Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 5

Fourier Analysis and Synthesis of Continuous Time Signals

Problem Set 5-1

Find the Fourier transform of the following signals using the FT analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

(a)

$$x(t) = \delta(t+1) + \delta(t-1)$$

(b)

$$x(t) = e^{-a|t|}, \ (a > 0)$$

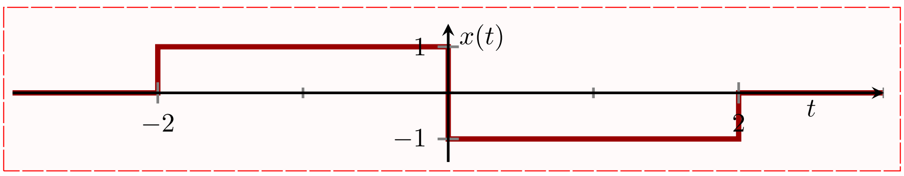
(c)

$$x(t) = e^{2t} u(-t)$$

(d)

$$x(t) = e^{-2t} u(t-1)$$

(e) For the signal $x(t)$ shown in the figure below:



Problem Set 5-2

Find the inverse Fourier transform of the following spectra using the FT synthesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

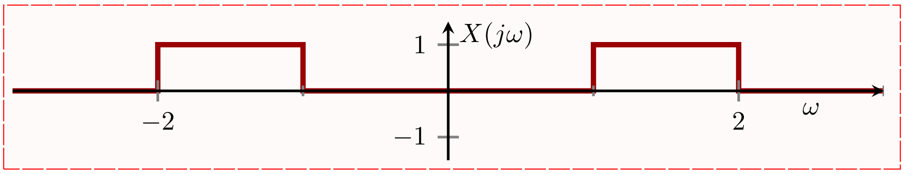
(a)

$$X(j\omega) = 3\delta(\omega - 4)$$

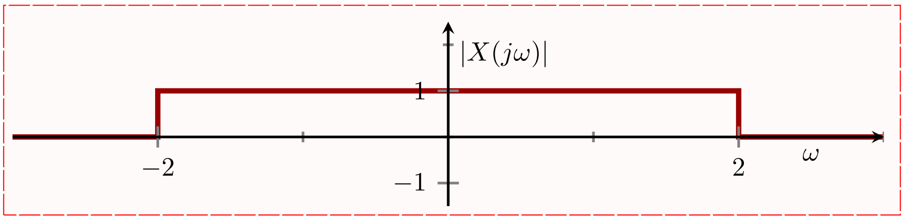
(b)

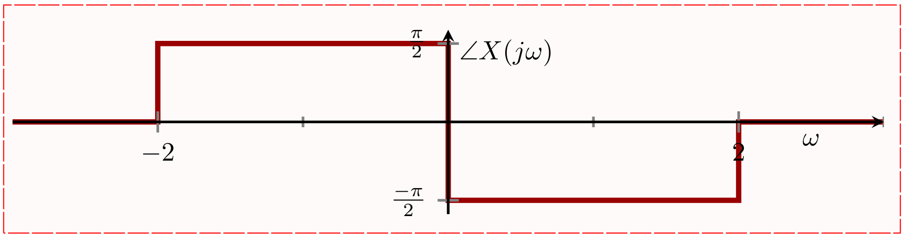
$$X(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$$

(c) For the spectrum $X(j\omega)$ shown in the figure below:



(d) For the spectrum $X(j\omega)$ shown in the figures below:





Fourier Transform Properties of CT Signals

Problem Set 5-3

Determine whether the Fourier transforms $X(j\omega)$ in Figure 1(a) and 1(b) correspond to real continuous time signal $x(t)$.

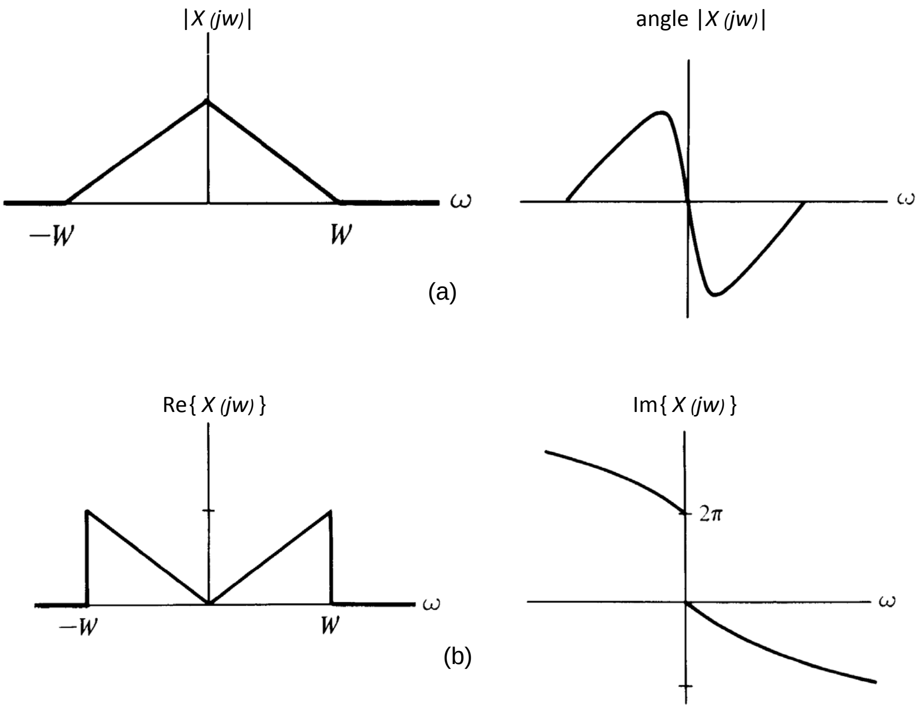


Figure 1: Problem 5-3

Problem Set 5-4

Given $x(t)$ in Figure 2, sketch $X(j\omega)$. If $y(t) = x(t/2)$, sketch both $y(t)$ and $Y(j\omega)$.

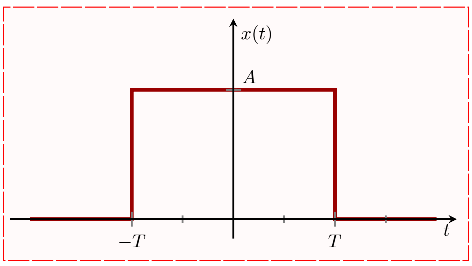


Figure 2: $x(t)$ for Problem 5-4.

Problem Set 5-5

For an input signal

$$x(t) = e^{-\alpha t}u(t), \quad \alpha > 0$$

to the continuous time LTI system with impulse response

$$h(t) = e^{-\beta t}u(t), \quad \beta > 0$$

find the output $y(t)$ of the LTI system using the convolution property of the Fourier transform. Also find the output $y(t)$ for the case when $\alpha = \beta$.

Problem Set 5-6

The following differential equation relates the output $y(t)$ of causal continuous LTI system to the input $x(t)$:

$$\frac{dy(t)}{dt} + 3\,y(t) = x(t).$$

(a) Determine the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ and sketch the magnitude of $H(j\omega)$.

(b) If $x(t) = e^{-t}\,u(t)$, determine $Y(j\omega)$ and $y(t)$.

Det

Problem Set 5-7

Consider two CT LTI systems with frequency responses

$$H_1(j\omega) = \frac{2 + j\omega}{(1 - 0.5\,j\omega)^2}$$

and

$$H_2(j\omega) = \frac{(1 + j\omega)^2}{(-1/2 + j\omega)(3/4 + j\omega)}.$$

(a) Find the differential equation describing $H_1(j\omega)$

(b) Find the differential equation describing $H_2(j\omega)$

(c) Find the differential equation describing the cascade of $H_1(j\omega)$ and $H_2(j\omega)$.

(d) Determine the impulse response of the cascade of $H_1(j\omega)$ and $H_2(j\omega)$.

Problem Set 5-8

A CT LTI system has input $x(t) = (e^{-t} + e^{-3t})u(t)$ and output $y(t) = (2e^{-t} + 2e^{-4t})u(t)$. Find the impulse response $h(t)$ of the LTI system.

Problem Set 5-9

Find the impulse response $h(t)$ of the CT LTI system described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dy}y(t) + 3y(t) = \frac{d}{dx}x(t) + 2x(t)$$

Fourier Analysis and Synthesis of Discrete-time Signals

Problem Set 5-10

Compute the DTFT of each of the following signals:

(a) $x[n] = \delta[n-1] + \delta[n+1]$

(b) $x[n] = \delta[n+2] - \delta[n-2]$

(c) $x[n] = u[n-2] - u[n-6]$

(d)

$$x[n] = \begin{cases} 2^n & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$x[n] = \Big(-\frac{1}{5}\Big)^n u[n] - 6\Big(-\frac{1}{5}\Big)^{n-2} u[n-2]$$

(f)

$$x[n] = \sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$$

(g)

$$x[n] = \sin\left(\frac{n\pi}{2}\right) + \cos(n)$$

(h)

$$x[n] = 3^n \sin\left(\frac{\pi}{4}n\right)u[-n]$$

Problem Set 5-11

The following are the DTFTs of DT signals. Determine the corresponding signals $x[n]$ in the time domain.

(a)

$$X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-2j\omega} - 4e^{-3j\omega} + e^{-10j\omega}$$

(b)

$$X(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & 0 \leq |\omega| < \frac{\pi}{4},\ \frac{3\pi}{4} \leq |\omega| < \pi \end{cases}$$

(c)

$$X(e^{j\omega}) = \cos^2 \omega + \sin^2 3\omega$$

(d)

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \{2\pi\delta(\omega - 2\pi l) + \pi\delta(\omega - \frac{\pi}{2} - 2\pi l) + \pi\delta(\omega + \frac{\pi}{2} - 2\pi l)\}, \quad -\infty < \omega < \infty$$

(e)

$$X(e^{j\omega}) = e^{\frac{-j\omega}{2}}, \quad -\pi \leq \omega \leq \pi$$

(f)

$$X(e^{j\omega}) = \frac{1 - \frac{1}{3}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$$

(g)

$$X(e^{j\omega}) = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)^2}$$

(h)

$$X(e^{j\omega}) = \frac{1 - \left(\frac{1}{3}\right)^6 e^{-j6\omega}}{1 - \frac{1}{3}e^{-j\omega}}$$

Properties Discrete-time Fourier Transform

Problem Set 5-12

Consider the DT LTI system with frequency response

$$H(e^{j\omega}) = \frac{1}{2 - e^{-j\omega}} \tag{3}$$

and the identity

$$x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega}) \tag{4}$$

(a) What is the DC gain (response at $\omega = 0$) for the DT LTI system in equation (3)?

(b) Sketch/plot the magnitude of the frequency response $\left|H(e^{j\omega})\right|$. How would you describe this system in terms of filtering?

(c) State in words - making reference to terms such as magnitude and phase, and delay - the meaning of identity (4).

(d) Using the identity (4), or otherwise, determine the DT difference equation corresponding to equation (3) that relates input $x[n]$ to output $y[n]$.

(e) Determine the impulse response $h[n]$ corresponding to frequency response (3).

(f) If we cascade two filters with the same frequency response $H(e^{j\omega})$, what is the overall frequency response and the overall impulse response?

Problem Set 5-13

Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2} y[n-1] = x[n] \tag{5}$$

(a) Provided $X(e^{j\omega})$ is the frequency response of discrete time signal $x[n]$, prove the following identity

$$x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$$

(b) Determine the frequency response $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega})$ of the system in (5).

(c) For the DT LTI system in (5), find the response $y[n]$ to the inputs $x[n]$ with the following Fourier transforms:

i)

$$X(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

ii)

$$X(e^{j\omega}) = \frac{1}{(1 - \frac{1}{4}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})}$$

Problem Set 5-14

Consider a discrete LTI system with input $x[n]$ and output $y[n]$ and is described by the following relation between $Y(e^{j\omega})$ and $X(e^{j\omega})$

$$Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}$$

(a) Is the system linear? Justify your answer

(b) Is the system time-invariant?

(c) Find impulse response of the system, that is, find $y[n]$ when $x[n] = \delta[n]$. Is the system causal?

Problem Set 5-15

Consider the DT LTI system with impulse response

$$h[n] = \frac{\sin Wn}{\pi n}$$

where $0 < W \leq \pi$. (Note that when $W = \pi$ we have $h[n] = \delta[n]$.)

(a) Determine the possible values of W, within the range $0 < W \leq \pi$, such that

$$h[5] = 0.$$

(b) Sketch the impulse response $h[n]$ and frequency response $H(e^{j\omega})$ for the system with the least value of $W > 0$ such that $h[5] = 0$.

(c) Consider the DT signal

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right).$$

i) Determine the Fourier transform of $x[n]$, $X(e^{j\omega})$, in the interval $-\pi \leq \omega \leq \pi$? (You can use the identity (5.24) given in Example 5.5 in the text. Delta functions should be labeled with their complex amplitude in round brackets.)

ii) Plot $X(e^{j\omega})$ in the range $-3\pi \leq \omega \leq 3\pi$. (This range of ω should include three periods of $X(e^{j\omega})$.)

(d) Suppose the signal $x[n]$, from part (c), is input to LTI systems with the following impulse responses. Determine the output in each case.

i) $h_1[n] = \dfrac{\sin(\pi n/6)}{\pi n}$

ii) $h_2[n] = \dfrac{\sin(\pi n/6)}{\pi n} + \dfrac{\sin(\pi n/2)}{\pi n}$

iv) $h_4[n] = \dfrac{\sin(\pi n/6)\,\sin(\pi n/3)}{\pi n}$

Convolution and Difference Equations in the Frequency Domain using DTFT

Problem Set 5-16

Consider two DT LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{1 + 2e^{-j\omega}}{1 - (1/3)e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 + (1/3)e^{-j\omega} + (1/9)e^{-j2\omega}}$$

The following properties may be useful:

$$z[n-k] \xleftrightarrow{\mathcal{F}} e^{-j\omega k}\, Z(e^{j\omega}), \quad k \in \mathbb{Z} \text{ (integer)}$$

$$z_{(k)}[n] = \begin{cases} z[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{\mathcal{F}} Z(e^{jk\omega}),$$

where

$$z[n] \xleftrightarrow{\mathcal{F}} Z(e^{j\omega}).$$

(a) Find the **difference equation** describing $H_1(e^{j\omega})$ where $x_1[n]$ is the input and $y_1[n]$ is the output.

(b) Find the **difference equation** describing $H_2(e^{j\omega})$ where $x_2[n]$ is the input and $y_2[n]$ is the output.

(c) Find the **difference equation** describing the cascade of $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$.

(d) Determine the **impulse response** of the cascade of $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$, that is,

$$h[n] \xleftrightarrow{\mathcal{F}} H_1(e^{j\omega})\, H_2(e^{j\omega}).$$

To solve this part you can use any method you like but remember that the impulse response is what comes out when the input is $\delta[n]$.

Problem Set 5-17

Find the convolution $y[n] = x[n] * h[n]$ where

$$X(e^{j\omega}) = 3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega}$$

$$H(e^{j\omega}) = -e^{j\omega} + 2e^{-j2\omega} + e^{j4\omega}$$

Sampling

Problem Set 5-18

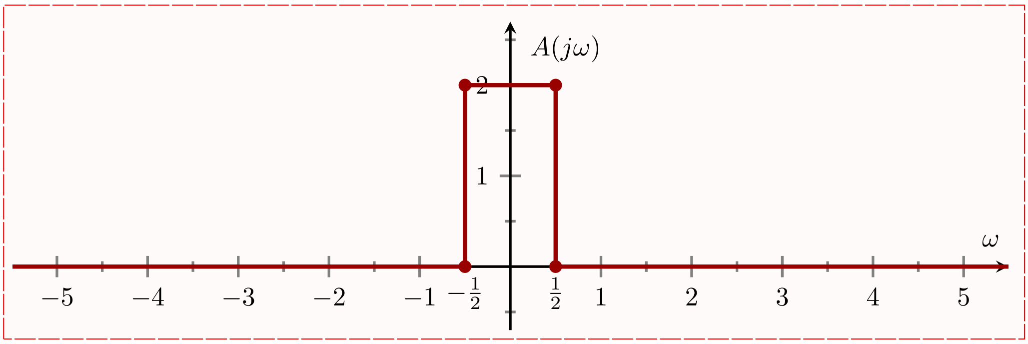
(a) The impulse response of an ideal low-pass filter with maximum frequency ω_M is given by

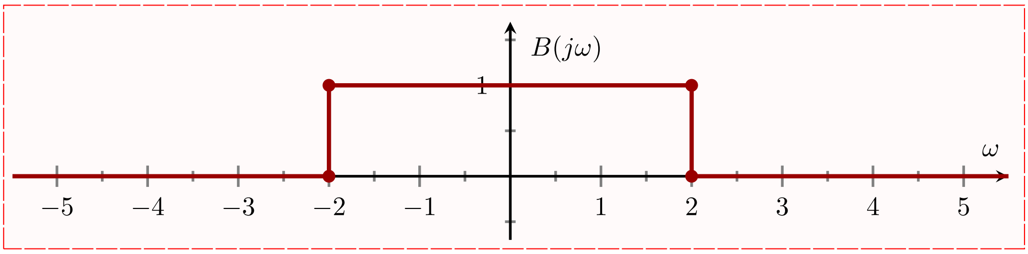
$$\frac{\sin \omega_M t}{\pi t} \stackrel{\mathcal{F}}{\longleftrightarrow} \chi_{[-\omega_M, +\omega_M]}(\omega), \quad 0 \leq \omega_M.$$

Infer or derive the **time domain representations** of the two signals,

$$a(t) \stackrel{\mathcal{F}}{\longleftrightarrow} A(j\omega) \qquad b(t) \stackrel{\mathcal{F}}{\longleftrightarrow} B(j\omega)$$

whose frequency contents are shown below (all frequency representations are real-valued and have zero phase).

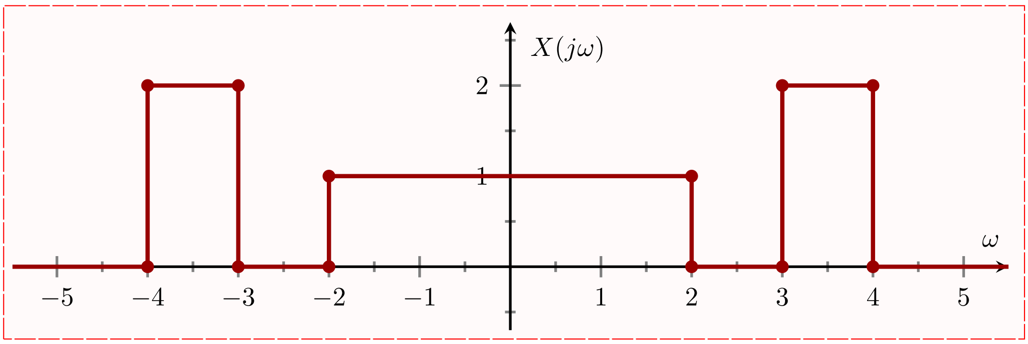




(b) Consider the signal

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

whose frequency content is shown in the figure below. Use the multiplication property and superposition to find the corresponding **time domain representation** using the results in (a).



(c) Consider a sampled version of the signal, $x(t)$ in part (b), given by

$$\begin{aligned} x_p(t) = x(t)\, p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT). \end{aligned}$$

Draw the **frequency content** of $x_p(t)$ when the sampling rate is

$$\omega_s = 10 \text{ rad/sec},$$

corresponding to $T = \pi/5$ sec.

(d) Consider the recovery of the sampled signal $x_p(t)$ in part (c), where the sampling rate is $\omega_s = 10$ rad/sec, with an ideal low-pass filter whose cutoff or bandwidth is given by ω_c rad/sec.

i) What is the **least bandwidth**, ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?

ii) What is the **maximum bandwidth**, ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?

Problem Set 5-19

(a)

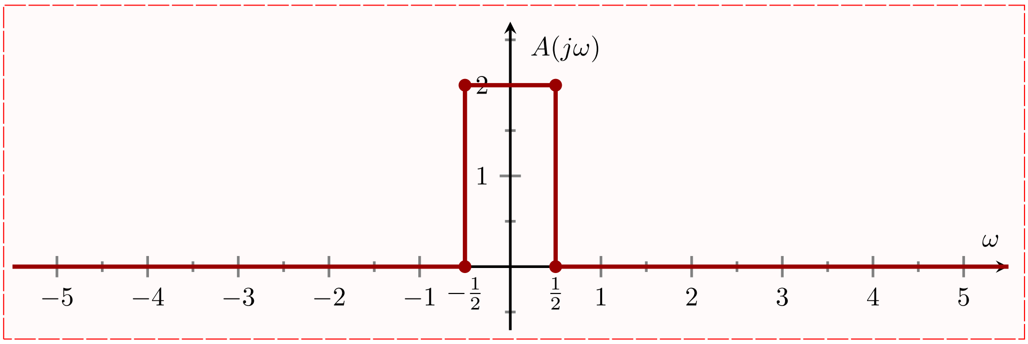
The impulse response of an ideal low-pass filter with maximum frequency ω_M is given by

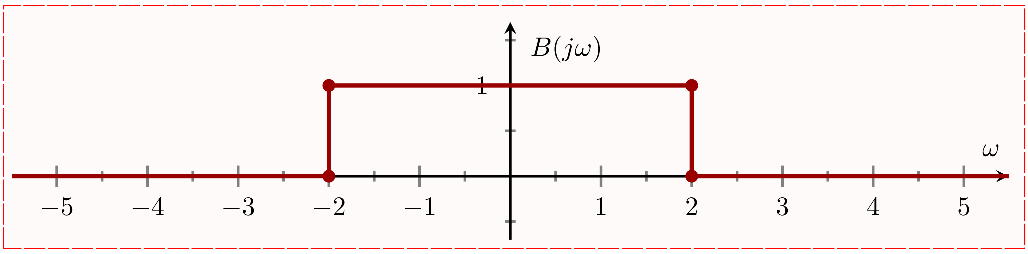
$$\frac{\sin \omega_M t}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \chi_{[-\omega_M, +\omega_M]}(\omega), \quad 0 \le \omega_M.$$

Infer or derive the **time domain representations** of the two signals,

$$a(t) \overset{\mathcal{F}}{\longleftrightarrow} A(j\omega) \qquad b(t) \overset{\mathcal{F}}{\longleftrightarrow} B(j\omega)$$

whose frequency contents are shown below (all frequency representations are real-valued and have zero phase).

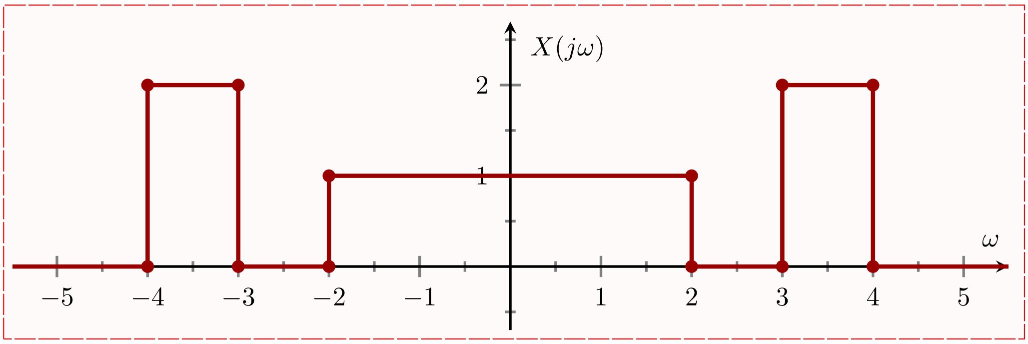




(b) Consider the signal

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

whose frequency content is shown in the figure below. Use the multiplication property and superposition to find the corresponding **time domain representation** using the results in (a).



`(c)` Consider a sampled version of the signal, $x(t)$ in part (b), given by

$$x_p(t) = x(t)\,p(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT).$$

Draw the **frequency content** of $x_p(t)$ when the sampling rate is

$$\omega_s = 10 \text{ rad/sec},$$

corresponding to $T = \pi/5$ sec.

`(d)` Consider recovery with an ideal pass filter whose cutoff or bandwidth is given by ω_c rad/sec. What are the minimum and maximum bandwidths for ω_c, that can be used to perfectly recover $x(t)$ from $x_p(t)$?