Signal Processing
ENGN2228

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Second Semester

CT Signals and Systems – Preamble

- By and large, features and properties of CT signals and systems look very similar or are the same as their DT counterparts. The terminology is largely identical.
- When a CT System is LTI then it can be characterized in terms of an impulse response which itself is in the form of a signal. That is, knowing the impulse response signal of a CT LTI System completely characterizes it (apart from initial conditions).

Part 6 Outline

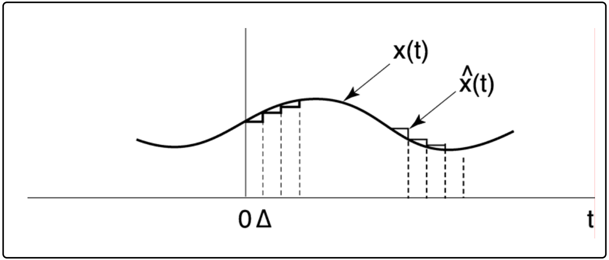
24 CT Signals and Systems

- Preamble
- Representation
- Response of a CT LTI System

25 CT System Properties

- Commutativity
- Sifting Property
- Integrator
- Step Response
- Causality
- Stability
- Memoryless System
- Distributivity
- Associativity

Approximate any CT signal $x(t)$ as a sum of shifted, scaled rectangular pulses:

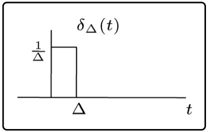


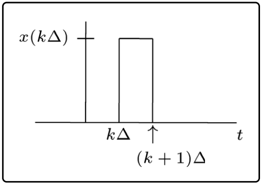
$$\widehat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$

(Here $\widehat{x}(t)$ denotes an approximation to $x(t)$.)

Define a rectangle pulse

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$$





$$= x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

Whence

$$\widehat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

In the limit as $\Delta \to 0$ we infer

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

which is the "Sifting Property" of the unit impulse:

- the integrand is zero for all $t-\tau \neq 0$, only when $\tau = t$ is the integrand non-zero,
- the unit impulse $\delta(t)$ has the area of unity; $x(\tau)\,\delta(t-\tau)$ has area of $x(t)$

Part 6 Outline

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

$$\delta_\Delta \longrightarrow h_\Delta(t)$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta}_{\widehat{x}(t)} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,h_\Delta(t-k\Delta)\Delta}_{\widehat{y}(t)}$$

$$\Downarrow \qquad (\text{as } \Delta \to 0)$$

$$\delta(t) \longrightarrow h(t)$$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}_{y(t)}$$

Convolution Integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Interpretation

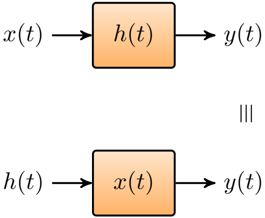
$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau)$$

$$h(-\tau) \xrightarrow{\text{Shift}} h(t-\tau)$$

$$h(t-\tau) \xrightarrow{\text{Multiply}} x(\tau)h(t-\tau)$$

$$x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

CT System Properties – Commutativity



$$
\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= h(t) \star x(t)
\end{aligned}
$$

- This follows from (change variables to $\sigma = t - \tau$)

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} x(t-\sigma) h(\sigma)\, d\sigma
\end{aligned}
$$

Part 6 Outline

CT System Properties – Sifting Property

Sifting Property:

$$x(t) \star \delta(t - t_0) = x(t - t_0)$$

That is, a system response $x(t) \xrightarrow{h(t)} y(t)$ of the form

$$h(t) = \delta(t - t_0),$$

acts as a time delay of t_0, that is,

$$y(t) = x(t - t_0).$$

CT System Properties – Sifting Property (cont'd)

Note that $x(t) \star \delta(t - t_0)$ is different from $x(t)\,\delta(t - t_0)$. The expression on the left is convolution of two signals and the expression on the right is pointwise multiplication.

- Convolution

$$\begin{aligned} x(t) \star \delta(t - t_0) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\, d\tau \\ &= x(t - t_0) \end{aligned}$$

- Pointwise multiplication

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

which is a scaled impulse response.

CT System Properties – Integrator

Integrator Property:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \qquad (2)$$

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow y(t)$$

So we need to determine the system response $h(t)$ to synthesize the integrator (2). But $y(t) = h(t)$ is output when $x(t) = \delta(t)$ is input, so substituting into (2)

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\, d\tau \\ &= u(t) \qquad \text{(step function)} \end{aligned}$$

That is, a system response of the form $h(t) = u(t)$ acts as an integrator.

Part 6 Outline

CT System Properties – Step Response

Step Response: Let $x(t) = u(t)$ be a step input signal to a CT LTI System with pulse response $h(t)$, then the output signal is given by

$$\begin{aligned} s(t) &= u(t) \star h(t) = h(t) \star u(t) \\ &= \int_{-\infty}^{t} h(\tau)\, d\tau \end{aligned}$$

This is a common test signal used in diagnosing physical systems (often called "plants").

CT System Properties – Causality

Theorem (Causal CT LTI System)

A CT LTI system is **causal** *if and only if its pulse response, $h(t)$, satisfies*

$$h(t) = 0, \quad \textit{for all } t < 0.$$

- If $h(t) \neq 0$ for at least one $t = -t_0$ $(t_0 > 0)$ then the output at time t, $y(t)$, would have an integrand term

$$h(-t_0)x(t + t_0)$$

and hence not be causal.

CT System Properties – Stability

Stability Property:

Definition (CT LTI System Stability)

CT LTI system, $x(t) \xrightarrow{h(t)} y(t)$, is **stable**, if and only if

$$\int_{-\infty}^{\infty} \left| h(\tau) \right| d\tau < \infty$$

Part 6 Outline

Definition (Memoryless CT System)

A CT system is **memoryless** if its output at time t depends only on the input at the same time t.

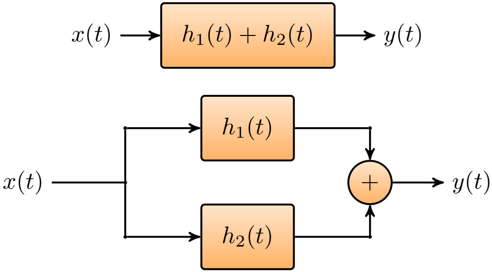
The following CT Systems $x(t) \xrightarrow{h(t)} y(t)$ are:

- **Memoryless**
  - $y(t) = 7x(t)$
  - $y(t) = \sqrt{x(t)} + 23$
  - $y(t) = t\,x(t)$
  - $y(t) = -5$ (even though independent of $x(t)$)
- **Not memoryless** (have memory)
  - $y(t) = 5\,x(t - 0.5)$
  - $y(t) = \frac{1}{C} \int_{-\infty}^{t} x(\tau)\,d\tau$

Definition (Distributivity Property of CT LTI Systems)

Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

$$\begin{aligned} y(t) &= x(t) \star \big(h_1(t) + h_2(t)\big) \\ &= x(t) \star h_1(t) + x(t) \star h_2(t) \end{aligned}$$



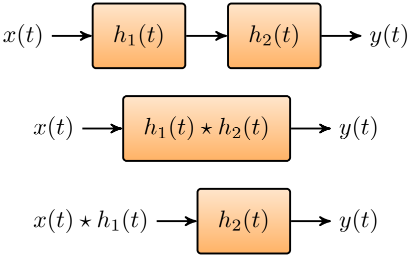
Two CT LTI systems in **parallel** implies we **add** their impulse responses.

CT System Properties – Associativity

Definition (Associativity Property of CT LTI Systems)

Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

$$y(t) = \big(x(t) \star h_1(t)\big) \star h_2(t)\big) = x(t) \star \big(h_1(t) \star h_2(t)\big)$$



Two CT LTI systems in **series** implies we **convolve** their impulse responses.

26 Impulses and More

Part 7 Outline

26 Impulses and More

- Unit Impulse
- Trivial System
- Delay System
- Additional Results
- Differentiator System
- Integrators
- Further Results
- Digression
- Tricks
- Unit Impulse Revisit
- Unit Doublet Revisit

$\delta(t)$ — unit area (that integrates to 1) pulse and is the limit of a narrow rectangular pulse with width, Δ, going to zero and height, $1/\Delta$, going to infinity.

- The shape of the (unit) impulse isn't important, that is, there is nothing special about the rectangular shape.
- When applied to a CT LTI System gives the output equal to the impulse response:

  $$\boxed{\delta(t) \star h(t) = h(t), \quad \text{for all } h(t).}$$

  This is a tautology of sorts, this says "the response to an unit impulse is the impulse response". Let's look at this next, mathematically.
- $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Impulses and More – Unit Impulse (cont'd)

LTI System General Input: Start with $x(t) \star h(t) = y(t)$, which means

$$\begin{aligned} y(t) &= x(t) \star h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \end{aligned}$$

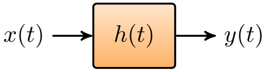


Fig: LTI System General Input

LTI System Special Input: Then set the input to an impulse, that is, set $x(t) = \delta(t)$, to yield

$$\int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau = h(t)$$

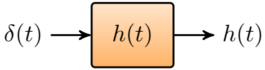
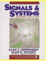


Fig: LTI System Special Input $x(t) = \delta(t)$

Here $\delta(\tau) = 0$ if $\tau \neq 0$ and it "sifts" the value of $h(t)$.

Impulses and More – Trivial System



Signals & Systems
section 2.5.2
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From commutativity $x(t) \star h(t) = h(t) \star x(t) = y(t)$:

$$
\begin{aligned}
y(t) &= h(t) \star x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau
\end{aligned}
$$

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

Fig: Flipped $x(t)$ and $h(t)$

Trivial System: With "impulse response" $x(t) = \delta(t)$ (the part inside the box) and "input" $h(t)$ (the part feeding the box)

$$\int_{-\infty}^{\infty} h(\tau)\, \delta(t-\tau)\, d\tau = h(t)$$

$$h(t) \longrightarrow \boxed{\delta(t)} \longrightarrow h(t)$$

Fig: Trivial System

Here $\delta(t-\tau) = 0$ if $t \neq \tau$ and it sifts the value $h(t)$ (under then integral sign). This system, with impulse response given by the impulse, has output signal equal to the input signal, that is, just passes the input to the output — called the **"Trivial System"**.

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Impulses and More – Delay System

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Now, with impulse response $h(t) = \delta(t - t_0)$ and input $x(t)$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t - t_0 - \tau)\, d\tau = x(t - t_0).$$

$$x(t) \longrightarrow \boxed{\delta(t - t_0)} \longrightarrow x(t - t_0)$$

Fig: Time Shift LTI System (Delay when $t_0 > 0$)

Here $\delta(t - t_0 - \tau) = 0$ if $t - t_0 \neq \tau$ and it "sifts" the value $h(t - t_0)$. This system, with impulse response given by the impulse with time shift, has output equal to the input with a time shift. Note that $t_0 > 0$ gives a delay and $t_0 < 0$ gives a time advance (which would be non-causal).

Note that

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which is the unit step signal.

Compare this with the fundamental theorem of calculus which asserts

$$\frac{d}{dt} \int_{-\infty}^{t} f(\tau)\, d\tau = f(t)$$

for sufficiently regular functions $f(t)$. (cont'd)

With the function $\delta(t)$ as the function in the fundamental theorem of calculus, we can infer

$$\frac{d}{dt}\underbrace{\int_{-\infty}^{t}\delta(\tau)\,d\tau}_{u(t)} = \delta(t).$$

So the $\delta(t)$ "is" the derivative of the unit step $u(t)$. (cont'd)

Impulses and More – Additional Results (cont'd)

We can represent this as:

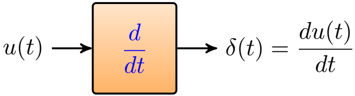


Fig: Differentiator with Unit Step Input

where the $\frac{d}{dt}$ inside the box is not the impulse response but denotes an operator. (cont'd)

Taking the derivative is actually a linear time-invariant (LTI) operator. It satisfies superposition (L)

$$\frac{d}{dt}\big(\alpha_1 x_1(t) + \alpha_2 x_2(t)\big) = \alpha_1 \frac{dx_1(t)}{dt} + \alpha_2 \frac{dx_2(t)}{dt}$$

and, with the notation,

$$x'(t) = \frac{dx(t)}{dt}$$

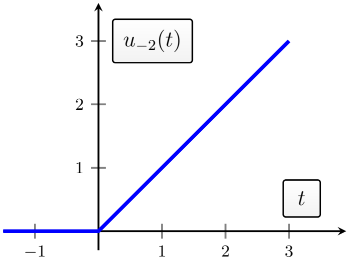
then by the chain rule we have the time-invariance (TI)

$$\frac{d}{dt} x(t - t_0) = x'(t - t_0)$$

So we conclude that the differentiator operator acts like a LTI system and so must have an **impulse response** which we'll consider shortly.

Now suppose we have a linear **unit ramp**

$$u_{-2}(t) \triangleq \begin{cases} t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



or, equivalently, $u_{-2}(t) \triangleq t\,u(t) = t\,u_{-1}(t)$.

We can also write this as

$$u_{-2}(t) = u(t) \star u(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$$

(picture in your mind, convolving two unit steps $u(t)$ together).

Impulses and More – Additional Results (cont'd)

Big picture:

$$\cdots \xrightarrow{\frac{d}{dt}} u_{-2}(t) \xrightarrow{\frac{d}{dt}} u(t) \xrightarrow{\frac{d}{dt}} \delta(t) \xrightarrow{\frac{d}{dt}} \ ? \xrightarrow{\frac{d}{dt}} \cdots$$

Can we continue? What is the derivative of the impulse $\delta(t)$? (And the second derivative, etc?)

Later picture (different notation only):

$$\cdots \xrightarrow{\frac{d}{dt}} u_{-2}(t) \xrightarrow{\frac{d}{dt}} u_{-1}(t) \xrightarrow{\frac{d}{dt}} u_0(t) \xrightarrow{\frac{d}{dt}} u_1(t) \xrightarrow{\frac{d}{dt}} \cdots$$

Here, evidently, $u_0(t) = \delta(t)$ and $u_{-1}(t) = u(t)$.

Impulses and More – Additional Results (cont'd)

We give a partial explanation of some of the previous weird notation. For the trivial system, the system that just passes the input to output without modification, we found that the impulse response is just $\delta(t)$, that is,

$$h(t) \longrightarrow \boxed{\delta(t)} \longrightarrow h(t)$$

Fig: Trivial System

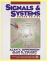
Then we can make some sense of the notation $u_0(t) = \delta(t)$

$$h(t) \longrightarrow \boxed{u_0(t)} \longrightarrow h(t)$$

Fig: Trivial System (Alternative Notation)

where 0 indicates a reference trivial (zero) system.

26 Impulses and More



Consider the differentiator (recall this is LTI)

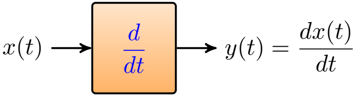


Fig: Differentiator System

What is the impulse response to describe this?

The answer is denoted $u_1(t)$, called the **unit doublet**, and has the property/definition:

$$\frac{dx(t)}{dt} = x(t) \star u_1(t)$$

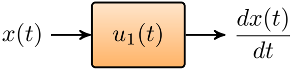


Fig: Differentiator System

We relate $u_1(t)$ to $\delta(t)$ later.

Further generalizing, for $k > 0$,

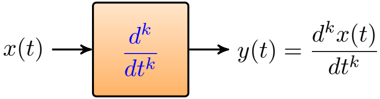


Fig: kth Differentiator System

Then, for $k > 0$,

$$\frac{d^k x(t)}{dt^k} = x(t) \star u_k(t)$$

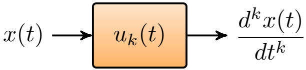


Fig: kth Differentiator System

and, by cascading differentiators we can get a nth order differentiator impulse response

$$u_n(t) \triangleq \underbrace{u_1(t) \star \cdots \star u_1(t)}_{n \text{ times}}, \quad n > 0$$

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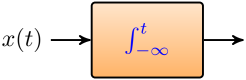
An integrator

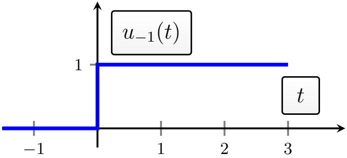


$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Fig: Integrator System

Impulse response is the step response:

$$u_{-1}(t) = u(t),$$



that is, the -1th derivative or simply the integral. We can write

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Further, cascading n integrators, leads to impulse response

$$u_{-n}(t) \triangleq \underbrace{u_{-1}(t) \star \cdots \star u_{-1}(t)}_{n \text{ times}}, \quad n > 0$$

Impulses and More – Integrators (cont'd)

Cascade of two integrators, with input $\delta(t)$:

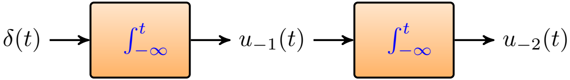


Fig: Cascade of Integrator Systems

Whence

$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)\, d\tau = \int_{-\infty}^{t} u(t)\, d\tau = t\, u(t)$$

which is the unit ramp. More generally,

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u(t), \quad n > 1$$

where $k! = k(k-1)(k-2)\cdots 2 \cdot 1$.

Impulses and More – Integrators (cont'd)

Further reflections on integrators. A differentiator operator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Fig: Differentiator System

"destroys" information, that is, a constant signal on input is mapped to zero (the zero signal) at the output.

The integrator

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Fig: Integrator System

would normally require some initial conditions to completely determine the output.

Part 7 Outline

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- Integrators
- **Further Results**
- Digression
- Tricks
- Unit Impulse Revisit
- Unit Doublet Revisit

Impulses and More – Further Results

We have,

$$u_0(t) = \delta(t)$$

which can be viewed as:

- the impulse response for the trivial system, $y(t) = x(t)$
- the zeroth derivative operator

Then

$$u_m(t) \star u_n(t) = u_{m+n}(t), \quad m, n \in \mathbb{Z} \text{ (integers)}$$

For example,

$$u_1(t) \star u_{-1}(t) = u_0(t) \quad \Rightarrow \quad \frac{d}{dt} u(t) = \delta(t).$$

Digression: (for those interested)

- **Functions** map scalars to scalars (points to points).
- **Operators** map functions to functions
- **Functionals** map functions to scalars

Examples:

- A signal is a **function** of time. For each number we put in, say $t = 3$, we get a number out, here $x(3)$.
- Computing the energy or power of a signal is an example of a **functional** — a whole function is gobbled up and a number spat out.
- A system is an **operator**. For each input function $x(t)$ we input a function $y(t)$ is output. Also called a "filter".

Previously,

$$u_1(t) \star u_{-1}(t) = u_0(t) \quad \Rightarrow \quad \frac{d}{dt}u(t) = \delta(t).$$

Here u is an input function and δ is an output function. The operator is $D \triangleq d/dt$ and we can write

$$\frac{d}{dt}u = \delta \quad \text{or} \quad Du = \delta$$

If you want to include the independent variable t then, by convention, this is written

$$\Big(\frac{d}{dt}u\Big)(t) = \delta(t) \quad \text{or} \quad \big(Du\big)(t) = \delta(t)$$

Part 7 Outline

Impulses and More – Tricks

Occasionally useful tricks:

$$
\begin{aligned}
x(t) \star h(t) &= x(t) \star \delta(t) \star h(t) \\
&= x(t) \star u_1(t) \star u_{-1}(t) \star h(t) \\
&= \big(x(t) \star u_1(t) \star h(t)\big) \star u_{-1}(t)
\end{aligned}
$$

Differentiate, then convolve then integrate. There are many other possibilities.

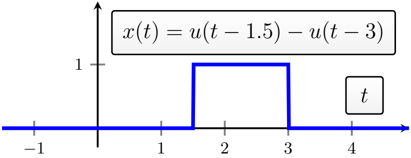
Consider unit height rectangular pulse with duration $[a, b)$ (here $a < b$ and are arbitrary, $a, b \in \mathbb{R}$):

$$x(t) = \begin{cases} 1 & a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

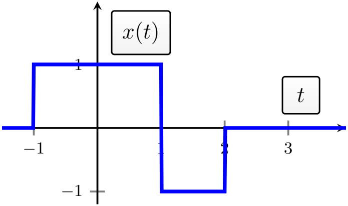
This can be written in terms of shifted $u(t)$:

$$x(t) = u(t - a) - u(t - b)$$

Example plotted on right for $a = 1.5$ and $b = 3$.



Consider the problem:



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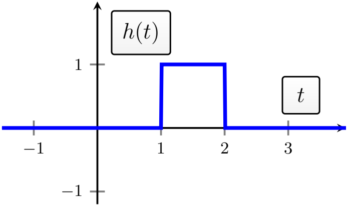


Let's use

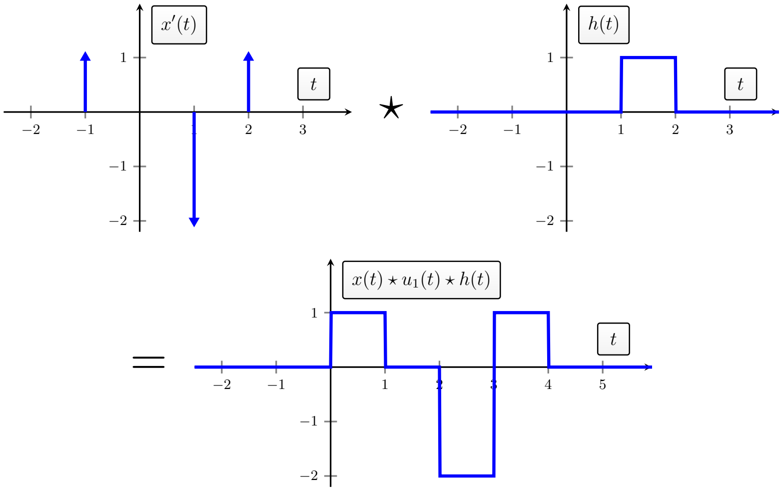
$$x(t) \star h(t) = \big(x(t) \star u_1(t) \star h(t)\big) \star u_{-1}(t)$$

to compute this convolution because it will be like having three impulses at the step transitions. First we have

$$x'(t) = x(t) \star u_1(t) = \delta(t+1) - 2\,\delta(t-1) + \delta(t-2)$$

this means, signal $x(t)$ is input to a 1st order differentiator (which is an LTI system with $u_1(t)$ as the impulse response).

Impulses and More – Tricks (cont'd)



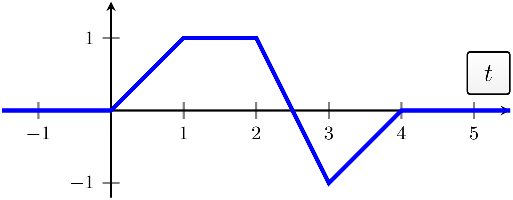
Impulses and More – Tricks (cont'd)

Finally,

$$x(t) \star h(t) = \int_{-\infty}^{t} \left(\frac{dx(\tau)}{d\tau} \star h(\tau) \right) d\tau$$

This can be drawn, by observation, as:

$$\boxed{\int_{-\infty}^{t} \left(\frac{dx(\tau)}{d\tau} \star h(\tau) \right) d\tau = x(t) \star u_1(t) \star h(t) \star u_{-1}(t)}$$



26 Impulses and More

Scaling properties of the $\delta(t)$

- It can be shown:

$$\delta(nt) = \frac{1}{n}\,\delta(t)$$

  Start with a rectangular approximation and consider the limit as the width goes to 0. Approximation of $\delta(nt)$ is n times thinner horizontally than approximation of $\delta(t)$.

- Recall $\delta(t)$ has unit area. The above function has area $1/n$.

26 Impulses and More

- Unit Impulse
- Trivial System
- Delay System
- Additional Results
- Differentiator System
- Integrators
- Further Results
- Digression
- Tricks
- Unit Impulse Revisit
- Unit Doublet Revisit

Revisit $u_1(t)$, called the **unit doublet**, which has the property/definition:

$$\frac{dx(t)}{dt} = x(t) \star u_1(t)$$

$$x(t) \longrightarrow \boxed{u_1(t)} \longrightarrow \frac{dx(t)}{dt}$$

Fig: Differentiator System

Signal $u_1(t)$ is the derivative of $\delta(t)$. What does this look like?

Advanced Note: This is quite mad in fact. Convolution is an integrating action and we need to significantly pre-distort the signal in the integrand to have the signal derivative come out as output.

Impulses and More – Unit Doublet Revisit (cont'd)

Consider the rectangular pulse

$$\delta_\Delta(t) = \begin{cases} 1 & 0 \le t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

which behaves like the impulse $\delta(t)$ as $\Delta \to 0$. Its derivative should behave like the unit doublet as $\delta(t)$ as $\Delta \to 0$. We have

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}\big(\delta(t) - \delta(t-\Delta)\big)$$

and this looks like the superposition of two delta functions of opposite sign and area $1/\Delta$. Further,

$$x(t) \star \frac{d\delta_\Delta(t)}{dt} = \frac{x(t) - x(t-\Delta)}{\Delta} \to \frac{dx(t)}{dt} \text{ as } \Delta \to \infty$$

Impulses and More – Unit Doublet Revisit (cont'd)

Two properties, we learnt earlier, of $u_0(t) = \delta(t)$ are:

$$\int_{-\infty}^{\infty} x(\tau) u_0(t-\tau)\, d\tau = x(t)$$

$$x(t)\, u_0(t) = x(0)\, \delta(t)$$

For $u_1(t)$ analogous properties (without proof) are:

$$\int_{-\infty}^{\infty} x(\tau) u_1(t-\tau)\, d\tau = -x'(t)$$

$$x(t)\, u_1(t) = x(0)\, u_1(t) - x'(0)\, \delta(t)$$

where $x'(t) \triangleq dx(t)/dt$.

27 **Fourier Series**

- Where we are heading?
- Digression
- Background Ideas
- Response to Complex Exponentials
- Matrix Digression
- Eigenfunctions of LTI Systems
- DT Case Eigenfunctions

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Fourier Series – Where we are heading?

DT and CT LTI systems admit a simple description in terms of the impulse response. The output signal can be computed in terms of the convolution of the input and impulse response. In essence, not much else is needed.

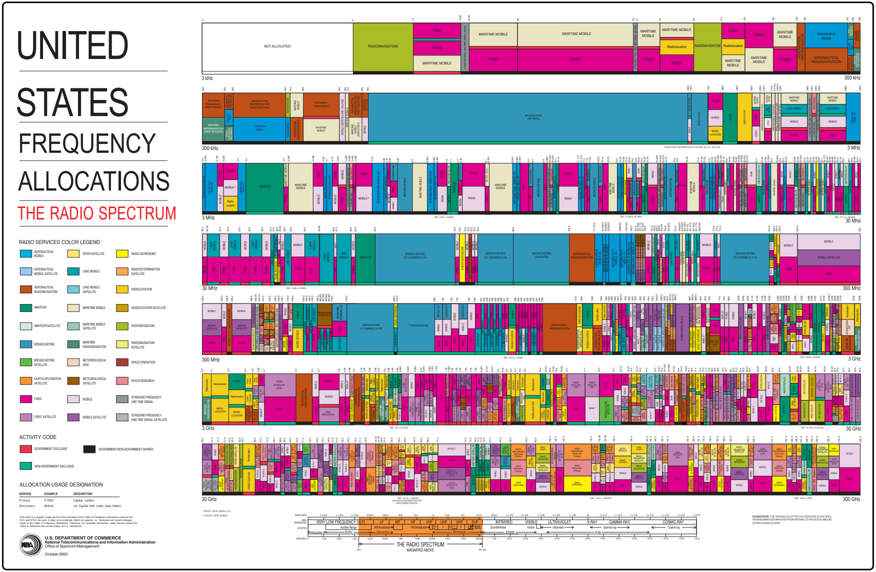
Fourier Series, the Fourier Transform and other "frequency domain" descriptions provide an alternative (but equivalent) viewpoint with a number of key advantages:

- Convolution is simplified using the Fourier representations. Convolution is "twisted and complicated" in the "time domain" but in the frequency domain it is "untwisted and simple".
- Studying the behavior of a LTI system in the frequency domain is actually natural and intuitive (after a few years).

To this point we haven't said much about **design**. We want to build LTI systems that achieve certain design goals. This is best done in the frequency domain.

Design and resource allocation specifications almost always have a frequency domain formulation:

Fourier Series – Where we are heading? (con't)



Part 8 Slide 316/many Convenor: R. A. Kennedy

Fourier Series – Where we are heading? (con't)

Consider

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

which is a moving average of three consecutive terms. So $y[n]$ looks like a smooth version of $x[n]$.

Note that we use $1/3$ weights because if $x[n]$ is very smooth to start with then $y[n]$ should be equal or close to $x[n]$. So if $x[n] = 1$ for all n, that is, it is a constant, then $y[n] = 1/3 + 1/3 + 1/3 = 1$.

So

$$y[n] = \frac{1}{5}x[n-2] + \frac{1}{5}x[n-1] + \frac{1}{5}x[n] + \frac{1}{5}x[n+1] + \frac{1}{5}x[n+2]$$

works as a smoother as well. It should smooth more.

Ditto

$$y[n] = \frac{1}{9}x[n-2] + \frac{2}{9}x[n-1] + \frac{1}{3}x[n] + \frac{2}{9}x[n+1] + \frac{1}{9}x[n+2]$$

works as a smoother too. It relies more on the current value $x[n]$ in forming $y[n]$.

Which smoother works best? What are we trying to achieve?

Fourier Series – Where we are heading? (con't)

We are lacking tools to analyze which smoother is best. We are lacking tools for design. This is one main reason to look at the description of LTI systems in the Frequency or Fourier domain.

Part 8 Outline

27 Fourier Series

- Where we are heading?
- Digression
- Background Ideas
- Response to Complex Exponentials
- Matrix Digression
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- DT Case Eigenfunctions

We can view the "slow" system

$$h_5[n] \triangleq \frac{1}{5}\delta[n-2] + \frac{1}{5}\delta[n-1] + \frac{1}{5}\delta[n] + \frac{1}{5}\delta[n+1] + \frac{1}{5}\delta[n+2]$$

as extracting the slow/smooth part of $x[n]$

$$x_s[n] \equiv x_5[n] \triangleq x[n] \star h_5[n]$$

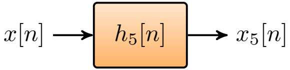

Fig: Smoother

This is called **filtering**. It is a **"low pass filter"**. It passes to the output the slowly varying parts of input $x[n]$ (mostly).

But what if I didn't want the slowly varying parts of input $x[n]$ but the opposite? Can we extract just the quickly varying parts (the parts blocked by $h_5[n]$)?

Fourier Series – Digression (con't)

Of course, the "fast" system is just the complement of the "slow" system

$$h_5^c[n] \triangleq \delta[n] - h_5[n] =$$

$$-\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]$$

and extracts the non-slow part of $x[n]$

$$x_s^c[n] \equiv x_5^c[n] \triangleq x[n] \star h_5^c[n]$$

$$x[n] \longrightarrow \boxed{h_s^c[n]} \longrightarrow x_5^c[n]$$

Fig: Complement of the Smoother

It is a **"high pass filter"**. It passes to the output the quickly varying parts of input $x[n]$ (mostly). Also evidently, $x[n]$ is perfectly split into the two parts $x[n] = x_5[n] + x_5^c[n]$.

Finally, Goldilocks says she doesn't want the fast nor the slow but just the parts of $x[n]$ that are varying not too fast nor too slow. Can we figure this out? (Or should we just feed her to the bears?)

The complement of the five term moving average

$$
\begin{aligned}
h_5^c[n] &\triangleq \delta[n] - h_5[n] \\
&= -\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]
\end{aligned}
$$

essentially blocks the very slow parts of $x[n]$ and lets through more than the complement of the three term moving average

$$h_3[n] \triangleq \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n+1]$$

so we can get the desired action via the **"band pass filter"**

$$h_5^c[n] \star h_3[n] = \text{whatever}$$

Fourier Series – Digression (con't)

$x[n]$ can be split into slow, medium (neither fast nor slow) and fast signals

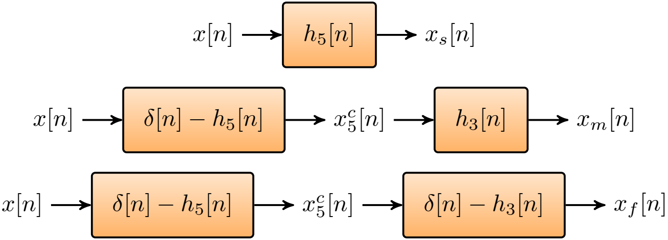

Fig: Slow, Medium and Fast Signal Processing

Here

$$x_m[n] + x_f[n] = x_5^c[n] \quad \text{and} \quad x_s[n] + x_5^c[n] = x[n]$$
$$x_s[n] + x_m[n] + x_f[n] = x[n]$$

Part 8 Outline

27 **Fourier Series**

- Where we are heading?
- Digression
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Evidently, it seems useful to be able to characterize how a system treats fast, medium and slow inputs (then we can broadly regard it as low pass, high pass, band pass or some blurring of the three).

The most obvious thing to do is see how the system responds to **complex exponentials** (of different "frequencies"). This leads us to study this problem more completely.

27 **Fourier Series**

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Prior to studying the set of complex exponentials we can lay down the desirable characteristics of a set of basic signals to probe LTI systems:

- We need to be able to represent any **signal** (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or scanning in some sense).
- The response of an LTI **system** to any of these basic signals should be simple, useful and insightful.

To emphasize, it has to meet demands of both: i) signal representation and ii) systems characterization.

Fourier Series – Response to Complex Exponentials

Previous focus was unit samples and impulses.

Alternative/new focus is "eigenfunctions of LTI systems" (what the?)

Fourier Series – Response to Complex Exponentials

Start with CT LTI system convolution equation. Let the input be

$$x(t) = e^{st}$$

where $s \in \mathbb{C}$ is complex. Then, with $h(t)$ the impulse response,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau, \quad \text{with } x(t) = e^{st} \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \end{aligned}$$

(. . . yawn)

Fourier Series – Response to Complex Exponentials

So $x(t) = e^{st}$ is a really good choice. Reflect on

$$y(t) = \underbrace{e^{st}}_{\text{the input}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\text{independent of } t}$$

The output equals the input apart from a complex multiplier

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \quad \in \mathbb{C}.$$

This is some sort of "transform" of the impulse response $h(t)$. What does this all mean?

Part 8 Outline

27 **Fourier Series**

- Where we are heading?
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Fourier Series – Matrix Digression

Suppose I have a $N \times N$ square matrix $\mathbf{M}$ then if there is a N-vector $\boldsymbol{\mu}$ satisfying

$$\boldsymbol{M}\boldsymbol{\mu} = \lambda\boldsymbol{\mu}, \quad \lambda \in \mathbb{C}$$

then $\boldsymbol{\mu} \in \mathbb{C}^N$ is an **eigenvector** and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

We could draw

$$\boldsymbol{\mu} \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda\boldsymbol{\mu}$$

which reveals that the output vector/signal is equal to input vector/signal apart from a complex scale factor ("gain"). The action of matrix $\boldsymbol{M}$ is simple when the input is an eigenvector.

Fourier Series – Matrix Digression (con't)

More generally, there are a set of eigenvectors

$$\boldsymbol{M}\boldsymbol{\mu}_i = \lambda_i \boldsymbol{\mu}_i, \quad i = 1, 2, \ldots, N$$

and for each we could draw

$$\boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda_i \boldsymbol{\mu}_i$$

So different eigenvectors have different gains.

This is powerful because, take any $\boldsymbol{v} \in \mathbb{C}^N$ not generally an eigenvector and assume the set of N eigenvectors $\boldsymbol{\mu}_i$ span $\mathbb{C}^N$.

Fourier Series – Matrix Digression (con't)

Then

$$\boldsymbol{v} = \sum_{i=1}^{N} a_i \boldsymbol{\mu}_i$$

and we can draw

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i$$

These eigenvectors are a special set (for this matrix $\boldsymbol{M}$).

Fourier Series – Matrix Digression (con't)

Recall previous guidelines for a desirable signal "set"

- We need to can able to represent any **signal**/vector (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or spanning in some sense).
- The response of an LTI **system**/matrix to any of these basic signals should be simple, useful and insightful.

In fact it is more than a coincidence, since a system is an operator (maps functions to functions) and a matrix is a finite dimensional operator (maps vectors in $\mathbb{C}^N$ to $\mathbb{C}^N$).

27 Fourier Series

- Where we are heading?
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In the back of our minds...

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i, \quad \lambda_i = \lambda_i(\boldsymbol{M})$$

Eigen-behavior of LTI Systems

$$\sum_{i=1}^{N} a_i e^{s_i t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i e^{s_i t}, \quad \lambda_i = H(s_i)$$

where

$$H(s_i) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s_i \tau}\, d\tau \in \mathbb{C}.$$

So being able to decompose signals into complex exponentials leads to a simple characterization of an LTI system.

Part 8 Outline

Fourier Series – DT Case Eigenfunctions

What can we use to in the DT case for $x[n]$ analogous to CT $x(t) = e^{st}$?

Let the input be

$$x[n] = z^n$$

where $z \in \mathbb{C}$ is complex. Then, with $h[n]$ the impulse response,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k], \quad x[n] = z^n \\ &= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \end{aligned}$$

Reflect

$$y[n] = \underbrace{z^n}_{\text{the input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{\text{independent of } n}$$

The output equals the input apart from a complex multiplier

$$\boxed{H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \;\in \mathbb{C}}$$

This is some sort of "transform" of the impulse response $h[k]$.

28 **Fourier Stuff**
- Review Key Observation
- Examples
- CT Periodic Signals
- Fourier Series
- Classical Fourier Series
- Fourier Coefficients
- Periodic Rectangular Wave

Part 9 Outline

28 **Fourier Stuff**

- Review Key Observation
- Examples
- CT Periodic Signals
- Fourier Series
- Classical Fourier Series
- Fourier Coefficients
- Periodic Rectangular Wave

CT LTI system response to **arbitrary** input $x(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

versus

CT LTI system response to **specific, complex exponential** input $x(t) = e^{st}$, for some complex $s \in \mathbb{C}$,

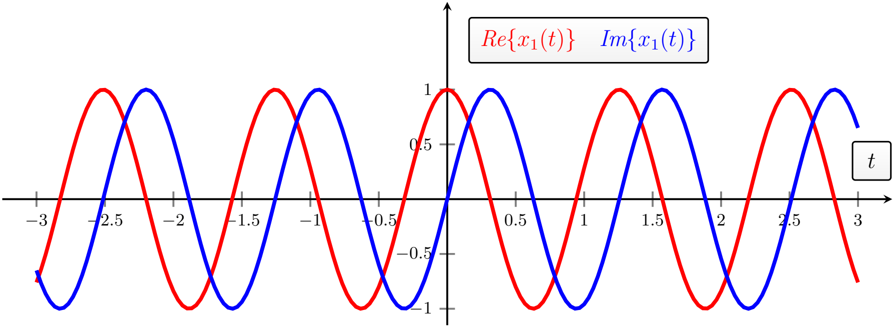
$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s) \in \mathbb{C}}$$

where $H(s)$ is just a complex number — call this the "complex gain".

28 Fourier Stuff

Example 1: Let

$$x_1(t) \triangleq e^{j\,5t}$$



be input to a LTI system with impulse response $h_1(t)$ and “complex gain”

$$H_1(j5) \triangleq 1 + j\sqrt{3}.$$

Find

$$y_1(t) = x_1(t) \star h_1(t), \quad \text{where } x_1(t) = e^{j\,5t}$$

For this input, we have

$$s = j5, \qquad \text{(recall } e^{st}\text{)}$$

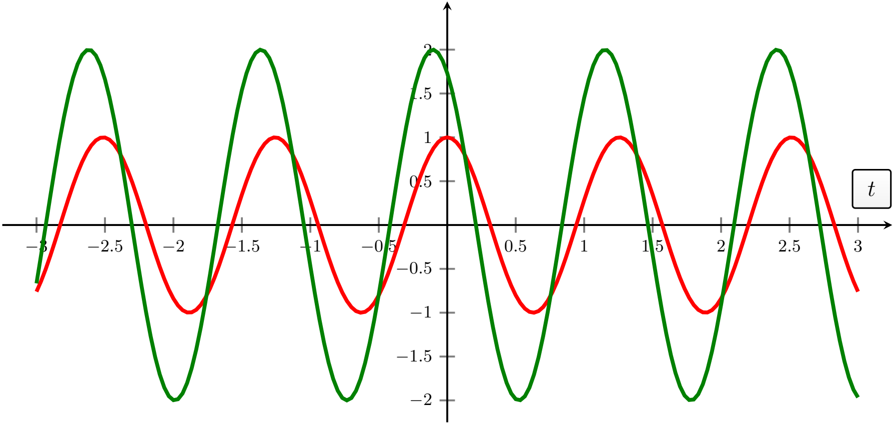
and the output is

$$\begin{aligned}
y_1(t) &= e^{j\,5t} \star h_1(t) \qquad &\text{(convolution)} \\
&= e^{j\,5t} H_1(j5) \qquad &\text{(multiplication)} \\
&= e^{j\,5t}(1 + j\sqrt{3}) \\
&= 2\,e^{j\,5t} e^{j\,\pi/6} \\
&= 2\,e^{j\,(5t+\pi/6)}
\end{aligned}$$

That is, the output is at the same frequency $\omega_1 = 5$ $(s = j\omega_1)$, is twice the amplitude of the input and is phase shifted by $\pi/6$.

Fourier Stuff – Examples

Plotting the real part of this output $y_1(t)$ (and input $x_1(t)$):



Example 2: Now let the input be only real, have a phase shift of $\pi/4$ and have magnitude 4:

$$x_1(t) \triangleq 4\cos(5t + \pi/4)$$

Let this be input to a LTI system with impulse response $h_1(t)$ which has gain

$$H_1(\pm j5) \triangleq 1 \pm j\sqrt{3}, \quad \text{(here } s = \pm j5\text{)},$$

(This notation means $H_1(j5) \triangleq 1 + j\sqrt{3}$ and $H_1(-j5) \triangleq 1 - j\sqrt{3}$ merged into one equation.) Find

$$y_1(t) = x_1(t) \star h_1(t)$$

Again we won't have to explicitly compute the convolution

$$y_1(t) = \int_{-\infty}^{\infty} 4\cos(5\tau + \pi/4)\, h_1(t-\tau)\, d\tau$$

and in fact we don't know $h_1(t)$ completely anyway.

For this input, we have

$$\begin{aligned} x_1(t) &= 4\cos(5t + \pi/4) \\ &= 4\frac{\left(e^{j\,(5t+\pi/4)} + e^{-j(5t+\pi/4)}\right)}{2} \\ &= (\sqrt{2} + \sqrt{2}j)e^{j\,5t} + (\sqrt{2} - \sqrt{2}j)e^{-j5t} \end{aligned}$$

So we can use the principle of superposition with values $s = j5$ and $s = -j5$, that is, $s = \pm j5$,

$$(\sqrt{2} \pm j\sqrt{2})e^{\pm j5t} \longrightarrow \boxed{h_1(t)} \longrightarrow (\sqrt{2} \pm j\sqrt{2})H_1(\pm j5)e^{\pm j5t}$$

Fourier Stuff – Examples

For input

$$x_1(t) = 4\cos(5t + \pi/4)$$

the output is then

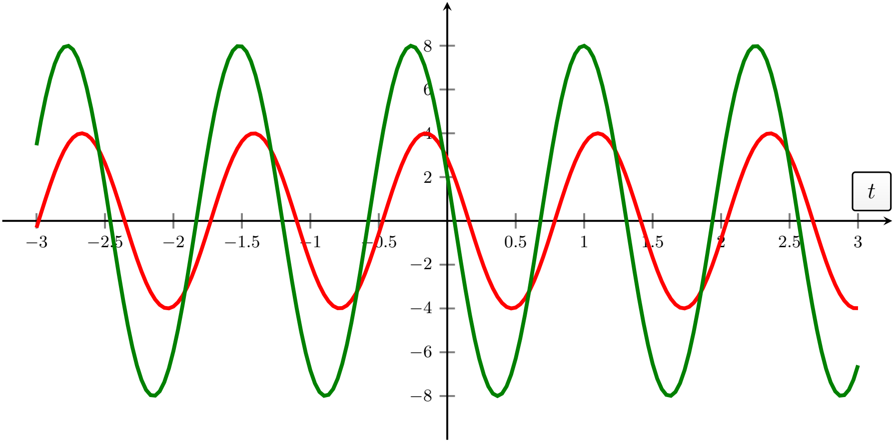
$$\begin{aligned} y_1(t) &= (\sqrt{2} + j\sqrt{2})(1 + j\sqrt{3})e^{+j5t} + (\sqrt{2} - j\sqrt{2})(1 - j\sqrt{3})e^{-j5t} \\ &= (2e^{j\,\pi/4})2e^{j\,\pi/6}e^{+j5t} + (2e^{-j\pi/4})2e^{-j\pi/6}e^{-j5t} \\ &= 8\,\frac{\left(e^{j\,(5t+5\pi/12)} + e^{-j(5t+5\pi/12)}\right)}{2} \\ &= 8\,\cos(5t + 5\pi/12) \end{aligned}$$

Or equivalently

$$\int_{-\infty}^{\infty} 4\,\cos(5\tau + \pi/4)\,h_1(t-\tau)\,d\tau = 8\,\cos(5t + 5\pi/12)$$

Fourier Stuff – Examples

Plotting input $x_1(t)$ and output $y_1(t)$:



This calculation can be considerably simplified. Note

- $\left|H_1(\pm j5)\right| = 2$ so the gain (magnitude) is just 2.
- The phase of $H_1(\pm j5)$ is $\pm\pi/4$ which is the **relative** phase shift of the output with respect to the input.
- Recall the system is time-invariant so the $\pi/4$ phase on the input is equivalent to a time shift and gets conveyed to the output.

Therefore,

$$x_1(t) = 4\cos(5t + \pi/4)$$

$$H_1(j5) = 2\, e^{j\,\pi/6} \quad \text{System}$$

$$y_1(t) = \underbrace{2 \times 4}_{8} \cos(5t + \underbrace{\pi/4 + \pi/6}_{5\pi/12})$$

Example 3: Consider the series/cascade connection

$$x_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow y_1(t) \longrightarrow \boxed{h_2(t)} \longrightarrow y_2(t)$$

with LTI system $h_1(t)$ such that

$$H_1(j5) = 0.5\,e^{j\,\pi/4} \quad \text{and} \quad H_1(-j5) = 0.5\,e^{-j\pi/4}$$

and LTI system $h_2(t)$ such that

$$H_2(j5) = 0.3\,e^{j\,2\pi/3} \quad \text{and} \quad H_2(-j5) = 0.3\,e^{-j2\pi/3}$$

Compute

$$y_2(t) = x_1(t) \star h_1(t) \star h_2(t)$$

for input signal

$$x_1(t) \triangleq 3\cos(5t - \pi/7).$$

First test understanding:

- $y_1(t)$ has to look like
$$\alpha \cos(5t + \beta)$$
because the input, $x_1(t)$, is of that form.
- $y_2(t)$ has to look like
$$\gamma \cos(5t + \delta)$$
because the input, $x_2(t) = y_1(t)$ is of that form.
- We only have to find two real numbers (magnitude and phase) or, equivalently, one complex number.

Overall gain is

$$3 \times \underbrace{0.5 \times 0.3}_{0.15} = 0.45$$

Overall phase

$$-\pi/7 + \underbrace{\pi/4 + 2\pi/3}_{11\pi/12} = 65\pi/84$$

Therefore,

$$y_2(t) = 0.45\,\cos(5t + 65\pi/84)$$

and we note that

$$H_1(j5)\,H_2(j5) = 0.15\,e^{j\,11\pi/12}$$

and

$$H_1(-j5)\,H_2(-j5) = 0.15\,e^{-j\,11\pi/12}.$$

Observation 1: This is revealing something very important. Cascading two LTI systems implies **convolving** their impulse responses. But for complex exponentials signals we only have to **multiply** complex gains. As we show much later this is the key property of frequency domain descriptions; series LTI systems lead to multiplications and parallel LTI systems lead to additions (superposition).

It seems like behavior of interconnections of LTI systems with complex exponential signals as inputs (and outputs) is relatively simple.

Observation 2: Houston we have a problem. OK so complex exponentials are simple to work with. What about more general signals?

Firstly we look at **periodic signals**. Fourier series, which we now consider, show that any periodic signal can be expressed into terms of appropriate linear combinations of complex exponential signals (which are themselves periodic). Then we can appeal to superposition to characterize the response of an LTI system to general (not necessarily complex exponential) periodic signals.

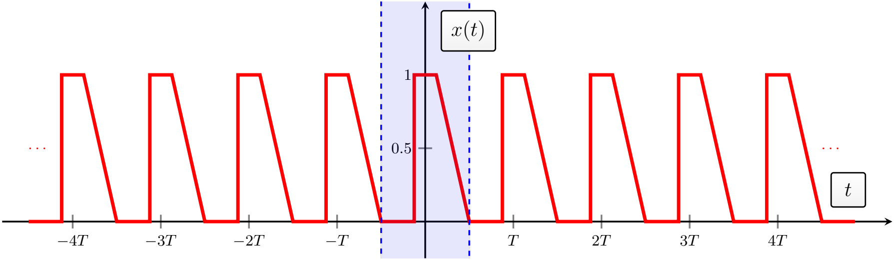
Fourier Stuff – CT Periodic Signals

CT periodic signals satisfy

$$\boxed{x(t) = x(t+T), \quad \text{for all } t}$$

where T is the **fundamental period** (smallest positive T).

An example periodic waveform with fundamental period T is shown below.



Define the **fundamental frequency** associated with fundamental period T

$$\omega_0 = \frac{2\pi}{T}$$

Prototypical periodic signal is

$$e^{j\,\omega_0 t}$$

What other related signals (to this prototypical signal) are also periodic with period T?

It is reasonably clear that ($\iff$ means if and only if)

$$e^{j\,\omega t} \text{ is periodic with period } T \iff \omega = k\omega_0, \quad k \in \mathbb{Z}$$

- For each $|k| \neq 1$, the period T is not fundamental.
- For example, with $k = -3$, $e^{-j3\omega_0 t}$ has fundamental period $T/3$ but it is still periodic with period T. It is periodic with periods: $T/3$, $2T/3$, T, $4T/3$, etc.
- Further $k = 0$, which implies $\omega = 0$, is weird, it leads to just constant function.
- $k = -1$ leads to the conjugate of $k = 1$ and has fundamental period T also.

28 Fourier Stuff

Fourier Stuff – Fourier Series

The infinite linear combination

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \, e^{jk2\pi t/T}$$

- is periodic with period T because all components are periodic with period T
- $k = 0$ term is the "DC" term ("direct current")
- $k = \pm 1$ terms are the first harmonic
- $k = \pm 2$ terms are the second harmonic
- The two kth terms are called the kth harmonic ($k > 0$) and have fundamental period kT.

In

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk2\pi t/T}}$$

the $\{a_k\}$ are the **Fourier Series coefficients**.

It's clear there are an infinite number of periodic functions that can be built this way but...

Question: Can any periodic function be expressed in such a way?

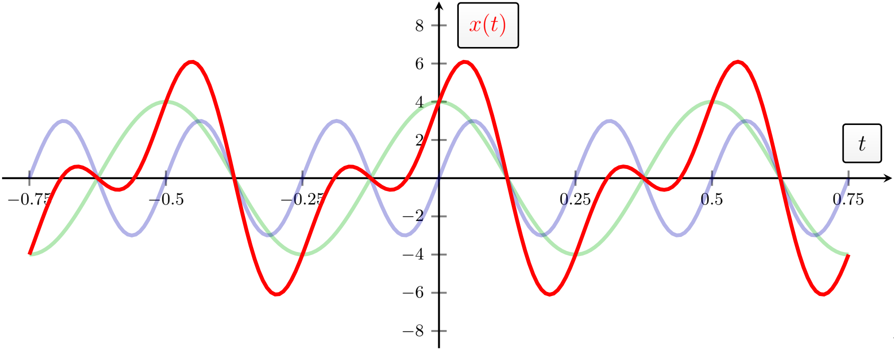
Answer: (Pretty well) yes.

Then we need to a way of determining the Fourier Series coefficients, that is:

$$\boxed{\text{Given } x(t) \text{ find } \{a_k\} \text{ for all } k \in \mathbb{Z}.}$$

(Simple) Example 1: If

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$



we find the fundamental frequency and fundamental period as

$$\omega_0 = 4\pi, \qquad T = \frac{2\pi}{\omega_0} = \frac{1}{2},$$

For

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

the Fourier Series coefficients are:

$$\begin{aligned} a_1 &= 2 \\ a_{-1} &= 2 \\ a_2 &= -3j/2 \\ a_{-2} &= 3j/2 \\ a_k &= 0, \quad \text{otherwise} \end{aligned}$$

meaning

$$x(t) = 2e^{j\,4\pi t} + 2e^{-j4\pi t} - \frac{3j}{2}e^{j\,8\pi t} + \frac{3j}{2}e^{-j8\pi t}$$

For **real-valued** signals, other Fourier Series expressions are possible

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} \big(\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t)\big)$$

or

$$x(t) = \gamma_0 + \sum_{k=1}^{\infty} \big(\gamma_k \cos(k\omega_0 t + \theta_k)\big)$$

We'll stick to the complex exponential form. Then both positive and negative frequencies need to be used:

$$e^{jk\omega_0 t} \qquad e^{-jk\omega_0 t}$$

Fourier Stuff – Fourier Coefficients

For (contrived)

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

we could read off (almost) the Fourier Series coefficients. But how do we compute them for a general function?

Given an arbitrary $x(t)$, how would we compute the coefficient a_{-37} in the expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \quad ?$$

Consider

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \int_T \overbrace{\Bigg(\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \Bigg)}^{x(t)} e^{-jn\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \Bigg(\int_T e^{j\,(k-n)\omega_0 t}\, dt \Bigg)$$

But

$$\int_T e^{j\,(k-n)\omega_0 t}\, dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

$$= T\,\delta[k-n]$$

This is orthogonality. Here T is just a constant equal to the fundamental period.

Fourier Stuff – Fourier Coefficients (cont'd)

Note we have written

$$\int_T$$

rather than

$$\int_0^T$$

because the integrand is periodic with period T. As such

$$\int_0^T \equiv \int_{-T/2}^{T/2} \equiv \int_{\epsilon}^{T+\epsilon}$$

(assuming the integrand is periodic with period T).

Fourier Stuff – Fourier Coefficients (cont'd)

Hence

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \Big(\int_T e^{\,j\,(k-n)\omega_0 t}\, dt \Big)$$

$$= T \sum_{k=-\infty}^{\infty} a_k\, \delta[k-n] = a_n T$$

Fourier Stuff – Fourier Coefficients (cont'd)

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t + T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

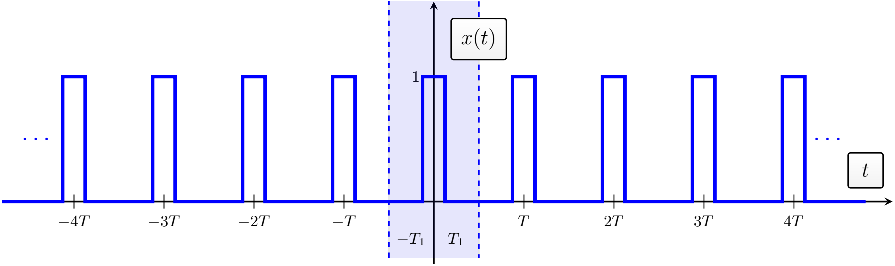
$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

28 Fourier Stuff

- Review Key Observation
- Examples
- CT Periodic Signals
- Fourier Series
- Classical Fourier Series
- Fourier Coefficients
- Periodic Rectangular Wave

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \le T/2$$



The Fourier coefficients can be calculated:

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt = \frac{2T_1}{T}$$

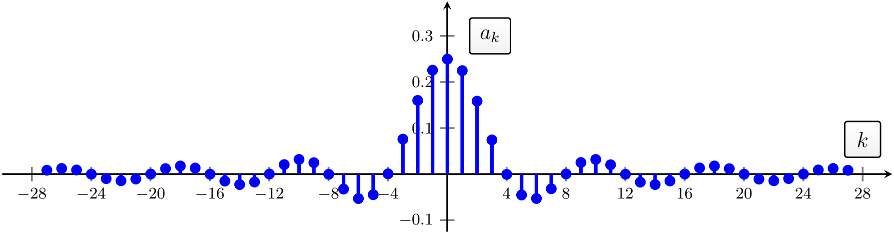
is the average or DC value, and

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad \text{recalling that} \quad \omega_0 = \frac{2\pi}{T}$$

Called a "sinc" function.

With $T_1 = T/8$ the periodic rectangular wave has Fourier coefficients, $\{a_k\}$, as follows



Fourier Stuff – Periodic Rectangular Wave (cont'd)

In the following we take more and more terms in the truncated Fourier series

$$\sum_{k=-L}^{L} a_k \, e^{jk\omega_0 t}$$

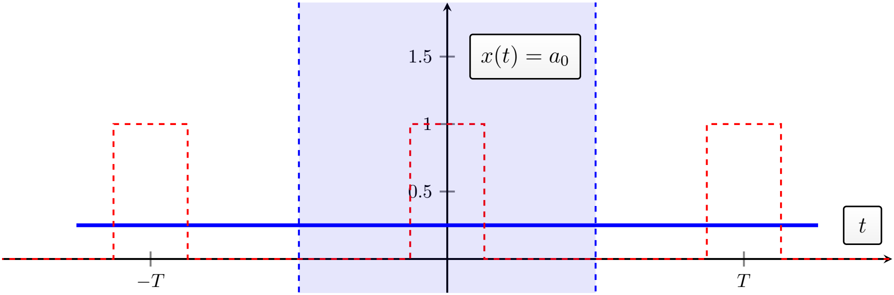
where

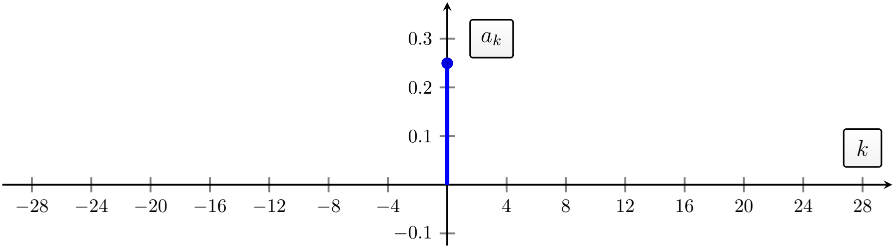
$$\begin{aligned} a_k &= \frac{\sin(k\,\omega_0 T_1)}{k\pi} = \frac{\sin(k\,2\pi T_1/T)}{k\pi} \\ &= \frac{\sin(k\,\pi/4)}{k\pi}, \end{aligned}$$

given $T_1 = T/8$, varying L as follows:

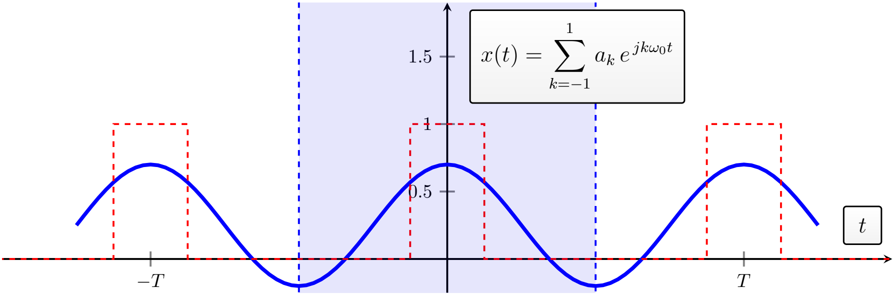
$$L = 0,\ 1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 9,\ 19,\ 27$$

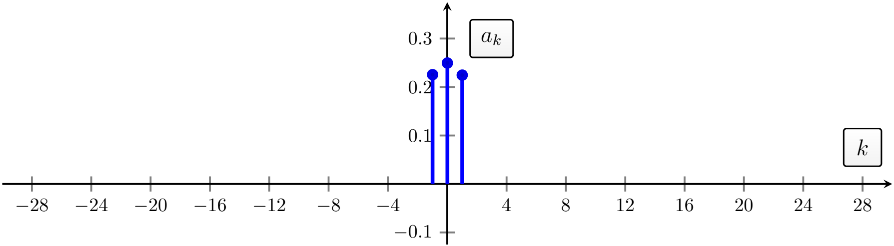
Fourier Stuff – Periodic Rectangular Wave



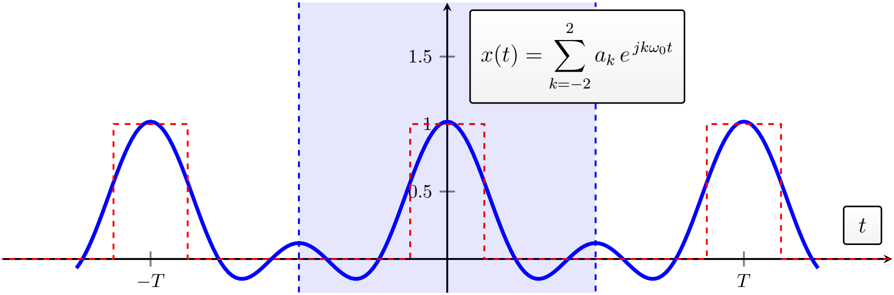


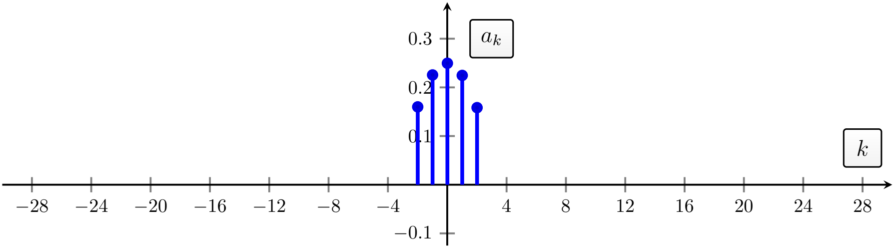
Fourier Stuff – Periodic Rectangular Wave



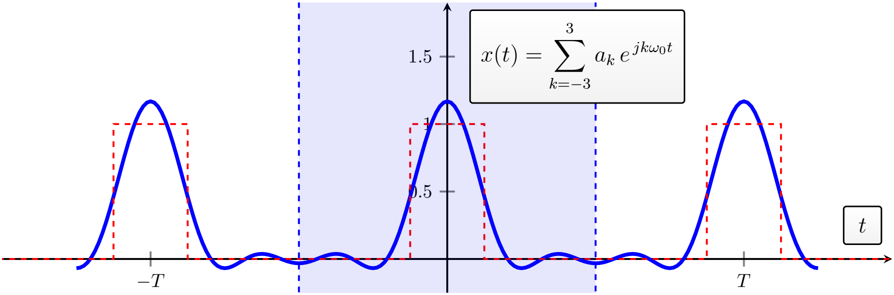


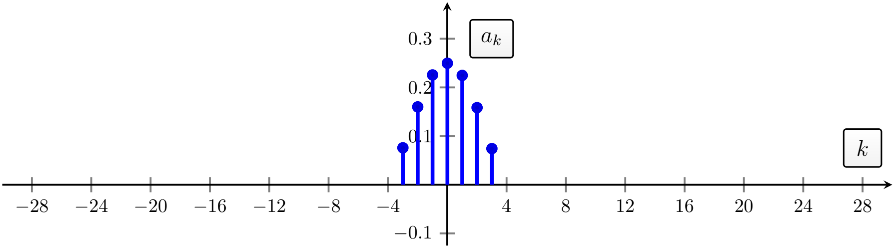
Fourier Stuff – Periodic Rectangular Wave



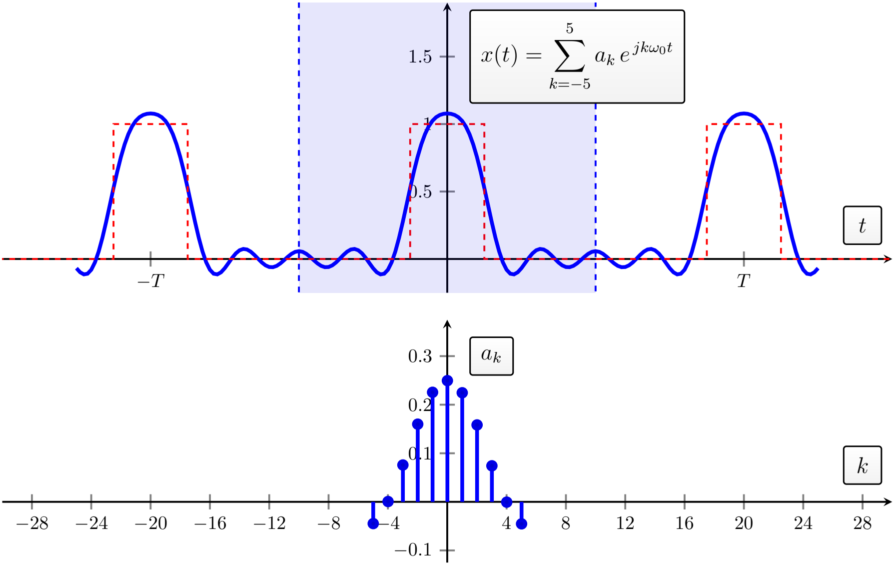


Fourier Stuff – Periodic Rectangular Wave

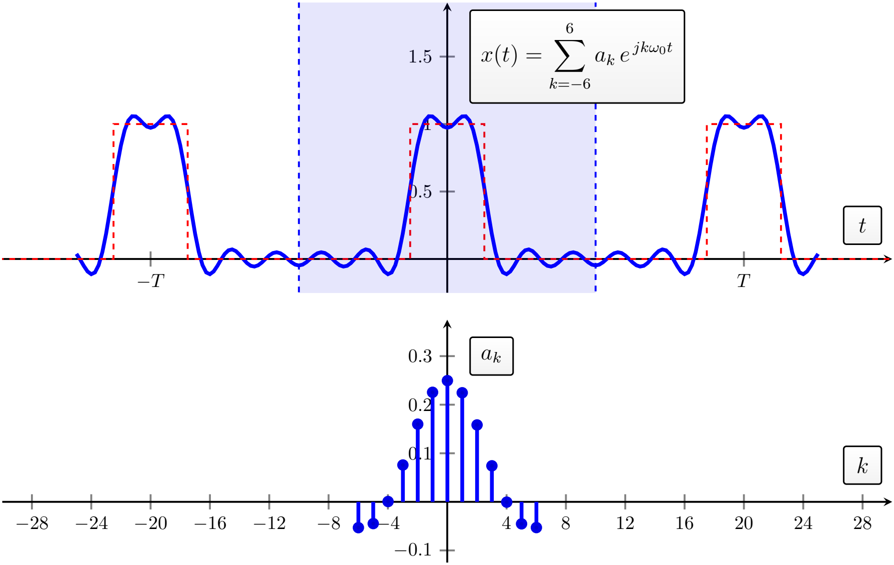




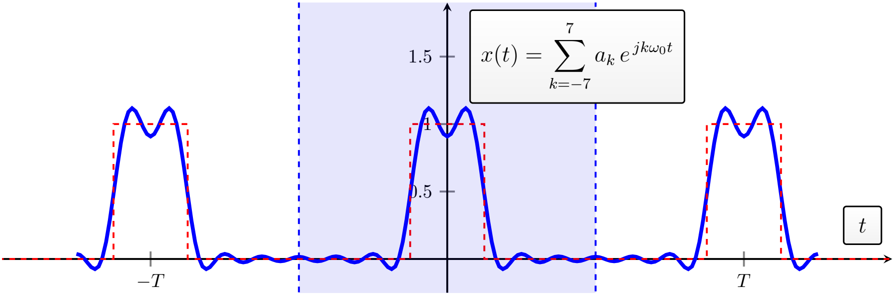
Fourier Stuff – Periodic Rectangular Wave

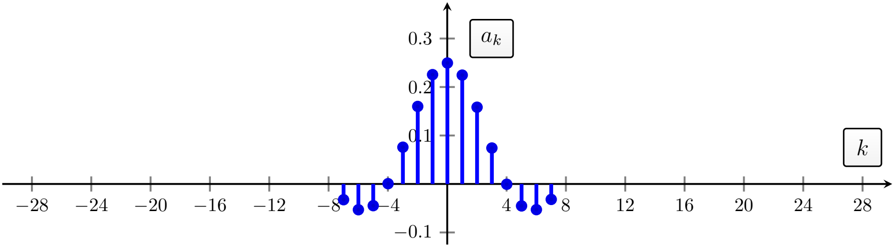


Fourier Stuff – Periodic Rectangular Wave

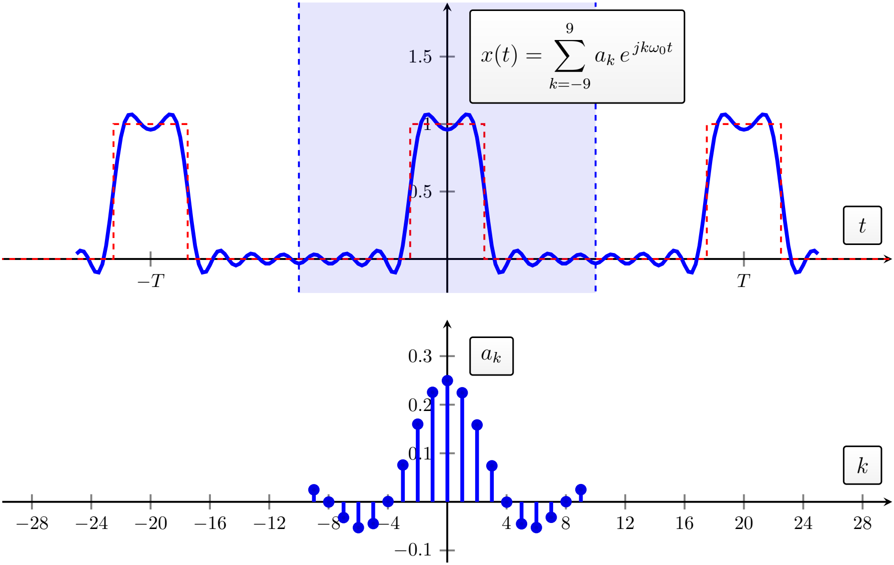


Fourier Stuff – Periodic Rectangular Wave



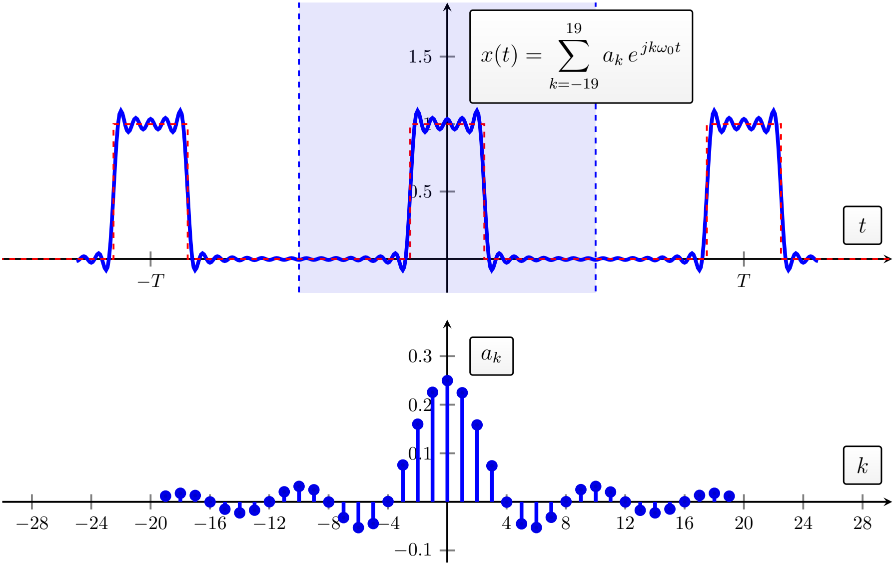


Fourier Stuff – Periodic Rectangular Wave

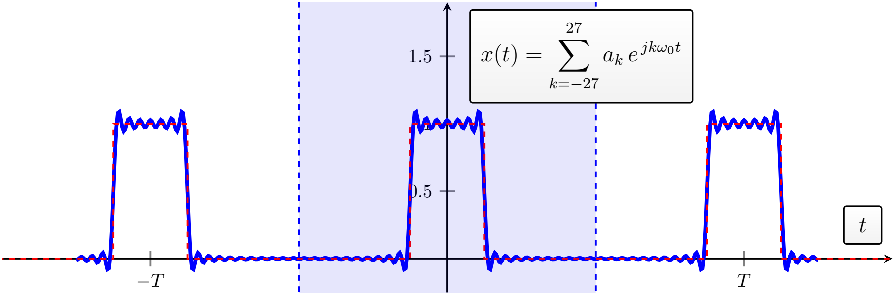


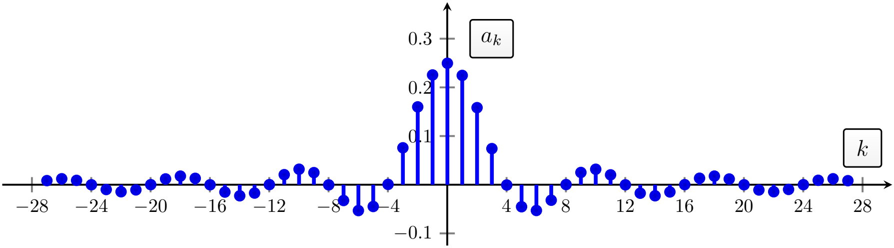
Fourier Stuff – Periodic Rectangular Wave

$$x(t) = \sum_{k=-19}^{19} a_k \, e^{jk\omega_0 t}$$



Fourier Stuff – Periodic Rectangular Wave





29 **Fourier Series Properties**
- Notation
- Motivation
- Linearity
- Conjugate Symmetry
- Time Shift
- Parseval's Relation
- Multiplication Property

30 **Periodic Impulse Train**
- Definition
- Time Shifted PIT

31 **Periodic Convolution**
- Definition
- Key Result

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- Definition
- Key Result

Fourier Series Properties – Notation

Previously we established:

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

We adopt the shorthand

$$x(t) \xleftrightarrow{\mathscr{F}} a_k$$

and use this extensively.

Part 10 Outline

29 Fourier Series Properties

Fourier Series Properties – Motivation

What's coming:

- We're lazy, we don't want to do work if we can avoid it. We don't want to do something complicated if there is an easy way or a trick available.
- Computing Fourier Series is not hard but can be tedious.
- With a few Fourier Series we can synthesize others. A signal derived from an original signal via a simply transformation such as time-shift, scaling, compression, etc., should have a Fourier Series some how related to the original signal.
- There are many of these, see Table 3.1 in text.

Motivation – Motivation (cont'd)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Fourier Series Properties – Linearity

If

$$x(t) \xleftrightarrow{\mathscr{F}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\mathscr{F}} b_k$$

then

$$\boxed{\alpha\, x(t) + \beta\, y(t) \xleftrightarrow{\mathscr{F}} \alpha\, a_k + \beta\, b_k}$$

> *"Linear combinations of signals leads to identical linear combinations of the Fourier coefficients."*

For example, this can be used with parallel connection of signals.

$x(t)$ real valued $\quad \Rightarrow \quad \boxed{a_{-k} = \overline{a_k} \equiv a_k^* \quad \text{(conjugate)}}$

"Negative index Fourier coefficients are the complex conjugates of the positive index Fourier coefficients whenever the time domain signal is real valued (zero imaginary part)."

This implies for real valued signals

$$\mathit{Re}\{a_k\} \text{ is even}, \quad \mathit{Im}\{a_k\} \text{ is odd}$$

$$|a_k| \text{ is even}, \quad \angle a_k \text{ is odd}$$

Note that with any complex number written as $b\,e^{j\,\omega}$ with real $b > 0$ and angle ω, then $\angle b\,e^{j\,\omega} = \omega$, and $|b\,e^{j\,\omega}| = b$.

Fourier Series Properties – Time Shift

Given

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k$$

then

$$\boxed{x(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow} a_k\, e^{-jk\omega_0 t_0} \equiv a_k\, e^{-jk\,2\pi t_0/T}}$$

“Time shifting a signal introduces a linear phase shift $\propto t_0$ in the Fourier coefficients.”

It can be shown:

$$\boxed{\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2}$$

"Average energy is the time domain is the same as the energy in the frequency domain."

Given

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} b_k$$

then

$$\boxed{x(t)\,y(t) \overset{\mathscr{F}}{\longleftrightarrow} c_k \triangleq \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell} = a_k \star b_k}$$

"The (pointwise) product of two periodic signals (of the same period, T) is the convolution of the Fourier coefficients."

For example, this can be used with serial/cascade connection of signals/systems.

Fourier Series Properties – Multiplication Property

With Fourier Series pairs (time-domain and frequency-domain) convolution in one domain is multiplication in the other.

At this point we have only considered the product in time-domain leading to convolution in frequency-domain.

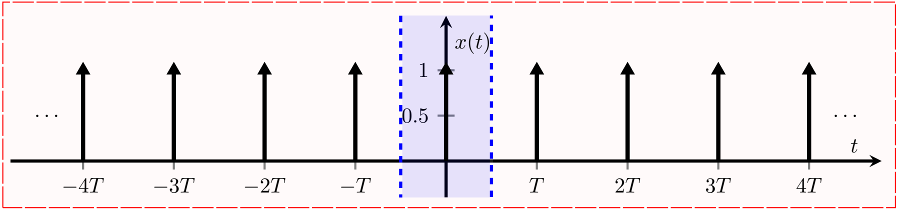
$$
\begin{aligned}
x(t)\,y(t) &= \sum_{\ell} a_\ell\, e^{j\,\ell\omega_0 t} \sum_{m} b_m\, e^{j\,m\omega_0 t} \\
&= \sum_{\ell}\sum_{m} a_\ell b_m\, e^{j\,(\ell+m)\omega_0 t} \\
&= \sum_{k} \underbrace{\Big(\sum_{\ell} a_\ell b_{k-\ell}\Big)}_{c_k} e^{jk\omega_0 t} \qquad (\ell + m = k)
\end{aligned}
$$

Part 10 Outline

Part 10 Outline

Periodic Impulse Train – Definition

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 0}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt$$

$$= \frac{1}{T} \quad \text{for all } k$$

Periodic Impulse Train – Definition (cont'd)

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k = \frac{1}{T},$$

that is,

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

has Fourier Series

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is known as the **sampling function**. As its name implies it is useful for sampling CT signals uniformly at time instants which are a multiple of T:

$$\begin{aligned} y_s(t) &\triangleq y(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} y(nT)\, \delta(t - nT) \end{aligned}$$

Part 10 Outline

Periodic Impulse Train – Time Shifted PIT

Consider time shifting an **arbitrary** signal $x(t)$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k$$

by the **specific** value of half the period, $t_0 = T/2$. By the "Time Shift" property, a linear phase shift is introduced relative to $x(t)$ in the Fourier coefficients

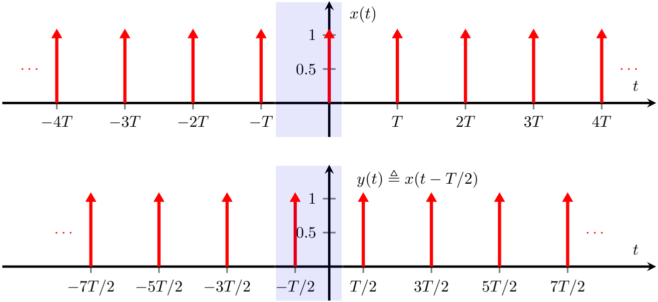
$$y(t) \triangleq x(t - T/2) \overset{\mathscr{F}}{\longleftrightarrow} a_k e^{-jk\pi} \equiv (-1)^k a_k$$

from

$$e^{-jk\omega_0 T/2} = e^{-jk\pi}, \qquad e^{-j\pi} = -1$$

Next we consider the case when $x(t)$ is the impulse train.

Periodic Impulse Train – Time Shifted PIT (cont'd)



$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{T} \quad \Rightarrow \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{(-1)^k}{T}$$

Periodic Impulse Train – Time Shifted PIT

- The Fourier coefficients for both the PIT and $T/2$ time shifted PIT are purely real. Note that in the time domain both the PIT and $T/2$ time shifted PIT are even functions.
- A more general delay, e.g., $t_0 = 0.3498T$, would lead to a time shifted PIT with complex Fourier coefficients.

Periodic Convolution – Definition

By "Periodic Convolution" we mean convolution of periodic functions (of the same period T) in the time domain. But we need further qualifications.

Previously we found that multiplication in the time domain led to "discrete" convolution in the Fourier series (frequency) domain.

Doing the obvious

$$x(t) \star y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$$

has the hazard of often being infinite. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) \star y(t) = \infty$$

So we define a modified periodic convolution.

Define Periodic Convolution to an integral over any one period, e.g., $-T/2$ to $T/2$. (The ends will justify the means.)

$$z(t) \triangleq \int_T x(\tau) y(t-\tau)\, d\tau$$

Let

$$x(t) \xleftrightarrow{\mathscr{F}} a_k, \quad y(t) \xleftrightarrow{\mathscr{F}} b_k, \quad z(t) \xleftrightarrow{\mathscr{F}} c_k$$

Part 10 Outline

Periodic Convolution – Key Result

Computing the Fourier coefficients of $z(t)$,

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T}\int_T \left(\int_{-T/2}^{T/2} x(\tau)y(t-\tau)\, d\tau \right) e^{-jk\omega_0 t}\, dt \\
&= \int_T \underbrace{\left(\frac{1}{T}\int_T y(t-\tau)\, e^{-jk\omega_0 (t-\tau)} dt \right)}_{b_k} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= b_k \int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= T\, a_k b_k
\end{aligned}
$$

Multiplication in frequency (Fourier series domain).

Periodic Convolution – Key Result (cont'd)

In summary, for signals sharing a common period T,

$$x(t) \xleftrightarrow{\mathscr{F}} a_k, \quad y(t) \xleftrightarrow{\mathscr{F}} b_k, \quad z(t) \xleftrightarrow{\mathscr{F}} c_k$$

related through Periodic Convolution

$$z(t) \triangleq \int_T x(\tau) y(t-\tau)\, d\tau$$

then

$$c_k = T\, a_k b_k$$

Alternatively

$$\left(\frac{1}{T}\int_T x(\tau) y(t-\tau)\, d\tau\right) \xleftrightarrow{\mathscr{F}} a_k b_k$$