Signal Processing

ENGN2228

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Second Semester

Lectures 10 and 11

Part 0 Slide 1/many Convenor: R. A. Kennedy

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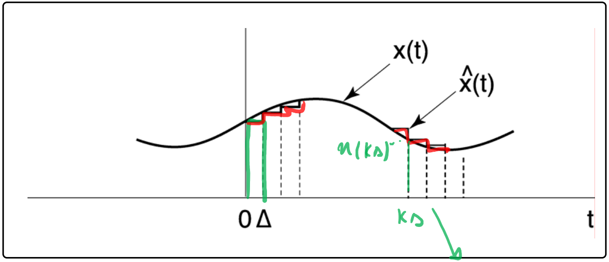
CT Signals and Systems – Preamble

- By and large, features and properties of CT signals and systems look very similar or are the same as their DT counterparts. The terminology is largely identical.
- When a CT System is LTI then it can be characterized in terms of an impulse response which itself is in the form of a signal. That is, knowing the impulse response signal of a CT LTI System completely characterizes it (apart from initial conditions).

CT Signals and Systems – Representation

Approximate any CT signal $x(t)$ as a sum of shifted, scaled rectangular pulses:



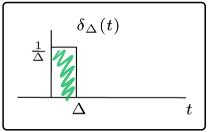
$$\widehat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$

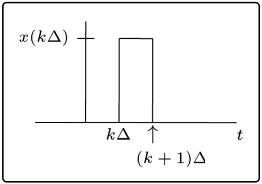
(Here $\widehat{x}(t)$ denotes an approximation to $x(t)$.)

CT Signals and Systems – Representation

Define a rectangle pulse

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$$





$$= x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

Whence

$$\widehat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

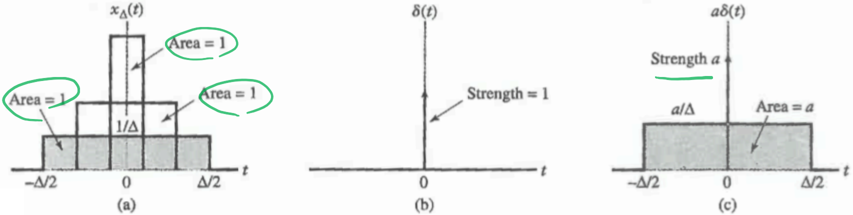
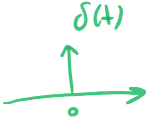


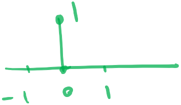
FIGURE 1.42 (a) Evolution of a rectangular pulse of unit area into an impulse of unit strength (i.e., unit impulse). (b) Graphical symbol for unit impulse. (c) Representation of an impulse of strength a that results from allowing the duration Δ of a rectangular pulse of area a to approach zero.

$\Delta \to 0$

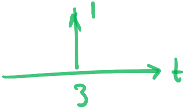
$\delta_\Delta(t) \longrightarrow$



$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

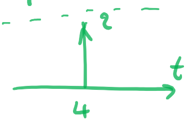


$\delta(t-3)$



$$\int_{-\infty}^{\infty} \delta(t-3)\,dt = 1$$

$2\delta(t-4) =$



$$\int_{-\infty}^{\infty} 2\delta(t-4) = 2$$

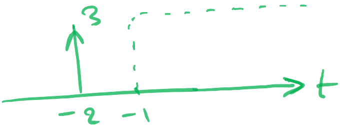
CT Signals and Systems – Representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

In the limit as $\Delta \to 0$ we infer

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Example: $\int_{-1}^{\infty} 3\,\delta(t+2) = 0$



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CT Signals and Systems – Response of a CT LTI System

$x[n] \longrightarrow \boxed{\text{DT LTI}} \longrightarrow y[n]$

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

$$\delta_\Delta \longrightarrow h_\Delta(t)$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta}_{\widehat{x}(t)} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,h_\Delta(t-k\Delta)\Delta}_{\widehat{y}(t)}$$

$$\Downarrow \qquad (\text{as } \Delta \to 0)$$

$$\delta(t) \longrightarrow h(t)$$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \boxed{\underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}_{y(t)}}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

DT impulse res.

Convolution Integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Interpretation

I $h(\tau) \xrightarrow{\text{Flip}} h(-\tau)$

II $h(-\tau) \xrightarrow{\text{Shift}} h(t-\tau)$

III $h(t-\tau) \xrightarrow{\text{Multiply}} x(\tau)h(t-\tau)$

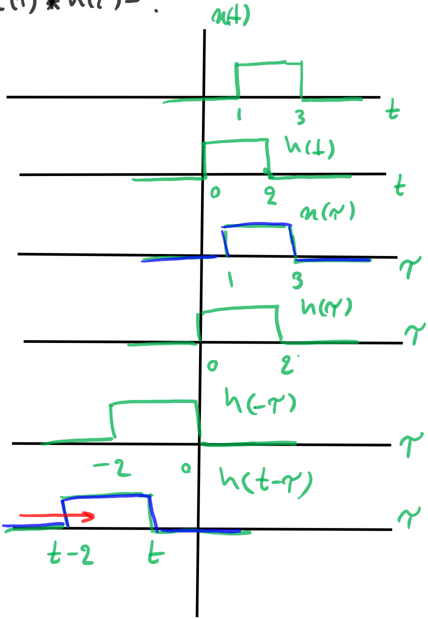
IV $x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = y(t)$

Example:

$x(t) = u(t-1) - u(t-3)$

$h(t) = u(t) - u(t-2)$

$x(t) * h(t) = ?$



For $t < 1$: $y(t) = 0$

there is no overlap

For $1 \leq t \leq 3$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

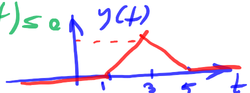
$$= \int_{1}^{t} (1)(1) d\tau = \tau \Big|_{1}^{t} = t - 1$$

For $3 \leq t \leq 5$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau =$$

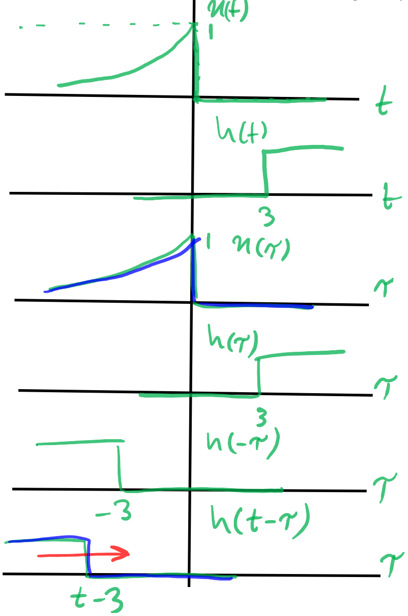
$$= \int_{t-2}^{3} (1)(1) d\tau = \tau \Big|_{t-2}^{3} = 5 - t$$

For $t \geq 5$ $y(t) = 0$



Another example: $x(t) = e^{2t}u(-t)$

$h(t) = u(t-3)$



For $t \leqslant 3$:

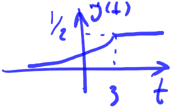
$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\,d\tau = \frac{1}{2} e^{2\tau}\Big|_{-\infty}^{t-3}$$

$$= \frac{1}{2}\left(e^{2(t-3)} - \underbrace{\cancel{e^{-\infty}}}_{0}\right) = \frac{1}{2} e^{2(t-3)}$$

For $3 \leqslant t$

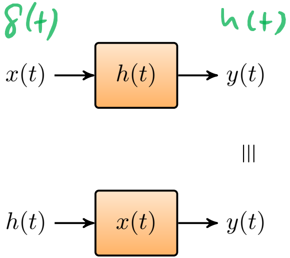
$$y(t) = \int_{-\infty}^{0} e^{2\tau}\,d\tau = \frac{1}{2} e^{2\tau}\Big|_{-\infty}^{0}$$

$$= \frac{1}{2}$$



$$y(t) = \begin{cases} \frac{1}{2} e^{2(t-3)} & t \leqslant 3 \\ \frac{1}{2} & t \geqslant 3 \end{cases}$$

CT System Properties – Commutativity

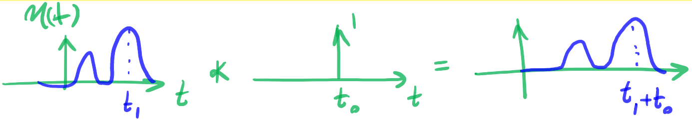


$$
\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= h(t) \star x(t)
\end{aligned}
$$

- This follows from (change variables to $\sigma = t - \tau$)

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} x(t-\sigma) h(\sigma)\, d\sigma
\end{aligned}
$$

CT System Properties – Sifting Property



Sifting Property:

$$x(t) \star \delta(t-t_0) = x(t-t_0)$$

That is, a system response $x(t) \xrightarrow{h(t)} y(t)$ of the form

$$h(t) = \delta(t-t_0),$$

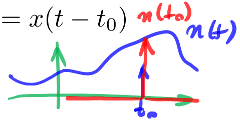
acts as a time delay of t_0, that is,

$$y(t) = x(t-t_0).$$

Note that $x(t) \star \delta(t - t_0)$ is different from $x(t)\,\delta(t - t_0)$. The expression on the left is convolution of two signals and the expression on the right is pointwise multiplication.

- Convolution

I
$$x(t) \star \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\, d\tau$$
$$= x(t - t_0)$$



- Pointwise multiplication

II
$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

which is a scaled impulse response.

III
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)\,dt = x(t_0)$$

Examples:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-5)\,dt = x(5)$$

III $t_0 = 5$

$$\int_{-1}^{1} \cos(t)\,\delta(t)\,dt = \cos(0)$$

$$\int_{1}^{3} \cos(t)\,\delta(t-2)\,dt = \cos(2)$$

$\cos(2)\,\delta(t-2)$

$$\int_{-\infty}^{\infty} e^{-4(t-1)}\,\delta(t-2)\,dt = e^{-4(2-1)} = e^{-4}$$

$$\int_{-\infty}^{\infty} u(t-3)\,\delta(t+4)\,dt = u(-4-3)$$

$$= u(-7)$$

$\delta(t) \longrightarrow \boxed{} \longrightarrow y(t) = ?$

Integrator Property:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \qquad (2)$$

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow y(t)$$

So we need to determine the system response $h(t)$ to synthesize the integrator (2). But $y(t) = h(t)$ is output when $x(t) = \delta(t)$ is input, so substituting into (2)

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\, d\tau \\ &= u(t) \quad \text{(step function)} \end{aligned}$$

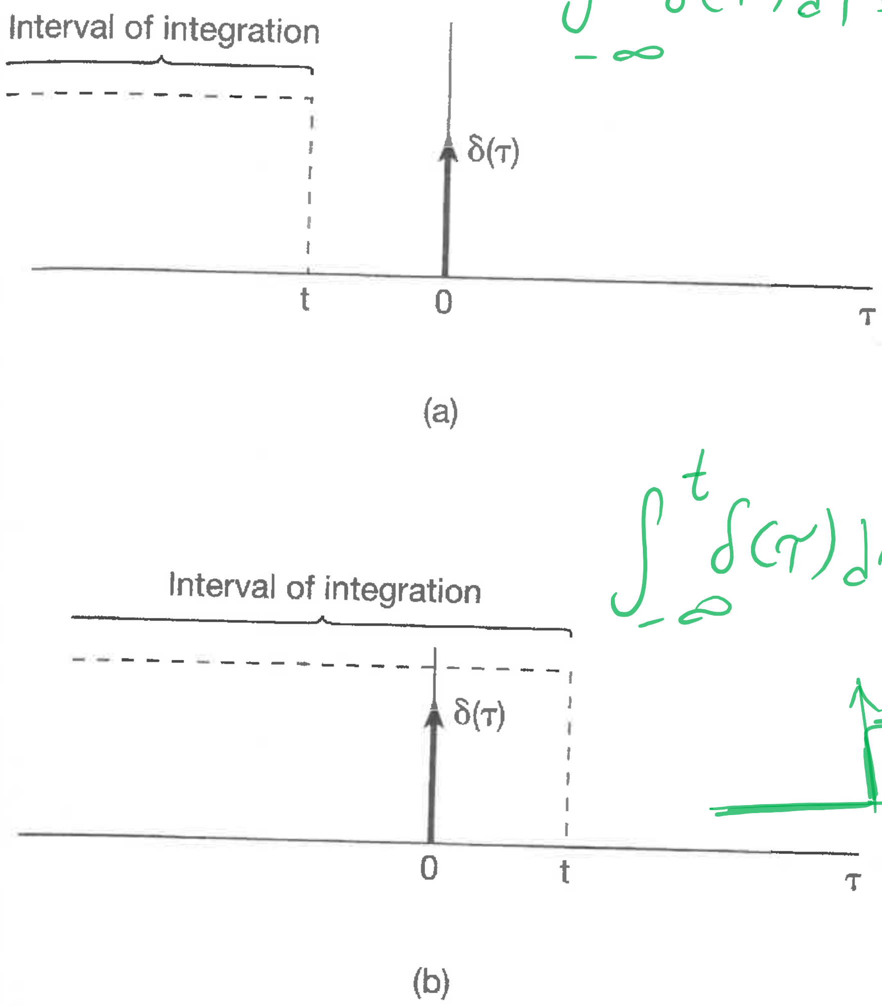
$h(t) = u(t)$

That is, a system response of the form $h(t) = u(t)$ acts as an integrator.

Section 1.4.2 (pages 32 - 36 in textbook)

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = 0 \quad t<0$$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = 1 \quad t \geq 0$$



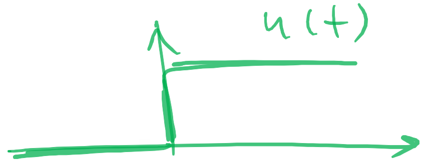


Figure 1.37 Running integral given in eq. (1.71): (a) $t < 0$; (b) $t > 0$.

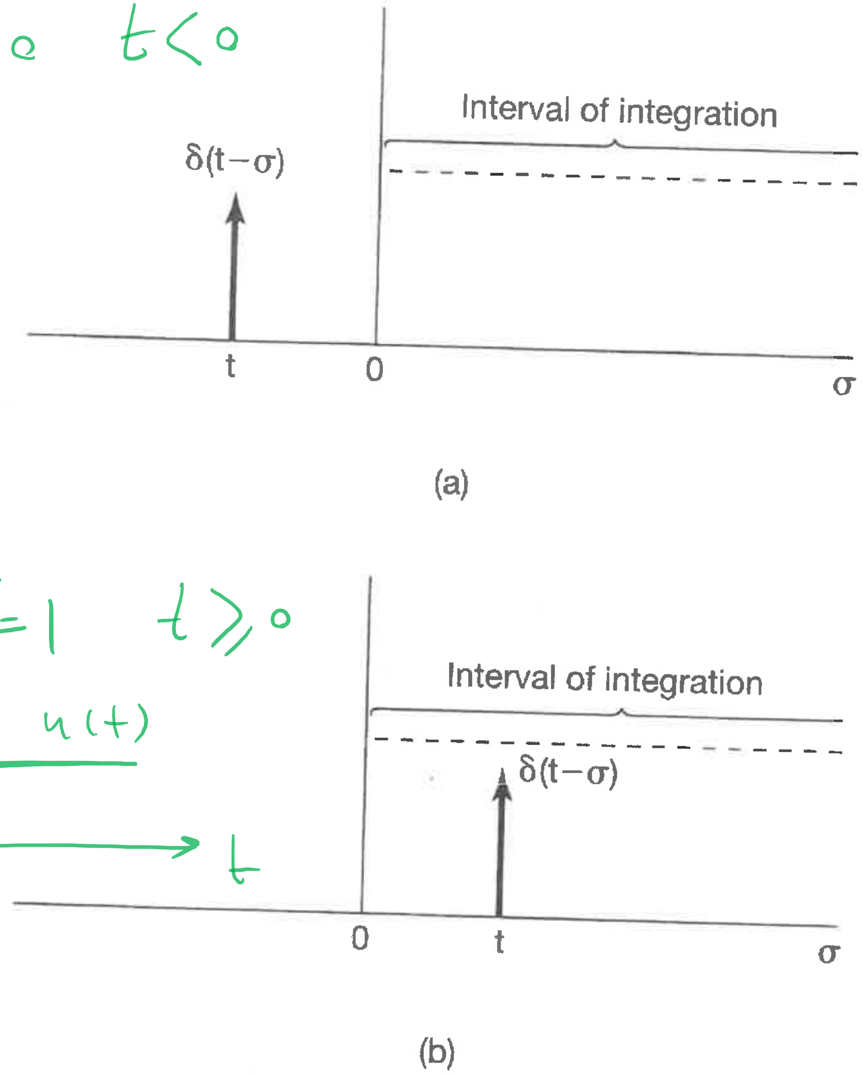


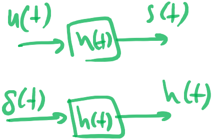
Figure 1.38 Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

CT System Properties – Step Response

$h(t) \rightarrow$ impulse response

$s(t) \rightarrow$ step response

Step Response: Let $x(t) = u(t)$ be a step input signal to a CT LTI System with pulse response $h(t)$, then the output signal is given by



$$s(t) = u(t) \star h(t) = h(t) \star u(t)$$
$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

This is a common test signal used in diagnosing physical systems (often called "plants").

For DT LTI systems:

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

Examples:

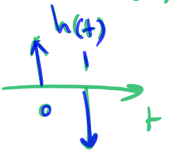
$h(t) = \delta(t) - \delta(t-1)$

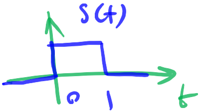
$s(t) = ?$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$= \int_{-\infty}^{t} \delta(\tau)\,d\tau - \int_{-\infty}^{t} \delta(\tau-1)\,d\tau$$

$$= u(t) - u(t-1)$$





$h[n] = (\tfrac{1}{2})^n u[n]$

$s[n] = ?$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$= \sum_{k=-\infty}^{n} (\tfrac{1}{2})^k u[k] = \sum_{k=0}^{n} (\tfrac{1}{2})^k$$

$$* \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$ ← formula sheet

$$= \frac{1-(\tfrac{1}{2})^{n+1}}{\tfrac{1}{2}} = 2 - (\tfrac{1}{2})^n \quad n \geq 0$$

$$s[n] = \left\{2 - (\tfrac{1}{2})^n\right\} u[n]$$

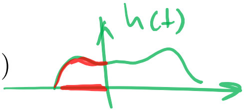
Theorem (Causal CT LTI System)

A CT LTI system is **causal** *if and only if its pulse response, $h(t)$, satisfies*

$$h(t) = 0, \quad \textit{for all } t < 0.$$

- If $h(t) \neq 0$ for at least one $t = -t_0$ $(t_0 > 0)$ then the output at time t, $y(t)$, would have an integrand term

$$h(-t_0)x(t + t_0)$$



and hence not be causal.

For DT LTI System: $h[n] = 0$ for all $n < 0$

Stability Property:

Definition (CT LTI System Stability)

CT LTI system, $x(t) \xrightarrow{h(t)} y(t)$, is **stable**, if and only if

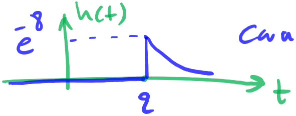
$$\int_{-\infty}^{\infty} \left|h(\tau)\right| d\tau < \infty$$

For DT LTI system:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Examples:

$h(t) = e^{-4t} u(t-2)$



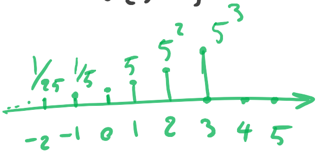
causal

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{2}^{\infty} e^{-4t}\, dt$$

$$= -\frac{1}{4} e^{-4t} \Big|_{2}^{\infty} = \frac{1}{4} e^{-8} < \infty$$

The system is stable.

$h[n] = 5^n u[3-n]$



The system is non-causal

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{3} 5^n$$

formula sheet

$$\ast \sum^{-\infty} \alpha^k = \alpha^{-k}\left(\frac{\alpha}{\alpha-1}\right)$$

$n=-k$

$$= 5^3\left(\frac{5}{5-1}\right) = \frac{625}{4} < \infty$$

The system is stable.