Notation

- LHS means left hand side and RHS means right hand side.
- CT means continuous time, and DT means discrete time.
- A system being LTI means the system is linear and time-invariant.
- The binary operator $\star$ denotes convolution for both CT and DT.
- The unit sample delta signal is given by

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- $\delta(t)$ represents the unit impulse and satisfies

$$x(t) \star \delta(t - t_0) = x(t - t_0)$$

- The CT unit step and DT unit step functions, $u(t)$ and $u[n]$, respectively:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Transform Definitions

1. *Fourier Series (FS):* For $x(t) = x(t + T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad \text{(Synthesis)}$$

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad \text{(Analysis)}$$

2. *Discrete Time Fourier Series (DTFS):* For $x[n] = x[n+N]$ periodic with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis)}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad \text{(Analysis)}$$

3. *Fourier Transform (FT):*

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega, \quad t \in \mathbb{R} \qquad \text{(Synthesis)}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt, \quad \omega \in \mathbb{R} \qquad \text{(Analysis)}$$

4. *Discrete Time Fourier Transform (DTFT):*

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \qquad \text{(Synthesis)}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \qquad \text{(Analysis)}$$

5. *CT Frequency Response:* The CT Frequency Response is defined by

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

6. *DT Frequency Response:* The DT Frequency Response is defined by

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Useful Formulas

Complex Numbers and Complex Exponentials

$$j = \sqrt{-1}$$
$$j^2 = -1$$
$$\frac{1}{j} = -j$$
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{j\pi n} = (-1)^n$$
$$e^{-j\pi n} = (-1)^n$$
$$e^{j2\pi n} = 1$$
$$e^{-j2\pi n} = 1$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Trigonometric Identities

$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right) \qquad \cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \qquad \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^3(\theta) = \frac{3\sin(\theta) - \sin(3\theta)}{4} \qquad \cos^3(\theta) = \frac{3\cos(\theta) + \cos(3\theta)}{4}$$

Geometric series

If α is a complex number then the following relationships hold:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \qquad |\alpha| < 1$$

$$\sum_{n=0}^{\infty} n\,\alpha^n = \frac{\alpha}{(1-\alpha)^2} \qquad |\alpha| < 1$$

$$\sum_{n=-k}^{-\infty} \alpha^n = \alpha^{-k}\left(\frac{\alpha}{\alpha - 1}\right) \qquad |\alpha| > 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \alpha = 1, \\ \dfrac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \end{cases}$$

$$\sum_{n=k}^{\ell} \alpha^n = \begin{cases} \ell - k + 1 & \alpha = 1, \\ \dfrac{\alpha^k - \alpha^{\ell+1}}{1-\alpha} & \alpha \neq 1 \end{cases}$$

Integration

The letters n, a and b are constants.

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax} \qquad \int xe^{ax}\,dx = \frac{ax-1}{a^2}\,e^{ax}$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) \qquad \int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} \qquad \int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\cos(ax)\,dx = \frac{\cos(ax) + ax\sin(ax)}{a^2} \qquad \int x\sin(ax)\,dx = \frac{\sin(ax) - ax\cos(ax)}{a^2}$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}\left(a\cos(bx) + b\sin(bx)\right)}{a^2+b^2} \qquad \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\left(a\sin(bx) - b\cos(bx)\right)}{a^2+b^2}$$

Differentiation

The letters f and g represent functions of x, whereas n, a and b are constants.

$$\frac{d}{dx}(af) = a\frac{df}{dx} \qquad \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}(e^{ax}) = a\ e^{ax} \qquad \frac{d}{dx}(x^n) = n\ x^{n-1}$$

$$\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right) g + f\left(\frac{dg}{dx}\right) \qquad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$$

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b) \qquad \frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$$

Properties of CT Fourier Series

Property	Periodic Signal, Period T Fundamental Frequency $\omega_0 = \frac{2\pi}{T}$	Fourier Series Coefficients
	$x(t)$ $y(t)$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Periodic Convolution	$x(t) \star y(t) = \int_T x(\tau)\, y(t-\tau)\, d\tau$	$Ta_k b_k$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0\, a_k$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd

Also note Parseval's relation:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Discrete-time Fourier Transform Properties

Property	Signal	Fourier Transform
	$x[n]$ $y[n]$	$X(e^{j\omega})$ $Y(e^{j\omega})$
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Convolution	$x[n] \star y[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$	$X(e^{j\omega})Y(e^{j\omega})$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$

Properties of Fourier Transform

Property	Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X(\frac{j\omega}{a})$
Convolution	$x(t) \star y(t) = \int x(\tau)\, y(t-\tau)\, d\tau$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)\, y(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)\, Y(j(\omega - \theta))\, d\theta$
Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd

Fourier Transform Pairs

Signal	Fourier Transform	Fourier Series Coefficients (if Periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$\cos \omega_0 t$	$\pi \Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big]$	$a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0$ otherwise
$\sin \omega_0 t$	$\frac{\pi}{j} \Big[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\Big]$	$a_1 = -a_{-1} = \frac{1}{2j}, \quad a_k = 0$ otherwise
$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	–
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	–
$e^{-at}u(t), \quad Re\{a\} > 0$	$\frac{1}{a + j\omega}$	–
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$	$a_k = \frac{1}{T}$ for all k