Signal Processing
ENGN2228

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Second Semester

Review of Module Two

At the end of the second module:

- Fourier series
  - For continous-time periodic signals
- Properties of Fourier series

Now we are going to look at:

- Discrete-time Fourier series

Part 1 Outline

1. **DT Periodic Signals**
   - Definition
   - Fourier Series
   - Recap to Here
   - Inspection Method to Calculate DTFS Coefficients
   - Using Synthesis Equation
   - Using Analysis Equation to Calculate DTFS Coefficients
   - Properties DTFT

Part 1 Outline

1. **DT Periodic Signals**
   - Definition
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DT Periodic Signals – Definition

Now we consider **DT periodic signals** which satisfy:

$$x[n+N] = x[n] \quad \text{and} \quad N = \frac{2\pi}{\omega_0} m$$

where $N_0 \in \mathbb{Z}$ is the fundamental period (smallest possible period N).

$m = 1, 2 \ldots$ such that N is an integer.

Periodicity DT and CT Complex Exponentials

Continuous-time exponentials $e^{j\omega_0 t}$:

- Periodic all ω_0
- $T = \frac{2\pi}{\omega_0}$
- Distinct functions for each ω_0
- Increasing ω_0 get a unique signal with faster oscillations

Discrete-time exponentials $e^{j\omega_0 n}$:

- Periodic only if ω_0 is a rational multiple of 2π
- $N = \frac{2\pi}{\omega_0}m$
- Functions not distinct for each ω_0
- Identical signals for values of ω_0 separated by multiples of 2π
- The rate of oscillation doesn't depend on $|\omega_0|$
- Fastest oscillation at $\omega_0 = \pi$, slowest $\omega_0 = 0$ and $\omega_0 = 2\pi$

DT Periodic Signals – Fourier Series

Definition (DT Fourier Series Pair)

For $x[n] = x[n+N]$ periodic with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad \textit{(Synthesis Equation)}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad \textit{(Analysis Equation)}$$

- Note that the a_k are N-periodic, $a_k = a_{k+N}$, and so only N consecutive a_k need be known/computed.
- Similarly, only N consecutive $x[n]$ are needed.

DT Periodic Signals – Fourier Series (cont'd)

Definition (DT Fourier Series Pair)

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

Definition (CT Fourier Analysis and Synthesis)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

DT Periodic Signals – Proof of Analysis Equation

To figure out the a_k, we have the (completeness) identity

$$\sum_{n=0}^{N-1} e^{jk\omega_0 n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$
$$= N\,\delta[n]$$

So

$$\begin{aligned}\sum_{n=0}^{N-1} x[n]\,e^{-jm\omega_0 n} &= \sum_{n=0}^{N-1} \Big(\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}\Big) e^{-jm\omega_0 n} \\ &= \sum_{k=0}^{N-1} a_k \underbrace{\Big(\sum_{n=0}^{N-1} e^{j(k-m)\omega_0 n}\Big)}_{N\,\delta[k-m]} \\ &= N\,a_m\end{aligned}$$

DT Periodic Signals – Fourier Series (cont'd)

Since the discrete time signal is N periodic, the Fourier Series coefficients are N periodic and the synthesis and analysis equations are linear then the conversion between

$$N \text{ consecutive } a_k \longleftrightarrow N \text{ consecutive } x[n]$$

is describable by a $N \times N$ (invertible) matrix. Either direction would require of the order of N^2 multiplications plus additions.

There is a cleverer way to do the matrix multiplication, it is called the **Fast Fourier Transform (FFT)**. It is pervasive throughout electronic systems and communications. Your digital TV wouldn't work without it. If $N = 1024$ then the FFT is $10\times$ faster than otherwise.

1. **DT Periodic Signals**
   - Definition
   - Fourier Series
   - Recap to Here
   - Inspection Method to Calculate DTFS Coefficients
   - Using Synthesis Equation
   - Using Analysis Equation to Calculate DTFS Coefficients
   - Properties DTFT

DT Periodic Signals – Recap to Here

This is getting confusing. There seems to be some patterns. What is going on?

Time-domain properties	Periodic	Non-periodic	
Continuous	Fourier series (FS)	Fourier transform (FT)	**Non-periodic**
Discrete	Discrete-time Fourier series (DTFS)	Discrete-time Fourier transform (DTFT)	**Periodic**
	Discrete	**Continuous**	**Frequency-domain Properties**

Inspection Method to Calculate DTFS Coefficients

- If $x[n]$ is composed of real or complex sinusoids, then it is better to determine a_k using the inspection method (rather than using the DTFS Analysis Equation).

- This involves expanding the sinusoids in terms of complex exponentials using Euler's formula and comparing each term with the terms in the DTFS Synthesis Equation.

Inspection Method to Calculate DTFS Coefficients

Example 1:

$$x[n] = \sin \frac{2\pi}{5} n$$

Inspection Method to Calculate DTFS Coefficients

Example 2:

$$x[n] = 1 + \sin\left(\frac{2\pi}{12}n + \frac{3\pi}{8}\right)$$

Inspection Method to Calculate DTFS Coefficients

Example 3:

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

Using Synthesis Equation

- Given a_k find $x[n]$
- Example 1:

Using Synthesis Equation

- Given a_k find $x[n]$
- Example 2:

Part 1 Outline

Using Analysis Equation to Calculate DTFS Coefficients

Analysis Equation:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z}$$

Using Analysis Equation to Calculate DTFS Coefficients

Using Analysis Equation to Calculate DTFS Coefficients

Using Analysis Equation to Calculate DTFS Coefficients

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients

DT Square Wave:

Let $x[n]$ be periodic with period N. With N_1 such that $2N_1 + 1 \leq N$, define $x[n]$ over N-length interval

$$[-N_1, N - N_1 - 1]$$

as follows

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & N_1 + 1 \leq n < N - N_1 - 1 \end{cases}$$

For other n we can just use

$$x[n] = x[n + N]$$

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients (cont'd)

Example 3.12 of text book page 218. Make sure can get this answer:

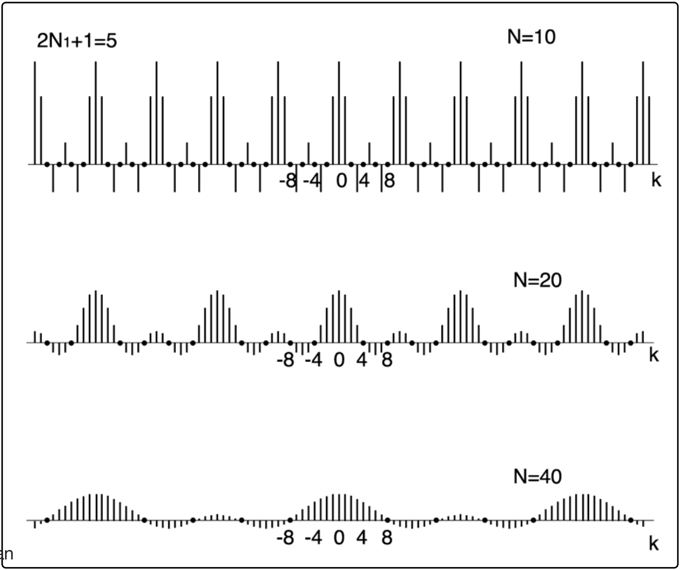
The (N-periodic) Fourier coefficients are

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \times \begin{cases} \dfrac{\sin\left(2\pi k(N_1 + 1/2)/N\right)}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ 2N_1 + 1, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

- This is like a "sampled" periodic sinc function.

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients (cont'd)



DT Periodic Signals – Properties DTFT

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Part 2 Outline

Part 2 Outline

Fourier Series and LTI Systems – Eigenfunctions Revisited

Representations of inputs and outputs LTI system:

- Previous focus was unit samples $\delta[n]$ and impulses $\delta(t)$ - convolution
- Alternative focus is eigenfunctions of LTI systems

Eigenfunction definition:

A signal for which the system output is a (possibly complex) constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is refereed to as the systems *eigenvalue*.

Fourier Series and LTI Systems – Eigenfunctions Revisited

- $e^{j\omega t}$, e^{st} are eigenfunctions of CT LTI systems
- $e^{j\omega n}$, z^n are eigenfunctions of DT LTI systems

- Study of CT LTI systems using $e^{j\omega t}$:
  - Fourier series (FS) - periodic signals
  - Fourier transform (FT) - general signals
- Study of CT LTI systems using e^{st}:
  - Laplace transform (outside the scope of this course)
- Study of DT LTI systems using $e^{j\omega n}$:
  - Discrete-time Fourier series (DTFS) - periodic signals
  - Discrete-time Fourier transform (DTFT) - general signals
- Study of DT LTI systems using z^n:
  - z-transform (to be covered in ENGN4537 - DT Signal Processing)

Fourier Series and LTI Systems – Eigenfunctions Revisited

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\text{CT LTI} \quad h(t)} \longrightarrow y(t) = ?$$

$$\begin{aligned}
y(t) &= x(t) * h(t) = h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau
\end{aligned}$$

Therefore

$$y(t) = e^{j\omega t} H(j\omega)$$

where $H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

Generalisation: If the input to an LTI system is expressed as a linear combination of periodic complex exponentials, the output can also be expressed in this form.

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\text{DT LTI} \quad h[n]} \longrightarrow y[n] = ?$$

$$
\begin{aligned}
y[n] &= x[n] * h[n] = h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\
&= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}
\end{aligned}
$$

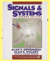
Therefore

$$y[n] = e^{j\omega n} H(e^{j\omega})$$

where $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

Part 2 Outline

Frequency Response of LTI System – Continuous Time



$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

Definition (CT Frequency Response)

The **CT Frequency Response** is defined by

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = H(j\omega)$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

Part 2 Outline

$$e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega})\,e^{j\omega n}$$

Definition (DT Frequency Response)

The **DT Frequency Response** is defined by

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\omega n} = H(e^{j\omega})$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

Part 2 Outline

Frequency Response of LTI System – Periodic Signals

Continuous Time

$$\left\langle \begin{array}{c}\textit{periodic}\\ \textit{signal}\end{array} \right\rangle x(t) \longrightarrow \boxed{h(t)} \longrightarrow \left\langle \begin{array}{c}\textit{periodic}\\ \textit{signal}\end{array} \right\rangle y(t)$$

Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{\,jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k \, \overbrace{H(\,jk\omega_0)}^{\text{eigenvalue}}}_{b_k} \, e^{\,jk\omega_0 t}$$

The Fourier coefficients map like

$$a_k \to b_k \triangleq H(\,jk\omega_0)\, a_k$$

Here the complex gain (eigenvalue) $H(\,jk\omega_0) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$, that is, $e^{\,jk\omega_0 t}$ (eigenfunction).

Frequency Response of LTI System – Periodic Signals

Discrete Time

$$\left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle x[n] \longrightarrow \boxed{h[n]} \longrightarrow \left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle y[n]$$

Fourier Series representation

$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y(t) = \sum_{k=0}^{N-1} \underbrace{a_k \, H(e^{jk\omega_0})}_{b_k} \, e^{jk\omega_0 n}$$

The Fourier coefficients map like

$$a_k \to b_k \triangleq H(e^{jk\omega_0}) \, a_k$$

Here complex gain $H(e^{jk\omega_0}) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$.

Part 2 Outline

Frequency Response of LTI System – Finding $y(t)$

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$, what is $y(t)$?

- Could find $y(t) = x(t) * h(t)$
- Use Fourier series to solve for $y(t)$ instead
- Output Fourier coefficients given by $b_k = H(jk\omega_0)a_k$
- First need $H(j\omega)$:

Frequency Response of LTI System –Finding $y(t)$ Cont.

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$, what is $y(t)$?

- $x(t) = \cos(2\pi 1000t)$ (1kHz)
- $R = \frac{1000}{2\pi}$
- C $= 10\ \mu F$
- Output Fourier coefficients given by $b_k = H(jk\omega_0)a_k$

Need to find $H(jk\omega_0)$:

Need to find FS coefficients $x(t)$, a_k:

Frequency Response of LTI System –Finding $y(t)$ Cont.

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$, what is $y(t)$?

- Output Fourier coefficients given by $b_k = H(jk\omega_0) a_k$
- Finally $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$

CT Example Summary

Frequency Response of LTI System –Given $h(t)$ find $H(j\omega)$

Remember:

- $h(t)$ impulse response $\delta(t) \to h(t)$
- $H(j\omega)$ frequency response LTI system, by definition $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$
- Example: $h(t) = e^{-t}u(t)$

Frequency Response of LTI System –Given $h[n]$ find $H(e^{j\omega})$

Remember:

- $h[n]$ impulse response $\delta[n] \to h[n]$
- $H(e^{j\omega})$ frequency response LTI system, by definition
  $H(j\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$
- Example: $h[n] = \alpha^n u[n], \quad -1 < \alpha < 1$

Frequency Response of LTI System –Given $h[n]$ find $H(e^{j\omega})$

Remember:

- $h[n]$ impulse response $\delta[n] \to h[n]$
- $H(e^{j\omega})$ frequency response LTI system, by definition
  $H(j\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$
- Example: $h[n] = \alpha^n u[n], \quad -1 < \alpha < 1$

Frequency Response of LTI System –Given $h[n]$ find $H(e^{j\omega})$

Difference equation: $y[n] - \frac{1}{4}y[n-1] = x[n]$
Sol. 1: Solve difference equation to find $h[n]$, then find $H(e^{j\omega})$

Frequency Response of LTI System –Given $h[n]$ find $H(e^{j\omega})$

Difference equation: $y[n] - \frac{1}{4}y[n-1] = x[n]$
Sol. 2: Avoids recursion

Part 2 Outline

2. Fourier Series and LTI Systems
   - Eigenfunctions Revisited
3. Frequency Response of LTI System
   - Continuous Time
   - Discrete Time
   - Periodic Signals
   - Examples using Frequency Response
4. **Freq Shaping and Filtering**
   - Quick Review of Analogue Filters (non-assessable)
   - Key Observation
   - CT Low Pass Filter
   - CT High Pass Filter
   - CT Band Pass Filter
   - DT Low Pass Filter
   - DT High Pass Filter
   - DT Band Pass Filter
   - Moving Average Filter
   - Other Types of Filters

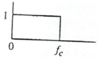
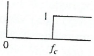
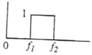
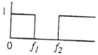
Part 2 Outline

Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Filter Circuits

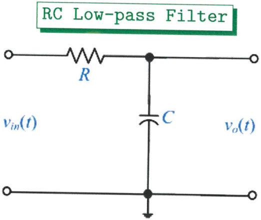
- A **Filter** is a circuit that passes certain frequencies and attenuates or rejects all other frequencies.

Filter Types

Type	Typical Ideal $\|H(f)\|$	Description	Example Uses
Lowpass		removes all frequency information above f_c	noise removal, interpolation, data smoothing
Highpass		removes all frequency information below f_c	removing DC or low freq drift, edge detection or enhancement
Bandpass		removes all frequency information outside of $f_1 \to f_2$	tuning in to one radio station, audio graphic equalizers
Notch		removes all frequency information between $f_1 \to f_2$	removing noise at a particular frequency, e.g. 60 Hz

RC Filter Circuits

- A **Filter** is a circuit that passes certain frequencies and attenuates or rejects all other frequencies.
- **Low-pass** filters allow low frequency (from DC to f_c) to pass and reject/attenuate (i.e. reduce amplitude of) all other frequencies.



Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Bode Plots

- A **Bode plot** shows the **magnitude of a transfer function** in decibels versus **frequency** using a logarithmic scale for frequency.
- Because it can clearly illustrate very large and very small magnitudes for a wide range of frequencies on one plot, the Bode plot is particularly useful for displaying transfer function magnitudes.
- **Bode plots** of filter circuits can be closely approximated by **straight line segments (asymptotes)**, so they are relatively easy to draw.

  Asymptote = a line or curve that approaches a given curve arbitrarily close

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Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

Bode Plot Approximation

- The two straight line asymptotes intersect at $f = f_B$ Hz. For this reason, f_B is called the **break** or the **corner** or **cut-off** frequency (also sometimes denoted as f_c). The asymptotes are in error by only 3 dB at the break frequency.

Low freq asymptote: $|H(f)|_{\mathrm{dB}} \approx 0\mathrm{dB}$

Actual response:

$|H(f)|_{\mathrm{dB}} = -20\log_{10}\left(\sqrt{1+\left(\frac{f}{f_B}\right)^2}\right)$

High frequency asumptote:

$|H(f)|_{\mathrm{dB}} \approx -20\log_{10}\left(\frac{f}{f_B}\right)^2$

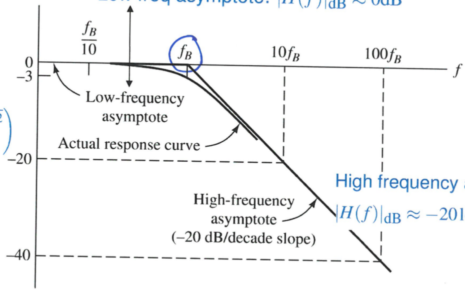
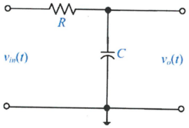


Figure 6.15 Magnitude Bode plot for the first-order lowpass filter.

Freq Shaping and Filtering –Quick Review of Analogue Filters (non-assessable)

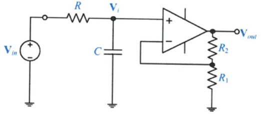
Summary of Low-pass Filter Circuits

$$\boxed{H(f) = A\left(\frac{1}{1 + j\frac{f}{f_B}}\right)}$$



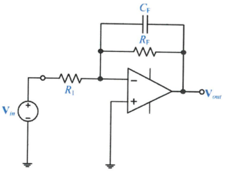
$$\boxed{A = 1}$$

$$\boxed{f_B = \frac{1}{2\pi RC}}$$



$$\boxed{A = 1 + \frac{R_2}{R_1}}$$

$$\boxed{f_B = \frac{1}{2\pi RC}}$$



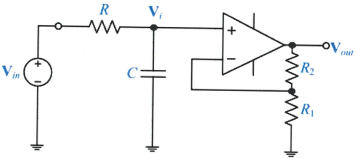
$$\boxed{A = -\frac{R_F}{R_1}}$$

$$\boxed{f_B = \frac{1}{2\pi R_F C_F}}$$

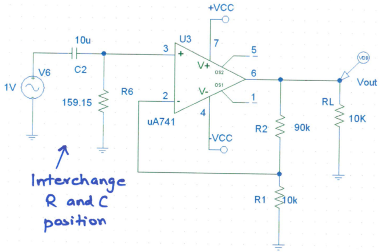
Active Highpass Filter Circuits

- Interchange resistor and capacitor to get highpass filter.

Active Lowpass Filter



Active Highpass Filter



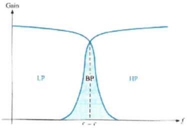
See: `Lec17_highpass.sch`

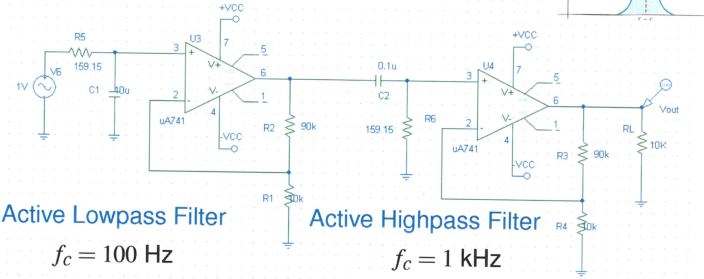
Active Bandpass Filter Circuits

- Cascade lowpass and highpass filter to get bandpass filter.

Active Bandpass Filter

(Passband: 100Hz - 1000Hz)





See: `Lec17_bandpass.sch`

Part 2 Outline

Freq Shaping and Filtering – Key Observation

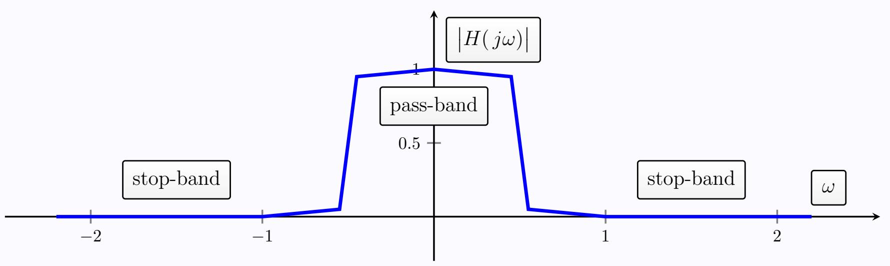
We can choose/design $H(j\omega)$, or $H(e^{j\omega})$, as a function of (radial) frequency ω to determine how frequency components of the input are passed/amplified/attenuated to the output.

For example, the bass, treble and mid-range control in an audio system:

- To boost bass say at 100 Hz or $\omega = 200\pi$ then we have $\left|H(\,j200\pi)\right| > 1$
- To attenuate treble say at 1 kHz or $\omega = 2000\pi$ then we have $\left|H(\,j2000\pi)\right| < 1$

Part 2 Outline

The CT "Low Pass" frequency response looks like:



- Conventionally look at the magnitude, $|H(j\omega)|$, to characterize the type of "filter".
- Defines "passband" (low or no attenuation) and "stopband" (high attenuation).

Part 2 Outline

Freq Shaping and Filtering – CT High Pass Filte

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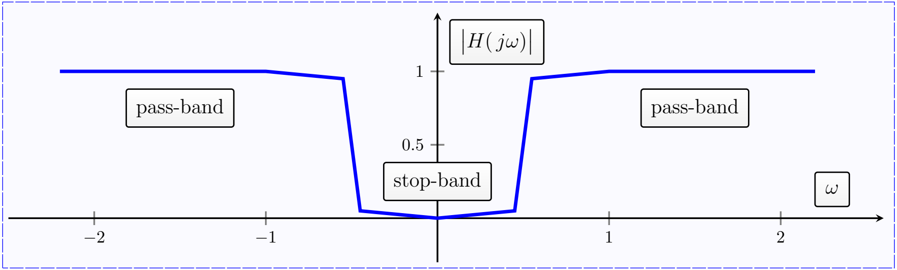
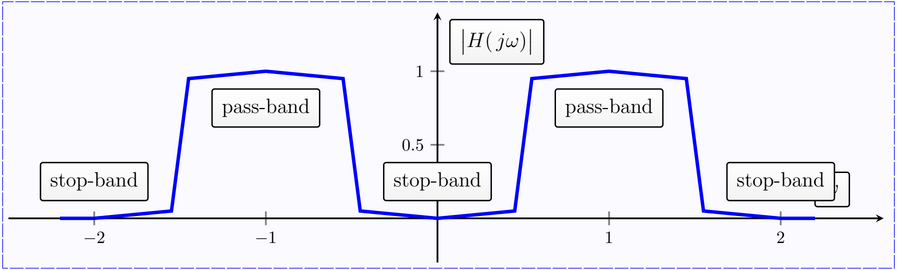
The CT “High Pass” frequency response looks like:



- DC gain should be small or zero.
- High frequencies are passed, low frequencies are blocked. The exact shape isn’t so important.

Part 2 Outline

The CT "Band Pass" frequency response looks like:

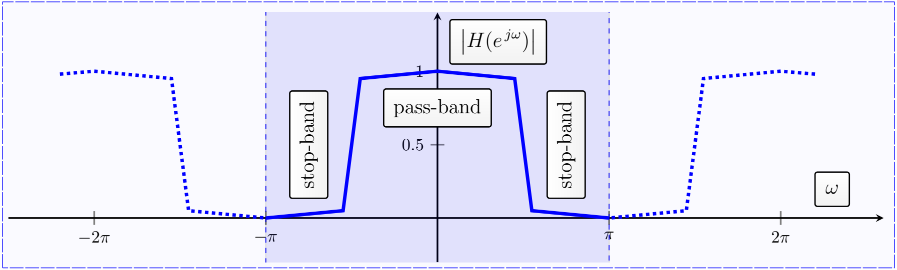


- DC gain should be small or zero. The high frequency gain should be small or zero.
- High frequencies are blocked, low frequencies are blocked, mid-frequencies are passed. The exact shape isn't so important.

Part 2 Outline

DT Low Pass Filter:

The DT "Low Pass" frequency response looks like:

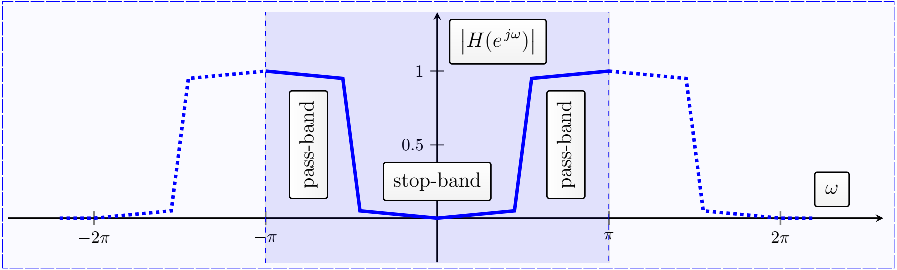


- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

Part 2 Outline

DT High Pass Filter:

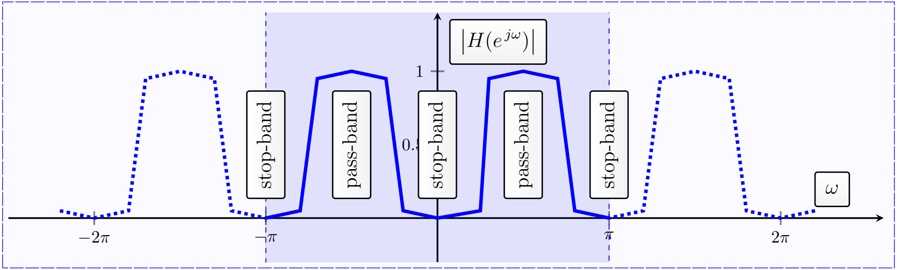
The DT "High Pass" frequency response looks like:



- Most often look at the magnitude, $|H(e^{j\omega})|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

DT Band Pass Filter:

The DT "Band Pass" frequency response looks like:



- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- The spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π.
- The frequency $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

Causal moving average of two terms:

$$\boxed{y[n] = \tfrac{1}{2}\,x[n] + \tfrac{1}{2}\,x[n-1]}$$

Freq Shaping and Filtering – Moving Average Filter

Alternative approach:

$$\boxed{y[n] = \tfrac{1}{2}\,x[n] + \tfrac{1}{2}\,x[n-1]}$$

Freq Shaping and Filtering – Magnitude and Phase Features

- Going from $H(e^{j\omega})$ to $\left|H(e^{j\omega})\right|$ introduces weirdness in phase such as apparent discontinuities (at frequencies ω where $H(e^{j\omega})$ gets small and passes through the origin in the complex plane).
- This phase flipping by π can seem like an apparent discontinuity in phase, where infact there is none.
- The phase flips can make the phase go beyond $-\pi$ and π.
- Can limit the phase "wind" to $-\pi < \angle H(e^{j\omega}) \leq \pi$.
- The Matlab angle() command gives the correct wrapped phase $-\pi \leq \theta \leq \pi$.
- Magnitude, $\left|H(e^{j\omega})\right|$, is an **even function** of ω for real $h(t)$.
- Phase, $-\pi < \angle H(e^{j\omega}) \leq \pi$, can be made an **odd function** of ω for real $h(t)$
- We tend to define the action of a filter in terms of the magnitude of the frequency response. This can be a little deceptive.

Freq Shaping and Filtering – Summary

Take home messages:

- To filter $x(t)$, we pass it through a physical circuit (e.g. containing R, C, op-amps)
- To filter $x[n]$, we pass it through a suitable difference equation
- We tend to define the action of a filter in terms of the magnitude of the frequency response
- $H(j\omega)$ is not periodic with period 2π (because $e^{j\omega t}$ is distinct for all ω) but $H(e^{j\omega})$ is periodic with period 2π (since $e^{j\omega n}$ is periodic with period 2π)
- Going from $H(e^{j\omega})$ to $|H(e^{j\omega})|$ introduces jumps in phase in $\angle H(e^{j\omega})$
- For $H(e^{j\omega})$, $\omega = 0$ is the DC or low frequency
- For $H(e^{j\omega})$, $\omega = \pm\pi$ is the highest frequency

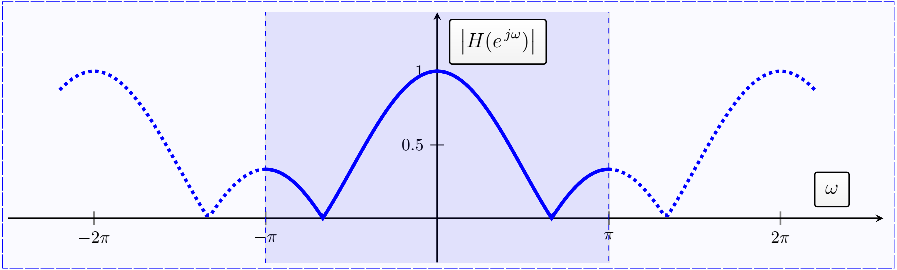
Freq Shaping and Filtering – Moving Average Filter

Causal moving average filters of different orders:

Order	$H(e^{j\omega})$	$\left\|H(e^{j\omega})\right\|$
1	1	1
2	$e^{-j\omega/2}\cos(\omega/2)$	$\left\|\cos(\omega/2)\right\|$
3	$e^{-j\omega}\left(\frac{1}{3}+\frac{2}{3}\cos(\omega)\right)$	$\left\|\frac{1}{3}+\frac{2}{3}\cos(\omega)\right\|$
4	$e^{-j3\omega/2}\left(\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right)$	$\left\|\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right\|$
5	$e^{-j2\omega}\left(\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right)$	$\left\|\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right\|$

3 term non-causal moving average:

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$



$$\left|\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)\right|$$

Freq Shaping and Filtering – Moving Average Filter

From

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$

we deduce

$$h[n] = \begin{cases} \tfrac{1}{3}, & n = -1, 0, +1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\omega n} = \tfrac{1}{3} \sum_{n=-1}^{1} e^{-j\omega n}$$

$$= \tfrac{1}{3} + \tfrac{2}{3}\,\frac{e^{j\omega} + e^{-j\omega}}{2} = \tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)$$

Note that this is purely real. The phase appears to be zero. In reality it flips between 0 and π — it is π at those frequencies ω where $\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega) < 0$.

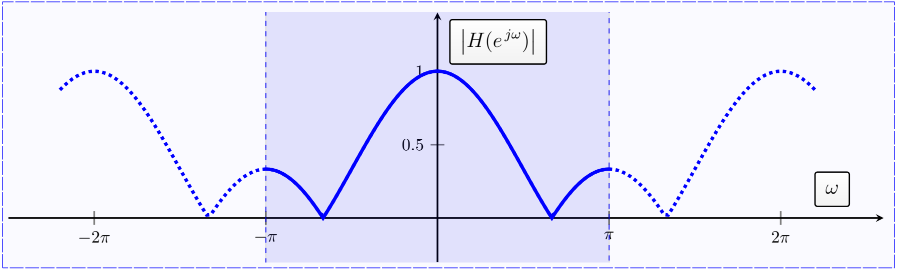
Alternative method:

$$\boxed{y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]}$$

Freq Shaping and Filtering – Moving Average Filter

3 term causal moving average (real-life system):

$$y[n] = \tfrac{1}{3}\, x[n] + \tfrac{1}{3}\, x[n-1] + \tfrac{1}{3}\, x[n-2]$$



$$\left|\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)\right|$$

Freq Shaping and Filtering – Moving Average Filter

From

$$\boxed{y[n] = \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1] + \tfrac{1}{3}\,x[n-2]}$$

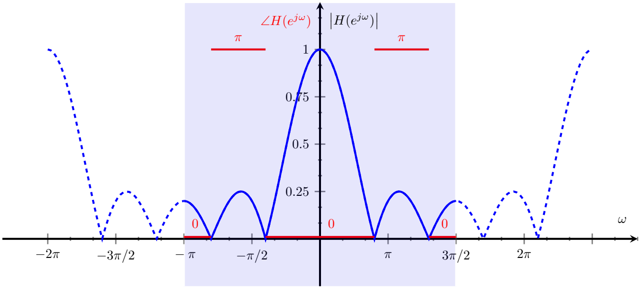
Freq Shaping and Filtering – Moving Average Filter

Alternative method:

$$\boxed{y[n] = \frac{1}{3}\,x[n+1] + \frac{1}{3}\,x[n] + \frac{1}{3}\,x[n-1]}$$

The 5 term non-causal moving average is:

$$\boxed{y[n] = \tfrac{1}{5}\,x[n+2] + \tfrac{1}{5}\,x[n+1] + \tfrac{1}{5}\,x[n] + \tfrac{1}{5}\,x[n-1] + \tfrac{1}{5}\,x[n-2]}$$



This has "flat" phase. This is because of the special form of the filter coefficients and a non-causal formulation.

For this example, the frequency response is:

$$\boxed{H(e^{j\omega}) = \tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega) \in \mathbb{R}}$$

is purely real, but for some values of ω the frequency response takes **negative** real values. For example, when $\omega = \pi/2$ then

$$\begin{aligned} H(e^{j\pi/2}) &= \tfrac{1}{5} + \tfrac{2}{5} \times 0 + \tfrac{2}{5} \times (-1) \\ &= -\tfrac{1}{5} \equiv \tfrac{1}{5}\, e^{j\pi} \end{aligned}$$

So the magnitude of $H(e^{j\pi/2})$ is $\frac{1}{5}$ and the phase is π. This is a little annoying as the phase can show an apparent discontinuity whereas there really is none in the frequency response. Sometimes engineers do dumb things.

Freq Shaping and Filtering – Moving Average Filter

For this example, the **magnitude** of the frequency response is:

$$\boxed{\left|H(e^{j\omega})\right| = \left|\tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega)\right|}$$

and the **phase** of the frequency response is:

$$\boxed{\angle H(e^{j\omega}) = \begin{cases} 0 & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ \pi & \text{otherwise} \end{cases}}$$

as plotted previously.

However, this phase is relatively boring and atypical. In contrast, this course is absolutely boring and typical.

- We also note that we only need to plot magnitude and phase over one period such as $\pi < \omega \leq \pi$.
- Of course, we could have taken the negating phase as $-\pi$ instead of π. This choice makes more sense because it lets us see whether the magnitude and phase are even, or odd, or neither even nor odd.
- For a **filter with real coefficients** the **magnitude is an even function** and the **phase is an odd function**.

Freq Shaping and Filtering – Moving Average Filter

The **causal** (implementable) 5 term moving average is:

$$\boxed{y[n] = \tfrac{1}{5}\,x[n] + \tfrac{1}{5}\,x[n-1] + \tfrac{1}{5}\,x[n-2] + \tfrac{1}{5}\,x[n-3] + \tfrac{1}{5}\,x[n-4]}$$

and had frequency response

$$\boxed{H(e^{j\omega}) = e^{-j2\omega}\big(\tfrac{1}{5} + \tfrac{2}{5}\cos(\omega) + \tfrac{2}{5}\cos(2\omega)\big), \quad \pi < \omega \le \pi}$$

The magnitude is as for the non-causal filter:

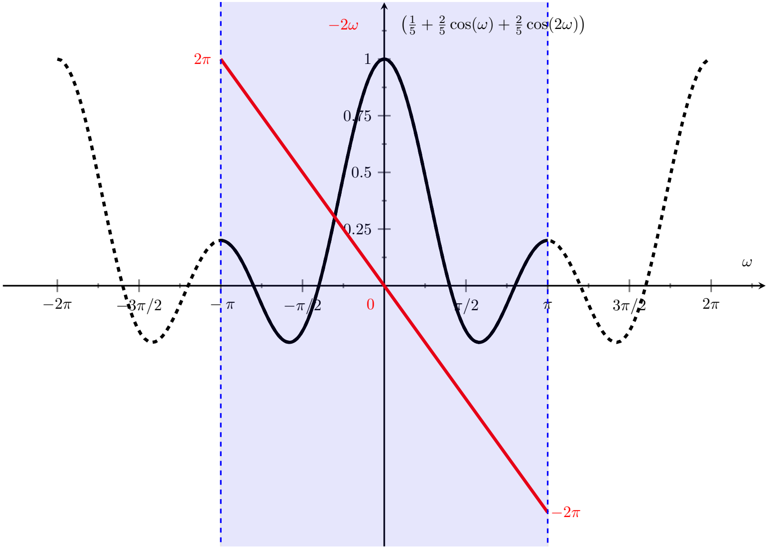
$$\boxed{\left|H(e^{j\omega})\right| = \left|\tfrac{1}{5} + \tfrac{2}{5}\cos(\omega) + \tfrac{2}{5}\cos(2\omega)\right|}$$

but the phase is linear (apart from the π discontinuities)

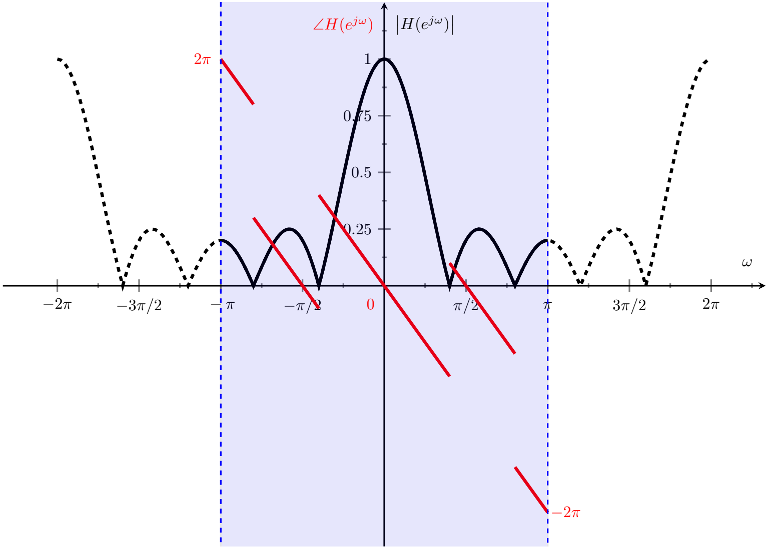
$$\boxed{\angle H(e^{j\omega}) = \begin{cases} -2\omega & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ -2\omega \pm \pi & \text{otherwise} \end{cases}}$$

Here one conventionally takes the sign of π such that $-\pi < -2\omega \pm \pi \leq \pi$.

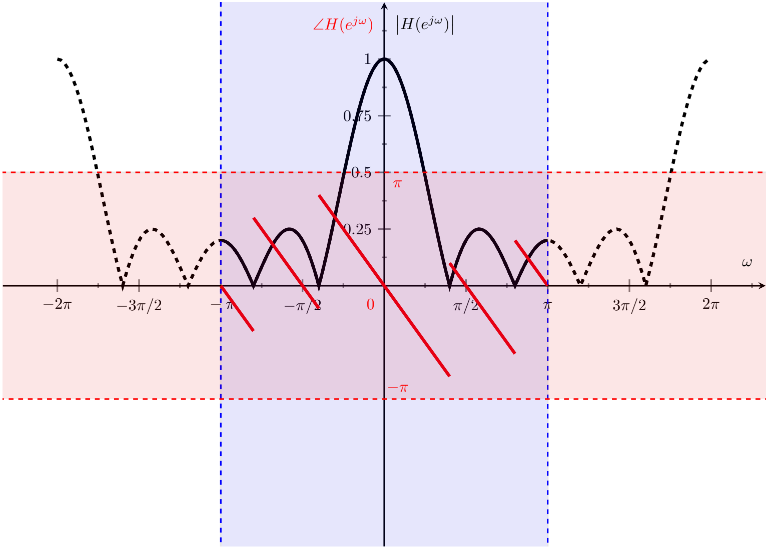
Freq Shaping and Filtering – Moving Average Filter



Freq Shaping and Filtering – Moving Average Filter



Freq Shaping and Filtering – Moving Average Filter



Summary:

- Phase is linear for causal filters (disregarding any discontinuities)
- Phase is flat for non-causal filters (disregarding the jumps between 0 and π)
- Phase is π for non-causal MA filter at frequencies where $H(e^{j\omega}) < 0$

Discontinuties due to:

- $H(e^{j\omega})$ going from positive to negative or vice versa
    - Causes jumps in phase of size π
- The convention of taking the phase $-\pi < \angle H(e^{j\omega}) \leq \pi$

High pass filter (edge detector):

$$\boxed{y[n] = \frac{1}{2}(x[n] - x[n-1])}$$

High pass filter (edge detector):

$$\boxed{y[n] = \frac{1}{2}(x[n] - x[n-1])}$$

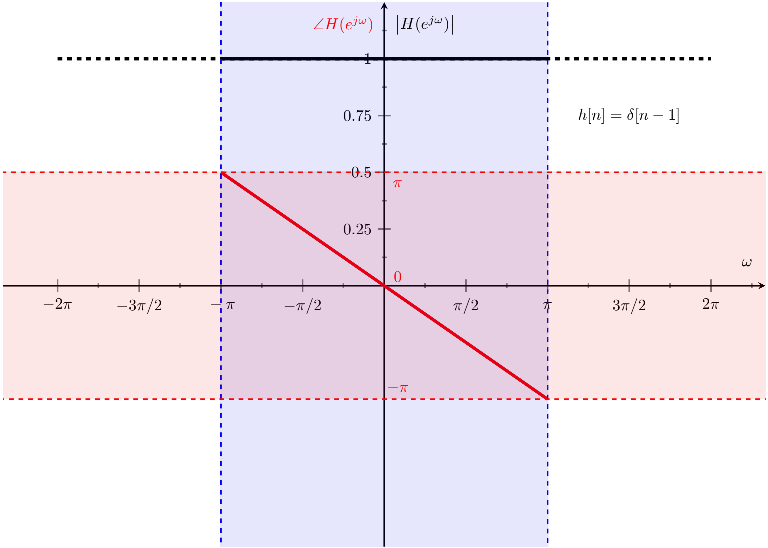
Alternative approach:

An all-pass filter satisfies

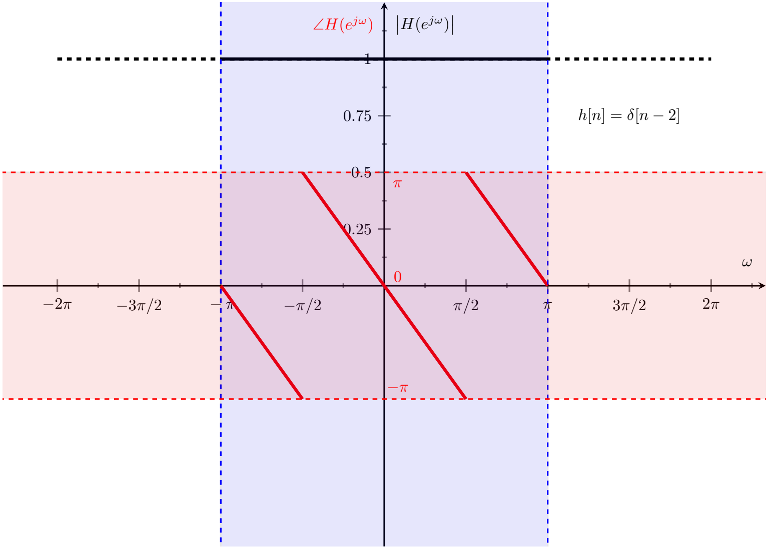
$$\boxed{\left|H(e^{j\omega})\right| = 1, \quad \text{for all } \omega}$$

- The trivial filter $h[n] = \delta[n]$ with frequency response $H(e^{j\omega}) = 1$ is **all-pass**.
- A delay filter $h[n] = \delta[n - n_0]$ ($n_0 \in \mathbb{Z}$) with frequency response $H(e^{j\omega}) = e^{-jn_0\omega}$ is **all-pass**.
- All frequencies at the input are passed to the output with no change in amplitude/magnitude. But the phase can be modified.
- Not every all-pass filter is a delay filter.
- A delay filter has (negative) linear phase. The phase is a straight line with slope proportional to n_0. The delay is encoded in the slope.

Freq Shaping and Filtering – Other Types of Filters

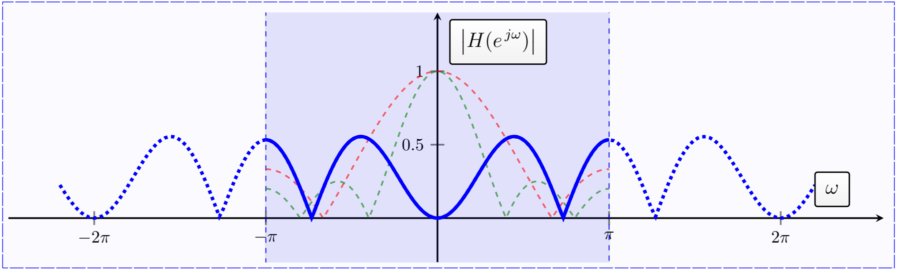


Freq Shaping and Filtering – Other Types of Filters



DT Band Pass Filter example (3 and 5 term non-causal moving average difference):

$$y[n] = \tfrac{1}{3}\,x[n-1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n+1]$$
$$- \tfrac{1}{5}\,x[n-2] - \tfrac{1}{5}\,x[n-1] - \tfrac{1}{5}\,x[n] - \tfrac{1}{5}\,x[n+1] - \tfrac{1}{5}\,x[n+2]$$



Part 3 Outline

5. **CT Non-Periodic Signals**
   - Up to this Point

6. **Fourier Transforms**
   - Fourier Analysis and Synthesis
   - Examples
   - Ideal Low Pass Filter

Part 3 Outline

CT Non-Periodic Signals – Up to this Point

Time-domain properties	Periodic	Non-periodic	
Continuous	Fourier series (FS)	Fourier transform (FT)	**Non-periodic**
Discrete	Discrete-time Fourier series (DTFS)	Discrete-time Fourier transform (DTFT)	**Periodic**
	Discrete	**Continuous**	**Frequency-domain Properties**

Time-domain Property	Frequency-domain Property
continuous	non-periodic
discrete	periodic
periodic	discrete
non-periodic	continuous

CT Non-Periodic Signals – Up to this Point

For Example:

- $x[n] = \cos(\frac{n}{6})$ - DTFT (discrete and non-periodic)
- $x(t) = \cos(2\pi t)$ - FS (continuous and periodic)
- $x(t) = u(t)$ - FT (continuous and non-periodic)
- $x[n] = \cos(\frac{\pi n}{6})$ - DTFS (discrete and periodic)

CT Non-Periodic Signals – Up to this Point

In the end, **Fourier series** only describe/represent **periodic** time domain signals (continuous or discrete).

- But not all signals are periodic (most aren't).
- Can we generalize the Fourier series for non-periodic time domain signals?
- Leads to the Fourier transform. It's not too much different from Fourier series.

Part 3 Outline

5. CT Non-Periodic Signals
   - Up to this Point

6. **Fourier Transforms**
   - Fourier Analysis and Synthesis
   - Examples
   - Ideal Low Pass Filter

Definition (Fourier Analysis and Synthesis)

Provided the following integrals are finite/exist

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

- $X(j\omega)$ is called the (frequency) spectrum
- $|X(j\omega)|$ is the magnitude spectrum
- $\angle X(j\omega)$ is the phase spectrum

Fourier Transforms – Fourier Analysis and Synthesis

Definition (Fourier Transform Pairs)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

Definition (Fourier Series Pairs)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T}\int_{T} x(t)\,e^{-jk\omega_0 t}\,dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

Fourier transform is valid for periodic and non-periodic signals. Fourier series is only valid for periodic signals.

Comments:

- Both time t and frequency ω are continuous. Going from periodic to non-periodic means frequency goes from discrete to continuous (as a rule).
- Synthesis and analysis equations are virtually identical in form. There is a -1 in one exponent, t and ω are interchanged and there is a leading constant term.
- Also called the Fourier transform and inverse Fourier transform.

Key Questions:

- Fourier transform: what is the frequency content of an "arbitrary" time domain signal.
- Inverse Fourier transform: what time domain signal corresponds to a given frequency domain description.

Fourier transform: What is the frequency content of an "arbitrary" time domain signal.

- An LTI system is a "filter". It has a frequency response. A frequency response tells you how the frequencies in a signal are modified (magnitude and phase).
- But the action of an LTI filter is to convolve. The frequency content of the filter output which is the frequency content of the filter input modified by the filter frequency response must be the Fourier Transform of the convolution of input and system impulse response.

Inverse Fourier transform: What time domain signal corresponds to a given frequency domain description.

- Engineering specifications are in the frequency domain most often. In communications and broadcast radio/TV each transmitter needs to confine the transmissions into tightly controlled frequency ranges. (Bandwidth is very expensive, want to limit interference to neighbouring frequency bands, improve efficiency, etc.)
- If I know the frequency shape then the inverse Fourier transform tells me how to design the time domain filter for implementation.

Part 3 Outline

5 CT Non-Periodic Signals
- Up to this Point

6 **Fourier Transforms**
- Fourier Analysis and Synthesis
- **Examples**
- Ideal Low Pass Filter

Fourier Transforms – Examples

Transform Pair 10: Let $x(t) = \delta(t)$ then

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$
$$= 1, \quad \text{for all } \omega$$

This is the constant function equal to 1 for all ω.

That is, we have the synthesis:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

Could say that $\delta(t)$ is unusual in that it contains equal portions of all frequencies (phase aligned).

Fourier Transforms – Examples

Recap:

$$\delta(t) \xleftrightarrow{\mathscr{F}} 1$$

$$\mathscr{F}\{\delta(t)\} = 1$$

$$\mathscr{F}^{-1}\{1\} = \delta(t)$$

- Frequency domain function 1 is the *Fourier Transform* of $\delta(t)$
- Time domain function $\delta(t)$ is the *Inverse Fourier Transform* of 1

Transform Pair 12: Let $x(t) = \delta(t - t_0)$, which as an impulse response gives an LTI system acting as a delay of t_0, then

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$
$$= e^{-j\omega t_0}$$

This is the a linear phase, with slope $-t_0$. Of course, the previous example is the special case $t_0 = 0$, no delay (as an impulse response, the trivial do-nothing system).

That is, we have the synthesis:

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t - t_0)}\, d\omega$$

Fourier Transforms – Examples

Recap:

$$\delta(t - t_0) \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0}$$

$$\mathscr{F}\left\{\delta(t - t_0)\right\} = e^{-j\omega t_0}$$

$$\mathscr{F}^{-1}\left\{e^{-j\omega t_0}\right\} = \delta(t - t_0)$$

- Frequency domain function $e^{-j\omega t_0}$ is the *Fourier Transform* of $\delta(t - t_0)$
- Time domain function $\delta(t - t_0)$ is the *Inverse Fourier Transform* of $e^{-j\omega t_0}$

Fourier Transforms – Digression

Generally with Fourier Transforms:

- Well known and useful signals are tabulated. No need to compute the Fourier transform (except in introductory courses like this).
- Can use superposition to combine Fourier transforms of different component signals. Why? The Fourier transform is a **linear** operator (functions to functions) since integration is linear.
- Also want to be able to stretch and shrink, multiply and other operations to be able to use the raw tabulated Fourier transforms. We'll consider these later.

Fourier Transforms – Digression

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier Transforms – Digression

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transforms – Examples

Transform Pair 13: Let $x(t) = e^{-at}\, u(t)$, for $a > 0$ and $u(t)$ is the unit step, then

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Recap:

$$e^{-at}\,u(t) \xleftrightarrow{\mathscr{F}} \frac{1}{a+j\omega}$$

$$\mathscr{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{a+j\omega}$$

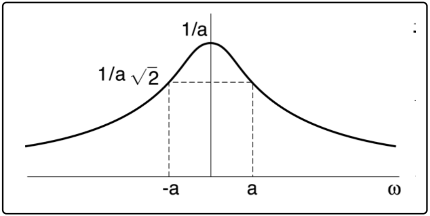
$$\mathscr{F}^{-1}\left\{\frac{1}{a+j\omega}\right\} = e^{-at}\,u(t)$$

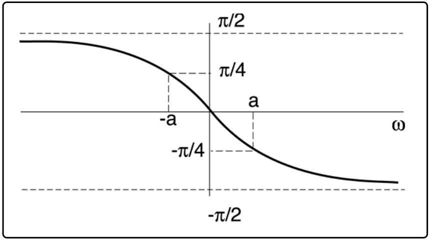
- Frequency domain function $\frac{1}{a+j\omega}$ is the *Fourier Transform* of $e^{-at}\,u(t)$
- Time domain function $e^{-at}\,u(t)$ is the *Inverse Fourier Transform* of $\frac{1}{a+j\omega}$

Fourier Transforms – Examples

Magnitude and phase are:

$$\left|X(j\omega)\right| = (a^2 + \omega^2)^{-1/2}$$
$$\angle X(j\omega) = -\tan^{-1}(\omega/a)$$





Fourier Transforms – Examples

$e^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$

Gaussian: Let $x(t) = e^{-at^2}$, for $a > 0$, then

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2} \, e^{-j\omega t} \, dt \\
&= \int_{-\infty}^{\infty} e^{-a[t^2 + j\omega t/a + (j\omega/2a)^2] + a(j\omega/2a)^2} \, dt \\
&= \int_{-\infty}^{\infty} e^{-a(t + j\omega/2a)^2} \, dt \cdot e^{-\omega^2/4a} \\
&= \sqrt{\tfrac{\pi}{a}} \, e^{-\omega^2/4a}
\end{aligned}
$$

So the Fourier Transform of a Gaussian is a compressed (or expanded) and scaled Gaussian. Weird but important in applications and the “uncertainty principle”.

Recap:

$$e^{-at^2} \overset{\mathscr{F}}{\longleftrightarrow} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}\left\{e^{-at^2}\right\} = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\right\} = e^{-at^2}$$

- Frequency domain function $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$ is the *Fourier Transform* of e^{-at^2}
- Time domain function e^{-at^2} is the *Inverse Fourier Transform* of $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$

Fourier Transforms – Examples

Transform Pair 5: Let $x(t) = 1$

Transform Pair 5: Start instead with $X(e^{j\omega}) = 2\pi\delta(\omega)$:

Transform Pair 8: The Fourier Transform of a rectangular "brickwall" pulse of width $2T_1$, that is,

$$x(t) = \chi_{[-T_1, T_1]}(t) \triangleq \begin{cases} 1 & |t| \leq T_1 \\ 0 & \text{otherwise} \end{cases}$$

Fourier Transforms – Examples

Recap:

$$\chi_{[-T_1,T_1]}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{2\sin(\omega T_1)}{\omega}$$

$$\mathscr{F}\left\{\chi_{[-T_1,T_1]}(t)\right\} = \frac{2\sin(\omega T_1)}{\omega}$$

$$\mathscr{F}^{-1}\left\{\frac{2\sin(\omega T_1)}{\omega}\right\} = \chi_{[-T_1,T_1]}(t)$$

- Frequency domain function $\frac{2\sin(\omega T_1)}{\omega}$ is the *Fourier Transform* of $\chi_{[-T_1,T_1]}(t)$
- Time domain function $\chi_{[-T_1,T_1]}(t)$ is the *Inverse Fourier Transform* of $\frac{2\sin(\omega T_1)}{\omega}$

Fourier Transforms – Examples

Transform Pair 9: Consider a low pass (rectangular) spectrum in the frequency domain:

$$X(j\omega) = \chi_{[-W,W]}(\omega) \triangleq \begin{cases} 1 & |\omega| \le W \\ 0 & \text{otherwise} \end{cases}$$

Recap:

$$\frac{\sin(Wt)}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-W,W]}(\omega)$$

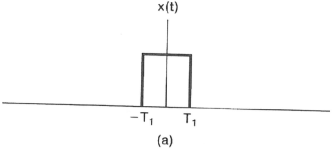
$$\mathscr{F}\left\{\frac{\sin(Wt)}{\pi t}\right\} = \chi_{[-W,W]}(\omega)$$

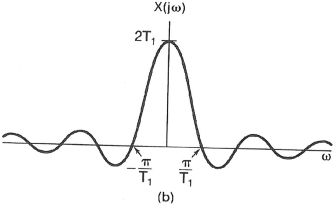
$$\mathscr{F}^{-1}\left\{\chi_{[-W,W]}(\omega)\right\} = \frac{\sin(Wt)}{\pi t}$$

- Frequency domain function $\chi_{[-W,W]}(\omega)$ is the *Fourier Transform* of $\frac{\sin(Wt)}{\pi t}$
- Time domain function $\frac{\sin(Wt)}{\pi t}$ is the *Inverse Fourier Transform* of $\chi_{[-W,W]}(\omega)$

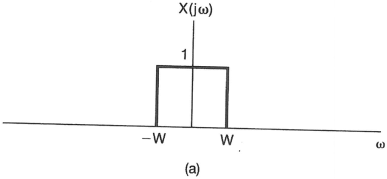
A signal which is concentrated in one domain is spread out in the other domain.

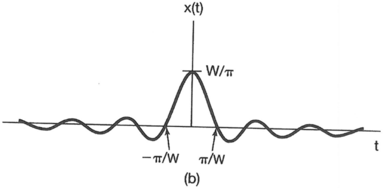
$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \longleftrightarrow \frac{2\sin(\omega T_1)}{\omega}$$





$$\frac{1}{\pi t}\sin Wt \longleftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$





Part 3 Outline

Ideal LPF: Passes only frequencies between $[-\omega_c, +\omega_c]$, where ω_c is the cut-off frequency.

The frequency domain specification is

$$\boxed{H(j\omega) = \chi_{[-\omega_c, +\omega_c]}(\omega)}$$

(RHS is the characteristic function, just some shorthand) where the phase is zero for all ω (our choice here). Then

$$\begin{aligned} h(t) = \mathscr{F}^{-1}\{H(j\omega)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-\omega_c, +\omega_c]}(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega \\ &= \frac{\sin \omega_c t}{\pi t} \\ &= \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \end{aligned}$$

where the "sinc function" is defined as

$$\operatorname{sinc}(\theta) \triangleq \frac{\sin \pi\theta}{\pi\theta}$$

Fourier Transforms – Ideal Low Pass Filter

Definition (Ideal Low Pass Filter)

The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin \omega_c t}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-\omega_c, +\omega_c]}(\omega)}$$

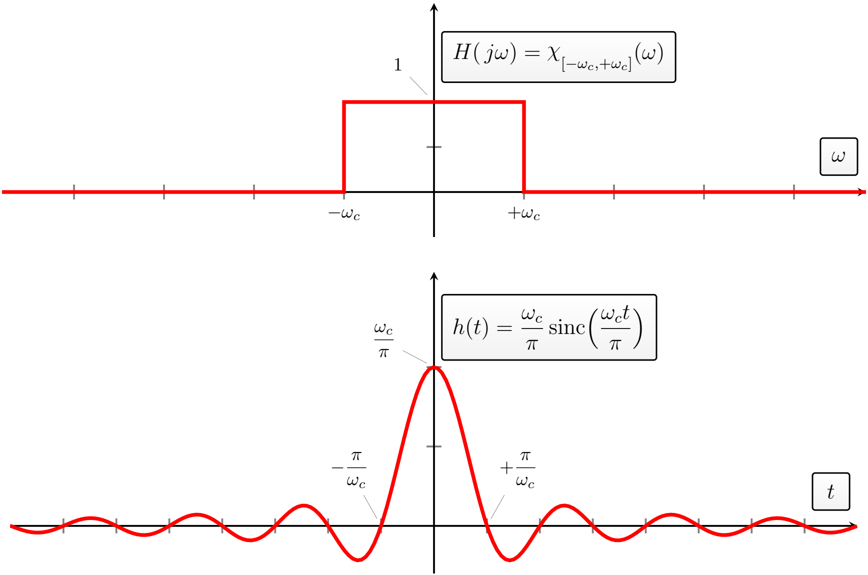
or

Definition (Ideal Low Pass Filter)

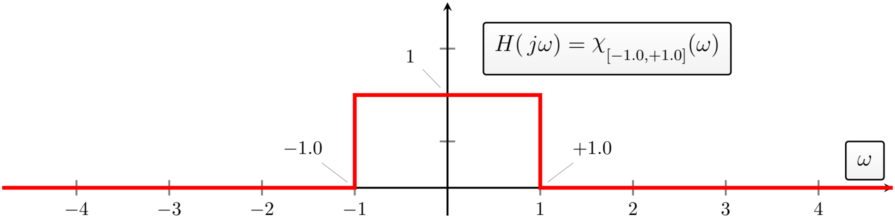
The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

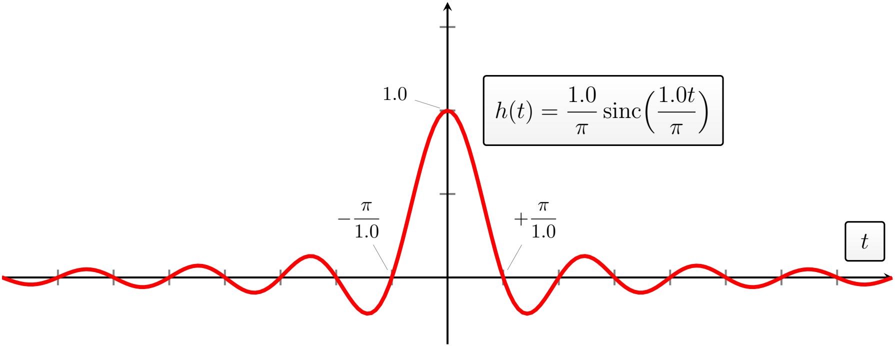
$$\boxed{\frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-\omega_c, +\omega_c]}(\omega)}$$

Fourier Transforms – Ideal Low Pass Filter



Fourier Transforms – Ideal Low Pass Filter





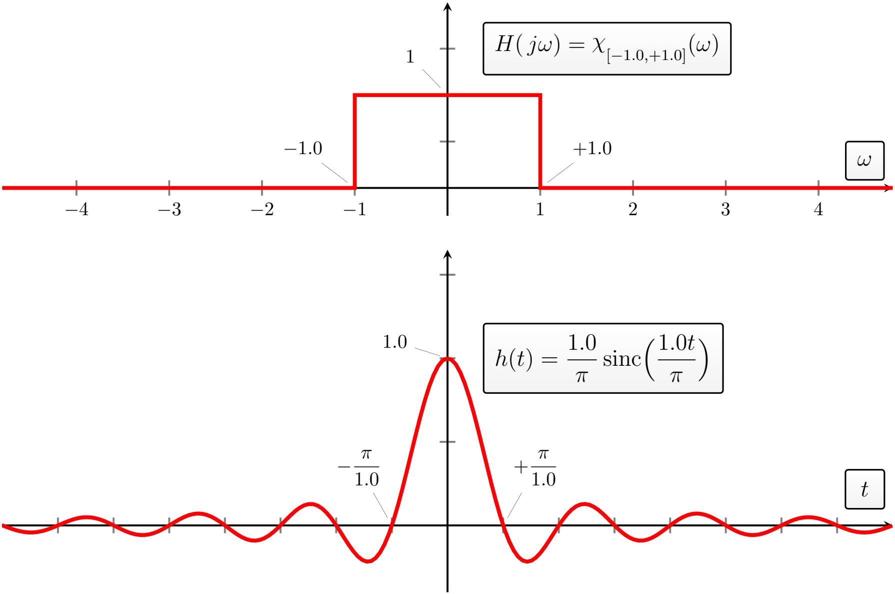
Cutoff Variations: The following frames show the effects of varying the cutoff frequency, increasing the bandwidth of an ideal LPF which contracts the sinc function in time. This can be explained in terms of

$$x(a\,t) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|}\,X\Big(\frac{j\omega}{a}\Big)$$

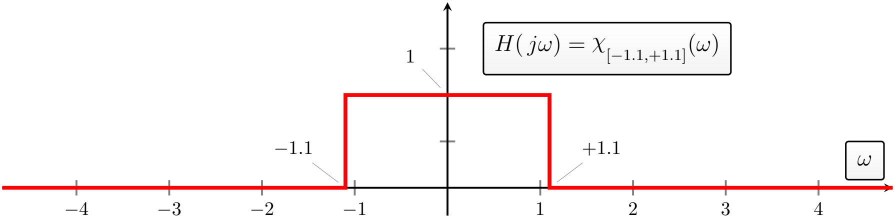
The plots show a sequence with $a = 1.0, 1.1, 1.2, 1.3, 1.4$ and 1.5

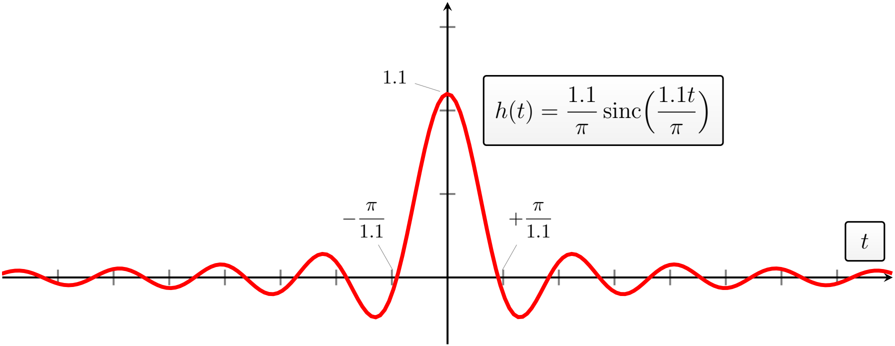
Followed by $a = 1.0, 0.9, 0.5$ and 0.25

Fourier Transforms – Ideal Low Pass Filter

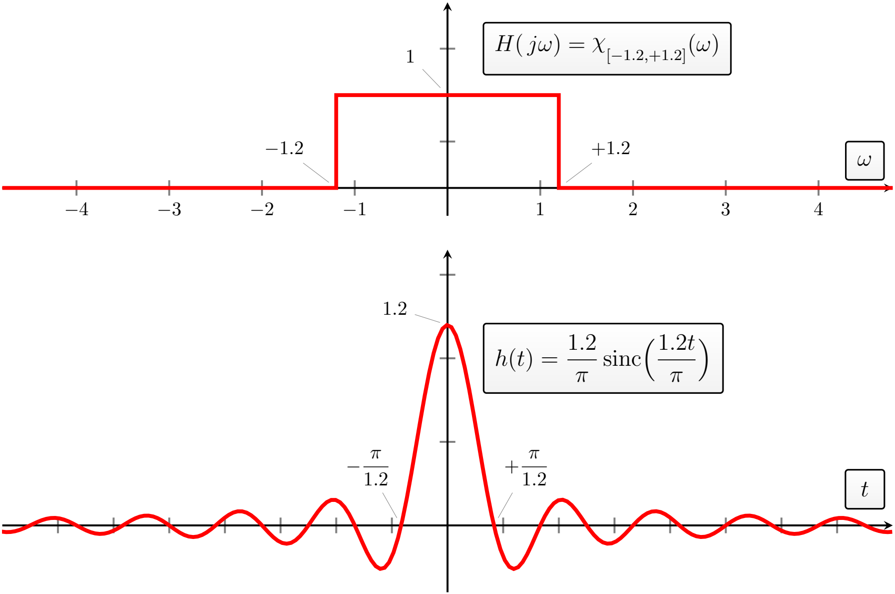


Fourier Transforms – Ideal Low Pass Filter

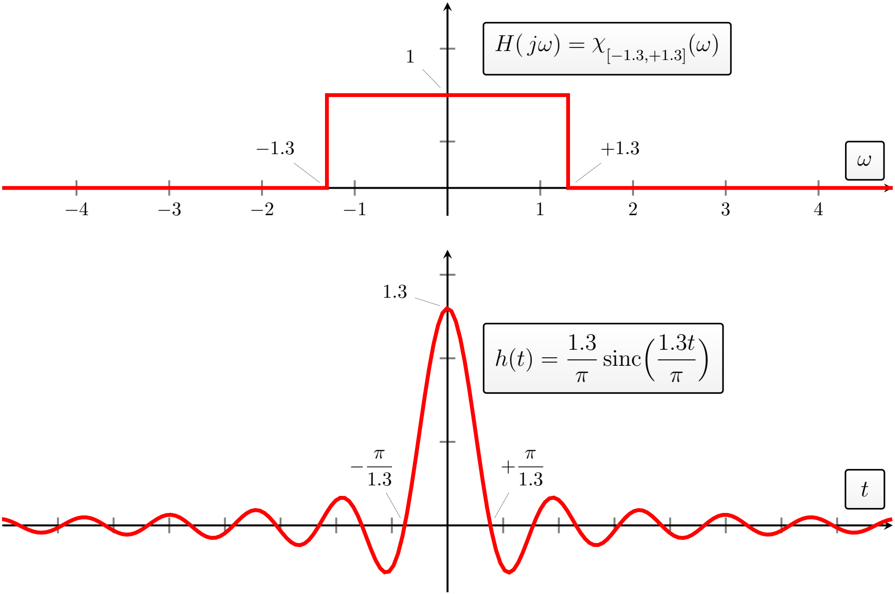


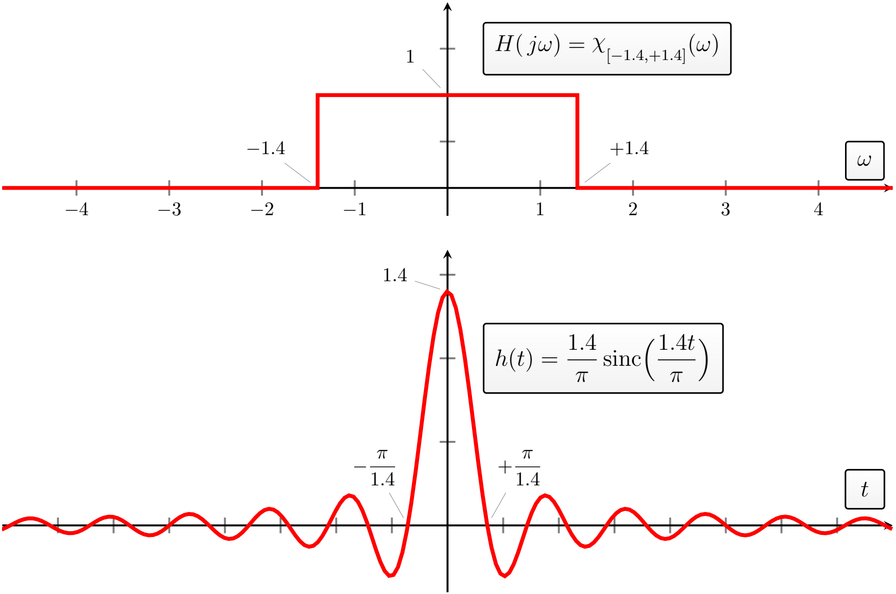


Fourier Transforms – Ideal Low Pass Filter



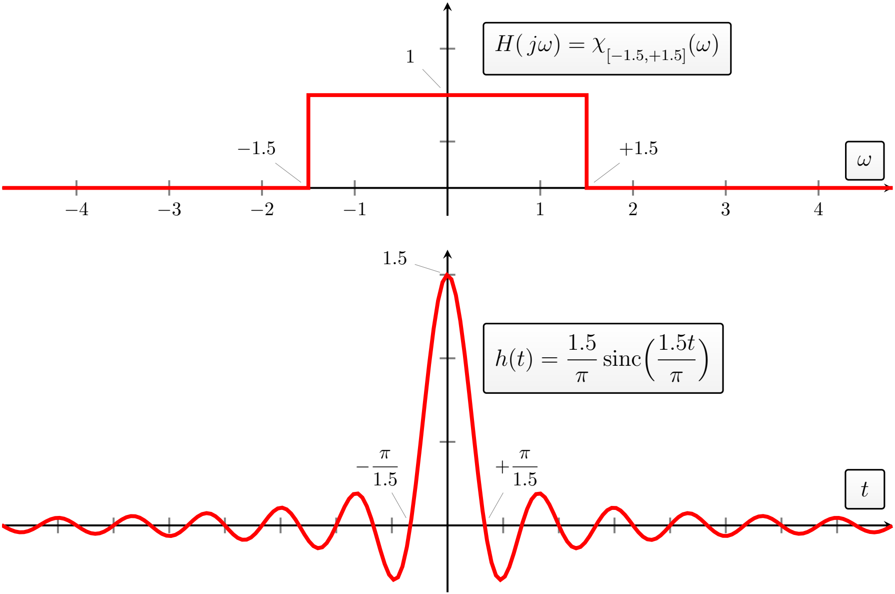
Fourier Transforms – Ideal Low Pass Filter



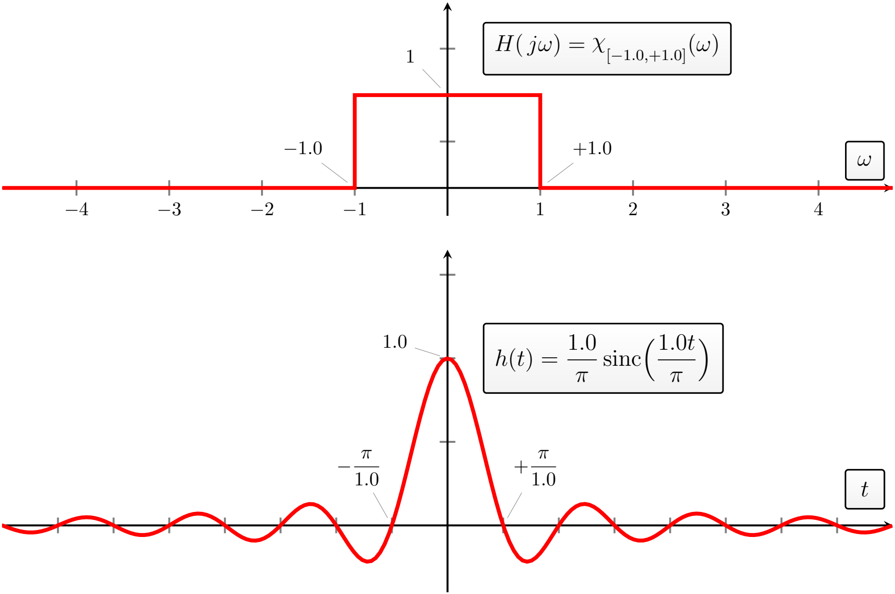


$H(j\omega) = \chi_{[-1.4,+1.4]}(\omega)$
1
-1.4
$+1.4$
ω
-4 -3 -2 -1 1 2 3 4
1.4
$h(t) = \frac{1.4}{\pi}\operatorname{sinc}\left(\frac{1.4t}{\pi}\right)$
$-\frac{\pi}{1.4}$
$+\frac{\pi}{1.4}$
t

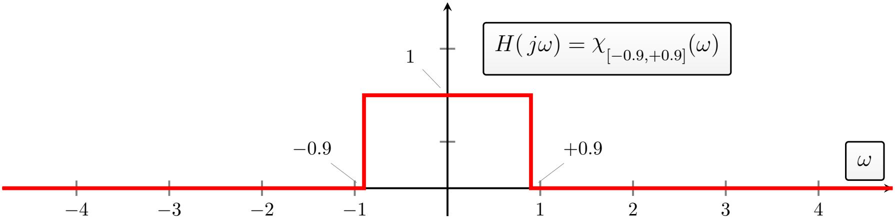
Fourier Transforms – Ideal Low Pass Filter

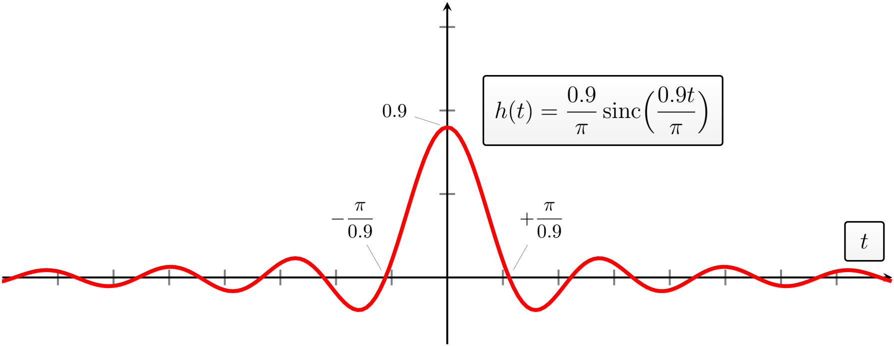


Fourier Transforms – Ideal Low Pass Filter

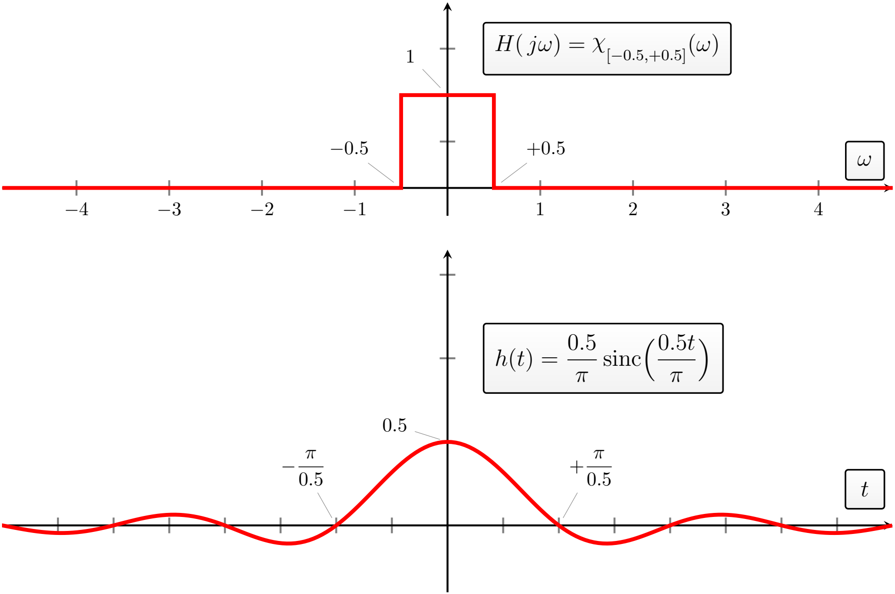


Fourier Transforms – Ideal Low Pass Filter

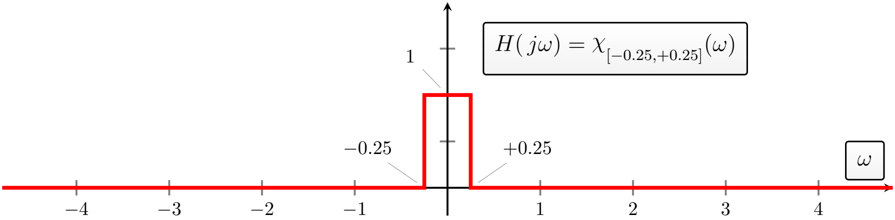


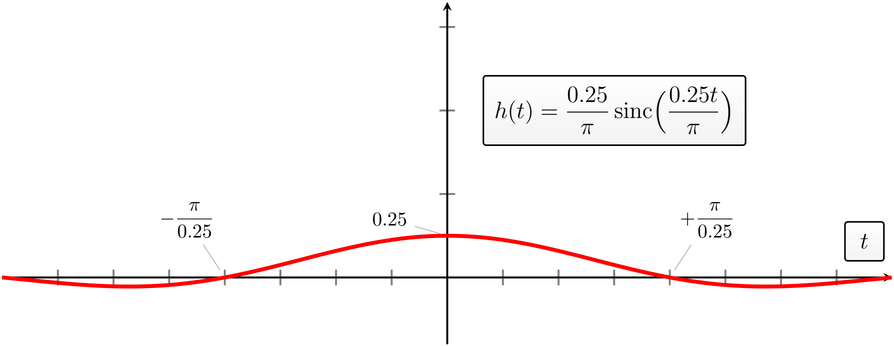


Fourier Transforms – Ideal Low Pass Filter



Fourier Transforms – Ideal Low Pass Filter





Fourier Transforms – FT Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\,\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Part 4 Outline

7. **Fourier Transforms**
    - Periodic Signals
    - Properties
    - Convolution
    - Transform Method
    - Partial Fractions
    - Filter Cascade
    - Multiplication Property
    - Differential Equations
    - Differentiator

7. **Fourier Transforms**
   - Periodic Signals
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations
   - Differentiator

Key Questions:

What if the CT signals are **periodic** in time?

Can the Fourier transform sensibly handle this case or do we need to revert to the Fourier series approach?

Answer: impulses (in the frequency domain) to the rescue.

Transform Pair 2: Begin with

$$X(j\omega) = \delta(\omega - \omega_0)$$

Recap:

$$\frac{1}{2\pi}e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} \delta(\omega - \omega_0)$$

$$\mathscr{F}\left\{\frac{1}{2\pi}e^{j\omega_0 t}\right\} = \delta(\omega - \omega_0)$$

$$\mathscr{F}^{-1}\left\{\delta(\omega - \omega_0)\right\} = \frac{1}{2\pi}e^{j\omega_0 t}$$

- Frequency domain function $\delta(\omega - \omega_0)$ is the *Fourier Transform* of $\frac{1}{2\pi}e^{j\omega_0 t}$
- Time domain function $\frac{1}{2\pi}e^{j\omega_0 t}$ is the *Inverse Fourier Transform* of $\delta(\omega - \omega_0)$

We have for all ω_0,

$$e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega - \omega_0),$$

therefore

$$e^{jk\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega - k\omega_0), \quad \forall k.$$

If $x(t)$ is **periodic** then it has Fourier Series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

where a_k are the Fourier Series coefficients. Then the Fourier Transform, by linearity, is

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

This has energy at only discrete frequencies $k\omega_0$ where $k \in \mathbb{Z}$. **This is Transform Pair 1.**

Recap:

$$\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

$$\mathscr{F}\left\{\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

$$\mathscr{F}^{-1}\left\{2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)\right\} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

- Frequency domain function $2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$ is the *Fourier Transform* of $\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$
- Time domain function $\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$ is the *Inverse Fourier Transform* of $2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$

This is an important general result for periodic signals:

- Fourier transform of a periodic signal $x(t)$ with Fourier series coefficients a_k can be interpreted as a train of impulses occurring at harmonically related frequencies and for which the area of the impulse at the k-th harmonic frequency $k\omega_0$ is 2π times the k-th Fourier series coefficient a_k.

- If already know Fourier series coefficient a_k of periodic signal can substitute a_k into this relationship to get the Fourier transform of the signal.

- Means that we can avoid integration.

Example:

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

has Fourier Transform

$$X(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

The only non-zero Fourier Series coefficients are $a_{-1} = 1/2$ and $a_1 = 1/2$. In the Fourier Transform representation, $X(j\omega)$, these just get multiplied by 2π (an artefact of definitions) and becomes the weights of delta function in continuous ω frequency domain.

This is Transform Pair 3.

Recap:

$$\cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

$$\mathscr{F}\left\{\cos(\omega_0 t)\right\} = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

$$\mathscr{F}^{-1}\left\{\pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)\right\} = \cos(\omega_0 t)$$

- Frequency domain function $\pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$ is the *Fourier Transform* of $\cos(\omega_0 t)$
- Time domain function $\cos(\omega_0 t)$ is the *Inverse Fourier Transform* of $\pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$

Example:

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

This is Transform Pair 4.

Fourier Transforms – Periodic Signals

Recap:

$$\sin(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$$

$$\mathscr{F}\left\{\sin(\omega_0 t)\right\} = \frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$$

$$\mathscr{F}^{-1}\left\{\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}\right\} = \sin(\omega_0 t)$$

- Frequency domain function $\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$ is the *Fourier Transform* of $\sin(\omega_0 t)$
- Time domain function $\sin(\omega_0 t)$ is the *Inverse Fourier Transform* of $\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$

Fourier Transforms – FT Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier Transforms – Periodic Signals

Recall the sampling function which is periodic with (fundamental) period T:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Then $\omega_0 = 2\pi/T$ and Fourier Series coefficients are, for all integer k, given by

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jw_0 t}\, dt = \frac{1}{T}$$

Whence, from earlier (4 slides ago),

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{T}}^{a_k} \delta\Big(\omega - \overbrace{\frac{2\pi k}{T}}^{k\omega_0}\Big) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\Big(\omega - k\,\omega_0\Big)$$

which is a sampling function, periodic in ω with period ω_0 and scaled by $2\pi/T$, in the frequency domain.

Recap:

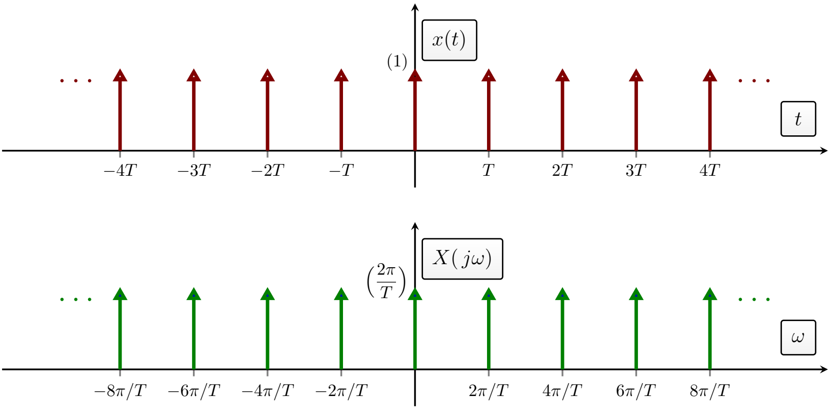
$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\mathscr{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\mathscr{F}^{-1}\left\{\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)\right\} = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

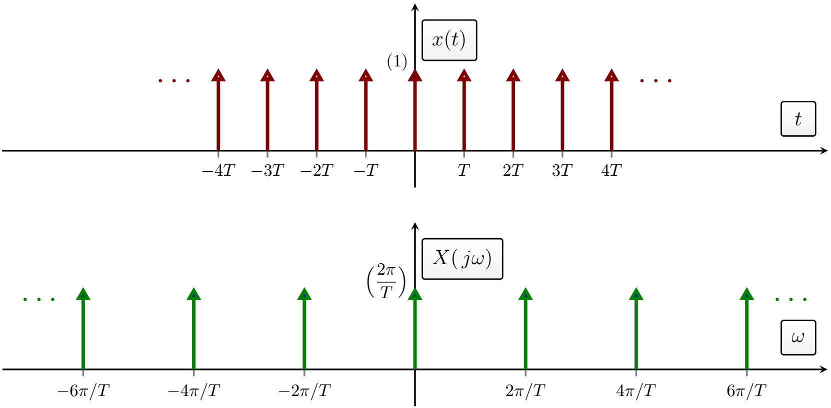
- Frequency domain function $\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ is the *Fourier Transform* of $\sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Time domain function $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ is the *Inverse Fourier Transform* of $\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$

Fourier Transforms – Periodic Signals



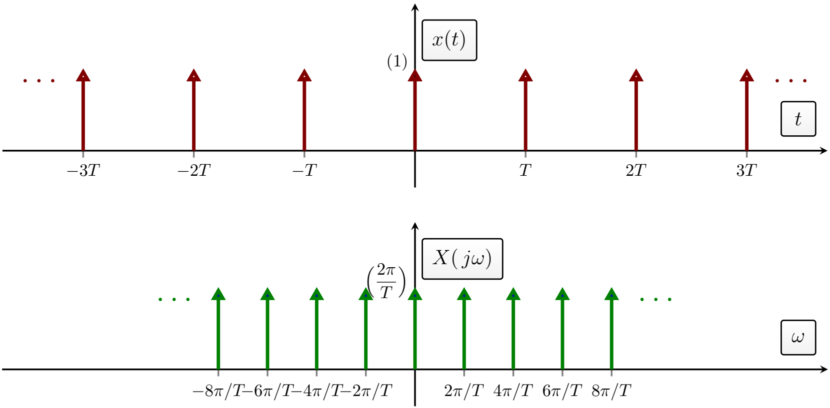
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Fourier Transforms – Periodic Signals



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

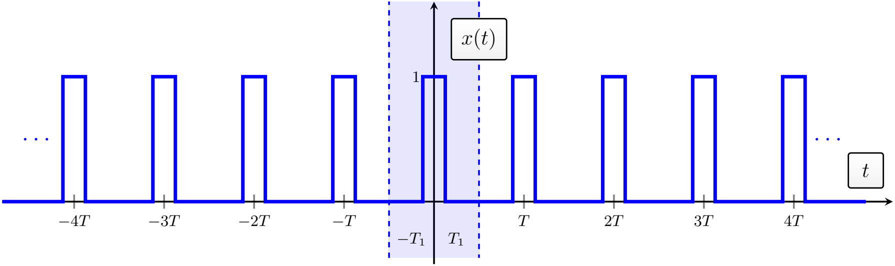
Fourier Transforms – Periodic Signals



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \le T/2$$



With Fourier series coefficients:

$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

This is Transform Pair 6.

7. **Fourier Transforms**
   - Periodic Signals
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations
   - Differentiator

Fourier Transforms – Properties

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transforms – Properties

The differentiation property: if

$$\boxed{x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega)}$$

then

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathscr{F}}{\longleftrightarrow} j\omega\, X(j\omega)}$$

That is, the operation of differentiation in the time domain is replaced by multiplication by $j\omega$ in the frequency domain.

The integration property:

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)}$$

The impulse term on the RHS of the equation reflects the DC or average value that can result from integration.

CT unit step $u(t)$ can be written as:

$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$ running integral of the unit impulse.
$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$ superposition of an infinite number of delayed impulses.

Use the first relationship and the integration property to work out the FT of $u(t)$:

This is Transform Pair 11.

Fourier Transforms – Properties

Recap:

$$u(t) \xleftrightarrow{\mathscr{F}} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\mathscr{F}\left\{u(t)\right\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\mathscr{F}^{-1}\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} = u(t)$$

- Frequency domain function $\frac{1}{j\omega} + \pi\delta(\omega)$ is the *Fourier Transform* of $u(t)$
- Time domain function $u(t)$ is the *Inverse Fourier Transform* of $\frac{1}{j\omega} + \pi\delta(\omega)$

Fourier Transforms – Properties

- Have now gone through all the transform pairs in the table.
- Can use these transform pairs and the properties of the FT to calculate the FT of many signals.
- Much easier than applying the analysis equation directly.
- Need to be able to select which transform pair to use and which properties to apply.

Fourier Transforms – Examples

Example 1: $x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$

Example 2: $x(t) = e^{-2t}u(t-3)$

From simple manipulations of the Fourier Transform integral:

Linearity/Superposition:

$$\boxed{a\,x(t) + b\,y(t) \xleftrightarrow{\mathscr{F}} a\,X(j\omega) + b\,Y(j\omega)}$$

Time Shift:

$$\boxed{x(t - t_0) \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0}\,X(j\omega)}$$

Time Scale:

$$\boxed{x(at) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|}\,X(j\omega/a)}$$

Fourier Transforms – Properties

In the CT non-periodic (in general) signal case:

Definition (Parseval's Relation)

The total energy in time domain equals total energy in frequency domain:

$$\boxed{\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| X(j\omega) \right|^2 d\omega}$$

The term

$$\boxed{\frac{1}{2\pi} \left| X(j\omega) \right|^2}$$

is called the spectral density (energy per unit frequency).

7 **Fourier Transforms**

Convolution:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\boxed{y(t) = h(t) \star x(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)\, X(j\omega)}$$

where

$$h(t) \overset{\mathscr{F}}{\longleftrightarrow} H(j\omega)$$

This follows from the eigenfunction property of the $e^{j\omega t}$ which is central to the definition of the Fourier Transform (see over).

Terminology, $H(j\omega)$ is the **frequency response**.

Recall the eigenfunction property:

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

$$\begin{aligned} x(t) &= \mathscr{F}^{-1}\{X(j\omega)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \end{aligned}$$

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) H(j\omega)\, e^{j\omega t}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(j\omega)\, X(j\omega)}_{Y(j\omega)}\, e^{j\omega t}\, d\omega \\ &= \mathscr{F}^{-1}\{H(j\omega)\, X(j\omega)\} \equiv \mathscr{F}^{-1}\{Y(j\omega)\} \end{aligned}$$

Hence

$$\boxed{y(t) = h(t) \star x(t) \iff Y(j\omega) = H(j\omega)\, X(j\omega)}$$

7. **Fourier Transforms**
   - Periodic Signals
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations
   - Differentiator

Convolution via transform techniques:

- Do Fourier Transform to convert problem to Fourier/frequency domain.
- Do convolution via multiplication of Fourier Transforms.
- Do algebraic manipulations, e.g., partial fractions.
- Do Inverse Fourier Transform to get back time domain signal.

Part 4 Outline

7 **Fourier Transforms**

- Periodic Signals
- Properties
- Convolution
- Transform Method
- Partial Fractions
- Filter Cascade
- Multiplication Property
- Differential Equations
- Differentiator

Example 1: Consider

$$h(t) = e^{-t}u(t) \quad \text{and} \quad x(t) = e^{-2t}u(t)$$

What is $y(t) = h(t) \star x(t)$?

Obviously could grind through a convolution integral. But generally going via the Fourier Transform is quicker and easier.

$X(j\omega) =$

$H(j\omega) =$

$$
\begin{aligned}
Y(j\omega) &= H(j\omega)\,X(j\omega) \\
&= \frac{1}{(1+j\omega)}\,\frac{1}{(2+j\omega)} = \frac{1}{(1+j\omega)(2+j\omega)}
\end{aligned}
$$

Do we know what the Inverse Fourier Transform of this is? Not really. However, we can do a “partial fraction expansion” (trick):

Example 2: Consider

$$H(j\omega) = \frac{(j\omega + 2)}{(j\omega + 1)^2(j\omega + 3)}$$

What is $h(t)$?

Example 3: Consider

$$x(t) = e^{-2t}u(t) \quad \text{and} \quad y(t) = e^{-t}u(t)$$

What is $h(t)$? Deconvolution example - solve in the frequency domain.
$X(j\omega) =$
$Y(j\omega) =$

Fourier Transforms – Partial Fractions

$$Y(j\omega) = H(j\omega)\, X(j\omega) \;\; H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

What is $y(t)$? If tried to solve in the time domain:

Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

What is $y(t)$? Solving in the frequency domain:

Example 4: Consider

$$x(t) = \frac{\sin(\omega_i t)}{\pi t} \quad \text{and} \quad h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

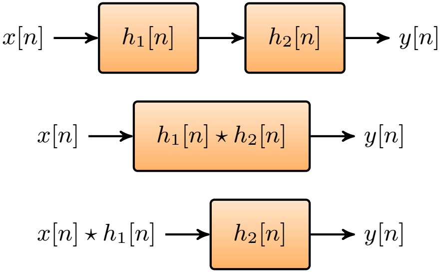
What is $y(t)$? Solving in the frequency domain:

7 Fourier Transforms

Fourier Transforms – Filter Cascade

Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

$$\boxed{x[n] \star \big(h_1[n] \star h_2[n]\big) = \big(x[n] \star h_1[n]\big) \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).

Fourier Transforms – Filter Cascade

Example 1: Cascading filters

$$x(t) \longrightarrow \boxed{H_1(j\omega)} \longrightarrow \boxed{H_2(j\omega)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

Then the convolution theorem gives

$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

Note that if $H_1(j\omega) = H_2(j\omega)$ then $H_1^2(j\omega) \equiv \big(H_1(j\omega)\big)^2$ tends to have a sharper frequency selectivity/cutoff, for example, as some frequency, attenuation by 10% becomes attenuation by almost 20%, $0.9^2 = 0.81$.

Example 2: Cascading ideal LPF filters. What is

$$\underbrace{\frac{\sin(4\pi t)}{\pi t}}_{x(t)} \star \underbrace{\frac{\sin(8\pi t)}{\pi t}}_{h(t)} = ?$$

Think about $x(t)$ as the input into LTI system with impulse response $h(t)$.

Note that: $x(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 4\pi$; and $h(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 8\pi$

Answer:

$$\frac{\sin(4\pi t)}{\pi t} \star \frac{\sin(8\pi t)}{\pi t} = \frac{\sin(4\pi t)}{\pi t}$$

That's weird. Imagine the pain to calculate:

$$\int_{-\infty}^{\infty} \frac{\sin(4\pi\tau)}{\pi\tau} \frac{\sin(8\pi(t-\tau))}{\pi(t-\tau)} \, d\tau$$

But the result is clear if we consider the frequency domain.

Useful result: If we cascade (ideal) LPFs the effect is the same as just applying the (ideal) LPF of the least bandwidth (least cut-off ω_c).

Fourier Transforms – Filter Cascade

$$\frac{\sin(3.4\pi t)}{\pi t} \star \frac{\sin(2.7\pi t)}{\pi t} \star \frac{\sin(4.1\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$
$$\frac{\sin(3.7\pi t)}{\pi t} \star \frac{\sin(5.7\pi t)}{\pi t} \star \frac{\sin(5.1\pi t)}{\pi t} \star \frac{\sin(4.0\pi t)}{\pi t} \star$$
$$\frac{\sin(3.3\pi t)}{\pi t} \star \frac{\sin(2.8\pi t)}{\pi t} \star \frac{\sin(4.9\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$
$$\frac{\sin(7.4\pi t)}{\pi t} \star \frac{\sin(8.7\pi t)}{\pi t} \star \frac{\sin(5.3\pi t)}{\pi t} \star \frac{\sin(6.1\pi t)}{\pi t} \star$$
$$\frac{\sin(5.2\pi t)}{\pi t} \star \frac{\sin(8.8\pi t)}{\pi t} \star \frac{\sin(9.1\pi t)}{\pi t} \star \frac{\sin(8.2\pi t)}{\pi t} \star$$
$$\frac{\sin(4.5\pi t)}{\pi t} \star \frac{\sin(3.5\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} = \frac{\sin(2.7\pi t)}{\pi t}$$

7 Fourier Transforms

- Periodic Signals
- Properties
- Convolution
- Transform Method
- Partial Fractions
- Filter Cascade
- **Multiplication Property**
- Differential Equations
- Differentiator

Fourier Transforms – Multiplication Property

Convolution theory states

$$\boxed{x(t) \star y(t) \xleftrightarrow{\mathscr{F}} X(j\omega)\, Y(j\omega)}$$

then by "symmetry" (and a bit of book-keeping)

$$\boxed{x(t) \cdot y(t) \xleftrightarrow{\mathscr{F}} \frac{1}{2\pi} X(j\omega) \star Y(j\omega)}$$

where, convolution in frequency,

$$\frac{1}{2\pi} X(j\omega) \star Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\, j\zeta)\, Y\big(\, j(\omega - \zeta)\big)\, d\zeta$$

Example: Suppose $s(t)$ is a signal whose spectrum is $S(j\omega)$.
Sketch the spectrum of $r(t) = s(t) \times \cos(\omega_0 t)$:

Fourier Transforms – Multiplication Property

Real-world Example: Amplitude modulation (AM):

> *AM or Amplitude Modulation is a method of radio broadcasting where the frequency is modulated or varied by its changing amplitude. Radio frequencies for AM broadcasts are expressed in kilohertz (kHz).*

ABC Canberra 666 means the centre frequency is 666 kHz. To express in terms of ω (radians per second):

$$\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots \quad \text{radians per second}$$

What is the Fourier Transform theory behind 666 Canberra?

First we take an audio signal, albeit a rather boring audio signal, $s(t)$. The Fourier Transform yields:

$$s(t) \xleftrightarrow{\mathscr{F}} S(j\omega)$$

Since $s(t)$ is audio its frequencies are limited to what people can hear. The human range, for teenagers and younger is roughly 1 Hz to 20 kHz. For various reasons AM audio is further limited to 10 kHz, so $s(t)$ is low pass and occupies the frequency range

$$|\omega| \leq 10 \text{ kHz}$$

Imagine, we could transmit such a signal directly over radio (called baseband). There would be a number of problems: interference from other baseband transmitters, ridiculously huge antennas, etc.

So ABC Canberra 666 really means $666,000 \pm 10,000$ Hz. Different stations are centred on different frequencies so that they don't interfere.

So how do we translate a baseband audio signal, $S(j\omega)$, to be centred on 666 kHz?

Fourier Transforms – Multiplication Property

Use time-domain multiplication `O&W 4.5 pp.323-324`

$$r(t) = s(t) \cdot p(t) \overset{\mathscr{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi} S(j\omega) \star P(j\omega)$$

with "modulator"

$$p(t) = \cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} P(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

where $\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots$. This implies for any audio signal $s(t)$ that

$$R(j\omega) = \frac{1}{2}\,S\big(j(\omega - \omega_0)\big) + \frac{1}{2}\,S\big(j(\omega + \omega_0)\big)$$

The frequency range for the AM modulated signal $r(t)$ is non-zero in

$$\big||\omega| - 4,184,601.41\ldots\big| \leq 10 \text{ kHz}$$

which is in RF.

At the receiver, you can multiply again by $\cos(\omega_0 t)$ to move the signal back to baseband and low pass filter to move components that go to $2\omega_0$ `O&W 4.5 pp.323-324`

$$\cos^2(\omega_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)$$

This being the combination of the modulator (transmitter) and demodulator (receiver).

7. **Fourier Transforms**
   - Periodic Signals
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations
   - Differentiator

Differential Equation of CT System

General form:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

N-th order differential equation

For example:

$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Fourier Transforms – Differential Equations

Consider the Fourier Transform of the derivative, $x'(t) = \frac{dx(t)}{dt}$ of a signal $x(t)$:

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathscr{F}} j\omega\, X(j\omega)}$$

Further, the Fourier Transform of the kth derivative, $x^{(k)}(t)$ of a signal $x(t)$:

$$\boxed{\frac{d^k x(t)}{dt^k} \xleftrightarrow{\mathscr{F}} (j\omega)^k\, X(j\omega)}$$

Fourier Transforms – Differential Equations

LCC Differential Equations: `O&W 4.7 pp.330-333`

Now solve the linear, constant coefficient differential equation:

$$\boxed{\sum_{k=0}^{K} a_k \, \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \, \frac{d^m x(t)}{dt^m}}$$

where we can interpret this as describing a system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fourier Transform both side of the differential equation to yield (see next slide)

$$\boxed{\sum_{k=0}^{K} a_k \, (j\omega)^k Y(j\omega) = \sum_{m=0}^{M} b_m \, (j\omega)^m X(j\omega)}$$

Fourier Transforms – Differential Equations

That is, take Fourier Transforms of both sides of

$$\sum_{k=0}^{K} a_k \, \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \, \frac{d^m x(t)}{dt^m}$$

to yield

$$\mathscr{F}\left\{\sum_{k=0}^{K} a_k \, \frac{d^k y(t)}{dt^k}\right\} = \mathscr{F}\left\{\sum_{m=0}^{M} b_m \, \frac{d^m x(t)}{dt^m}\right\}$$

$$\sum_{k=0}^{K} a_k \, \mathscr{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{m=0}^{M} b_m \, \mathscr{F}\left\{\frac{d^m x(t)}{dt^m}\right\}$$

$$\sum_{k=0}^{K} a_k \, (j\omega)^k Y(j\omega) = \sum_{m=0}^{M} b_m \, (j\omega)^m X(j\omega)$$

Fourier Transforms – Differential Equations

$$Y(j\omega) = \left(\frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k} \right) X(j\omega)$$

So

$$\boxed{H(j\omega) = \frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k}}$$

is the frequency response of the linear, constant coefficient differential equation system. Amazing.

Called a "rational function of $j\omega$", ratio of polynomials in $j\omega$ or ω.

Filter Design: Can get different shaped $H(j\omega)$ by choosing different values for the a_k and b_m. That is, we can **design** different filters and we can implement them in practice by implementing, somehow, the differential equation. Not these days because we now use, with care, discrete time/digital techniques. This course is not about design, however.

Example: An LTI system is described by the DE
$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$, what is $h(t)$?

Alternative Method: An LTI system is described by the DE
$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t)$, what is $h(t)$?

Part 4 Outline

7 **Fourier Transforms**

- Periodic Signals
- Properties
- Convolution
- Transform Method
- Partial Fractions
- Filter Cascade
- Multiplication Property
- Differential Equations
- Differentiator

Fourier Transforms – Differentiator

Differentiator: A differentiator is a linear time invariant system and, therefore, has an impulse response. Call this impulse response $h(t)$ and its Fourier Transform $H(j\omega)$.

Let $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$ be an signal to the differentiator. Then the output is

$$\boxed{\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega\, X(j\omega)}$$

This is a standard Fourier Transform property (given earlier but proved later).

Consider the differentiator LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \frac{dx(t)}{dt}$$

then we can infer that

$$h(t) \overset{\mathscr{F}}{\longleftrightarrow} j\omega \quad \text{or} \quad H(j\omega) = j\omega.$$

Look at this in more detail (over).

Fourier Transforms – Differentiator

Starting from

$$H(j\omega) = j\omega$$

we observe this amplifies high frequencies, and kills DC because the magnitude and phase are:

$$\big|H(j\omega)\big| = |\omega| \quad \text{and} \quad \angle H(j\omega) = \overbrace{\mathsf{sgn}(\omega)}^{\pm 1}\,\pi/2$$

noting that $j = e^{j\pi/2}$.

Is this reasonable? Yes, consider

$$\begin{aligned}\frac{d}{dt}\sin(\omega_c t) &= \omega_c\,\cos(\omega_c t)\\ &= \omega_c\,\sin(\omega_c t + \pi/2)\end{aligned}$$

where ω_c is a multiplier (gain) and there is a $\pi/2$ phase shift.