Part 2 Outline

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- Discrete Time
- Periodic Signals
- Examples using Frequency Response

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- CT Band Pass Filter
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- DT High Pass Filter
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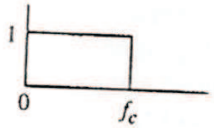
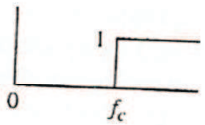
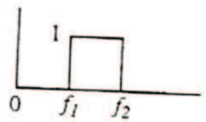
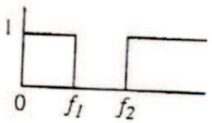
Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Filter Circuits

- A **Filter** is a circuit that passes certain frequencies and attenuates or rejects all other frequencies.

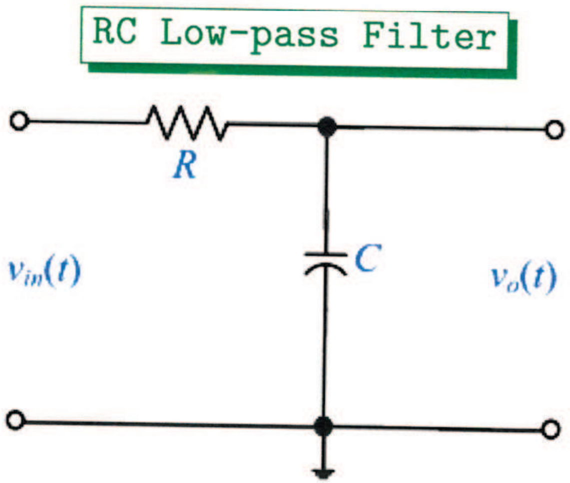
Filter Types

Type	Typical Ideal $\|H(f)\|$	Description	Example Uses
Lowpass		removes all frequency information above f_c	noise removal, interpolation, data smoothing
Highpass		removes all frequency information below f_c	removing DC or low freq drift, edge detection or enhancement
Bandpass		removes all frequency information outside of $f_1 \rightarrow f_2$	tuning in to one radio station, audio graphic equalizers
Notch		removes all frequency information between $f_1 \rightarrow f_2$	removing noise at a particular frequency, e.g. 60 Hz

Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

RC Filter Circuits

- A **Filter** is a circuit that passes certain frequencies and attenuates or rejects all other frequencies.
- **Low-pass** filters allow low frequency (from DC to f_c) to pass and reject/attenuate (i.e. reduce amplitude of) all other frequencies.



Bode Plots

- A **Bode plot** shows the **magnitude of a transfer function** in decibels versus **frequency** using a logarithmic scale for frequency.
- Because it can clearly illustrate very large and very small magnitudes for a wide range of frequencies on one plot, the Bode plot is particularly useful for displaying transfer function magnitudes.
- **Bode plots** of filter circuits can be closely approximated by **straight line segments (asymptotes)**, so they are relatively easy to draw.

  Asymptote = a line or curve that approaches a given curve arbitrarily close

Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Bode Plot Approximation

- The two straight line asymptotes intersect at $f = f_B$ Hz. For this reason, f_B is called the **break** or the **corner** or **cut-off** frequency (also sometimes denoted as f_c). The asymptotes are in error by only 3 dB at the break frequency.

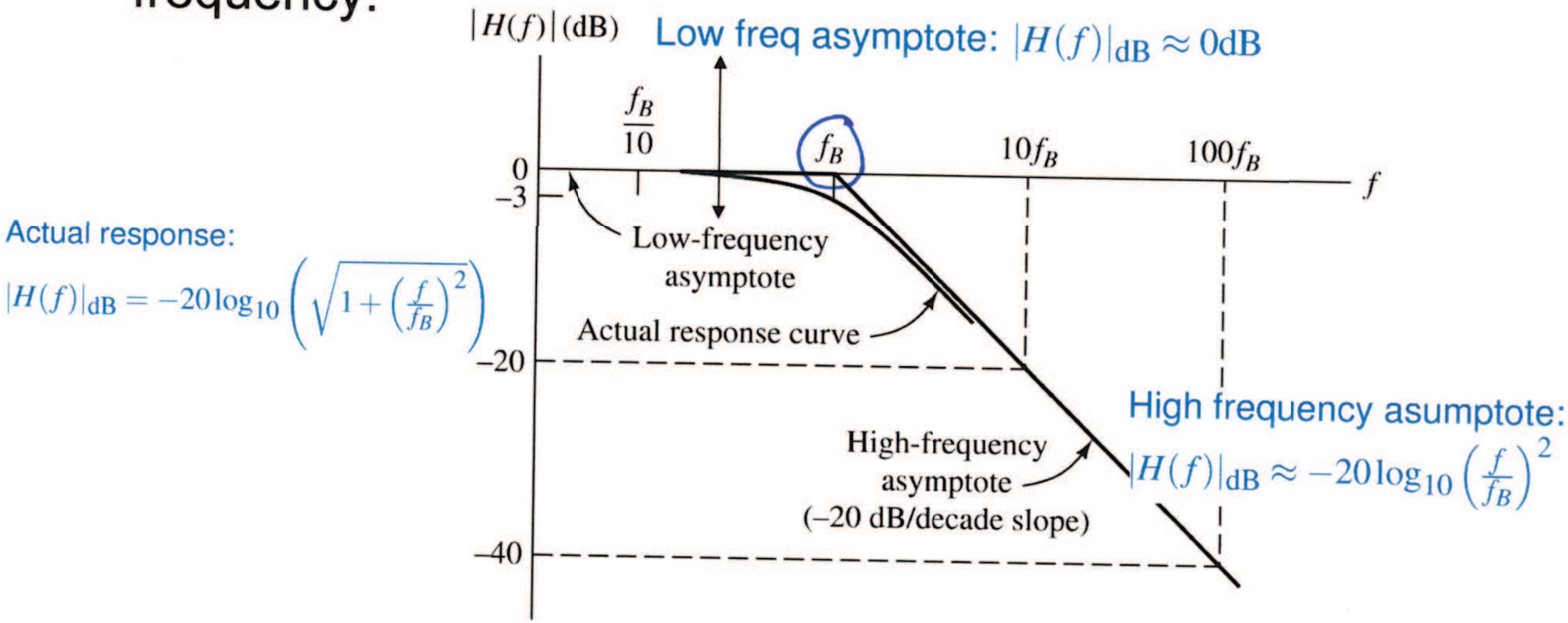
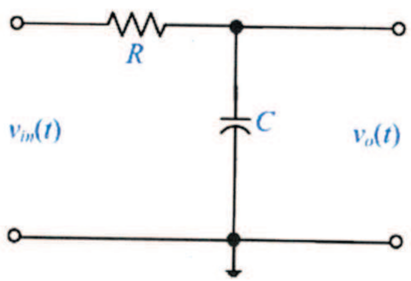


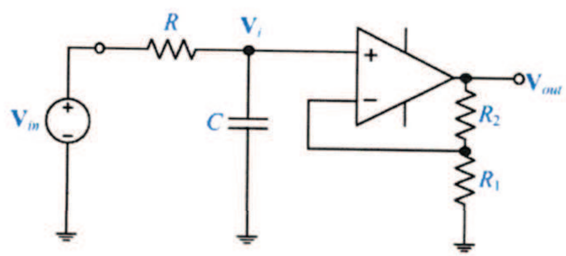
Figure 6.15 Magnitude Bode plot for the first-order lowpass filter.

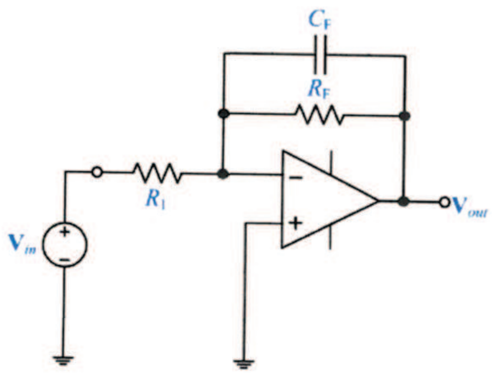
Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Summary of Low-pass Filter Circuits

$$H(f) = A\left(\frac{1}{1 + j\frac{f}{f_B}}\right)$$





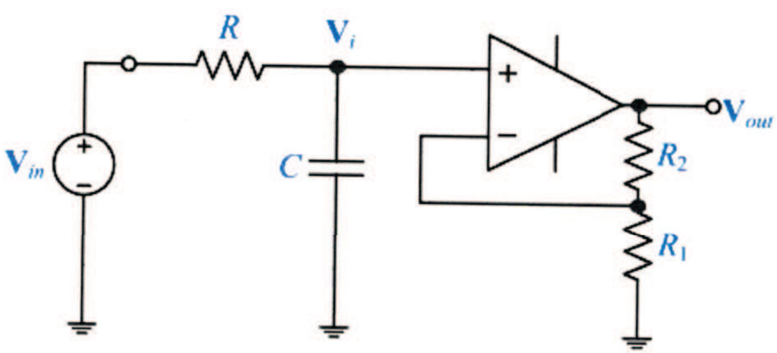


$A = 1$	$A = 1 + \frac{R_2}{R_1}$	$A = -\frac{R_F}{R_1}$
$f_B = \frac{1}{2\pi RC}$	$f_B = \frac{1}{2\pi RC}$	$f_B = \frac{1}{2\pi R_F C_F}$

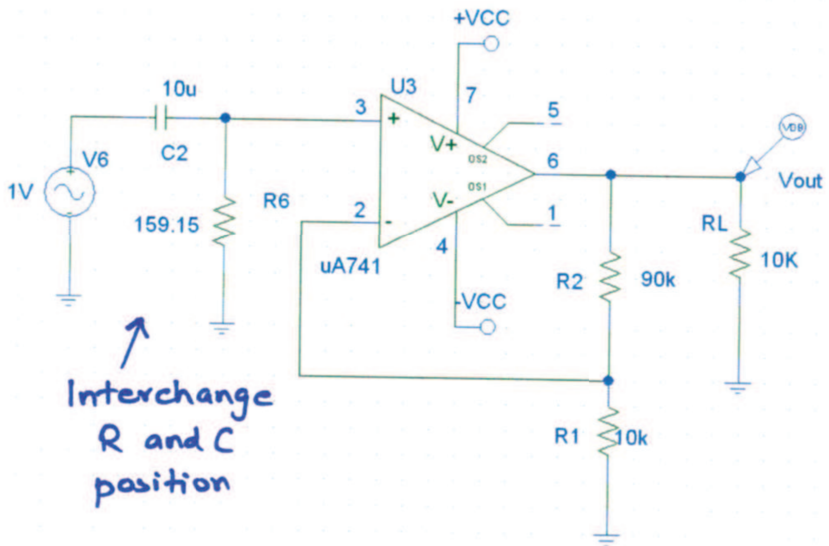
Active Highpass Filter Circuits

- Interchange resistor and capacitor to get highpass filter.

Active Lowpass Filter



Active Highpass Filter



See: Lec17_highpass.sch

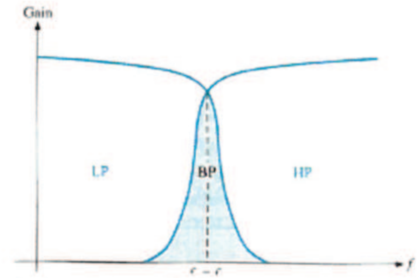
Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

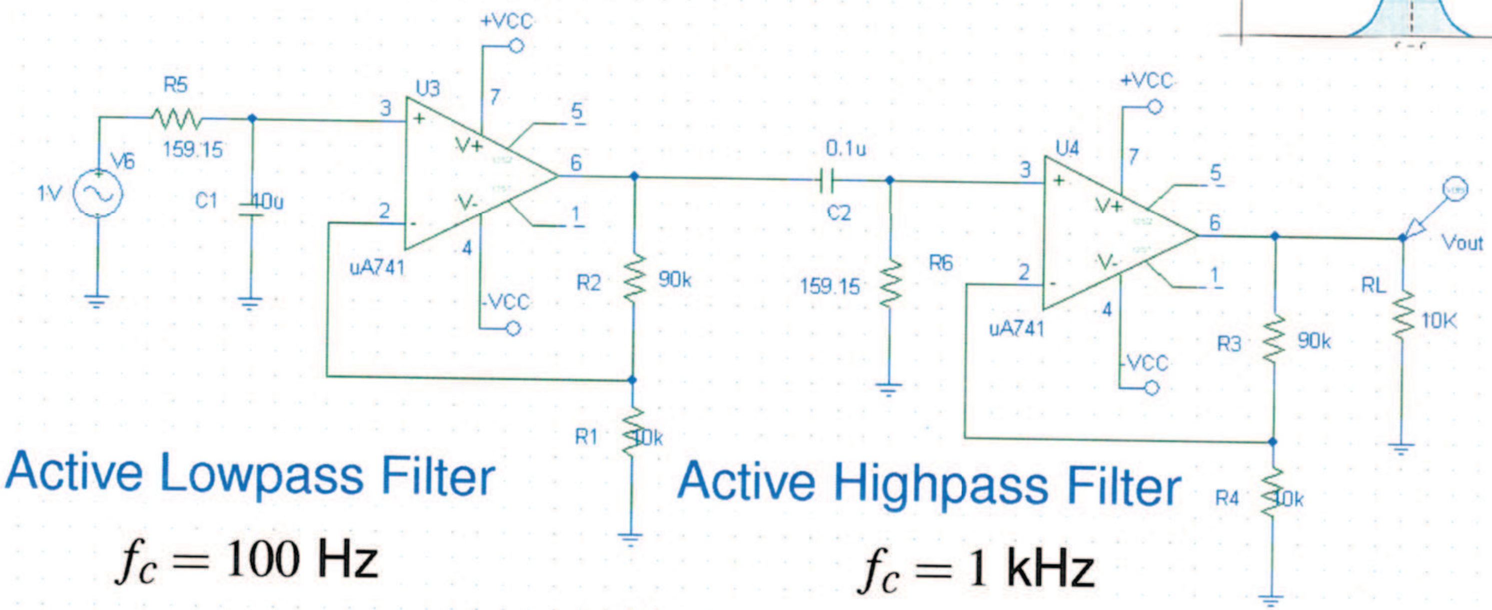
Active Bandpass Filter Circuits

- Cascade lowpass and highpass filter to get bandpass filter.

Active Bandpass Filter

(Passband: 100Hz - 1000Hz)





Active Lowpass Filter

$f_c = 100$ Hz

Active Highpass Filter

$f_c = 1$ kHz

See: `Lec17_bandpass.sch`

Freq Shaping and Filtering – Key Observation

We can choose/design $H(j\omega)$, or $H(e^{j\omega})$, as a function of (radial) frequency ω to determine how frequency components of the input are passed/amplified/attenuated to the output.

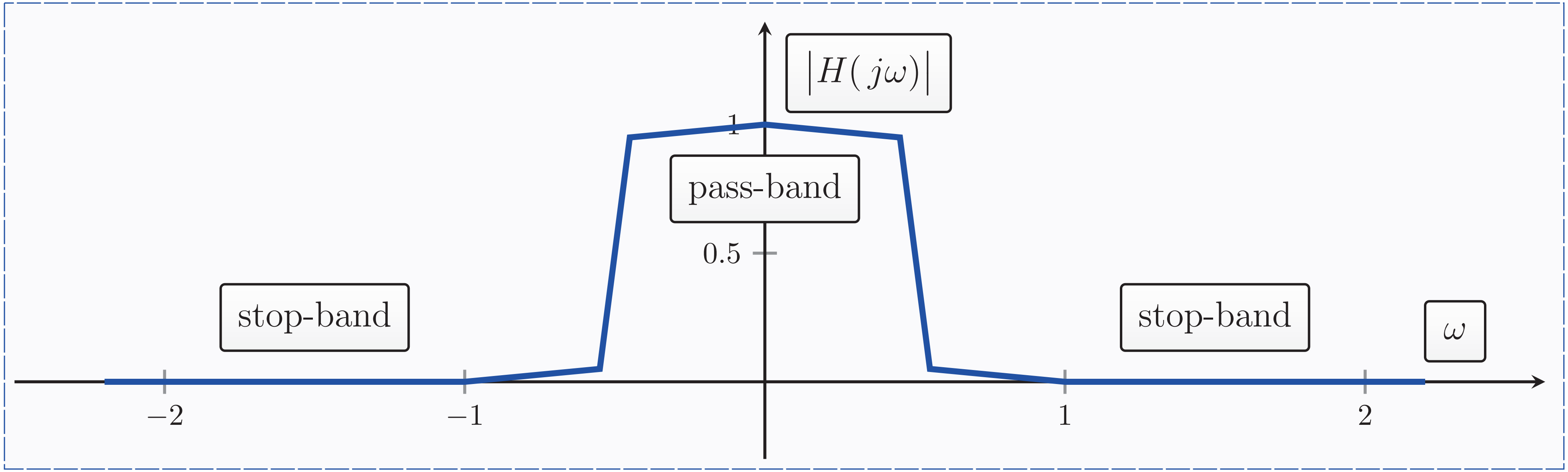
For example, the bass, treble and mid-range control in an audio system:

- To boost bass say at 100 Hz or $\omega = 200\pi$ then we have $\left|H(j200\pi)\right| > 1$
- To attenuate treble say at 1 kHz or $\omega = 2000\pi$ then we have $\left|H(j2000\pi)\right| < 1$

Part 2 Outline

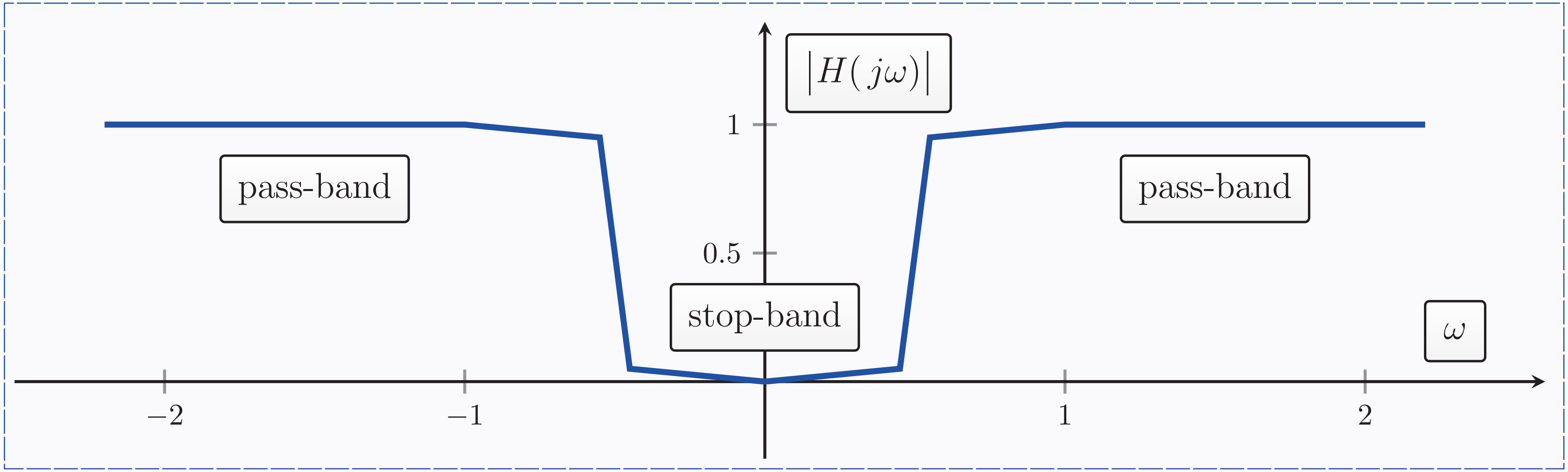
Freq Shaping and Filtering – CT Low Pass Filter

The CT "Low Pass" frequency response looks like:



- Conventionally look at the magnitude, $|H(j\omega)|$, to characterize the type of "filter".
- Defines "passband" (low or no attenuation) and "stopband" (high attenuation).

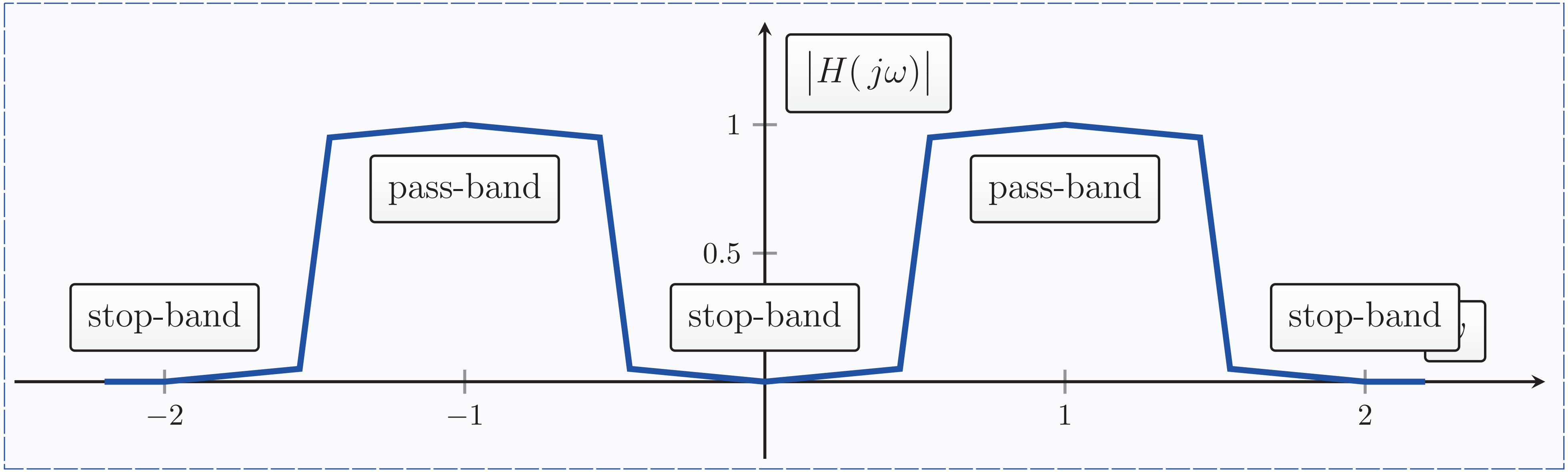
The CT "High Pass" frequency response looks like:



- DC gain should be small or zero.
- High frequencies are passed, low frequencies are blocked. The exact shape isn't so important.

Part 2 Outline

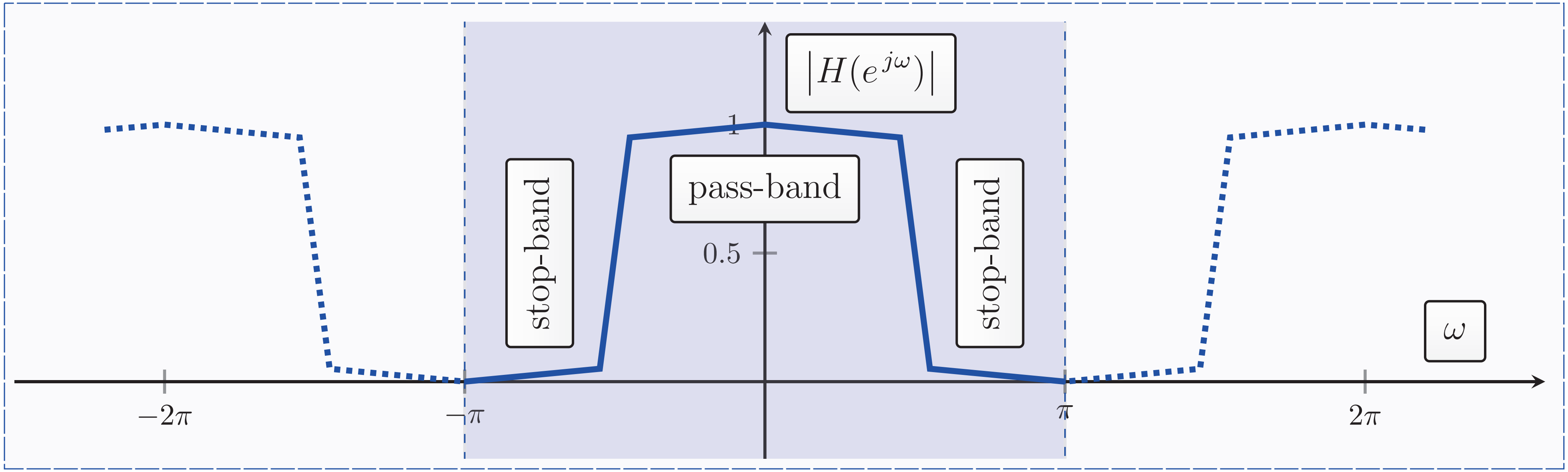
The CT "Band Pass" frequency response looks like:



- DC gain should be small or zero. The high frequency gain should be small or zero.
- High frequencies are blocked, low frequencies are blocked, mid-frequencies are passed. The exact shape isn't so important.

DT Low Pass Filter:

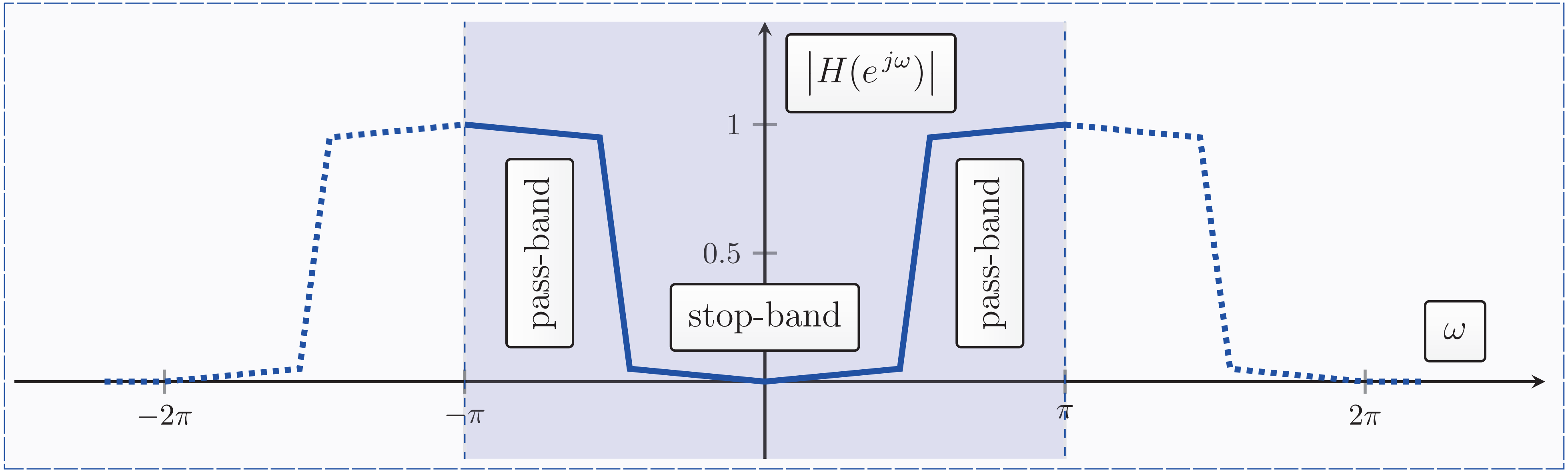
The DT "Low Pass" frequency response looks like:



- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

DT High Pass Filter:

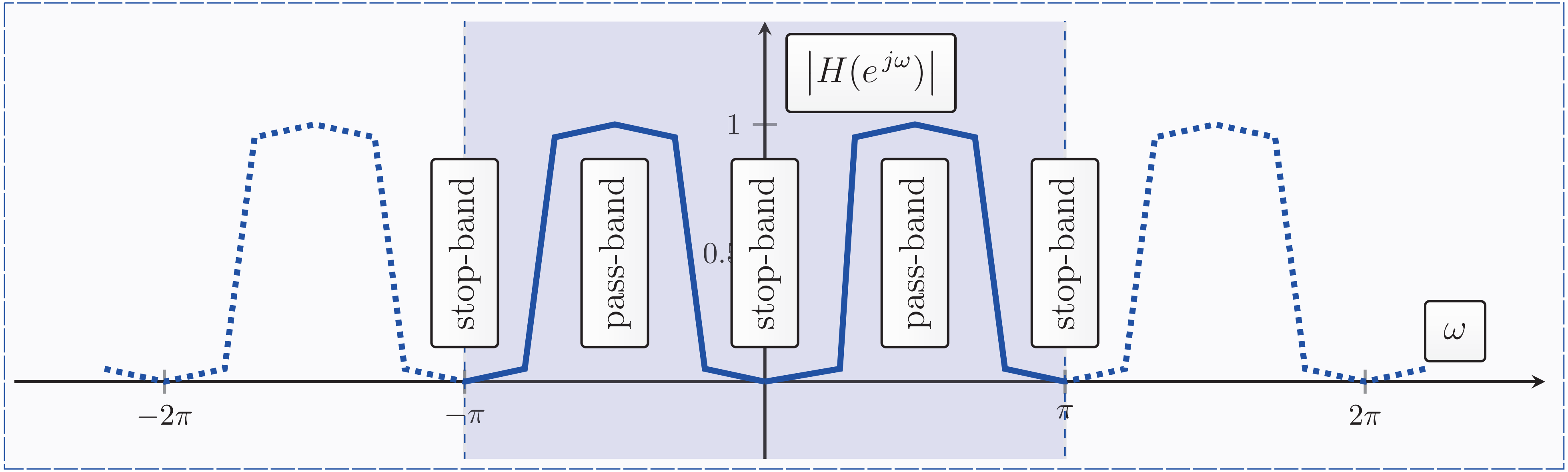
The DT "High Pass" frequency response looks like:



- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

DT Band Pass Filter:

The DT "Band Pass" frequency response looks like:



- Most often look at the magnitude, $|H(e^{j\omega})|$, to characterize the type of "filter".
- The spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π.
- The frequency $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Find the frequency response $H(e^{j\omega})$ for a

Causal moving average of two terms:

$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$

$$\boxed{y[n] = \frac{1}{2}\,x[n] + \frac{1}{2}\,x[n-1]}$$

$h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$= h[0] + h[1] e^{-j\omega} = \frac{1}{2} + \frac{1}{2} e^{-j\omega}$$

$$= \frac{1}{2} e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)$$

$$= e^{-j\omega/2} \cos\left(\frac{\omega}{2}\right)$$

$x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

Alternative approach:

$$\boxed{y[n] = \frac{1}{2}\,x[n] + \frac{1}{2}\,x[n-1]}$$

$$H(e^{j\omega})\cancel{e^{j\omega n}} = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{j\omega(n-1)}$$

$$= \frac{1}{2}\cancel{e^{j\omega n}} + \frac{1}{2}\cancel{e^{j\omega n}}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1}{2} + \frac{1}{2}e^{-j\omega}$$

$$= e^{-j\omega/2}\cos\left(\frac{\omega}{2}\right)$$

Freq Shaping and Filtering – Magnitude and Phase Features

- Going from $H(e^{j\omega})$ to $\left|H(e^{j\omega})\right|$ introduces weirdness in phase such as apparent discontinuities (at frequencies ω where $H(e^{j\omega})$ gets small and passes through the origin in the complex plane).
- This phase flipping by π can seem like an apparent discontinuity in phase, where infact there is none.
- The phase flips can make the phase go beyond $-\pi$ and π.
- Can limit the phase "wind" to $-\pi < \angle H(e^{j\omega}) \leq \pi$.
- The Matlab angle() command gives the correct wrapped phase $-\pi \leq \theta \leq \pi$.
- Magnitude, $\left|H(e^{j\omega})\right|$, is an **even function** of ω for real $h(t)$.
- Phase, $-\pi < \angle H(e^{j\omega}) \leq \pi$, can be made an **odd function** of ω for real $h(t)$
- We tend to define the action of a filter in terms of the magnitude of the frequency response. This can be a little deceptive.

Freq Shaping and Filtering – Summary

Take home messages:

- To filter $x(t)$, we pass it through a physical circuit (e.g. containing R, C, op-amps)

- To filter $x[n]$, we pass it through a suitable difference equation

- We tend to define the action of a filter in terms of the magnitude of the frequency response

- $H(j\omega)$ is not periodic with period 2π (because $e^{j\omega t}$ is distinct for all ω) but $H(e^{j\omega})$ is periodic with period 2π (since $e^{j\omega n}$ is periodic with period 2π)

- Going from $H(e^{j\omega})$ to $|H(e^{j\omega})|$ introduces jumps in phase in $\angle H(e^{j\omega})$

- For $H(e^{j\omega})$, $\omega = 0$ is the DC or low frequency

- For $H(e^{j\omega})$, $\omega = \pm\pi$ is the highest frequency

Freq Shaping and Filtering – Moving Average Filter

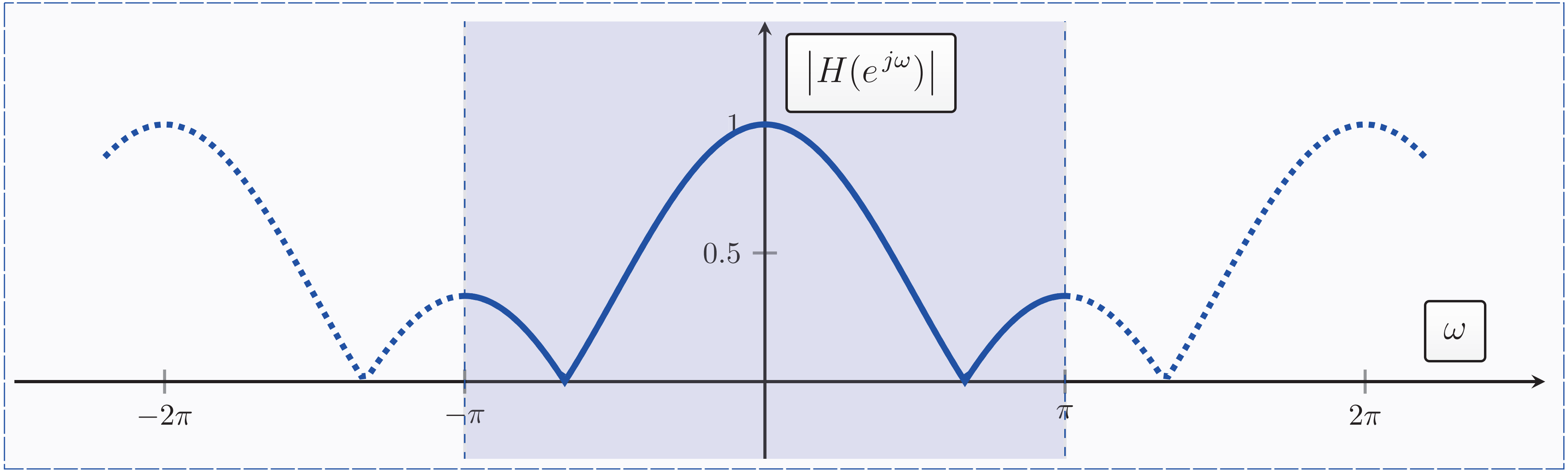
Causal moving average filters of different orders:

Order	$H(e^{j\omega})$	$\left\lvert H(e^{j\omega})\right\rvert$
1	1	1
2	$e^{-j\omega/2}\cos(\omega/2)$	$\left\lvert\cos(\omega/2)\right\rvert$
3	$e^{-j\omega}\left(\frac{1}{3}+\frac{2}{3}\cos(\omega)\right)$	$\left\lvert\frac{1}{3}+\frac{2}{3}\cos(\omega)\right\rvert$
4	$e^{-j3\omega/2}\left(\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right)$	$\left\lvert\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right\rvert$
5	$e^{-j2\omega}\left(\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right)$	$\left\lvert\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right\rvert$

Freq Shaping and Filtering – Moving Average Filter

3 term non-causal moving average:

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$



$$\left|\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)\right|$$

From

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$

we deduce

$$h[n] = \begin{cases} \frac{1}{3}, & n = -1, 0, +1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\omega n} = \frac{1}{3} \sum_{n=-1}^{1} e^{-j\omega n}$$

$$= \tfrac{1}{3} + \tfrac{2}{3}\,\frac{e^{j\omega} + e^{-j\omega}}{2} = \tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)$$

Note that this is purely real. The phase appears to be zero. In reality it flips between 0 and π — it is π at those frequencies ω where $\frac{1}{3} + \frac{2}{3}\cos(\omega) < 0$.

Freq Shaping and Filtering – Moving Average Filter

Alternative method: $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

$$\boxed{y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]}$$

$$H(e^{j\omega})\cancel{e^{j\omega n}} = \frac{1}{3}e^{j\omega(n+1)} + \frac{1}{3}e^{j\omega n} + \frac{1}{3}e^{j\omega(n-1)}$$

$$= \frac{1}{3}\cancel{e^{j\omega n}}e^{j\omega} + \frac{1}{3}\cancel{e^{j\omega n}} + \frac{1}{3}\cancel{e^{j\omega n}}e^{-j\omega}$$

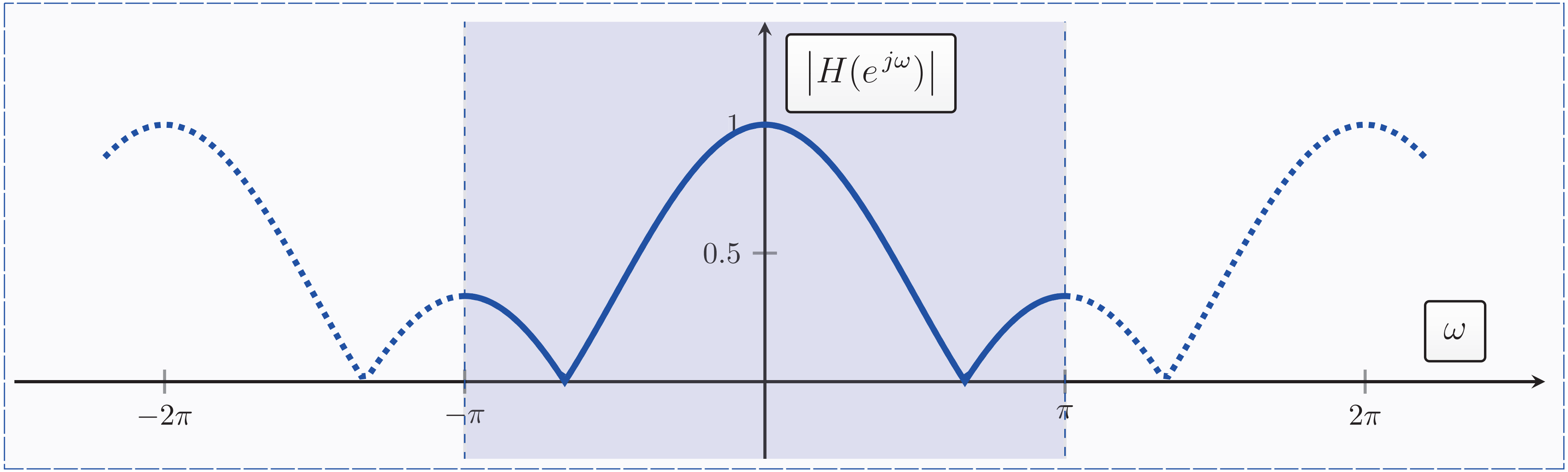
$$H(e^{j\omega}) = \frac{1}{3} + \frac{2}{3}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)$$

$$= \frac{1}{3} + \frac{2}{3}\cos\omega$$

Freq Shaping and Filtering – Moving Average Filter

3 term causal moving average (real-life system):

$$y[n] = \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1] + \tfrac{1}{3}\,x[n-2]$$



$$\left|\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)\right|$$

Find $H(e^{j\omega})$

$n-1=0$
$n=1$

$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$

From

$$y[n] = \frac{1}{3}\,x[n] + \frac{1}{3}\,x[n-1] + \frac{1}{3}\,x[n-2]$$

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} = \frac{1}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}$$

$$= \frac{1}{3}\left(1 + e^{-j\omega} + e^{-j2\omega}\right) = \frac{e^{-j\omega}}{3}\left(e^{j\omega} + 1 + e^{-j\omega}\right) = \frac{e^{-j\omega}}{3}\left(1 + 2\cos\omega\right)$$

Alternative method: $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

$$y[n] = \frac{1}{3}\,x[n-2] + \frac{1}{3}\,x[n] + \frac{1}{3}\,x[n-1]$$

$$H(e^{j\omega})\cancel{e^{j\omega n}} = \frac{1}{3}e^{j\omega(n-2)} + \frac{1}{3}e^{j\omega n} + \frac{1}{3}e^{j\omega(n-1)}$$

$$= \frac{1}{3}\cancel{e^{j\omega n}}e^{-j\omega 2} + \frac{1}{3}\cancel{e^{j\omega n}} + \frac{1}{3}\cancel{e^{j\omega n}}e^{-j\omega}$$

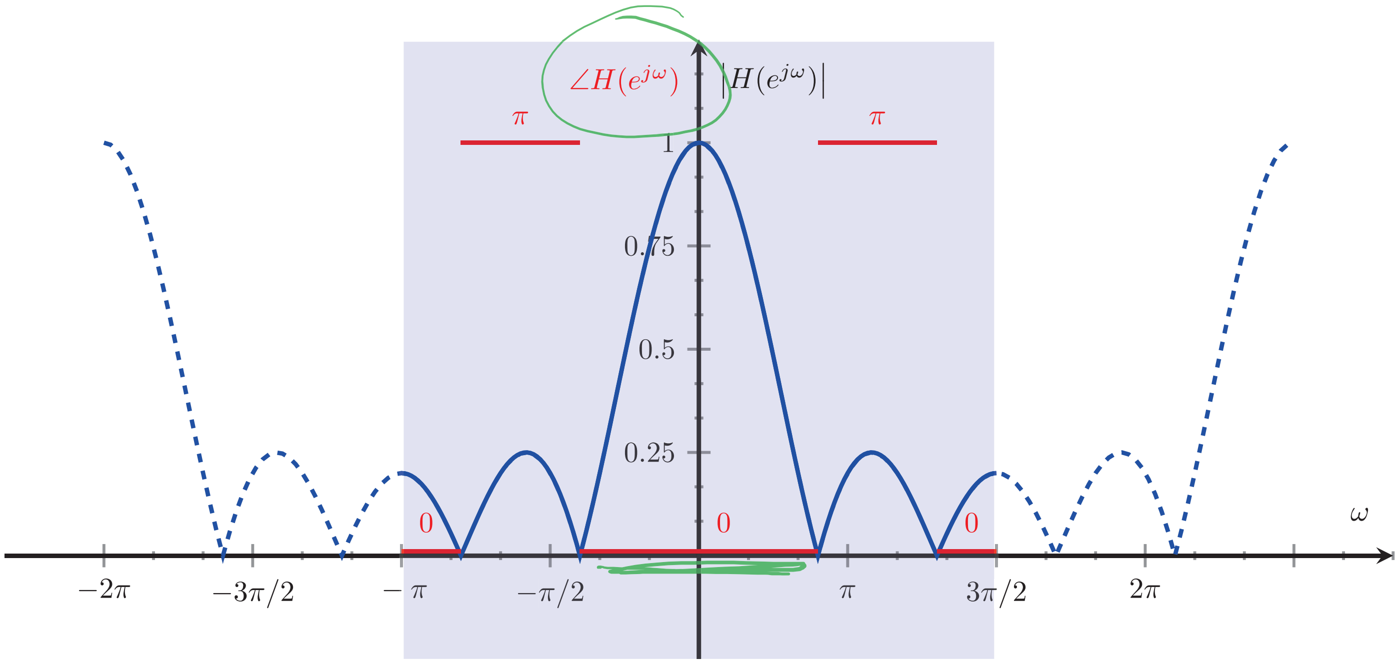
$$= \frac{1}{3}\left(1 + e^{-j\omega} + e^{-j2\omega}\right)$$

$$= \frac{1}{3}e^{-j\omega}\left(e^{j\omega} + 1 + e^{-j\omega}\right) = \frac{1}{3}e^{-j\omega}\left(1 + 2\cos\omega\right)$$

as before

The 5 term non-causal moving average is:

$$y[n] = \tfrac{1}{5}\, x[n+2] + \tfrac{1}{5}\, x[n+1] + \tfrac{1}{5}\, x[n] + \tfrac{1}{5}\, x[n-1] + \tfrac{1}{5}\, x[n-2]$$



This has "flat" phase. This is because of the special form of the filter coefficients and a non-causal formulation.

For this example, the frequency response is:

$$\boxed{H(e^{j\omega}) = \tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega) \in \mathbb{R}}$$

is purely real, but for some values of ω the frequency response takes **negative** real values. For example, when $\omega = \pi/2$ then

$$\begin{aligned} H(e^{\,j\pi/2}) &= \tfrac{1}{5} + \tfrac{2}{5} \times 0 + \tfrac{2}{5} \times (-1) \\ &= -\tfrac{1}{5} \equiv \tfrac{1}{5}\,e^{\,j\pi} \end{aligned}$$

So the magnitude of $H(e^{\,j\pi/2})$ is $\frac{1}{5}$ and the phase is π. This is a little annoying as the phase can show an apparent discontinuity whereas there really is none in the frequency response.

Freq Shaping and Filtering – Moving Average Filter

For this example, the **magnitude** of the frequency response is:

$$\left|H(e^{j\omega})\right| = \left|\frac{1}{5} + \frac{2}{5}\cos(\omega) + \frac{2}{5}\cos(2\omega)\right|$$

and the **phase** of the frequency response is:

$$\angle H(e^{j\omega}) = \begin{cases} 0 & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ \pi & \text{otherwise} \end{cases}$$

as plotted previously.

- We also note that we only need to plot magnitude and phase over one period such as $-\pi < \omega \leq \pi$.
- Of course, we could have taken the negating ω as $-\pi$ instead of π. This choice makes more sense because it lets us see whether the magnitude and phase are even, or odd, or neither even nor odd.
- For a **filter with real coefficients** the **magnitude is an even function** and the **phase is an odd function**.

The **causal** (implementable) 5 term moving average is:

$$y[n] = \tfrac{1}{5}\,x[n] + \tfrac{1}{5}\,x[n-1] + \tfrac{1}{5}\,x[n-2] + \tfrac{1}{5}\,x[n-3] + \tfrac{1}{5}\,x[n-4]$$

and had frequency response

$$H(e^{j\omega}) = e^{-j2\omega}\Big(\tfrac{1}{5} + \tfrac{2}{5}\cos(\omega) + \tfrac{2}{5}\cos(2\omega)\Big), \qquad \pi < \omega \leq \pi$$

Freq Shaping and Filtering – Moving Average Filter

The magnitude is as for the non-causal filter:

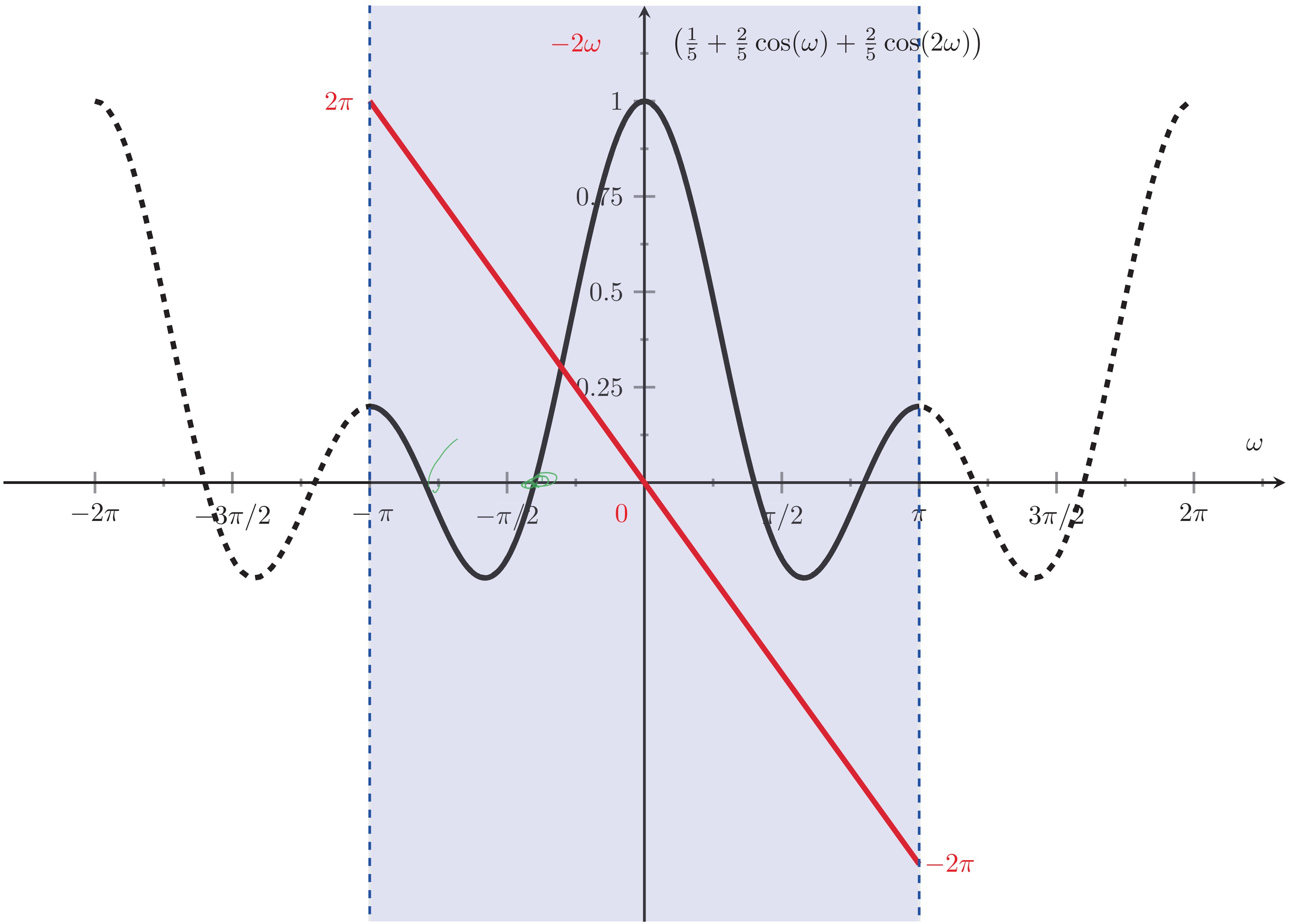
$$\left|H(e^{j\omega})\right| = \left|\tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega)\right|$$

but the phase is linear (apart from the π discontinuities)

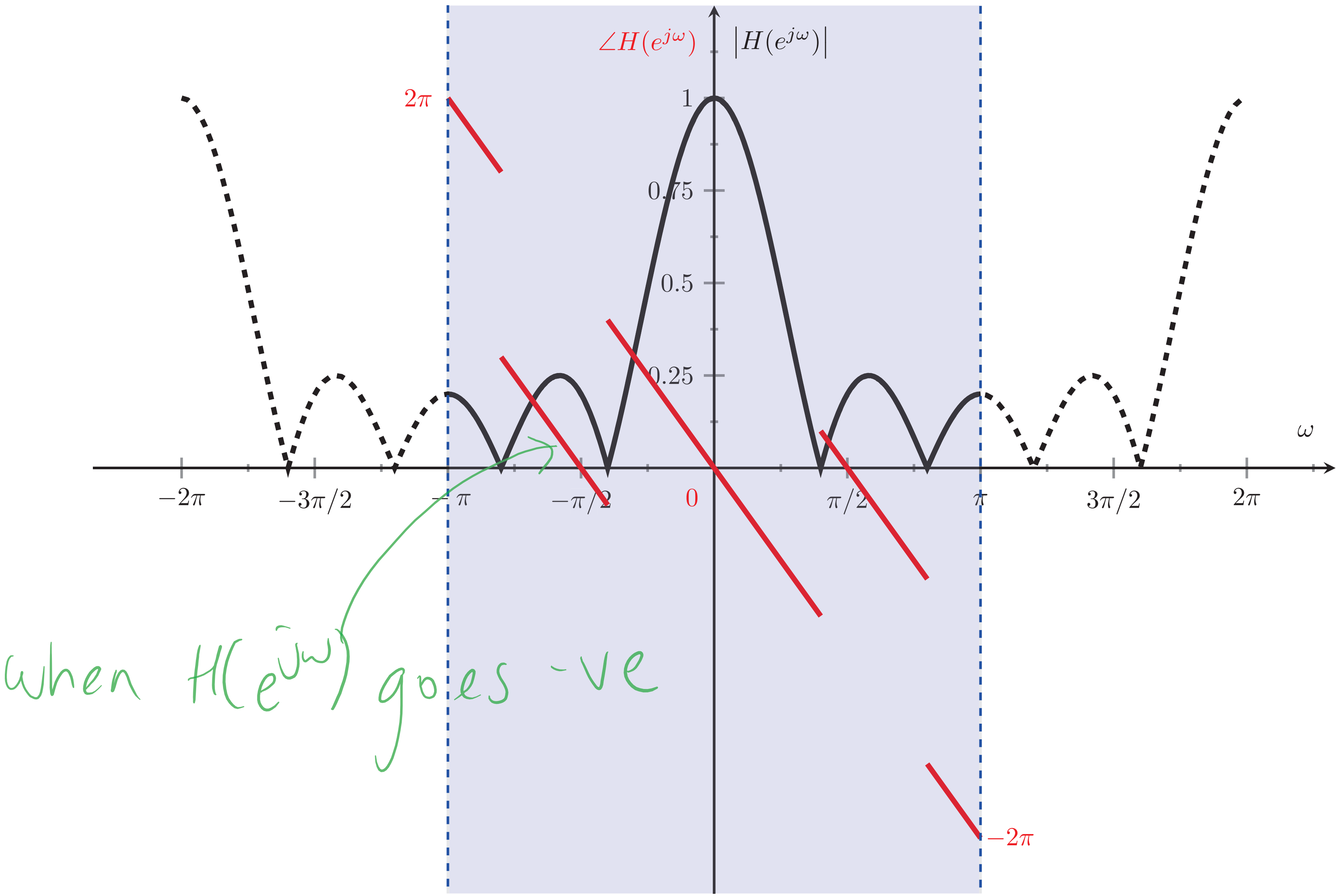
$$\angle H(e^{j\omega}) = \begin{cases} -2\omega & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ -2\omega \pm \pi & \text{otherwise} \end{cases}$$

Here one conventionally takes the sign of π such that $-\pi < -2\omega \pm \pi \leq \pi$.

Freq Shaping and Filtering – Moving Average Filter

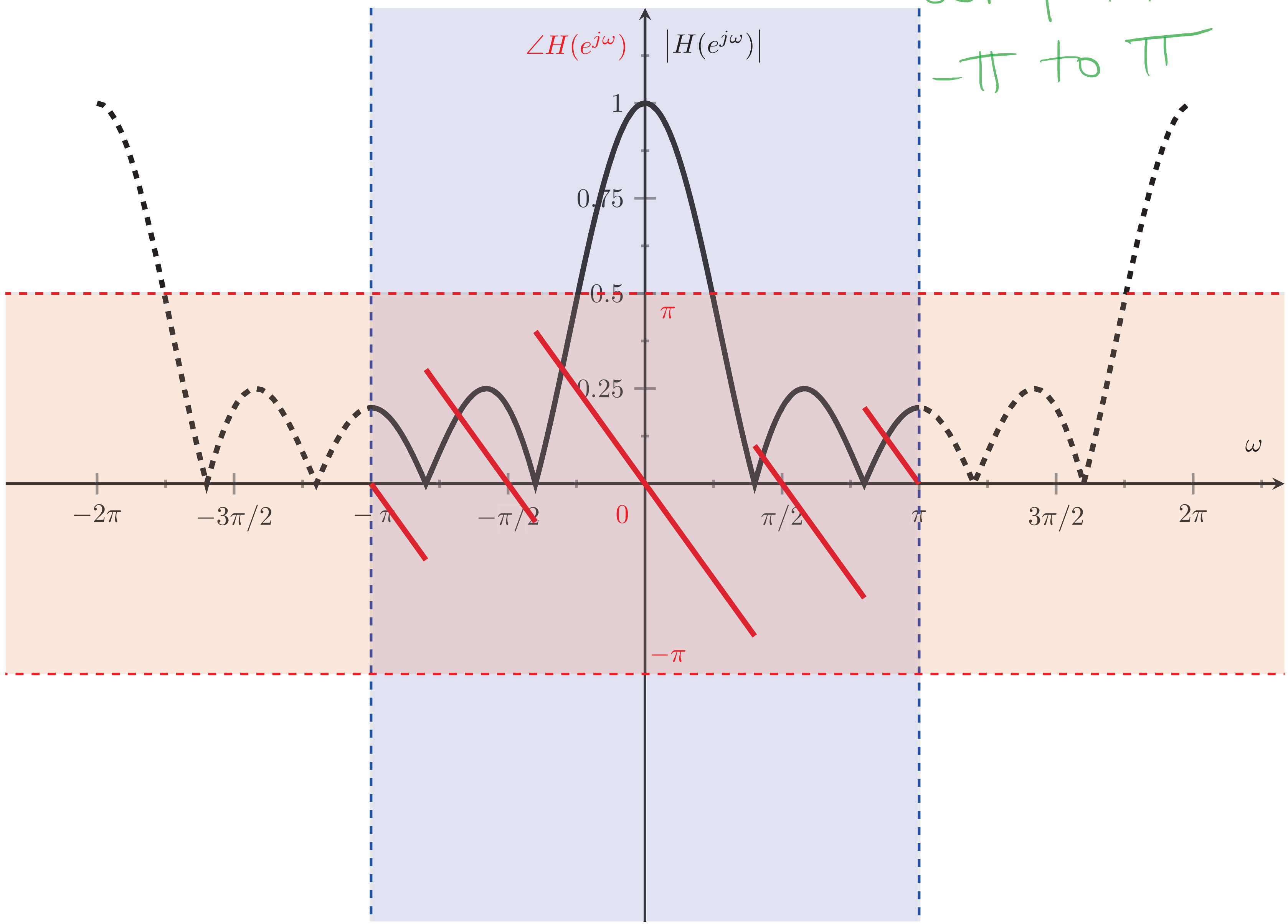


Freq Shaping and Filtering – Moving Average Filter



Freq Shaping and Filtering – Moving Average Filter

wrap into range
$-\pi$ to π



Freq Shaping and Filtering – Moving Average Filter

Summary:

- Phase is linear for causal filters (disregarding any discontinuities)
- Phase is flat for non-causal filters (disregarding the jumps between 0 and π)
- Phase is π for non-causal MA filter at frequencies where $H(e^{j\omega}) < 0$

Discontinuities due to:

- $H(e^{j\omega})$ going from positive to negative or vice versa
  - Causes jumps in phase of size π
- The convention of taking the phase $-\pi < \angle H(e^{j\omega}) \leq \pi$

Part 2 Outline

Find frequency response $H(e^{j\omega})$ for a

High pass filter (edge detector):

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\boxed{y[n] = \tfrac{1}{2}(x[n] - x[n-1])}$$

$$h[n] = \frac{1}{2}(\delta[n] - \delta[n-1])$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = h[0] + h[1]e^{-j\omega n}$$

$$= \frac{1}{2} - \frac{1}{2}e^{-j\omega} = j\frac{e^{-j\omega/2}}{2j}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= je^{-j\omega/2}\sin\left(\frac{\omega}{2}\right)$$

Freq Shaping and Filtering – Other Types of Filters

High pass filter (edge detector):

$$x[n] = e^{j\omega n}$$

$$\boxed{y[n] = \frac{1}{2}(x[n] - x[n-1])}$$

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

Alternative approach:

$$H(e^{j\omega})\cancel{e^{j\omega n}} = \frac{1}{2}\left(e^{j\omega n} - e^{j\omega(n-1)}\right)$$

$$= \frac{1}{2}\cancel{e^{j\omega n}} - \frac{1}{2}\cancel{e^{j\omega n}}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1}{2} - \frac{1}{2}e^{-j\omega}$$

$$= je^{-j\omega/2}\sin\left(\frac{\omega}{2}\right)$$

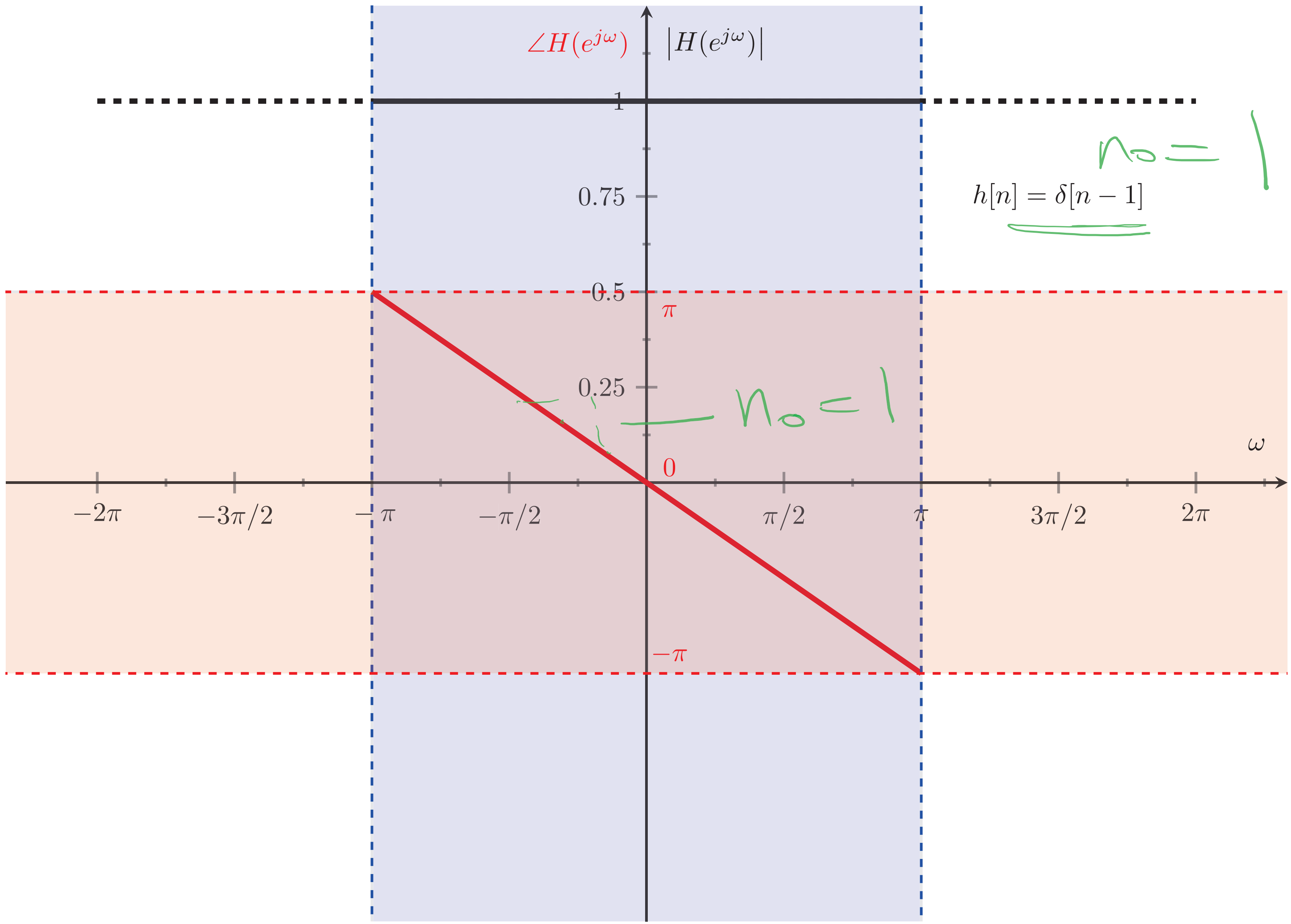
Part 2 Slide 102/many Convenor: R. A. Kennedy

An all-pass filter satisfies

$$\boxed{\left|H(e^{j\omega})\right| = 1, \quad \text{for all } \omega}$$

- The trivial filter $h[n] = \delta[n]$ with frequency response $H(e^{j\omega}) = 1$ is **all-pass**.
- A delay filter $h[n] = \delta[n - n_0]$ $(n_0 \in \mathbb{Z})$ with frequency response $H(e^{j\omega}) = e^{-jn_0\omega}$ is **all-pass**.
- All frequencies at the input are passed to the output with no change in amplitude/magnitude. But the phase can be modified.
- Not every all-pass filter is a delay filter.
- A delay filter has (negative) linear phase. The phase is a straight line with slope proportional to n_0. The delay is encoded in the slope.

Freq Shaping and Filtering – Other Types of Filters



$h[n] = \delta[n-1]$

$n_0 = 1$

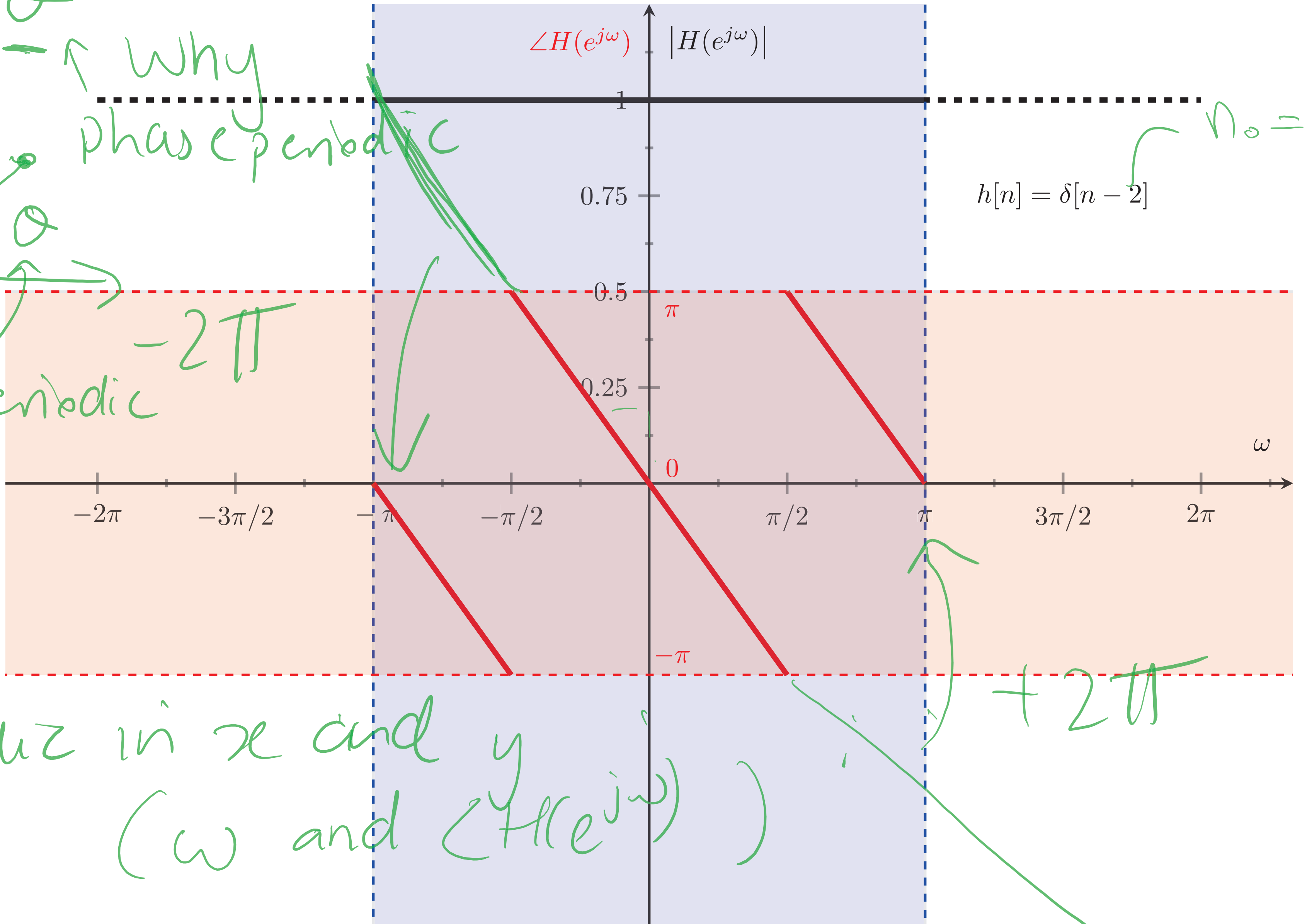
Freq Shaping and Filtering – Other Types of Filters

$re^{j\theta}$ ← why

phase periodic

θ periodic 2π

-2π



$n_0 = 2$

$+2\pi$

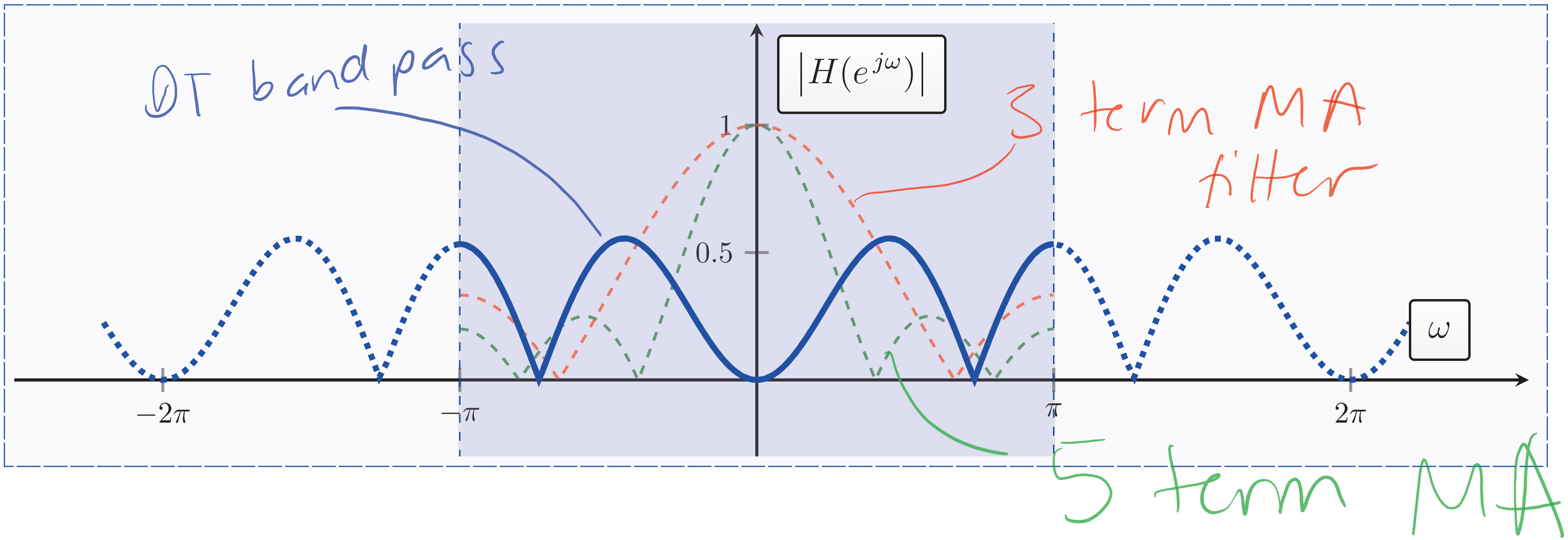
periodic in x and y

(ω and $\angle H(e^{j\omega})$)

Freq Shaping and Filtering – Other Types of Filters

DT Band Pass Filter example (3 and 5 term non-causal moving average difference):

$$
\begin{aligned}
y[n] = \tfrac{1}{3}\,x[n-1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n+1] \\
- \tfrac{1}{5}\,x[n-2] - \tfrac{1}{5}\,x[n-1] - \tfrac{1}{5}\,x[n] - \tfrac{1}{5}\,x[n+1] - \tfrac{1}{5}\,x[n+2]
\end{aligned}
$$



Filter summary

$x[n]$ → **DT filter** $h[n]$ or $H(e^{j\omega})$ → $y[n] = \ldots$

complex number

difference equation

How filter classified:

- LP
- HP
- BP
- Notch

→ $|H(e^{j\omega})|$

$\angle H(e^{j\omega})$

$y[n] = x[n] * h[n]$

$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

will come to later

Part 3 Outline

5 **CT Non-Periodic Signals**
- Up to this Point

6 **Fourier Transforms**
- Fourier Analysis and Synthesis
- Examples
- Ideal Low Pass Filter

Part 3 Outline

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CT Non-Periodic Signals – Up to this Point

$x(t)$ or $x[n]$

Time-domain properties	Periodic	Non-periodic	
Continuous	Fourier series (FS)	Fourier transform (FT)	Non-periodic
Discrete	Discrete-time Fourier series (DTFS)	Discrete-time Fourier transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

a_k or $X(j\omega)$ or $X(e^{j\omega})$

Time-domain Property	Frequency-domain Property
continuous	non-periodic
discrete	periodic
periodic	discrete
non-periodic	continuous

CT Non-Periodic Signals – Up to this Point

For Example:

- $x[n] = \cos(\frac{n}{6})$ - DTFT (discrete and non-periodic)
- $x(t) = \cos(2\pi t)$ - FS (continuous and periodic)
- $x(t) = u(t)$ - FT (continuous and non-periodic)
- $x[n] = \cos(\frac{\pi n}{6})$ - DTFS (discrete and periodic)

CT Non-Periodic Signals – Up to this Point

In the end, **Fourier series** only describe/represent **periodic** time domain signals (continuous or discrete).

- But not all signals are periodic (most aren't).
- Can we generalize the Fourier series for non-periodic time domain signals?
- Leads to the Fourier transform. It's not too much different from Fourier series.

Part 3 Outline

5. **CT Non-Periodic Signals**
   - Up to this Point

6. **Fourier Transforms**
   - Fourier Analysis and Synthesis
   - Examples
   - Ideal Low Pass Filter

Part 3 Outline

5 CT Non-Periodic Signals
- Up to this Point

6 Fourier Transforms
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Fourier Transforms – Fourier Analysis and Synthesis

Definition (Fourier Analysis and Synthesis)

Provided the following integrals are finite/exist

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

- $X(j\omega)$ is called the (frequency) spectrum
- $|X(j\omega)|$ is the magnitude spectrum
- $\angle X(j\omega)$ is the phase spectrum

inverse FT

Definition (Fourier Transform Pairs)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

Fourier transform (FT)

Definition (Fourier Series Pairs)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

Fourier transform is valid for periodic and non-periodic signals. Fourier series is only valid for periodic signals.

Comments:

- Both time t and frequency ω are continuous. Going from periodic to non-periodic means frequency goes from discrete to continuous (as a rule).
- Synthesis and analysis equations are virtually identical in form. There is a -1 in one exponent, t and ω are interchanged and there is a leading constant term.
- Also called the Fourier transform and inverse Fourier transform.

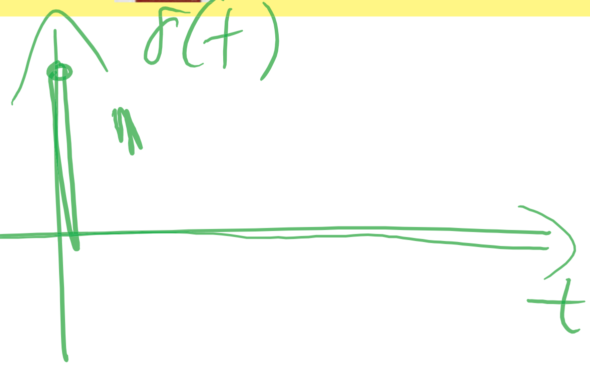
Fourier Transforms – Fourier Analysis and Synthesis

Key Questions:

- Fourier transform: what is the frequency content of an “arbitrary” time domain signal.
- Inverse Fourier transform: what time domain signal corresponds to a given frequency domain description.

5 CT Non-Periodic Signals
- Up to this Point

6 **Fourier Transforms**
- Fourier Analysis and Synthesis
- **Examples**
- Ideal Low Pass Filter



Transform Pair 10: Let $x(t) = \delta(t)$ then

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt}$$

$$= 1, \quad \text{for all } \omega$$

sifting property

This is the constant function equal to 1 for all ω.

That is, we have the synthesis:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

Could say that $\delta(t)$ is unusual in that it contains equal portions of all frequencies (phase aligned).

Fourier Transforms – Examples

Recap:

$$\delta(t) \xleftrightarrow{\mathscr{F}} 1$$

$$\mathscr{F}\{\delta(t)\} = 1$$

$$\mathscr{F}^{-1}\{1\} = \delta(t)$$

- Frequency domain function 1 is the *Fourier Transform* of $\delta(t)$
- Time domain function $\delta(t)$ is the *Inverse Fourier Transform* of 1

Fourier Transforms – Examples

Transform Pair 12: Let $x(t) = \delta(t - t_0)$, which as an impulse response gives an LTI system acting as a delay of t_0, then

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$
$$= e^{-j\omega t_0}$$

This is a linear phase, with slope $-t_0$. Of course, the previous example is the special case $t_0 = 0$, no delay (as an impulse response, the trivial do-nothing system).

That is, we have the synthesis:

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t - t_0)}\, d\omega$$

Fourier Transforms – Examples

Recap:

$$\delta(t-t_0) \stackrel{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0}$$

$$\mathscr{F}\left\{\delta(t-t_0)\right\} = e^{-j\omega t_0}$$

$$\mathscr{F}^{-1}\left\{e^{-j\omega t_0}\right\} = \delta(t-t_0)$$

- Frequency domain function $e^{-j\omega t_0}$ is the *Fourier Transform* of $\delta(t-t_0)$
- Time domain function $\delta(t-t_0)$ is the *Inverse Fourier Transform* of $e^{-j\omega t_0}$

Fourier Transforms – Digression

Generally with Fourier Transforms:

- Well known and useful signals are tabulated. No need to compute the Fourier transform (except in introductory courses like this).
- Can use superposition to combine Fourier transforms of different component signals. Why? The Fourier transform is a **linear** operator (functions to functions) since integration is linear.
- Also want to be able to stretch and shrink, multiply and other operations to be able to use the raw tabulated Fourier transforms. We'll consider these later.

Fourier Transforms – Digression

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier Transforms – Digression

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\}=\mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\}=-\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$