Determine the frequency response $G(e^{j\omega})$ and impulse response $g[n]$.

Solution: Let $F(e^{j\omega})$ denote the response of filter $y[n] = x[n] + \frac{1}{3}\,x[n-1]$, then

$$F(e^{j\omega}) = 1 + 1/3e^{-j2\omega}.$$

As the two filters are in parallel, $G(e^{j\omega}) = H(e^{j\omega}) - F(e^{j\omega})$. Therefore,

$$G(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}.$$

which is the desired frequency response, and

$$g[n] = \big(\frac{1}{3}\big)^n u[n]$$

is the desired impulse response. □

— End of Assignment —

Question	Points	Score
1	7	
2	5	
3	20	
4	20	
5	24	
6	12	
7	12	
Total:	100	