Signal Processing
ENGN2228

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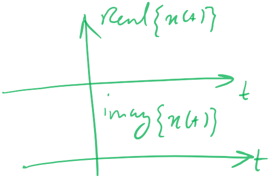
Second Semester

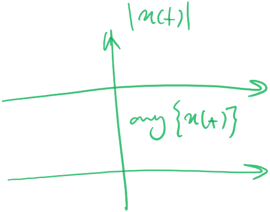
Lecture 18

$$x(t) = \alpha(t)e^{j\beta(t)} = \alpha(t)\cos(\beta(t)) + j\alpha(t)\sin(\beta(t))$$

$$\left\{\begin{array}{l} Real\{x(t)\} = \alpha(t)\cos(\beta(t)) \\ Imag\{x(t)\} = \alpha(t)\sin(\beta(t)) \end{array}\right.$$



$$\left\{\begin{array}{l} |x(t)| = \alpha(t) \\ \arg\{x(t)\} = \beta(t) \end{array}\right.$$



Example:

$x(t) + x^*(t) = 2\left[\text{Real}\{x(t)\}\right]$

Let $x(t) = \sqrt{2}(1 + j)e^{j\pi/4}e^{(-1+j2\pi)t}$. Sketch and label the following:

(a) $Re\{x(t)\}$ $= -e^{-t}\sin(2\pi t)$

(b) $Im\{x(t)\}$ $= e^{-t}\cos(2\pi t)$

(c) $x(t + 2) + x^*(t + 2)$ $= 2\left[\text{Real}\{x(t+2)\}\right] = 2\left[-e^{-t-2}\sin(2\pi(t+2))\right]$

(d) $|x(t)| = e^{-t}$

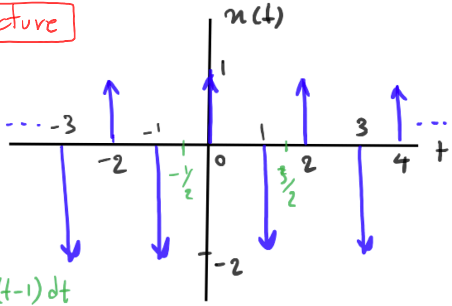
(e) $\arg\{x(t)\} = 2\pi t + \pi/2$

$$x(t) = \sqrt{2}\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)e^{j\pi/4}e^{-t}e^{j2\pi t} = \underline{e^{-t}}\,e^{j(2\pi t + \pi/2)}$$

$$= e^{-t}\left[\underbrace{\cos(2\pi t + \pi/2)}_{-\sin(2\pi t)} + j\underbrace{\sin(2\pi t + \pi/2)}_{\cos(2\pi t)}\right]$$

Example:

$\boxed{\text{From the last lecture}}$



$T = 2s,\ \omega_0 = \frac{2\pi}{T} = \pi \text{ rad/s}$

$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

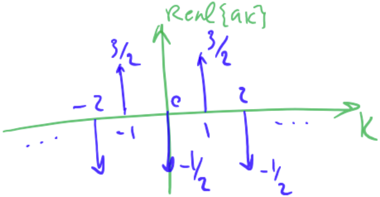
$a_0 = \frac{1}{2}\int_{-1/2}^{3/2} x(t)\, dt = \frac{1}{2}\int_{-1/2}^{3/2} \delta(t) - 2\delta(t-1)\, dt$

$= \frac{1}{2}(1-2) = -\frac{1}{2}$

$a_k = \frac{1}{2}\int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{2}\int_{-1/2}^{3/2} \delta(t) - 2\delta(t-1)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{2} - e^{-jk\omega_0} = \frac{1}{2} - \left(e^{-j\pi}\right)^k = \frac{1}{2} - (-1)^k$



$\frac{1}{2j} = \frac{1}{2}j$

$\frac{1}{2} e^{-j\pi/2}$

Signals Classifications:

$x(t)$

- Continuous or discrete $x[n]$

- Even, odd or none

$x(t) = \text{even}\{x(t)\} + \text{odd}\{x(t)\}$

$\text{even}\{x(t)\} = \frac{x(t)+x(-t)}{2}$

- Periodic or non-periodic

$\text{odd}\{x(t)\} = \frac{x(t)-x(-t)}{2}$

$\frac{2\pi}{\omega_0} = T$

$\cos(\alpha\omega_0 t) + \sin(\beta\omega_0 t)$

$\underbrace{T_1 \qquad T_2}_{T}$

$\frac{2\pi}{\omega_0} m$

$\cos(\omega_0 n)$

- Energy or power

$E_\infty\{x(t)\} = \int_{-T}^{T} |x(t)|^2 dt < \infty$

$T \to \infty$

Example:

Let $x[n] = \cos(\Omega_x(n + P_x) + \theta_x)$.

(a) Determine the period of $x[n]$ for each of the following three cases:

	Ω_x	P_x	θ_x
(i)	$\pi/3$	0	2π
(ii)	$3\pi/4$	2	$\pi/4$
(iii)	$3/4$	1	$1/4$

(i) $x[n] = \cos\left(\pi/3\,(n+0) + 2\pi\right) = \cos(\pi/3\, n)$

$$\frac{2\pi}{\omega_0} m = \frac{2\cancel{\pi}}{\cancel{\pi}/3} m = 6m \xRightarrow{m=1} N_0 = 6$$

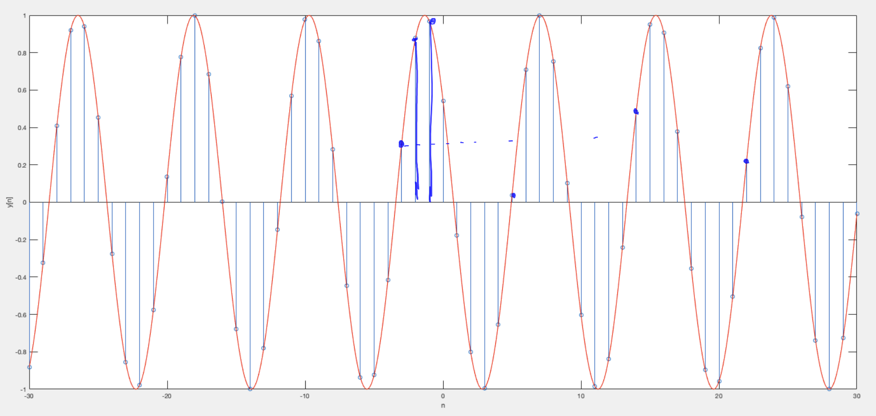
(ii) $x[n] = \cos\left(3\pi/4\,(n+2) + \pi/4\right) = \cos\left(3\pi/4\, n + 7\pi/4\right)$

$$\frac{2\cancel{\pi}}{3\cancel{\pi}/4} m = 8/3\, m \xRightarrow{m=3} N_0 = 8$$

(iii) $x[n] = \cos\left(3/4\,(n+1) + 1/4\right) = \cos\left(3/4\, n + 1\right)$

$$\frac{2\pi}{3/4} m = 8/3\, \pi m \longrightarrow$$ this signal is not periodic

Example (con'd):



System Properties :

- Causal or non-causal
- With memory or memory-less
- * Time-invariant or time-variant
- Linear or non-linear
- Stable or unstable

$$y(t) = f_t(x(t))$$

$$y(t-t_0) = f_{t-t_0}(x(t-t_0)) \quad \text{I}$$

$$x_1(t) = x(t-t_0)$$

$$y_1(t) = f_t(x_1(t)) = f_t(x(t-t_0)) \quad \text{II}$$

$$y(t) = f(x(t))$$

$$f(a x_1(t) + B x_2(t)) = a f(x_1(t)) + b f(x_2(t))$$

$$h(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Example:

$y(t), y[n]$	Properties: Memoryless	Linear	Time-Invariant	Causal
(a) $(2 + \sin t)x(t)$	✓	✓	✗	
(b) $x(2t)$		✓	✗	✗
(c) $\sum_{k=-\infty}^{\infty} x[k]$		✓	✓	
(d) $\sum_{k=-\infty}^{n} x[k]$	✗	✓	✓	✓
(e) $\frac{dx(t)}{dt}$		✓		
(f) $\max\{x[n], x[n-1], \ldots, x[-\infty]\}$	✗	✗		

$$y[n] = f[x[n]]$$

$$f[ax_1[n] + bx_2[n]] = \max\{ax_1[n] + bx_2[n], \ldots, ax_1[-\infty] + bx_2[-\infty]\}$$

$$\neq a\max\{x_1[n], \ldots, x_1[-\infty]\} + b\max\{x_2[n], \ldots, x_2[-\infty]\}$$

Solutions:

Memoryless:

(a) $y(t) = (2 + \sin t)x(t)$ is memoryless because $y(t)$ depends only on $x(t)$ and not on prior values of $x(t)$.

(d) $y[n] = \Sigma_{k=-\infty}^{n} x[n]$ is not memoryless because $y[n]$ does depend on values of $x[\cdot]$ before the time instant n.

(f) $y[n] = \max\{x[n], x[n-1], \ldots, x[-\infty]\}$ is clearly not memoryless.

Linear:

(a)

$$\begin{aligned} y(t) &= (2 + \sin t)x(t) = T[x(t)], \\ T[ax_1(t) + bx_2(t)] &= (2 + \sin t)[ax_1(t) + bx_2(t)] \\ &= a(2 + \sin t)x_1(t) + b(2 + \sin t)x_2(t) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

Therefore, $y(t) = (2 + \sin t)x(t)$ is linear.

(b)

$$\begin{aligned} y(t) &= x(2t) = T[x(t)], \\ T[ax_1(t) + bx_2(t)] &= ax_1(2t) + bx_2(t) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

Therefore, $y(t) = x(2t)$ is linear.

Solutions:

(c)
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] = T[x[n]],$$
$$T[ax_1[n] + bx_2[n]] = a\sum_{k=-\infty}^{\infty} x_1[k] + b\sum_{k=-\infty}^{\infty} x_2[k]$$
$$= aT[x_1[n]] + bT[x_2[n]]$$
Therefore, $y[n] = \Sigma_{k=-\infty}^{\infty} x[k]$ is linear.

(d) $y[n] = \sum_{k=-\infty}^{n} x[k]$ is linear (see part c).

(e)
$$y(t) = \frac{dx(t)}{dt} = T[x(t)],$$
$$T[ax_1(t) + bx_2(t)] = \frac{d}{dt}[ax_1(t) + bx_2(t)]$$
$$= a\frac{dx_1(t)}{dt} + b\frac{dx_2(t)}{dt} = aT[x_1(t)] + bT[x_2(t)]$$
Therefore, $y(t) = dx(t)/dt$ is linear.

(f)
$$y[n] = \max\{x[n], \ldots, x[-\infty]\} = T[x[n]],$$
$$T[ax_1[n] + bx_2[n]] = \max\{ax_1[n] + bx_2[n], \ldots, ax_1[-\infty] + bx_2[-\infty]\}$$
$$\neq a\max\{x_1[n], \ldots, x_1[-\infty]\} + b\max\{x_2[n], \ldots, x_2[-\infty]\}$$
Therefore, $y[n] = \max\{x[n], \ldots, x[-\infty]\}$ is not linear.

Solutions:

Time-invariant:

(a)
$$y(t) = (2 + \sin t)x(t) = T[x(t)],$$
$$T[x(t - T_0)] = (2 + \sin t)x(t - T_0)$$
$$\neq y(t - T_0) = (2 + \sin(t - T_0))x(t - T_0)$$

Therefore, $y(t) = (2 + \sin t)x(t)$ is not time-invariant.

(b)
$$y(t) = x(2t) = T[x(t)],$$
$$T[x(t - T_0)] = x(2t - 2T_0) \neq x(2t - T_0) = y(t - T_0)$$

Therefore, $y(t) = x(2t)$ is not time-invariant.

(c)
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] = T[x[n]],$$
$$T[x[n - N_0]] = \sum_{k=-\infty}^{\infty} x[k - N_0] = y[n - N_0]$$

Therefore, $y[n] = \sum_{k=-\infty}^{\infty} x[k]$ is time-invariant.

(d)
$$y[n] = \sum_{k=-\infty}^{n} x[k] = T[x[n]],$$
$$T[x[n - N_0]] = \sum_{k=-\infty}^{n} x[k - N_0] = \sum_{l=-\infty}^{n-N_0} x[l] = y[n - N_0]$$

Therefore, $y[n] = \sum_{k=-\infty}^{n} x[k]$ is time-invariant.

Solutions:

(e) $$y(t) = \frac{dx(t)}{dt} = T[x(t)],$$

$$T[x(t - T_0)] = \frac{d}{dt}x(t - T_0) = y(t - T_0)$$

Therefore, $y(t) = dx(t)/dt$ is time-invariant.

Causal:

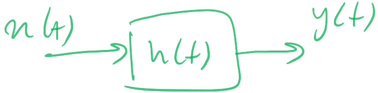
(b) $y(t) = x(2t)$,
$y(1) = x(2)$

The value of $y(\cdot)$ at time $= 1$ depends on $x(\cdot)$ at a future time $= 2$. Therefore, $y(t) = x(2t)$ is not causal.

(d) $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

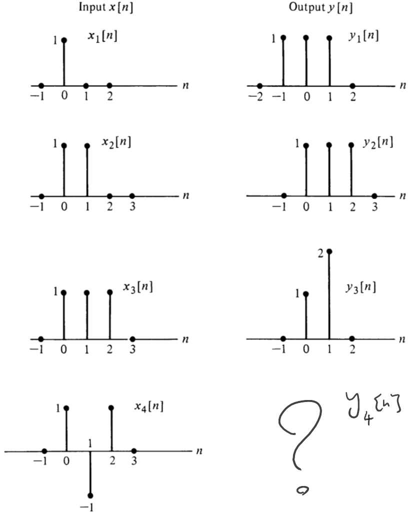
Yes, $y[n] = \Sigma_{k=-\infty}^{n} x[k]$ is causal because the value of $y[\cdot]$ at any instant n depends only on the previous (past) values of $x[\cdot]$.

LTI Systems:



LTI Systems Example:

The system is linear:



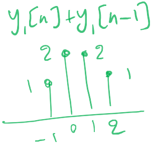
Is this system time-invariant?

$$x_4[n] = 2x_1[n] - 2x_2[n] + x_3[n]$$

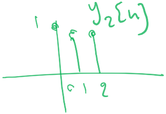
$$y_4[n] = 2y_1[n] - 2y_2[n] + y_3[n]$$

$$\underbrace{x_1[n] + x_1[n-1]} = x_2[n]$$

$y_1[n] + y_1[n-1] \Rightarrow$



$y_2[n] \Rightarrow$



Since $y_2[n] \neq y_1[n] + y[n-1]$

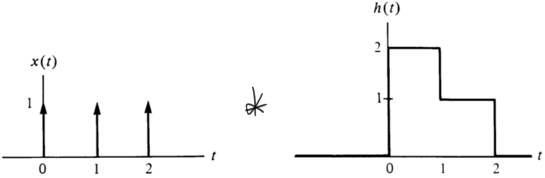
the system is not time-invariant

Convolution:

CT $\quad y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau$

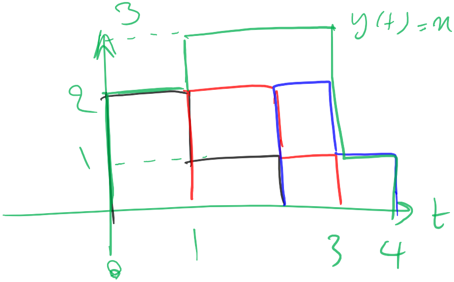
DT $\quad y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

Example:



$x(t) * h(t) = ?$

$y(t) = x(t) * h(t)$



Fourier Series:

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

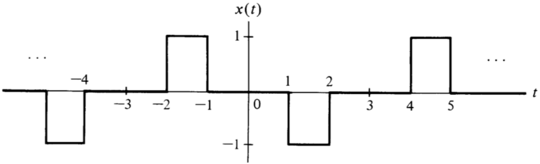
Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\, dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example:

$a_k = ?$



$$T_0 = 6 \qquad \frac{2\pi}{T} = \omega_0 \rightarrow \omega_0 = \pi/3$$

$$a_0 = \frac{1}{T}\int_T x(t) = 0$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{6}\int_{-3}^{3} x(t) e^{-jk\omega_0 t}\, dt = +\frac{1}{6}\frac{1}{jk\omega_0}\left[e^{-jk\omega_0 t}\right]_{-2}^{-1} - \frac{1}{6jk\omega_0}\left[e^{-j\omega_0 k t}\right]_{1}^{2} =$$

$$= \frac{1}{j2\pi k}\left[e^{+j\frac{\pi}{3}k} - e^{-j\frac{2\pi}{3}k} - e^{+j\frac{2\pi}{3}k} + e^{-j\frac{\pi}{3}k}\right] = \frac{1}{j\pi k}\left[\cos\left(\pi/3\, k\right) - \cos\left(\frac{2\pi}{3}k\right)\right]$$

Time vs. Frequency domain:

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties