Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 4 – SOLUTIONS

Fourier Analysis and Synthesis of Periodic Continuous Time Signals

Problem Set 4-1

Using the inspection method, determine the Fourier Series coefficients a_k of the signal

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t)$$

Solution:

$$a_k = \begin{cases} \frac{j}{2} & k = -3 \\ je^{j3} & k = -1 \\ -je^{-j3} & k = 1 \\ \frac{-j}{2} & k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Problem Set 4-2

Find the Fourier coefficients for each of the following signals if $\omega_0 = 2\pi$:

(a) $x(t) = 1 + \cos(2\pi t)$

Solution: Euler says $\cos(2\pi t) = (e^{j2\pi t} + e^{-j2\pi t})/2$ and so

$$a_k = \begin{cases} 1 & k = 0 \\ 1/2 & k \in \{-1, +1\} \\ 0 & \text{otherwise} \end{cases}$$

□

(b) $y(t) = \sin(10\pi t + \pi/6)$

Solution: OK this is slightly weird because ω_0 is not the fundamental frequency. What we have is $\omega_0 k = 10\pi$ so the two complex exponentials correspond to $k \in \{-5, +5\}$.
Euler says $\sin(10\pi t + \pi/6) = (e^{j\pi/6}e^{j10\pi t} - e^{-j\pi/6}e^{-j10\pi t})/2j$ and so

$$b_k = \begin{cases} \frac{1}{2}e^{-j\pi/3} & k = 5 \\ \frac{1}{2}e^{j\pi/3} & k = -5 \\ 0 & \text{otherwise} \end{cases},$$

where we have absorbed $j = e^{j\pi/2}$ into the overall phase $e^{j\pi/6} \times e^{-j\pi/2} = e^{-j\pi/3}$ (draw the unit circle in the complex plane). □

(c) $z(t) = \big(1 + \cos 2\pi t\big)\sin(10\pi t + \pi/6)$

Solution: We are multiplying two signals in the time domain so we convolve the Fourier Series coefficients computed in the two previous parts. That is,

$$\begin{aligned} c_k &= a_k \star b_k \\ &= \{\ldots, 0, 0.5, 1, 0.5, 0, \ldots\} \star \{\ldots, 0, b_{-5}, 0, \cdots, 0, b_5, 0, \ldots\} \\ &= \{\ldots, 0, 0.5\, b_{-5}, b_{-5}, 0.5\, b_{-5}, 0, \cdots, 0, 0.5\, b_5, b_5, 0.5\, b_5, 0, \ldots\} \\ &= \{\ldots, 0, c_{-6}, c_{-5}, c_{-4}, 0, \cdots, 0, c_4, c_5, c_6, 0, \ldots\} \end{aligned}$$

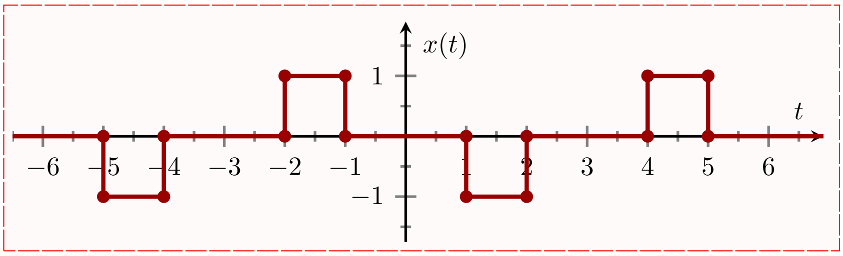
In matlab you could do this

```
conv((0.5, 1, 0.5), (exp(i * pi/3), 0, 0, 0, 0, 0, 0, 0, 0, 0, exp(-i * pi/3))).
```

□

Problem Set 4-3

Determine the Fourier series of following signal $x(t)$ by



(a) using analysis equation

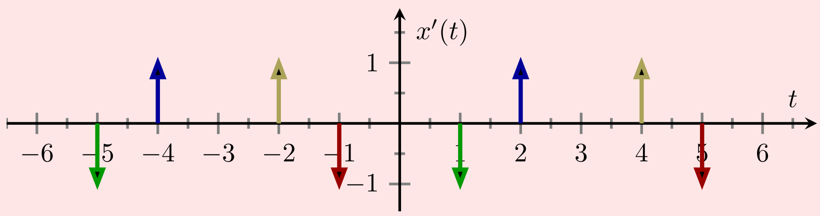
Solution: Unless you have a letter from your mummy saying that this problem is giving you a headache and could you please be excused from thinking too hard, then we have the following. Here $T = 6$, $\omega_0 = \pi/3$ and $x(t)$ will have Fourier coefficients a_k representing the weights at frequencies $\omega = k\omega_0 = k\pi/3$, for $k = 0, \pm1, \pm2, \ldots$. So the analysis equation is

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{6}\int_{-2}^{-1} e^{-jk(\pi/3)t}\, dt - \frac{1}{6}\int_{-1}^{2} e^{-jk(\pi/3)t}\, dt \\ &= \frac{1}{6}\left(\frac{e^{-jk(\pi/3)t}}{-jk\pi/3}\Bigg|_{-2}^{-1}\right) - \frac{1}{6}\left(\frac{e^{-jk(\pi/3)t}}{-jk\pi/3}\Bigg|_{1}^{2}\right) \\ &= \cdots = \frac{j}{2k\pi}\left(e^{jk\pi/3} + e^{-jk\pi/3} - e^{j2k\pi/3} - e^{-j2k\pi/3}\right) \\ &= \frac{j}{k\pi}\Big(\cos(k\pi/3) - \cos(2k\pi/3)\Big). \end{aligned}$$

These coefficients are purely imaginary because Captain Hindsight observes that $x(t)$ is an odd function. □

(b) combinations of derivatives, impulse trains, linearity, hallucinogenic drugs, etc.

Solution: Take the derivative of $x(t)$



Evidently it is the superposition of 4 impulse trains (each delta function has area +1 or −1) shown in hallucinogenic colors. If the standard even positive impulse strain is given by $p(t)$, with Fourier coefficients $p_k = 1/T = 1/6$ for all k, then we have

$$x'(t) = -p(t-1) + p(t-2) + p(t-4) - p(t-5)$$

So the Fourier coefficients of $x'(t)$ are

$$\begin{aligned} b_k &= p_k\left(-e^{jk\pi/3} - e^{-jk\pi/3} + e^{j2k\pi/3} + e^{j2k\pi/3}\right) \\ &= \frac{1}{3}\Big(-\cos(k\pi/3) + \cos(2k\pi/3)\Big) \end{aligned}$$

But $b_k = jk\omega_0 a_k = jk(\pi/3)a_k$ from earlier in the tut. So

$$a_k = \frac{j}{k\pi}\Big(\cos(k\pi/3) - \cos(2k\pi/3)\Big)$$

as before. □

Problem Set 4-4

The Fourier series coefficient of a continuous time signal with period $T = 4$ seconds is specified below.

$$a_k = \begin{cases} 0, & k = 0 \\ (-1)^k \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

(a) Determine and sketch the signal $x(t)$ using the properties of Fourier series (Module 2: slide 398) and the result of Example 3.5 in the textbook (this periodic rectangular wave example was also solved in Module 2: slide 379,380). Hint: $(-1)^k = e^{j\pi k}$.

Solution: For convenience, let us define the following Fourier series pairs,

$x(t) \longleftrightarrow a_k$
$y(t) \longleftrightarrow b_k$
$z(t) \longleftrightarrow b_k(-1)^k$
$g(t) \longleftrightarrow c_k$

First, consider $b_k = \frac{\sin(k\pi/4)}{k\pi}$.

Comparing b_k with the Fourier series coefficients for the periodic square wave, i.e., $\frac{\sin(k\omega_0 T_1)}{k\pi}$, we have $\omega_0 T_1 = \frac{\pi}{4}$. This implies $\frac{T_1}{T} = \frac{1}{8}$. Since it is given that $T = 4$, we have $T_1 = 1/2$ and $\omega_0 = \pi/2$.

Hence $y(t)$ is a periodic square wave as shown on L17:slide 2 with $T_1 = 1/2$ and $T = 4$.

Second, consider $y(t)$ for which $b_0 = \frac{2T_1}{T} = \frac{1}{4}$. However, for $x(t)$, $a_0 = 0$. Hence, we define $c_k = -\frac{1}{4}$ for $k = 0$ and $c_k = 0$ for $k \neq 0$. This implies $g(t)$ is a DC signal with value $g(t) = -\frac{1}{4}$.

Last, consider $b_k(-1)^k = b_k e^{jk\pi} = b_k e^{jk\frac{\pi}{2}2} \equiv b_k e^{-jk\omega_0 t_0}$. From the table of properties of Fourier series, this corresponds to a time shift of $t_0 = -2$, i.e., $z(t) = y(t - t_0) = y(t + 2)$.

Hence, $x(t) = z(t) + g(t) = y(t + 2) + g(t)$, where $y(t)$ is a periodic square wave as shown on Module 2:slide 379 with $T_1 = 1/2$ and $T = 4$.
The plot is shown below.

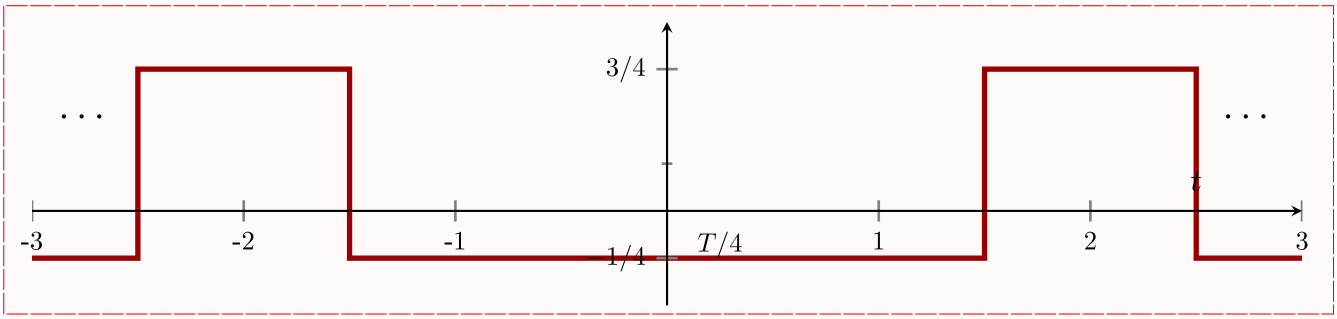


Figure 1: Periodic signal $x(t)$.

Problem Set 4-5

Suppose $x(t)$ is a periodic signal as given in Fig. 2 below with period $T = 6$ seconds.

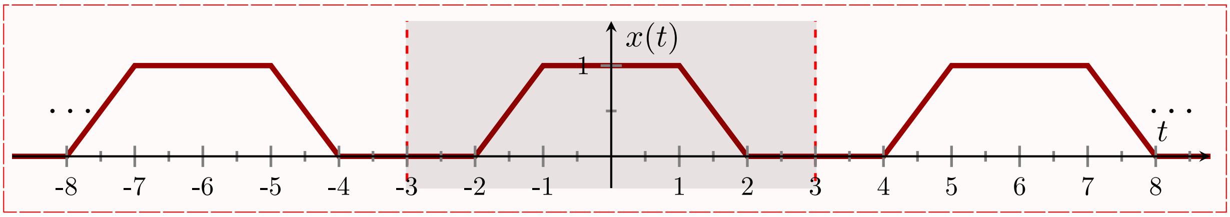


Figure 2: Periodic signal $x(t)$ for Problem 1.

Here,

$$x(t) = \begin{cases} 0, & -3 \le t \le -2 \\ t+2, & -2 \le t \le -1 \\ 1, & -1 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & 2 \le t \le 3 \end{cases}$$

(a) Find the value of a_0, that is,

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

Write a sentence to intuitively explain your answer.

Solution:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_T x(t)dt \\ &= \frac{1}{6}\left[\int_{-2}^{-1}(t+2)dt + \int_{-1}^{1}(1)dt + \int_{1}^{2}(2-t)dt\right] \\ &= \frac{1}{6}\left[\left|\frac{t^2}{2}+2t\right|_{-2}^{-1} + |t|_{-1}^{1} + \left|2t-\frac{t^2}{2}\right|_{1}^{2}\right] \\ &= \frac{1}{2} \end{aligned} \tag{1}$$

a_0 is the DC component of the signal and from the plot of $x(t)$ given, it is obvious that the average value if 1/2. □

(b) Determine the Fourier series coefficients for this signal, that is,

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Show that

$$a_k = \frac{6}{k^2\pi^2}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi k}{6}\right)$$

You must show your intermediate steps. Do not substitute the value of $\omega_0 = \frac{2\pi}{T}$ until the final step. You may wish to use all or some of the following results to help with your derivation:

$$\int te^{-jk\omega_0 t}dt = \frac{e^{-jkt\omega_0}(1+jkt\omega_0)}{k^2\omega_0^2}$$

$$\int e^{-jk\omega_0 t}dt = \frac{je^{-jkt\omega_0}}{k\omega_0}$$

$$\int_a^b e^{-jk\omega_0 t}dt = -\frac{j(e^{-jak\omega_0} - e^{-jbk\omega_0})}{k\omega_0}$$

$$\int_a^b (t+c)e^{-jk\omega_0 t}dt = \frac{e^{-ik\omega_0(a+b)}\left(e^{iak\omega_0}(1+ik\omega_0(b+c)) - ie^{ibk\omega_0}(k\omega_0(a+c)-i)\right)}{k^2\omega_0^2}$$

$$\int_a^b (-t+c)e^{-jk\omega_0 t}dt = \frac{e^{-ik\omega_0(a+b)}\left(e^{ibk\omega_0}(1+ik\omega_0(a-c))+e^{iak\omega_0}(-1-ik\omega_0(b-c))\right)}{k^2\omega_0^2}$$

$$\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Solution: $\omega_0 = \frac{\pi}{3}$. Abbreviated solution showing the main steps is:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-jk\omega_0 t}dt + \int_{-1}^{1} e^{-jk\omega_0 t}dt + \int_1^2 (2-t)e^{-jk\omega_0 t}dt\right] \\
&= \frac{e^{jk\omega_0 t}(1-e^{jk\omega_0 t}+jk\omega_0)}{6k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} + \frac{e^{-j2k\omega_0 t}(-1+e^{jk\omega_0 t}-jk\omega_0 e^{jk\omega_0 t})}{6k^2\omega_0^2} \\
&= \left(\frac{e^{jk\omega_0 t}(1-e^{jk\omega_0 t}+jk\omega_0)}{6k^2\omega_0^2} + \frac{e^{-j2k\omega_0 t}(-1+e^{jk\omega_0 t}-jk\omega_0 e^{jk\omega_0 t})}{6k^2\omega_0^2}\right) + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{e^{jk\omega_0 t}-e^{j2k\omega_0 t}+jk\omega_0 e^{jk\omega_0 t}-e^{-j2k\omega_0 t}+e^{-jk\omega_0 t}-jk\omega_0 e^{-jk\omega_0 t}}{6k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{(e^{jk\omega_0 t}+e^{-jk\omega_0 t})-(e^{j2k\omega_0 t}+e^{-j2k\omega_0 t})+jk\omega_0(e^{jk\omega_0 t}+e^{-jk\omega_0 t})}{6k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2}\right)-\left(\frac{e^{j2k\omega_0 t}+e^{-j2k\omega_0 t}}{2}\right)+jk\omega_0\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2}\right)}{3k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2}\right)-\left(\frac{e^{j2k\omega_0 t}+e^{-j2k\omega_0 t}}{2}\right)+j^2k\omega_0\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2j}\right)}{3k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2}\right)-\left(\frac{e^{j2k\omega_0 t}+e^{-j2k\omega_0 t}}{2}\right)-k\omega_0\left(\frac{e^{jk\omega_0 t}+e^{-jk\omega_0 t}}{2j}\right)}{3k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{\cos(k\omega_0)-\cos(2k\omega_0)-k\omega_0\sin(k\omega_0)}{3k^2\omega_0^2} + \frac{\sin(k\omega_0)}{3k\omega_0} \\
&= \frac{\cos(k\omega_0)-\cos(2k\omega_0)}{3k^2\omega_0^2} \\
&= \frac{2}{3k^2\omega_0^2}\sin\left(\frac{3k\omega_0}{2}\right)\sin\left(\frac{k\omega_0}{2}\right) = \frac{6}{k^2\pi^2}\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi k}{6}\right)
\end{aligned}$$

□

Fourier Series Properties of CT Periodic Signals

Problem Set 4-6

Suppose $x(t)$ is a periodic signal given as in Fig. 3 below with period T.

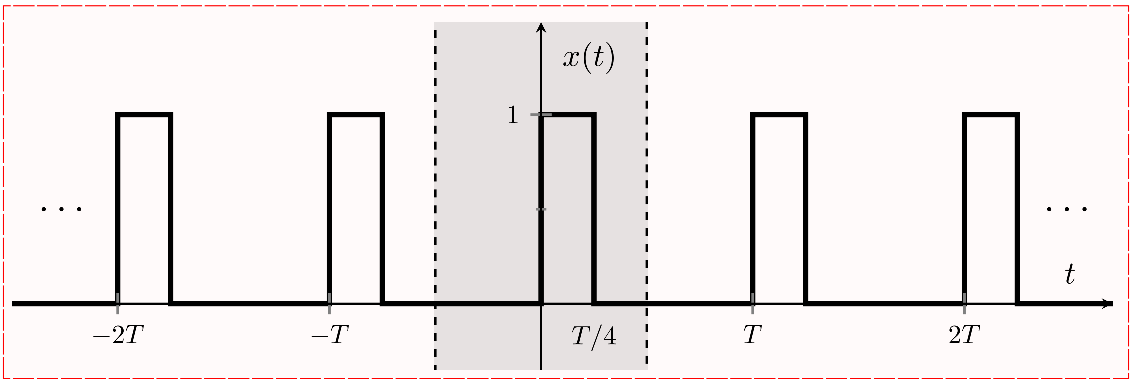


Figure 3: Periodic signal $x(t)$.

(a) Determine the Fourier series coefficients for this signal, that is,

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

using two different strategies. For example, you can use the direct integration as the first strategy and transform the periodic rectangular waveform from the lectures Part 9 as the second strategy.

Solution: *Strategy 1:* Directly, we have

$$\begin{aligned} a_k = \frac{1}{T}\int_0^{T/4} e^{-jk\,2\pi t/T}\, dt &= \frac{1}{-jk\,2\pi} e^{-jk\,2\pi t/T}\Big|_0^{T/4} \\ &= \frac{1}{2j\,k\pi}\left(1 - e^{-jk\,\pi/2}\right) \\ &= \frac{1}{k\pi} e^{-jk\,\pi/4} \frac{1}{2j}\left(e^{jk\,\pi/4} - e^{-jk\,\pi/4}\right) \end{aligned}$$

We identify the $\sin(k\pi/4)$ term to obtain

$$\boxed{a_k = \frac{\sin(k\pi/4)}{k\pi} e^{-jk\,\pi/4}}$$

Just for information: the exponential part encodes the delay. When written this way we can see the angle is $\pi/4$, which is $1/8$ of a full period 2π. So the delay is $T/8$.

Strategy 2: We use the Fourier Series time shift result (Part 10 page 353 in the lecture notes) on the standard periodic rectangular wave (Part 9 page 329 in the lecture notes). The standard periodic rectangular wave, call this $z(t)$, is an even function with $T_1 = T/8$ (meaning the pulse width is $2\,T_1 = T/4$) and, therefore, has Fourier Series coefficients ($z(t) \longleftrightarrow c_k$):

$$c_k = \frac{\sin(k\pi/4)}{k\pi}$$

noting that $k\omega_0 T_1 = k\pi/4$. The Fourier Series time shift says that if $z(t) \longleftrightarrow c_k$ then $z(t - t_0) \longleftrightarrow c_k e^{-jk\,2\pi\,t_0/T}$. We observe that $x(t) = z(t - T/8)$ and, therefore,

$$\boxed{a_k = \frac{\sin(k\pi/4)}{k\pi} e^{-jk\,\pi/4}}$$

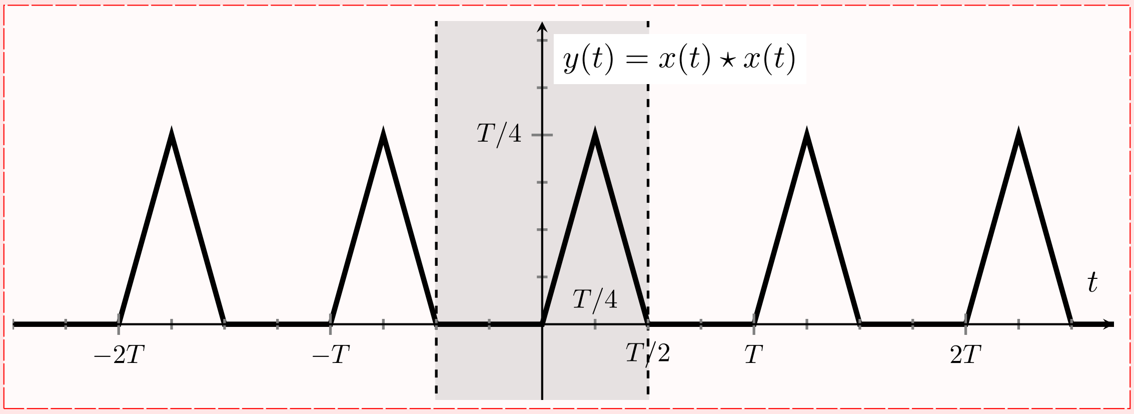
as before. □

(b) Consider the periodic convolution of $x(t)$ with itself, that is,

$$y(t) = x(t) \star x(t)$$

Determine the signal $y(t)$, plot and compare with signal $x(t)$. Part 10 of the lectures should be useful here.

Solution: The convolution of a rectangular pulse with itself is a triangular pulse of twice the width and its height is the area of a rectangle: height $1^2 = 1$ times width $T/4$ equals $T/4$. The rectangular pulses in the pulse train $x(t)$ are skinny enough not to interfere with each other so we end up with a triangular pulse train $y(t)$ shown below. The centre of the triangular pulse $y(t)$ offset $T/4$ is twice that of the rectangular pulse $x(t)$ offset $T/8$.



Explicitly

$$y(t) = \int_T x(\tau)\, x(t - \tau)\, d\tau$$

and one can compute $y(t)$ using this as an alternative. □

(c) For the signal $y(t)$ in part (b) determine its Fourier series.

Solution: Generally, convolution in time yields multiplication in frequency. Let $y(t) \longleftrightarrow c_k$ then since $y(t) = x(t) \star x(t)$ and $x(t) \longleftrightarrow a_k$, so we just need to multiply each Fourier series coeffiicents with itself (and multiply by T, which is the only part not obvious),

$$\boxed{c_k = T\,(a_k)^2 = T\,\frac{\sin^2(k\pi/4)}{(k\pi)^2} e^{-jk\,\pi/2}.}$$

using Part 10 page 373 in the lecture notes.

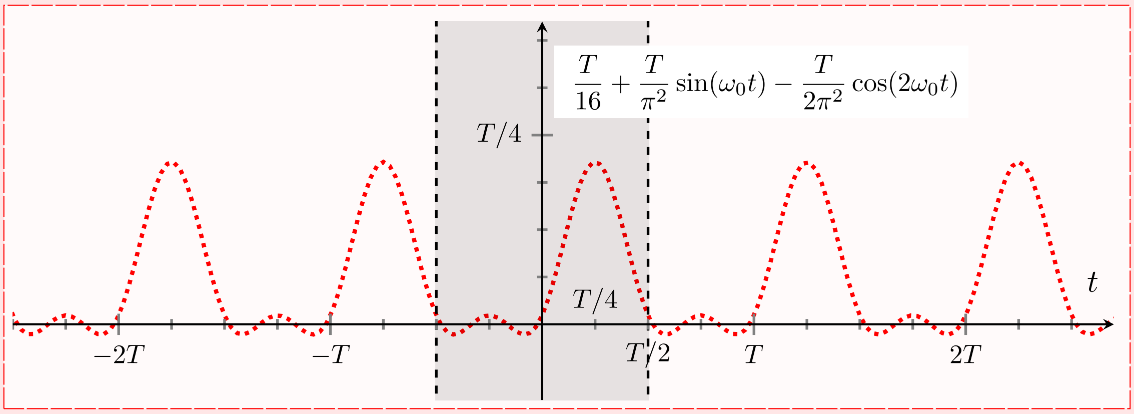
Embellishment: This is not part of the required answer but we might like to confirm these Fourier coefficients work as expected. Note that

$$\begin{aligned} c_0 &= T/16 \\ c_1 &= -jT/(2\pi^2) \\ c_{-1} &= jT/(2\pi^2) \\ c_2 &= -T/(4\pi^2) \\ c_{-2} &= -T/(4\pi^2), \text{ etc.} \end{aligned}$$

So we have

$$\begin{aligned} y(t) &= \frac{T}{16} - \frac{jT}{2\pi^2} e^{jw_0 t} + \frac{jT}{2\pi^2} e^{-jw_0 t} - \frac{T}{4\pi^2} e^{j2w_0 t} - \frac{T}{4\pi^2} e^{-j2w_0 t} \\ &= \frac{T}{16} + \frac{T}{\pi^2}\sin(\omega_0 t) - \frac{T}{2\pi^2}\cos(2\omega_0 t) + \cdots \end{aligned}$$

and the sum of up to and including the second harmonic is plotted below and indeed already begins to approximate $y(t)$ well.



A simple MATLAB script, `foursynth.m`, for doing similar plots with any number of harmonics and any period T:

```
function foursynth( kmax , T )
    k=[-kmax:kmax]; % Fourier coefficient index span
    omega0=2*pi/T; % fundamental frequency (rad/sec); T - period

    k=k+0.0001; % lazy way to handle sinc at zero
    ak=sin(k*pi/4)./(k*pi).*exp(-j*k*pi/4);
    ck=T*ak.*ak;

    t=T*[-2.5:0.01:2.5]; % time scan for plot (5 periods)
    ekt=exp(j*omega0*kron(k,t')); % k horz; t vert

    yt=ck*ekt.'; % sum the series; .' means non-conjugate transpose

    plot(t,real(yt)); shg
end
```

This is called using `foursynth(2,1)` — yielding something analogous to plot above — or try `foursynth(10,1)` to get something much closer to $y(t)$. □

Problem Set 4-7

Suppose we are given following information about a signal $x(t)$

1. $x(t)$ is real and odd
2. $x(t)$ is periodic with period $T = 2$
3. The Fourier coefficients are a_k, such that $a_k = 0$ for $k > 1$
4. $\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$

Specify two different signals that satisfy these conditions.

Solution: Since $x(t)$ is real and odd its Fourier coefficients a_k are purely imaginary and odd. Hence, $a_{-k} = -a_k$ and $a_0 = 0$. As $a_k = 0$ for $k > 0$, the only unknown coefficients are a_1 and a_{-1} with the property $a_{-1} = -a_1$.

The Parseval relation for continuous-time periodic signals yields

$$\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1 = \sum_{k=-\infty}^{\infty} |a_k|^2 = |a_{-1}|^2 + |a_1|^2$$

So the two signals are

$$x_1(t) = \sqrt{2}\sin(\pi t)$$

and

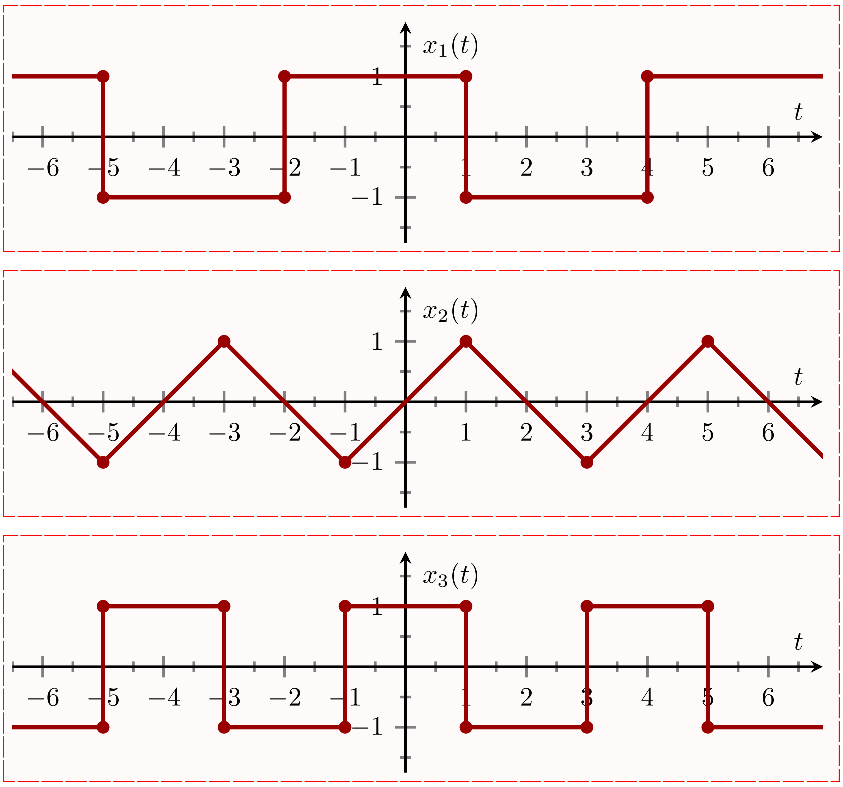
$$x_2(t) = -\sqrt{2}\sin(\pi t),$$

noting that $\omega_0 = \pi$ for $T = 2$. □

Problem Set 4-8

Without evaluating the Fourier series coefficients, find which of the following periodic signals have Fourier coefficients with the following properties:

1. Only odd harmonics
2. Only real harmonics
3. Only imaginary harmonics



Solution:

1. $x_2(t)$
2. $x_3(t)$
3. $x_2(t)$

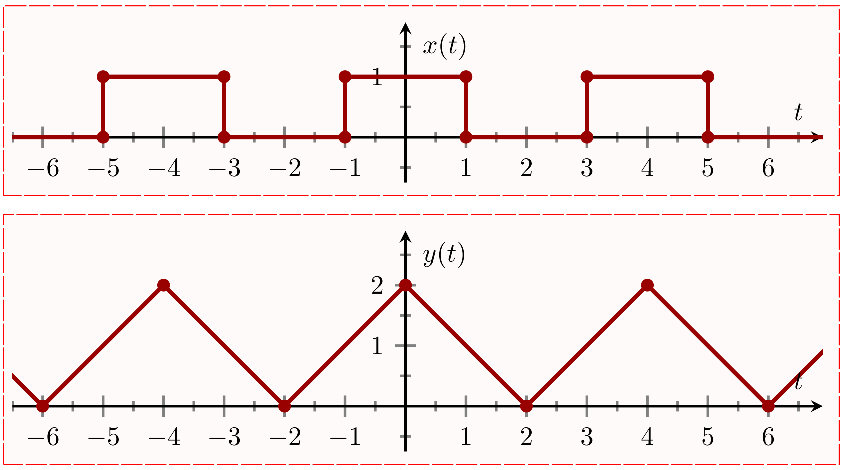
$\square$

Problem Set 4-9

In the figure below $x(t)$ is a periodic rectangular wave with period $T = 4$ and has the Fourier series coefficients

$$a_0 = \frac{1}{2}, \qquad a_k = \frac{\sin(k\pi/2)}{k\pi}.$$

Using these Fourier series coefficients of $x(t)$, find the Fourier series coefficients, b_k, of the triangular wave with period 4, $y(t)$, as shown in the figure.



Solution: Here $y(t) = x(t) \star x(t)$, $T = 4$ and so

$$\begin{aligned} b_k &= T\, a_k a_k = 4\, a_k^2 \\ &= 4\frac{\sin^2(k\pi/2)}{k^2\pi^2} \end{aligned}$$

and Bob's your uncle. Check the scaling: from the Figure for $y(t)$ the DC value is $b_0 = 1$, and from the algebra we use $\sin(x)/x = 1$ in the limit as x approaches 0 and confirm that $b_0 = 1$. $\square$

Problem Set 4-10

Let $x(t)$ be a periodic signal with fundamental frequency ω_0 and Fourier coefficients a_k, that is,

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{j\omega_0 t}.$$

Similarly for periodic

$$y(t) = \sum_{k=\infty}^{\infty} b_k e^{j\omega_0 t},$$

where the coefficients are b_k.

Find the Fourier coefficients b_k in terms of the Fourier coefficients a_k for the following signals.

(a) $y(t) = -2\,x(t) + j\,x(t)$

Solution: $b_k = (-2 + j)\,a_k$ by linearity/superposition of Fourier Series. You can write it out in full using the same method as shown in the next answers below. Also note that we should check that $y(t)$ is periodic with fundamental frequency ω_0 (or a multiple of ω_0). $\square$

(b) $y(t) = x(t-1)$

Solution:

$$\begin{aligned} x(t-1) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-1)} \\ &= \sum_{k=-\infty}^{\infty} \underbrace{e^{-jk\omega_0} a_k}_{b_k} e^{jk\omega_0 t} = y(t) \end{aligned}$$

Even that dunderhead who sits next to you in the lectures could get this right. $\square$

(c) $y(t) = x'(t) = \dfrac{d}{dt}x(t)$

Solution: Take the time-derivative of the Fourier Series synthesis equation

$$\begin{aligned} x'(t) &= \sum_{k=\infty}^{\infty} a_k \frac{d}{dt} e^{jk\omega_0 t} \\ &= \sum_{k=\infty}^{\infty} \underbrace{jk\omega_0\, a_k}_{b_k} e^{jk\omega_0 t} = y(t) \end{aligned}$$

Remember a_k is the kth harmonic which means its frequency is k times the fundamental frequency ω_0. That is, a_k is the weight for a frequency at $\omega = k\omega_0$.
Later we'll see that the frequency weighting when taking the derivative is always $j\omega$

$$\begin{aligned} a_k\delta(\omega - k\omega_0) \longrightarrow b_k\delta(\omega - k\omega_0) &= j\omega\, a_k\delta(\omega - k\omega_0) \\ &= j\omega_0\, a_k\delta(\omega - k\omega_0) \end{aligned}$$

The energy in this kth harmonic is $|a_k|^2$ and having gone through a differentiator (system) the energy becomes $|b_k|^2 = (k\omega_0)^2 |a_k|^2$. $\square$

(d) $y(t) = x(1-t)$

Solution: Same strategy as $y(t) = x(t-1)$...

$$\begin{aligned} x(t-1) &= \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0(1-t)} \\ &= \sum_{k=-\infty}^{\infty} e^{jk\omega_0} a_k e^{-jk\omega_0 t} \end{aligned}$$

Hmmmmm, what to do with the minus in the exponent? Just reverse it.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} e^{jk\omega_0} a_k e^{-jk\omega_0 t} &= \sum_{k=\infty}^{-\infty} e^{-jk\omega_0} a_{-k} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} \underbrace{e^{-j\omega_0} a_{-k}}_{b_k} e^{j\omega_0 t} = y(t) \end{aligned}$$

Someone is bored. $\square$

(e) $y(t) = x^2(t)$

Solution: Hmmmmm, convolution thingy in the other domain because we have multiply thingy in the time-domain. $x^2(t) = x(t) \times x(t)$. We can use the "Multiplication Property"

$$b_k = a_k \star a_k$$

what does that mean exactly? It's shorthand.

$$\begin{aligned} a_k \star a_k &= \sum_{\ell=-\infty}^{\infty} a_\ell a_{k-\ell} \\ &= \cdots + a_{-2}a_{k+2} + a_{-1}a_{k+1} + a_0 a_k + a_1 a_{k-1} + a_2 a_{k-2} + \cdots = b_k \end{aligned}$$

OK now I like the shorthand. □

Problem Set 4-11

A normal mains voltage waveform versus time is shown as $x(t)$ in the figure below.

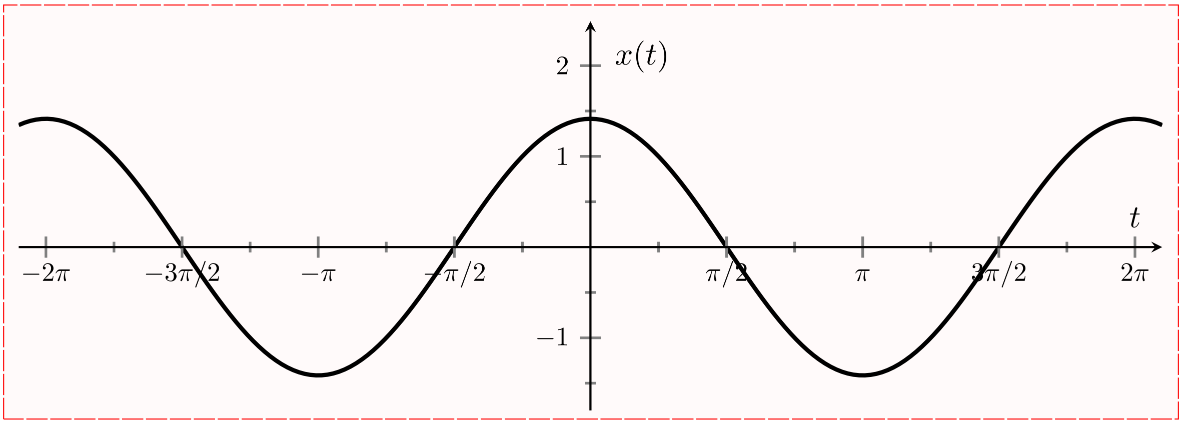


Figure 4: Normal mains voltage waveform (normalized).

Normally the voltage is 230 volts which is an RMS measure the peak voltage is thereby $230\sqrt{2}$ and the frequency of oscillation is 50 Hz or $\omega_0 = 100\pi$ rad/sec. For simplicity for this question the peak value is taken as $\sqrt{2}$, the fundamental period is $T_0 = 2\pi$ and fundamental frequency $\omega_0 = 1$.

(a) With $x(t) = \sqrt{2}\cos(t)$, show the average power per period of $x(t)$ is 1.

Solution:

$$\begin{aligned} \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sqrt{2}\cos(t)\right)^2 dt \\ &= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} 2\,\frac{1}{2}\,dt}_{1} + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} 2\,\frac{1}{2}\cos(2t)\,dt}_{0} = 1 \end{aligned}$$

since $\cos^2(t) = (1/2) + (1/2)\cos(2t)$. □

(b) Modern light dimmers work by gating (or chopping up) the main voltage waveform $x(t)$. Normally these are called trailing-edge and leading edge dimmers. For this problem we simplify the action of the dimmer to be like a combination of both trailing and leading edge dimmers, generating the periodic signal $y_\alpha(t)$ as shown in Fig. 5.

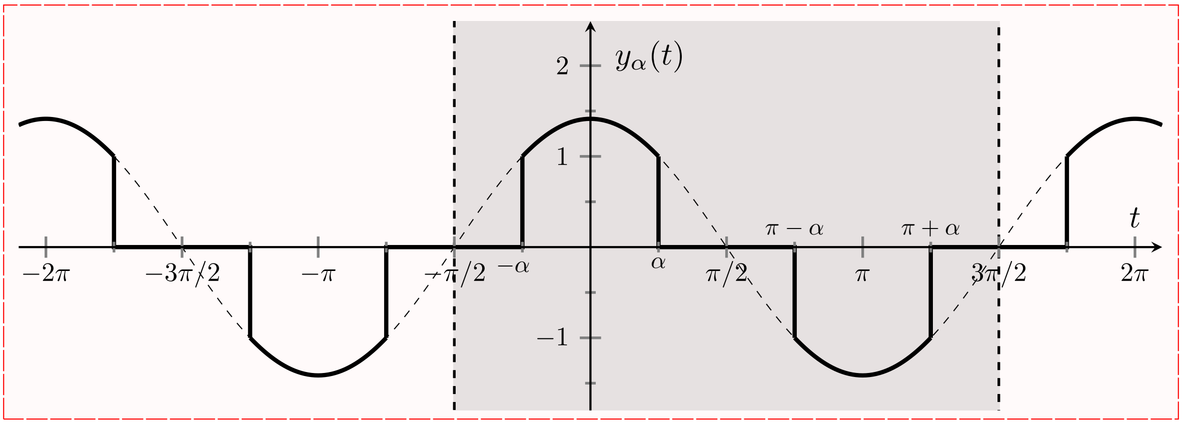


Figure 5: Gated mains voltage waveform for dimming, where $\alpha \in [0, \pi/2]$ adjusts the dimming.

Mathematically we can define $y_\alpha(t)$ over one period, and it is convenient to take the interval as $[-\pi/2, 3\pi/2]$ (shown shaded in Fig. 5):

$$y_\alpha(t) = \begin{cases} x(t) & t \in [-\alpha, \alpha] \cup [\pi - \alpha, \pi + \alpha] \\ 0 & \text{otherwise} \end{cases}, \quad t \in [-\pi/2, 3\pi/2]$$

and $y_\alpha(t + 2\pi) = y_\alpha(t)$. A valid range of values for the parameter α is

$$0 \le \alpha \le \pi/2 \quad \text{or} \quad \alpha \in [0, \pi/2]$$

and corresponds to the dimmer dial setting.

Find as a function of α the average power per period

$$P(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \left|y_\alpha(t)\right|^2 dt$$

and confirm that $P(0) = 0$ and $P(\pi/2) = 1$.

Solution:

$$\begin{aligned}
P(\alpha) &= \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \left|y_\alpha(t)\right|^2 dt \\
&= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left(\sqrt{2}\cos(t)\right)^2 dt + \frac{1}{2\pi} \int_{\pi-\alpha}^{\pi+\alpha} \left(\sqrt{2}\cos(t)\right)^2 dt \\
&= 2 \times \frac{1}{2\pi} \int_{-\alpha}^{\alpha} 2\cos^2(t)\, dt \quad \text{(equal contributions from each integral)} \\
&= \frac{1}{\pi} \int_{-\alpha}^{\alpha} 1 + \cos(2t)\, dt \\
&= \frac{2\alpha}{\pi} + \frac{1}{\pi}\left[\frac{1}{2}\sin(2t)\Big|_{-\alpha}^{\alpha}\right. \\
&= \frac{2\alpha}{\pi} + \frac{1}{\pi}\left[\frac{1}{2}\sin(2\alpha) - \frac{1}{2}\sin(-2\alpha)\right]
\end{aligned}$$

That is,

$$\boxed{P(\alpha) = \frac{2\alpha}{\pi} + \frac{\sin(2\alpha)}{\pi}, \quad \alpha \in [0, \pi/2].}$$

If $\alpha = 0$ then indeed $P(0) = 0$. If $\alpha = \pi/2$ then indeed $P(\pi/2) = 1$. $P(\alpha)$ is monotonically increasing on its domain $\alpha \in [0, \pi/2]$. □

(c) Both $x(t) = \sqrt{2}\cos(t)$ and $y_\alpha(t)$ are periodic with the same fundamental frequency $\omega_0 = 1$ and both have zero DC component. The *total harmonic distortion* (THD) is the ratio of the power per period of the harmonics $|k| > 1$ divided by the power per period in the first harmonic components $k = \pm 1$ (or $|k| = 1$).

Compute the total harmonic distortion (THD) as a function of α of $y_\alpha(t)$, that is,

$$\mathrm{THD}(\alpha) = \frac{\sum_{k=-\infty}^{-2}\left|b_k(\alpha)\right|^2 + \sum_{k=2}^{\infty}\left|b_k(\alpha)\right|^2}{\left|b_{-1}(\alpha)\right|^2 + \left|b_1(\alpha)\right|^2}$$

where the Fourier Series coefficients are given by

$$b_k(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} y_\alpha(t)\, e^{-jk\omega_0 t}\, dt.$$

where $\omega_0 = 1$ and $b_0(\alpha) = 0$.

(You should probably want to use Parseval's Relation, as given in Module 2: slide 398, unless you are a glutton for punishment. Also note that both $x(t)$ and $y_\alpha(t)$ are even real-valued functions.)

Solution: Parseval's relation tells us the time domain power per period is equal to the sum power in the Fourier coefficients. So we can infer

$$\boxed{P(\alpha) = \frac{2\alpha}{\pi} + \frac{\sin(2\alpha)}{\pi} = \sum_{k=-\infty}^{\infty} \left|b_k(\alpha)\right|^2.}$$

Clearly we have zero DC:

$$P_0(\alpha) \triangleq b_0(\alpha) = 0.$$

So to answer the question we just need to determine the first harmonic(s) of $y_\alpha(t)$, which are (note that $\omega_0 = 1$)

$$b_{\pm 1}(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} y_\alpha(t)\, e^{\mp jt}\, dt.$$

Now $y_\alpha(t)$ is an even real-valued function and the integration can be over any single period and the clever choice is $[-\pi, \pi]$. Evidently, the two coefficients are given by

$$\begin{aligned} b_{\pm 1}(\alpha) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} y_\alpha(t) \big(\cos(t) \mp j \sin(t)\big)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} y_\alpha(t)\, \cos(t)\, dt \\ &= 2 \times \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \sqrt{2} \cos(t)\, \cos(t)\, dt \\ &= \frac{1}{\pi\sqrt{2}} \int_{-\alpha}^{\alpha} 1 + \cos(2t)\, dt = \frac{2\alpha}{\pi\sqrt{2}} + \frac{\sin(2\alpha)}{\pi\sqrt{2}}. \end{aligned}$$

That is, the first harmonics are real-valued and equal, and given by

$$\boxed{b_1(\alpha) = b_{-1}(\alpha) = \frac{1}{\pi\sqrt{2}} \big(2\alpha + \sin(2\alpha)\big).}$$

Hence the total power in the first harmonic(s) is

$$\begin{aligned} P_1(\alpha) &\triangleq \big|b_{-1}(\alpha)\big|^2 + \big|b_1(\alpha)\big|^2 \\ &= 2\Big(\frac{1}{\pi\sqrt{2}}\Big)^2 \big(2\alpha + \sin(2\alpha)\big)^2 \end{aligned}$$

That is,

$$\boxed{P_1(\alpha) = \frac{1}{\pi^2} \big(2\alpha + \sin(2\alpha)\big)^2.}$$

Next we can rewrite the total harmonic distortion as

$$\begin{aligned} \mathrm{THD}(\alpha) &= \frac{P(\alpha) - P_1(\alpha)}{P_1(\alpha)} \\ &= \frac{P(\alpha)}{P_1(\alpha)} - 1 \\ &= \frac{\frac{1}{\pi}\big(2\alpha + \sin(2\alpha)\big)}{\frac{1}{\pi^2}\big(2\alpha + \sin(2\alpha)\big)^2} - 1 \end{aligned}$$

Hence

$$\boxed{\mathrm{THD}(\alpha) = \frac{\pi}{2\alpha + \sin(2\alpha)} - 1, \quad \alpha \in [0, \pi/2].}$$

For $\alpha = \pi/2$, $\mathrm{THD}(\pi/2) = 0$ because it is a pure sinusoid with no higher harmonics. For $\alpha \to 0$, $\mathrm{THD}(\alpha) \to \infty$ because $y_\alpha(t)$ starts looking like an impulse train, which tends to distribute power equally across all harmonics. If $\alpha = 0$ one can argue $y_0(t) = 0$ and regard the THD as being undefined. □

Problem Set 4-12

Find the output $y(t)$ of a causal LTI system for the periodic input $x(t) = \cos(2\pi t)$, where

$$\frac{d}{dt} y(t) + 4y(t) = x(t)$$

Solution: Solution in frequency domain:

Let $x(t) = e^{j\omega t}$ (eigenfunction) therefore $y(t) = H(j\omega)e^{j\omega t}$.

Differentiating

$$\frac{d}{dt} y(t) = j\omega e^{j\omega t} H(j\omega)$$

Substituting into the differential equation:

$$j\omega e^{j\omega t}H(j\omega) + 4H(j\omega)e^{j\omega t} = e^{j\omega t}$$
$$j\omega H(j\omega) + 4H(j\omega) = 1$$

Therefore

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

Finding coefficients of $x(t)$:

$x = \cos(2\pi t)$, $\omega_0 = 2\pi$ and $T = 1$

$$x = \cos(2\pi t) = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2}$$

Comparing with $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$:

$a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$ and $a_k =$ otherwise

Finding coefficients of $y(t)$:

$b_k = a_k H(jk\omega_0)$ therefore:

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(j2\pi + 4)}$$

$$b_{-1} = a_{-1} H(-j2\pi) = \frac{1}{2(-j2\pi + 4)}$$

Now

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\ &= \frac{1}{2(j2\pi + 4)} e^{j2\pi t} + \frac{1}{2(-j2\pi + 4)} e^{-j2\pi t} \\ &= 0.0671 e^{-j1.004} e^{j2\pi t} + 0.0671 e^{j1.004} e^{-j2\pi t} \\ &= 0.134 \cos(2\pi t - 1.004) \\ &= 0.134 \cos(2\pi t)\cos(1.004) + 0.134 \sin(2\pi t)\sin(1.004) \\ &= 0.072 \cos(2\pi t) + 0.113 \sin(2\pi t) \end{aligned}$$

This is the steady state output.

Time domain solution:

In the time domain, we would have to solve the differential equation to find $h(t)$ (there seems no obvious way to do so) and then convolve $h(t)$ with $x(t) = \cos(2\pi t)$

Fourier Analysis and Synthesis for Discrete-time Periodic Signals

Problem Set 4-13

Suppose $x[n]$ is a periodic signal as given in Fig. 6 below with period $N = 10$ seconds.

(a) Show that

$$a_k = -\frac{j}{5} \sin\left(\frac{\pi k}{2}\right) \frac{\sin\left(\frac{2\pi k}{5}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$

Hint: Introduce a +ve and a -ve unit sample at $n = 0$ so series sums can be calculated using the series sum formulas.

Solution: $N = 10$ and $\omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$. Abbreviated solution showing the main steps is:

$$\begin{aligned}
a_k &= \frac{1}{10}\sum_{n=-4}^{5} x[n]e^{-jk\pi n/5} \\
&= -\frac{1}{10}\sum_{n=-4}^{0} e^{-jk\pi n/5} + \frac{1}{10}\sum_{n=0}^{4} e^{-jk\pi n/5} \\
&= \frac{1}{10}\sum_{n=0}^{4} e^{-jk\pi n/5} - \frac{1}{10}\sum_{n=-4}^{0} e^{-jk\pi n/5} \quad \text{(just swapping the order in which the terms are written)} \\
&= \frac{1}{10}\sum_{n=0}^{4} e^{-jk\pi n/5} - \frac{1}{10}\sum_{m=0}^{4} e^{-jk\pi(m-4)/5} \quad \text{(substituting } m = n+4\text{, i.e., } n = m-4\text{)} \\
&= \frac{1}{10}\sum_{n=0}^{4} e^{-jk\pi n/5} - \frac{1}{10}e^{jk\pi 4/5}\sum_{m=0}^{4} e^{-jk\pi m/5} \\
&= \left(\frac{1}{10}\right)\left(\frac{1-e^{-j\pi k}}{1-e^{-jk\pi/5}}\right) - \left(\frac{1}{10}\right)(e^{jk\pi 4/5})\left(\frac{1-e^{-j\pi k}}{e^{-jk\pi/5}}\right) \\
&= \left(\frac{1}{10}\right)\left(\frac{1-e^{-j\pi k}}{1-e^{-jk\pi/5}}\right)\left(1-e^{jk\pi 4/5}\right) \\
&= \left(\frac{1}{10}\right)\left(\frac{e^{-jk\pi/2}(e^{jk\pi/2}-e^{-jk\pi/2})}{e^{-jk\pi/10}(e^{jk\pi/10}-e^{-jk\pi/10})}\right)\left(\frac{e^{+j2k\pi/5}(e^{-j2k\pi/5}-e^{+j2k\pi/5})}{2j}(2j)\right) \\
&= \left(\frac{1}{10}\right)\left(\frac{e^{-jk\pi/2}(e^{jk\pi/2}-e^{-jk\pi/2})}{e^{-jk\pi/10}(e^{jk\pi/10}-e^{-jk\pi/10})}\right)\left(\frac{e^{+j2k\pi/5}(e^{+j2k\pi/5}-e^{-j2k\pi/5})}{2j}(-2j)\right) \\
&= \left(\frac{1}{10}\right)\left(\frac{e^{-jk\pi/2}\sin\left(\frac{\pi k}{2}\right)}{e^{-jk\pi/10}\sin\left(\frac{\pi k}{10}\right)}\right)\left((e^{+j2k\pi/5})\sin\left(\frac{2\pi k}{5}\right)(-2j)\right) \\
&= -\frac{j}{5}\frac{\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{2\pi k}{5}\right)}{\sin\left(\frac{\pi k}{10}\right)}(e^{-jk\pi/2}e^{+jk\pi/10}e^{+j2k\pi/5}) \\
&= -\frac{j}{5}\frac{\sin\left(\frac{\pi k}{2}\right)\sin\left(\frac{2\pi k}{5}\right)}{\sin\left(\frac{\pi k}{10}\right)} \quad \text{(Since } e^{jk(-\pi/2+\pi/10+2\pi/5)} = e^{jk(0)} = 1\text{)}
\end{aligned}$$

Problem Set 4-14

Determine the DTFS coefficients of the following periodic signals using inspection method:

(a)

$$x[n] = 1 + \sin\left(\frac{n\pi}{12} + \frac{3\pi}{8}\right)$$

Solution: The DTFS of $x(t)$ a_k are:

$$a_k = \begin{cases} \frac{-e^{-j\frac{3\pi}{8}}}{2j} & k = -1 \\ 1 & k = 0 \\ \frac{e^{-j\frac{3\pi}{8}}}{2j} & k = 1 \\ 0 & \text{otherwise on} -11 \leq \text{k} \leq 12 \end{cases}$$

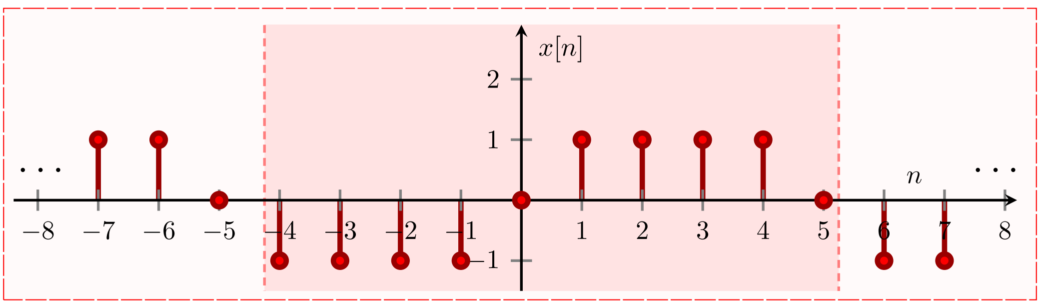


Figure 6: Signal $x[n]$ with period $N = 10$ for Problem 3.

(b)

$$x[n] = 1 + \cos\left(\frac{n\pi}{30}\right) + 2\sin\left(\frac{n\pi}{90}\right)$$

Solution: The DTFS of $x(t)$ a_k are:

$$a_k = \begin{cases} \frac{-1}{j} & k = -1 \\ \frac{1}{j} & k = 1 \\ \frac{1}{2} & k = \pm 3 \\ 0 & \text{otherwise on} -89 \leq \text{k} \leq 90 \end{cases}$$

Problem Set 4-15

Using Matlab, find the time domain signal $x[n]$ corresponding to the DTFS coefficients

$$a_k = \cos\left(\frac{k4\pi}{11}\right) + 2j\sin\left(\frac{k6\pi}{11}\right)$$

- This problem is just too cumbersome to solve by hand.
- Hints: You have to find N first. Show that a_k is periodic with period $N = 11$. Use the DTFS synthesis equation, summing from $k = -5$ to 5. Evaluate for each value of n.
- For some values of n, due to finite machine precision, Matlab may give an answer which is very very small (1e-15 or 1e-16) which means the value is 0.

Solution:

Matlab script:

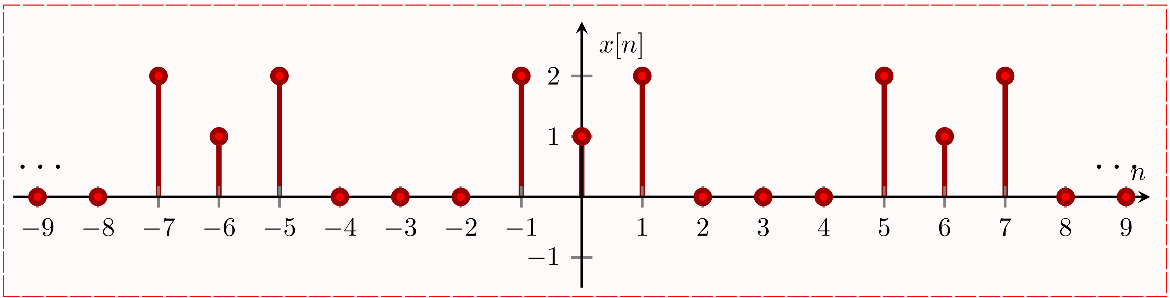
```
clc
clear all
%% change value of n in the range -5<n<5 below to see value of xn
n = -3;
N = 11;
w0 = 2*pi/N;
k= -5:1:5;
ak = cos(4.*pi.*k./11)+2.*j.*sin(6.*pi.*k./11);
temp = exp(j.*n.*k.*w0);
xn= sum(ak.*temp)
```

which gives:

$x[n] = \frac{11}{2}$, $n = 2$ or -2 $\quad x[n] = -11$, $n = 3$ $\quad x[n] = 11$, $n = -3$ $\quad x[n] = 0$ otherwise on $-5 < n < 5$

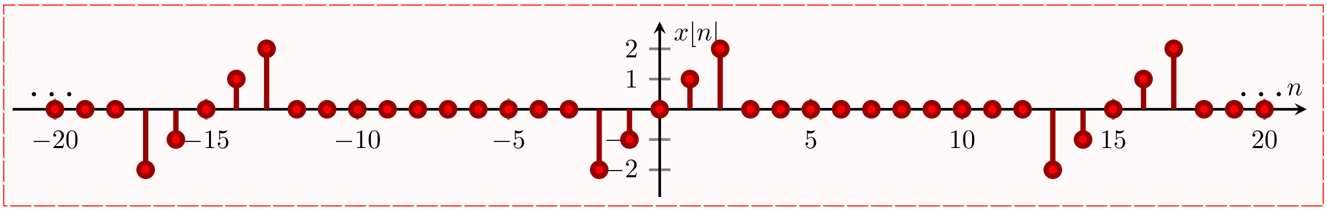
Problem Set 4-16

Determine the DTFS coefficients of the periodic signals depicted in the figures below using the DTFS analysis equation, determine N for each plot first.



(a) **Solution:**

$$a_k = \frac{1}{6} + \frac{2}{3}\cos\left(\frac{k\pi}{3}\right)$$



(b) **Solution:**

$$a_k = \frac{-2j}{15}\left(\sin\left(\frac{k2\pi}{15}\right) + 2\sin\left(\frac{k4\pi}{15}\right)\right)$$

Fourier Series Properties of DT Periodic Signals

Problem Set 4-17

Find the output $y[n]$ of a causal LTI system for the periodic input $x[n] = \cos\frac{n\pi}{6}$, where

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Solution: Finding the coefficients of $x[n]$:

$\omega_0 = \frac{\pi}{6}$ and $N = \frac{2\pi m}{\omega_0} = 12m$. For $m = 1$, $N_0 = 12$ samples (fundamental period)

$$x[n] = \frac{e^{j\frac{\pi n}{6}} + e^{-j\frac{\pi n}{6}}}{2}$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=-5}^{6} a_k e^{jk\frac{\pi}{6}n}$$

Comparing: $a_1 = a_{-1} = \frac{1}{2}$

Finding the coefficients of $y[n]$:

$$b_k = a_k H(e^{jk\omega_0})$$

$$b_1 = \frac{1}{2}\frac{1}{1 - \frac{1}{2}e^{-j\frac{\pi}{6}}} = \frac{1}{2 - e^{-j\frac{\pi}{6}}}$$

$$b_{-1} = \frac{1}{2}\frac{1}{1 - \frac{1}{2}e^{j\frac{\pi}{6}}} = \frac{1}{2 - e^{j\frac{\pi}{6}}}$$

Now:

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N-1} b_k e^{jk\omega_0 n} \\
&= \frac{1}{2 - e^{-j\frac{\pi}{6}}} e^{j\frac{\pi}{6}n} + \frac{1}{2 - e^{j\frac{\pi}{6}}} e^{-j\frac{\pi}{6}n} \\
&= 0.8069 e^{-j0.4152} e^{j\frac{\pi}{6}n} + 0.8069 e^{j0.4152} e^{-j\frac{\pi}{6}n} \\
&= 0.8069 e^{j\left(\frac{\pi}{6}n - 0.4152\right)} + 0.8069 e^{-j\left(\frac{\pi}{6}n - 0.4152\right)} \\
&= 1.6138\cos\left(\frac{\pi}{6}n - 0.4152\right)
\end{aligned}$$

For DT, output is expressed in the following form using trig identity:

$$y[n] = 1.47668\cos\left(\frac{\pi}{6}n\right) + 0.65\sin\left(\frac{\pi}{6}n\right)$$

In the time domain:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \cos\left(\frac{\pi}{6}n\right) * \left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

This is too combersome to evaluate by hand. Students can try it out or use a computer program to evaluate to get the same answer as using the frequency domain method.

Frequency Response of Discrete-time Filters

The following problems involve the $N = 4$ DT periodic signal $x[n]$ and DT pulse response $h[n]$ of some LTI system, shown in the figures below. For $x[n]$ the values in the shaded region covers one period and are repeated indefinitely for both positive and negative n.

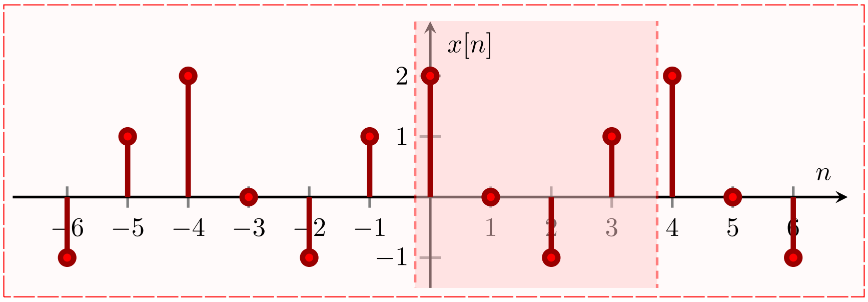


Figure 7: Signal $x[n]$ with period $N = 4$.

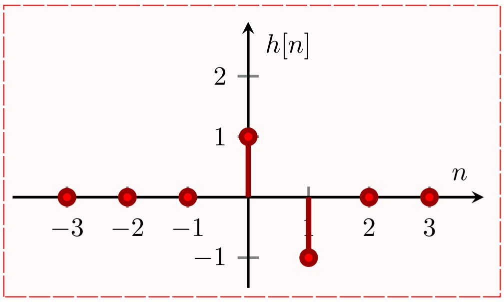


Figure 8: Pulse response $h[n]$ of DT LTI system.

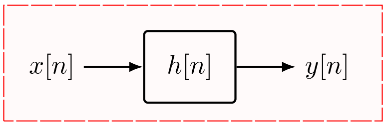


Figure 9: DT LTI system with pulse response $h[n]$, input $x[n]$ and output $y[n]$.

Problem Set 4-18

Questions on Expressing the Signals Algebraically:

(a) Express $h[n]$ in terms of a superposition of time-shifted unit pulse signals $\delta[n]$.

Solution: From the formula you could write

$$\begin{aligned} h[n] &= \sum_{k=-\infty}^{\infty} h[k]\,\delta[n-k] \\ &= h[0]\,\delta[n-0] + h[1]\,\delta[n-1] \end{aligned}$$

since there are only these two non-zero terms. Hence

$$\boxed{h[n] = \delta[n] - \delta[n-1].}$$

But of course you could just do this by observation. □

(b) Express $x[n]$ in terms of a superposition of shifted unit pulse signals $\delta[n]$.

Solution: Here the shaded portion corresponds to $k' = 0$, and k' indexes the length-4 time-span block that is repeated:

$$\boxed{x[n] = \sum_{k'=-\infty}^{\infty} \left(2\,\delta[n-4k'] - \delta[n-2-4k'] + \delta[n-3-4k']\right).}$$

□

Problem Set 4-19

Questions on the Convolution Output:

(a) Compute the DT convolution of $x[n]$ and $h[n]$

$$\begin{aligned} y[n] &= x[n] \star h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \end{aligned}$$

and give the answer in terms of a superposition of shifted unit pulse signals $\delta[n]$.

Solution: We can use $\delta[n-n_0] \star \delta[n-n_1] = \delta[n-n_0-n_1]$, which means a cascade of delay of n_0 and a delay of n_1 is equivalent to a single delay of n_0+n_1, or do it by observation:

$$x[n] \star h[n] = \sum_{k=-\infty}^{\infty} \big(2\,\delta[n-4k] - \delta[n-2-4k] + \delta[n-3-4k]\big) \star \big(\delta[n] - \delta[n-1]\big)$$

Hence

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} \big(\delta[n-4k] - 2\,\delta[n-1-4k] - \delta[n-2-4k] + 2\,\delta[n-3-4k]\big).}$$

Ugly but that's the answer. □

(b) Plot $y[n]$ using the template shown in Figure 10 (in a manner similar to Figure 7)

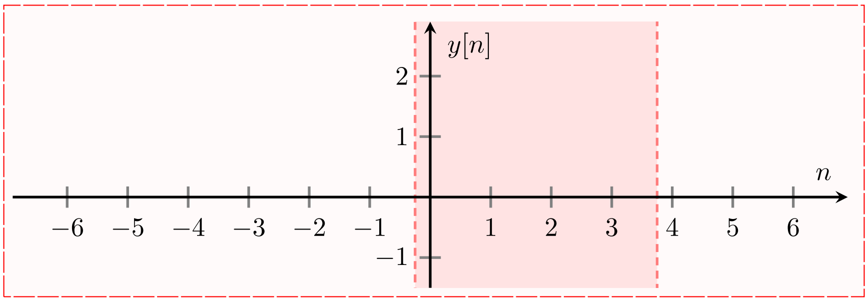
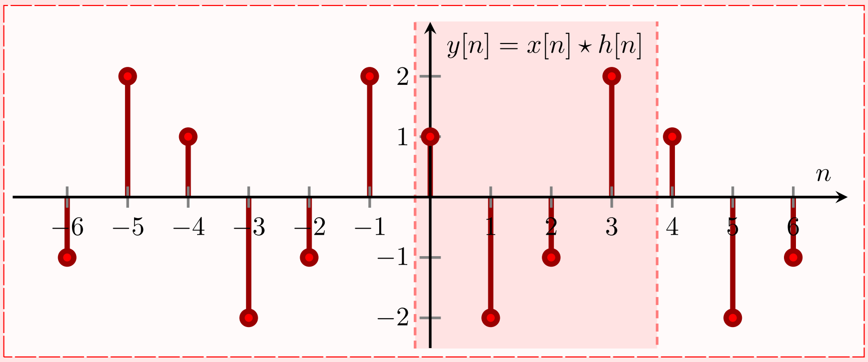


Figure 10: (Template) Signal $y[n]$ with period $N = 4$.

Solution:



□

Problem Set 4-20

Questions on the DC Gain of the System:

(a) What is the DC (zero frequency response) value of $x[n]$?

Solution: Average of $x[n]$ over one period, $N = 4$, is the sum of the values being $2 - 1 + 1 = 2$ divided by the period 4. That is, the DC value is $1/2$. □

(b) What can you say about the DC value of $y[n]$ and how does it relate to the DC gain of $h[n]$?

Solution: The DC value of $y[n]$ is zero because the frequency response of $h[n]$ is 0 at DC ($h[0] + h[1] = 0$).$\square$

Problem Set 4-21

Questions on DT Fourier Series:

(a) Four distinct complex exponentials that have period $N = 4$ are given by

$$\phi_k[n] = e^{j\pi kn/2}, \quad k = 0, 1, 2, 3,$$

and the Fourier series synthesis equation for $x[n]$ is then given by

$$x[n] = \sum_{k=0}^{3} a_k\, e^{j\pi kn/2}.$$

Determine the Fourier series coefficients a_0, a_1, a_2 and a_3 corresponding to $x[n]$ in Figure 7.

Solution: The analysis equation is

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi kn/2}, \quad k = 0, 1, 2, 3.$$

Then we get

$$4\,a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = (2 + 0 - 1 + 1) = 2$$

$$4\,a_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n/2} = 2 \times 1 + 0 - 1 \times e^{-j\pi} + 1 \times e^{-j\pi 3/2} = 3 + j$$

$$4\,a_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n} = 2 \times 1 + 0 - 1 \times e^{-j2\pi} + 1 \times e^{-j\pi 3} = 0$$

$$4\,a_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi 3n/2} = 2 \times 1 + 0 - 1 \times e^{-j\pi 3} + 1 \times e^{-j\pi 9/2} = 3 - j = \overline{4\,a_1}$$

So

$$\boxed{a_0 = \frac{1}{2},\ a_1 = \frac{3+j}{4},\ a_2 = 0,\ \text{and}\ \ a_3 = \frac{3-j}{4}.}$$

$\square$

(b) In the above, the $N = 4$ periodic signal $x[n]$ is characterized by 4 numbers, which by convention are taken as the four values shown in the shaded portion of Figure 7, and can be written as a 4-vector

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Similarly the Fourier series coefficients can be written as a 4-vector

$$\boldsymbol{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Determine the 16 entries, $\phi_{i,j}$, in the following 4×4 (analysis equation) matrix, $\boldsymbol{\Phi}$, that relates these two 4-vectors through the matrix equation

$$\boldsymbol{a} = \frac{1}{4}\boldsymbol{\Phi}\,\boldsymbol{x},$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \phi_{0,2} & \phi_{0,3} \\ \phi_{1,0} & \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,0} & \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,0} & \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \phi_{0,2} & \phi_{0,3} \\ \phi_{1,0} & \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,0} & \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,0} & \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Solution: The analysis equation with the above notation is

$$a_k = \frac{1}{4}\sum_{n=0}^{3} e^{-j\pi kn/2}\, x_n, \quad k \in \{0,1,2,3\}.$$

but this is just the equation for matrix multiplication with the matrix elements given by

$$\phi_{k,n} = e^{-j\pi kn/2}, \quad k,n \in \{0,1,2,3\},$$

and

$$\boxed{\mathbf{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \phi_{0,2} & \phi_{0,3} \\ \phi_{1,0} & \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,0} & \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,0} & \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}$$

□

(c) Confirm the values you got for the Fourier coefficients of $x[n]$ in the previous part by using the new matrix calculation. That is, compute $\frac{1}{4}\mathbf{\Phi}\,\boldsymbol{x}$.

Solution:

$$\boxed{\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/4 + j/4 \\ 0 \\ 3/4 - j/4 \end{bmatrix}.}$$

In fact if we write $\omega = \phi_{1,1}$ then

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix}$$

where $\omega = -j$. □

(d) One of the grim realities of life is trying to make sense of poorly documented material or material that uses different notations and conventions. Review the following documentation:

`http://www.mathworks.com.au/help/matlab/ref/fft.html`

and determine how the analysis equation calculations

$$\boldsymbol{a} = \frac{1}{4}\mathbf{\Phi}\,\boldsymbol{x},$$

performed above, are related to the MATLAB functions `Y=fft(x)` and/or `y=ifft(X)`.

Solution: The analysis equation

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi kn/2}, \quad k = 0,1,2,3.$$

has negative exponents and there is a $(1/N) = 1/4$ factor. Reviewing the documentation, this is the same as running the MATLAB function `fft(x)` on vector `x` and dividing by 4. For example,

```
>> fft( [2 0 -1 1]' )/4

ans =

   0.5000
   0.7500 + 0.2500i
        0
   0.7500 - 0.2500i
```

gives our previous $\boldsymbol{a} = [a_0\ a_1\ a_2\ a_3]'$ vector. That is, MATLAB function `fft(x)` implements a matrix multiplication by the matrix we have defined as $\mathbf{\Phi}$, or in pseudo-notation

$$\boxed{\texttt{fft}(\boldsymbol{x}) = \mathbf{\Phi}\boldsymbol{x}.}$$

□

Problem Set 4-22

Questions on DT Frequency Response:

(a) For the pulse response $h[n]$ in Figure 8 determine its frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n},$$

and simplify the expression in the form of a complex exponential times a real function of ω.

Solution:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= h[0] + h[1]e^{-j\omega} \\ &= 1 - e^{-j\omega} \\ &= e^{-j\omega/2} 2j \frac{(e^{j\omega/2} - e^{-j\omega/2})}{2j} \end{aligned}$$

That is,

$$\boxed{H(e^{j\omega}) = \underbrace{e^{-j(\omega-\pi)/2}}_{\text{complex exponential}} \times \underbrace{2\,\sin(\omega/2)}_{\text{real function}}.}$$

(b) Determine $|H(e^{j\omega})|$ and plot it in the range $\omega \in [-2\pi, 2\pi]$ using the template shown in Figure 11.

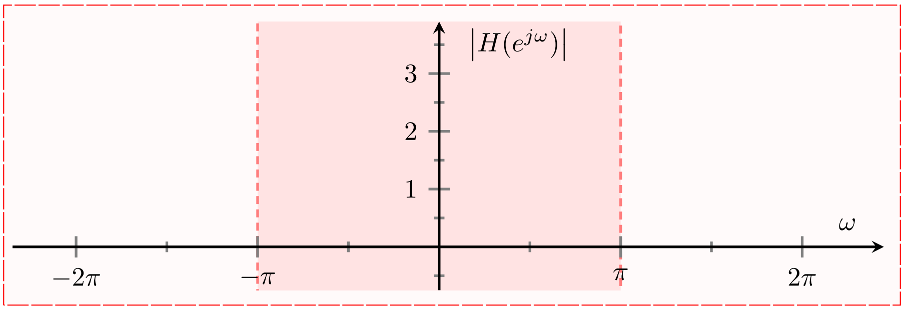
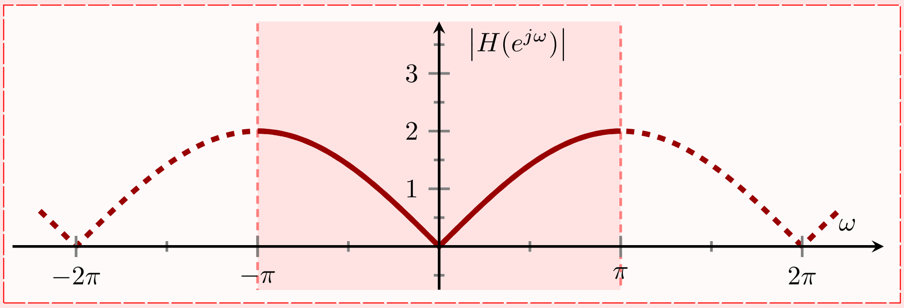


Figure 11: (Template) Frequency response $|H(e^{j\omega})|$ over range -2π to 2π.

Solution: We have

$$\boxed{|H(e^{j\omega})| = 2|\sin(\omega/2)|.}$$

and this is plotted below.



□

(c) What type of filter is it and why? (low-pass, band-pass, high-pass, all-pass)

Solution: High-pass, the gain at $\omega = 0$ is zero and the gain is maximum at the maximum frequency of $\omega = \pi$. □

Problem Set 4-23

Questions on System, Input and Output:

(a) Redraw the frequency response plot for $H(e^{j\omega})$ from the previous problem. Now add arrows to that plot to indicate the frequencies (within the frequency range $[-\pi, \pi]$) present in the period-4 signal $x[n]$ given in Figure 7. Such a plot should show which frequencies are input to the system defined by $h[n]$.

(Figure 12 gives an example of how to indicate these frequencies in the case where the frequencies ω equal $-4\pi/5$ and $+4\pi/5$, and a nonsense $|H(e^{j\omega})| = 1.5 + 0.5\cos(3\pi\omega)$.)

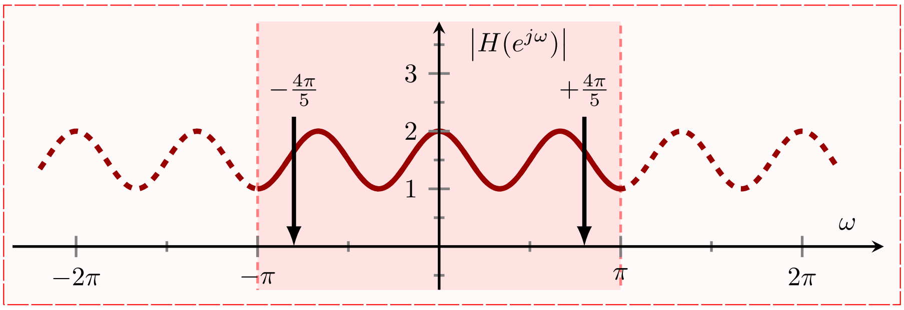
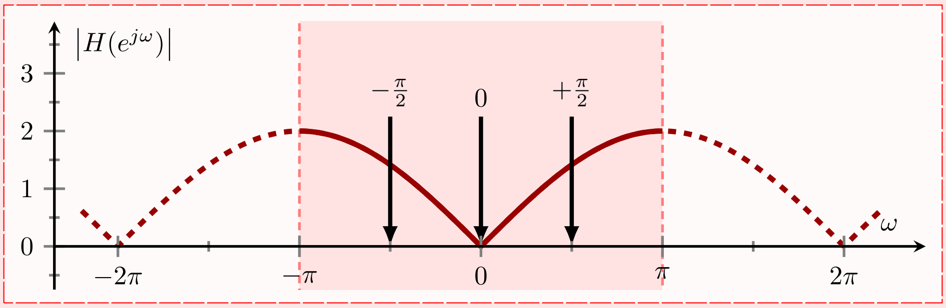


Figure 12: (Template) Example frequency response $|H(e^{j\omega})| = 1.5 + 0.5\cos(3\pi\omega)$ (which is not the right answer) over range -2π to 2π. Example of how to show the input frequencies of $x[n]$, in this case two frequencies at ω equal $-4\pi/5$ and $+4\pi/5$ (which is not the right answer).

Solution: The frequencies in $x[n]$ are at integer multiples of the fundamental frequency $\omega_0 = \pi/2$ ($2\pi/N$ where $N = 4$). The multiples are indexed in the integers $k \in \mathbb{Z}$, where $k = 0$ is DC, $k = -1, +1$ are the first harmonic, etc. So there are frequencies at multiples of $\pi/2$. However, since there is no energy at $k = 2$, since $a_2 = 0$ for $x[n]$, then this frequency is absent.



□

(b) For the period $N = 4$ signal $y[n]$ find its Fourier coefficients b_0, b_1, b_2 and b_3.

Solution:

$$\boldsymbol{b} = \frac{1}{4}\boldsymbol{\Phi}\,\boldsymbol{y},$$

where

$$\boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}.$$

So,

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 + j \\ 0 \\ 1/2 - j \end{bmatrix}$$

and, therefore,

$$b_0 = 0,\ b_1 = \frac{1+2j}{2},\ b_2 = 0,\ \text{and}\ \ b_3 = \frac{1-2j}{2}.$$

Of course, given $H(1) = 0$ then $b_0 = 0$ and given $a_2 = 0$ then $b_2 = 0$. Further, $b_{-1} = \overline{b_1}$ because $y[n]$ is real-valued. So we really only needed to compute b_1. □

Problem Set 4-24

Questions on Basic Filter Design:

(a) Design or find a new causal LTI system, $g[n]$ that produces zero output for the period-4 input $x[n]$ input shown in Figure 7. That is, find a non-trivial (non-zero) $g[n]$ such that

$$x[n] \star g[n] = 0.$$

Solution: We know that $x[n+4] = x[n]$ for all n, so we can have a simple filter that destructively combines two values $n = 4$ time units apart. A simple non-trivial example is

$$\boxed{g[n] = \delta[n] - \delta[n-4]}$$

and this gives

$$\begin{aligned} y[n] = x[n] \star g[n] &= x[n] - x[n-4] \\ &= x[n] - x[n] = 0, \quad \text{for all } n. \end{aligned}$$

The zero filter $g[n] = 0$ also solves the problem but it is a trivial example. □

(b) Plot the frequency response $\left|G(e^{j\omega})\right|$ and explain why your design works.

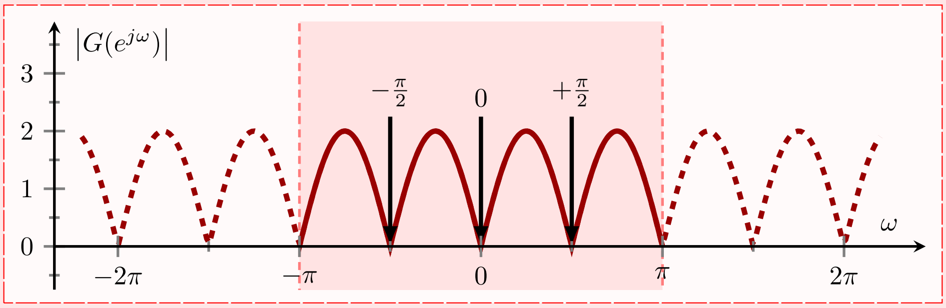
Solution: This is very similar to the calculation we gave for $H(e^{j\omega})$ earlier:

$$\begin{aligned} G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} &= g[0] + g[4]e^{-j4\omega} \\ &= 1 - e^{-j4\omega} \\ &= e^{-j2\omega} \frac{2j(e^{j2\omega} - e^{-j2\omega})}{2j} \\ &= \underbrace{e^{-j(2\omega - \pi/2)}}_{\text{complex exponential}} \times \underbrace{2\,\sin(2\omega)}_{\text{real function}} \end{aligned}$$

This works because it places notches at multiples of the frequency $\pi/2$, which are the zeros of $\sin(2\omega)$. In fact it wipes out all period-4 signals. Note that we didn't need to notch at $\omega = \pm\pi$ because $x[n]$ has no energy there anyway, but the design here is very simple.

We plot below the magnitude of the frequency response,

$$\boxed{\left|G(e^{j\omega})\right| = \left|2\,\sin(2\omega)\right|.}$$



□

(c) [Difficult] It is likely that your design for $g[n]$ above filters out all period-4 signals and not just $x[n]$. Design a new filter, $p[n]$ that filters out $x[n]$ but has a non-zero output for other (more general) period-4 signals.

Solution: Period-4 signals may also have components at $\omega = +\pi$ and $\omega = -\pi$ for which $x[n]$ does not. So we need $\left|P(e^{\pm j\pi})\right| \neq 0$. We can cascade our $h[n]$, which notches $\omega = 0$, with a second filter, say $q[n]$, that notches $\omega = \pm\pi/2$. In the frequency domain these multiply. A suitable $q[n]$ in the time domain is the convolution

$$q[n] = \delta[n] + \delta[n-2]$$

and recall $h[n] = \delta[n] - \delta[n-1]$. So our filter is (easy to show)

$$\begin{aligned} p[n] &= h[n] \star q[n] \\ &= \big(\delta[n] - \delta[n-1]\big) \star \big(\delta[n] + \delta[n-2]\big) \end{aligned}$$

Therefore, a suitable new filter is

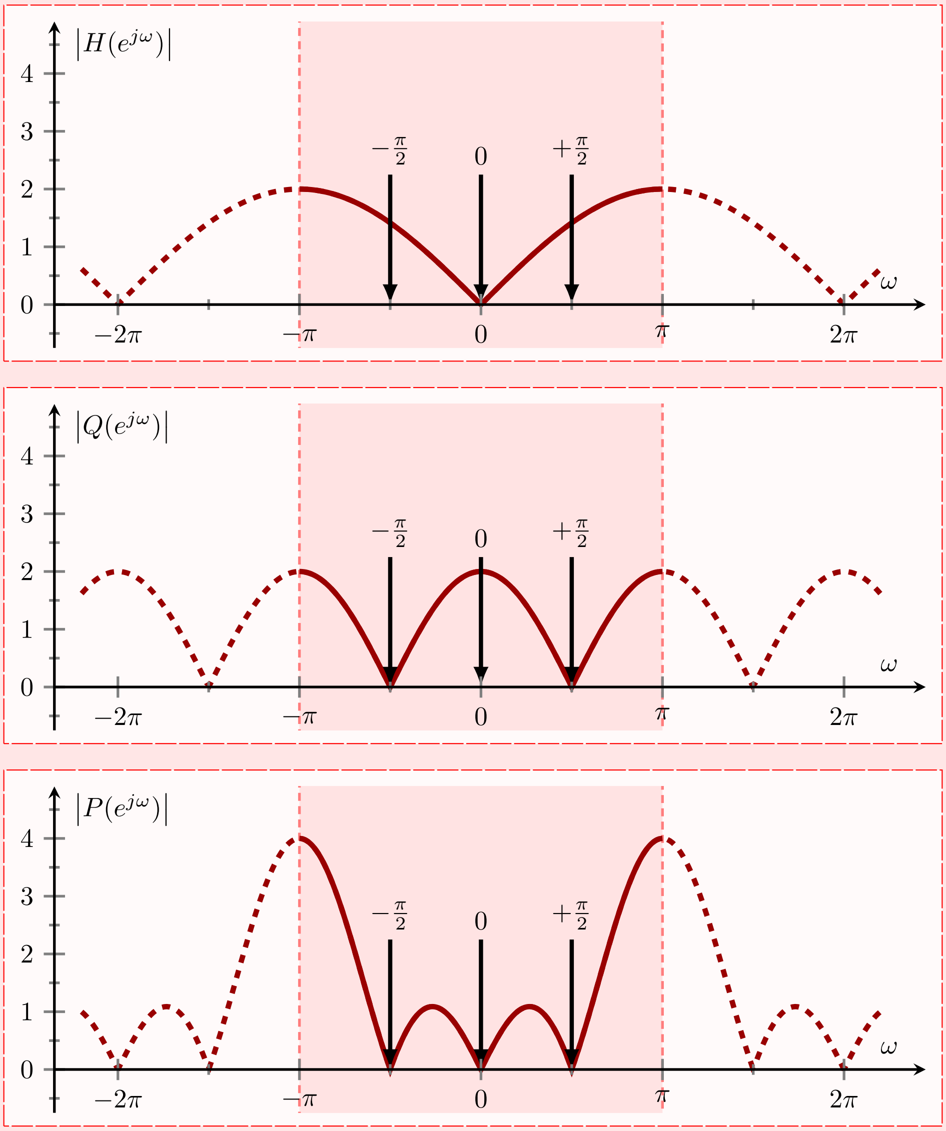
$$\boxed{p[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3],}$$

which has frequency response magnitude:

$$\left|P(e^{j\omega})\right| = \left|H(e^{j\omega})\,Q(e^{j\omega})\right|$$

$$\boxed{\left|P(e^{j\omega})\right| = 4\left|\sin(\omega/2)\,\cos(\omega)\right|.}$$

The plots below show what is going on:



□

Problem Set 4-25

Consider the following pairs of signal $x[n]$ and $y[n]$, which are the input and output of a system shown in Fig. 13. For each pair, determine whether there is a discrete-time LTI system for which $y[n]$ is the output when the corresponding $x[n]$ is the input. If such a system exists, determine whether the system is unique (i.e., whether there is more than one LTI system with the given input-output pair). Also determine the frequency response of an LTI system with the desired behaviour. If no such LTI system exists for a given $x[n]$, $y[n]$ pair, explain why.

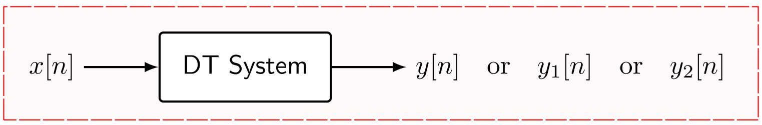


Figure 13: System with input $x[n]$ and output $y[n]$ or $y_1[n]$ (in part 2(h)) or $y_2[n]$ (in part 2(i)).

(a) $x[n] = (0.5)^n$ and $y[n] = (0.25)^n$

Solution: There is no LTI system because there is an eigenfunction in the output that is not present in the input, which violates the eigenfunction property of LTI systems.

Proof: The output is of the form of an eigenfunction, $(1/4)^n$, but this is not present in the input, which violates the eigenfunction property of LTI systems. $\square$

(b) $x[n] = (0.5)^n u[n]$ and $y[n] = (0.25)^n u[n]$

Solution: There exists an LTI system and it is unique. The pulse response is

$$h[n] = 2\,\delta[n] - (1/4)^n u[n]$$

and the frequency response is its Fourier transform

$$\boxed{H(e^{j\omega}) = 2 - \frac{1}{1-(1/4)\,e^{-j\omega}} \equiv \frac{1-(1/2)\,e^{-j\omega}}{1-(1/4)\,e^{-j\omega}}.}$$

(Neither input nor output are eigenfunctions so we need a different approach.)

Proof 1: Note that $\delta[n] = x[n] - (1/2)\,x[n-1]$ and by the LTI property therefore

$$h[n] = y[n] - (1/2)\,y[n-1] = \delta[n] - (1/4)^n u[n-1] = 2\,\delta[n] - (1/4)^n u[n].$$

This is unique and its Fourier transform is the unique frequency response. Since

$$a^n u[n] \longleftrightarrow 1 + a\,e^{-j\omega} + a^2\,e^{-j2\omega} + \cdots = \frac{1}{1-a\,e^{-j\omega})}, \quad |a| < 1,$$

and $2\,\delta[n] \longleftrightarrow 2$, then with $a = (1/4)$:

$$H(e^{j\omega}) = 2 - \frac{1}{1-(1/4)\,e^{-j\omega}} \equiv \frac{1-(1/2)\,e^{-j\omega}}{1-(1/4)\,e^{-j\omega}}.$$

$\square$

Proof 2: We use properties of linear difference equations from first principles.

The LTI system can be shown to be causal (see Proof 1 or you can regard it as obvious) so we attempt to determine $h[n]$ for $n \in \{0, 1, 2, \dots\}$. At $n = 0$ we have $y[0] = h[0]x[0]$, which implies $h[0] = 1$. At $n = 1$ we have $y[1] = h[0]x[1] + h[1]x[0]$ and only $h[1]$ is unknown and so is uniquely determined: $(1/4) = 1 \cdot (1/2) + h[1] \cdot 1$, which implies $h[1] = -(1/4)$. At $n = 2$ we have $y[2] = h[0]x[2] + h[1]x[1] + h[2]x[0]$ and only $h[2]$ is unknown and so is uniquely determined: $(1/4)^2 = 1 \cdot (1/2)^2 + (-1/4)(1/2) + h[2] \cdot 1$, which implies $h[2] = -(1/4)^2$. At $n = 3$, we can determine $h[3] = -(1/4)^3$ similarly. For each n the output $y[n]$ enables us to determine $h[n]$ (linear equation in one unknown at each step)

$$h[n] = \delta[n] - (1/4)^n u[n-1] = 2\,\delta[n] - (1/4)^n u[n]$$

(by induction), as before. $\square$

Proof 3: With later material from the course we have

$$X(e^{j\omega}) = \frac{1}{1-(1/2)\,e^{-j\omega}},\; Y(e^{j\omega}) = \frac{1}{1-(1/4)\,e^{-j\omega}},\; H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \equiv \frac{1-(1/2)\,e^{-j\omega}}{1-(1/4)\,e^{-j\omega}}.$$

$\square$

(c) $x[n] = 0.5^n u[n]$ and $y[n] = 4^n u[-n]$

Solution: There exists a non-causal LTI system and it is unique and has frequency response given by

$$\boxed{H(e^{j\omega}) = \frac{1-(1/2)\,e^{-j\omega}}{1-(1/4)\,e^{j\omega}}}$$

Proof: As in (b) the pulse response is $h[n] = y[n] - (1/2)\,y[n-1]$, that is,

$$h[n] = 4^n u[-n] - (1/2)\,4^{n-1} u[-n+1] = (7/8)4^n u[-n] - (1/2)\,\delta[n-1]$$

(to see this you should draw $y[n] = 4^n u[-n]$ and $y[n-1]$ and combine the two) and its Fourier transform is the frequency response, which results in

$$\begin{aligned} H(e^{j\omega}) &= (7/8)\big(1 + (1/4)e^{j\omega} + (1/4)^2 e^{j2\omega} + \cdots\big) - (1/2)e^{-j\omega} \\ &= \frac{(7/8)}{1-(1/4)e^{j\omega}} - (1/2)e^{-j\omega} = \frac{1-(1/2)\,e^{-j\omega}}{1-(1/4)\,e^{j\omega}} \end{aligned}$$

$\square$

(d) $x[n] = e^{jn/8}$ and $y[n] = 2\,e^{jn/8}$

Solution: There exists an LTI system, it is not unique, and any LTI system with the appropriate frequency response $H(e^{j\omega})$ such that

$$\boxed{H(e^{j/8}) = 2}$$

is suitable.

Proof: There is an eigenfunction at the output that has a corresponding eigenfunction at the input, which is consistent with an LTI system. It is not unique because the behaviour is constrained only for the single $z = e^{j/8}$ where $\omega = 1/8$ and no other frequency, that is, we only need $H(e^{j/8}) = 2$. $H(e^{j\omega})$ is unconstrained for other frequencies. The simplest possibility is $y[n] = 2\,x[n]$ for which $H(e^{j\omega}) = 2$ for all ω. □

(e) $x[n] = e^{jn/8}u[n]$ and $y[n] = 2\,e^{jn/8}u[n]$

Solution: There exists an LTI system and it is unique with frequency response

$$\boxed{H(e^{j\omega}) = 2.}$$

Proof: This is very similar to (b) and (c), and the same techniques can be applied. For example, $\delta[n] = x[n] - e^{j8}x[n-1]$ and this implies $h[n] = y[n] - e^{j8}y[n-1]$, which reduces to $h[n] = 2\,\delta[n]$, and therefore

$$H(e^{j\omega}) = 2.$$

Of course we might have seen directly that $y[n] = 2\,x[n]$ is an LTI system that works. So the additional calculation we are doing above, which proves $h[n] = 2\,\delta[n]$, demonstrates uniqueness. □

In case you are wondering, there is a big difference between signal $e^{jn/8}$ and signal $e^{jn/8}u[n]$. The former has a single frequency and the latter has all (an infinite number of non-zero) component frequencies.

(f) $x[n] = j^n$ and $y[n] = 2\,j^n(1-j)$

Solution: There exists an LTI system, it is not unique, and any LTI system with the appropriate frequency response $H(e^{j\omega})$ satisfying

$$\boxed{H(e^{j\pi/2}) = 2(1-j)}$$

is suitable.

Proof: This is pretty well the same situation as in Part (d). The input is an eigenfunction with $z = j$ or $z = e^{j\pi/2}$ and we only require $H(e^{j\pi/2}) = 2(1-j)$. Otherwise $H(e^{j\omega})$ is unconstrained for all $\omega \neq \pi/2$ and so is not unique. □

(g) $x[n] = \cos(\pi n/3)$ and $y[n] = \cos(\pi n/3) + \sqrt{3}\sin(\pi n/3)$

Solution: There exists an LTI system, it is not unique, and any LTI system with the appropriate frequency response $H(e^{j\omega})$ satisfying

$$\boxed{H(e^{j\pi/3}) = (1 - j\sqrt{3}) \quad \text{and} \quad H(e^{-j\pi/3}) = (1 + j\sqrt{3})}$$

is suitable.

Proof: Again this turns out to be similar to Part (d). The input can be written

$$x[n] = (1/2)\,e^{j\pi/3} + (1/2)\,e^{-j\pi/3}.$$

The output is

$$y[n] = (1/2 + \sqrt{3}/(2j))\,e^{j\pi/3} + (1/2 - \sqrt{3}/(2j))\,e^{-j\pi/3}.$$

So we only require $H(e^{j\pi/3}) = (1 - j\sqrt{3})$ and $H(e^{-j\pi/3}) = (1 + j\sqrt{3})$ to explain this input output behaviour. It is clearly not unique given $H(e^{j\omega})$ is unconstrained at all other frequencies. □

(h) $x[n]$ and $y_1[n]$ shown in Fig. 14.

Solution: There exists an LTI system, it is not unique, and its frequency response $H(e^{j\omega})$ is only constrained for frequencies ω that are integer multiples of $\pi/6$:

$$\boxed{H(e^{jk\pi/6}) = \begin{cases} 0 & k \text{ odd} \\ \dfrac{b_{k/2}}{a_k} & k \text{ even} \end{cases}}$$

where a_k are the Fourier series coefficients of $x(t)$ and b_m are the the Fourier series coefficients of $y_1(t)$.

Proof: $x[n]$ has frequencies (rad/sec) at

$$0 \text{ (DC)}, \pm\pi/6 \text{ (its 1st harmonic)}, \pm\pi/3, \pm\pi/2, \pm 2\pi/3, \pm 5\pi/6, \pm\pi, \ldots.$$

$y_1[n]$ has frequencies (rad/sec) at

$$0 \text{ (DC)}, \pm\pi/3 \text{ (its 1st harmonic)}, \pm 2\pi/3, \pm\pi, \ldots.$$

So all frequencies at the output are present at the input (all complex exponentials in $y_1[n]$ are present in the input and so act as eigenfunctions). An LTI system needs to only apply the correct complex weights to the eigenfunctions to convert $x[n]$ to $y_1[n]$ — these weights are just the eigenvalues. For example, the first harmonic of $x(t)$ at $\omega = \pm\pi/6$ needs to be filtered out ($H(e^{j\pm\pi/6}) = 0$), and the second harmonic of $x(t)$ needs to be scaled to match the first harmonic of $y_1(t)$ (both at frequencies $\omega = \pm\pi/3$); the third harmonic of $x(t)$ at $\omega = \pm\pi/2$ needs to be filtered out ($H(e^{j\pm\pi/2}) = 0$), etc.

Let $H(e^{j\omega})$ be the frequency response. Then we are only specifying $H(e^{j\omega})$ at integer multiples of $\pi/6$. (Note that need $H(e^{j\omega})$ needs to be zero at $\omega = \pm\pi/6, \pm\pi/2, \ldots$, to filter-out those frequencies, which are present in the input but not in the output.)

That the LTI system is not unique is obvious given $H(e^{j\omega})$ is unconstrained for other values of ω (not equal to multiples of $\pi/6$).

(Note that, as a technicality, we should also check that all the harmonic frequencies of $x[n]$, which are present in the output $y_1[n]$, have non-zero energy. You can't create energy at a frequency from nothing.) □

(i) $x[n]$ and $y_2[n]$ shown in Fig. 14.

Solution: There is no LTI system because there are frequencies in the output that are not present in the input, which violates the eigenfunction property of LTI systems.

Proof: $y_2[n]$ has frequencies (rad/sec) at

$$0 \text{ (DC)}, \pm 2\pi/9 \text{ (its 1st harmonic)}, \pm 4\pi/9, \pm 2\pi/3, \pm 8\pi/9, \ldots.$$

Output $y_2[n]$ contains many frequencies, for example, $\pm 2\pi/9$, not present in the input $x[n]$, which violates the eigenfunction property of LTI systems. □

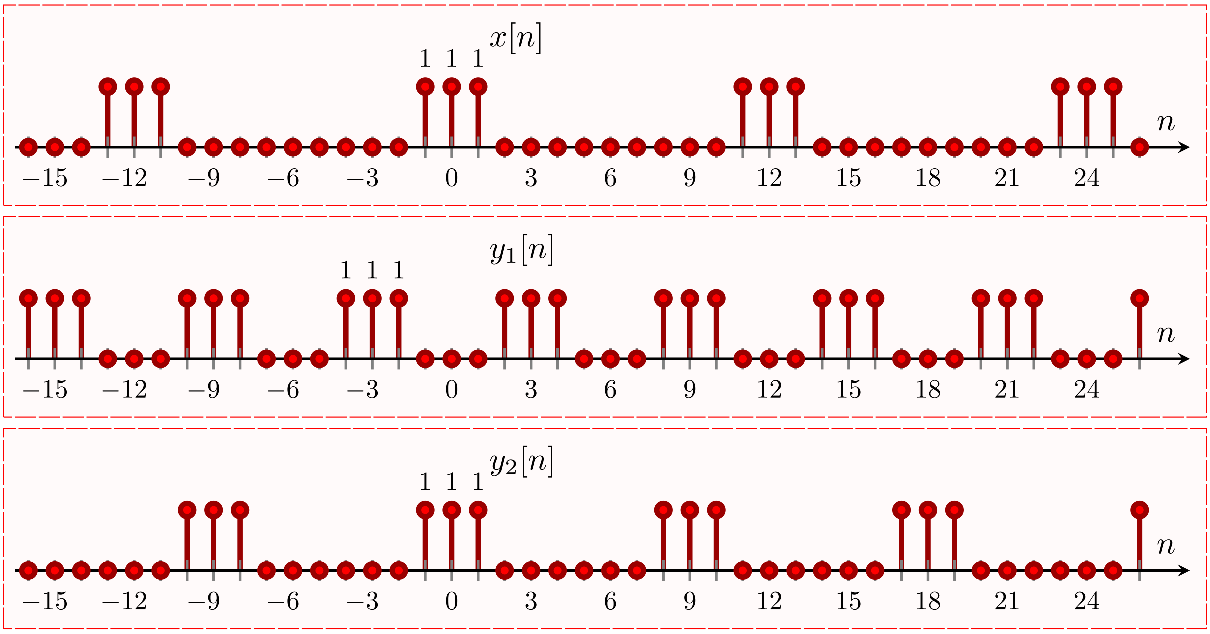


Figure 14: Periodic input signal $x[n]$ and two periodic output signals $y_1[n]$ and $y_2[n]$.