System Properties – Examples

$x(t-t_0) \to y(t-t_0)$

do proof

System	Linear	Time-Invariant	Causal	Memoryless
$y[n] = 2x[n]$	✓	✓	✓	✓
$y[n] = 2x[n] + 3$	✗	✓	✓	✓
$y[n] = x[-n]$	✓	✗	✗	✗
$y(t) = tx(t)$	✓	✗	✓	✓
$y(t) = \cos(3t)x(t)$	✓	✗	✓	✓
$y(t) = \sin(x(t))$	✗	✓	✓	✓
$y(t) = t^2 x(t-1)$	✓	✗	✓	✗

$\alpha_1 x_1(t) + \alpha_2 x_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$

$y[n] = x[-n]$

time-invariant: $\underbrace{x[n-n_0]}\rightarrow \underbrace{y[n-n_0]}$

1. $y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$

2. $x_1[n] = x[n-n_0]$

$\rightarrow y_1[n] = x_1[-n] = x[-n-n_0] \neq y[n-n_0]$

$\Rightarrow$ system is time varying

linear: $\alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \underbrace{\alpha_1 y_1[n] + \alpha_2 y_2[n]}$

1. $x_1[n] \rightarrow y_1[n] = x_1[-n]$, $x_2[n] \rightarrow y_2[n] = x_2[-n]$

2. $\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1 x_1[-n] + \alpha_2 x_2[-n]$

3. $x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ $\quad$)) $\Rightarrow$ linear

$y_3[n] = x_3[-n] = \alpha_1 x_1[-n] + \alpha_2 x_2[-n]$

$y[n] = 2x[n] + 3$

time-invariant $\quad x[n-n_0] \rightarrow y[n-n_0]$

1. $y[n-n_0] = 2x[n-n_0] + 3$
2. $x_1[n] = x[n-n_0] \Rightarrow y_1[n] = 2x_1[n] + 3 = 2x[n-n_0] + 3$

$= y[n-n_0] \Rightarrow$ time-invariant

linear $\quad \alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$

1. $x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$

   $x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3$
2. $\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1 (2x_1[n] + 3) + \alpha_2 (2x_2[n] + 3)$
3. $x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$\rightarrow y_3[n] = 2x_3[n] + 3 = 2(\alpha_1 x_1[n] + \alpha_2 x_2[n]) + 3$

$\neq \alpha_1 y_1[n] + \alpha_2 y_2[n] \Rightarrow$ non-linear

Part 4 Outline

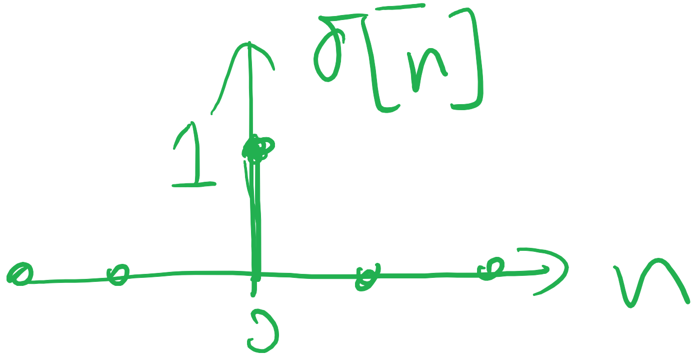
18 Summary to this Point

19 **DT LTI Systems**

- Using DT LTI Property
- Superposition + Time-Invariance
- Signal Representation
- Unit Pulse Response
- Examples
- More General TV Case
- Commutative Property

LTI

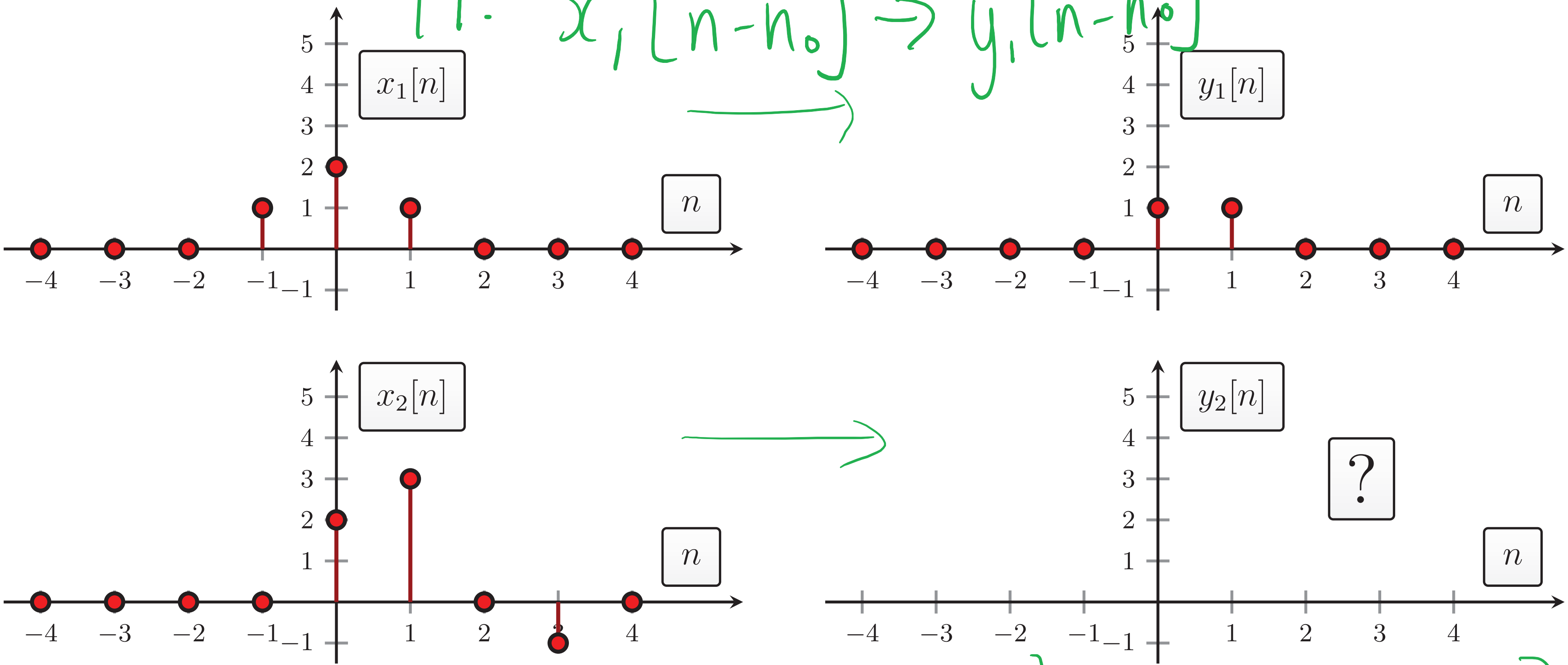
- Linear Time-Invariant (LTI) Systems
- That is systems which have the properties of linearity and time-invariance
- Practically important - model many physical processes
- Extensive analysis tools for LTI systems - e.g. convolution
- Response of an LTI system can be understand from the response to a special type of input signal
- LTI systems can be characterized by a "signal" called a unit sample (DT) or an impulse (CT)



DT LTI Systems – Using DT LTI Property

A system is linear and time-invariant. It is unknown except that output $y_1[n]$ occurs for input $x_1[n]$, that is, $x_1[n] \longrightarrow y_1[n]$. Can we work out the output $y_2[n]$ of this system when the input is $x_2[n]$ ($x_2[n] \longrightarrow y_2[n]$)?

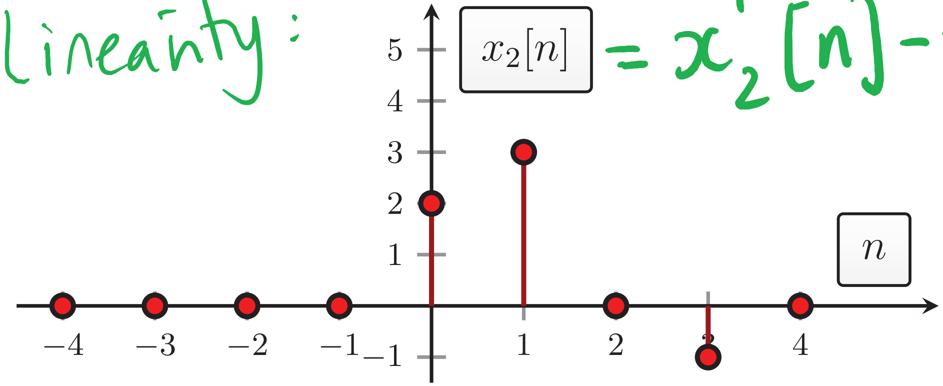
TI: $x_1[n-n_0] \to y_1[n-n_0]$

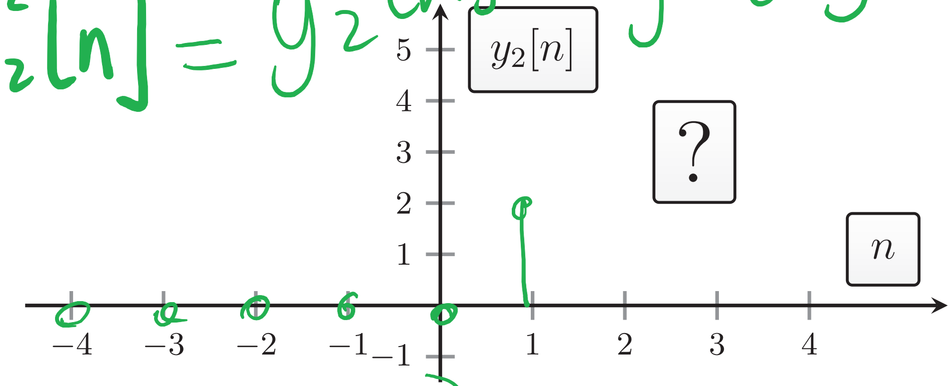


L: $\alpha_1 x_1[n-n_0^1] + \alpha_2 x_1[n-n_0^2] \to \alpha_1 y_1[n-n_0^1] + \alpha_2 y_2[n-n_0^2]$

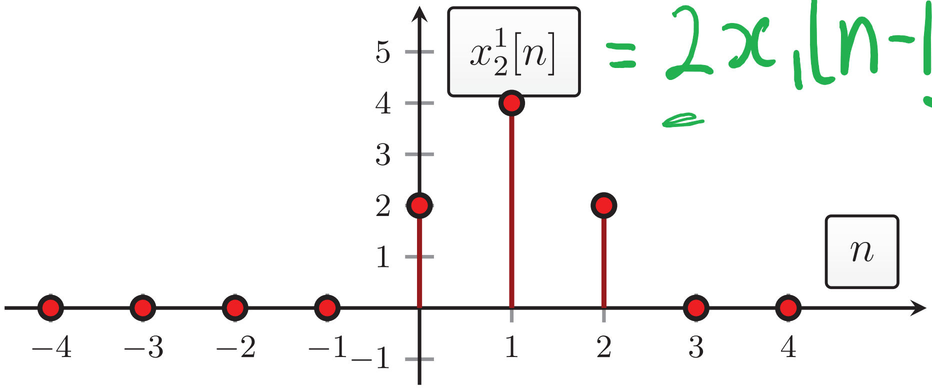
DT LTI Systems – Using DT LTI Property

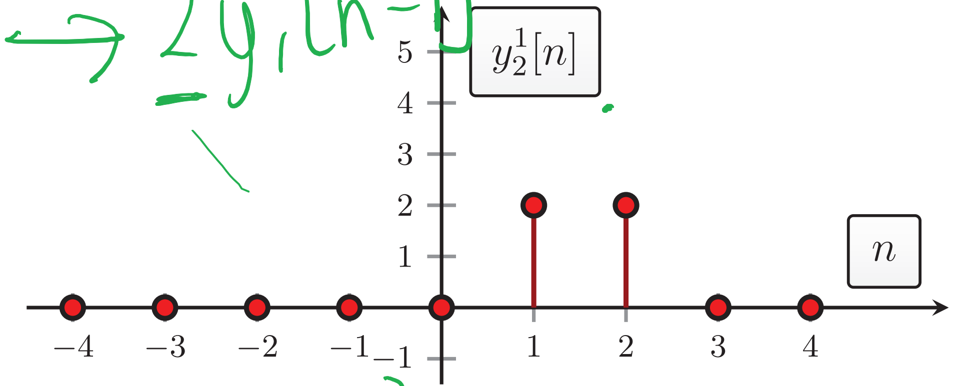
Linearity: $x_2[n] = x_2^1[n] - x_2^2[n] = y_2^1[n] - y_2^2[n]$



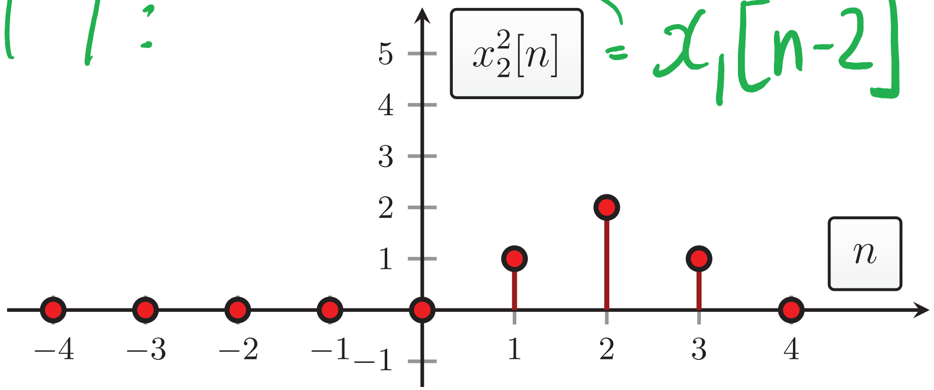


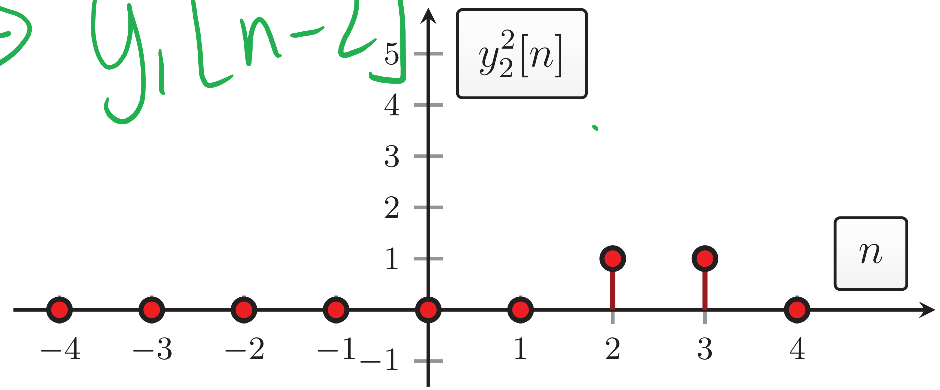
$x_2^1[n] = 2x_1[n-1] \rightarrow 2y_1[n-1]$





TI: $x_2^2[n] = x_1[n-2] \rightarrow y_1[n-2]$





DT LTI Systems – Using DT LTI Property

Note that $x_2^1[n]$ and $x_2^2[n]$ are shifted and scaled versions of $x_1[n]$, and further that

$$x_2[n] = x_2^1[n] - x_2^2[n]$$

So by linearity and time-invariance

$$y_2[n] = y_2^1[n] - y_2^2[n]$$

Part 4 Outline

DT LTI Systems – Superposition + Time-Invariance

e.g. scaled $x_1[n]$

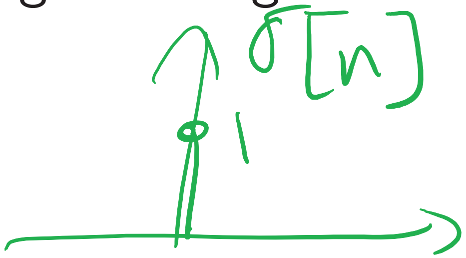
- By superposition and time-invariance

$$x_k[n] \longrightarrow y_k[n]$$

$$\boxed{x[n] = \sum_k a_k x_k[n] \longrightarrow y[n] = \sum_k a_k y_k[n]}$$

$x_k[n]$ is an indexed set of candidate building blocks.

- Seek basic building blocks to represent any signal, that is, linear combinations of these building blocks.
- The response of LTI systems to these basic building blocks, if properly chosen, is elegant and powerful.
- In DT the natural choice of building block signals are **time shifted unit samples**.



DT LTI Systems – Signal Representation

DT

Finding the output of a linear system:

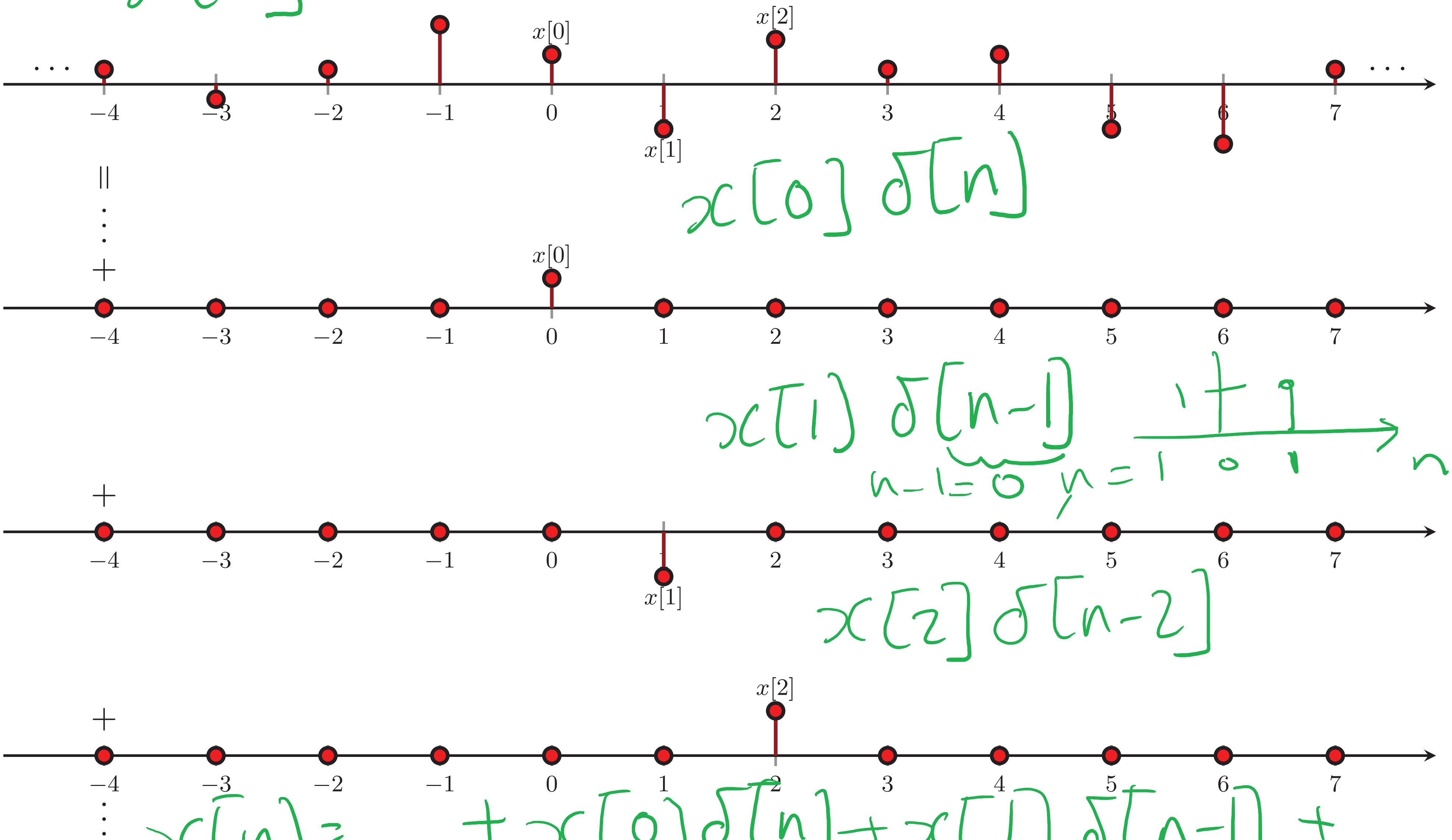
1 • **Any signal can be represented as a combination of scaled and shifted DT unit impulses.**

2 • Hence, find the response of a system to a unit impulse.

3 • By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.

DT LTI Systems – Signal Representation (cont'd)

$x[n]$



$x[0]\,\delta[n]$

$x[1]\,\delta[\underbrace{n-1}]$

$n-1=0 \quad n=1$

$x[2]\,\delta[n-2]$

$x[n] = \ldots + x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \ldots$

DT LTI Systems – Signal Representation

- Unit impluse/sample (this is a signal)

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- Expand $x[n]$ in terms of shifted unit samples

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + \\ x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[\underbrace{n-k}]}$$

"Sifting Property" of the unit sample $\delta[n]$ $n-k=0$

Part 4 Outline

18 Summary to this Point

19 **DT LTI Systems**

- Using DT LTI Property
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- Examples
- More General TV Case
- Commutative Property

DT LTI Systems – Unit Pulse Response h[n]

DT

Finding the output of a linear system:

1 • Any signal can be represented as a combination of scaled and shifted DT unit impulses.

2 • **Hence, find the response of a system to a unit impulse.**

3 • By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.

DT LTI Systems – Unit Pulse Response h[n]

definition :

$$\delta[n] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow h[n]$$

y(n) =

time -invariance:

$$\delta[n-k] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow h[n-k]$$

DT LTI Systems – Unit Pulse Response

Finding the output of a linear system:

- Any signal can be represented as a combination of scaled and shifted DT unit impulses.

- Hence, find the response of a system to a unit impulse.

- **By superposition, we can find the DT output for any DT input, using the impulse response. This becomes the convolution sum.**

DT LTI Systems – Unit Pulse Response

$$x[n] \longrightarrow \boxed{\text{DT LTI System } h[n]} \longrightarrow y[n]$$

$$\boxed{= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]} \qquad \boxed{= \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]}$$

convolution

- For an LTI system, define **unit sample response** $h[n]$

$$\delta[n] \longrightarrow h[n],$$

that is, $h[n]$ is the response of the LTI system to a kick at $\overset{n}{\cancel{k}} = 0$.

- But from time-invariance

$$\delta[n-k] \longrightarrow h[n-k],$$

that is, kick at k is the same as a kick at 0 shifted by k. Hence, $h[n]$ completely characterizes a LTI System !!!

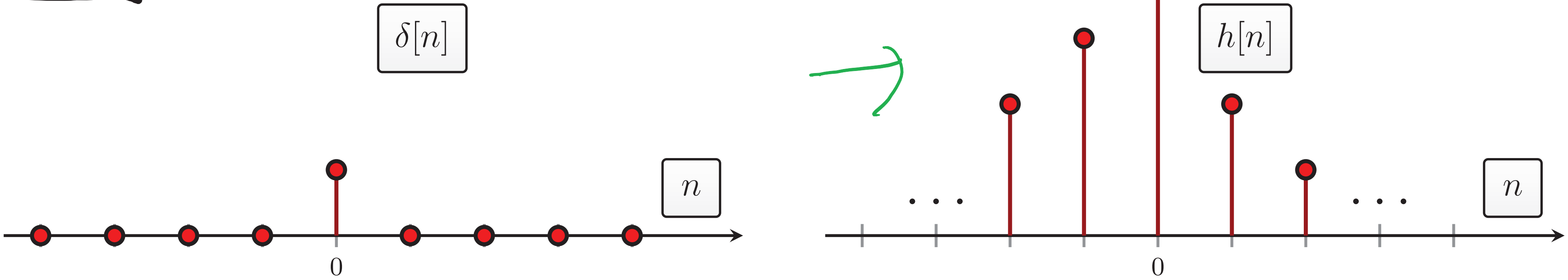
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$

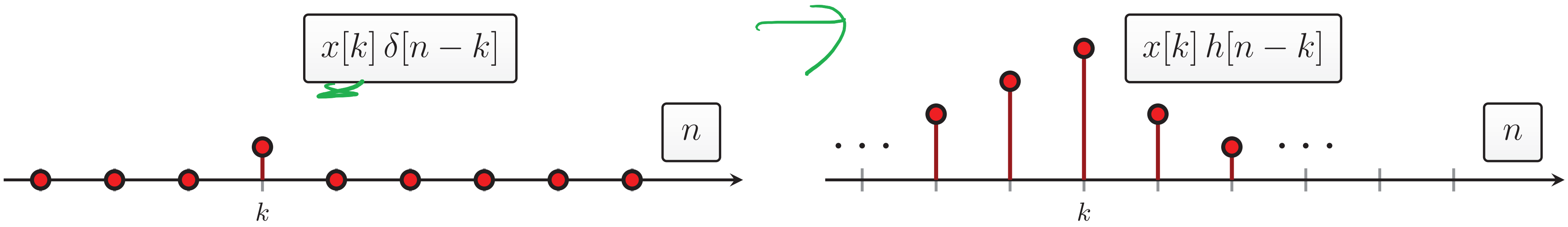
DT LTI Systems – Unit Pulse Response

Interpretation:

Input

output





An accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is causal LTI.

- Unit sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

  where $u[n]$ is the unit step.

- In convolution form

h[n]

$$x[n] \star u[n] = \sum_{k=-\infty}^{n} x[k]$$

Part 4 Outline

DT LTI Systems – More General TV Case

$\delta[n-k] \not\to h[n-k]$

- Response (output $y[n]$) of a **time-varying (TV) linear system** to $x[n]$ can be thought of in terms of superposition of unit sample responses.
- So we want to figure out the response in $y[n]$ to the $\delta[n-k]$ components in $x[n]$.
- Define $h_k[n]$ as the response to $\delta[n-k]$:

$$\delta[n-k] \longrightarrow h_k[n]$$

- By superposition, this linear (generally time-varying) system is given by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

- We need to know the response of a time-varying linear system to an **infinite** number of shifted unit samples to fully characterize it.

DT LTI Systems – More General TV Case

- If I give a linear system a kick then it has a response.
- But the response may be different depending on when I kick it. It can still be linear.
- If the response is different depending on when I kick it it is called **time-varying (TV)**. This is why we wrote

$$h_k[n]$$

which is the result of a kick at time k, $\delta[n-k]$.

Part 4 Outline

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DT LTI Systems – Commutative Property

From the definition of convolution it can be shown (we do this later in Part 5)

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$

- Can be used to reinterpret LTI systems. $x[n]$ as input signal to a system with unit sample response $h[n]$ is the same as $h[n]$ as input signal to a system with unit sample response $x[n]$.
- Can be used as a tool or trick:

$$s[n] = u[n] \star h[n] = h[n] \star u[n]$$

the second is the "accumulator" system. Hence

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

Part 5 Outline

20. **DT Convolution**
    - Street Version
    - Graphical Flip and Shift
    - Other DT Convolution Methods
    - Convolution with Impulses
    - Commutative Property
    - Distributive Property
    - Associative Property
21. **DT System Properties**
    - Causality Property
    - Stability Property
    - Review of System Properties
22. **Difference Equation of DT System**
    - Direct-Form I implementation
23. **Finding the Impulse Response of a DT System**

Part 5 Outline

DT Convolution – Street Version

Dictionary | convolution

con•vo•lu•tion |ˌkänvəˈlo͞o sн ən|

noun

1 (often **convolutions**) a coil or twist, esp. one of many : *crosses adorned with elaborate convolutions.*

- a thing that is complex and difficult to follow : *the convolutions of farm policy.*
- a sinuous fold in the surface of the brain.
- the state of being coiled or twisted, or the process of becoming so : *the flexibility of the polymer chain allows extensive convolution.*

2 (also **convolution integral**) Mathematics a function derived from two given functions by integration that expresses how the shape of one is modified by the other.

- a method of determination of the sum of two random variables by integration or summation.

DERIVATIVES

con•vo•lu•tion•al |-sн ənl| |ˈkɑnvəˈluʃənl| |ˈkɑnvəˈluʃnəl| adjective

ORIGIN mid 16th cent.: from medieval Latin ***convolutio(n-)***, from ***convolvere 'roll together'*** (see CONVOLVE).

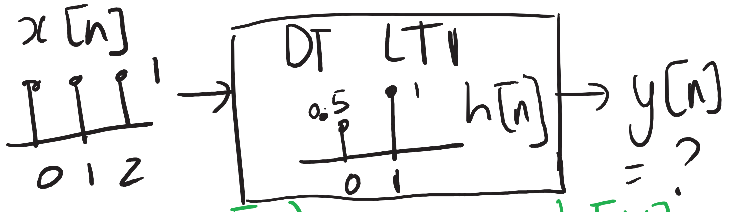
Part 5 Outline

DT Convolution – Graphical Flip and Shift

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$

Looking at the equation for convolution we can arrive at the following procedure:

1. Choose one signal to be $x[n]$, the other is then $h[n]$
2. Draw them on the k-axis $x[k]$ and $h[k]$
3. Flip $h[k]$ around $k = 0$ (the y-axis) $h[-k]$
4. Shift the flipped version of h to the right by n $h[n-k]$
5. Multiply $x[k]$ by flipped/shifted version of $h[k]$ and sum over all values of k
6. The summation only gives you $y[n]$ for one value of n



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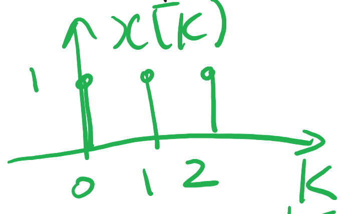

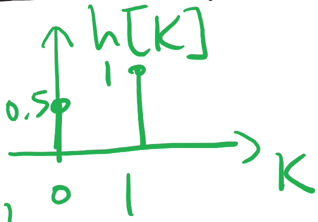

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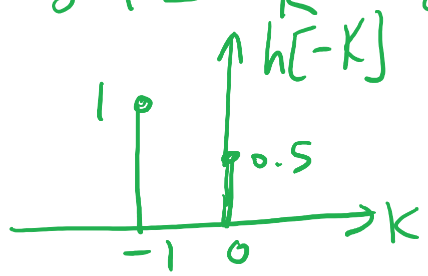
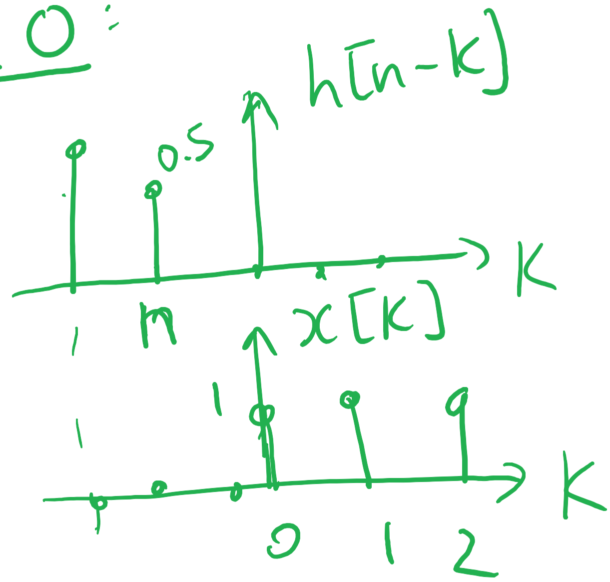


$n<0$:

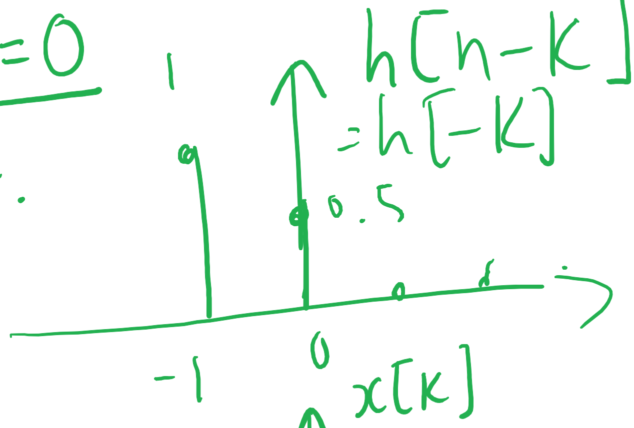
4.

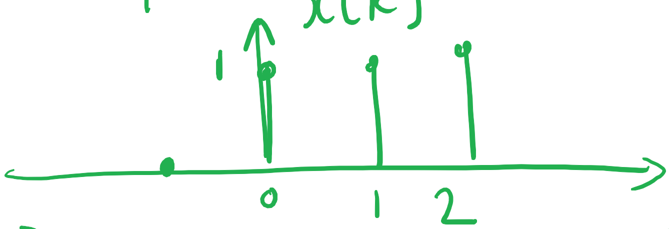


5. $y[n]=0 \quad \forall\, n<0$

$n=0$

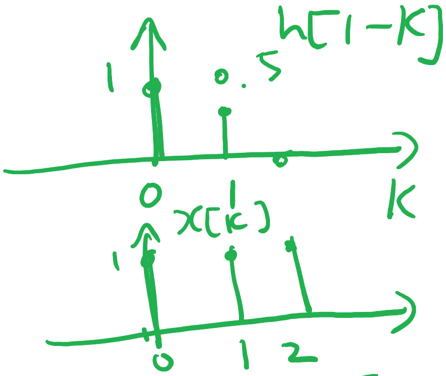
4.





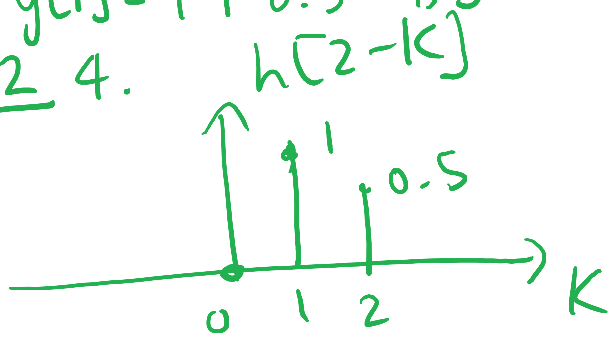
5. $y[0]=0+0.5\times 1+0+0=0.5$

$n=1$ 4.



5. $y[1]=1+0.5=1.5$

$n=2$ 4.



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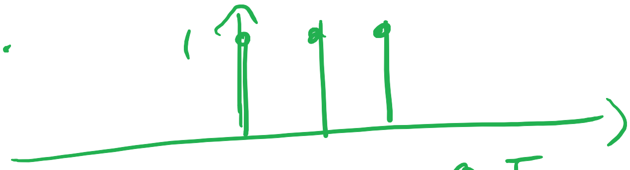
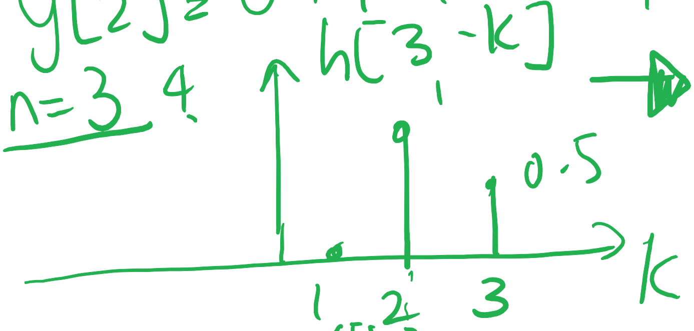


$y[2]=0+1+0.5=1.5$

$n=3$ 4.



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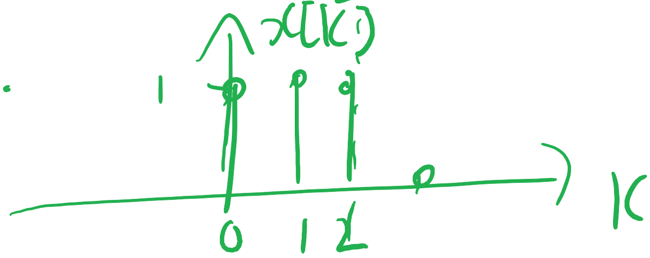
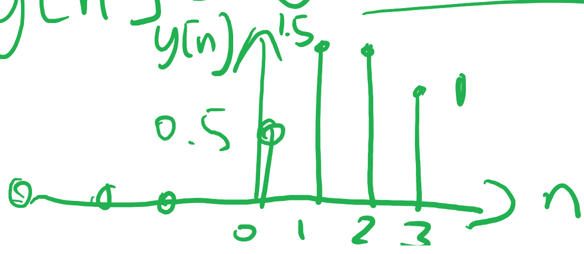


$y[3]=1$

$n>3$

$y[n]=0$

$$\boxed{y[n]=0.5\delta[n]+1.5\delta[n-1]+1.5\delta[n-2]+\delta[n-3]}$$



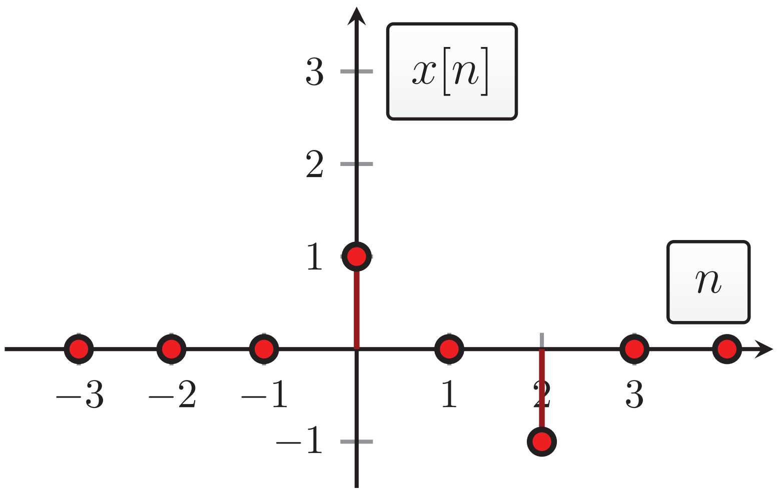
DT Convolution – Graphical Flip and Shift

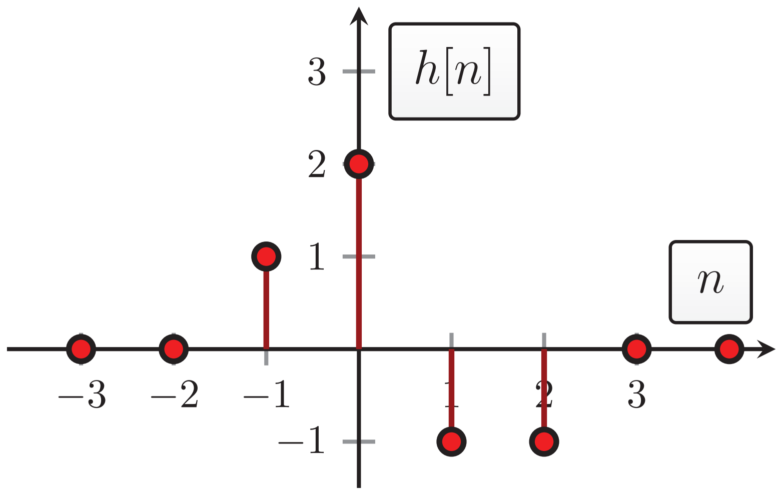
Consider the convolution of

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < 0, \text{ or } n > 2) \end{cases}$$

and (non-causal) pulse response

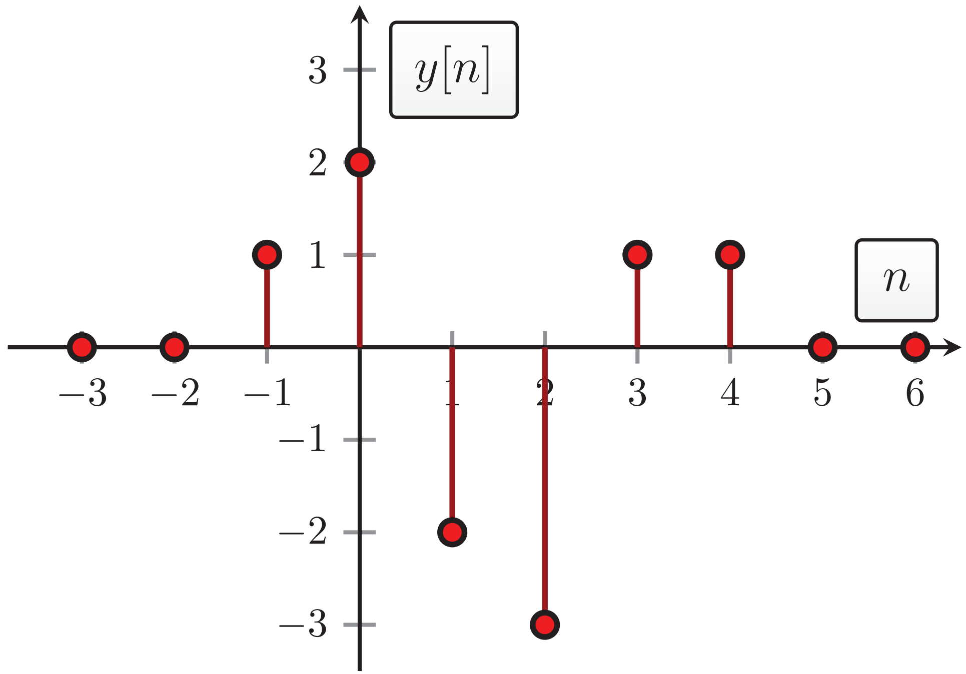
$$h[n] = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ -1 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < -1, \text{ or } n > 2) \end{cases}$$

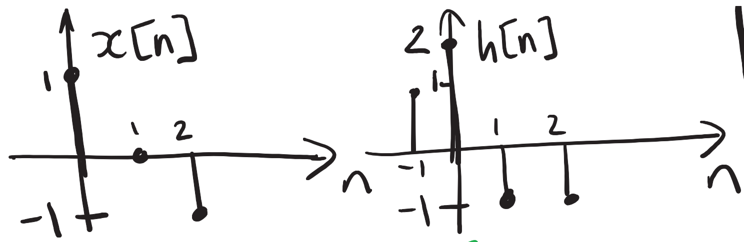




The convolution is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ -1 - 1 = -2 & n = 1 \\ -1 - 2 = -3 & n = 2 \\ 1 & n = 3 \\ 1 & n = 4 \\ 0 & \text{otherwise } (n < -1, \text{ or } n > 4) \end{cases}$$





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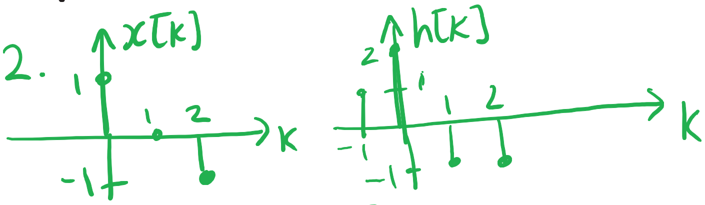

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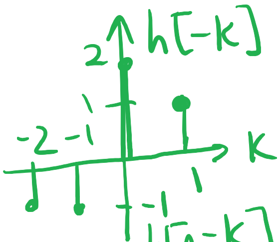
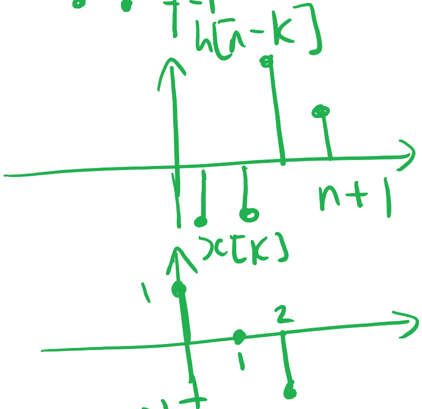


4.

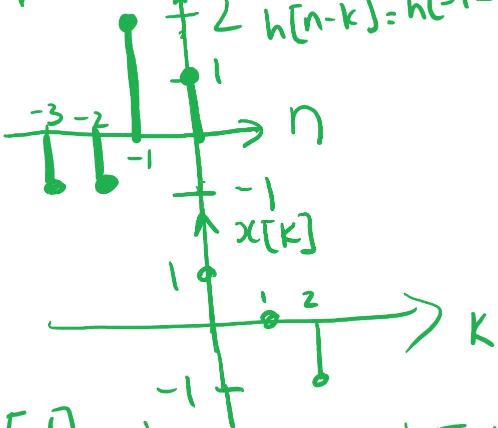


5. no overlap for $n+1 < 0$

$\Rightarrow y[n] = 0 \quad \forall\, n < -1$ ($\forall$ = for all)

4. for $n = -1$

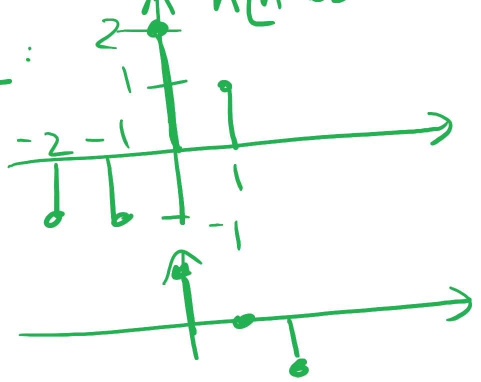
$h[n-k] = h[-1-k]$



5. $y[-1] = 1$

4. for $n = 0$:

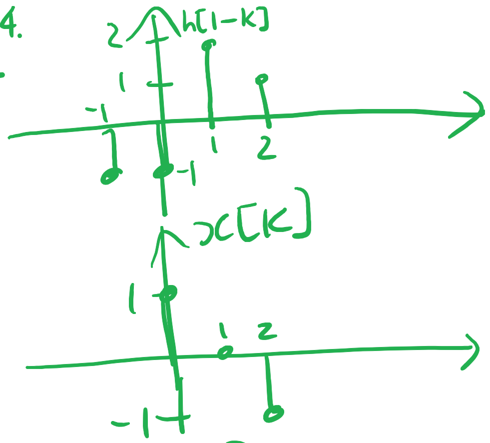
$h[n-k] = h[-k]$



5. $y[0] = 2 + 0 = 2$

for $n = 1$

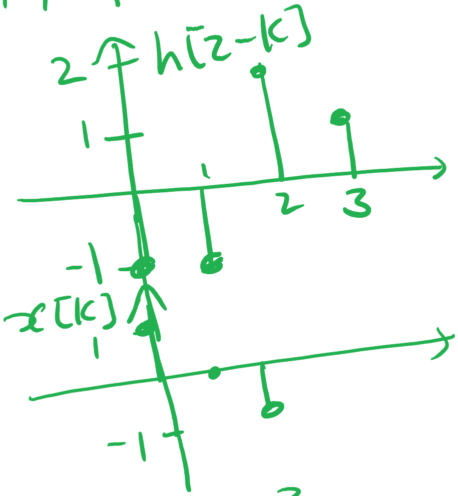
4.



5. $y[1] = -1 + -1 = -2$

for $n = 2$

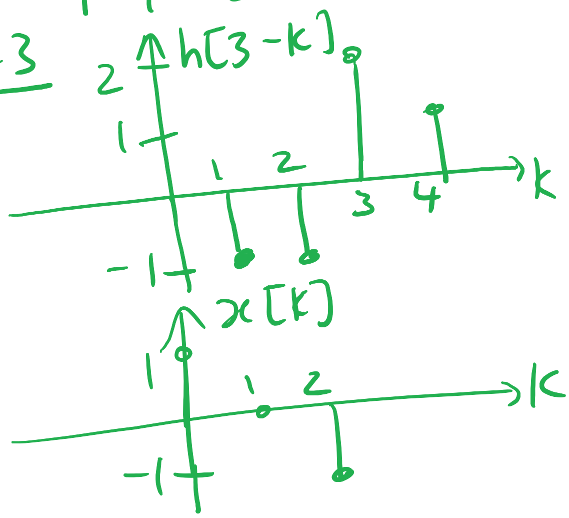
4.



5. $y[2] = -1 + -2 = -3$

for $n = 3$

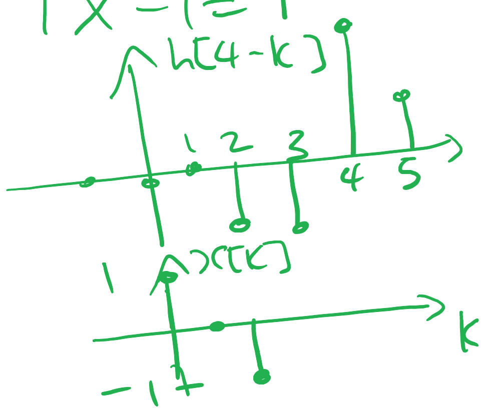
4.



5. $y[3] = -1 \times -1 = 1$

for $n = 4$

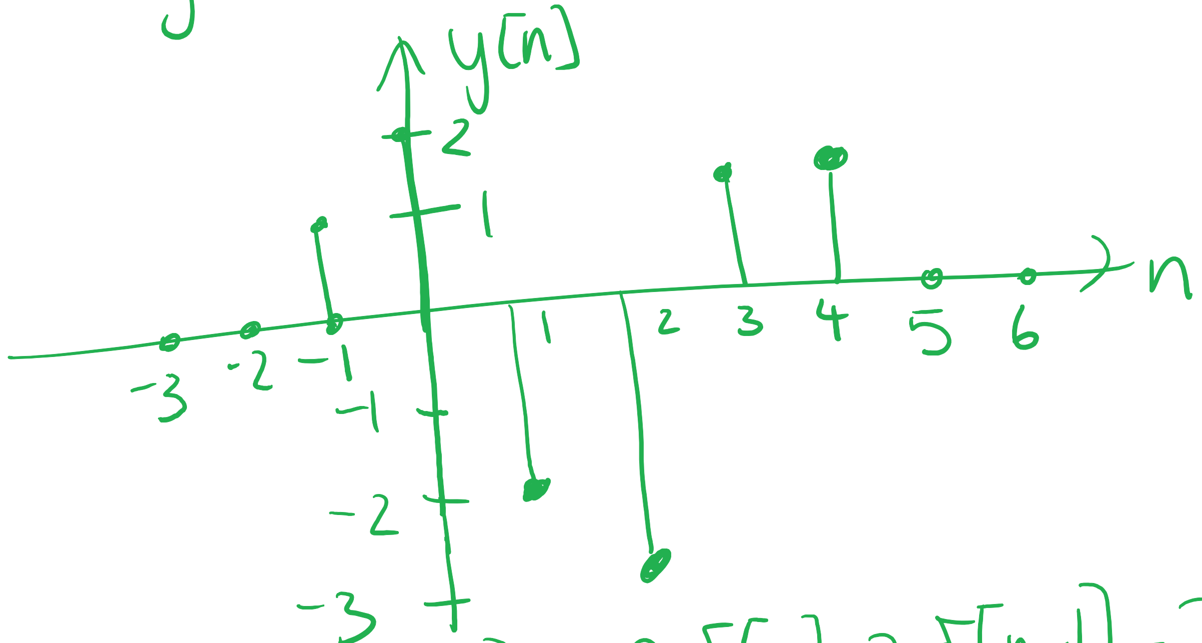
4.



5. $y[4] = -1 \times -1 = 1$

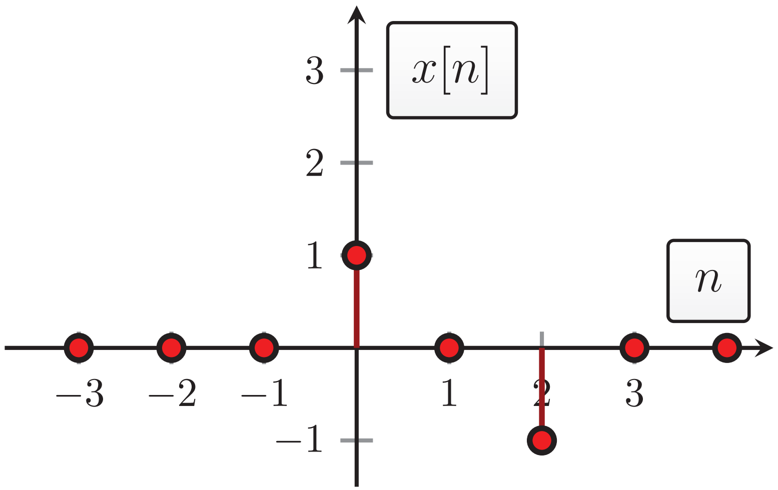
for $n > 4$ no overlap $\Rightarrow y[n] = 0 \ \forall n > 4$

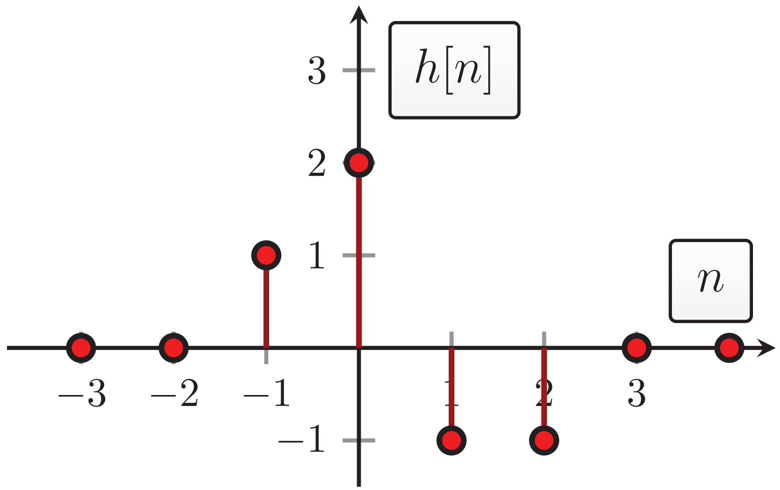
combining our results

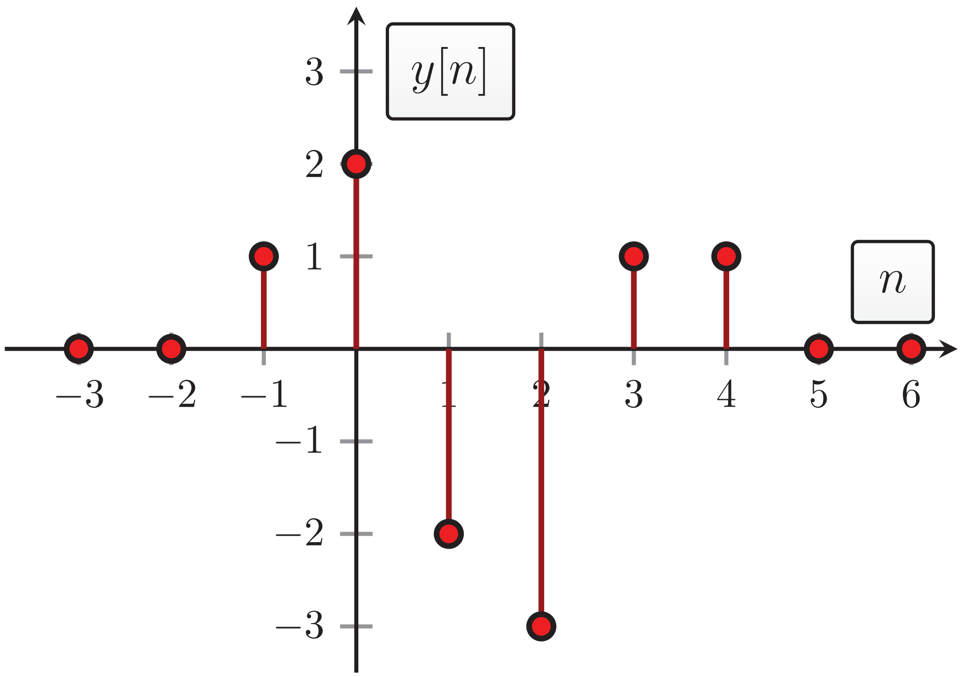


or $y[n] = \delta[n+1] + 2\delta[n] - 2\delta[n-1] - 3\delta[n-2] + \delta[n-3] + \delta[n-4]$

DT Convolution – Graphical Flip and Shift (cont'd)





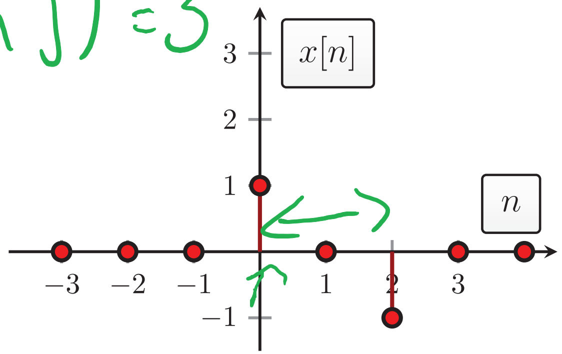


DT Convolution – Graphical Flip and Shift (cont'd)

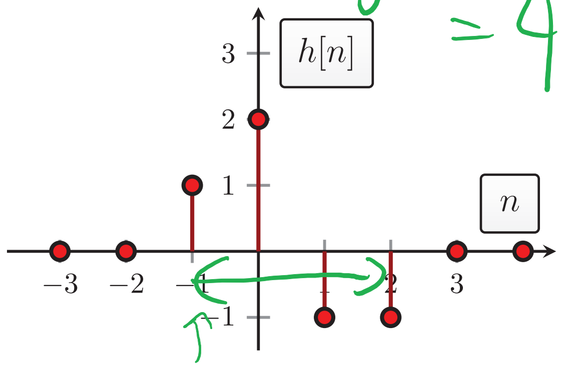
Checks: DT finite length convolution

- length($y[n]$) $=$ length($x[n]$) $+$ length($h[n]$) -1
- $y[n]$ will begin at start($x[n]$)+start($h[n]$)
- $y[n]$ will end at end($x[n]$) $+$ end($h[n]$)

length($x[n]$) $= 3$

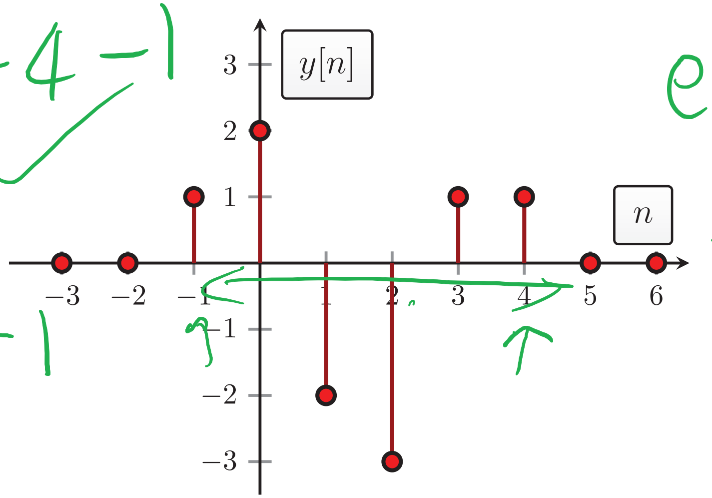


length($h[n]$) $= 4$



length($y[n]$) $= 3 + 4 - 1$
$= 6$ ✓

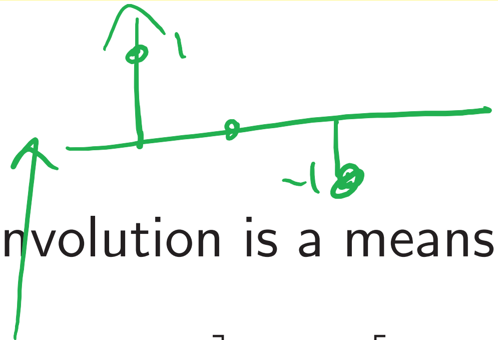
start($y[n]$) $= 0 + -1$
$= -1$ ✓



end($y[n]$)
$= 2 + 2$
$= 4$ ✓

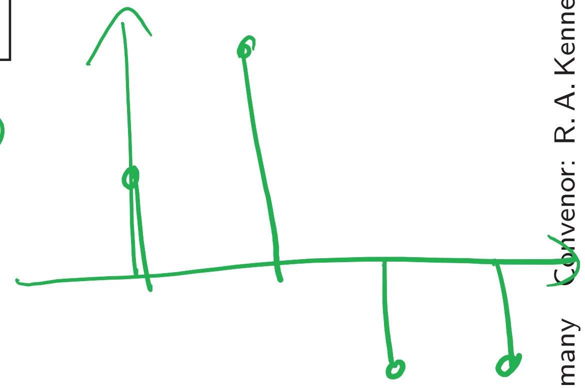
Part 5 Outline

Other DT Convolution Methods – Matlab



The calculation of DT convolution is a means to combine two vectors/sequences

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & -1 & -1 \end{bmatrix}$$



to generate a new vector/sequence

$$\underline{\begin{bmatrix} 1 & 2 & -2 & -3 & 1 & 1 \end{bmatrix}}$$

and indeed this is what MATLAB command

```
y = conv( x, h );
```

does. Where else does this exact type of calculation appear?

Other DT Convolution Methods – Polynomial Version

$[1\ 0\ -1]$ $[1\ 2\ -1\ -1]$

Consider the polynomial multiplication:

$$(1x^2 - 1)(x^3 + 2x^2 - x - 1) = x^5 + 2x^4 - 2x^3 - 3x^2 + x + 1$$

$[1\ 2\ -2\ -3\ 1\ 1]$

- $x[n]$ and $h[n]$ provide the LHS polynomials' coefficients.
- $y[n]$ provides the RHS polynomial's coefficients.
- So DT convolution appears to be related to polynomial multiplication.
- Here the non-causal shift in $h[n]$ has not been factored in (but can be).
- x is the polynomial indeterminate, it can be thought of as a unit time shift.

Other DT Convolution Methods – Matrix Version

Matrix version of convolution

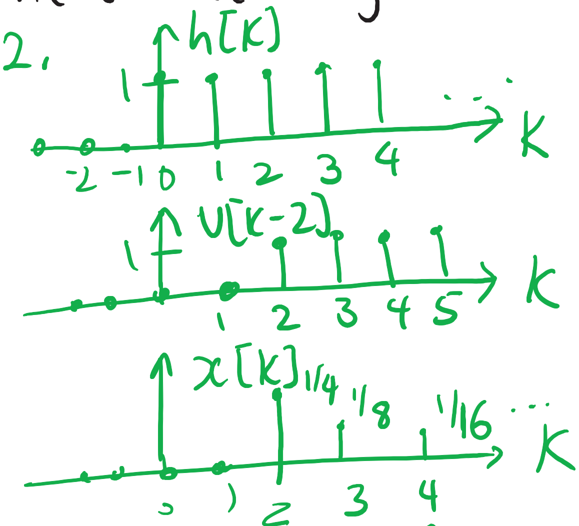
$$
\begin{array}{rrr}
-1 & -1 & 2 \\
0 & -1 & -1 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 2 & 1 \\
-1 & -1 & 2 \\
0 & -1 & -1 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}
=
\begin{bmatrix}
1 \\
2 \\
-2 \\
-3 \\
1 \\
1
\end{bmatrix}
$$

- Here $h[n]$ is implemented as a 6×3 Toeplitz matrix. The rows are formed by reversing $h[n]$ and shifting.
- The orange portions above are just an aid to understand and not part of the matrix.

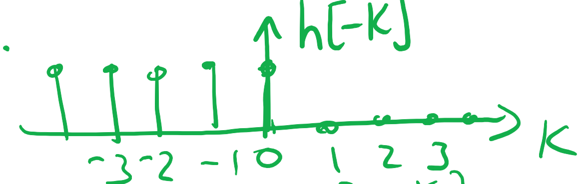
$x[n] = \left(\frac{1}{2}\right)^n u[n-2]$

$h[n] = u[n]$, $y[n] = x[n] * h[n]$

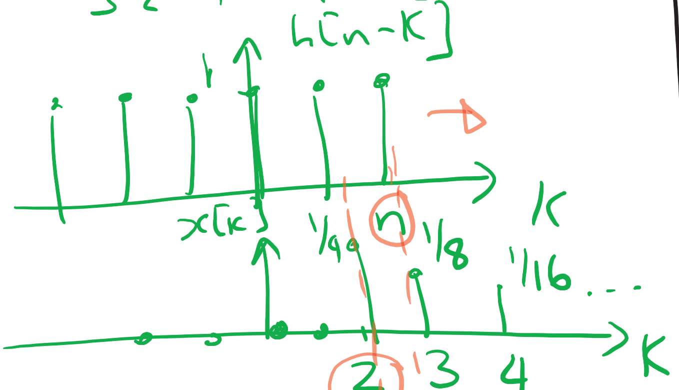
2.



3.



4.



5. for $n<2$ no overlap

$\Rightarrow y[n] = 0$ for $n<2$

$n \geq 2$ overlap

$$y[n] = \sum_{k=\cancel{-\infty}\,2}^{\cancel{\infty}\,n} x[k]\, h[n-k]$$

$$= \sum_{k=2}^{n} \left(\frac{1}{2}\right)^k 1$$

in formula

$K=2$, $\ell = n$, $\alpha = 1/2$

$$y[n] = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{n+1}}{1 - 1/2}$$

$$= \frac{\frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}}{\cancel{1/2}} \quad \left(\cancel{\tfrac{1}{2}}\right)\left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^n$$

Therefore

$$y[n] = \begin{cases} 0, & n<2 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^n, & n \geq 2 \end{cases}$$

Part 5 Outline

20 DT Convolution

- Street Version
- Graphical Flip and Shift
- Other DT Convolution Methods
- Convolution with Impulses
- Commutative Property
- Distributive Property
- Associative Property

21 DT System Properties

- Causality Property
- Stability Property
- Review of System Properties

22 Difference Equation of DT System

- Direct-Form I implementation

23 Finding the Impulse Response of a DT System

DT Convolution – Convolution with Impulses

Convolution of a signal with a shifted impulse shifts the signal to the location of the impulse — can use this property rather than doing an infinite length convolution by hand

- $u[n] * \delta[n-1] = u[n-1]$

- $\delta[n] * \delta[n-4] = \delta[n-4]$

- $\delta[n-1] * \delta[n-4] = \delta[n-5]$

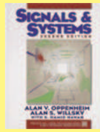
$\delta[n-4-1]$ =

- $(\delta[n] + 2\delta[n-1] - \delta[n-2]) * 2\delta[n-4] = 2\delta[n-4] + 4\delta[n-5] - 2\delta[n-6]$

$= \delta[n] * 2\delta[n-4] + 2\delta[n-1] * 2\delta[n-4]$

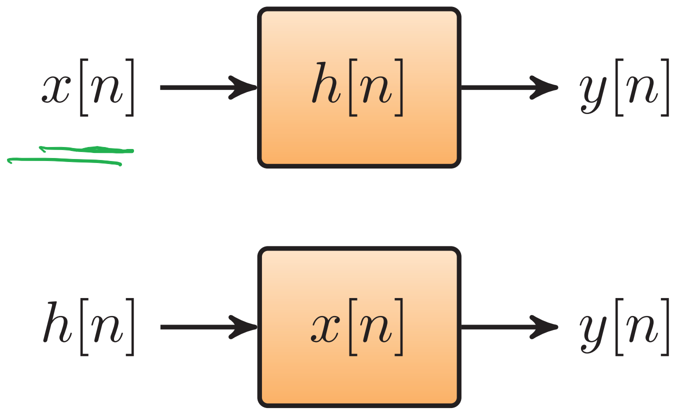
Part 5 Outline

DT Convolution – Commutative Property



Previously we noted the **Commutative Property**:

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$



DT Convolution – Commutative Property

- This follows from (change variables to $\ell = n - k$)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{\ell=-\infty}^{\infty} x[n-\ell]\, h[\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, x[n-\ell]$$

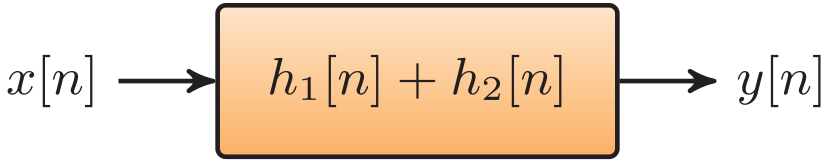
- Alternatively, if we have two polynomials say $p(x)$ and $q(x)$ then $p(x)q(x) = q(x)p(x)$. Polynomial multiplication is commutative. Convolution is commutative.

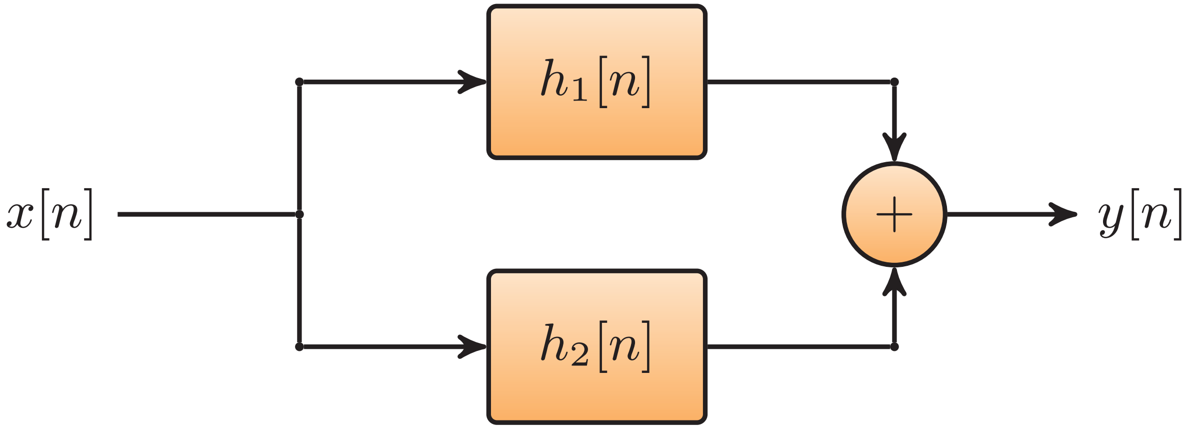
Part 5 Outline

DT Convolution – Distributive Property

Consider an input signal $x[n]$ and two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **parallel**, then we have the **Distributive Property**:

$$\boxed{x[n] \star (h_1[n] + h_2[n]) = x[n] \star h_1[n] + x[n] \star h_2[n]}$$





- This implies that we can combine two DT LTI systems in parallel into a single equivalent DT LTI system (by **adding** the pulse responses).

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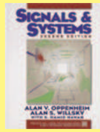
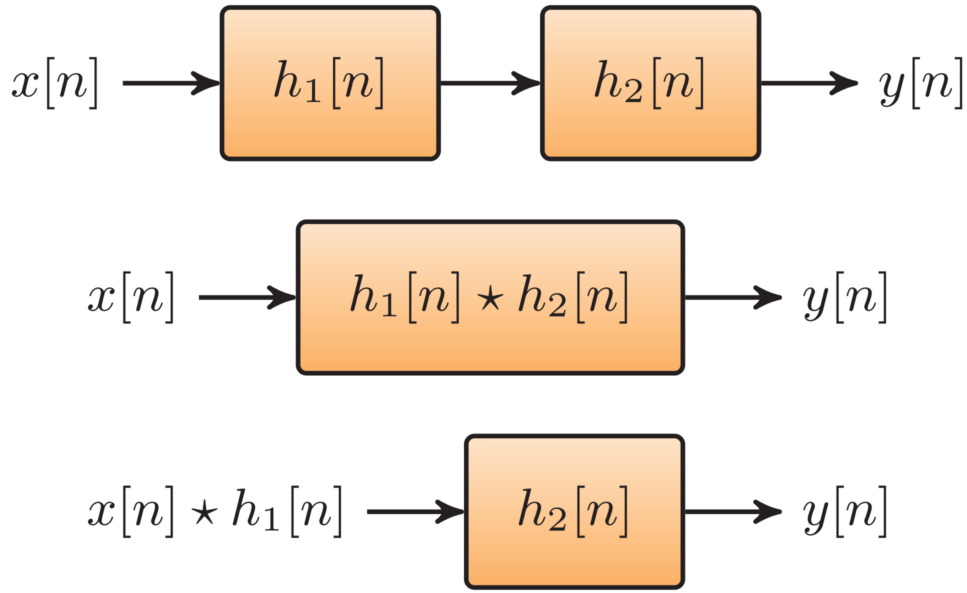
DT Convolution – Associative Property



Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

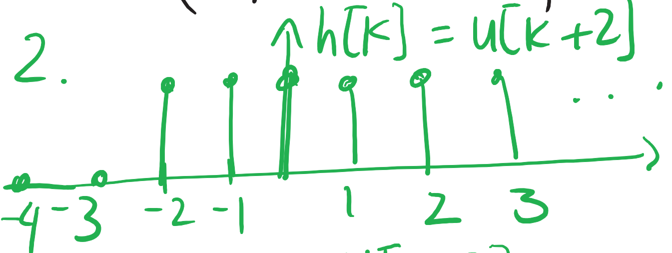
$$\boxed{x[n] \star \big(h_1[n] \star h_2[n]\big) = \big(x[n] \star h_1[n]\big) \star h_2[n]}$$

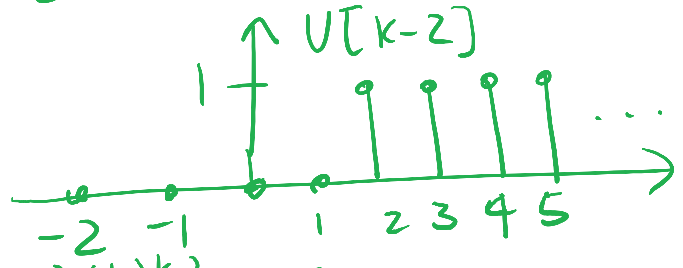


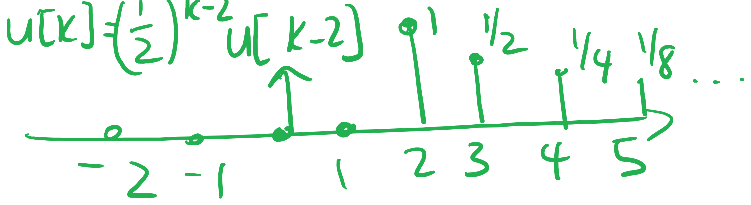
- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).

$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$, $h[n] = u[n+2]$

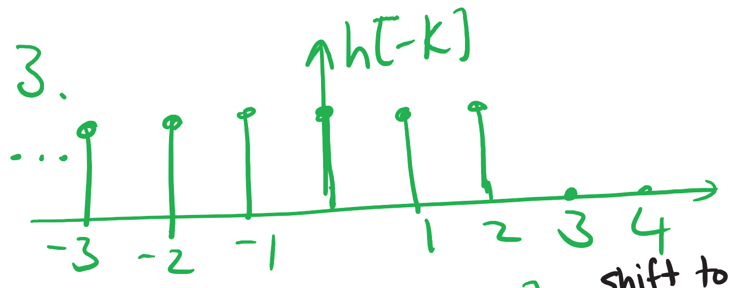
2.



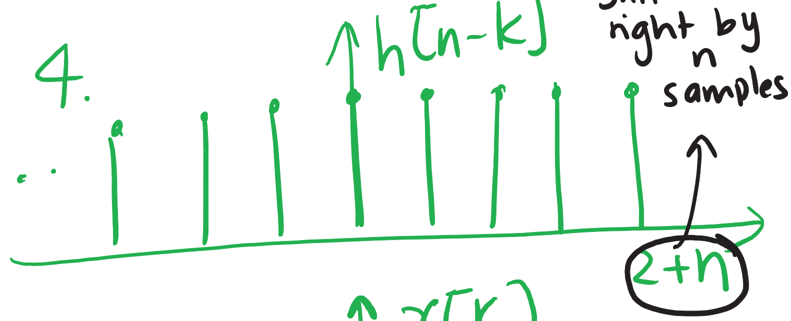


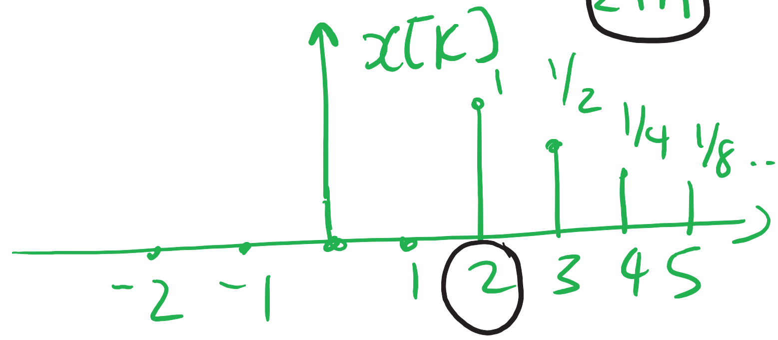


3.



4.





5. $2+n < 2$ no overlap

$\Rightarrow y[n] = 0 \ \forall \ n < -4$

4. for $n \geq -4$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

as $x[k]h[n-k]$ is 0 outside of these limits

$$= \sum_{k=2}^{2+n} \left(\frac{1}{2}\right)^{k-2}$$

$$= \sum_{k=2}^{2+n} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{-2}$$

$$= 4 \sum_{k=2}^{2+n} \left(\frac{1}{2}\right)^{k}$$

use formula sheet

From formula sheet:

$\alpha = \frac{1}{2}$, $l = 2+n$

$k = 2$. Putting these values into the formula:

$$y[n] = 4 \left(\frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{2+n+1}}{1 - 1/2} \right)$$

$$= 2\left(\frac{1}{4} - \left(\frac{1}{2}\right)^{3+n}\right)$$

$$= 2 - \left(\frac{1}{2}\right)^{2+n}$$

Therefore

$$y[n] = \begin{cases} 0, & n < -4 \\ 2 - \left(\frac{1}{2}\right)^n, & n \geq -4 \end{cases}$$