Part 1 Outline

1. **Exam Revision**
   - Fourier Series Question
   - Fourier Transform Question

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a) The analysis and synthesis equations for a periodic CT signal $x(t)$ and its Fourier series coefficients a_k are given below:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad \text{(Synthesis Equation)}$$

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad \text{(Analysis Equation)}$$

where T is the fundamental period and $\omega_0 = 2\pi/T$ is the fundamental frequency in rad/sec.

i) [3 marks] Can we find the Fourier transform of $x(t)$? If so how?

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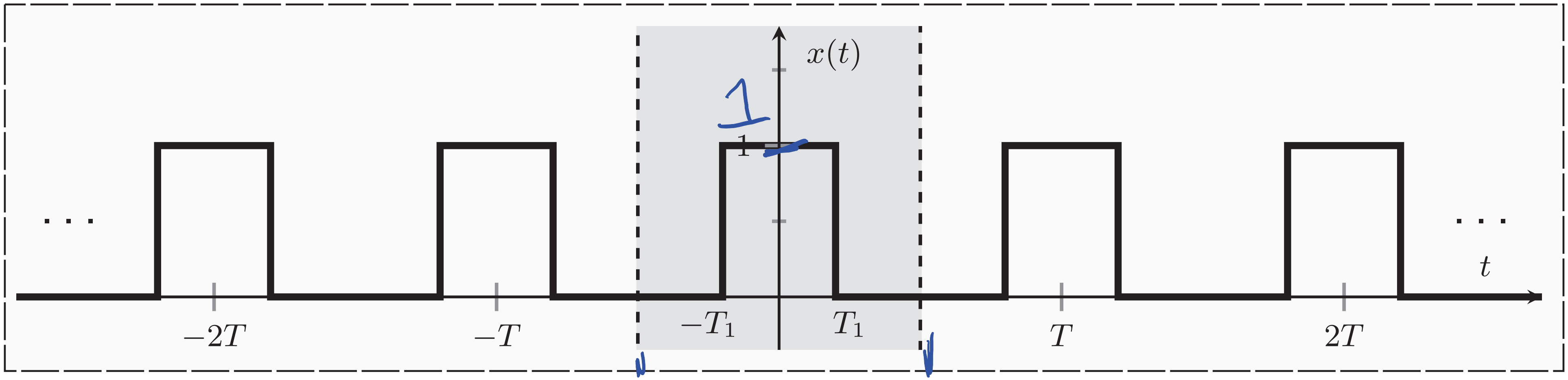
i) [3 marks] Can we find the Fourier transform of $x(t)$? If so how?

Solution: Yes Fourier transform (FT) is defined for any CT signal. Can use the Fourier transform analysis equation but if have Fourier series coefficient easiest to use the FT pair for periodic signals.

Exam Revision – Fourier Series Question

b) Consider the periodic CT signal $x(t)$, shown below:



$-T/2$ $\quad$ $T/2$

i) [2 marks] Determine the Fourier series coefficient

$$a_0 = \frac{1}{T}\int_T x(t)\,dt.$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt$$

$$= \frac{1}{T}\left[t\right]_{-T_1}^{T_1} = \frac{1}{T}\left(T_1 - -T_1\right) = \frac{2T_1}{T}$$

ii) [3 marks] Determine the Fourier series of $x(t)$ using the analysis equation.

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{-1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} = \frac{-1}{jk\omega_0 T}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{1}{jk\frac{2\pi}{\cancel{T}}\cancel{T}}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right) = \frac{1}{k\pi}\sin(k\omega_0 T_1)$$

iii) [2 marks] Find the Fourier series coefficients of the signal $a(t) = x(t-1)$.

using time-shift property FS:

$$x(t-1) \longleftrightarrow a_k e^{-jk\omega_0}$$

b_k to denote FS of $a(t)$

$$b_k = \frac{1}{k\pi} \sin(k\omega_0 T_1) e^{-jk\omega_0}$$

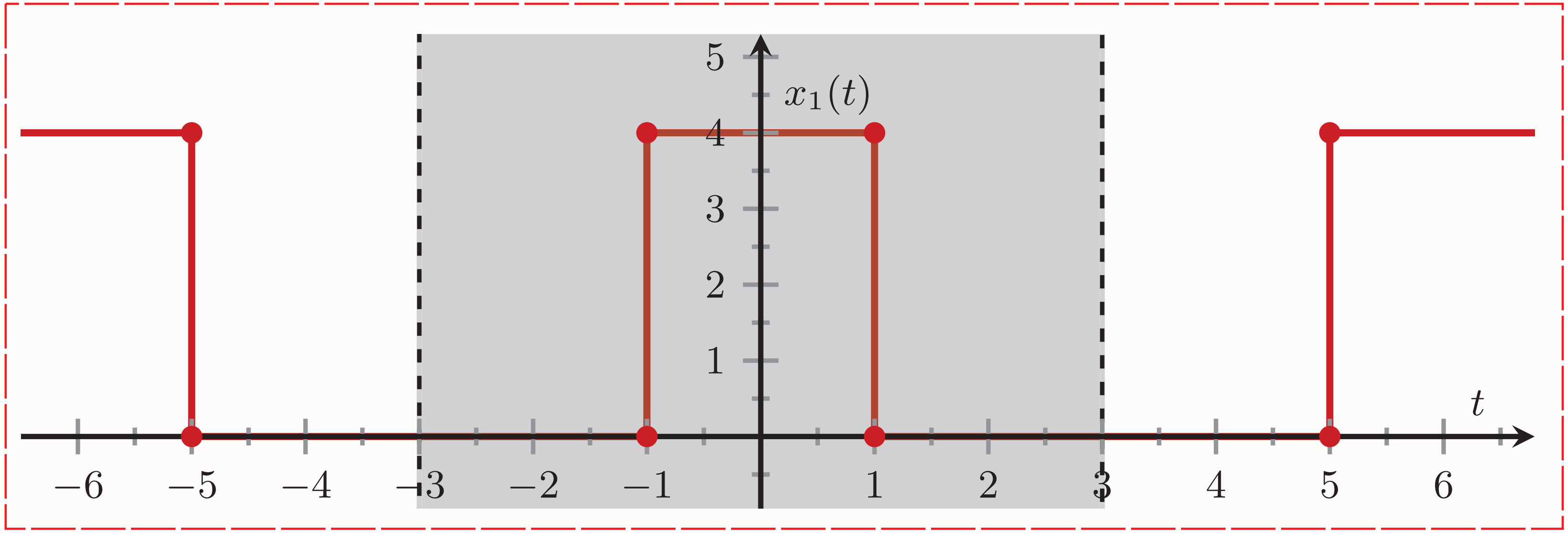
iv) [2 marks] Find the Fourier series coefficients of the signal
$b(t) = x(t) \star a(t) = \int_T x(\tau)\, a(t-\tau)\, d\tau$..

using convolution property FS

$$T a_k b_k = \frac{T}{k\pi} \sin(k\omega_0 T_1) \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$e^{-jk\omega_0}$

v) [2 marks] Using your answer from part (b) ii), determine the Fourier series coefficients for the periodic signal $x_1(t)$:



One period of width 6 has been shaded.

$T=6$, $T_1=1$ $\quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{6} = \pi/3$

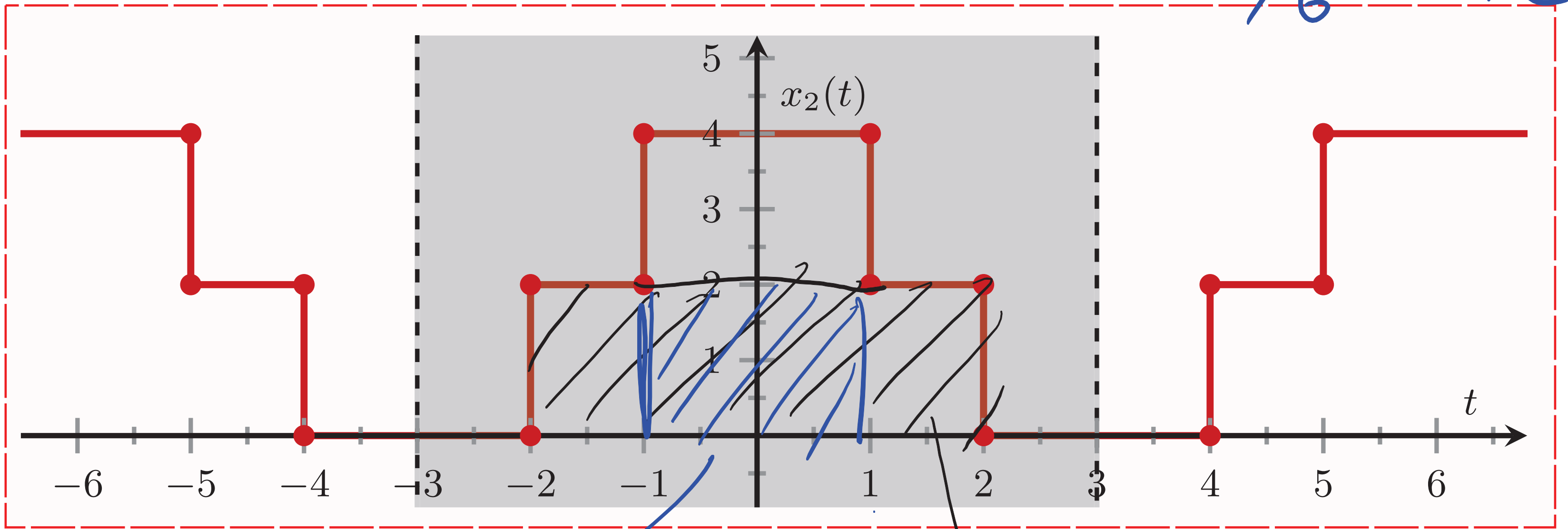
$$a_k = 4\,\frac{\sin(k\omega_0 T_1)}{k\pi} = 4\,\frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}$$

linearity property FS

Exam Revision – Fourier Series Question

vi) [4 marks] Determine the Fourier Series coefficients for the periodic signal $x_2(t)$:

$\omega_0 = 2\pi/6 = \pi/3$



One period of width 6 has been shaded.

$$\frac{2}{k\pi}\sin\left(\frac{k\pi}{3}\right) + \frac{2}{k\pi}\sin\left(\frac{2k\pi}{3}\right)$$

linearity property FS

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Exam Revision – Fourier Transform Question

a) [2 marks] The analysis and synthesis equations for a CT signal $x(t)$ and its Fourier transform $X(j\omega)$ are given below:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

Express in a few sentences your understanding of these two equations.

Exam Revision – Fourier Transform Question

a) [2 marks] The analysis and synthesis equations for a CT signal $x(t)$ and its Fourier transform $X(j\omega)$ are given below:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

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Express in a few sentences your understanding of these two equations.

Solution: The synthesis equation shows that in the time-domain the signal $x(t)$ can be built-up or synthesized by a superposition (linear combination) of complex exponentials.

The analysis equation shows the frequency-domain represenation of the signal $X(j\omega)$, i.e. it shows the frequency components (which ones and in what amounts) that make-up the signal.

b) [8 marks] Show, using any method, that the FT of the signal

$$x(t) = e^{-t/10}\cos(10t)u(t)$$

$= \cos(10t) \times e^{-t/10} u(t)$

can be expressed in the form

$$X(j\omega) = \frac{j\omega + \frac{1}{10}}{(j\omega + \frac{1}{10})^2 + 10^2}$$

Note: You must show all steps and name any properties used to arrive at the correct answer.

Exam Revision – Fourier Transform Question

$$x(t) = e^{-t/10}\cos(10t)u(t) = \cos 10t \times e^{-t/10}u(t)$$

has

$$X(j\omega) = \frac{j\omega + \frac{1}{10}}{(j\omega + \frac{1}{10})^2 + 10^2}$$

Proof:

$$\cos 10t \longleftrightarrow \pi\delta(\omega+10) + \pi\delta(\omega-10)$$

$$e^{-t/10}u(t) \longleftrightarrow \frac{1}{j\omega + 1/10}$$

using multiplication property FT:

$$X(j\omega) = \frac{1}{2\pi}\left(\frac{1}{j\omega+10} * \left(\pi\left(\delta(\omega+10) + \pi\delta(\omega-10)\right)\right)\right)$$

$$= \frac{1}{2\cancel{\pi}}\left(\frac{\cancel{\pi}}{j(\omega+10)+10} + \frac{\cancel{\pi}}{j(\omega-10)+10}\right)$$

if simplify get expression asked to proove

c) [8 marks] Find, using any method, the time domain signal corresponding to the following Fourier representation

$$X(j\omega) = \frac{1}{2\pi}\Big(\frac{2\sin(\omega-2)}{(\omega-2)} \star \frac{e^{-j2\omega}2\sin(2\omega)}{\omega}\Big)$$

where $\star$ denotes the convolution. Note: You must show all steps and name any properties used in your working.

Exam Revision – Fourier Transform Question

$$X(j\omega) = \frac{1}{2\pi}\Big(\underbrace{\frac{2\sin(\omega-2)}{(\omega-2)}}_{Y(j\omega)} \star \underbrace{\frac{e^{-j2\omega}2\sin(2\omega)}{\omega}}_{Z(j\omega)}\Big)$$

Find $x(t)$: using multiplication property FT

$$x(t) = y(t)\, z(t)$$

$$a(t) = \begin{cases} 1, & |t| < T_1 = 1 \\ 0, & |t| > T_1 = 1 \end{cases} \longleftrightarrow \frac{2\sin\omega}{\omega} = A(j\omega)$$

$$A(j(\omega-2)) = \frac{2\sin(\omega-2)}{\omega-2} = Y(j\omega)$$ frequency shift property

$$e^{j2t}a(t)$$

$$\Rightarrow y(t) = \begin{cases} e^{j2t}, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

$$X(j\omega) = \frac{1}{2\pi}\Big(\overbrace{\frac{2\sin(\omega-2)}{(\omega-2)}}^{Y(j\omega)} \star \overbrace{\frac{e^{-j2\omega}2\sin(2\omega)}{\omega}}^{Z(j\omega)}\Big)$$

Find $x(t)$: $y(t) = e^{j2t}[u(t+1) - u(t-1)]$

Now to find $z(t)$:

$$b(t) = \begin{cases} 1, & |t|<2 \\ 0, & |t|>2 \end{cases} \Longleftrightarrow 2\sin\frac{2\omega}{2} = B(j\omega)$$

$$Z(j\omega) = e^{-j2\omega}B(j\omega) \Longleftrightarrow \underbrace{b(t-2)}_{z(t)}$$

time shift property F.T.

$$\Rightarrow z(t) = \begin{cases} 1, & 0<t<4 \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = y(t)z(t) = e^{j2t}[u(t) - u(t-1)]$$