Fourier Transforms – Digression

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

	Signal	Fourier transform	Fourier series coefficients (if periodic)
1	$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
2	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
3	$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
4	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
5	$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\ a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
6	Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
7	$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
8	$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
9	$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
10	$\delta(t)$	1	—
11	$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
12	$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
13	$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
14	$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
15	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier Transforms – Digression

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

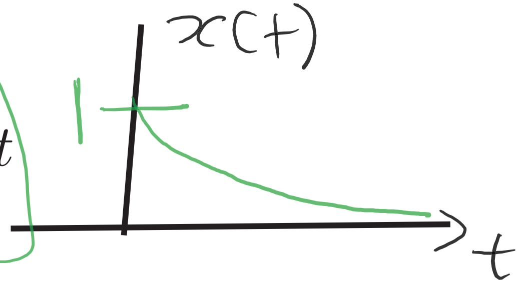
Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Transform Pair 13: Let $x(t) = e^{-at}\, u(t)$, for $a > 0$ and $u(t)$ is the unit step, then

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t(a+j\omega)}\, dt = \frac{-1}{a+j\omega}\left[e^{-t(a+j\omega)}\right]_0^{\infty}$$

$$= \frac{-1}{a+j\omega}\left(e^{-\infty} - e^{0}\right) = \frac{1}{a+j\omega}$$

Fourier Transforms – Examples

Recap:

$$\underbrace{e^{-at}u(t)}_{x(t)} \overset{\mathscr{F}}{\longleftrightarrow} \underbrace{\frac{1}{a+j\omega}}_{X(j\omega)}$$

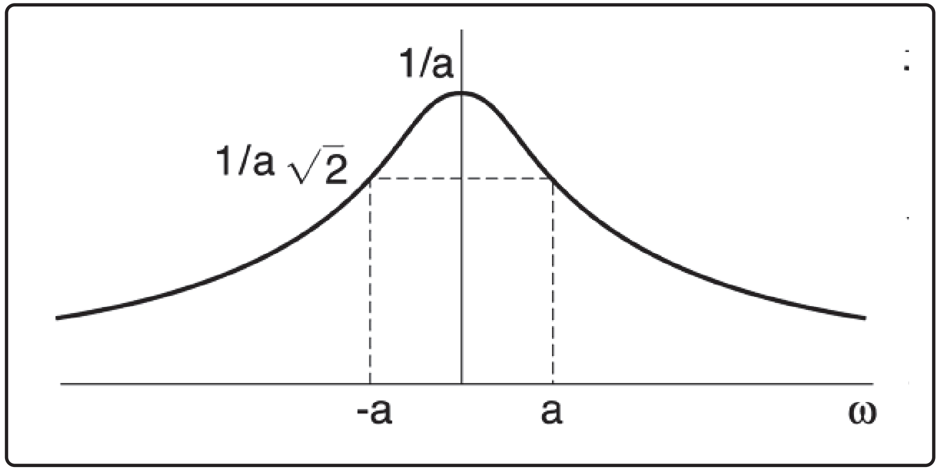
$$\mathscr{F}\left\{e^{-at}u(t)\right\} = \frac{1}{a+j\omega}$$

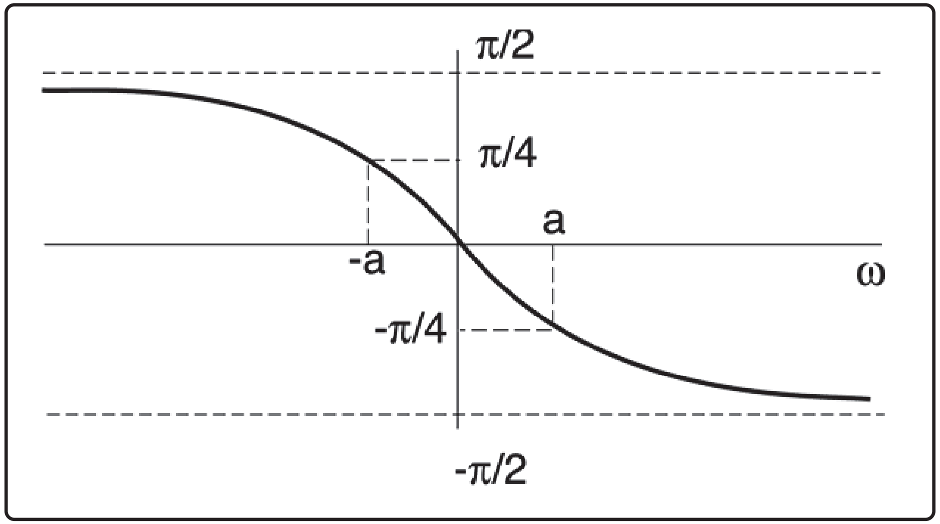
$$\mathscr{F}^{-1}\left\{\frac{1}{a+j\omega}\right\} = e^{-at}u(t)$$

- Frequency domain function $\frac{1}{a+j\omega}$ is the *Fourier Transform* of $e^{-at}u(t)$
- Time domain function $e^{-at}u(t)$ is the *Inverse Fourier Transform* of $\frac{1}{a+j\omega}$

Magnitude and phase are:

$$\left|X(j\omega)\right| = (a^2 + \omega^2)^{-1/2}$$
$$\angle X(j\omega) = -\tan^{-1}(\omega/a)$$





$x(t)$	$X(j\omega)$
$e^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$

don't have to proove

Gaussian: Let $x(t) = e^{-at^2}$, for $a > 0$, then

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a[t^2 + j\omega t/a + (j\omega/2a)^2] + a(j\omega/2a)^2}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a(t + j\omega/2a)^2}\, dt \cdot e^{-\omega^2/4a} \\
&\boxed{= \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}}
\end{aligned}
$$

So the Fourier Transform of a Gaussian is a compressed (or expanded) and scaled Gaussian. Weird but important in applications and the "uncertainty principle".

Fourier Transforms – Examples

Recap:

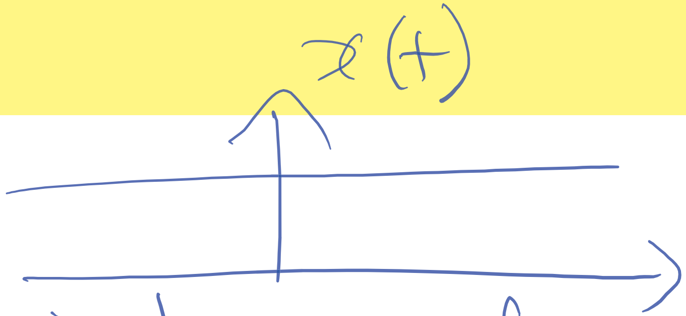
$$e^{-at^2} \overset{\mathscr{F}}{\longleftrightarrow} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}\left\{e^{-at^2}\right\} = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\right\} = e^{-at^2}$$

- Frequency domain function $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$ is the *Fourier Transform* of e^{-at^2}
- Time domain function e^{-at^2} is the *Inverse Fourier Transform* of $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$

Fourier Transforms – Examples



Transform Pair 5: Let $x(t) = 1$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j\omega t} dt$$

$$= \frac{-1}{j\omega} \left[e^{-j\omega t} \right]_{-\infty}^{\infty} = \frac{-1}{j\omega} \left(e^{-\infty} - e^{\infty} \right)$$

This integral is undefined

Transform Pair 5: Start instead with $X(\cancel{e}^{j\omega}) = 2\pi\delta(\omega)$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(w)\, e^{jwt}\, dw$$

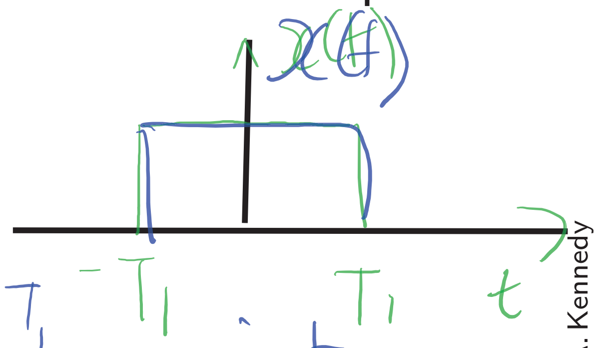
$$= e^{j0t} = 1$$

sifting property $\delta(w)$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Transform Pair 8: The Fourier Transform of a rectangular "brickwall" pulse of width $2T_1$, that is,

$$x(t) = \chi_{[-T_1,T_1]}(t) \triangleq \begin{cases} 1 & |t| \le T_1 \\ 0 & \text{otherwise} \end{cases}$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T_1}^{T_1} = \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right)$$

$$= \frac{2}{2j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) = \frac{2\sin(\omega T_1)}{\omega}$$

Fourier Transforms – Examples

Recap:

$$\chi_{[-T_1,T_1]}(t) \xleftrightarrow{\mathscr{F}} \frac{2\sin(\omega T_1)}{\omega}$$

$$\mathscr{F}\left\{\chi_{[-T_1,T_1]}(t)\right\} = \frac{2\sin(\omega T_1)}{\omega}$$

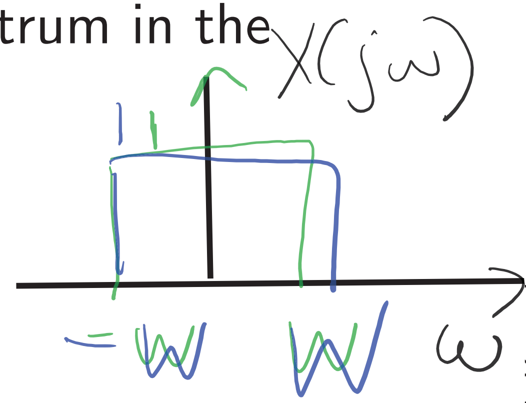
$$\mathscr{F}^{-1}\left\{\frac{2\sin(\omega T_1)}{\omega}\right\} = \chi_{[-T_1,T_1]}(t)$$

- Frequency domain function $\frac{2\sin(\omega T_1)}{\omega}$ is the *Fourier Transform* of $\chi_{[-T_1,T_1]}(t)$
- Time domain function $\chi_{[-T_1,T_1]}(t)$ is the *Inverse Fourier Transform* of $\frac{2\sin(\omega T_1)}{\omega}$

Fourier Transforms – Examples

Transform Pair 9: Consider a low pass (rectangular) spectrum in the frequency domain:

$$X(j\omega) = \chi_{[-W,W]}(\omega) \triangleq \begin{cases} 1 & |\omega| \le W \\ 0 & \text{otherwise} \end{cases}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi j t}\left[e^{j\omega t}\right]_{-W}^{W}$$

$$= \frac{1}{2\pi j t}\left(e^{jWt} - e^{-jWt}\right)$$

$$= \frac{1}{\pi t}\sin(Wt)$$

Fourier Transforms – Examples

$x(t)$ $X(j\omega)$

Recap:

$$x(t) = \frac{\sin(Wt)}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-W,W]}(\omega) = X(j\omega)$$

$$\mathscr{F}\left\{\frac{\sin(Wt)}{\pi t}\right\} = \chi_{[-W,W]}(\omega)$$

$$\mathscr{F}^{-1}\left\{\chi_{[-W,W]}(\omega)\right\} = \frac{\sin(Wt)}{\pi t}$$

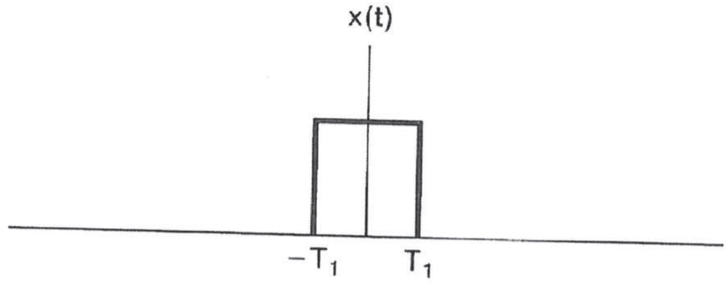
- Frequency domain function $\chi_{[-W,W]}(\omega)$ is the *Fourier Transform* of $\frac{\sin(Wt)}{\pi t}$
- Time domain function $\frac{\sin(Wt)}{\pi t}$ is the *Inverse Fourier Transform* of $\chi_{[-W,W]}(\omega)$

Fourier Transforms – Examples

A signal which is concentrated in one domain is spread out in the other domain.

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \longleftrightarrow \frac{2\sin(\omega T_1)}{\omega}$$

$$\frac{1}{\pi t}\sin Wt \longleftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$



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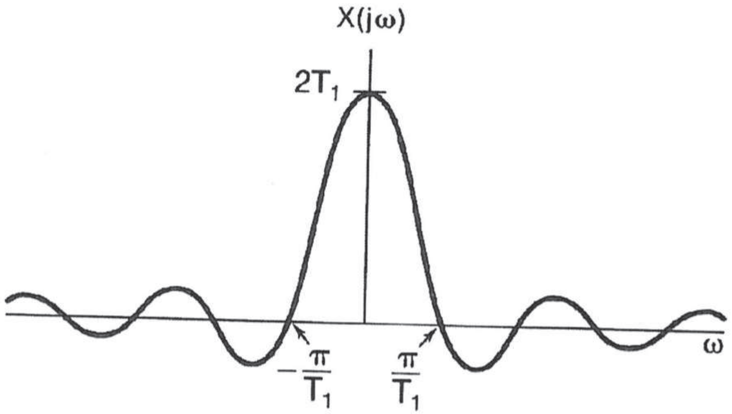

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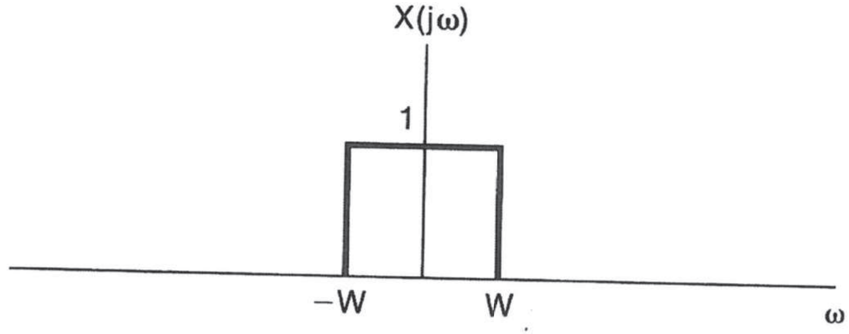

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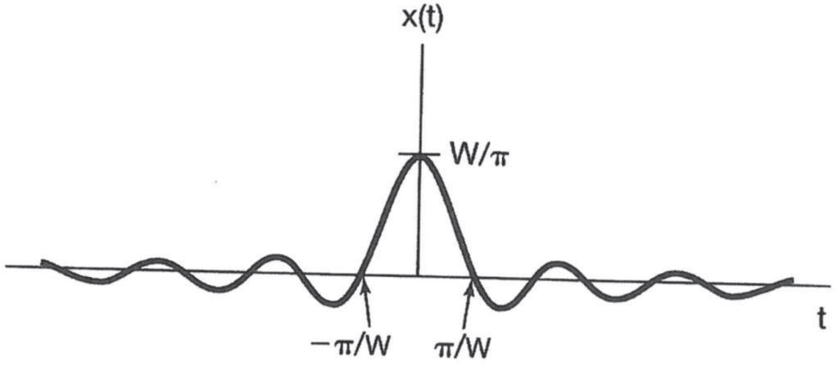


(b)

Part 3 Outline

5 CT Non-Periodic Signals
- Up to this Point

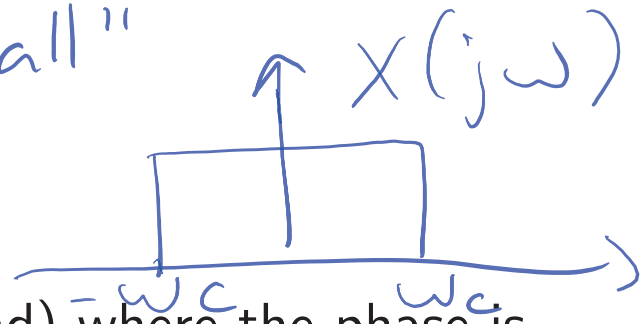
6 **Fourier Transforms**
- Fourier Analysis and Synthesis
- Examples
- Ideal Low Pass Filter

Ideal LPF: Passes only frequencies between $[-\omega_c, +\omega_c]$, where ω_c is the cut-off frequency.

The frequency domain specification is

notation for "brickwall"

$$\boxed{H(j\omega) = \chi_{[-\omega_c,+\omega_c]}(\omega)}$$



(RHS is the characteristic function, just some shorthand) where the phase is zero for all ω (our choice here). Then

$$h(t) = \mathscr{F}^{-1}\{H(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-\omega_c,+\omega_c]}(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega$$

$$\boxed{= \frac{\sin \omega_c t}{\pi t}}$$

$$= \frac{\omega_c}{\pi}\,\mathrm{sinc}\!\left(\frac{\omega_c t}{\pi}\right)$$

where the "sinc function" is defined as

$$\boxed{\mathrm{sinc}(\theta) \triangleq \frac{\sin \pi\theta}{\pi\theta}}$$

θ = independent variable

Definition (Ideal Low Pass Filter)

The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin \omega_c t}{\pi t} \stackrel{\mathscr{F}}{\longleftrightarrow} \chi_{[-\omega_c, +\omega_c]}(\omega)}$$

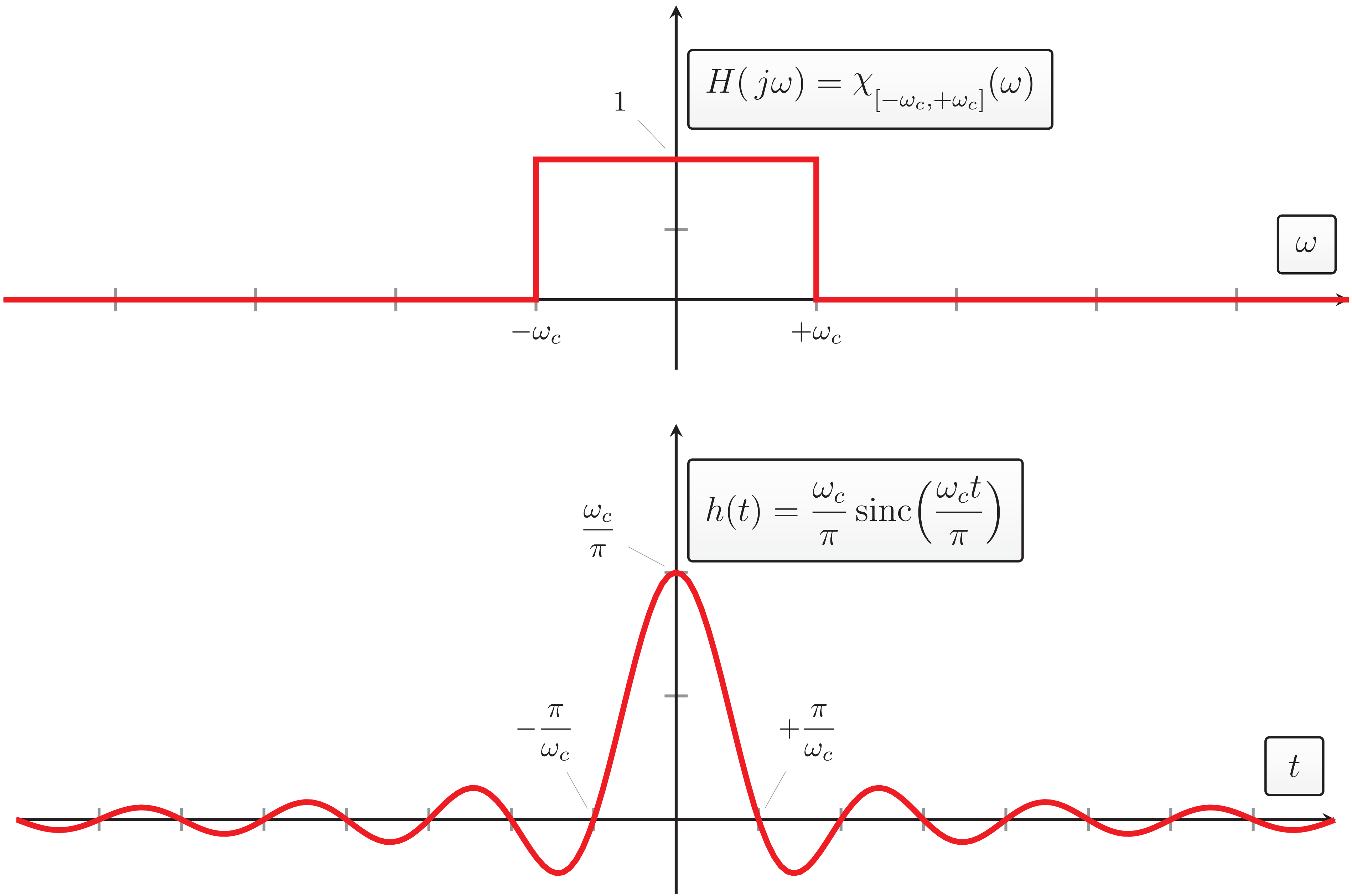
or

Definition (Ideal Low Pass Filter)

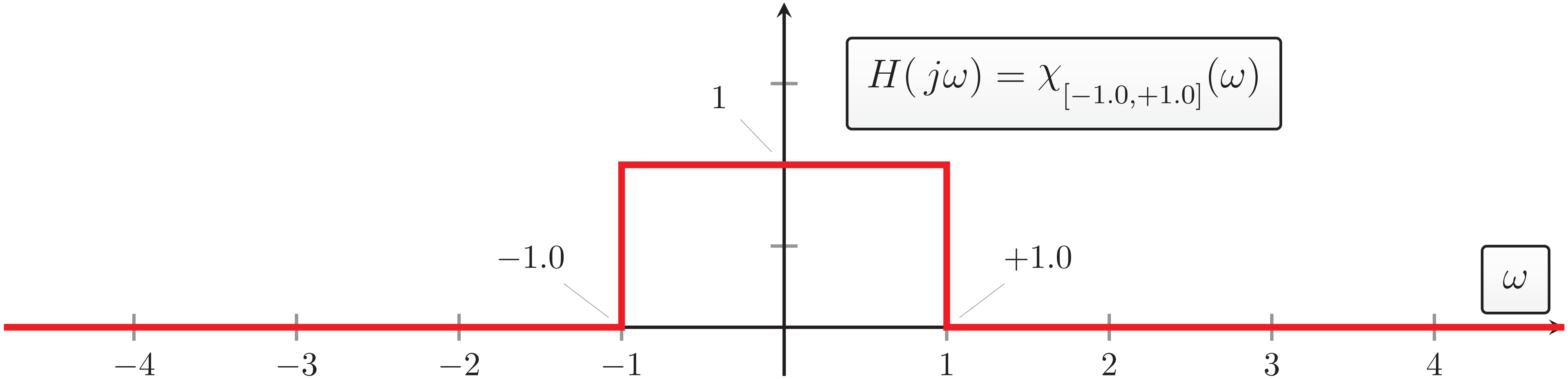
The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

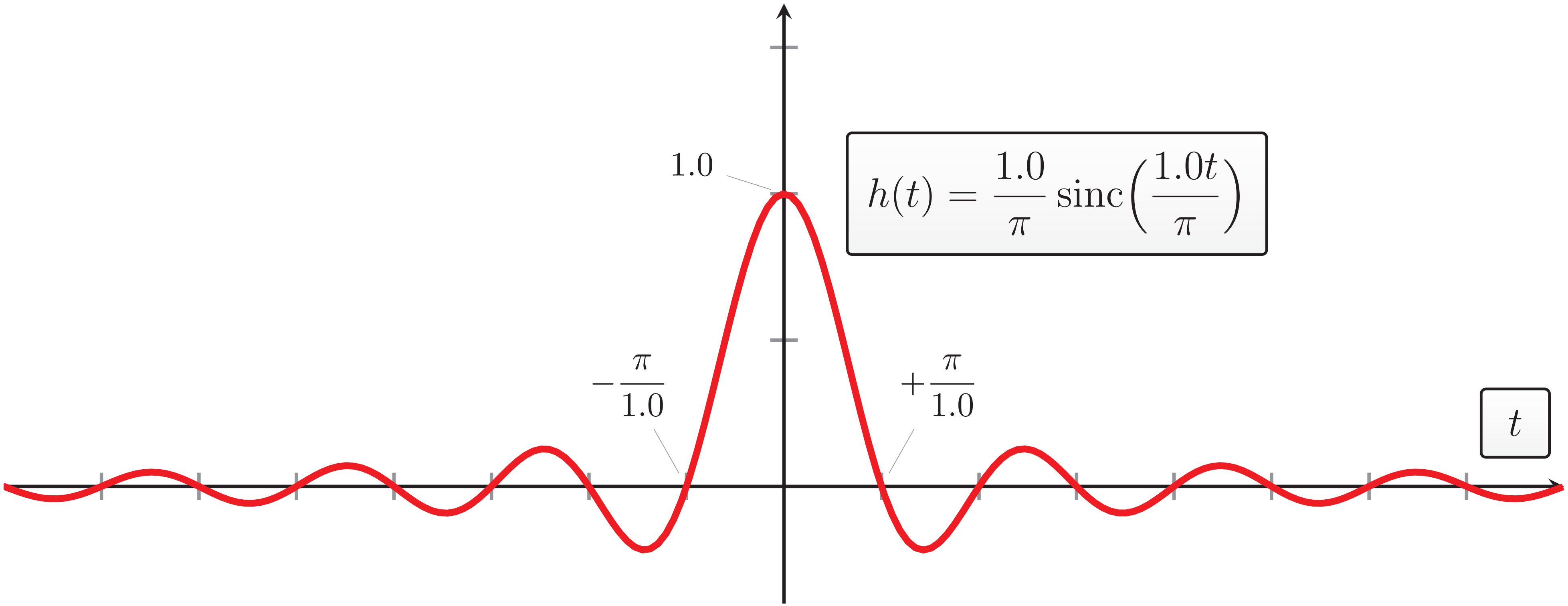
$$\boxed{\frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \stackrel{\mathscr{F}}{\longleftrightarrow} \chi_{[-\omega_c, +\omega_c]}(\omega)}$$

Fourier Transforms – Ideal Low Pass Filter



Fourier Transforms – Ideal Low Pass Filter





Cutoff Variations: The following frames show the effects of varying the cutoff frequency, increasing the bandwidth of an ideal LPF which contracts the sinc function in time. This can be explained in terms of

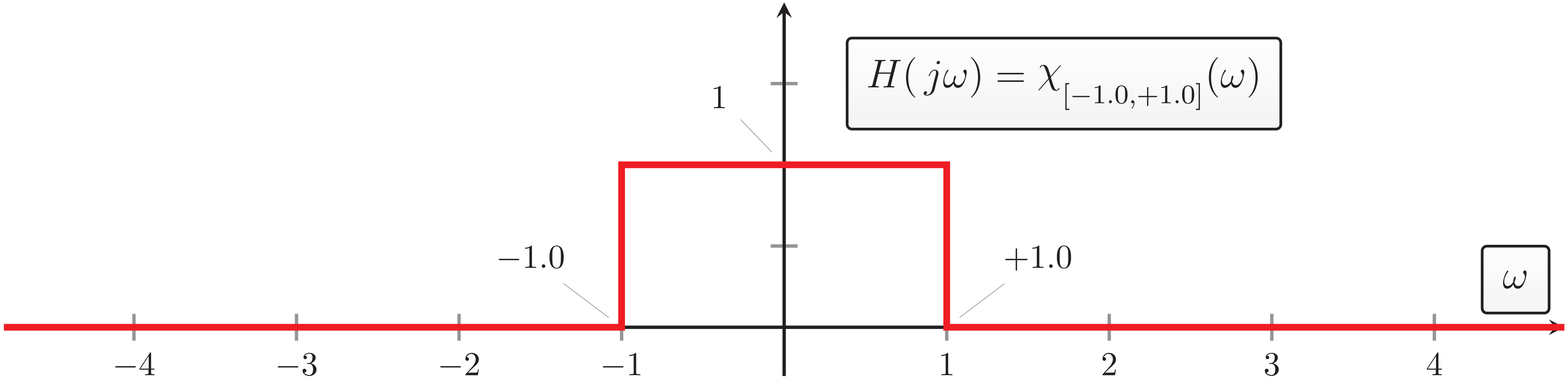
$$x(a\,t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}\,X\Big(\frac{j\omega}{a}\Big)$$

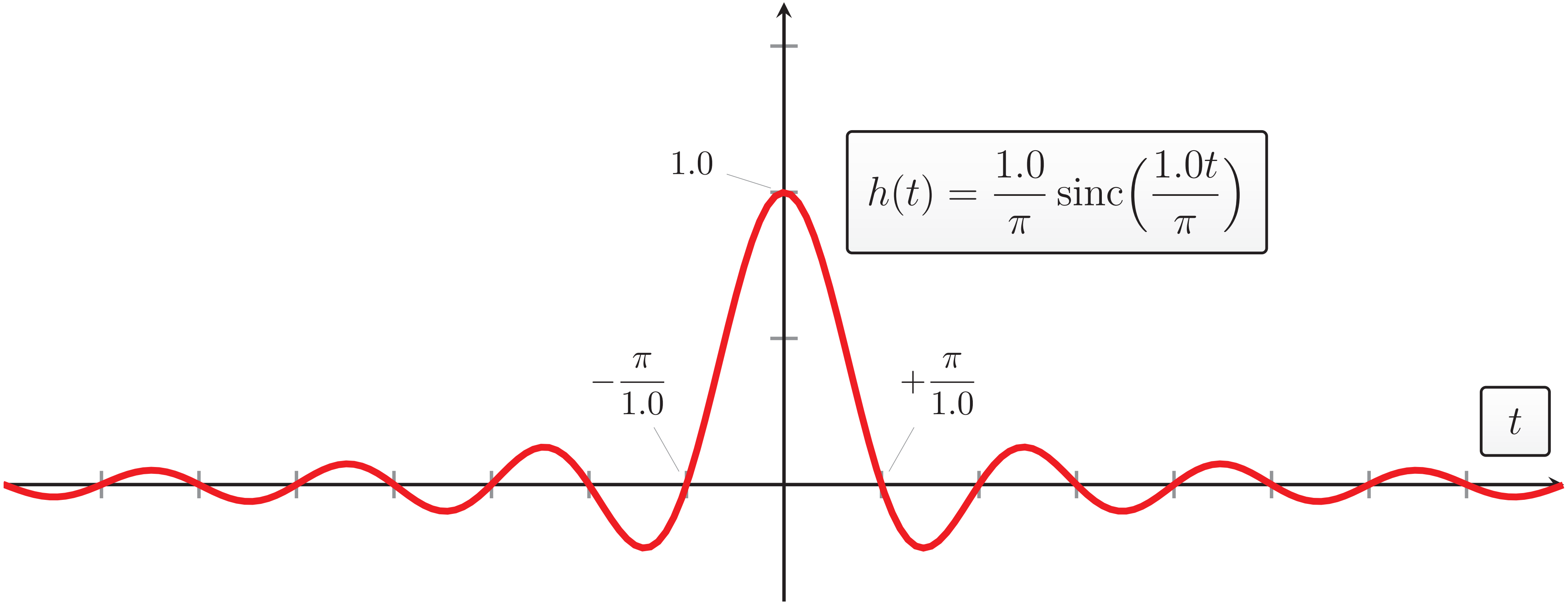
The plots show a sequence with $a = 1.0, 1.1, 1.2, 1.3, 1.4$ and 1.5

Followed by $a = 1.0, 0.9, 0.5$ and 0.25

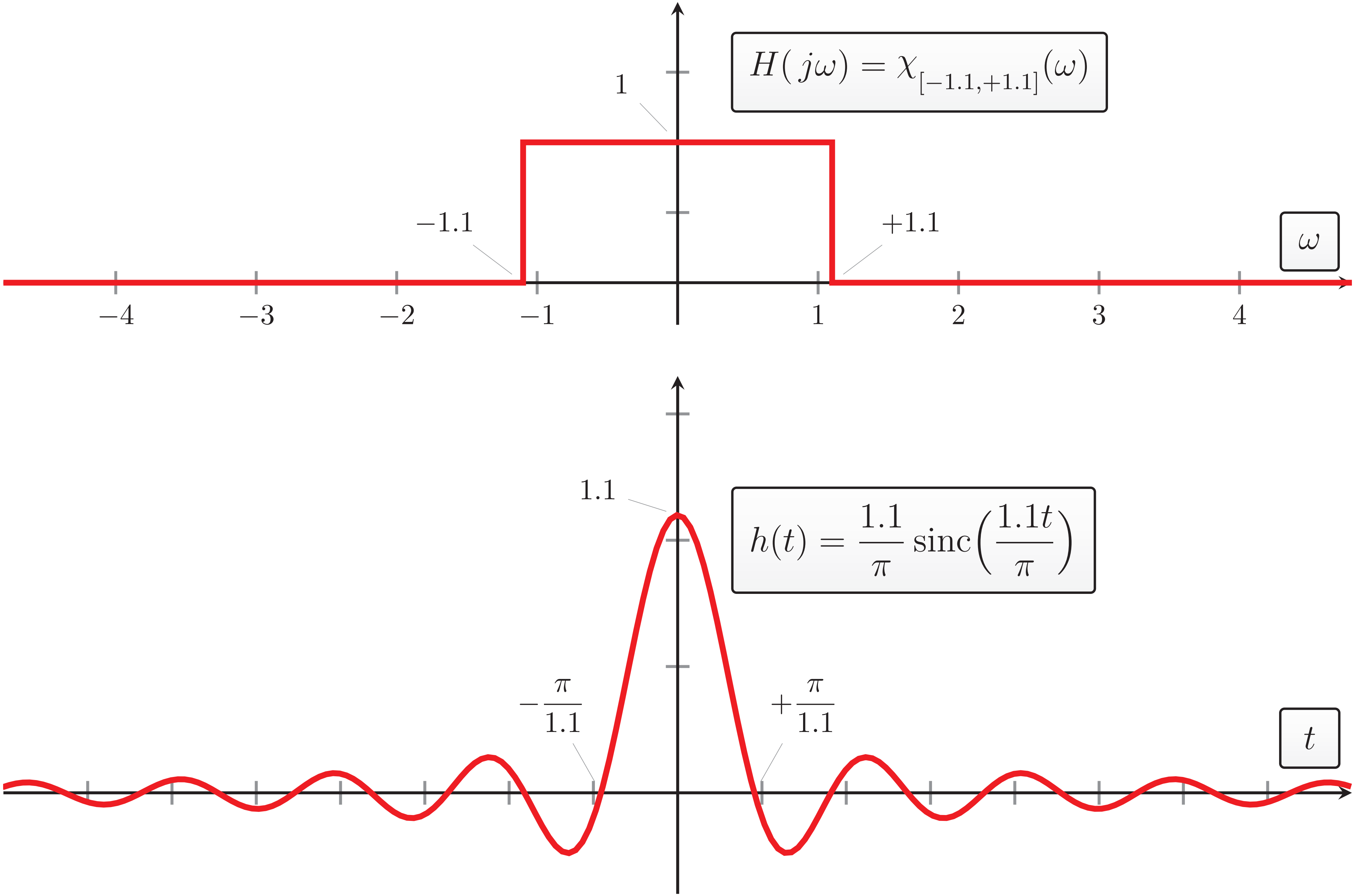
$a > 1$ = compressed
$a < 1$ = expanded in the time domain

Fourier Transforms – Ideal Low Pass Filter



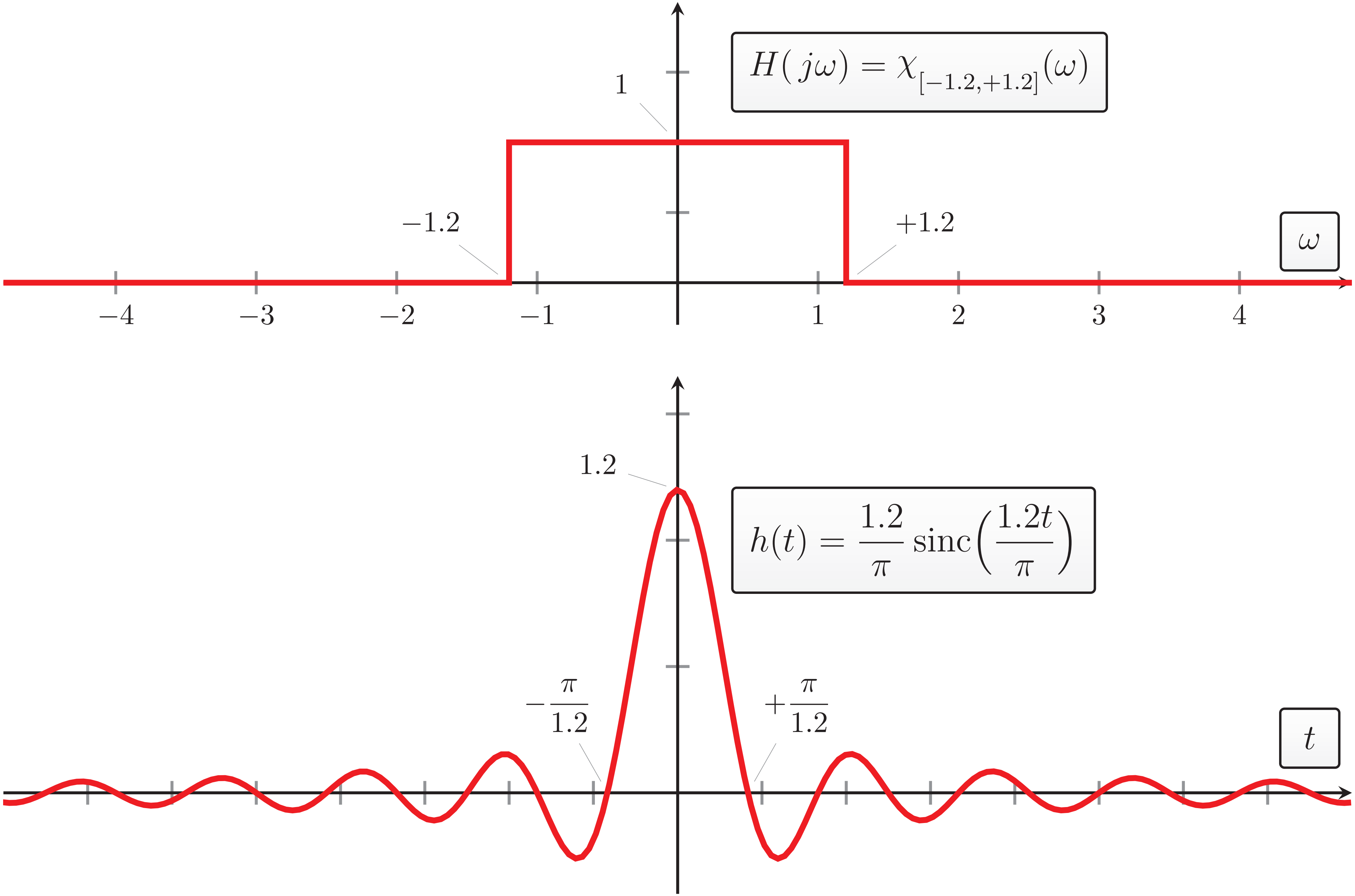


Fourier Transforms – Ideal Low Pass Filter

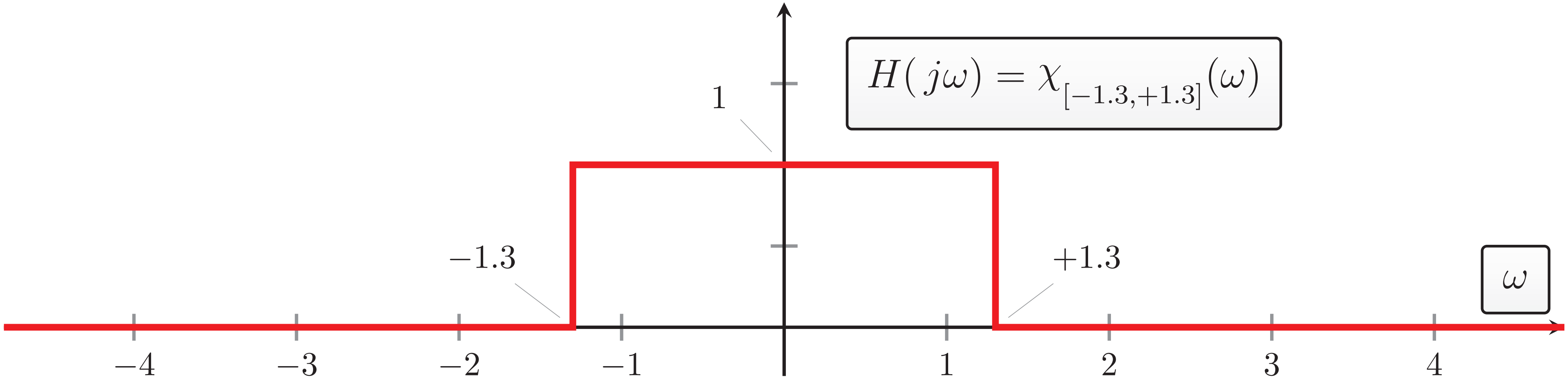


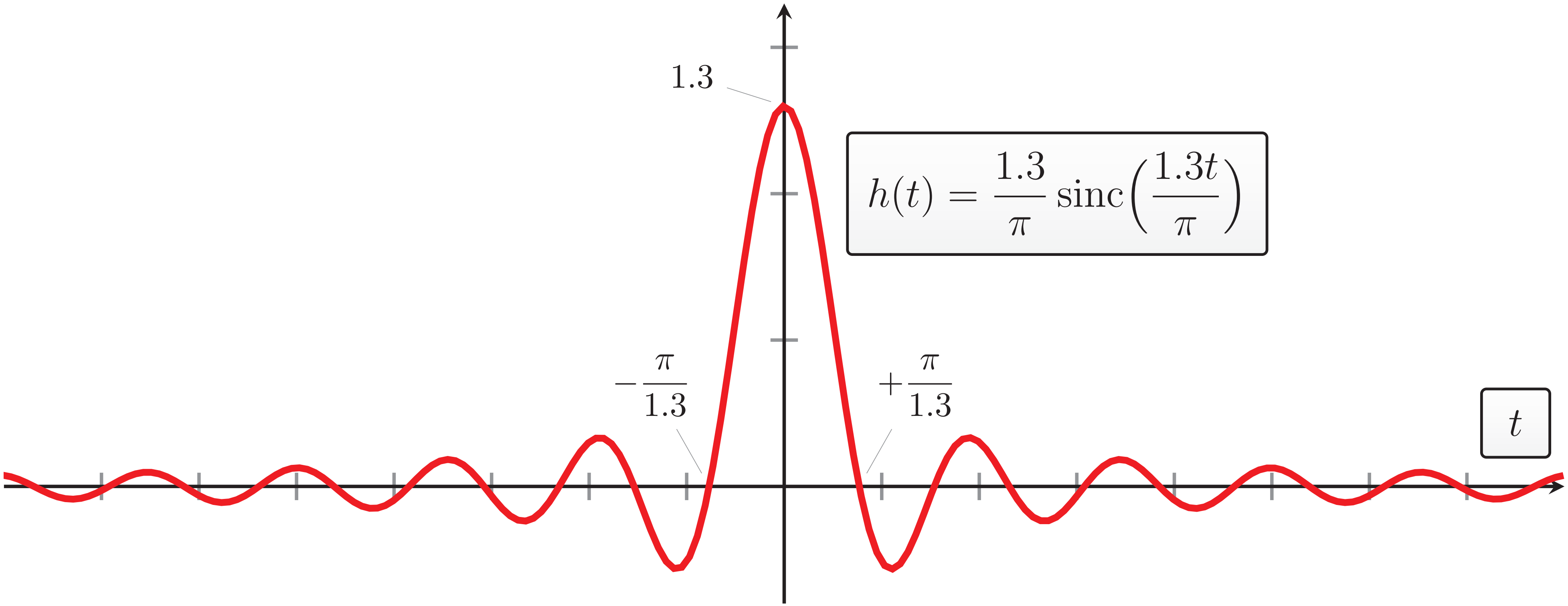
Part 3 Slide 148/many Convenor: R. A. Kennedy

Fourier Transforms – Ideal Low Pass Filter

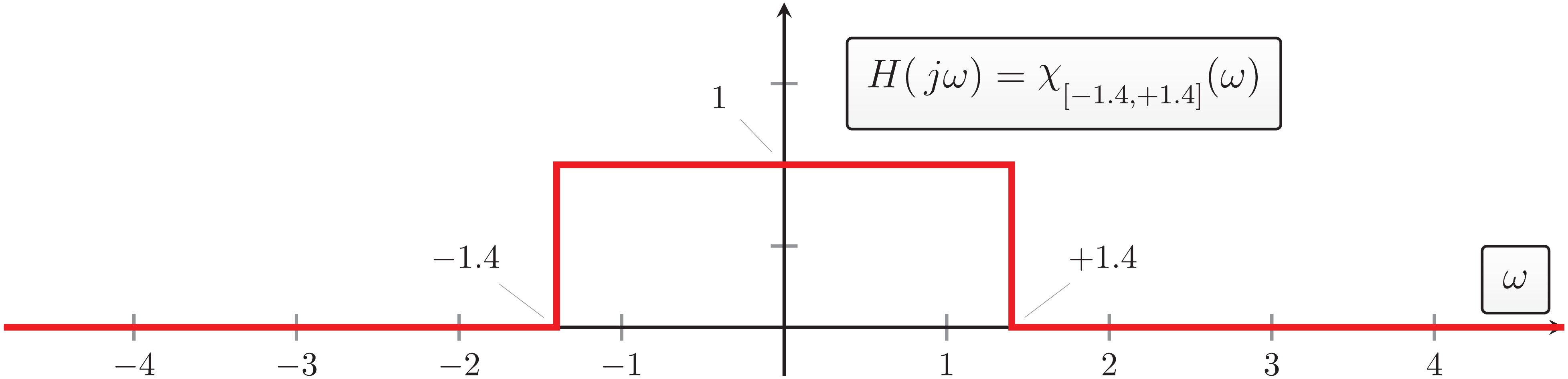


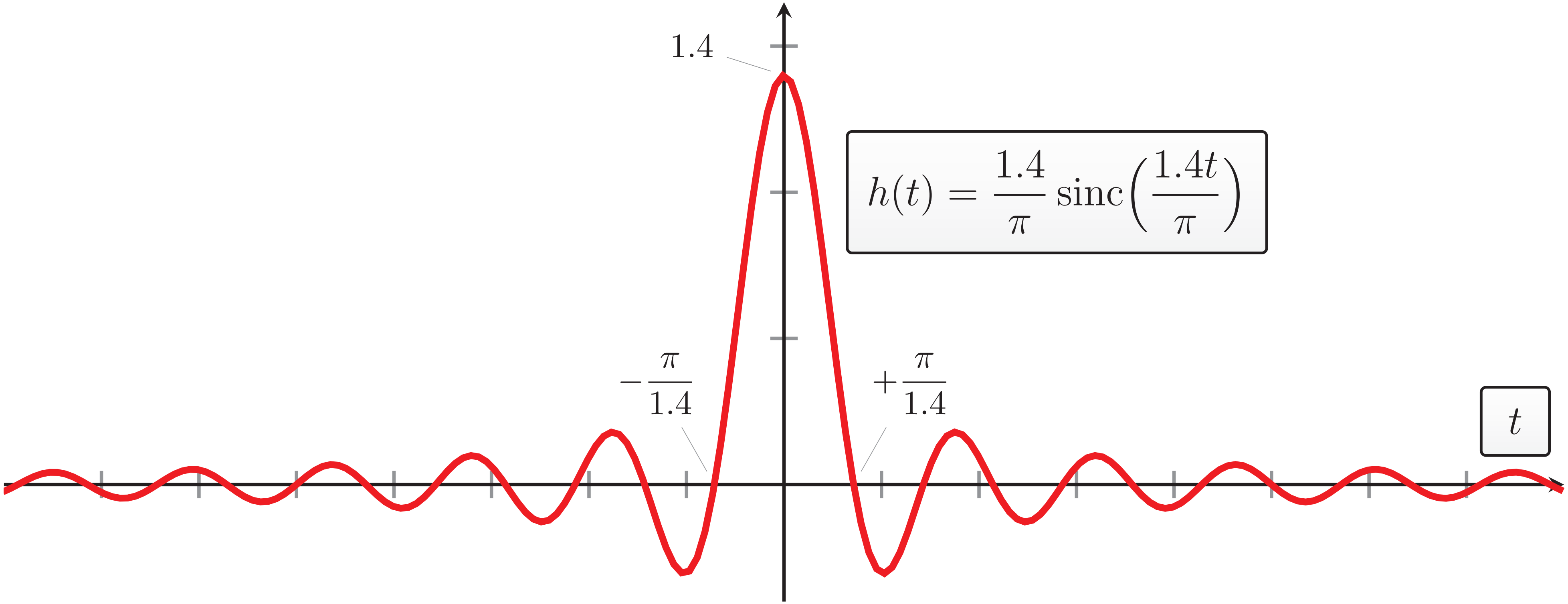
Fourier Transforms – Ideal Low Pass Filter



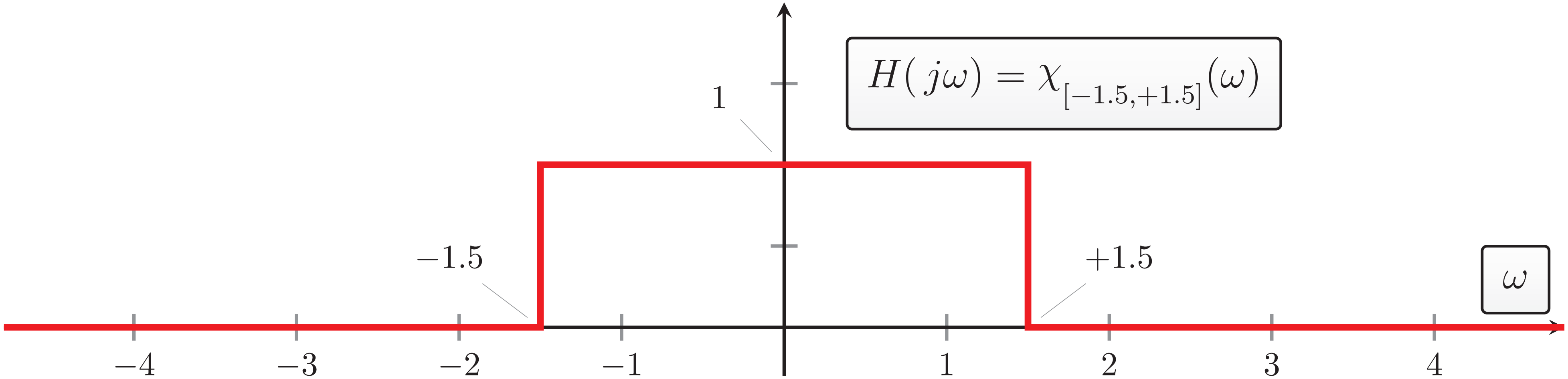


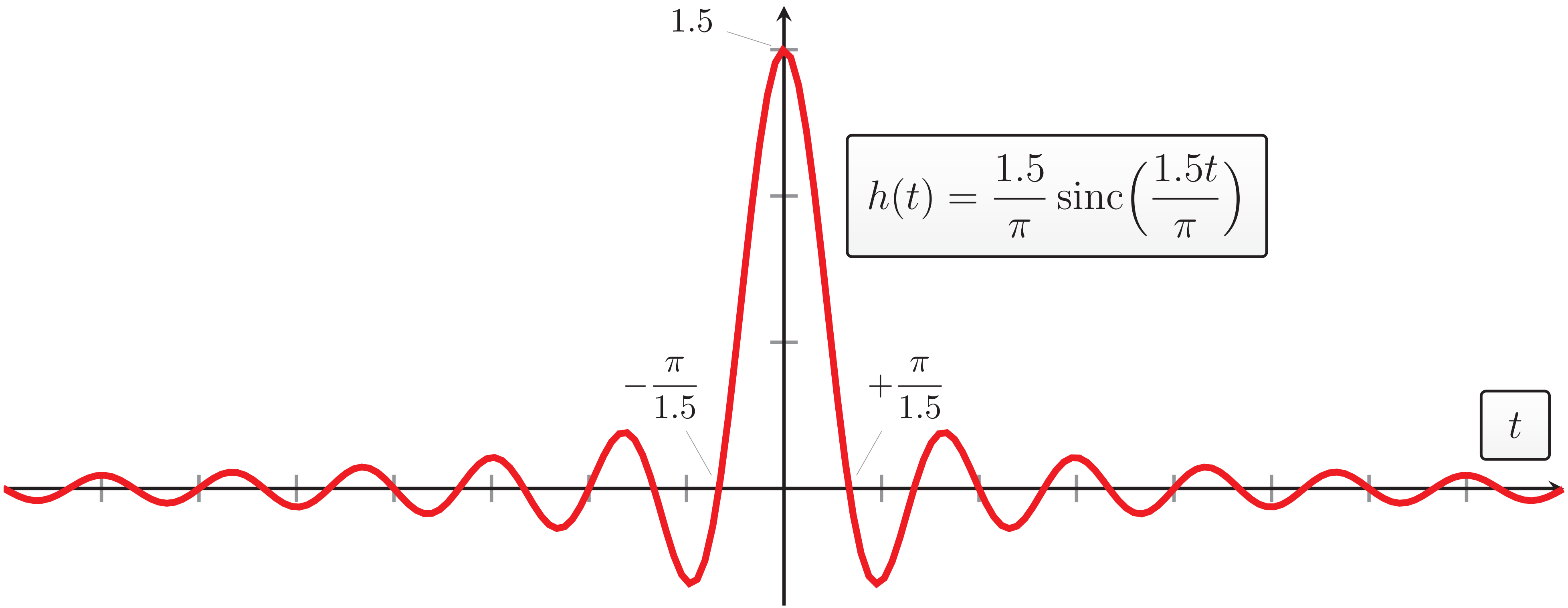
Fourier Transforms – Ideal Low Pass Filter



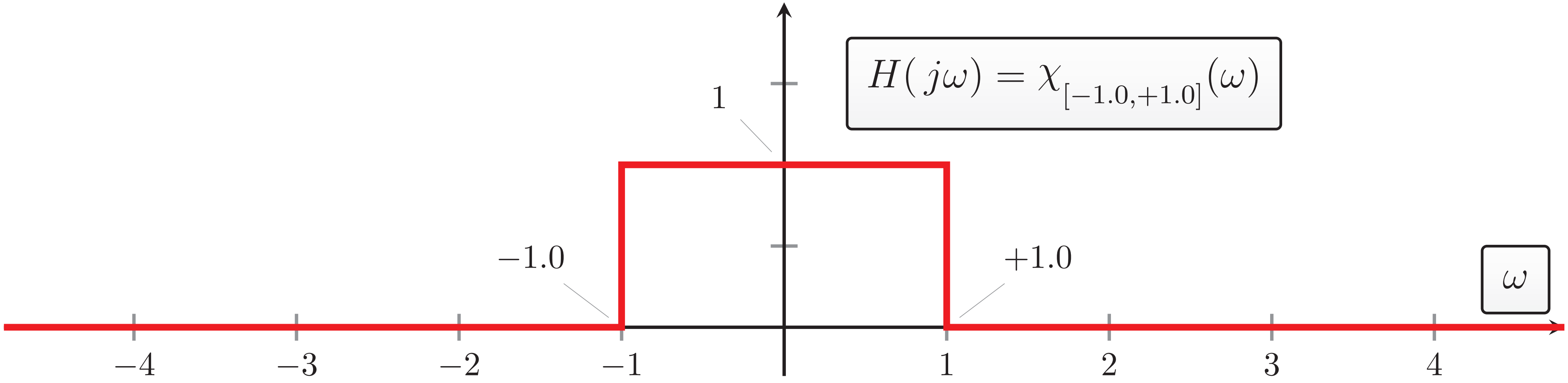


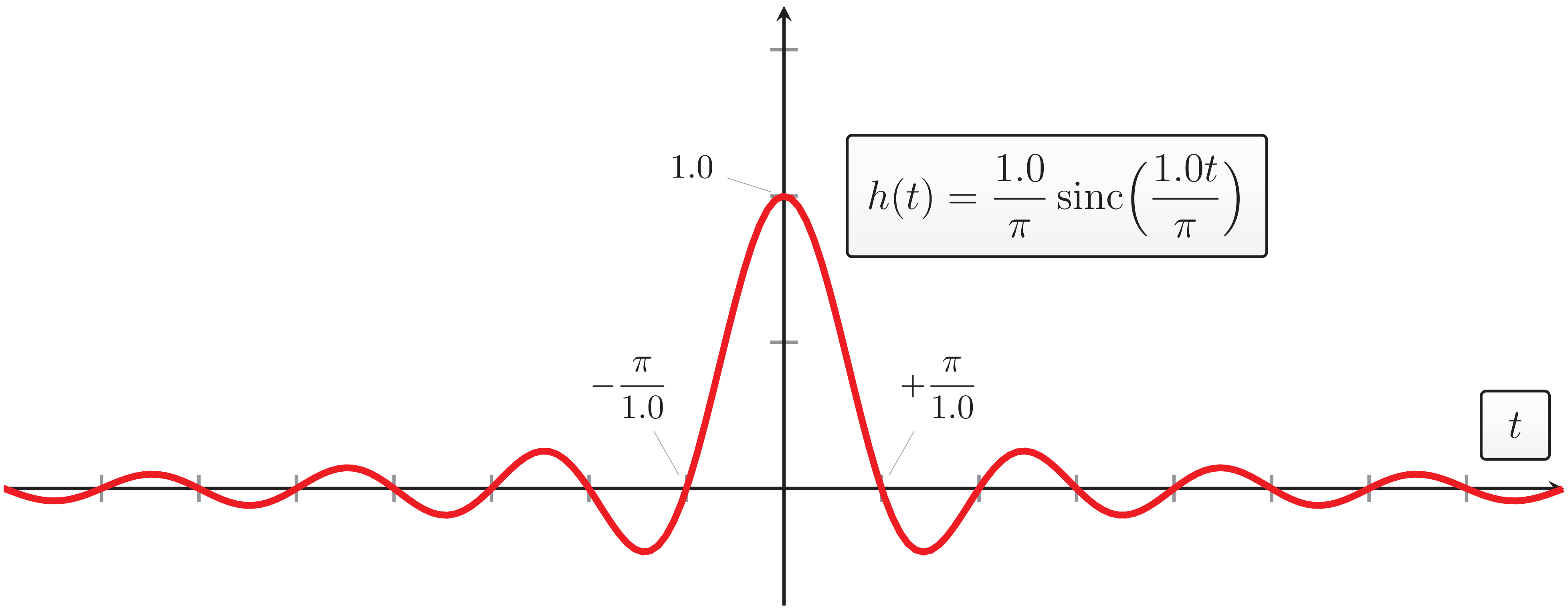
Fourier Transforms – Ideal Low Pass Filter



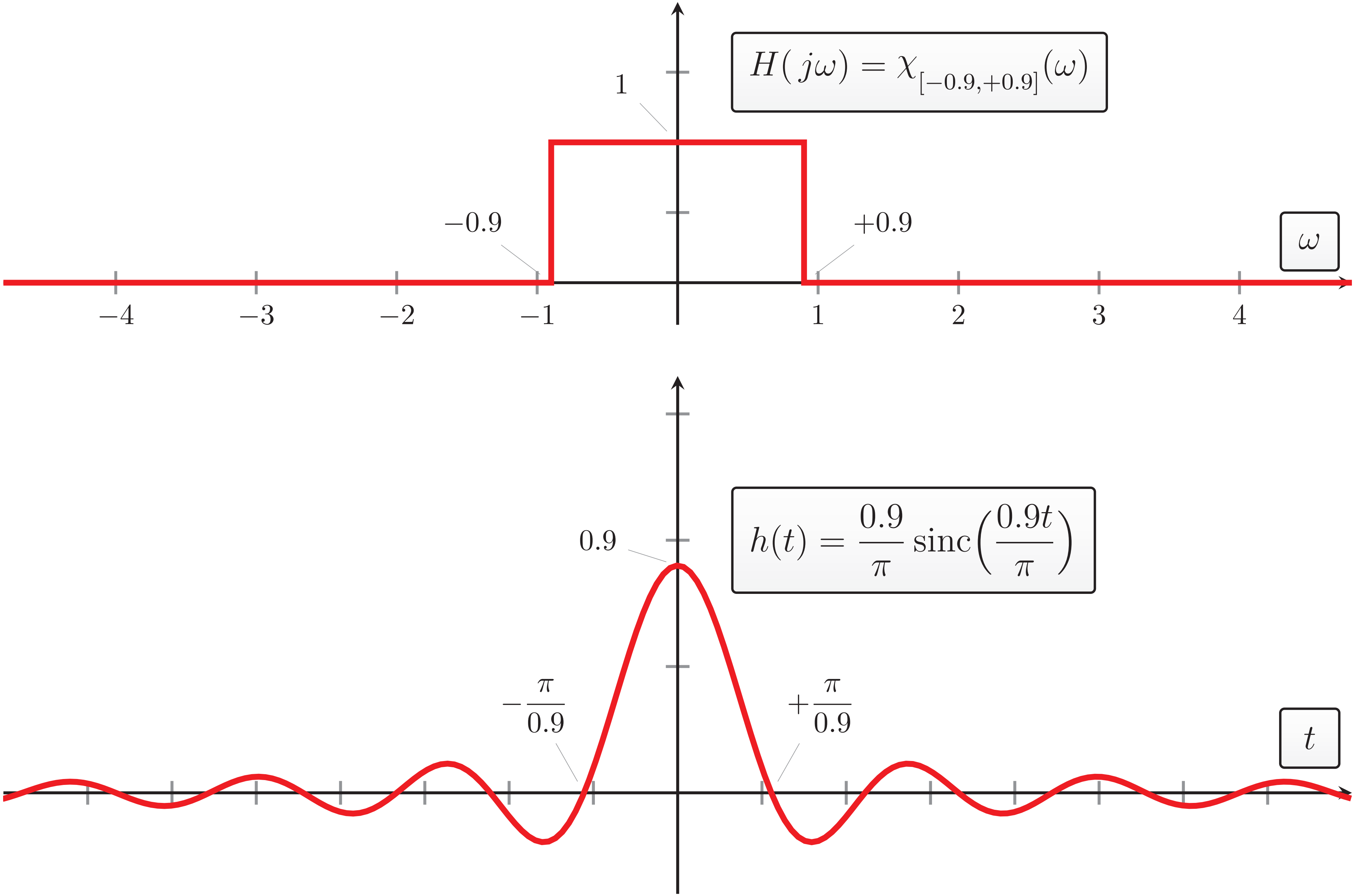


Fourier Transforms – Ideal Low Pass Filter

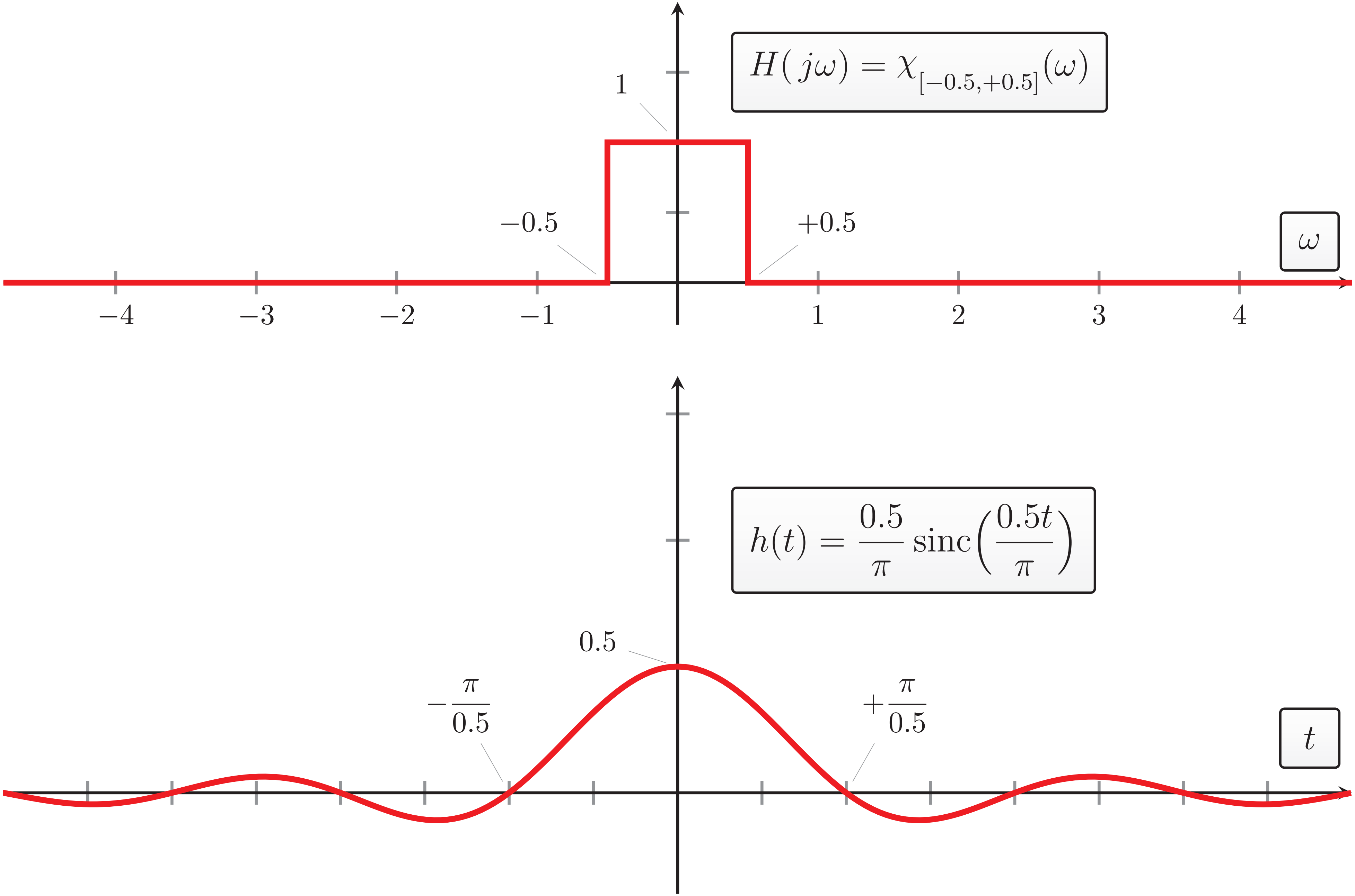




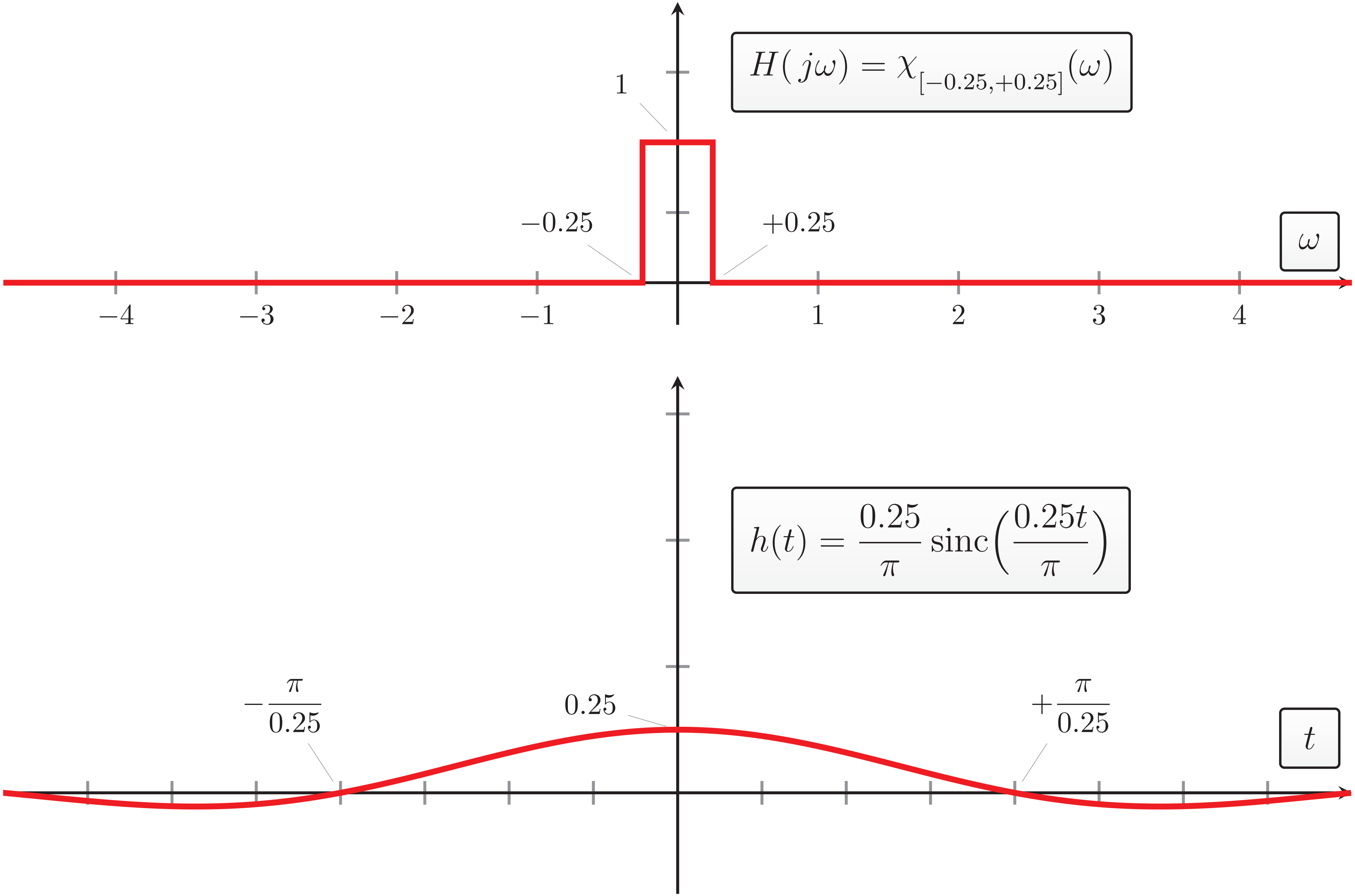
Fourier Transforms – Ideal Low Pass Filter



Fourier Transforms – Ideal Low Pass Filter



Fourier Transforms – Ideal Low Pass Filter



Part 4 Outline

7 Fourier Transforms

- Periodic Signals
- Properties
- Convolution
- Transform Method
- Partial Fractions
- Filter Cascade
- Multiplication Property
- Differential Equations
- Differentiator

Part 4 Outline

7. **Fourier Transforms**
   - Periodic Signals
   - Properties
   - Convolution
   - Transform Method
   - Partial Fractions
   - Filter Cascade
   - Multiplication Property
   - Differential Equations
   - Differentiator

Fourier Transforms – Periodic Signals

Key Questions:

What if the CT signals are **periodic** in time?

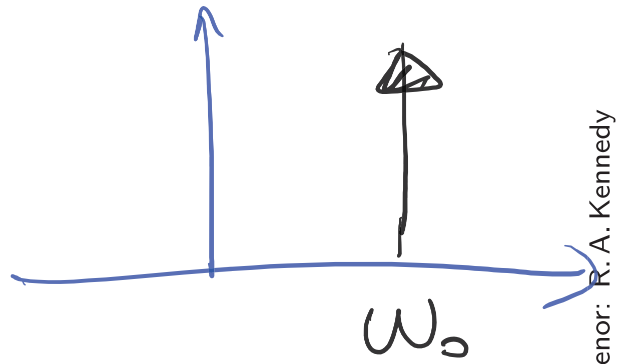
Can the Fourier transform sensibly handle this case or do we need to revert to the Fourier series approach?

Answer: impulses (in the frequency domain) to the rescue.

$$x(t) \to e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0) \quad X(j\omega)$$

Transform Pair 2: Begin with

$$X(j\omega) = \overset{2\pi}{}\delta(\omega - \omega_0)$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\delta(\omega-\omega_0)e^{j\omega t}\,d\omega$$

$$= e^{j\omega_0 t}$$

sifting property impulse function

Fourier Transforms – Periodic Signals

Recap:

$$\frac{1}{2\pi}e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} \delta(\omega - \omega_0)$$

$$\mathscr{F}\left\{\frac{1}{2\pi}e^{j\omega_0 t}\right\} = \delta(\omega - \omega_0)$$

$$\mathscr{F}^{-1}\left\{\delta(\omega - \omega_0)\right\} = \frac{1}{2\pi}e^{j\omega_0 t}$$

- Frequency domain function $\delta(\omega - \omega_0)$ is the *Fourier Transform* of $\frac{1}{2\pi}e^{j\omega_0 t}$
- Time domain function $\frac{1}{2\pi}e^{j\omega_0 t}$ is the *Inverse Fourier Transform* of $\delta(\omega - \omega_0)$

Fourier Transforms – Periodic Signals

We have for all ω_0,

$$e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega - \omega_0),$$

therefore

$$e^{jk\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega - k\omega_0), \quad \forall k.$$

If $x(t)$ is **periodic** then it has Fourier Series expansion

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}}$$

where a_k are the Fourier Series coefficients. Then the Fourier Transform, by linearity, is

F.S. coefficients

$$\boxed{X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)}$$

This has energy at only discrete frequencies $k\omega_0$ where $k \in \mathbb{Z}$. **This is Transform Pair 1.**

Fourier Transforms – Periodic Signals

Recap:

$$\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

$$\mathscr{F}\left\{\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

$$\mathscr{F}^{-1}\left\{2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)\right\} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

- Frequency domain function $2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$ is the *Fourier Transform* of $\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$
- Time domain function $\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$ is the *Inverse Fourier Transform* of $2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$

Fourier Transforms – Periodic Signals

This is an important general result for periodic signals:

- Fourier transform of a periodic signal $x(t)$ with Fourier series coefficients a_k can be interpreted as a train of impulses occurring at harmonically related frequencies and for which the area of the impulse at the k-th harmonic frequency $k\omega_0$ is 2π times the k-th Fourier series coefficient a_k.

- If already know Fourier series coefficient a_k of periodic signal can substitute a_k into this relationship to get the Fourier transform of the signal.

- Means that we can avoid integration.

$$F\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} = \boxed{2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)}$$

Example:

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

has Fourier Transform

$$X(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

The only non-zero Fourier Series coefficients are $a_{-1} = 1/2$ and $a_1 = 1/2$. In the Fourier Transform representation, $X(j\omega)$, these just get multiplied by 2π (an artefact of definitions) and becomes the weights of delta function in continuous ω frequency domain.

This is Transform Pair 3.

Recap:

$$\cos(\omega_0 t) \stackrel{\mathscr{F}}{\longleftrightarrow} \pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$$

$$\mathscr{F}\left\{\cos(\omega_0 t)\right\} = \pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$$

$$\mathscr{F}^{-1}\left\{\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)\right\} = \cos(\omega_0 t)$$

- Frequency domain function $\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$ is the *Fourier Transform* of $\cos(\omega_0 t)$
- Time domain function $\cos(\omega_0 t)$ is the *Inverse Fourier Transform* of $\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Example:

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$$

$$\mathcal{F}\{\sin \omega_0 t\} = 2\pi\left(a_{-1}\delta(\omega + \omega_0) + a_1 \delta(\omega - \omega_0)\right) = 2\pi\left(\frac{-1}{2j}\delta(\omega+\omega_0) + \frac{1}{2j}\delta(\omega-\omega_0)\right)$$

This is Transform Pair 4.

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Recap:

$$\sin(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$$

$$\mathscr{F}\left\{\sin(\omega_0 t)\right\} = \frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$$

$$\mathscr{F}^{-1}\left\{\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}\right\} = \sin(\omega_0 t)$$

- Frequency domain function $\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$ is the *Fourier Transform* of $\sin(\omega_0 t)$
- Time domain function $\sin(\omega_0 t)$ is the *Inverse Fourier Transform* of $\frac{\pi\,\delta(\omega-\omega_0)}{j} - \frac{\pi\,\delta(\omega+\omega_0)}{j}$

Fourier Transforms – Periodic Signals

Recall the sampling function which is periodic with (fundamental) period T:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Then $\omega_0 = 2\pi/T$ and Fourier Series coefficients are, for all integer k, given by

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jw_0 t}\, dt = \frac{1}{T}$$

$$F\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Whence, from earlier (4 slides ago),

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{T}}^{a_k}\, \delta\Big(\omega - \overbrace{\frac{2\pi k}{T}}^{k\omega_0}\Big) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\Big(\omega - k\,\omega_0\Big)$$

which is a sampling function, periodic in ω with period ω_0 and scaled by $2\pi/T$, in the frequency domain.

Fourier Transforms – Periodic Signals

Recap:

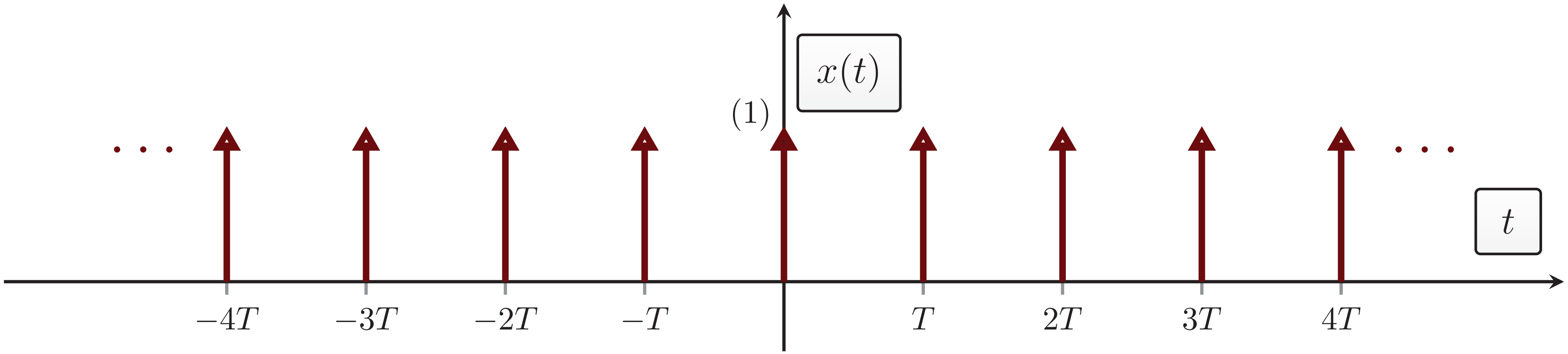
$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

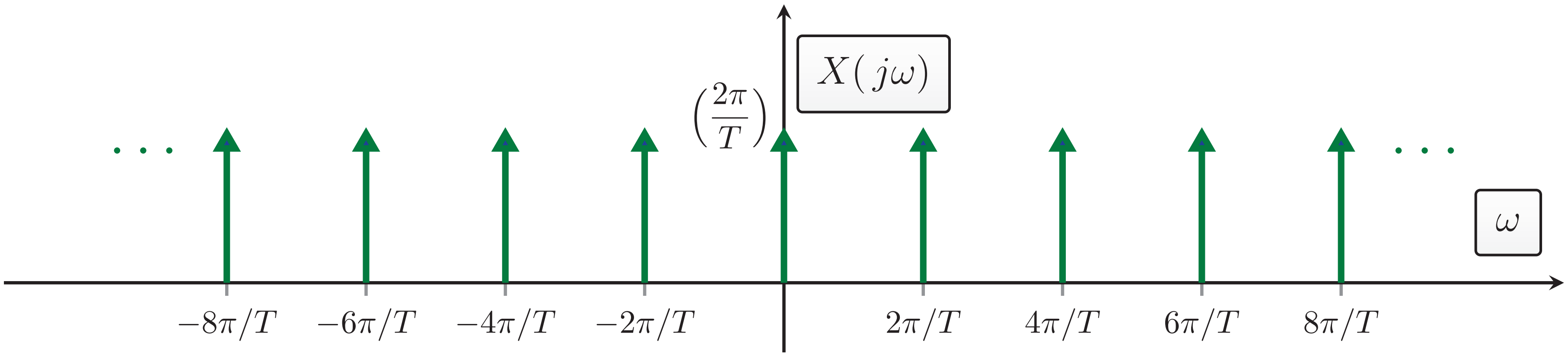
$$\mathscr{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\mathscr{F}^{-1}\left\{\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)\right\} = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Frequency domain function $\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ is the *Fourier Transform* of $\sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Time domain function $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ is the *Inverse Fourier Transform* of $\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$

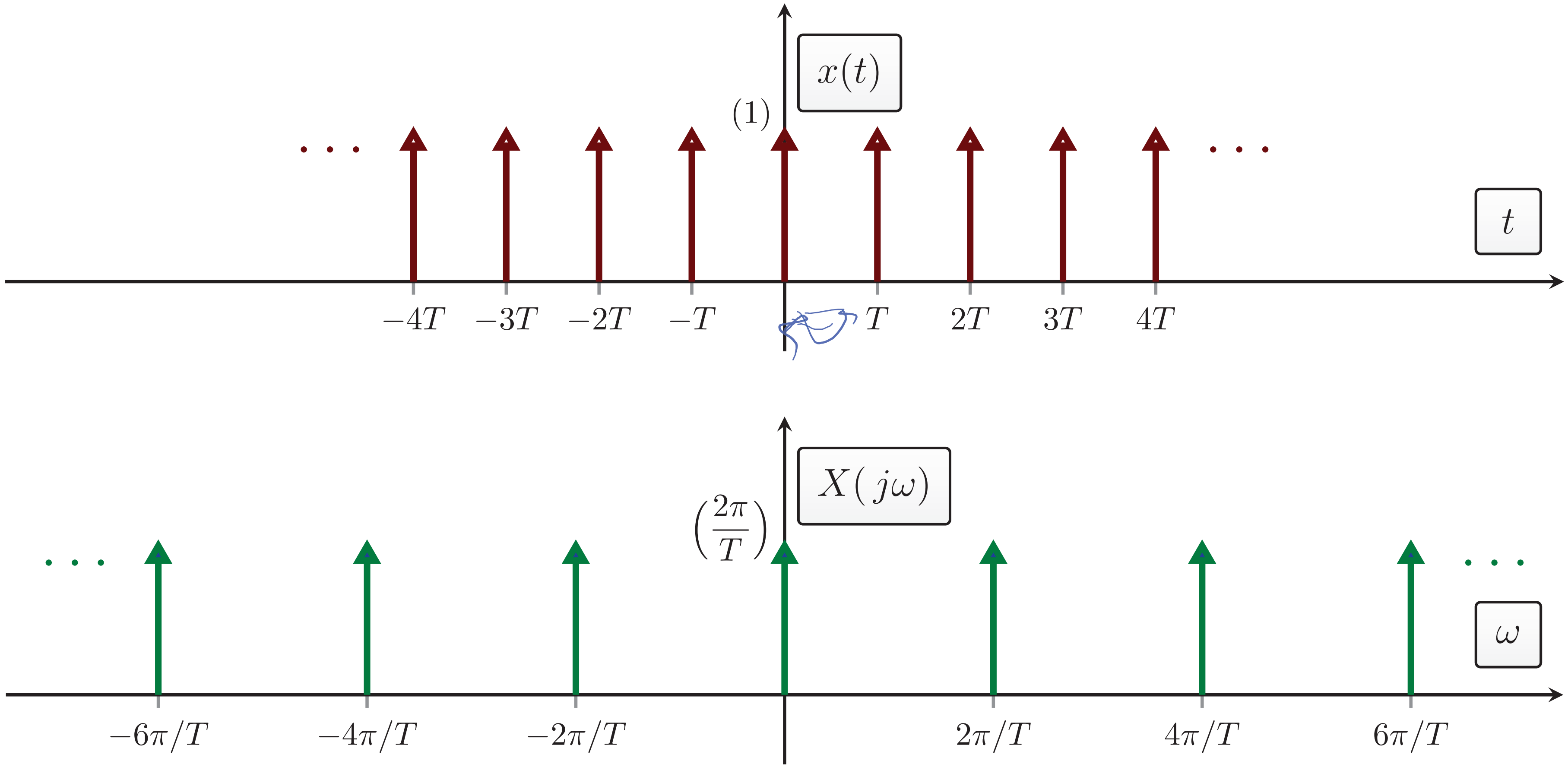
Fourier Transforms – Periodic Signals





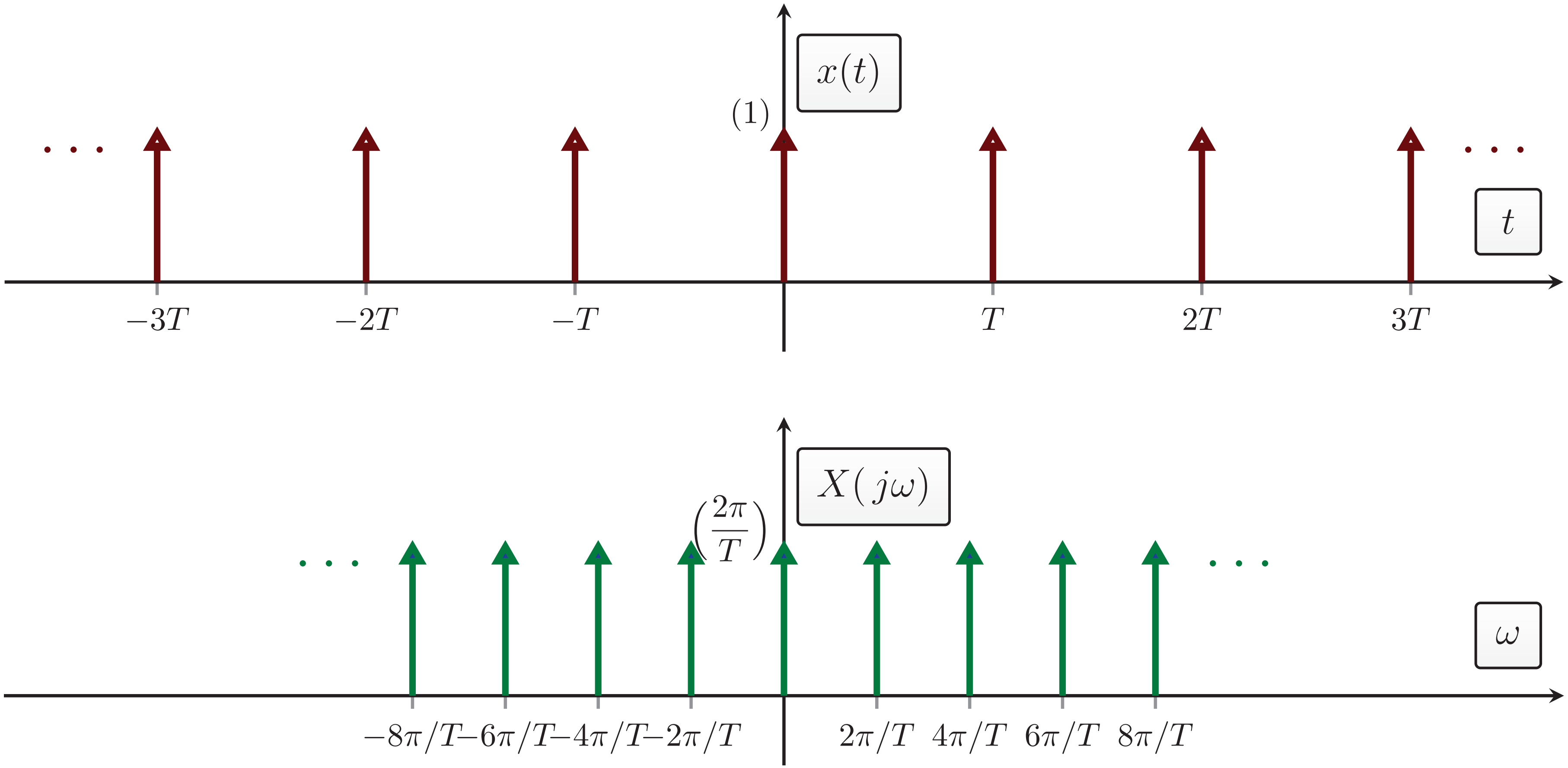
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Fourier Transforms – Periodic Signals



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

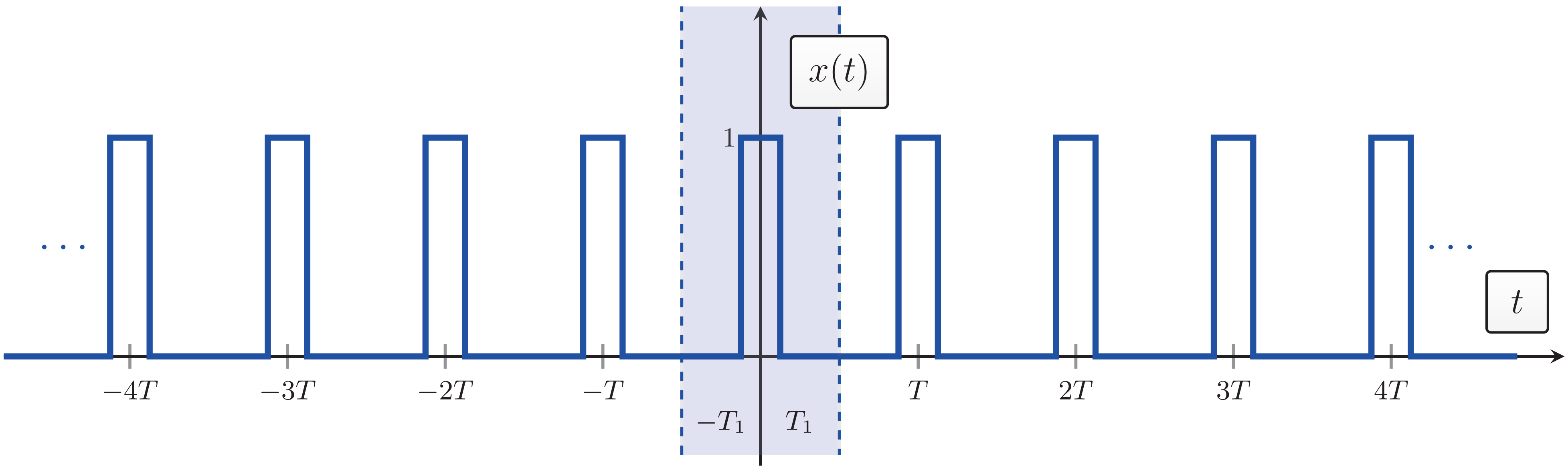
Fourier Transforms – Periodic Signals



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \le T/2$$



Fourier Transforms – Periodic Signals

$$F\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

With Fourier series coefficients:

$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = 2\cancel{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_0 T_1)}{\cancel{\pi} k} \delta(\omega - k\omega_0)$$

$$\sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$$

This is Transform Pair 6.