Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

ASSIGNMENT 1 – SOLUTIONS

Due Date: Friday, 1 September 2017, 11:55 PM (Friday Week 6)

Late Submission Policy: Submit by the due date and time. Late assessment is *not accepted* for this course. That is, late submissions will get 0 marks.

This policy is to support the majority of students who complete and submit on time. This hard deadline enables the quick release of the assignment solutions.

Wattle submission: Upload your report as PDF format as a *multipage, single file* in Wattle. Use a scanner or scan-type smartphone app to create the PDF from your handwritten solutions.

Assignment format: 5 problems, for a total of 75 marks.

Value: 7% of total course assessment.

Solution: Will be posted on Wattle by Saturday, 2 September 2017. Marked assignments will be returned back in Wattle within 10–14 days.

Relationship to textbook: This assignment is related to Chapters 1–2 in the textbook, and Problem Sets 1, 2 and 3. It is intended to aid you in your preparation for the mid-semester exam.

Declaration by the Student: All assessment task submissions, regardless of mode of submission, require agreement to the following declaration by the student:

> "I declare that this work:
> - upholds the principles of academic integrity, as defined in the University Academic Misconduct Rules;
> - is original, except where collaboration (for example group work) has been authorised in writing by the course convener in the course outline and/or Wattle site;
> - is produced for the purposes of this assessment task and has not been submitted for assessment in any other context, except where authorised in writing by the course convener;
> - gives appropriate acknowledgement of the ideas, scholarship and intellectual property of others insofar as these have been used;
> - in no part involves copying, cheating, collusion, fabrication, plagiarism or recycling."

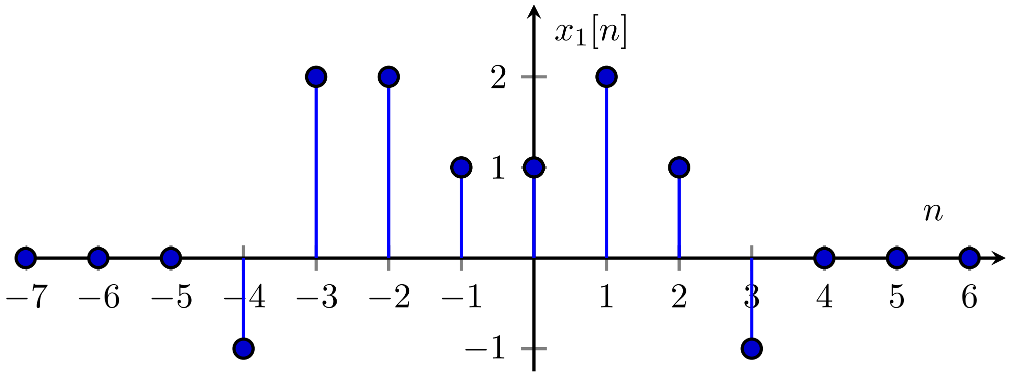
Release Date: Monday, 31 July 2017 (Monday Week 2)

In the following: $\delta[n]$ and $u[n]$ represent the Dirac and unit step functions for discrete-time (DT). Similarly $\delta(t)$ and $u(t)$ for continuous-time (CT). Convolution of signals is written $x[n] \star h[n]$ or $x(t) \star h(t)$. Please indicate any identities or formulas used in the simplification of the results.

Problem 1

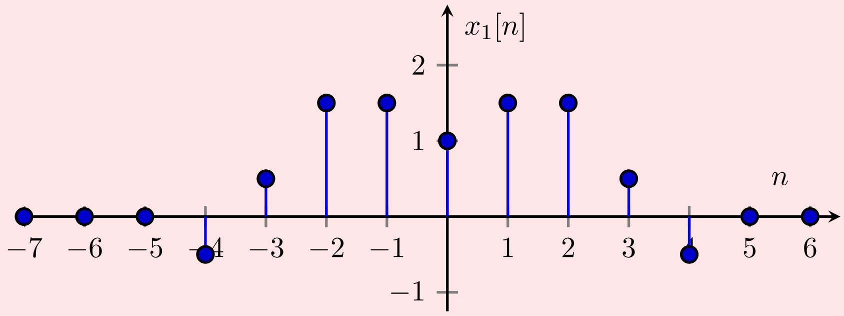
Determine and sketch the even and odd parts of the DT signals depicted below:

(a) **[2 marks]** For $x_1[n]$ shown below:

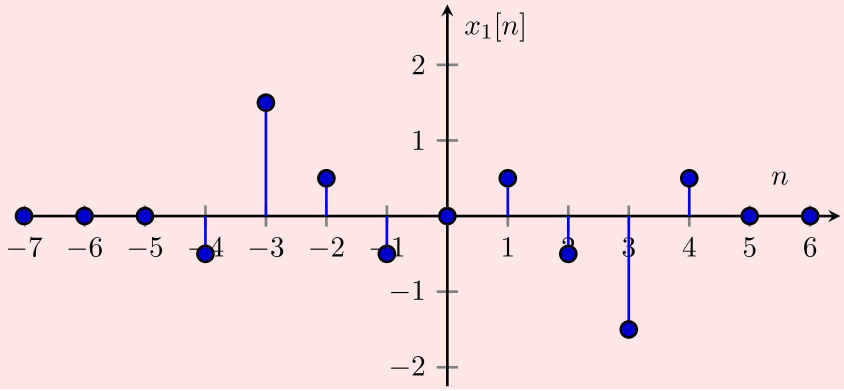


Solution:

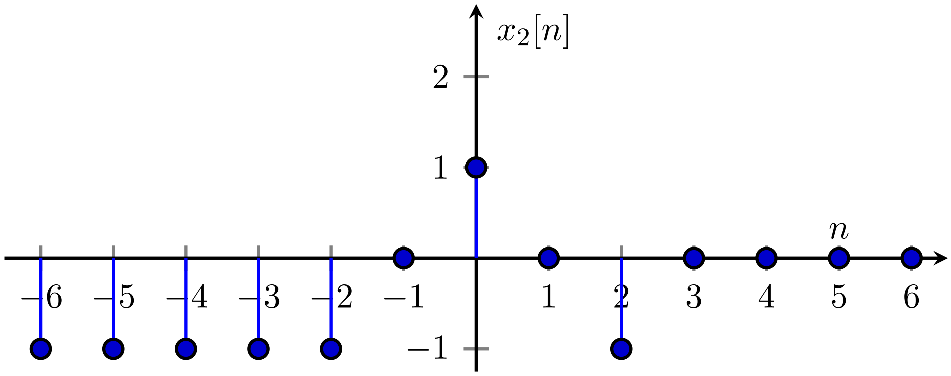
$Ev\{x_1[n]\}$



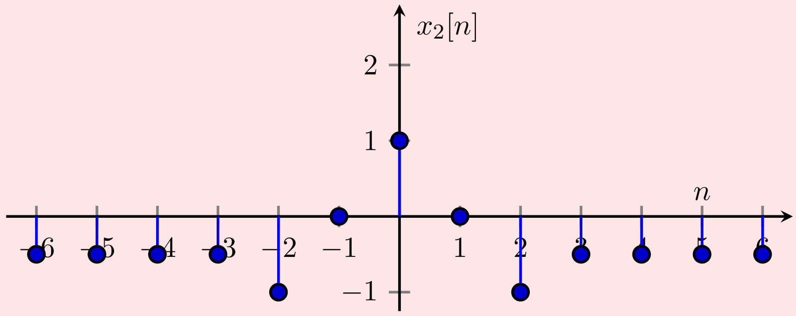
$Od\{x_1[n]\}$



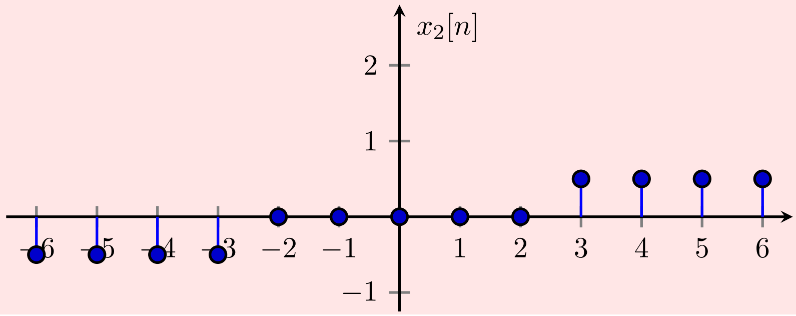
(b) **[2 marks]** For $x_2[n]$ shown below:



Solution:
$Ev\{x_2[n]\}$



$Od\{x_2[n]\}$



Problem 2

Determine whether or not each of the signals is periodic. If a signal is periodic, determine the fundamental period.

(a) **[2 marks]**

$$x(t) = Ev\big\{\sin(4\pi\, t)\, u(t)\big\}$$

Solution: It can be easily shown that $x(t) = \frac{1}{2}\big(\sin(4\pi\, t)\, u(t) + \sin(-4\pi\, t)\, u(-t)\big) = \frac{1}{2}\big(\sin(4\pi\, t)\, u(t) - \sin(4\pi\, t)\, u(-t)\big)$ is not periodic. Draw it.

At $t = 0$ there is a discontinuous derivative that is not repeated at any other time. □

(b) **[2 marks]**

$$x(t) = \sum_{n=-\infty}^{\infty} e^{(2t-n)}$$

Solution: The signal is not periodic. (There is no j in the exponent.) □

(c) **[2 marks]**

$$x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$$

Solution: We express $x[n]$ as (sums and differences of the frequencies)

$$x[n] = \frac{1}{2}\big(\cos\frac{3\pi n}{4} + \cos\frac{\pi n}{4}\big).$$

Both $\cos(3\pi n/4)$ and $\cos(\pi n/4)$ are periodic with period 8. Therefore $x[n]$ is periodic with period $N = 8$. □

(d) **[2 marks]**

$$x[n] = e^{j10n}\cos(\pi n/22)$$

Solution: Since e^{j10n} has period $\frac{2\pi}{\omega}m = \frac{2\pi}{10}m$ which is not an integer (so is aperodic) the whole function will be aperiodic.

(e) **[2 marks]**

$$x[n] = 3\,e^{j0.6(n+0.5)}$$

Solution: $N = \frac{2\pi}{3/5}(m)$, so aperiodic because can't find any integer m such that $\frac{2\pi}{\omega_0}(m)$ is also an integer.

(f) **[2 marks]**

$$x(t) = \sin(4t) + \sin(5t)$$

Solution: The period for $\sin(4t)$ is $\frac{2\pi}{4} = \frac{\pi}{2}$. The period for $\sin(5t)$ is $\frac{2\pi}{5}$. So the period for $x(t)$ will be the lowest common multiple of the two periods $= 2\pi$.

Problem 3

Determine whether the following systems are: i) time-invariant, ii) linear and iii) casual:

(a) **[3 marks]** The CT system:

$$y(t) = x(t+3) - x(1-t),$$

with input $x(t)$ and output $y(t)$.

Solution: Time-varying because of the time reversal about $t = 1$ of the second component; linear (linear combination of two linear systems); not causal since, for example, $y(-5)$ needs future input $x(6)$. □

(b) **[3 marks]** The DT system:

$$y[n] = \begin{cases} (-1)^n x[n] & \text{if } x[n] \geq 0 \\ 2\,x[n] & \text{if } x[n] < 0 \end{cases},$$

with input $x[n]$ and output $y[n]$.

Solution: Time-varying due to $(-1)^n$ weighting changing with time n; non-linear because scaling an input by -1 toggles the condition and the output won't scale by -1, for example, if $x[3] = -2$ then $y[3] = -4$, whereas $x[3] = 2$ (scaling by -1) then $y[3] = -2$ (which is not 4); causal (in fact it is memoryless). □

(c) **[3 marks]** The DT system :

$$y[n] = \sum_{k=n}^{\infty} x[k],$$

with input $x[n]$ and output $y[n]$.

Solution: This is the same as

$$y[n] = x[n] + x[n+1] + x[n+2] + \cdots$$

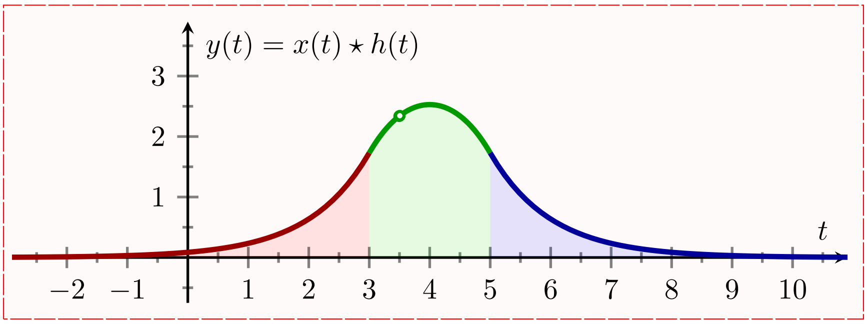
Time-invariant; linear but not causal. □

Problem 4

The impulse response of a system is given by $h(t) = 2\,u(t-3) - 2\,u(t-5)$. For an input of $x(t) = e^{-|t|}$, the system produces the output

$$y(t) = \begin{cases} 2(e^{t-3} - e^{t-5}) & t < 3 \\ 4 - 2\,e^{t-5} - 2\,e^{3-t} & 3 \leq t \leq 5 \\ -2(e^{3-t} - e^{5-t}) & t > 5 \end{cases}$$

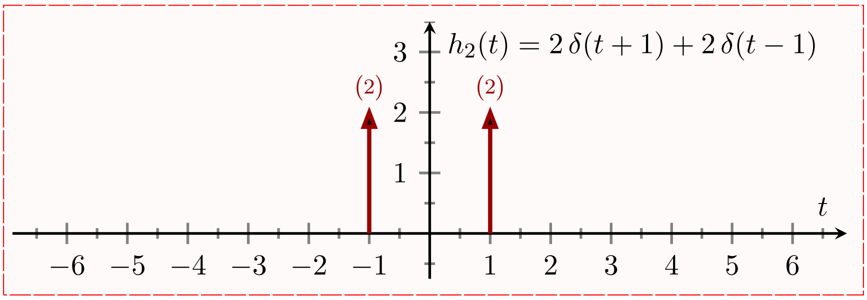
shown in the figure below:



We want to approximate the system with that of a simpler system that gives approximately $y(t)$ for the input of interest $x(t) = e^{-|t|}$. Let's try a new "system" called $h_2(t)$ consisting of two Dirac delta functions:

$$h_2(t) = 2\,\delta(t+1) + 2\,\delta(t-1),$$

which is shown below:



The value of the delta function is presented in round brackets and that number represents the area, here (2) means the delta function has area 2 (so $h_2(t)$ has total area 4).

(a) [**7 marks**] Compute the following convolution

$$y_2(t) = x(t) \star h_2(t),$$

where $x(t) = e^{-|t|}$ as before.

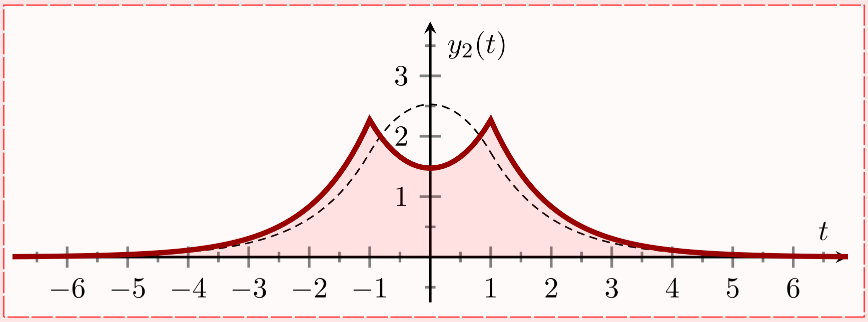
Solution:

$$\begin{aligned} y_2(t) &= e^{-|t|} \star \big(2\,\delta(t+1) + 2\,\delta(t-1)\big) \\ &= 2\,e^{-|t+1|} + 2\,e^{-|t-1|}. \end{aligned}$$

□

(b) [**7 marks**] Plot $y_2(t) = x(t) \star h_2(t)$ and compare with $y(t) = x(t) \star h(t)$.

Solution:

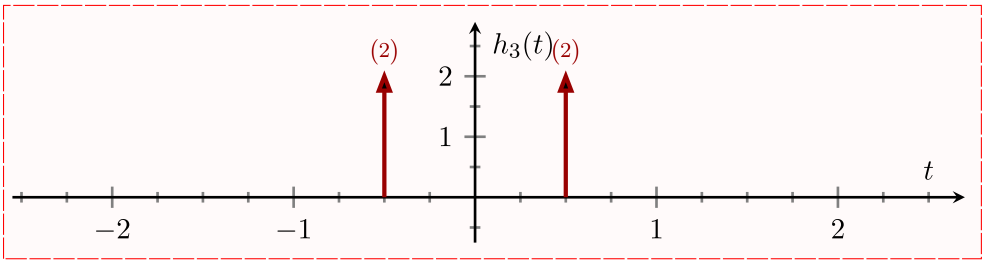


Here $y(t)$ is shown dashed. □

(c) [**5 marks**] It is clear that $h_2(t)$ is a rough approximation to $h(t)$ and $y_2(t)$ is a (less) rough approximation to $y(t)$. Show and argue why

$$h_3(t) = 2\,\delta(t+1/2) + 2\,\delta(t-1/2),$$

which is shown below, is a better approximation to $h(t)$ than $h_2(t)$.

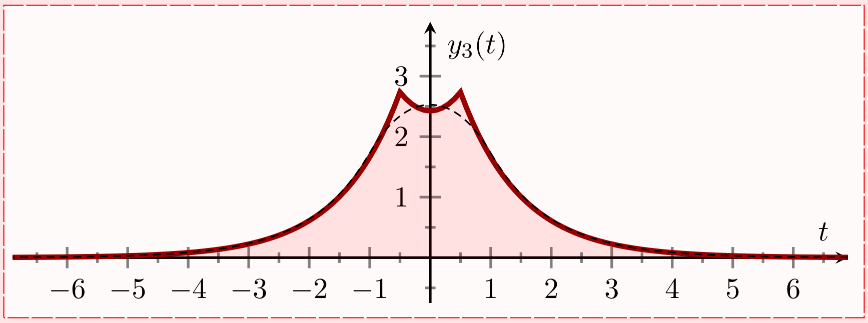


Compute and plot

$$y_3(t) = x(t) \star h_3(t),$$

where $x(t) = e^{-|t|}$ as before. Compare $y_3(t)$ with $y_2(t)$ and $y(t)$.

Solution: In a sense $h_3(t)$ is a better approximation to $h(t)$ than $h_2(t)$ with its mass more uniformly spread over the range where $h(t)$ is non-zero.

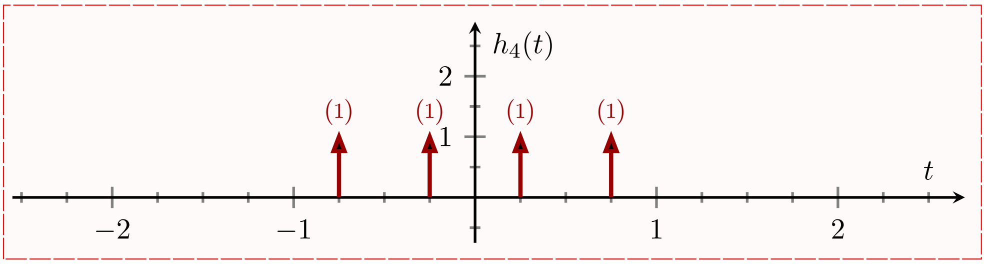


Here $y(t)$ is shown dashed. □

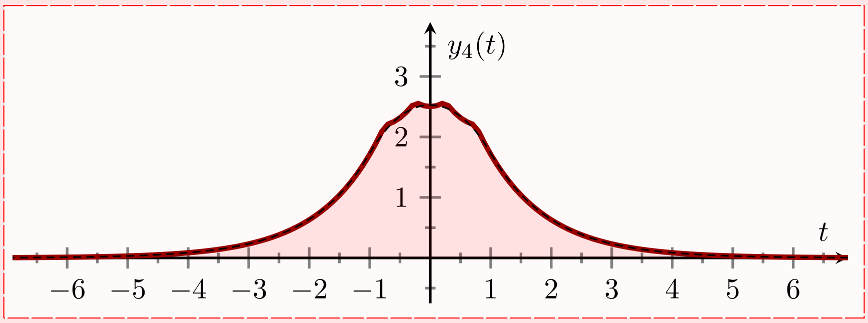
(d) **[5 marks]** Repeat (c) again for $y_4(t) = x(t) \star h_4(t)$ where

$$h_4(t) = \delta(t + 3/4) + \delta(t + 1/4) + \delta(t - 1/4) + \delta(t - 3/4),$$

which is shown below:



Solution: Here $y_4(t)$ is getting very close to $y(t)$.



Here $y(t)$ is shown dashed (barely visible). □

Problem 5 5 marks

Consider an RC circuit shown in Fig. 1, which is an example of a linear time-invariant (LTI) system. The impulse response of this circuit is given by

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

Assume that the capacitor is initially uncharged and the circuit's time constant is $RC = 1$ seconds. Use convolution to determine the voltage across the capacitor, $y(t)$, resulting from an input voltage $x(t) = u(t) - u(t-2)$ volts.

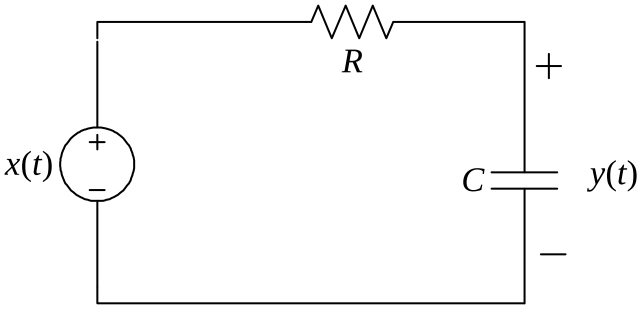
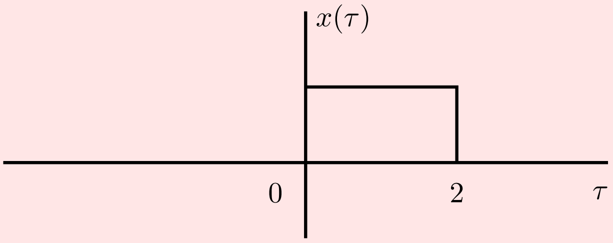
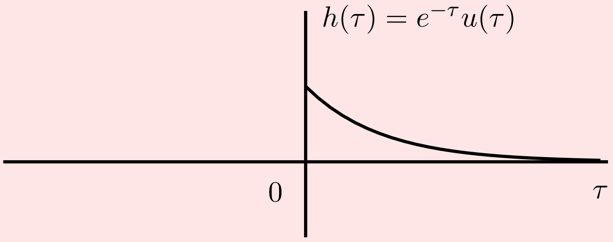
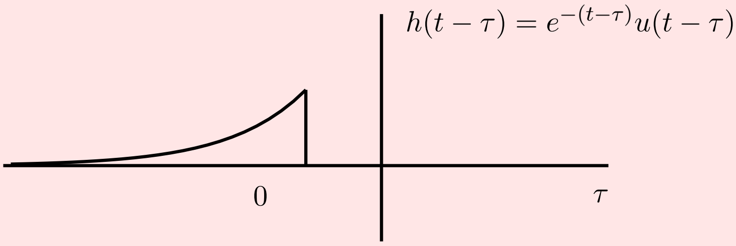


Figure 1: RC circuit which is an LTI system.

Solution: Suppose $\tau = RC = 1$. Then $h(t) = e^{-t}u(t)$. We have $x(t) = u(t) - u(t-2)$.







For $t < 0$, $y(t) = 0$.

For $0 \le t < 2$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^t (1)e^{-(t-\tau)}d\tau \\ &= e^{-t}\int_0^t e^{\tau}d\tau \\ &= e^{-t}|e^{\tau}|_0^t = 1 - e^{-t} \end{aligned}$$

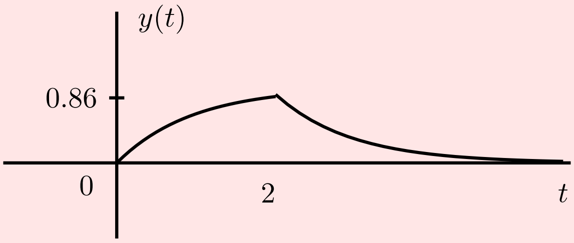
For $t > 2$,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_0^2 e^{-(t-\tau)}d\tau \\
&= e^{-t}\int_0^2 e^{\tau}d\tau \\
&= e^{-t}|e^{\tau}|_0^2 = (e^2-1)e^{-t}
\end{aligned}$$

Thus, overall we have

$$y(t) = \begin{cases} 0 & t < 0 \\ 1-e^{-t} & 0 \le t < 2 \\ (e^2-1)e^{-t} & t \ge 2 \end{cases}$$

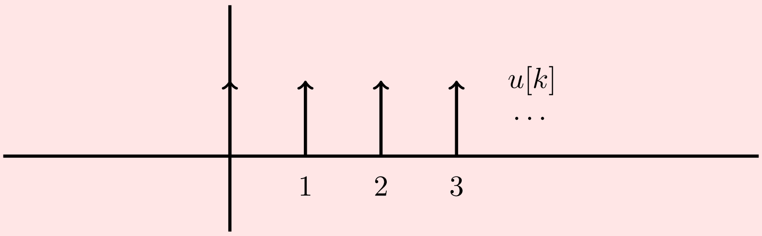
A plot of $y(t)$ is shown in the following figure.

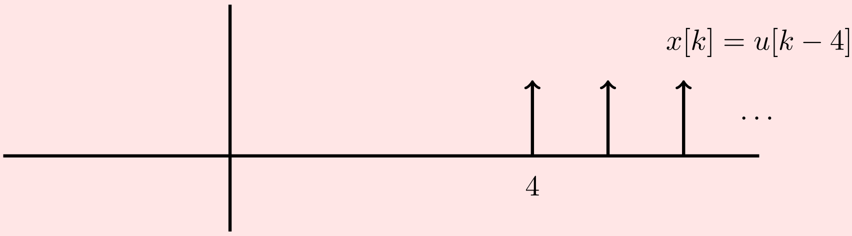


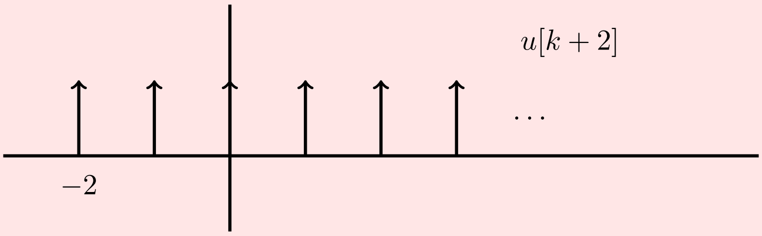
Problem 6 5 marks

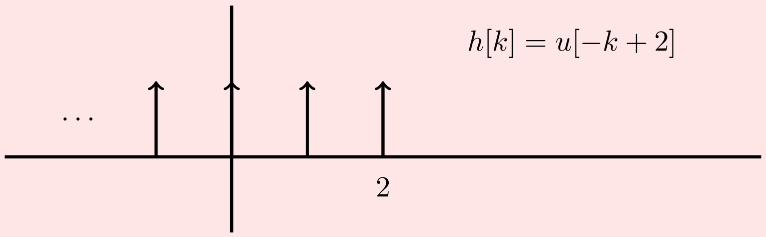
Compute the convolution $y[n] = x[n] * h[n]$ when $x[n] = \left(-\frac{1}{2}\right)^n u[n-4]$ and $h[n] = 4^n\, u[2-n]$.

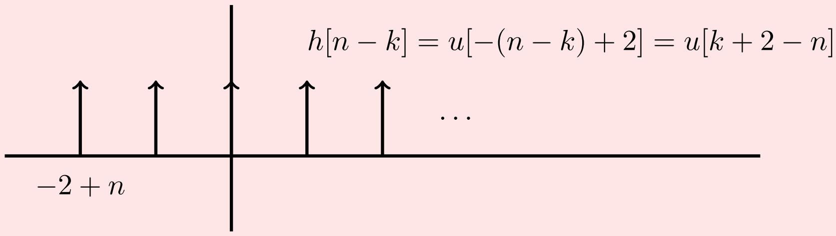
Solution:

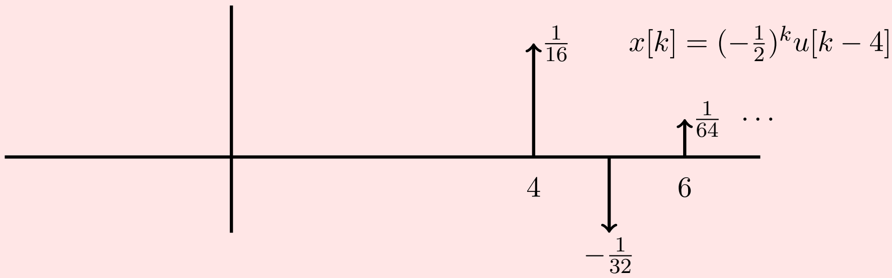


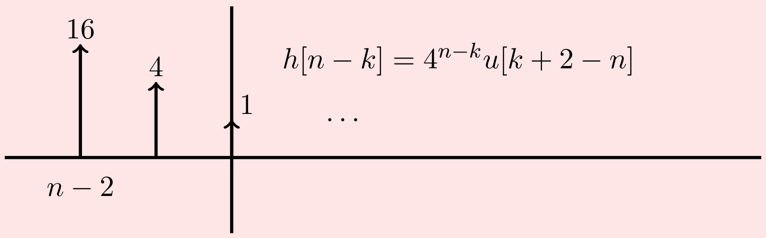












Here, we have two cases.
For $n-2 \leq 4 \Rightarrow n \leq 6$:

$$\begin{aligned} y[n] &= \sum_{k=4}^{\infty} 4^{n-k}(-\frac{1}{2})^k \\ &= \frac{1}{9}2^{-9+2n} = \frac{1}{4608}4^n \end{aligned}$$

For $n-2 > 4 \Rightarrow n > 6$:

$$\begin{aligned} y[n] &= \sum_{k=n-2}^{\infty} 4^{n-k}(-\frac{1}{2})^k \\ &= \frac{1}{9}(-1)^n 2^{9-n} = \frac{2^9}{9}(-1)^n(2)^{-n} \\ &= \frac{2^9}{9}(-1)^n(\frac{1}{2})^n = \frac{2^9}{9}(-\frac{1}{2})^n \end{aligned}$$

Thus, overall we have

$$y[n] = \begin{cases} \frac{1}{4608}4^n & n \leq 6 \\ \frac{2^9}{9}(-\frac{1}{2})^n & n > 6 \end{cases}$$

Problem 7

Let $h_1[n]$, $h_2[n]$ and $h_3[n]$ be the unit pulse responses of three systems S1, S2 and S3, respectively.

(a) **[5 marks]** Relate $y[n]$ and $x[n]$ in terms of h_1, h_2 and h_3 for the system configuration in Fig. 2.

Solution: Since convolution is a linear operation, the output $y[n]$ and the input $x[n]$ are related by

$$y[n] = x[n] \star (h_2[n] \star (h_1[n] - h_3[n]) - h_3[n])$$

□

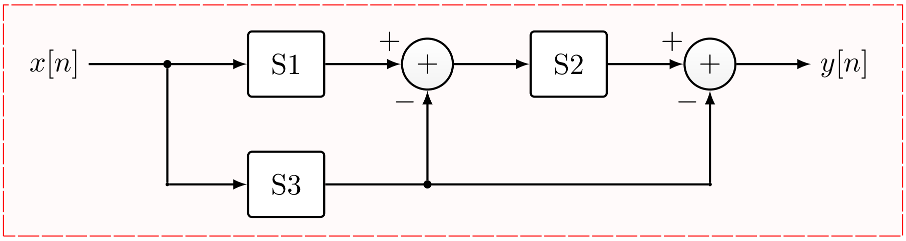


Figure 2: Interconnection of systems.

(b) **[8 marks]** Given

$$h_1[n] = \delta[n-1], \quad h_2[n] = n\,\alpha^n\,u[n], \quad \text{and} \quad h_3[n] = \beta^n\,u[n-2],$$

where $|\alpha| < 1$ and $|\beta| < 1$, determine $h[n]$ such that $y[n] = x[n] \star h[n]$.

Solution: From part (a), we see that $h[n]$ is given by

$$\begin{aligned} h[n] &= h_2[n] \star (h_1[n] - h_3[n]) - h_3[n] \\ &= h_2[n] \star h_1[n] - h_2[n] \star h_3[n] - h_3[n]. \end{aligned} \tag{1}$$

We first compute

$$\begin{aligned} h_2[n] \star h_1[n] &= \sum_{k=-\infty}^{\infty} h_1[n-k] h_2[k] = \sum_{k=-\infty}^{\infty} \delta[n-k-1]\, k\alpha^k\, u[k] \\ &= (n-1)\,\alpha^{n-1}\,u[n-1], \end{aligned} \tag{2}$$

where the summation is simplified using the fact that $\delta[n-k-1] = 0$ for $n-k-1 \neq 0$.

Now we compute

$$\begin{aligned} h_2[n] \star h_3[n] &= \sum_{k=-\infty}^{\infty} h_3[n-k] h_2[k] = \sum_{k=-\infty}^{\infty} \beta^{n-k}\, u[n-k-2]\, k\alpha^k\, u[k] \\ &= \sum_{k=0}^{n-2} k\,\alpha^k\,\beta^{n-k} = \beta^n \sum_{k=0}^{n-2} k\left(\frac{\alpha}{\beta}\right)^k, \end{aligned} \tag{3}$$

where the limits on the summation are simplified using the definition of the Heaviside (unit step) function: $u[n] = 0$ for $n < 0$.

Using (2) and (3), we write (1) as

$$h[n] = (n-1)\,\alpha^{n-1}\,u[n-1] - \beta^n \sum_{k=0}^{n-2} k\left(\frac{\alpha}{\beta}\right)^k - \beta^n\,u[n-2]. \tag{4}$$

Note that the summation on right hand side can be simplified using the following series expansion

$$\sum_{k=0}^{n} k\,x^k = \frac{x^{n+1}\big(n(x-1)-1\big)+x}{(1-x)^2},$$

which converges to $x/(1-x)^2$ in the limit $n \to \infty$ for $|x| < 1$. □

(c) **[3 marks]** Is the overall system causal? Why?

Solution: The impulse response $h[n]$ of an overall system is given in (4). Since $h[n] = 0$ for $n < 0$, the overall system is causal. $\square$

Another reason is that overall system consists of series connection of two causal systems and parallel connection of one causal system, therefore the overall system has to be causal.

— End of Assignment —

Question	Points	Score
1	4	
2	12	
3	9	
4	24	
5	5	
6	5	
7	16	
Total:	75	