Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 5 – SOLUTIONS

Fourier Analysis and Synthesis of Continuous Time Signals

Problem Set 5-1

Find the Fourier transform of the following signals using the FT analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

(a)

$$x(t) = \delta(t+1) + \delta(t-1)$$

Solution:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} (\delta(t+1) + \delta(t-1))e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \delta(t+1)e^{-j\omega t}dt + \int_{-\infty}^{\infty} \delta(t-1)e^{-j\omega t}dt \\ &= e^{-j\omega(-1)} + e^{-j\omega(1)} \\ &= 2\frac{e^{j\omega} + e^{-j\omega}}{2} \\ &= 2\cos(\omega) \end{aligned}$$

(b)

$$x(t) = e^{-a|t|},\ (a > 0)$$

Solution:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt \\ &= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\ &= \frac{1}{a-j\omega}\left|e^{(a-j\omega)t}\right|_{-\infty}^{0} + \frac{1}{a+j\omega}\left|e^{-(a+j\omega)t}\right|_{0}^{\infty} \\ &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

(c)

$$x(t) = e^{2t}u(-t)$$

Solution:

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{2t}u(-t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{2t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(2-j\omega)t}dt \\
&= \frac{1}{2-j\omega}\left| e^{(2-j\omega)t} \right|_{-\infty}^{0} \\
&= \frac{1}{2-j\omega}
\end{aligned}
$$

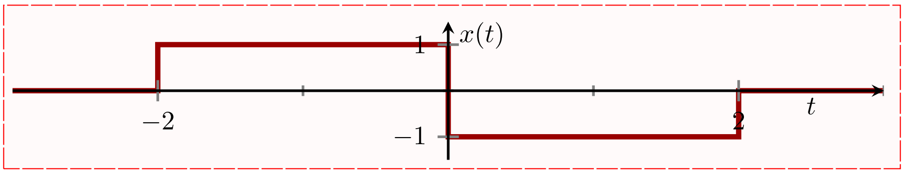
(d)

$$x(t) = e^{-2t}u(t-1)$$

Solution:

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-2t}u(t-1)e^{-j\omega t}dt \\
&= \int_{1}^{\infty} e^{-2t}e^{-j\omega t}dt \\
&= \int_{1}^{\infty} e^{-(2+j\omega)t}dt \\
&= -\frac{1}{2+j\omega}\left| e^{-(2+j\omega)t} \right|_{1}^{\infty} \\
&= \frac{e^{-(j\omega+2)}}{j\omega+2}
\end{aligned}
$$

(e) For the signal $x(t)$ shown in the figure below:



Solution:

$$
\begin{aligned}
X(j\omega) &= \int_{-2}^{0}(1)e^{-j\omega t}dt + \int_{0}^{2}(-1)e^{-j\omega t}dt \\
&= -\frac{1}{j\omega}\left| e^{-j\omega t} \right|_{-2}^{0} + \frac{1}{j\omega}\left| e^{-j\omega t} \right|_{0}^{2} \\
&= -\frac{1}{j\omega}[e^{0} - e^{j2\omega}] + \frac{1}{j\omega}[e^{-j2\omega} - e^{0}] \\
&= \frac{2j(1-\cos(2\omega))}{\omega}
\end{aligned}
$$

Problem Set 5-2

Find the inverse Fourier transform of the following spectra using the FT synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

(a)

$$X(j\omega) = 3\delta(\omega-4)$$

Solution:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 3\delta(\omega-4)e^{j\omega t}d\omega \\
&= \frac{3}{2\pi}e^{j4t}
\end{aligned}
$$

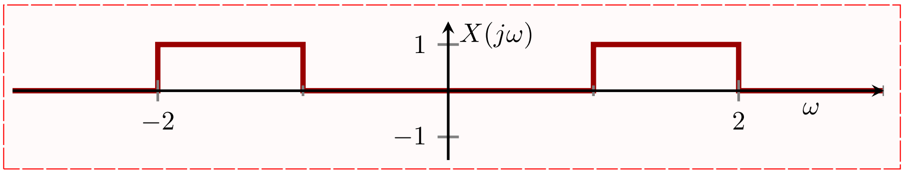
(b)

$$X(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega-4\pi) + \pi\delta(\omega+4\pi)$$

Solution:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (2\pi\delta(\omega) + \pi\delta(\omega-4\pi)\pi\delta(\omega+4\pi))e^{j\omega t}d\omega \\
&= e^{jt(0)} + \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} \\
&= 1 + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) \\
&= 1 + \cos(4\pi t)
\end{aligned}
$$

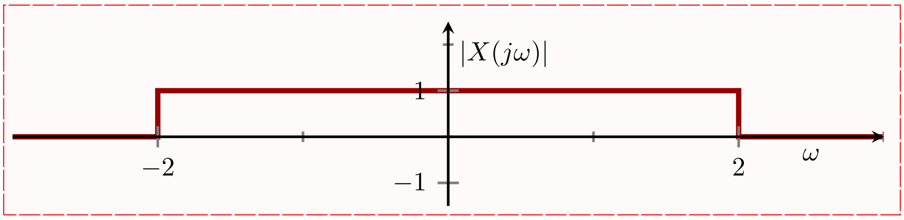
(c) For the spectrum $X(j\omega)$ shown in the figure below:

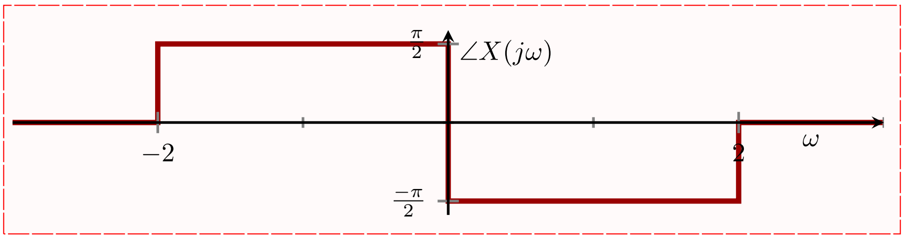


Solution: $X(j\omega)$ is real valued so 1 plot is sufficient

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-2}^{-1}(1)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{1}^{2}(1)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\frac{1}{jt}\Big|e^{j\omega t}\Big|_{-2}^{-1} + \frac{1}{2\pi}\frac{1}{jt}\Big|e^{j\omega t}\Big|_{1}^{2} \\
&= \frac{e^{-jt} - e^{-j2t}}{j2\pi t} - \frac{e^{jt} - e^{-jt}}{j2\pi t} \\
&= \frac{\sin(2t) - \sin(t)}{\pi t}
\end{aligned}
$$

(d) For the spectrum $X(j\omega)$ shown in the figures below:





Solution: $X(j\omega)$ is complex valued so two plots given

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-2}^{0}(1e^{j\frac{\pi}{2}})e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{2}(1e^{-j\frac{\pi}{2}})e^{j\omega t}d\omega \\ &= \frac{e^{j\frac{\pi}{2}}}{2\pi}\frac{1}{jt}\Big|e^{j\omega t}\Big|_{-2}^{0} + \frac{e^{-j\frac{\pi}{2}}}{2\pi}\frac{1}{jt}\Big|e^{j\omega t}\Big|_{0}^{2} \\ &= \frac{e^{j\frac{\pi}{2}}}{2j\pi t}[-e^{-j2t}+e^{0}] + \frac{e^{-j\frac{\pi}{2}}}{2j\pi t}[e^{j2t}+e^{0}] \\ &= \frac{1-\cos(2t)}{\pi t} \end{aligned}$$

Fourier Transform Properties of CT Signals

Problem Set 5-3

Determine whether the Fourier transforms $X(j\omega)$ in Figure 1(a) and 1(b) correspond to real continuous time signal $x(t)$.

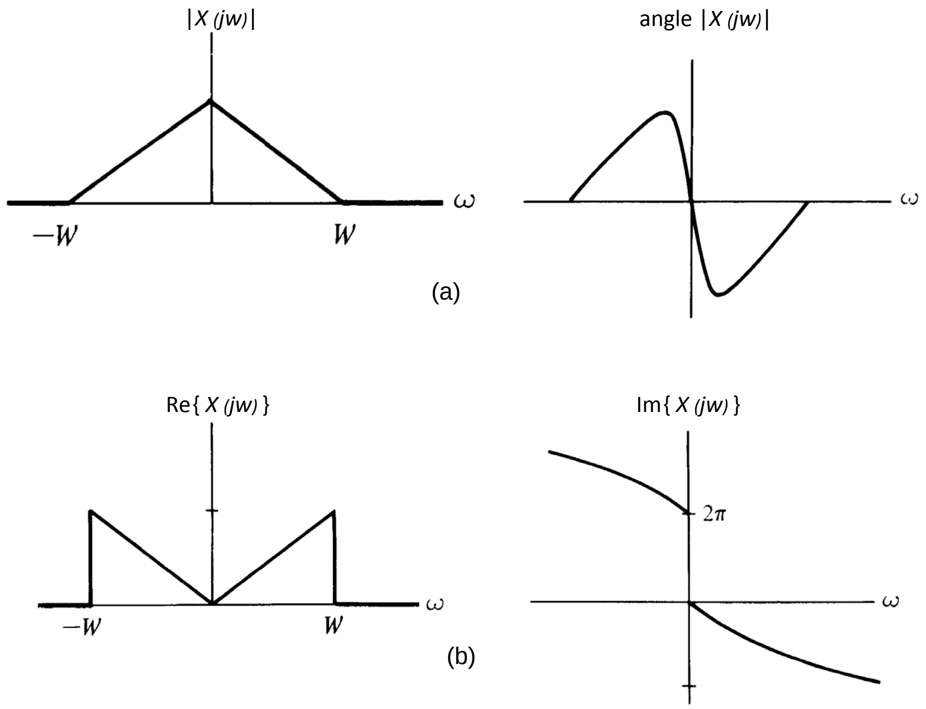


Figure 1: Problem 5-3

Solution: For a real signal $x(t)$, $X(-j\omega) = \overline{X(j\omega)}$.

(a) We are given magnitude $X(j\omega)$ and phase $\angle X(j\omega)$ of $X(j\omega)$.

We can write $X(j\omega) = |X(j\omega)|\, e^{j\angle X(j\omega)}$. Since

$$X(-j\omega) = |X(-j\omega)|\, e^{j\angle X(-j\omega)} = |X(j\omega)|\, e^{-j\angle X(j\omega)} = \overline{X(j\omega)},$$

the signal $x(t)$ is real.

(b) We are given real part $\mathrm{Re}\{X(j\omega)\}$ and imaginary part $\mathrm{Im}\{X(j\omega)\}$.

We can write $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\mathrm{Im}\{X(j\omega)\}$.

Since

$$\begin{aligned} X(-j\omega) &= \mathrm{Re}\{X(-j\omega)\} + j\mathrm{Im}\{X(-j\omega)\} \\ &= \mathrm{Re}\{X(j\omega)\} + j\left(\mathrm{Im}\{-X(j\omega)\} + 2\pi\right) \\ &\neq \mathrm{Re}\{X(j\omega)\} - j\mathrm{Im}\{X(j\omega)\} = \overline{X(j\omega)}, \end{aligned}$$

the signal $x(t)$ is not real.

Problem Set 5-4

Given $x(t)$ in Figure 2, sketch $X(j\omega)$. If $y(t) = x(t/2)$, sketch both $y(t)$ and $Y(j\omega)$.

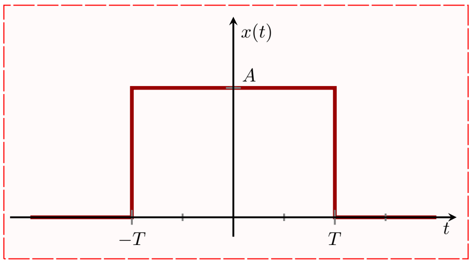
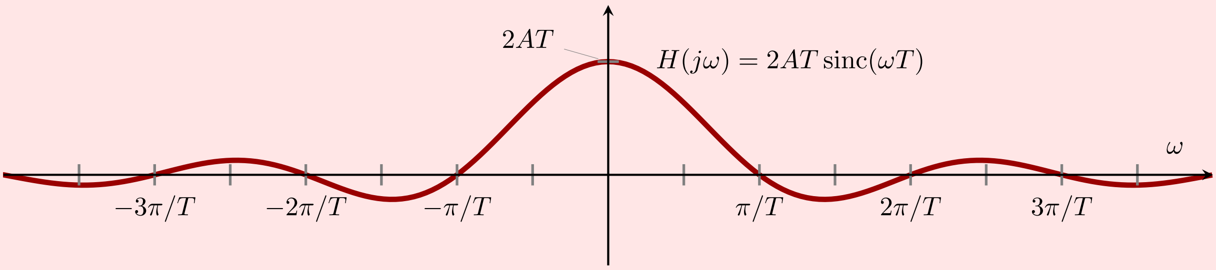


Figure 2: $x(t)$ for Problem 5-4.

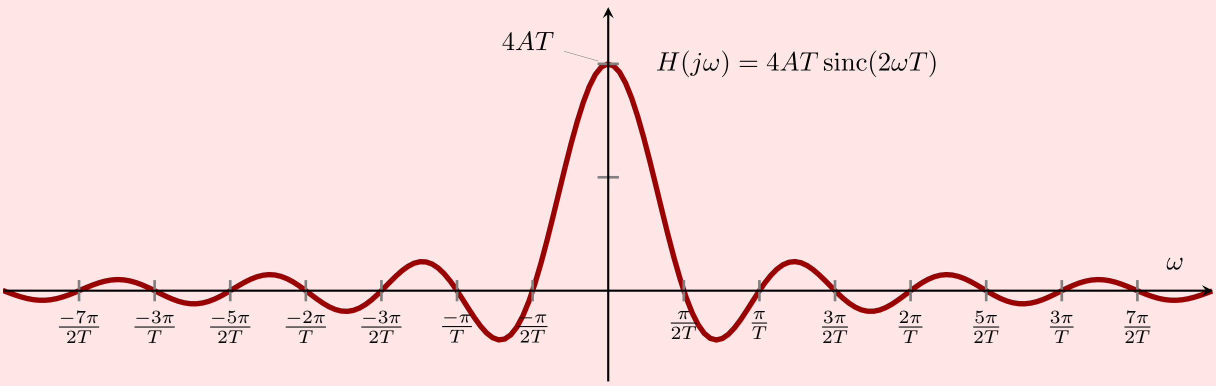
Solution: First we compute $X(j\omega)$ as follows:

$$\begin{aligned} X(j\omega) &= A\int_{-T}^{T} e^{-j\omega t}dt \\ &= \frac{A}{-j\omega}(e^{-j\omega T} - e^{j\omega T}) \\ &= 2AT\,\frac{\sin(\omega T)}{\omega T} \\ &= 2AT\,\mathrm{sinc}(\omega T). \end{aligned}$$



For $y(t) = x(t/2)$, we now compute $Y(j\omega)$ as follows:

$$Y(j\omega) = 2X(j\omega 2) = 4AT\,\mathrm{sinc}(2\omega T).$$



Problem Set 5-5

For an input signal

$$x(t) = e^{-\alpha t}u(t), \quad \alpha > 0$$

to the continuous time LTI system with impulse response

$$h(t) = e^{-\beta t}u(t), \quad \beta > 0$$

find the output $y(t)$ of the LTI system using the convolution property of the Fourier transform. Also find the output $y(t)$ for the case when $\alpha = \beta$.

Solution: The convolution property states that

$$y(t) = x(t) \star h(t) \longleftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$$

We first determine $X(j\omega)$:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-\alpha t}\, u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt \\ &= \frac{1}{\alpha+j\omega}e^{-(\alpha+j\omega)t}\Big|_0^\infty \\ &= \frac{1}{\alpha+j\omega}. \end{aligned}$$

Similarly, we determine $H(j\omega)$:

$$H(j\omega) = \frac{1}{\beta+j\omega}.$$

When $\alpha \neq \beta$, we write $Y(j\omega)$ using convolution property as:

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{\alpha+j\omega}\frac{1}{\beta+j\omega} = \frac{A}{\alpha+j\omega} + \frac{B}{\beta+j\omega}, \tag{1}$$

where

$$A = \frac{1}{\beta-\alpha}, \quad B = \frac{-1}{\beta-\alpha}.$$

Apply inverse Fourier transform to (1) to determine $y(t)$

$$y(t) = Ae^{-\alpha t}\, u(t) + Be^{-\beta t}\, u(t) = \frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})u(t).$$

When $\alpha = \beta$,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(\alpha+j\omega)^2} = j\frac{d}{d\omega}\left(\frac{1}{\alpha+j\omega}\right) \tag{2}$$

The partial fraction approach cannot be applied here. Using the following derivative property of Fourier transform,

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
$$t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\big(X(j\omega)\big)$$

we determine $y(t)$ as

$$y(t) = t\,e^{-\alpha t}u(t).$$

□

Problem Set 5-6

The following differential equation relates the output $y(t)$ of causal continuous LTI system to the input $x(t)$:

$$\frac{dy(t)}{dt} + 3\,y(t) = x(t).$$

(a) Determine the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ and sketch the magnitude of $H(j\omega)$.

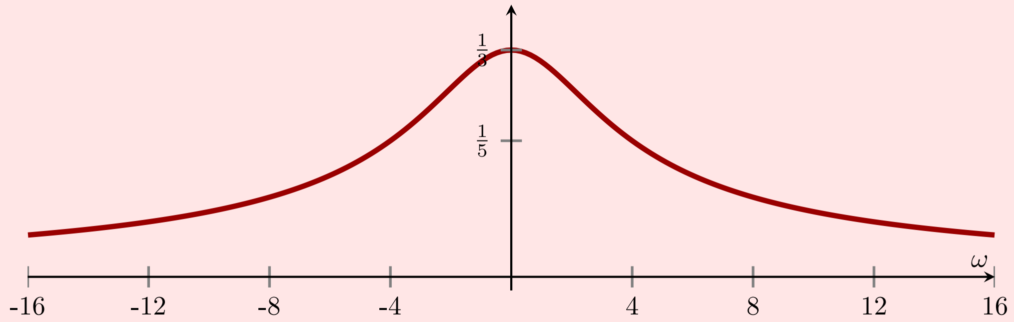
Solution: By taking Fourier transform of the following given differential equation, we obtain:

$$\begin{aligned} j\omega Y(j\omega) + 3Y(j\omega) &= X(j\omega) \\ Y(j\omega)(3+j\omega) &= X(j\omega) \\ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} &= \frac{1}{3+j\omega} \end{aligned}$$

The magnitude $|H(j\omega)|$ given by

$$|H(j\omega)| = \frac{1}{\sqrt{9+\omega^2}}$$

is sketched below.



□

(b) If $x(t) = e^{-t}\, u(t)$, determine $Y(j\omega)$ and $y(t)$.

Solution: We first determine $X(j\omega)$:

$$X(j\omega) = \frac{1}{1+j\omega}$$

Now, determine $Y(j\omega)$ as follows:

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{1+j\omega}\frac{1}{3+j\omega} = \left(\frac{1}{2}\right)\left(\frac{1}{1+j\omega}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{3+j\omega}\right).$$

Take inverse Fourier transform to determine $y(t)$

$$y(t) = \frac{1}{2}e^{-t}\, u(t) - \frac{1}{2}e^{-3t}\, u(t) = \frac{1}{2}(e^{-t} - e^{-3t})u(t).$$

□

Problem Set 5-7

Consider two CT LTI systems with frequency responses

$$H_1(j\omega) = \frac{2+j\omega}{(1-0.5\, j\omega)^2}$$

and

$$H_2(j\omega) = \frac{(1+j\omega)^2}{(-1/2+j\omega)(3/4+j\omega)}.$$

(a) Find the differential equation describing $H_1(j\omega)$

Solution: We have

$$H_1(j\omega) = \frac{2+j\omega}{(1-0.5j\omega)^2} = \frac{2+j\omega}{1-j\omega+\frac{1}{4}(j\omega)^2}$$

and

$$\frac{Y_1(j\omega)}{X_1(j\omega)} = H_1(j\omega)$$

therefore,

$$\frac{1}{4}\frac{d^2}{dt^2}y_1(t) - \frac{d}{dt}y_1(t) + y_1(t) = \frac{d}{dt}x_1(t) + 2x_1(t).$$

□

(b) Find the differential equation describing $H_2(j\omega)$

Solution: We have

$$H_2(j\omega) = \frac{(1+j\omega)^2}{(-\frac{1}{2}+j\omega)(\frac{3}{4}+j\omega)} = \frac{1+2j\omega+(j\omega)^2}{-\frac{3}{8}+\frac{1}{4}j\omega+(j\omega)^2} = \frac{Y_2(j\omega)}{X_2(j\omega)}$$

therefore,

$$\frac{d^2}{dt^2}y_2(t) + \frac{1}{4}\frac{d}{dt}y_2(t) - \frac{3}{8}y_2(t) = \frac{d^2}{dt^2}x_2(t) + 2\frac{d}{dt}x_2(t) + x_2(t).$$

□

(c) Find the differential equation describing the cascade of $H_1(j\omega)$ and $H_2(j\omega)$.

Solution: Let

$$H(j\omega) = H_1(j\omega)H_2(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

then

$$H(j\omega) = \frac{2 + 5j\omega + 4(j\omega)^2 + (j\omega)^3}{-\frac{3}{8} + \frac{5}{8}j\omega + \frac{21}{32}(j\omega)^2 - \frac{15}{16}(j\omega)^3 + \frac{1}{4}(j\omega)^4}$$

therefore

$$\frac{1}{4}\frac{d^4}{dt^4}y(t) - \frac{15}{16}\frac{d^3}{dt^3}y(t) + \frac{21}{32}\frac{d^2}{dt^2}y(t) + \frac{5}{8}\frac{d}{dt}y(t) - \frac{3}{8}y(t) = \frac{d^3}{dt^3}x(t) + 4\frac{d^2}{dt^2}x(t) + 5\frac{d}{dt}x(t) + 2x(t).$$

□

(d) Determine the impulse response of the cascade of $H_1(j\omega)$ and $H_2(j\omega)$.

Solution:

$$H(j\omega) = \frac{(2+j\omega)(1+j\omega)^2}{(1-\frac{1}{2}j\omega)^2(-\frac{1}{2}+j\omega)(-\frac{3}{4}+j\omega)} = \frac{A}{1-\frac{1}{2}j\omega} + \frac{B}{(1-\frac{1}{2}j\omega)^2} + \frac{C}{-\frac{1}{2}+j\omega} + \frac{D}{\frac{3}{4}+j\omega},$$

where $A = -3.96$, $B = \frac{384}{11}$, $C = 8$ and $D = -0.0331$. Therefore

$$h(t) = Ae^{2t}u(-t) + Bte^{2t}u(-t) + Ce^{0.5t}u(-t) + De^{-0.75t}u(t).$$

□

Problem Set 5-8

A CT LTI system has input $x(t) = (e^{-t} + e^{-3t})u(t)$ and output $y(t) = (2e^{-t} + 2e^{-4t})u(t)$. Find the impulse response $h(t)$ of the LTI system.

Solution: We know

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

Therefore

$$e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega}$$

and

$$e^{-3t}u(t) \longleftrightarrow \frac{1}{3+j\omega}.$$

So

$$x(t) \longleftrightarrow X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{3+j\omega} = \frac{4+2j\omega}{(1+j\omega)(3+j\omega)}.$$

For $y(t)$

$$y(t) \longleftrightarrow Y(j\omega) = \frac{2}{(j\omega+1)} - \frac{2}{(j\omega+4)} = \frac{6}{(1+j\omega)(4+j\omega)}$$

By definition

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}$$

Using partial fraction expansion

$$H(j\omega) = \frac{\frac{3}{2}}{j\omega+4} + \frac{\frac{3}{2}}{j\omega+2}$$

Using

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

$$h(t) = \frac{3}{2}e^{-4t}u(t) + \frac{3}{2}e^{-2t}u(t)$$

Problem Set 5-9

Find the impulse response $h(t)$ of the CT LTI system described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dy}y(t) + 3y(t) = \frac{d}{dx}x(t) + 2x(t)$$

Solution: Let

$$x(t) = e^{j\omega t}$$

then

$$\frac{d}{dx}x(t) = j\omega e^{j\omega t}$$

then

$$y(t) = H(j\omega)e^{j\omega t}$$

so

$$\frac{d}{dt}y(t) = j\omega e^{j\omega t}H(j\omega)$$

and

$$\frac{d^2}{dt^2}y(t) = (j\omega)^2 e^{j\omega t}H(j\omega)$$

substituing

$$(j\omega)^2 e^{j\omega t}H(j\omega) + 4j\omega e^{j\omega t}H(j\omega) + 3e^{j\omega t}H(j\omega) = j\omega e^{j\omega t} + 2e^{j\omega t}$$

Cancelling $e^{j\omega t}$ from both sides and rearranging gives

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

We know

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

therefore

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Fourier Analysis and Synthesis of Discrete-time Signals

Problem Set 5-10

Compute the DTFT of each of the following signals:

(a) $x[n] = \delta[n-1] + \delta[n+1]$

Solution: We know that $\delta[n] \longleftrightarrow 1$ and $\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$.
Hence $\delta[n-1] \longleftrightarrow e^{-j\omega}$ and $\delta[n+1] \longleftrightarrow e^{j\omega}$.
Using linearity,

$$\delta[n-1] + \delta[n+1] \longleftrightarrow e^{-j\omega} + e^{j\omega}$$

$$e^{-j\omega} + e^{j\omega} = 2\cos\omega$$

Hence

$$x[n] \longleftrightarrow 2\cos\omega$$

(b) $x[n] = \delta[n+2] - \delta[n-2]$

Solution:
We know that $\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$.
Hence $\delta[n-2] \longleftrightarrow e^{-j2\omega}$ and $\delta[n+2] \longleftrightarrow e^{j2\omega}$.
Using linearity,

$$\delta[n+2] - \delta[n-2] \longleftrightarrow e^{j2\omega} - e^{-j2\omega}$$

$$e^{j2\omega} - e^{-j2\omega} = j2\sin 2\omega$$

Hence

$$x[n] \longleftrightarrow j2\sin 2\omega$$

(c) $x[n] = u[n-2] - u[n-6]$

Solution:
We can write $x[n]$ as:

$$x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

We know that $\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$.
Hence using linearity,

$$X(e^{j\omega}) = e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega}$$

(d)

$$x[n] = \begin{cases} 2^n & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=0}^{9} 2^n e^{-j\omega n} \\ &= \sum_{n=0}^{9} (2e^{-j\omega})^n \end{aligned}$$

We know

$$\sum_{n=0}^{M-1} \alpha^n = \frac{1-\alpha^M}{1-\alpha}, \quad \alpha \neq 1$$

therefore

$$X(e^{j\omega}) = \frac{1-(2e^{-j\omega})^{10}}{1-(2e^{-j\omega})}$$

(e)

$$x[n] = \Big(-\frac{1}{5}\Big)^n u[n] - 6\Big(-\frac{1}{5}\Big)^{n-2} u[n-2]$$

Solution: We know that

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

therefore

$$\Big(-\frac{1}{5}\Big)^n u[n] \longleftrightarrow \frac{1}{1+\frac{1}{5}e^{-j\omega}}$$

and the time-shifting property

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

therefore

$$\Big(-\frac{1}{5}\Big)^{n-2} u[n-2] \longleftrightarrow \frac{e^{-j\omega 2}}{1+\frac{1}{5}e^{-j\omega}}$$

Finally using linearity

$$\Big(-\frac{1}{5}\Big)^n u[n] - 6\Big(-\frac{1}{5}\Big)^{n-2} u[n-2] \longleftrightarrow \frac{1}{1+\frac{1}{5}e^{-j\omega}} - 6\frac{e^{-j\omega 2}}{1+\frac{1}{5}e^{-j\omega}} = X(e^{j\omega})$$

Simplifying

$$X(e^{j\omega}) = \frac{1-6e^{-j2\omega}}{1+\frac{1}{5}e^{-j\omega}}$$

(f)

$$x[n] = \sin\big(\frac{\pi}{3}n + \frac{\pi}{4}\big)$$

Solution: $\omega_0 = \frac{\pi}{3}$ therefore $N = \frac{m2\pi}{\omega_0} = 6m$. For $m = 1$ fundamental period of $N_0 = 6$ samples. Therefore $x[n]$ is periodic.
First we use inspection method to find its DTFT coefficients a_k.

$$\begin{aligned} x[n] &= \frac{e^{j\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)} - e^{-j\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)}}{2j} \\ &= \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{3}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{3}n} \end{aligned}$$

We write the DTFS synthesis equation by summing from $k = -2$ to 3.

$$\begin{aligned} x[n] &= \sum_{k=-2}^{3} a_k e^{jk\omega_0 n} \\ &= a_{-2}e^{-j\frac{2\pi}{3}n} + a_{-1}e^{-j\frac{\pi}{3}n} + a_0 + a_1 e^{j\frac{\pi}{3}n} + a_2 e^{j\frac{2\pi}{3}n} + a_3 e^{j\pi n} \end{aligned}$$

comparing coefficients: $a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$ and $a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$
Now we use the general relationship in the range $-\pi \le \omega \le \pi$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

This summation has non-zero value only for $k = \pm 1$. Hence,

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1 \delta(\omega-\omega_0) + 2\pi a_{-1}\delta(\omega+\omega_0) \\ &= \frac{2\pi}{2j}e^{j\frac{\pi}{4}}\delta(\omega-\omega_0) + \frac{2\pi}{2j}e^{-j\frac{\pi}{4}}\delta(\omega+\omega_0) \\ &= \frac{\pi}{j}\Big\{e^{j\frac{\pi}{4}}\delta(\omega-\frac{\pi}{3}) - e^{-j\frac{\pi}{4}}\delta(\omega+\frac{\pi}{3})\Big\} \end{aligned}$$

(g)

$$x[n] = \sin\left(\frac{n\pi}{2}\right) + \cos(n)$$

Solution: We know

$$\sin(\omega_0 n) \longleftrightarrow \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\Big\}$$

For $\cos(n)$ we have

$$\cos(n) = \frac{e^{jn} + e^{-jn}}{2}$$

We know

$$e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - \omega_0 - 2\pi l)\Big\}$$

but we cannot use this as $\cos(n)$, e^{jn} and e^{-jn} are not periodic.
Implicit in the last relationship is the fact that if ω_0 is a rational multiple of 2π, then in the time domain we have periodicity.
We can rewrite it as (for non-periodic complex exponentials) as

$$e^{j\omega' n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - \omega' - 2\pi l)\Big\}$$

Therefore

$$e^{jn} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - 1 - 2\pi l)\Big\}$$

$$e^{-jn} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega + 1 - 2\pi l)\Big\}$$

So

$$\cos(n) \longleftrightarrow \pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - 1 - 2\pi l) + \delta(\omega + 1 - 2\pi l)\Big\}$$

Overall

$$X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - 1 - 2\pi l) + \delta(\omega + 1 - 2\pi l)\Big\} + \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \Big\{\delta(\omega - \frac{\pi}{2} - 2\pi l) - \delta(\omega + \frac{\pi}{2} - 2\pi l)\Big\}$$

(h)

$$x[n] = 3^n \sin(\frac{\pi}{4}n)u[-n]$$

Solution: The signal is not periodic because of $u[n]$.

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} 3^n \sin(\frac{\pi}{4}n)u[-n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0} 3^n \sin(\frac{\pi}{4}n)e^{-j\omega n} \end{aligned}$$

Let $m = -n$ therefore

$$\begin{aligned} X(e^{j\omega}) &= \sum_{m=0}^{\infty} 3^{-m} \sin(\frac{\pi}{4}(-m))e^{j\omega m} \\ &= -\sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \frac{e^{j\frac{\pi}{4}m} - e^{-j\frac{\pi}{4}m}}{2j} e^{j\omega m} \\ &= -\frac{1}{2j} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \left(e^{j\frac{\pi}{4}}\right)^m \left(e^{j\omega m}\right)^m + \frac{1}{2j} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \left(e^{-j\frac{\pi}{4}}\right)^m \left(e^{j\omega m}\right)^m \end{aligned}$$

Using $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \quad |\alpha| < 1$

$$\begin{aligned} X(e^{j\omega}) &= -\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}e^{j\omega}} + \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}e^{j\omega}} \\ &= \frac{\frac{3}{\sqrt{2}}e^{j\omega}}{-9+3\sqrt{2}e^{j\omega}-e^{-j2\omega}} \end{aligned}$$

Problem Set 5-11

The following are the DTFTs of DT signals. Determine the corresponding signals $x[n]$ in the time domain.

(a)

$$X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-2j\omega} - 4e^{-3j\omega} + e^{-10j\omega}$$

Solution: We know $\delta[n] \longleftrightarrow 1$. Therefore

$$\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$$

$$\delta[n-1] \longleftrightarrow e^{-j\omega}$$

$$\delta[n-2] \longleftrightarrow e^{-j2\omega}$$

etc. Therefore,

$$x[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \delta[n-10]$$

(b)

$$X(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & 0 \leq |\omega| < \frac{\pi}{4},\ \frac{3\pi}{4} \leq |\omega| < \pi \end{cases}$$

Solution:

$$\begin{aligned} x[n] &= \int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}}(1)e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(1)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\frac{1}{jn}\Big|e^{j\omega n}\Big|_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\frac{1}{jn}\Big|e^{j\omega n}\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \frac{1}{\pi n}\{\sin(\frac{3\pi n}{4}) - \sin(\frac{\pi n}{4})\} \end{aligned}$$

(c)

$$X(e^{j\omega}) = \cos^2\omega + \sin^2 3\omega$$

Solution:

$$\begin{aligned} X(e^{j\omega}) &= \cos^2\omega + \sin^2 3\omega \\ &= \frac{1+\cos 2\omega}{2} + \frac{1-\cos 6\omega}{2} \\ &= 1 + \frac{1}{2}\cos 2\omega - \frac{1}{2}\cos 6\omega \\ &= 1 + \frac{1}{4}(e^{j2\omega} + e^{-j2\omega}) - \frac{1}{4}(e^{j6\omega} + e^{-j6\omega}) \end{aligned}$$

We know that

$$e^{-j\omega n_0} \longleftrightarrow \delta[n-n_0]$$

Therefore

$$x[n] = \delta[n] + \frac{1}{4}\delta[n+2] + \frac{1}{4}\delta[n-2] - \frac{1}{4}\delta[n+6] - \frac{1}{4}\delta[n-6]$$

(d)

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \{2\pi\delta(\omega - 2\pi l) + \pi\delta(\omega - \frac{\pi}{2} - 2\pi l) + \pi\delta(\omega + \frac{\pi}{2} - 2\pi l)\}, \quad -\infty < \omega < \infty$$

Solution: We can rewrite this in $-pi < \omega \leq \pi$ as

$$X(e^{j\omega}) = 2\pi\delta(\omega) + \pi\delta(\omega - \frac{\pi}{2}) + \pi\delta(\omega + \frac{\pi}{2})$$

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big\{2\pi\delta(\omega) + \pi\delta(\omega - \frac{\pi}{2}) + \pi\delta(\omega + \frac{\pi}{2})\Big\}e^{j\omega n}d\omega \\
&= e^{jn(0)} + \frac{1}{2}e^{jn\frac{\pi}{2}} + \frac{1}{2}e^{-jn\frac{\pi}{2}} \\
&= 1 + \cos\Big(n\frac{\pi}{2}\Big)
\end{aligned}$$

(e)

$$X(e^{j\omega}) = e^{\frac{-j\omega}{2}}, \quad -\pi \leq \omega \leq \pi$$

Solution:

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{-j\omega}{2}}e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-\frac{1}{2})}d\omega \\
&= \frac{1}{2\pi}\frac{1}{j(n-\frac{1}{2}})\Big|e^{-j\omega(n-\frac{1}{2})}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{\pi(n-\frac{1}{2})}\Big\{\frac{e^{j(n-\frac{1}{2})\pi} - e^{-j(n-\frac{1}{2})\pi}}{2j}\Big\} \\
&= \frac{\sin\big((n-\frac{1}{2})\pi\big)}{\pi(n-\frac{1}{2})} \\
&= \frac{1}{\pi(n-\frac{1}{2})}\big\{\sin(n\pi)\cos(\frac{\pi}{2}) - \cos(n\pi)\sin(\frac{\pi}{2})\big\} \\
&= \frac{-\cos(n\pi)}{\pi(n-\frac{1}{2})} \\
&= \frac{-(-1)^n}{\pi(n-\frac{1}{2})} \\
&= \frac{(-1)^{n+1}}{\pi(n-\frac{1}{2})}
\end{aligned}$$

(f)

$$X(e^{j\omega}) = \frac{1 - \frac{1}{3}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$$

Solution: Using partial fractions

$$X(e^{j\omega}) = \frac{\frac{2}{9}}{1 - \frac{1}{2}e^{j\omega}} + \frac{\frac{7}{9}}{1 + \frac{1}{4}e^{j\omega}}$$

We know that

$$a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

Therefore

$$x[n] = \frac{2}{9}\Big(\frac{1}{2}\Big)^n u[n] + \frac{7}{9}\Big(-\frac{1}{4}\Big)^n u[n]$$

(g)

$$X(e^{j\omega}) = \frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

Solution: Using partial fractions

$$X(e^{j\omega}) = \frac{-4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

We know that

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

Therefore

$$x[n] = \Big[-4\Big(\frac{1}{4}\Big)^n - 2(n+1)\Big(\frac{1}{4}\Big)^n + 8\Big(\frac{1}{2}\Big)^n\Big]u[n]$$

(h)

$$X(e^{j\omega}) = \frac{1-\left(\frac{1}{3}\right)^6 e^{-j6\omega}}{1-\frac{1}{3}e^{-j\omega}}$$

Solution:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1-\left(\frac{1}{3}\right)^6 e^{-j6\omega}}{1-\frac{1}{3}e^{-j\omega}} \\ &= \frac{1-\left(\frac{e^{-j\omega}}{3}\right)^6}{1-\frac{e^{-j\omega}}{3}} \end{aligned}$$

As

$$\sum_{n=0}^{M-1} \alpha^n = \frac{1-\alpha^M}{1-\alpha}, \quad \alpha \neq 1$$

therefore

$$\begin{aligned} X(e^{j\omega}) &= \sum_{l=0}^{5} \left(\frac{e^{-j\omega}}{3}\right)^l \\ &= 1 + \frac{1}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega} + \frac{1}{27}e^{-j3\omega} + \frac{1}{81}e^{-j4\omega} + \frac{1}{243}e^{-j5\omega} \end{aligned}$$

We know that

$$e^{-j\omega n_0} \longleftrightarrow \delta[n-n_0]$$

Therefore

$$x[n] = \delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{9}\delta[n-2] + \frac{1}{27}\delta[n-3] + \frac{1}{81}\delta[n-4] + \frac{1}{243}\delta[n-5]$$

Properties Discrete-time Fourier Transform

Problem Set 5-12

Consider the DT LTI system with frequency response

$$H(e^{j\omega}) = \frac{1}{2 - e^{-j\omega}} \tag{3}$$

and the identity

$$x[n - n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}) \tag{4}$$

(a) What is the DC gain (response at $\omega = 0$) for the DT LTI system in equation (3)?

Solution: DC gain is the response of a system for $\omega = 0$. For the given LTI system, DC gain $= H(0) = 1$.

(b) Sketch/plot the magnitude of the frequency response $\left|H(e^{j\omega})\right|$. How would you describe this system in terms of filtering?

Solution:

(c) State in words - making reference to terms such as magnitude and phase, and delay - the meaning of identity (4).

(d) Using the identity (4), or otherwise, determine the DT difference equation corresponding to equation (3) that relates input $x[n]$ to output $y[n]$.

(e) Determine the impulse response $h[n]$ corresponding to frequency response (3).

(f) If we cascade two filters with the same frequency response $H(e^{j\omega})$, what is the overall frequency response and the overall impulse response?

Problem Set 5-13

Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2}\, y[n-1] = x[n] \tag{5}$$

(a) Provided $X(e^{j\omega})$ is the frequency response of discrete time signal $x[n]$, prove the following identity

$$x[n - n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}\, X(e^{j\omega})$$

(b) Determine the frequency response $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega})$ of the system in (5).

(c) For the DT LTI system in (5), find the response $y[n]$ to the inputs $x[n]$ with the following Fourier transforms:

i)

$$X(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

ii)

$$X(e^{j\omega}) = \frac{1}{(1 - \frac{1}{4}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})}$$

Problem Set 5-14

Consider a discrete LTI system with input $x[n]$ and output $y[n]$ and is described by the following relation between $Y(e^{j\omega})$ and $X(e^{j\omega})$

$$Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}$$

(a) Is the system linear? Justify your answer

(b) Is the system time-invariant?

(c) Find impulse response of the system, that is, find $y[n]$ when $x[n] = \delta[n]$. Is the system causal?

Problem Set 5-15

Consider the DT LTI system with impulse response

$$h[n] = \frac{\sin Wn}{\pi n}$$

where $0 < W \leq \pi$. (Note that when $W = \pi$ we have $h[n] = \delta[n]$.)

(a) Determine the possible values of W, within the range $0 < W \leq \pi$, such that

$$h[5] = 0.$$

Solution: The zeroes of $\sin(x)$ are at multiples of π. We require

$$\sin(5W) = 0$$

which implies $W = k\pi/5$ where k is an integer. If $0 < W \leq \pi$ then there are 5 possible values:

$$\boxed{W \in \left\{\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi\right\}.}$$

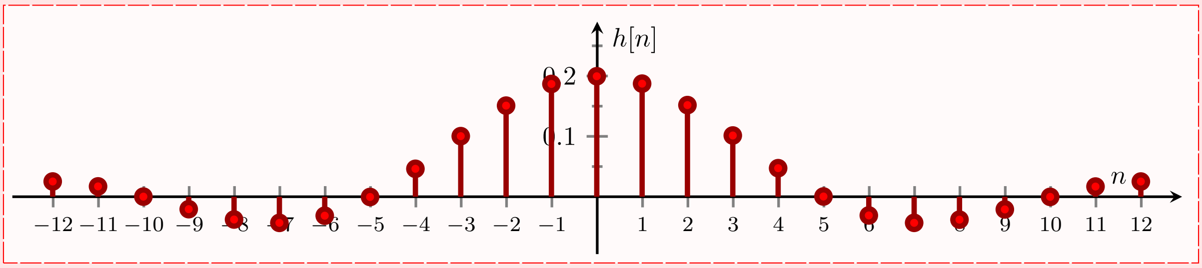
□

(b) Sketch the impulse response $h[n]$ and frequency response $H(e^{j\omega})$ for the system with the least value of $W > 0$ such that $h[5] = 0$.

Solution: The least value is $W = \pi/5$, so

$$h[n] = \frac{\sin(\pi n/5)}{\pi n},$$

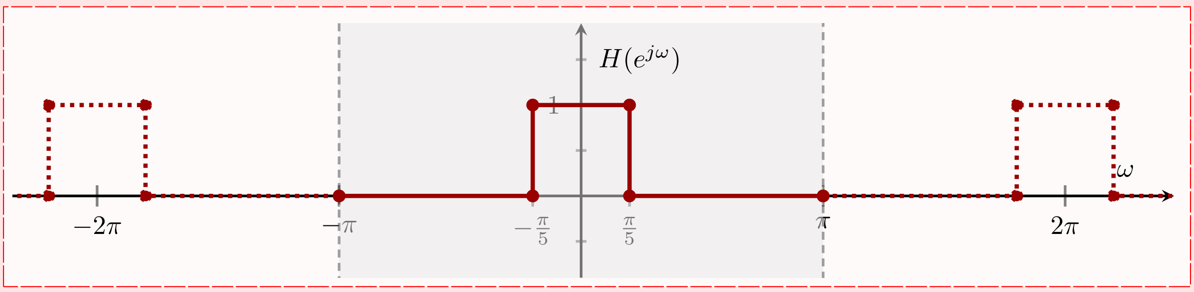
where we note that $h[0] = 1/5$, and is plotted below:



The frequency response is given by

$$H(e^{j\omega}) = \chi_{[-\pi/5,+\pi/5]}(\omega) = \begin{cases} 1 & |\omega| \leq \pi/5 \\ 0 & \pi/5 < |\omega| \leq \pi \end{cases}, \quad |\omega| \leq \pi,$$

noting it is periodic with period 2π, and is plotted below:



□

(c) Consider the DT signal

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right).$$

i) Determine the Fourier transform of $x[n]$, $X(e^{j\omega})$, in the interval $-\pi \leq \omega \leq \pi$? (You can use the identity (5.24) given in Example 5.5 in the text. Delta functions should be labeled with their complex amplitude in round brackets.)

Solution: Signal $x[n]$ has frequencies at $\pm\pi/8$ and $\pm\pi/4$ within the range $|\omega| \le \pi$. Naturally these frequencies are repeated given $X(e^{j\omega})$ is periodic with period 2π — in all that follows we only specify the Fourier transform within the range $|\omega| \le \pi$ to avoid clutter.

$$\sin\left(\frac{\pi n}{8}\right) \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}\,\delta(\omega - \pi/8) - \frac{\pi}{j}\,\delta(\omega + \pi/8)$$

$$\cos\left(\frac{\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} \pi\,\delta(\omega - \pi/4) + \pi\,\delta(\omega + \pi/4)$$

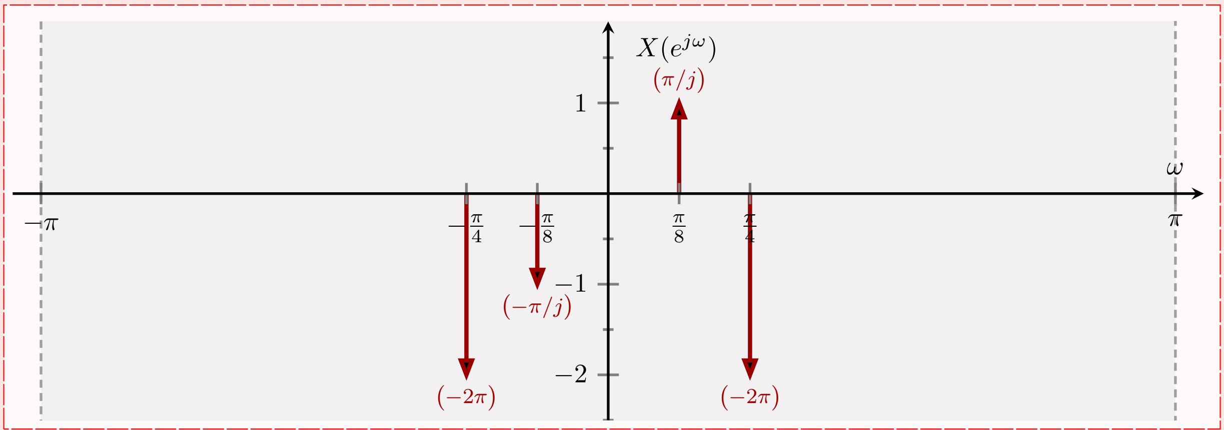
and therefore

$$x[n] \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}\,\delta(\omega - \pi/8) - \frac{\pi}{j}\,\delta(\omega + \pi/8) - 2\pi\,\delta(\omega - \pi/4) - 2\pi\,\delta(\omega + \pi/4).$$

That is

$$\boxed{X(e^{j\omega}) = \frac{\pi}{j}\,\delta(\omega - \pi/8) - \frac{\pi}{j}\,\delta(\omega + \pi/8) - 2\pi\,\delta(\omega - \pi/4) - 2\pi\,\delta(\omega + \pi/4),}$$

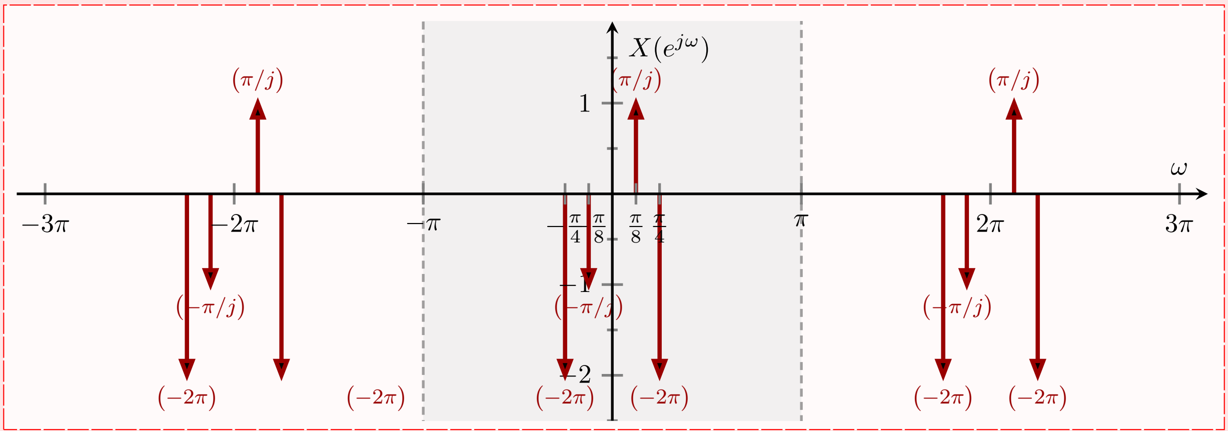
which is shown below:



□

ii) Plot $X(e^{j\omega})$ in the range $-3\pi \le \omega \le 3\pi$. (This range of ω should include three periods of $X(e^{j\omega})$.)

Solution: The periodicity of $X(e^{j\omega})$ makes this easy.



Not sure why I asked this question — encroaching dementia no doubt. □

(d) Suppose the signal $x[n]$, from part (c), is input to LTI systems with the following impulse responses. Determine the output in each case.

i) $h_1[n] = \dfrac{\sin(\pi n/6)}{\pi n}$

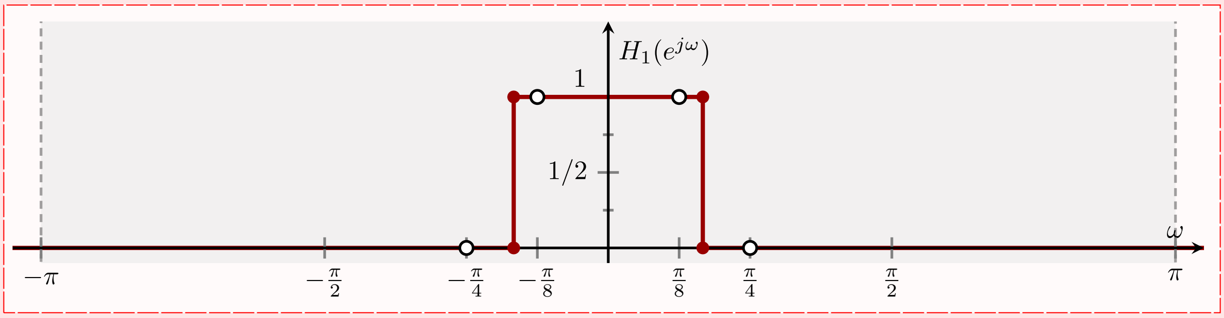
Solution:

$$\boxed{x[n] \star h_1[n] = \sin\left(\frac{\pi n}{8}\right).}$$

An ideal low-pass filter with cut-off $\omega_c > 0$ is given by:

$$\frac{\sin \omega_c t}{\pi t} \xleftrightarrow{\mathcal{F}} \chi_{[-\omega_c, +\omega_c]}(\omega).$$

So in this case the cut-off is $\omega_c = \pi/6$. This removes the higher frequency components in $x[n]$ at $\pm\pi/4$.



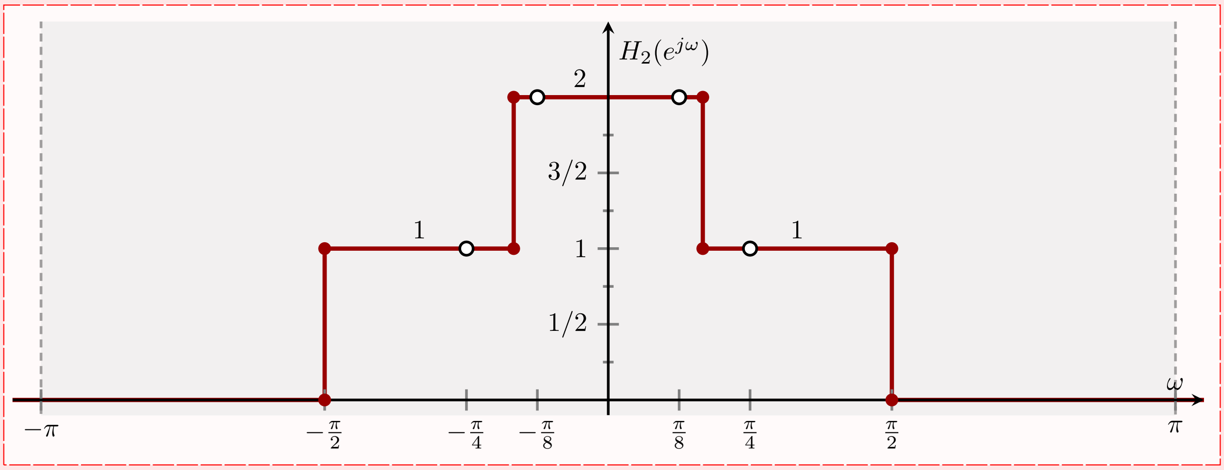
The open circles tag the input frequencies and the corresponding filter gains. □

ii) $h_2[n] = \dfrac{\sin(\pi n/6)}{\pi n} + \dfrac{\sin(\pi n/2)}{\pi n}$

Solution:

$$\boxed{x[n] \star h_2[n] = 2\,\sin\left(\frac{\pi n}{8}\right) - 2\,\cos\left(\frac{\pi n}{4}\right).}$$

Impulse response $h_2[n]$ is the sum (parallel connection) of two ideal low-pass filters with cutoffs $\pi/6$, as in part (a), and $\pi/2$. The lower frequency components, at $\pm\pi/8$, are passed by both filters, which each have gain 1, and thus get multiplied by 2. The higher frequency components in $x[n]$ at $\pm\pi/4$ get passed by one filter, and thus get multiplied by 1 (unchanged).



The open circles tag the input frequencies and the corresponding filter gains. □

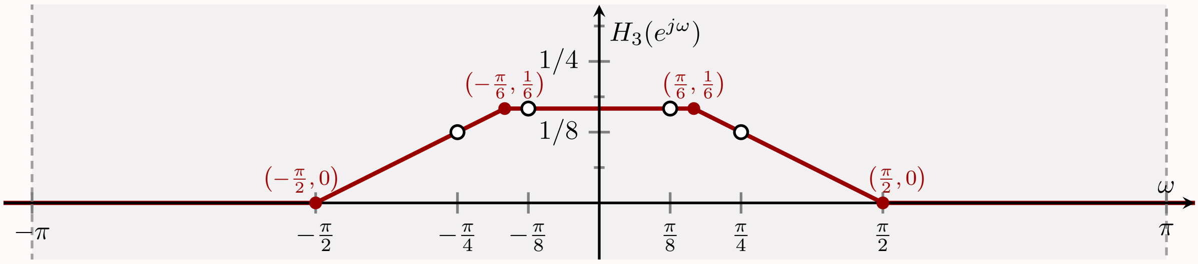
iii) **Solution:**

$$\boxed{x[n] \star h_3[n] = \frac{1}{6}\,\sin\left(\frac{\pi n}{8}\right) - \frac{1}{4}\,\cos\left(\frac{\pi n}{4}\right).}$$

Impulse response $h_3[n]$ is the product of two ideal low-pass filter impulse responses with cutoffs $\pi/6$, and $\pi/3$. (This is not a cascade or series connection as that would result in a convolution of the two impulse responses.) What we have is a convolution of purely-real two brick-wall frequency responses of different widths, formally

$$z_1[n]z_2[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} Z_1(e^{j\omega}) \star Z_2(e^{j\omega}),$$

which gives a trapezoid shape in this case. So we just need to find the corners.
The point where there is no overlap is at $\omega = \pm\pi/2$ because $\pi/2 = \pi/6 + \pi/3$. That is, there are corners at $(\pm\pi/2, 0)$.
There is complete overlap whilst the narrow cut-off (width $2\times\pi/6$) fits inside the wide cut-off (width $2\times\pi/3$), so the flat top section is when $\omega \in [-\pi/6, \pi/6]$. The height (DC gain) is $2 \times (\pi/6)/(2\pi) = 1/6$ from the convolution expression. That is, there are corners at $(\pm\pi/6, 1/6)$.



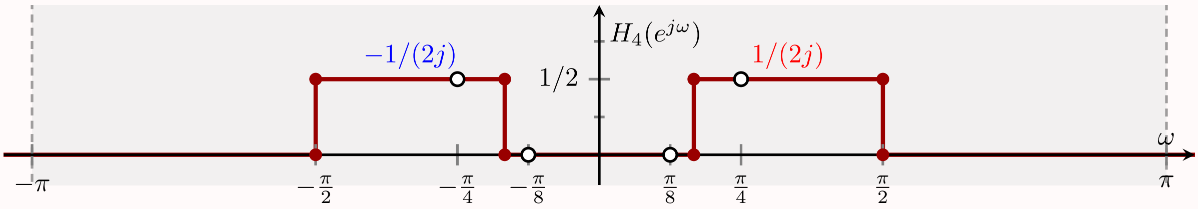
So we can see the effect of the filter is to attenuate the $\pm\pi/8$ frequencies by $1/6$ (inner open circles) and attenuate the $\pm\pi/4$ frequencies by $1/8$ (outer open circles). $\square$

iv) $h_4[n] = \dfrac{\sin(\pi n/6)\,\sin(\pi n/3)}{\pi n}$

Solution:

$$\boxed{x[n] \star h_4[n] = -\sin\left(\frac{\pi n}{4}\right).}$$

Impulse response $h_4[n]$ is sine modulated low-pass filter and the figure below shows the resulting $H_4(e^{j\omega})$. In fact this can be taken either way: a higher frequency carrier modulated lower cutoff frequency low-pass filter (no overlap), or a lower carrier frequency modulated higher cutoff frequency low-pass filter (where the inner overlap cancels) — they yield, of course, the same result:



This band-pass filter passes the higher frequency $\pm\pi/4$ and nulls the lower frequency $\pm\pi/8$. The high frequency on input is

$$-2\,\cos\left(\frac{\pi n}{4}\right) \overset{\mathcal{F}}{\longleftrightarrow} -2\pi\,\delta(\omega-\pi/4) - 2\pi\,\delta(\omega+\pi/4)$$

and the band-pass filter applies the color coded gains to reveal the answer:

$$\boxed{-\sin\left(\frac{\pi n}{4}\right) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}(-2\pi)\,\delta(\omega-\pi/4) + \frac{-1}{2j}(-2\pi)\,\delta(\omega+\pi/4).}$$

$\square$

Convolution and Difference Equations in the Frequency Domain using DTFT

Problem Set 5-16

Consider two DT LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{1 + 2e^{-j\omega}}{1 - (1/3)e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 + (1/3)e^{-j\omega} + (1/9)e^{-j2\omega}}$$

The following properties may be useful:

$$z[n-k] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega k}\, Z(e^{j\omega}), \quad k \in \mathbb{Z} \text{ (integer)}$$

$$z_{(k)}[n] = \begin{cases} z[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} Z(e^{jk\omega}),$$

where

$$z[n] \overset{\mathcal{F}}{\longleftrightarrow} Z(e^{j\omega}).$$

(a) Find the **difference equation** describing $H_1(e^{j\omega})$ where $x_1[n]$ is the input and $y_1[n]$ is the output.

Solution: The frequency response is the ratio of the output signal Fourier transform to the input signal Fourier transform:

$$H_1(e^{j\omega}) = \frac{Y_1(e^{j\omega})}{X_1(e^{j\omega})} = \frac{1 + 2e^{-j\omega}}{1 - (1/3)e^{-j\omega}}$$

That is,

$$\Big(1 - (1/3)e^{-j\omega}\Big) Y_1(e^{j\omega}) = \Big(1 + 2e^{-j\omega}\Big) X_1(e^{j\omega})$$

thus

$$Y_1(e^{j\omega}) - (1/3)e^{-j\omega} Y_1(e^{j\omega}) = X_1(e^{j\omega}) + 2e^{-j\omega} X_1(e^{j\omega})$$

and using the time-shift identity gives

$$\boxed{y_1[n] - (1/3)y_1[n-1] = x_1[n] + 2x_1[n-1].}$$

□

(b) Find the **difference equation** describing $H_2(e^{j\omega})$ where $x_2[n]$ is the input and $y_2[n]$ is the output.

Solution: Similar to part (a) we have

$$H_2(e^{j\omega}) = \frac{Y_2(e^{j\omega})}{X_2(e^{j\omega})} = \frac{1}{1 + (1/3)\, e^{-j\omega} + (1/9)\, e^{-j2\omega}}$$

from which we get

$$\boxed{y[n] + (1/3\,)y[n-1] + (1/9)\, y[n-2] = x[n].}$$

□

(c) Find the **difference equation** describing the cascade of $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$.

Solution: When cascading the frequency responses multiply

$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})\, H_2(e^{j\omega}) \\ &= \frac{1 + 2e^{-j\omega}}{1 - (1/3)e^{-j\omega}} \; \frac{1}{1 + (1/3)\, e^{-j\omega} + (1/9)\, e^{-j2\omega}} \\ &= \frac{1 + 2e^{-j\omega}}{1 - (1/3)\, e^{-j\omega} + (1/3)\, e^{-j\omega} - (1/9)\, e^{-j2\omega} + (1/9)\, e^{-j2\omega} - (1/27)\, e^{-j3\omega}} \end{aligned}$$

which simplifies to

$$\boxed{H(e^{j\omega}) = \frac{1 + 2\,e^{-j\omega}}{1 - (1/27)\,e^{-j3\omega}}}$$

from which we get

$$\boxed{y[n] - (1/27)\,y[n-3] = x[n] + 2\,x[n-1].}$$

□

(d) Determine the **impulse response** of the cascade of $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$, that is,

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H_1(e^{j\omega})\,H_2(e^{j\omega}).$$

To solve this part you can use any method you like but remember that the impulse response is what comes out when the input is $\delta[n]$.

Solution: *Proof 1:* Run the difference equation from part (c) rewritten for ease of computation:

$$(1/27)\,y[n-3] + x[n] + 2\,x[n-1] = y[n]$$

with the special input $x[n] = \delta[n]$ and output $y[n] = h[n]$, and record what is going on:

Time n	$(1/27)\,h[n-3]$	$\delta[n]$	$2\,\delta[n-1]$	Output $h[n]$
0	0	1	0	1
1	0	0	2	2
2	0	0	0	0
3	$(1/27)$	0	0	$(1/27)$
4	$(2/27)$	0	0	$(2/27)$
5	0	0	0	0
6	$(1/27)^2$	0	0	$(1/27)^2$
7	$2(1/27)^2$	0	0	$2(1/27)^2$
8	0	0	0	0
9	$(1/27)^3$	0	0	$(1/27)^3$
10	$2(1/27)^3$	0	0	$2(1/27)^3$
11	0	0	0	0
12	$(1/27)^4$	0	0	$(1/27)^4$
13	$2(1/27)^4$	0	0	$2(1/27)^4$

The pattern is evident. Firstly we note that there is no contribution from $x[n]$ or $x[n-1]$ after time $n = 1$ so the output can be written

$$h[n] = (1/27)\,h[n-3], \quad \text{for } n > 1.$$

We can write the answer

$$\boxed{h[n] = \begin{cases} (1/27)^{n/3} & n = 0, 3, 6, 9, 12, \ldots \\ 2\,(1/27)^{(n-1)/3} & n = 1, 4, 7, 10, 13, \ldots \\ 0 & \text{otherwise} \end{cases}}$$

Noting that $h[n]$ is causal (implying $h[n] = 0$ for $n < 0$). □

Proof 2: Another solution starts from

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 + 2\,e^{-j\omega}}{1 - (1/27)\,e^{-j3\omega}} \\ &= \frac{1}{1 - (1/27)\,e^{-j3\omega}} + \frac{2\,e^{-j\omega}}{1 - (1/27)\,e^{-j3\omega}} \end{aligned}$$

We have

$$z[n] \triangleq (1/27)^n\,u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - (1/27)\,e^{-j\omega}}$$

and, from the provided identity in the question statement,

$$\left\{ z_{(3)}[n] = \begin{cases} z[n/3] & \text{if } n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases} \right\} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - (1/27)\,e^{-j3\omega}}.$$

For the second component in $H(e^{j\omega})$, we observe that $2\,e^{-j\omega}$ is a delay of one and a gain of two. Thus we end up with the previous answer. □

Problem Set 5-17

Find the convolution $y[n] = x[n] * h[n]$ where

$$X(e^{j\omega}) = 3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega}$$

$$H(e^{j\omega}) = -e^{j\omega} + 2e^{-j2\omega} + e^{j4\omega}$$

Solution: Using convolution in the frequency domain

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\ &= (3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega})(-e^{j\omega} + 2e^{-j2\omega} + e^{j4\omega}) \\ &= 3e^{j5\omega} + e^{j4\omega} - e^{j3\omega} - 3e^{j2\omega} + e^{j\omega} + 1 + 6e^{-j\omega} - 2e^{-j3\omega} + 4e^{-j5\omega} \end{aligned}$$

Therefore,

$$y[n] = 3\delta[n+5] + \delta[n+4] - \delta[n+3] - 3\delta[n+2] + \delta[n+1] + \delta[n] + 6\delta[n-1] - 2\delta[n-3] + 4\delta[n-5].$$

Sampling

Problem Set 5-18

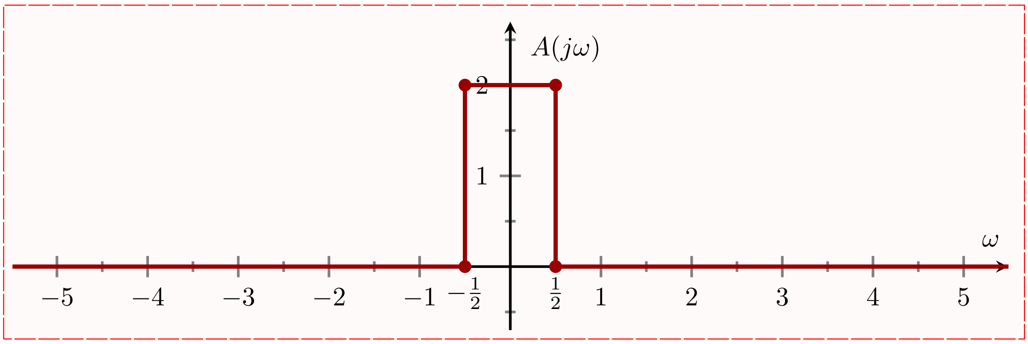
(a) The impulse response of an ideal low-pass filter with maximum frequency ω_M is given by

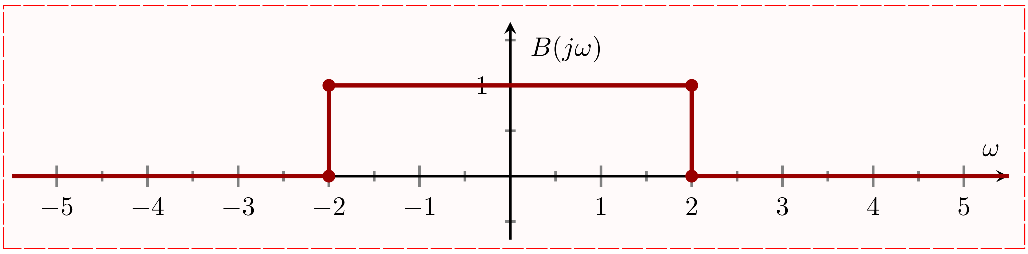
$$\frac{\sin \omega_M t}{\pi t} \xleftrightarrow{\mathcal{F}} \chi_{[-\omega_M,+\omega_M]}(\omega), \quad 0 \leq \omega_M.$$

Infer or derive the **time domain representations** of the two signals,

$$a(t) \xleftrightarrow{\mathcal{F}} A(j\omega) \qquad b(t) \xleftrightarrow{\mathcal{F}} B(j\omega)$$

whose frequency contents are shown below (all frequency representations are real-valued and have zero phase).





Solution: For $A(j\omega)$ we see that $\omega_M = 1/2$ and we can write

$$A(j\omega) = 2\,\chi_{[-1/2,+1/2]}(\omega)$$

and so

$$\boxed{a(t) = 2\,\frac{\sin(t/2)}{\pi t}.}$$

For $B(j\omega)$ we see that $\omega_M = 2$ and we can write

$$B(j\omega) = \chi_{[-2,+2]}(\omega)$$

and so

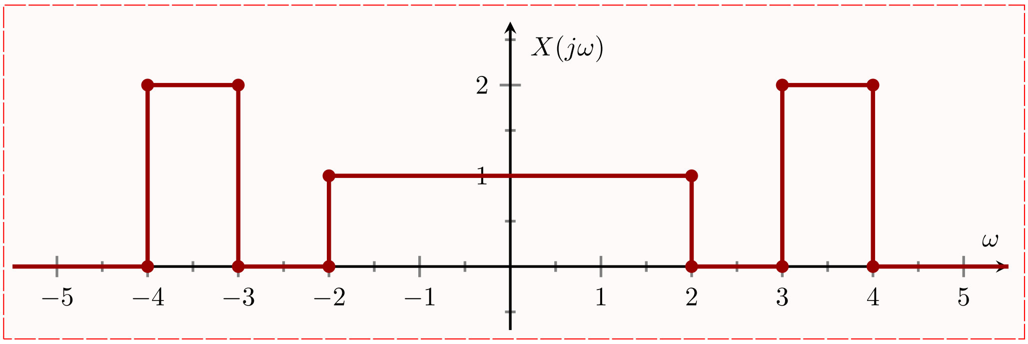
$$\boxed{b(t) = \frac{\sin(2t)}{\pi t}.}$$

□

(b) Consider the signal

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

whose frequency content is shown in the figure below. Use the multiplication property and superposition to find the corresponding **time domain representation** using the results in (a).



Solution: By observation we have

$$X(j\omega) = A\big(j(\omega + 3.5)\big) + B(j\omega) + A\big(j(\omega - 3.5)\big).$$

where the order of the terms matches those in the figure going left to right.

Now translation in frequency, which is a form of "Complex Exponential AM — Modulation", is achieved in the time-domain by multiplying by $e^{j\omega_c t}$ where ω_c is the frequency offset (slides 758–759):

$$a(t)\, e^{j\omega_c t} \overset{\mathcal{F}}{\longleftrightarrow} A\big(j(\omega - \omega_c)\big)$$

Therefore,

$$\begin{aligned} x(t) &= a(t)\, e^{-j3.5t} + b(t) + a(t)\, e^{j3.5t} \\ &= 2\, a(t) \cos(3.5t) + b(t) \end{aligned}$$

We can write this as;

$$\boxed{x(t) = 4\, \cos(3.5t)\, \frac{\sin(t/2)}{\pi t} + \frac{\sin(2t)}{\pi t},}$$

or

$$\boxed{x(t) = \frac{1}{\pi t}\Big(4\, \cos(3.5t)\, \sin(t/2) + \sin(2t)\Big).}$$

□

(c) Consider a sampled version of the signal, $x(t)$ in part (b), given by

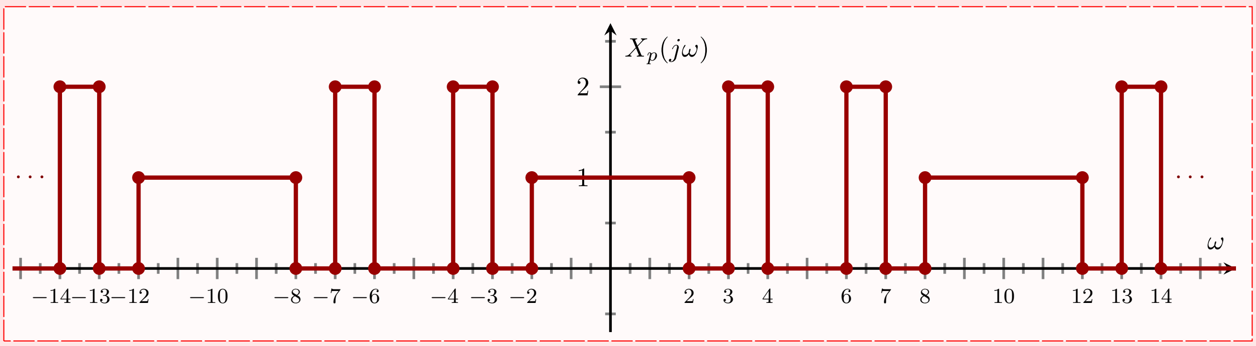
$$\begin{aligned} x_p(t) = x(t)\, p(t) = x(t) &\sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT). \end{aligned}$$

Draw the **frequency content** of $x_p(t)$ when the sampling rate is

$$\omega_s = 10 \text{ rad/sec},$$

corresponding to $T = \pi/5$ sec.

Solution: Images of the spectrum $X(j\omega)$ are centered at integer multiples of $\omega_s = 10$ as shown below (the centre 3 images are shown, the spectrum is periodic with frequency interval 10):



□

(d) Consider the recovery of the sampled signal $x_p(t)$ in part (c), where the sampling rate is $\omega_s = 10$ rad/sec, with an ideal low-pass filter whose cutoff or bandwidth is given by ω_c

rad/sec.

i) What is the **least bandwidth**, ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?

Solution:

$$\boxed{\omega_c = 4 \text{ rad/sec}}$$

Clearly, $\omega_c = 4$ is the least bandwidth given that is the bandwidth of $x(t)$. □

ii) What is the **maximum bandwidth**, ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?

Solution:

$$\boxed{\omega_c = 6 \text{ rad/sec}}$$

With $\omega_c = 6$ we still remove all higher frequency images.

Now you might suspect there is an issue with what happens with the edge at $\omega = 6$, is there some residual? Whatever there is there is no energy. The only way to get something non-zero at a single frequency is to have at least a delta function at the point. The residual is not a delta function and has zero area and zero squared area. □

Problem Set 5-19

(a)

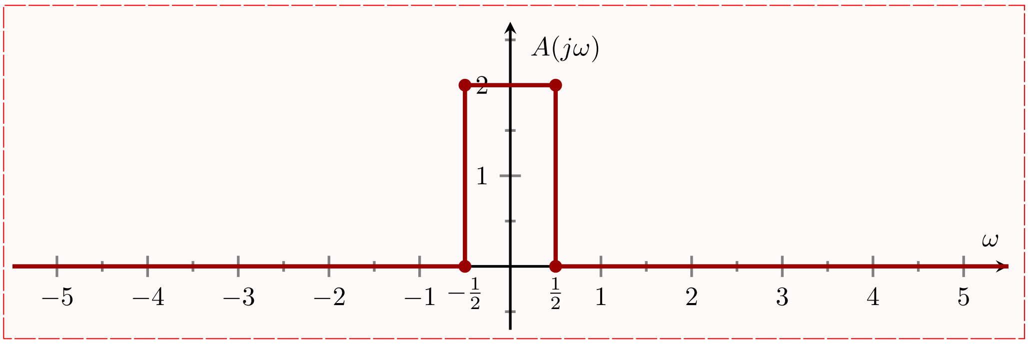
The impulse response of an ideal low-pass filter with maximum frequency ω_M is given by

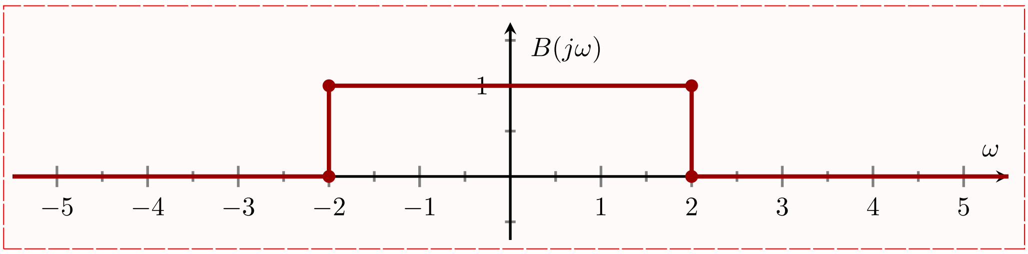
$$\frac{\sin \omega_M t}{\pi t} \stackrel{\mathcal{F}}{\longleftrightarrow} \chi_{[-\omega_M, +\omega_M]}(\omega), \quad 0 \le \omega_M.$$

Infer or derive the **time domain representations** of the two signals,

$$a(t) \stackrel{\mathcal{F}}{\longleftrightarrow} A(j\omega) \qquad b(t) \stackrel{\mathcal{F}}{\longleftrightarrow} B(j\omega)$$

whose frequency contents are shown below (all frequency representations are real-valued and have zero phase).





Solution: Solution a) For $A(j\omega)$ we see that $\omega_M = 1/2$ and we can write

$$A(j\omega) = 2\,\chi_{[-1/2,+1/2]}(\omega)$$

and so

$$\boxed{a(t) = 2\,\frac{\sin(t/2)}{\pi t}.}$$

For $B(j\omega)$ we see that $\omega_M = 2$ and we can write

$$B(j\omega) = \chi_{[-2,+2]}(\omega)$$

and so

$$\boxed{b(t) = \frac{\sin(2t)}{\pi t}.}$$

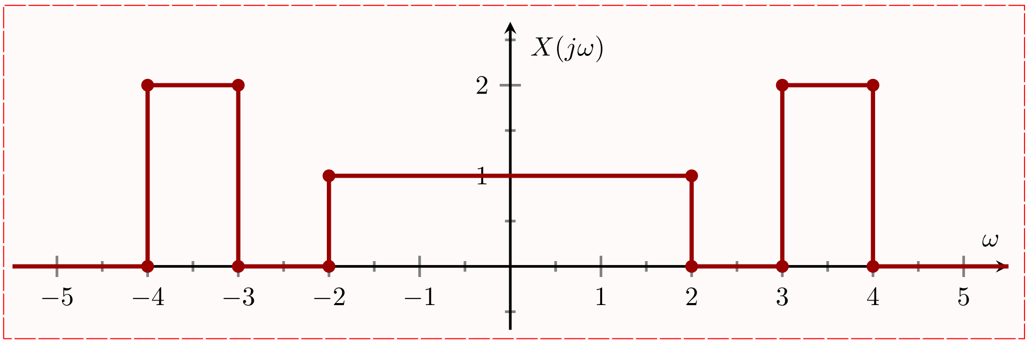
Solution b)
Use inverse Fourier transform of the frequency response to calculate the impulse response of the two signals. See page 516 of lecture slides.

□

(b) Consider the signal

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

whose frequency content is shown in the figure below. Use the multiplication property and superposition to find the corresponding **time domain representation** using the results in (a).



Solution: By observation we have

$$X(j\omega) = A\big(j(\omega + 3.5)\big) + B(j\omega) + A\big(j(\omega - 3.5)\big).$$

where the order of the terms matches those in the figure going left to right.
Now translation in frequency, which is a form of "Complex Exponential AM — Modulation", is achieved in the time-domain by multiplying by $e^{j\omega_c t}$ where ω_c is the frequency offset (slides 758–759):

$$a(t)\, e^{j\omega_c t} \stackrel{\mathcal{F}}{\longleftrightarrow} A\big(j(\omega - \omega_c)\big)$$

Therefore,

$$\begin{aligned} x(t) &= a(t)\, e^{-j3.5t} + b(t) + a(t)\, e^{j3.5t} \\ &= 2\, a(t) \cos(3.5t) + b(t) \end{aligned}$$

We can write this as;

$$\boxed{x(t) = 4\, \cos(3.5t)\, \frac{\sin(t/2)}{\pi t} + \frac{\sin(2t)}{\pi t},}$$

or

$$\boxed{x(t) = \frac{1}{\pi t}\Big(4\, \cos(3.5t)\, \sin(t/2) + \sin(2t)\Big).}$$

□

(c) Consider a sampled version of the signal, $x(t)$ in part (b), given by

$$\begin{aligned} x_p(t) = x(t)\, p(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT). \end{aligned}$$

Draw the **frequency content** of $x_p(t)$ when the sampling rate is

$$\omega_s = 10 \text{ rad/sec},$$

corresponding to $T = \pi/5$ sec.

Solution:

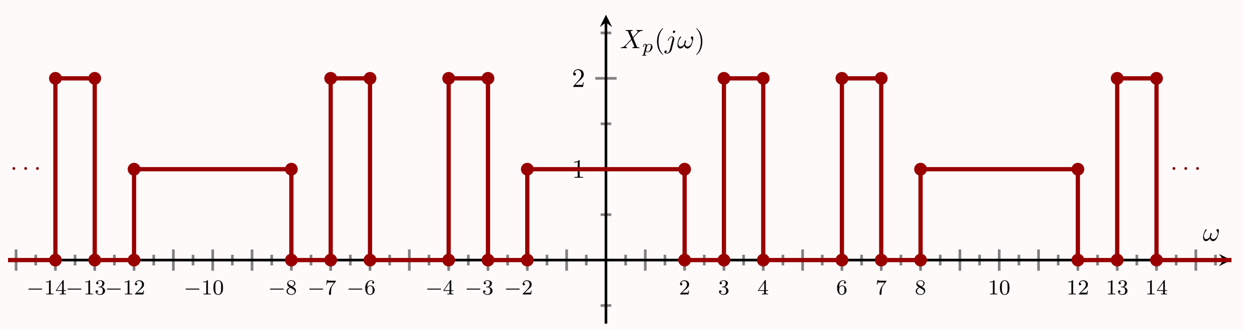
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The Fourier transform of $p(t)$ is

$$P(j\omega) = \frac{2\pi}{T} \sum \delta(\omega - n\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi}[X(j\omega)P(j\omega)]$$

Images of the spectrum $X(j\omega)$ are centered at integer multiples of $\omega_s = 10$ as shown below (the centre 3 images are shown, the spectrum is periodic with frequency interval 10):



□

(d) Consider recovery with an ideal pass filter whose cutoff or bandwidth is given by ω_c rad/sec. What are the minimum and maximum bandwidths for ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?

Solution:

Perfect recovery is possible if $\omega_M < \omega_c < (\omega_s - \omega_M)$, where $\omega_M = 4$ and $\omega_s = 10$. Therefore $4 < \omega_c < 6$ are the minimum and maximum bandwidths for ω_c.

□