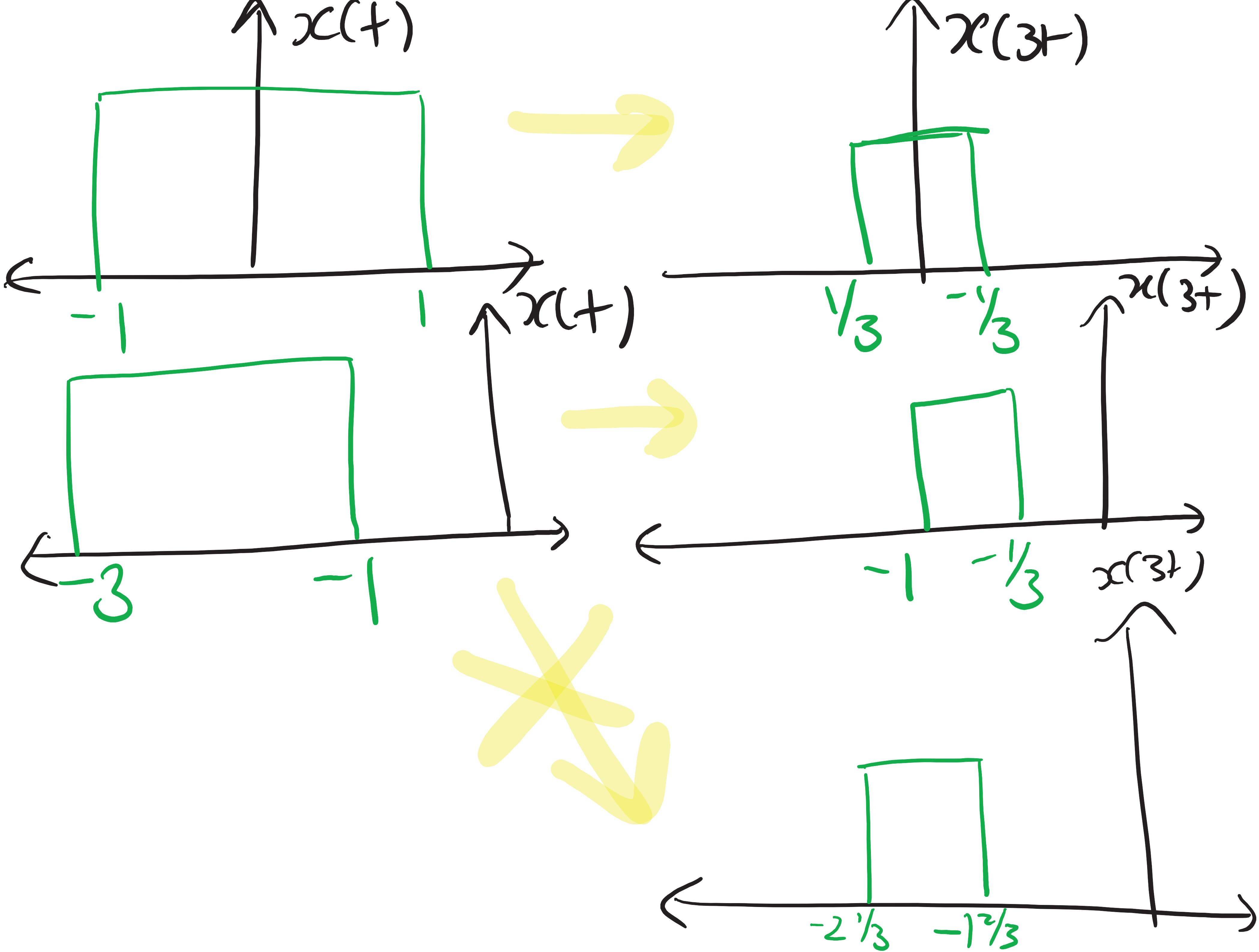
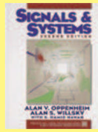

x(t)
-1
1
x(3t)
1/3
-1/3
x(t)
-3
-1
x(3t)
-1
-1/3
x(3t)
-2 1/3
-1 2/3

Part 3 Outline

System Properties



Signals & Systems
section 1.6
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Why study system properties?

- important practical / physical implications
- system properties imply structure that we can exploit to analyse and understand systems more deeply

Part 3 Outline

System Properties – Causality

not future

Definition (Causality)

A system is **causal** if the output at any time depends on values of the input at only the present and past times.

- All real time-based physical systems are causal. Time flows in one direction. Effect occurs after cause.
- Non-causal systems are the play thing of science fiction. (Don't murder any of your ancestors.)
- Causality relates to time. For other independent variables, like space, there need not be such a constraint. We can approach a point in space from any direction in general without pondering the consequences of strangling an unsuspecting ancestor.

System Properties – Causality

Terminology: causal, non-causal, anti-causal and strictly causal

- "Non-causal" means there is some output that anticipates the input for some input. For other input-output combinations the system may appear causal. (The set of numbers $\{0, -3, 7, 3, 4, 2\}$ is not positive, since at least one and not all elements are negative.)
- "Strictly causal" means the output depends on the past but not the present nor future. For example, $y[n]$ can be a function of $x[n-1]$, $x[n-2]$, ...but not a function of $x[n]$ nor $x[n+1]$, $x[n+2]$, ...
- "Anti-causal" systems always violate causality (output depends only on the future of the input). They are a type of time reversal of a strictly causal system.

System Properties – Causality

Examples: Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = \big(x(\underline{t-1})\big)^2$$

  is causal, e.g., $y(10)$ depends on $x(9)$, $y(t)$ depends strictly on past $x(t)$.
- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = x(t+1)$$

  is non-causal, e.g., $y(13) = x(14)$, $y(t)$ depends on strictly future $x(t)$.
- Note a CT system is non-causal even if it is only non-causal at one time instant.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = x[-n]$$

  is non-causal, e.g., $y[-5] = x[5]$ (but not anti-causal, as $y[5] = x[-5]$). $y[n]$ is the time-reversal of $x[n]$.

System Properties – Causality

Examples (cont'd): Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = x(-t)$$

  is non-causal. That is, the system that time reverses an input signal is a non-causal system.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$\boxed{y[n] = \left(\frac{1}{2}\right)^{n+1} \left(x[n-1]\right)^3}$$

is causal. The weighting $(1/2)^{n+1}$ is decaying with time n increasing but this is independent of signal $x[n]$.

System Properties – Causality

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[-\infty] + x[-\infty+1] + \dots + x[n]$$

is ...

Part 3 Outline

System Properties – Memory

Definition (Memory)

A system is said to be memoryless if its output for each value of t or n at a given time is dependent on input at only the same time.

For example:

- $v(t) = Ri(t)$ - memoryless (resistor is memoryless)
- $y[n] = (2x[n] - x^2[n])$ - memoryless
- $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$ - memory (capacitor has memory)
- $y[n] = x[n] + \underline{y[n-1]}$ - memory
- $y[n] = \frac{1}{3}\left(\underline{x[n+1]} + x[n] + \underline{x[n-1]}\right)$ - memory

A system is said to possess memory if its output signal depends on past or future values of the input signal.
All memoryless systems are causal, vice versa is not true.

Part 3 Outline

System Properties – Time-Invariance (cont'd)

Definition (CT System Time-Invariance)

A CT system is **time-invariant** if

$$x(t) \longrightarrow y(t)$$

then

$$\boxed{x(t - t_0) \longrightarrow y(t - t_0)}$$

for all $t_0 \in \mathbb{R}$.

- Time-Invariance means "doesn't change with time". It is a property of a system and not of the signals input and output (which are obviously functions of time). It means that if a caveman put a signal through a TI system then the output would be the same as the same signal today.
- Only a system can be time-invariant. It is senseless to say a signal is time-invariant.

Definition (DT System Time-Invariance)

A DT system is **time-invariant** if

$$x[n] \longrightarrow y[n]$$

then

$$\boxed{x[n-n_0] \longrightarrow y[n-n_0]}$$

for all $n_0 \in \mathbb{Z}$.

System Properties – Time-Invariance (cont'd)

Examples:

- The CT system $x(t) \longrightarrow y(t)$ described by

  $x(t-t_0) \rightarrow$

  $$y(t) = \left(x(t+1)\right)^2$$

  $y(t-t_0)$

  is time-invariant (TI).

- The DT system $x[n] \longrightarrow y[n]$ described by

  $$y[n] = \left(\frac{1}{2}\right)^{n+1} \left(x[n-1]\right)^3$$

  is not time-invariant.

Not time-invariant is preferably called time-varying (don't use the expression "time variant").

Examples:

$$y(t) = \left(x(t+1)\right)^2$$

$x(t-t_0) \to y(t-t_0)$

1. $y(t-t_0) = \left(x(t-t_0+1)\right)^2$

2. $x_1(t) = x(t-t_0) \to$

$y_1(t) = \left(x_1(t+1)\right)^2 = \left(x(t+1-t_0)\right)^2$

$= y(t-t_0) \Rightarrow$ time-invariant

System Properties – Time-Invariance (cont'd)

Examples:

$$\boxed{y[n] = \left(\frac{1}{2}\right)^{n+1} \left(x[n-1]\right)^3}$$

$x[n-n_0] \rightarrow y[n-n_0]$

1. $y[n-n_0] = \left(\frac{1}{2}\right)^{n-n_0+1} \left(x(n-n_0-1)\right)^3$

2. $x_1[n] = x[n-n_0]$

$\rightarrow y_1[n] = \left(\frac{1}{2}\right)^{n+1} \left(x_1[n-1]\right)^3 =$

$\left(\frac{1}{2}\right)^{\underline{n+1}} \left(x[n-1-n_0]\right)^3 \neq y[n-n_0].$

$\Rightarrow$ time varying

Summary of steps for prooving time-invariance

$$\boxed{x(t-t_0) \rightarrow y(t-t_0)}$$

1. $y(t-t_0) = \ldots$

2. $x_1(t) = \underline{\underline{x(t-t_0)}} \rightarrow$

$y_1(t) =$ put $x_1(t)$ into system equation

$= y(t-t_0) \Rightarrow$ time-invariant

$\neq y(t-t_0) \Rightarrow$ time-varying

Part 3 Outline

System Properties – Linear & Nonlinear

- Many, some say most, systems are **nonlinear**. For example, diodes, car dynamics, etc.
- In this course we focus of **linear** systems.
- Don't confuse nonlinear with time-varying linear, e.g. $2x + 3$ is a linear equation but system non-linear.
- Linear models are a very important class of models because:
  - they are mathematically tractable
  - they can model small signal variations in nonlinear systems
  - they model accurately circuit elements such as resistors, capacitors, etc.
  - they can provide insights into the behaviour of more complex nonlinear systems

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t)$$

then

$$\boxed{\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)}$$

for complex scalars α_1 and α_2.

Definition (Nonlinear System)

A **nonlinear** system is a system which is not linear.

An equivalent definition:

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_k(t) \longrightarrow y_k(t)$$

then

$$\boxed{\sum_k \alpha_k x_k(t) \longrightarrow \sum_k \alpha_k y_k(t)}$$

for complex scalars α_k.

Definition (Linear System)

A DT system is **linear** if superposition holds. If

$$x_1[n] \longrightarrow y_1[n] \text{ and } x_2[n] \longrightarrow y_2[n]$$

then

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

for complex scalars α_1 and α_2.

Definition (Linear System)

A DT system is **linear** if superposition holds. If

$$x_k[n] \longrightarrow y_k[n]$$

then

$$\sum_k \alpha_k x_k[n] \longrightarrow \sum_k \alpha_k y_k[n]$$

for complex scalars α_k.

- For linear systems, zero input gives zero output.

System Properties – Examples

1. $y(t-t_0) = (x(t-t_0))^2$

2. $x_1(t) = x(t-t_0) \to y_1(t) = (x_1(t))^2$

$= (x(t-t_0))^2 = y(t-t_0) \Rightarrow$ time-invarian

causal/non-causal, linear/nonlinear, time-invariant/time varying:

$\boxed{y(t) = \big(x(t)\big)^2 = x^2(t)}$ is a square law and as a system is:

- Time-invariant, do proof
- Causal and memoryless (current output depends only on current input)
- Nonlinear (it is quadratic), do proof

$$y(t) = (x(t))^2 = x^2(t)$$

$$\underbrace{\alpha_1 x_1(t) + \alpha_2 x_2(t)} \rightarrow \underbrace{\alpha_1 y_1(t) + \alpha_2 y_2(t)}$$

1. $x_1(t) \rightarrow y_1(t) = (x_1(t))^2$

   $x_2(t) \rightarrow y_2(t) = (x_2(t))^2$

2. $\alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 (x_1(t))^2 + \alpha_2 (x_2(t))^2$

---

3. $x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

$$\rightarrow y_3(t) = (x_3(t))^2$$

$$= (\alpha_1 x_1(t) + \alpha_2 x_2(t))^2 = \alpha_1^2 x_1^2(t) +$$

$$2\alpha_1\alpha_2 x_1(t) x_2(t) + \alpha_2^2 x_2^2(t) \neq \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

$\Rightarrow$ non-linear

causal/non-causal, linear/nonlinear, time-invariant/time varying:

$\boxed{y(t) = x(2t)}$ is a compression in time and as a system is:

- Non-causal, since for $t > 0$ we have $2t > t$, for example, at time $t = 3$ we have $y(3) = x(6)$ which is a time advance of 3. Note for $t < 0$ we have $2t < t$, for example, at time $t = -3$ we have $y(-3) = x(-6)$ which is a delay of 3 (that is, it acts causally at time $t = -3$).
- Linear, do proof.
- Time-varying, do proof.

$y(t) = x(2t)$ $\quad \alpha_1 x_1(t) + \alpha_2 x_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$

1. $x_1(t) \to y_1(t) = x_1(2t)$

$x_2(t) \to y_2(t) = x_2(2t)$

2. $\alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(2t) + \alpha_2 x_2(2t)$

3. $x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

$\to y_3(t) = x_3(2t) = \alpha_1 x_1(2t) + \alpha_2 x_2(2t)$

$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$

$\Rightarrow$ systm linear

Summary of steps for prooving linearity

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

1. $x_1(t) \rightarrow y_1(t) = \ldots$

   $x_2(t) \rightarrow y_2(t) = \ldots$

2. $\alpha_1 y_1(t) + \alpha_2 y_2(t) = \ldots$

3. $x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

   $\rightarrow y_3(t) = \ldots$

   $= \alpha_1 y_1(t) + \alpha_2 y_2(t) \Rightarrow$ linear

   $\neq$ " " $\Rightarrow$ non-linear

System Properties – Examples

causal/non-causal, linear/nonlinear, time-invariant/time varying:
$y[n] = x[n+1] - x[n-1]$ as a system is:

- Non-causal because uses future input $x[n+1]$.
- Linear, do proof.
- Time-invariant, do proof.

$y[n] = x[n+1] - x[n-1]$

$\alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$

1. $x_1[n] \rightarrow y_1[n] = x_1[n+1] - x_1[n-1]$

   $x_2[n] \rightarrow y_2[n] = x_2[n+1] - x_2[n-1]$

2. $\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1 (x_1[n+1] - x_1[n-1])$
$+ \alpha_2 (x_2[n+1] - x_2[n-1])$

3. $x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$\rightarrow y_3[n] = x_3[n+1] - x_3[n-1]$

$= \alpha_1 x_1[n+1] + \alpha_2 x_2[n+1] - (\alpha_1 x_1[n-1] +$
$\alpha_2 x_2[n-1]) = \alpha_1 (x_1[n+1] - x_1[n-1]) +$
$\alpha_2 (x_2[n+1] - x_2[n-1]) = \alpha_1 y_1[n] + \alpha_2 y_2[n]$

$\Rightarrow$ linear

System Properties – Examples

do proof for

System	Linear	Time-Invariant	Causal	Memoryless
$y[n] = 2x[n]$	✓	✓	✓	✓
$y[n] = 2x[n] + 3$	✗	✓	✓	✓
$y[n] = x[-n]$	✓	✗	✗	✗
$y(t) = tx(t)$	✓	✗	✓	✓
$y(t) = \cos(3t)x(t)$	✓	✗	✓	✓
$y(t) = \sin\left(x(t)\right)$	✗	✓	✓	✓
$y(t) = t^2x(t-1)$	✓	✗	✓	✗

Make sure you can do all these

System Properties – Linear & Nonlinear

Are all these combinations possible?

- Linear, time-invariant and causal?
- Linear, time-invariant and non-causal?
- Linear, time-varying and causal?
- Linear, time-varying and non-causal?
- Nonlinear, time-invariant and causal?
- Nonlinear, time-invariant and non-causal?
- Nonlinear, time-varying and causal?
- Nonlinear, time-varying and non-causal?

Yes, all combinations are possible.

Homework Problem: generate system examples for each of the 8 cases above.