Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 1 – SOLUTIONS

Complex Number Refresher

A quick refresher on complex numbers. Complex numbers are used in much of engineering. They are an near ideal shorthand in signal representation and they simplify expressions.

Please read textbook Page 71: Mathematical Review of Complex numbers, before attempting this homework.

Problem Set 1-1

(a) Prove the Euler identity:

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

Solution: Write out the Taylor series about $x = 0$ for e^x, $\cos x$ and $\sin x$:

$$\begin{aligned} e^x &= 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots \\ \cos x &= 1 - x^2/2! + x^4/4! + \cdots \\ \sin x &= x - x^3/3! + x^5/5! + \cdots \end{aligned}$$

Then, with $x = j\theta$,

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + (j\theta)^2/2 + (j\theta)^3/3! + (j\theta)^4/4! + \cdots \\ &= (1 - \theta^2/2! + \theta^4/4! + \cdots) + j(\theta - \theta^3/3! + \theta^5/5! \cdots) \\ &= \cos\theta + j\sin\theta \end{aligned}$$

□

(b) Find expressions for $\cos\theta$ and $\sin\theta$ in terms of $e^{j\theta}$ and its conjugate $e^{-j\theta}$.

Solution: Combine $e^{j\theta}$ with its conjugate $e^{-j\theta}$ gives $2\cos\theta$:

$$\begin{aligned} e^{j\theta} + e^{-j\theta} &= \cos\theta + j\sin\theta + \cos\theta - j\sin\theta = 2\cos\theta \\ e^{j\theta} - e^{-j\theta} &= \cos\theta + j\sin\theta - \cos\theta + j\sin\theta = 2j\sin\theta \end{aligned}$$

then divide by 2 and $2j$, respectively. □

Problem Set 1-2

What is the difference between taking the conjugate of an expression and replacing every occurrence of j with $-j$?

Solution: Nothing. □

Problem Set 1-3

Write each of the following in polar form, that is, in $re^{j\theta}$ find r (such that $r \geq 0$) and θ (such that $-\pi < \theta \leq \pi$).

(a) $1+j\sqrt{3}$

Solution: $2(1/2+j\sqrt{3}/2)=2e^{j\pi/3}$. $\square$

(b) $(\sqrt{3}+j^3)(1-j)$

Solution: $2e^{-j\pi/6}\sqrt{2}e^{-j\pi/4}=2\sqrt{2}e^{-j5\pi/12}$. $\square$

(c) $\dfrac{e^{j\pi/3}-1}{1+j\sqrt{3}}$

Solution: $\dfrac{1/2+j\sqrt{3}/2-1}{1+j\sqrt{3}}=\dfrac{1}{2}e^{j\pi/3}$. $\square$

(d) $j^{1,222,444,667,099,987,676,222,091,345,222,822,822,282,228}$

Solution: The number in the power of j is a multiple of 4 and $j^4=1$
$\implies (j^4)^x=1^x=1$, where x is any real number.
$\therefore j^{1,222,444,667,099,987,676,222,091,345,222,822,822,282,228}=1=1e^{j0}$. $\square$

Problem Set 1-4

With $j=\sqrt{-1}$ what is j^j (j to the power j)?

Solution: $1/\sqrt{e^\pi}$ (which is real) since $j=e^{j\pi/2}$ then $j^j=e^{j^2\pi/2}=e^{-\pi/2}=1/\sqrt{e^\pi}$. $\square$

Problem Set 1-5

Solve Textbook Problems 1.1, 1.2.

Solution: Answers are provided in the textbook.

Problem Set 1-6

Solve Textbook Problem 1.49.

Solution: (a) $2e^{j\pi/3}$ (b) $5\ e^{j\pi}$ (c) $5\sqrt{2}e^{j5\pi/4}$ (d) $5e^{j0.927}$ (e) $8e^{-j\pi}$ (f) $4\sqrt{2}e^{j5\pi/4}$ (g) $2\sqrt{2}e^{-j5\pi/12}$ (h) $e^{-j2\pi/3}$ (i) $e^{-j\pi/6}$ (j) $\sqrt{2}e^{j5\pi/12}$ (k) $4\sqrt{2}e^{-j\pi/12}$ (l) $\frac{1}{2}e^{j\pi/3}$

Classification and Properties of Signals

Problem Set 1-7

Find the even and odd components of each of the following signals:

(a)

$$x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$$

Solution: Even: $\cos(t)$
Odd: $\sin(t)\big(1 + \cos(t)\big)$

(b)

$$x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$$

Solution: Even : $1 + 3t^2 + 9t^4$
Odd: $t + 5t^3$

(c)

$$x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$$

Solution: Even : $1 + t^3\sin(t)\cos(t)$
Odd: $t\cos(t) + t^2\sin(t)$

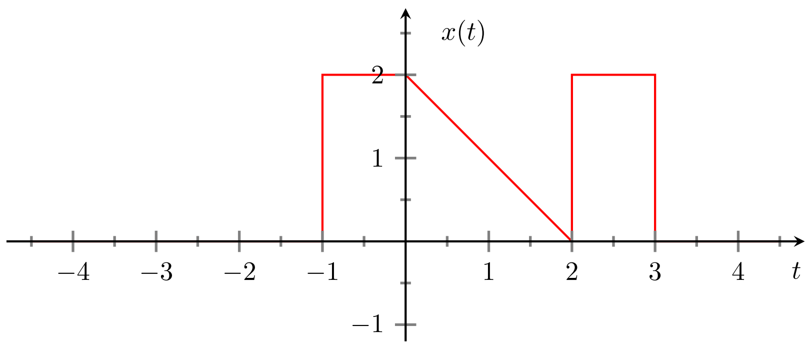
(d)

$$x(t) = (1 + t^3)\cos^3(10t)$$

Solution: Even : $\cos^3(10t)$
Odd: $t^3\cos^3(10t)$

Problem Set 1-8

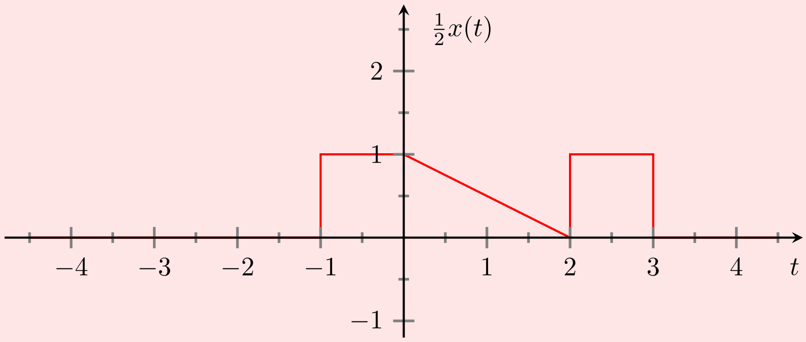
Determine and sketch the even and odd part of the following Continuous Time signal. Label your sketch carefully.



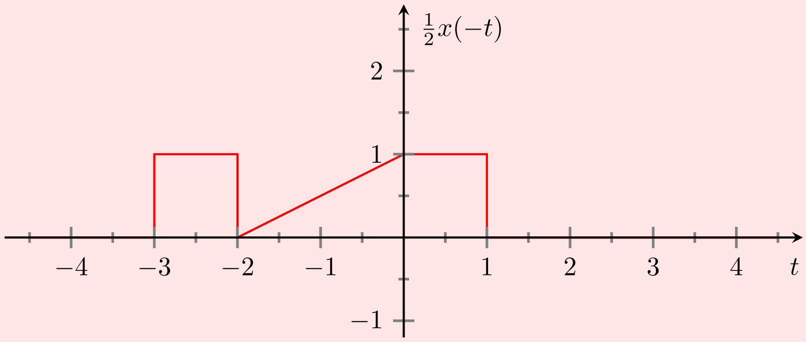
Solution:

$$Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

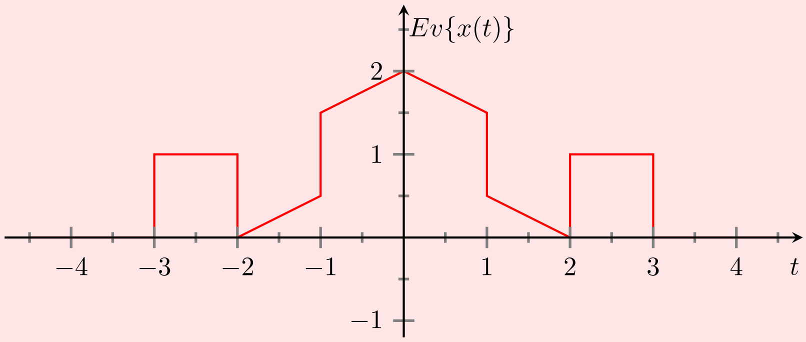
$\frac{1}{2}x(t)$ is



$\frac{1}{2}x(-t)$ is

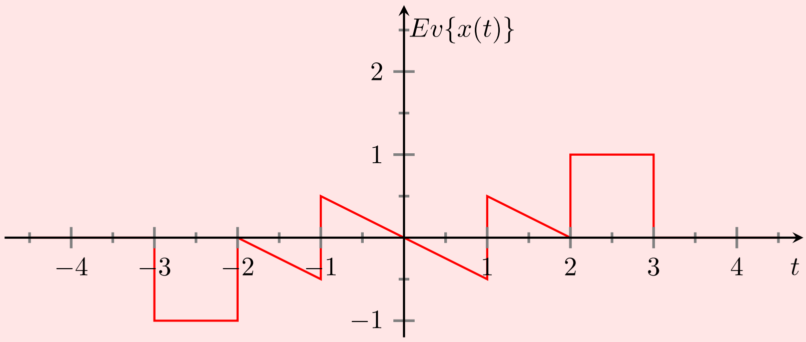


Therefore $Ev\{x(t)\}$ is



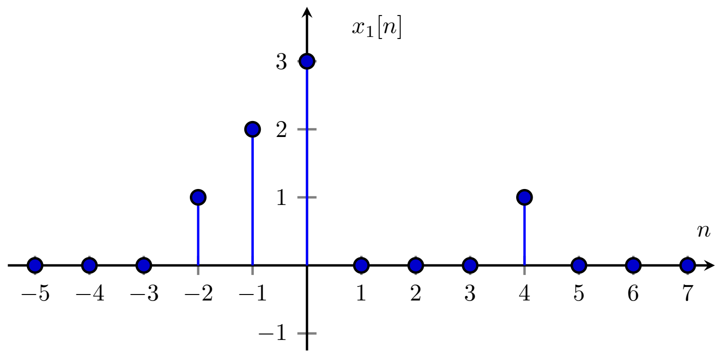
$$Od\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Therefore $Od\{x(t)\}$ is



Problem Set 1-9

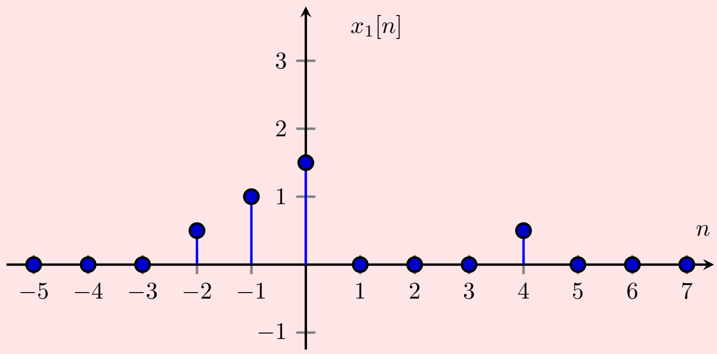
Determine and sketch the even and odd parts of the DT signal depicted below.



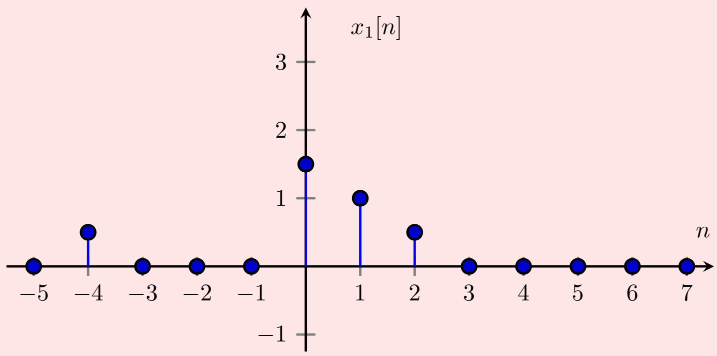
Solution:

$$Ev\{x[n]\} = \frac{1}{2}(x[n] + x[-n])$$

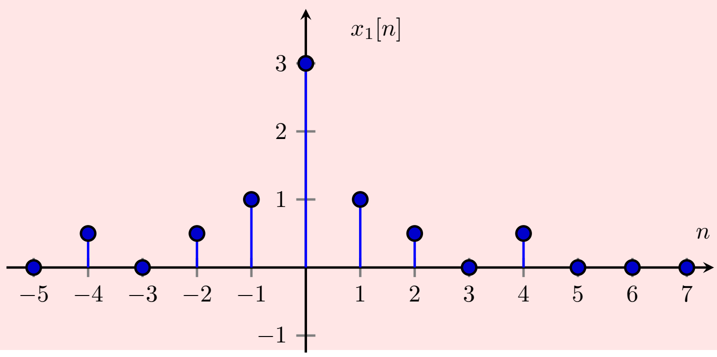
$\frac{1}{2}x[n]$ is



$\frac{1}{2}x[-n]$ is

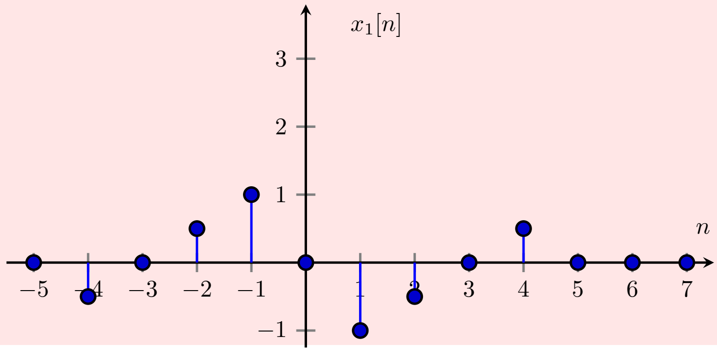


Therefore $Ev\{x[n]\}$ is



$$Od\{x[n]\} = \frac{1}{2}(x[n] - x[-n])$$

Therefore $Od\{x[n]\}$ is



Problem Set 1-10

Review phasors, see phasors on wikipedia and looks at phasor diagrams for one sinusoid and phasor diagram for sum of two sinusoids, where this only works because the sinusoids are the same frequency.

Problem Set 1-11

(Hard) Movies like two sinusoids show what happens when there is the sum of two sinusoids at the same frequency. The real part of the rotating phasor maps out a sinusoid. Using the phasor ideas, suppose there is the sum of two sinusoids at different frequencies, for example at ω and 3ω such as

$$\cos(\omega t) + \frac{1}{9}\cos(3\omega t)$$

What modification happens to the phasor picture?

Solution: Instead of a sinusoid a non-sinusoidal but periodic signal coming out. For $\cos(\omega t) + \frac{1}{9}\cos(3\omega t)$ you get something more triangular:

plot cos(t)+cos(3t)/9

and with more terms even more so triangular:

plot cos(t)+cos(3t)/9+cos(5t)/25

Plots can also be generated using WolframAlpha. □

Problem Set 1-12

Lecture slides 27 and 28 show that adding two periodic signals of different periods can give you an output signal that is periodic:

(a) When does this occur in general?

Solution: If T_1 is the period of the first signal and T_2 is the period of the second signal then T_1 and T_2 need to be rationally related, for example,

$$\frac{T_1}{T_2} = \frac{4099}{298}$$

means T_1 and T_2 are rationally related. □

(b) Assuming all periods under consideration are integer-valued (like 7 or 11 and not 4/5 nor $\sqrt{2}$), answer the following. If the output is periodic what is the relationship of the output period to the two input periods?

Solution: The output period is the least or lowest common multiple of T_1 and T_2, for example, if $T_1 = 4098$ and $T_2 = 18879$ then the LCM is $T_3 = 25788714$ (here T_1 and T_2 have a greatest common divisor of 3 and so the LCM is $T_1 \times T_2/3$.) □

(c) Repeat the previous part in the case when the periods are not integer-valued. (For example, suppose $T_1 = 4098\sqrt{2}$ and $T_2 = 18879\sqrt{2}$.)

Solution: Much the same as previously, T_1 and T_2 need to have a common not rational part that can be factored out. ☐

Problem Set 1-13

Review the type of calculation shown on lecture slide 62, repeated here

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t} \\ &= A\,\Re\{e^{j(\omega_0 t+\phi)}\} \end{aligned}$$

and

$$A\sin(\omega_0 t + \phi) = A\,\Im\{e^{j(\omega_0 t+\phi)}\},$$

which needs the Euler identity (here A is real-valued).

Problem Set 1-14

Roger Federer and Leonhard Euler were both born in Basel Switzerland but they never meet. Why?

Solution: It would be bad PR for Roger to exhume Euler's body. ☐

Problem Set 1-15

CT signal $x(t) = e^{j\omega t}$ is periodic for any choice of ω.

(a) True or false?

Solution: True ☐

(b) What is its fundamental period when $\omega = 4$?

Solution: $T_0 = \pi/2$, that is, $x(t + \pi/2) = x(t)$ for all t. ☐

(c) What is its fundamental period when $\omega = 19\pi$?

Solution: $T_0 = 2/19$ ☐

Problem Set 1-16

DT signal $x[n] = e^{j\omega n}$ is not periodic for every choice of ω.

(a) True or false?

Solution: True ☐

(b) Is it periodic when $\omega = 4$?

Solution: No, because otherwise we would need $4n$ to be a multiple of 2π which would imply π is rational , which is false ($\pi \neq 2n/k$ for integer k). ☐

(c) Is it periodic when $\omega = 19\pi$?

Solution: Yes, because it has fundamental period $N_0 = 2$. $x[0] = 1$, $x[1] = -1$, $x[2] = 1$, etc. ☐

(d) How does the DT signal $x[n] = e^{j19\pi n}$ differ from $x[n] = e^{j\pi n}$?

Solution: It doesn't differ. ☐

Problem Set 1-17

Solve Textbook Problem 1.9.

Solution: Answers are provided in the textbook. □

Problem Set 1-18

Determine whether or not each of the signals is periodic. If a signal is periodic, determine the fundamental period.

(a)

$$x[n] = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)$$

Solution: Periodic. $N_o = 16$ samples (Note: straight forward question similar to examples solved in class). □

(b)

$$x(t) = j\ e^{j10t}$$

Solution: Periodic. $T = \frac{\pi}{5}$ s. Note: $x(t) = j\ e^{j10t} = e^{j10t+\pi/2}$. Slightly tricky question. □

(c)

$$x(t) = 2e^{j\left(t+\frac{\pi}{4}\right)}u(t)$$

Solution: Not periodic because of $u(t)$ component in the signal, which makes it is zero for $t < 0$. □

(d)

$$x(t) = e^{(-1+j)t}$$

Solution: Not periodic because of decaying exponential component in the signal. This is easier to see if expand $x(t)$ as $x(t) = e^{-t}e^{jt}$. □

(e)

$$x(t) = \sqrt{\cos(2t)}$$

Solution:
Time period will be the same as for $\cos(2t)$ i.e. $\frac{\pi}{2} = \pi$

(f)

$$x(t) = \big(\cos(2t - 3\pi/4)\big)^2$$

Solution:

$$x(t) = \frac{1 + \cos\left(4t - 3\pi/2\right)}{2}$$

Therefore $T = \frac{2\pi}{4} = \frac{\pi}{2}$.

(g)

$$x(t) = \mathrm{Ev}\big\{\cos(4\pi t)\, u(t)\big\}$$

where $\mathrm{Ev}\{\cdot\}$ denotes the even part.

Solution:

$$\begin{aligned} x(t) = \mathrm{Ev}\big\{\cos(4\pi t)\, u(t)\big\} && (1)\\ = \frac{\cos\left(4\pi t\right)u(t) + \cos\left(4\pi t\right)u(-t)}{2} && (2)\\ = \frac{1}{2}\cos\left(4\pi t\right) && (3)\\ && (4) \end{aligned}$$

therefore $T = \frac{2\pi}{4\pi} = \frac{1}{2}$.

Problem Set 1-19

Determine whether or not each of the following signals is periodic. If the signal is periodic determine its fundamental period.

(a) $x(t) = \sin^2(4t) \equiv \big(\sin(4t)\big)^2$

Solution: Periodic. Normally $\sin(4t)$ has fundamental period $\pi/2$ but squaring does something unusual. From standard trig (or using Euler identity)

$$\sin^2(4t) = \big(1 - \cos(8t)\big)/2$$

so the fundamental period is actually $T_0 = \pi/4$. $T = \pi/2$ is still a period but twice the fundamental period. □

(b) $x[n] = \cos(4n + \pi/4)$

Solution: The phase shift $\pi/4$ does nothing, it is a red herring, so ignore it. We would need $4n$ to equal some multiple of 2π (the intrinsic period of cos). That is, $4n = 2\pi k$ where k is an integer. But this would mean π is rational. Therefore, $x[n]$ is not periodic. □

(c) $x[n] = (-1)^n \cos(2\pi n/7)$

Solution: As $(-1)^n \equiv e^{jn\pi}$, $x[n] = (-1)^n \cos(2\pi n/7) = e^{j\pi n}\cos(2\pi n/7)$. We need to expand $x[n]$ as the fundamental period of the product of two signals is not necessarily the lowest common multiple of the fundamental periods of the two signals. Hence, using Euler's rule $e^{j\pi n}\cos(2\pi n/7) = \Big(\cos(\pi n) + j\sin(\pi n)\Big)\cos(2\pi n/7) = \cos(\pi n)\cos(2\pi n/7)$. Using the product of two cosines trig identity, $\cos(\pi n)\cos(2\pi n/7) = \frac{1}{2}\Big[\cos\left(\pi n - \frac{2\pi}{7}\right) + \cos\left(\pi n + \frac{2\pi}{7}\right)\Big] = \frac{1}{2}\cos\left(\frac{5\pi}{7}\right) + \frac{1}{2}\cos\left(\frac{9\pi}{7}\right)$.
$\frac{1}{2}\cos\left(\frac{5\pi}{7}\right)$ has a fundamental period of 14, and $\frac{1}{2}\cos\left(\frac{9\pi}{7}\right)$ has a fundamental period of 14. So $x[n]$ is periodic with fundamental period $N_0 = 14$

$$x[n+14] = (-1)^{n+14}\cos(2\pi n/7 + 4\pi) = \cdots = (-1)^n\cos(2\pi n/7) = x[n],$$

Problem Set 1-20

If $x(t)$ and $x[n]$ are even signals, then show that

$$\int_{-a}^{a} x(t)\,dt = 2\int_0^a x(t)\,dt$$

and

$$\sum_{n=-k}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

Solution:

$$\int_{-a}^{a} x(t)\,dt = \int_{-a}^{0} x(t)\,dt + \int_0^a x(t)\,dt$$

let $t = -\lambda$:

$$\int_{-a}^{0} x(t)\,dt = -\int_a^0 x(-\lambda)\,d\lambda = \int_0^a x(-\lambda)\,d\lambda$$

$x(-\lambda) = x(\lambda)$ as $x(t)$ is even, so

$$\int_{-a}^{0} x(t)\,dt = \int_0^a x(\lambda)\,d\lambda = \int_0^a x(t)\,dt$$

Therefore

$$\int_{-a}^{a} x(t)\,dt = \int_0^a x(t)\,dt + \int_0^a x(t)\,dt = 2\int_0^a x(t)\,dt.$$

As for $\sum_{n=-k}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$:

$$\sum_{n=-k}^{k} x[n] = \sum_{n=-k}^{-1} x[n] + x[0] + \sum_{n=1}^{k} x[n]$$

let m=-n:

$$\sum_{n=-k}^{-1} x[n] = \sum_{m=1}^{k} x[-m]$$

$x[-m] = x[m]$ as $x[n]$ is odd, so

$$\sum_{m=1}^{k} x[-m] = \sum_{m=1}^{k} x[m] = \sum_{n=1}^{k} x[n]$$

Therefore

$$\sum_{n=-k}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

Problem Set 1-21

Categorise each of the following signals as an energy signal or power signal, and find the energy or the time-averaged power of the signal:

(a)

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases},$$

Solution: Energy signal. Energy $= \frac{2}{3}$.

(b)

$$x[n] = \begin{cases} n & 0 \le n < 5 \\ 10-n & 5 \le n \le 10 \\ 0 & \text{otherwise} \end{cases},$$

Solution:
Energy signal. Energy = 85.

(c)

$$x(t) = 5\cos(\pi t) + \sin(5\pi t)$$

Solution: Power signal. Power = 13.

(d)

$$x(t) = \begin{cases} 5\cos(\pi t) & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases},$$

Solution: Energy signal. Energy = 25

(e)

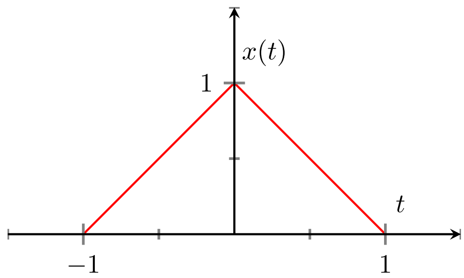
$$x[n] = \begin{cases} \cos(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases},$$

Solution: Energy signal. Energy = 9

Affine Transformations

Problem Set 1-22

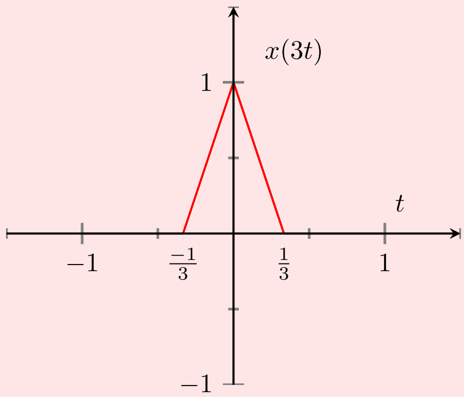
Sketch each of the following signals derived from the triangular pulse $x(t)$ shown below



(a)

$$x(3t)$$

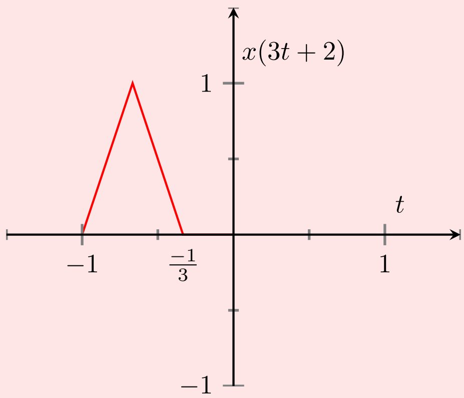
Solution:



(b)

$$x(3t+2)$$

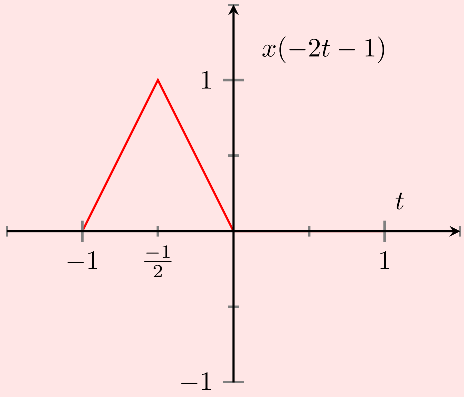
Solution:



(c)

$$x(-2t-1)$$

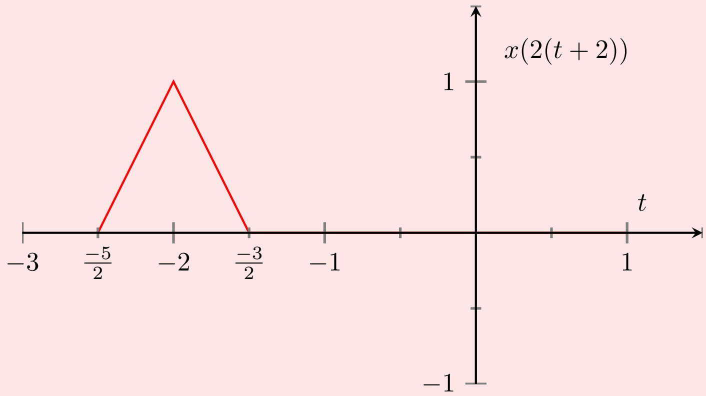
Solution:



(d)

$$x(2(t+2))$$

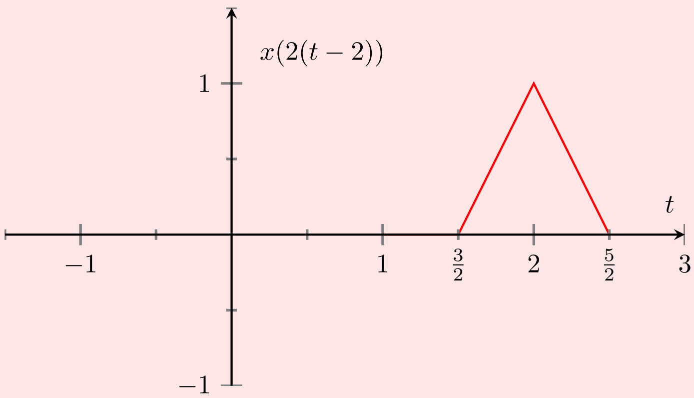
Solution:



(e)

$$x(2(t-2))$$

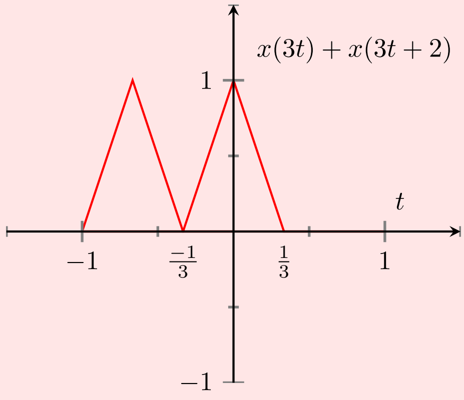
Solution:



(f)

$$x(3t) + x(3t+2)$$

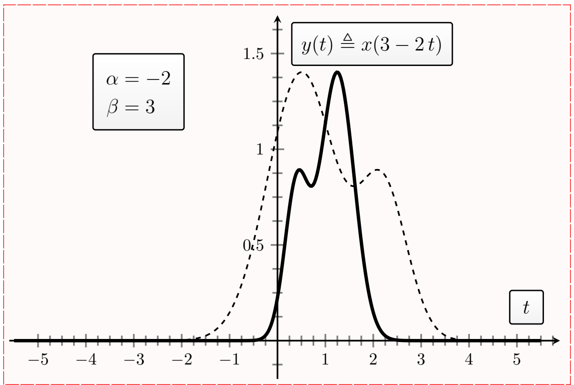
Solution:



Problem Set 1-23

Review the independent variable affine transformation examples on lecture slides 88 and 92, shown below.

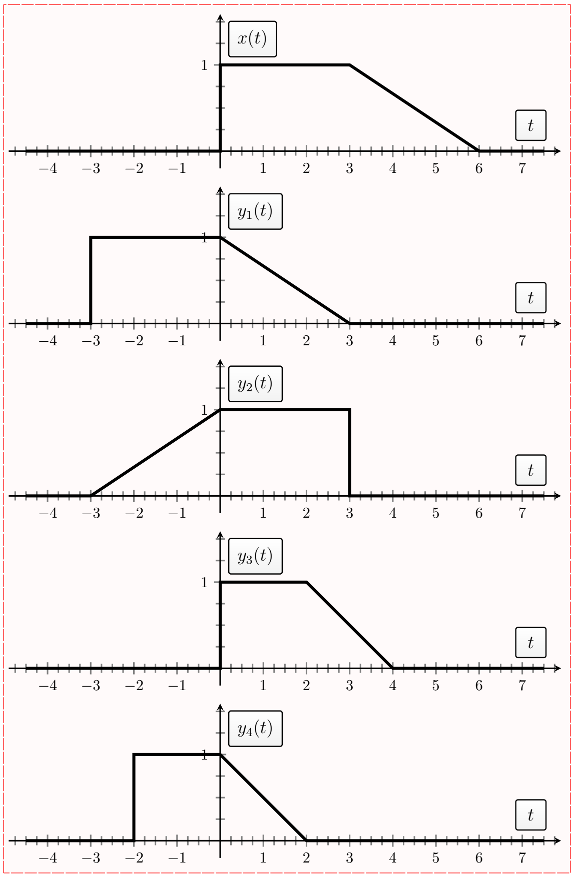
(a) Confirm the solid blue curve in the figure (blow) is indeed $y(t) = x(3 - 2t)$ where $x(t)$ is the dashed curve.



Solution: Test with different values of t. For example, for $t = 0.5$ the portion around $y(0.5)$ comes from the reversed dashed portion around $x(3 - 2 \times 0.5) = x(2)$. □

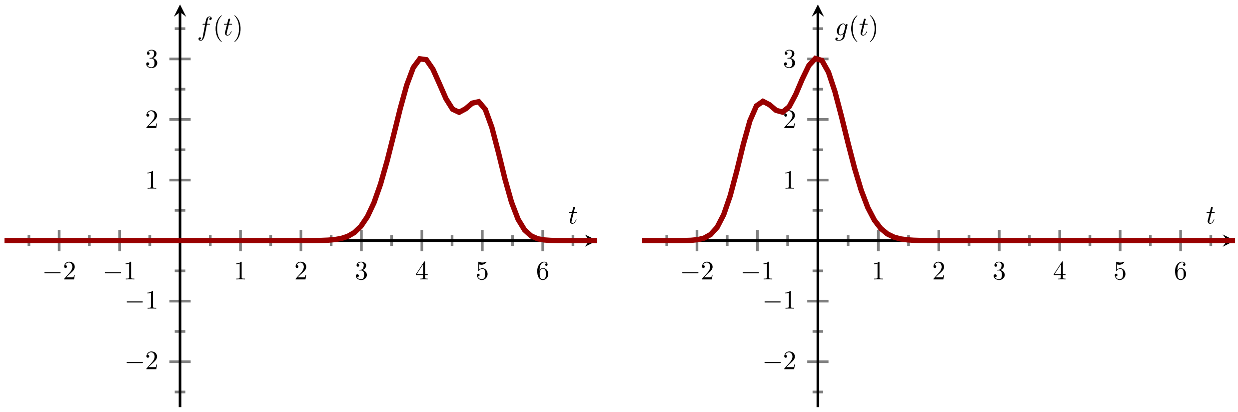
(b) Express each of $y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$ in terms of $x(t)$ (shown to the right).

Solution: $y_1(t) = x(t + 3)$, $y_2(t) = x(-t + 3)$, $y_3(t) = x(1.5\,t)$, and $y_4(t) = x(1.5\,t + 3)$. □



Problem Set 1-24

Two signals $f(t)$ and $g(t)$ are related through an affine transformation of their independent variable



a. $g(t) = f(t-4)$

b. $g(t) = f(4-t)$

c. $g(t) = f(t+4)$

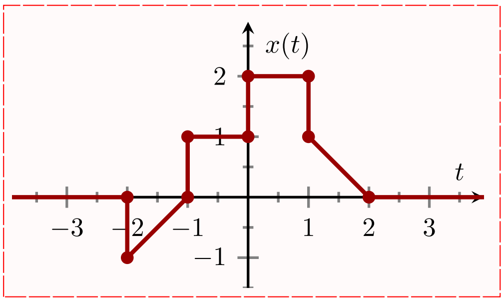
d. $g(t) = f(t+4)$

e. None of the above

Solution: b) $g(t) = f(4-t)$

Problem Set 1-25

For the CT signal $x(t)$ in the Text Figure P1.21 on page 60, shown below,

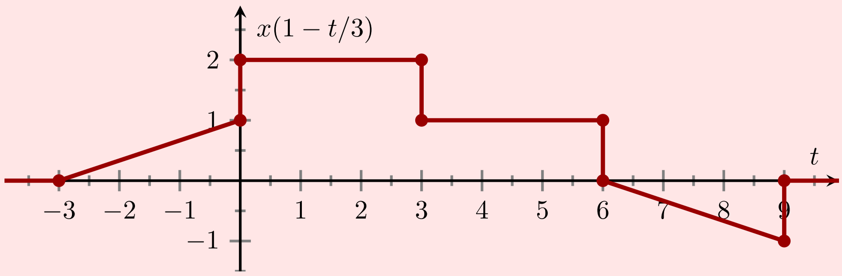


sketch and carefully label:

(a) $x(1-t/3)$

Solution: $y(t) = x(1-t/3)$ is a combination of time shift 1, time-scaling by $1/3$ (stretching by 3), and time reversal (-1) but we need to be careful because these terms can't be commuted.

$y(-3) = x(2) = 0$
$y(0^-) = x(1^+) = 1$
$y(0^+) = x(1^-) = 2$
$y(3^-) = x(0^+) = 2$
$y(3^+) = x(0^-) = 1$
$y(6^-) = x(-1^+) = 1$
$y(6^+) = x(-1^-) = 0$
$y(9^-) = x(-2^+) = -1$
$y(9^+) = x(-2^-) = 0$



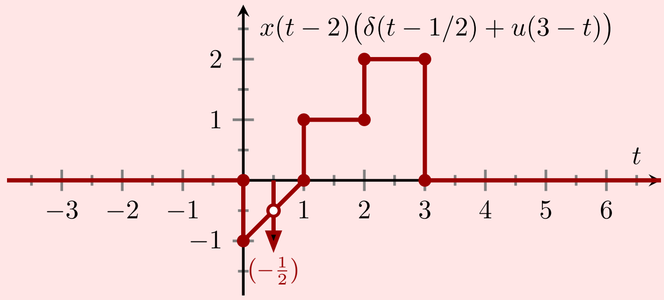
□

(b) $x(t-2)\big(\delta(t-1/2) + u(3-t)\big)$ where $\delta(t)$ is the delta function and $u(t)$ is the step function.

Solution: We have

$$x(t-2)\big(\delta(t-1/2)+u(3-t)\big) = x(-3/2)\,\delta(t-1/2) + x(t-2)\,u(3-t)$$
$$= -1/2\,\delta(t-1/2) + x(t-2)\,u(3-t),$$

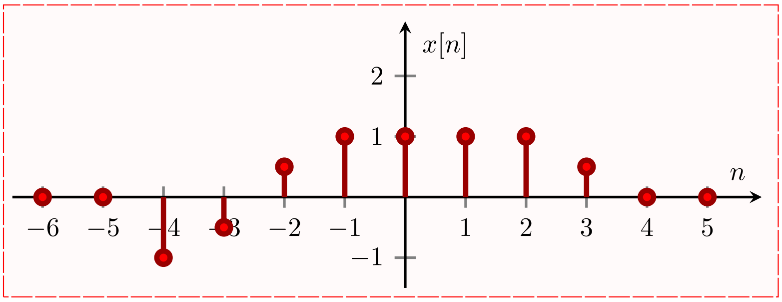
which is draw below. At $t = 1/2$ there is a delta function of area $-1/2$,



□

Problem Set 1-26

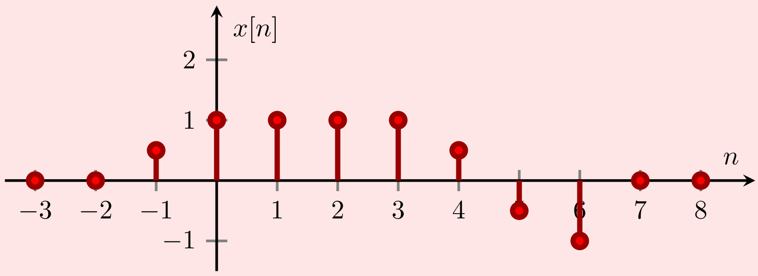
For the DT signal $x[n]$ from the Text Figure P1.22 on page 60, shown below,



sketch and carefully label:

(a) $x[2-n]$

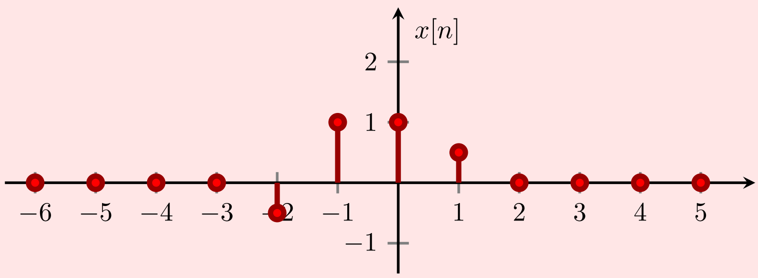
Solution: To draw this, shift to the right by 2 and then reverse the graph.



□

(b) $x[2n+1]$

Solution: As we step through integer n only every second sample (every odd sample) of $x[n]$ is used. The answer is: $-\frac{1}{2}\delta[n+2] + \delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$.



□

(c) $x[n/3]$

Solution: This is nonsense and cannot be sketched. ☐

Classification and Properties of Systems

Problem Set 1-27

Classify each of the following systems, where $x(t)$ or $x[n]$ is the input signal and $y(t)$ or $y[n]$ is the output signal, as:

i) linear or non-linear,

ii) time-invariant or time-varying,

iii) causal or non-causal.

For each choice, provide either a brief (typically one sentence) justification or a mathematical-checking procedure.

(a)

$$y[n] = 2\,x[n-1]$$

Solution:
(i) linear — the system satisfies all the conditions for linearity.
(ii) time-invariant — the system behaves the same for all time n.
(iii) causal — because it doesn't require future inputs (it requires a past inputs). □

(b)

$$y[n] = x[-n]$$

Solution:
(i) linear — the system satisfies all the conditions for linearity.
(ii) time-varying — the system behavior depends on how far from time $n = 0$ you are.
(iii) non-causal — for example $y[-3]$ requires future $x[3]$. □

(c)

$$y[n] = 0$$

Solution:
(i) linear — the system satisfies all the conditions for linearity. See also textbook equation 1.125 on page 53 and associated discussion.
(ii) time-invariant — the system behaves the same for all time n.
(iii) causal — because it doesn't require future inputs (in fact it requires no inputs). □

(d)

$$y[n] = -7$$

Solution: (i) The output of a system is always constant. If the input is scaled by zero, the output does not go to zero. Hence the homogeneity (scaling) property is not satisfied and the system is non-linear.
(ii) The system is time-invariant as the output is always constant.
(iii) This is causal as the output does not depend on the input (and so doesn't need future inputs). □

(e)

$$y(t) = 2.571489635789999999$$

Solution:
(Note that the output of a system is always constant. If the input is scaled by zero, the output does not go to zero. Hence the homogeneity (scaling) property is not satisfied and the system is non-linear.)
(i) non-linear — doubling the input doesn't double the output.
(ii) time-invariant — how the system behaves doesn't depend on time.
(iii) causal — it doesn't require future inputs (in fact it requires no inputs). □

(f)

$$y[n] = (\sin n\pi)\, x[n]$$

Solution: (Trick question. The short answer is this system is the same as $y[n] = 0$, which is the zero system and is linear, time-invariant and causal.)

(i) The system is linear as it satisfies both additivity and homogeneity (scaling) property.

(ii) The system is time-invariant. If you carefully look at the system, the output is always zero (constant) due to the factor $\sin n\pi = 0$, indicating that the shift in time in the input also shifts the output in time.

(iii) This is causal as output only depends on the current input (mathematically). In fact, output is always zero. □

(g)

$$y(t) = \int_{-\infty}^{4t} x(\tau)\, d\tau$$

Solution: (i) Since integration is a linear operator, the system is linear.

(ii) The system is time-varying as the shift in time in the input does not produce the same amount of time shift in the output.

(iii) The output at time t depends on the input over the time from $-\infty$ to $4t$. When $t > 0$ the system is not causal. For example for the output at $t = 1$ the input needs to be known for $t \leq 4$. Another way to see non-causality is to write

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau + \int_{t}^{4t} x(\tau)\, d\tau$$

where the first bit is causal stuff and the second but is strictly non-causal stuff. □

(h)

$$y(t) = \int_{-\infty}^{t+4} x(\tau)\, d\tau$$

Solution: (i) Since integration is a linear operator, the system is linear.

(ii) The system is time-invariant as the shift in time in the input also shifts the output by same amount. This can be shown as follows for any real α:

$$\int_{-\infty}^{t+4} x(\tau - \alpha)\, d\tau = \int_{-\infty}^{t-\alpha+4} x(\tau)\, d\tau = y(t-\alpha).$$

(iii) The systems is not causal as the output at t depends on the input over the time from $-\infty$ to $t+4$ (future). That is $y(t)$ needs up to $x(t+4)$. Another way, again, to see non-causality is to write

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau + \int_{t}^{4+t} x(\tau)\, d\tau$$

where the first bit is causal stuff and the second but is strictly non-causal stuff. □

(i)

$$y[n] = \left(\frac{n+0.5}{n-0.5}\right)^2 x[n]$$

Solution: (Note that because n is integer the coefficient is never infinite.)

(i) The system is linear as it satisfies both additivity and homogeneity (scaling) property.

(ii) The system is time varying due to the factor $\frac{n+0.5}{n-0.5}$. The shift in time in the input does not produce the same amount of time shift in the output.

(iii) This is causal as the output depends only on the current input. □

(j)

$$y(t) = \frac{dx(t)}{dt} + x(t-2)$$

Solution: Since both $\frac{dx(t)}{dt}$ and $x(t-2)$ represent linear, time-invariant and causal systems (see slide 233 of the lecture notes), the system is (i) linear, (ii) time-invariant and (iii) causal. □

(k)

$$y(t) = x^2(t) + 2x(t+1)$$

Solution: Easy to verify that it is non-linear (because of the squaring operation), time-invariant and not causal (because of $x(t+1)$ component). □

(l)

$$y[n] = \sum_{k=-\infty}^{n} x[k+3]$$

Solution:
(i) linear — the system satisfies all the conditions for linearity.
(ii) time-invariant — the system behaves the same for all time n.
(iii) non-causal as it requires future inputs. □

Problem Set 1-28

Consider the following four systems:

System A: $y[n] = \cos\big(2\pi\, x[n+1]\big) + x[n]$

System B: $y[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-2k]$

System C: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

System D: $y[n] = \Big(-\frac{1}{5}\Big)^{2n} \big(x[n] + 1.5\big)$

where $x(t)$ or $x[n]$ is the system input and $y(t)$ or $y[n]$ is the system output.

Answer yes or no for each of the following questions for each of the four systems. No explanation is required. Is the system:

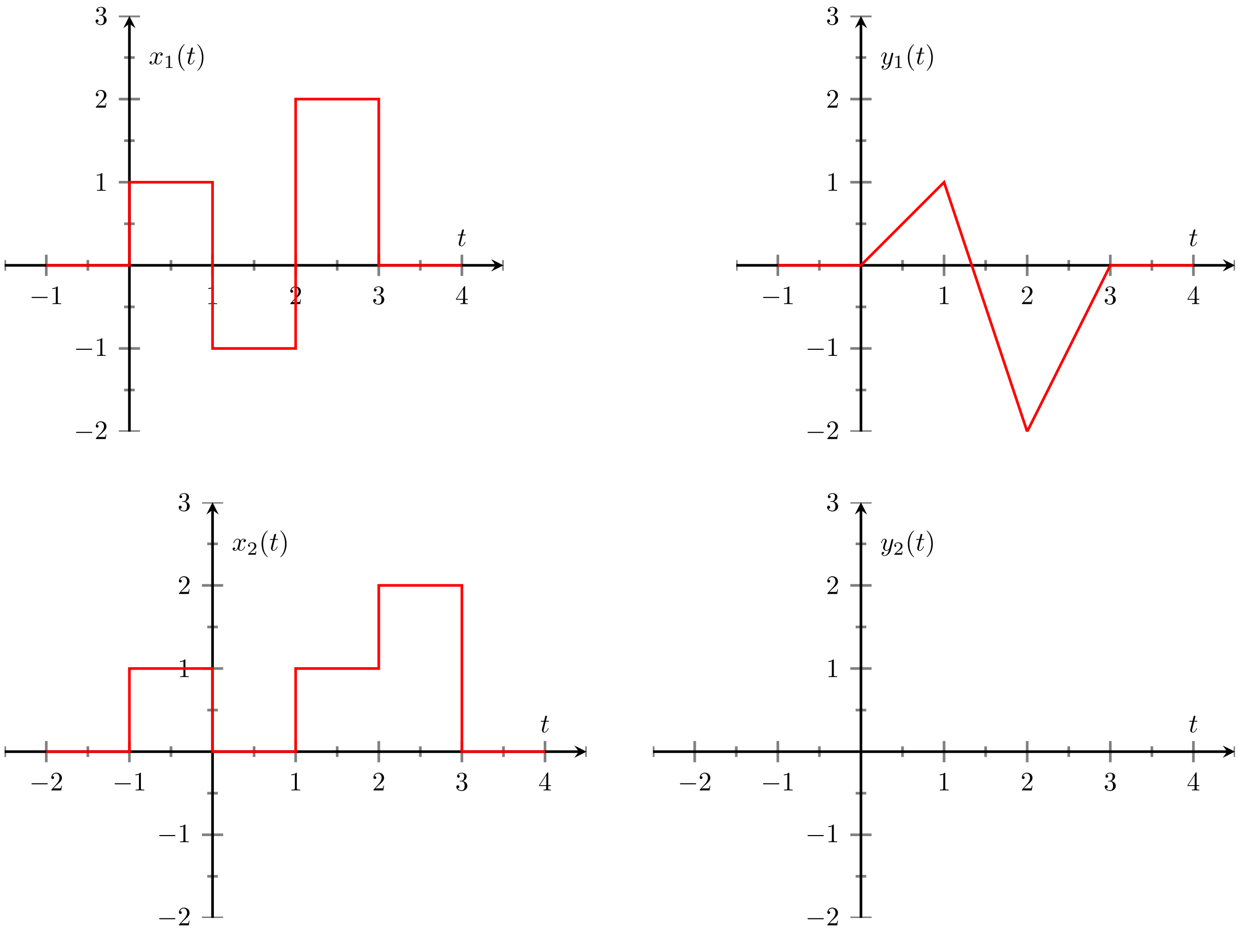
	System A	System B	System C	System D
memoryless ?	yes no	yes no	yes no	yes no
stable ?	yes no	yes no	yes no	yes no
casual ?	yes no	yes no	yes no	yes no
linear ?	yes no	yes no	yes no	yes no
time invariant ?	yes no	yes no	yes no	yes no

Solution:

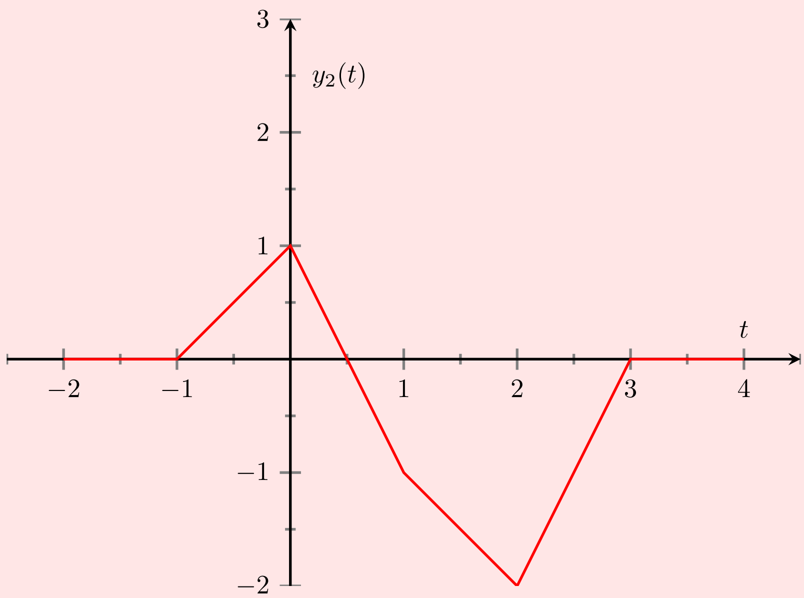
	System A	System B	System C	System D
memoryless	no	yes	no	yes
stable	yes	yes	no	no
casual	no	yes	yes	yes
linear	no	yes	yes	no
time invariant	yes	no	yes	no

Problem Set 1-29

Consider a LTI system whose reponse to the signal $x_1(t)$ is the signal $y_1(t)$ depicted below. Determine and provide a labelled sketch of the response $y_2(t)$ of the system to the input $x_2(t)$ depicted below. Provide concise statements to explain any reasoning. [Hint: $x_2(t) = x_1(t) + x_1(t+1)$]



Solution: Since the system is a LTI system and $x_2(t) = x_1(t) + x_1(t+1)$, the output of the system is $y_2(t) = y_1(t) + y_1(t+1)$. The response $y_2(t)$ is:



Problem 1-28   system C

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

$$x_1(t) = x(t-t_0)$$

$$\rightarrow y_1(t) = \int_{-\infty}^{t} x_1(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau - t_0)\, d\tau$$

set $z = \tau - t_0$, $\tau = z + t_0$ $\frac{d\tau}{dz} = 1 \Rightarrow d\tau = dz$

$$\Rightarrow y_1(t) = \int_{z=-\infty}^{t-t_0} x(z)\, dz = y(t-t_0)$$

$\Rightarrow$ time invariant

Problem 1-27 g)

$$y(t) = \int_{-\infty}^{4t} x(\tau)\, d\tau$$

$$y(t-t_0) = \int_{-\infty}^{4(t-t_0)} x(\tau)\, d\tau$$

$$x_1(t) = x(t-t_0)$$

$$\rightarrow y_1(t) = \int_{-\infty}^{4t} x(\tau - t_0)\, d\tau ,$$

set $z = \tau - t_0$

$$\Rightarrow y_1(t) = \int_{-\infty}^{4t-t_0} x(z)\, dz \neq y(t-t_0)$$

$\Rightarrow$ time varying

So even though the systems look similar one is time invariant the other is time varying.

For the first ... [illegible]