Signal Processing

ENGN2228

Lecturer: Dr. Amin Movahed

Research School of Engineering, CECS
The Australian National University
Canberra ACT 2601 Australia

Second Semester

Lecture 12

CT System Properties – Memoryless System

Definition (Memoryless CT System)

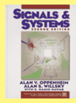
A CT system is **memoryless** if its output at time t depends only on the input at the same time t.

The following CT Systems $x(t) \xrightarrow{h(t)} y(t)$ are:

memoryless

- **Memoryless**
  - $y(t) = 7x(t)$
  - $y(t) = \sqrt{x(t)} + 23$
  - $y(t) = t\,x(t)$
  - $y(t) = -5$ (even though independent of $x(t)$)
- **Not memoryless** (have memory)
  - $y(t) = 5\,x(t - 0.5)$
  - $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

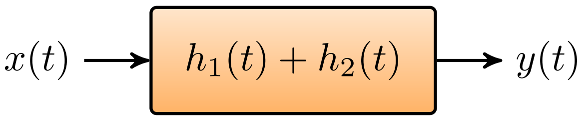
CT System Properties – Distributivity

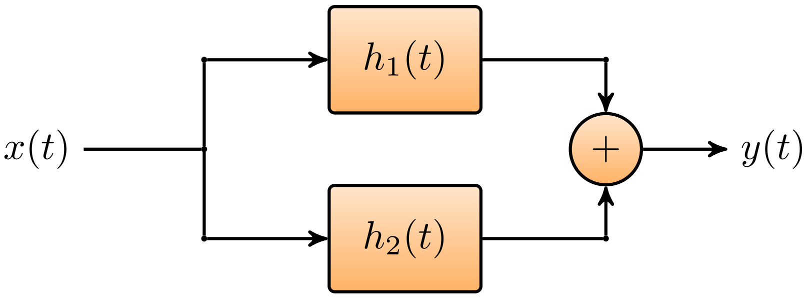


Definition (Distributivity Property of CT LTI Systems)

Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

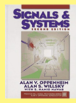
$$\begin{aligned} y(t) &= x(t) \star \big(h_1(t) + h_2(t)\big) \\ &= x(t) \star h_1(t) + x(t) \star h_2(t) \end{aligned}$$





Two CT LTI systems in **parallel** implies we **add** their impulse responses.

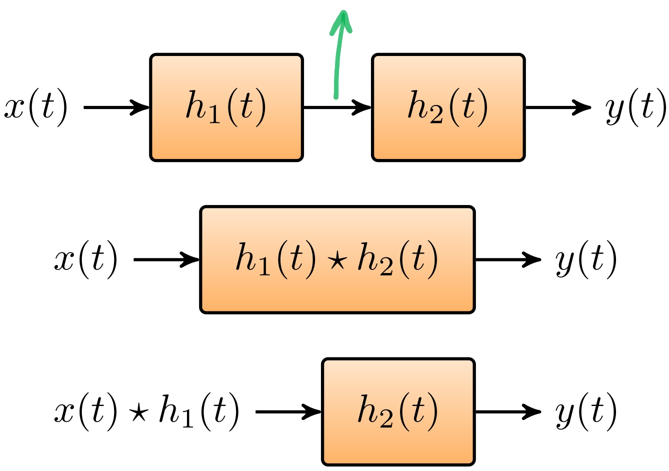
CT System Properties – Associativity



Definition (Associativity Property of CT LTI Systems)

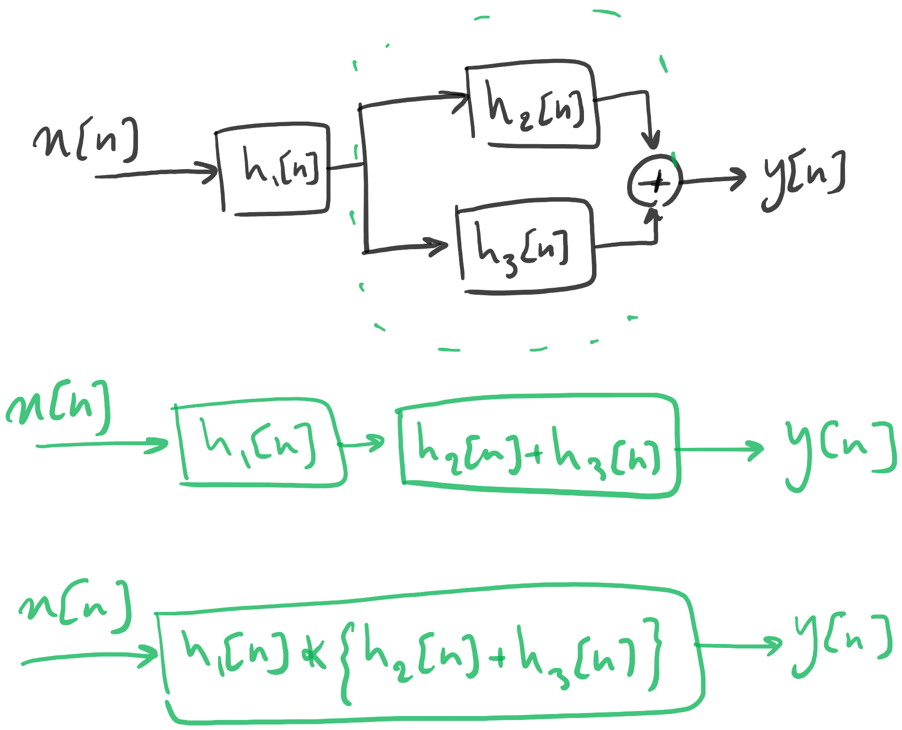
Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

$$y(t) = \big(x(t) \star h_1(t)\big) \star h_2(t)\big) = x(t) \star \big(h_1(t) \star h_2(t)\big)$$

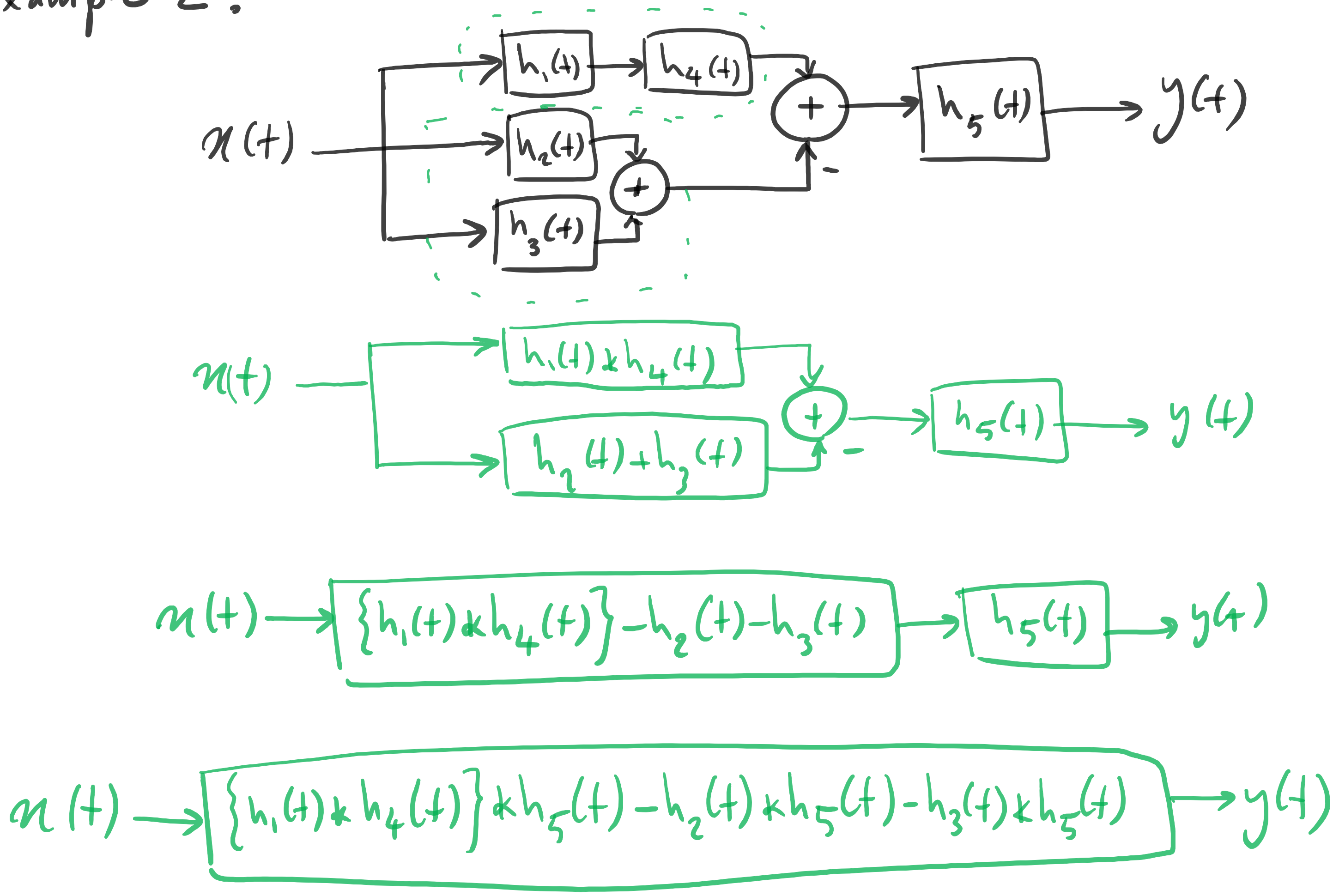


Two CT LTI systems in **series** implies we **convolve** their impulse responses.

Example 1:



Example 2:


x(t)
h1(t)
h4(t)
h2(t)
h3(t)
+
-
h5(t)
y(t)
x(t)
h1(t) * h4(t)
h2(t) + h3(t)
+
-
h5(t)
y(t)
x(t)
{h1(t) * h4(t)} - h2(t) - h3(t)
h5(t)
y(t)
x(t)
{h1(t) * h4(t)} * h5(t) - h2(t) * h5(t) - h3(t) * h5(t)
y(t)

Impulses and More – Unit Impulse

$\delta(t)$ — unit area (that integrates to 1) pulse and is the limit of a narrow rectangular pulse with width, Δ, going to zero and height, $1/\Delta$, going to infinity.

- The shape of the (unit) impulse isn't important, that is, there is nothing special about the rectangular shape.
- When applied to a CT LTI System gives the output equal to the impulse response:

$$\boxed{\delta(t) \star h(t) = h(t), \quad \text{for all } h(t).}$$

  This is a tautology of sorts, this says "the response to an unit impulse is the impulse response". Let's look at this next, mathematically.

-
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Impulses and More – Unit Impulse (cont'd)

LTI System General Input: Start with $x(t) \star h(t) = y(t)$, which means

$$
\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau
\end{aligned}
$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fig: LTI System General Input

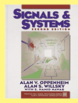
LTI System Special Input: Then set the input to an impulse, that is, set $x(t) = \delta(t)$, to yield

$$\int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau = h(t)$$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

Fig: LTI System Special Input $x(t) = \delta(t)$

Here $\delta(\tau) = 0$ if $\tau \neq 0$ and it "sifts" the value of $h(t)$.



Impulses and More – Trivial System

From commutativity $x(t) \star h(t) = h(t) \star x(t) = y(t)$:

$$\begin{aligned} y(t) &= h(t) \star x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \end{aligned}$$

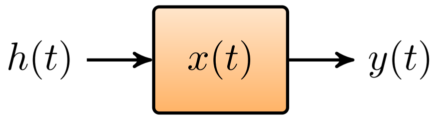


Fig: Flipped $x(t)$ and $h(t)$

Trivial System: With "impulse response" $x(t) = \delta(t)$ (the part inside the box) and "input" $h(t)$ (the part feeding the box)

$$\int_{-\infty}^{\infty} h(\tau)\, \delta(t-\tau)\, d\tau = h(t)$$

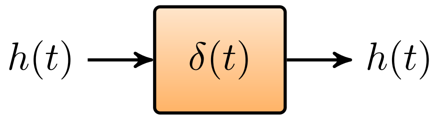
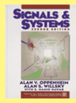


Fig: Trivial System

Here $\delta(t-\tau) = 0$ if $t \neq \tau$ and it sifts the value $h(t)$ (under then integral sign). This system, with impulse response given by the impulse, has output signal equal to the input signal, that is, just passes the input to the output — called the **"Trivial System"**.

Impulses and More – Delay System



Signals & Systems
section 2.5.2
pages 131-132

Now, with impulse response $h(t) = \delta(t - t_0)$ and input $x(t)$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t - t_0 - \tau)\,d\tau = x(t - t_0).$$

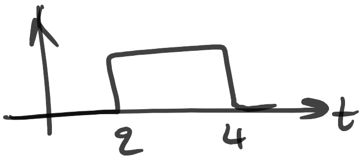
$$x(t) \longrightarrow \boxed{\delta(t - t_0)} \longrightarrow x(t - t_0)$$

Fig: Time Shift LTI System (Delay when $t_0 > 0$)

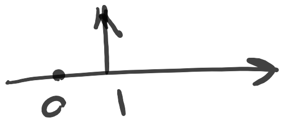
Here $\delta(t - t_0 - \tau) = 0$ if $t - t_0 \neq \tau$ and it "sifts" the value $h(t - t_0)$. This system, with impulse response given by the impulse with time shift, has output equal to the input with a time shift. Note that $t_0 > 0$ gives a delay and $t_0 < 0$ gives a time advance (which would be non-causal).

Examples:

$x(t) = u(t-2) - u(t-4)$ $\qquad\qquad$ $\delta(t-1)$



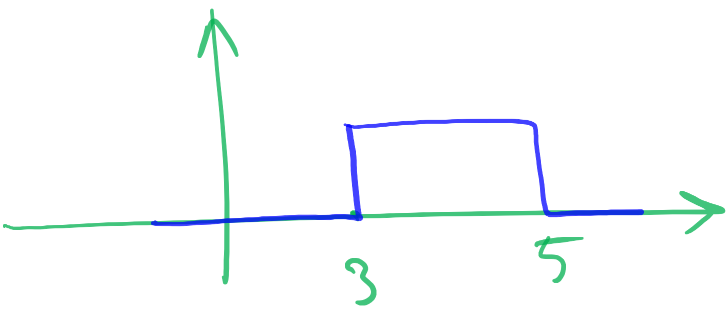
$*$



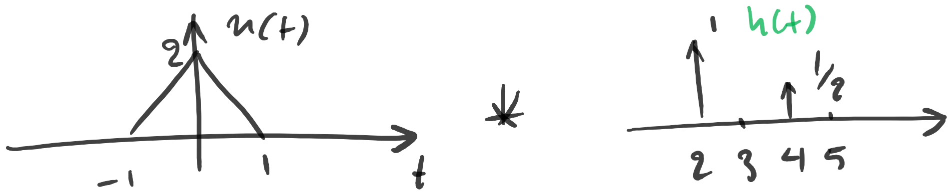
$$x(t) * \delta(t-1) = \{u(t-2) - u(t-4)\} * \delta(t-1)$$

$$= u(t-2) * \delta(t-1) - u(t-4) * \delta(t-1)$$

$$= u(t-2-1) - u(t-4-1) = \underbrace{u(t-3) - u(t-5)}$$

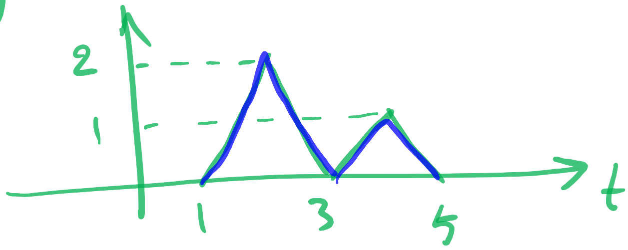


Example:



$$x(t) * \overbrace{\left\{\delta(t-2)+\tfrac{1}{2}\delta(t-4)\right\}}^{h(t)} = x(t)*\delta(t-2)+\tfrac{1}{2}\,x(t)*\delta(t-4)$$

$$= x(t-2)+\tfrac{1}{2}\,x(t-4)$$



$$h(t)=\delta(t+1)-\delta(t)+2\delta(t-2)$$

$$x(t)=e^{-t}u(t)$$

$$y(t)=x(t)*h(t)=e^{-(t+1)}u(t+1)-e^{-t}u(t)+2e^{-(t-2)}u(t-2)$$

Impulses and More – Additional Results

Signals & Systems
section 2.5.2
pages 131-136

Note that

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which is the unit step signal.

Compare this with the fundamental theorem of calculus which asserts

$$\frac{d}{dt} \int_{-\infty}^{t} f(\tau)\, d\tau = f(t)$$

for sufficiently regular functions $f(t)$. (cont'd)

Impulses and More – Additional Results (cont'd)

With the function $\delta(t)$ as the function in the fundamental theorem of calculus, we can infer

$$\frac{d}{dt}\underbrace{\int_{-\infty}^{t}\delta(\tau)\,d\tau}_{u(t)} = \delta(t).$$

So the $\delta(t)$ "is" the derivative of the unit step $u(t)$. (cont'd)

Impulses and More – Additional Results (cont'd)

We can represent this as:

$$u(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \delta(t) = \frac{du(t)}{dt}$$

Fig: Differentiator with Unit Step Input

where the $\frac{d}{dt}$ inside the box is not the impulse response but denotes an operator. (cont'd)

Impulses and More – Additional Results (cont'd)

Taking the derivative is actually a linear time-invariant (LTI) operator. It satisfies superposition (L)

$$\frac{d}{dt}\big(\alpha_1 x_1(t) + \alpha_2 x_2(t)\big) = \alpha_1 \frac{dx_1(t)}{dt} + \alpha_2 \frac{dx_2(t)}{dt}$$

and, with the notation,

$$x'(t) = \frac{dx(t)}{dt}$$

then by the chain rule we have the time-invariance (TI)

$$\frac{d}{dt} x(t - t_0) = x'(t - t_0)$$

So we conclude that the differentiator operator acts like a LTI system and so must have an **impulse response** which we'll consider shortly.

More examples:

$$\int_{-5}^{5} 7e^{t^2}\cos(t)\,\delta(t)\,dt = \int_{-5}^{5} 7e^{0}\cos(0)\,\delta(t)\,dt = 7$$

$$\int_{-5}^{5} 7e^{t^2}\cos(t)\,\delta(t-2)\,dt = \int_{-5}^{5} 7e^{4}\cos(2)\,\delta(t-2)\,dt = 7e^{4}\cos(2)$$

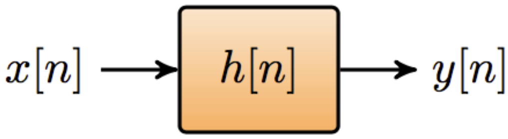
$$\int_{-5}^{1} 7e^{t^2}\cos(t)\,\delta(t-2)\,dt = \int_{-5}^{1} 7e^{4}\cos(2)\,\delta(t-2)\,dt = 0$$

$$\int_{-3}^{1} (x^3-3x^2+2x-1)\,\delta(x+2)\,dx = \int_{-3}^{1} (-8-12-4-1)\,\delta(x+2)\,dx = -25$$

$$\int_{0}^{\infty} \{\cos(3x)+2\}\,\delta(x-\pi)\,dx = \int_{0}^{\infty} \{\cos(3\pi)+2\}\,\delta(x-\pi)\,dx = -1+2=1$$

Two ways to describe DT LTI systems

- Difference equations provide an implicit specification of the system.
- Impulse response provides an explicit specification of the system.



$$y[n] = h[n] \star x[n]$$

Causal DT LTI Systems Described by Difference Equations

A general N-th order linear constant-coefficient difference equation is given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

$$b = [b_0, b_1, b_2, \ldots, b_M]$$
$$a = [a_0, a_1, a_2, \ldots, a_N]$$

Example:

$$y[n] - \tfrac{1}{2}y[n-1] = x[n-1]$$

$$b = ? \quad [0, 1]$$
$$a = ? \quad [1, -1/2]$$

MATLAB commands:

```
h=impz(b,a,n)
y=filter(b,a,x)
```

FIR vs. IIR DT LTI Systems

- If a DT LTI system has a finite duration impulse response (i.e, h[n] is non-zero only over a finite time interval), then the system is called a FINITE IMPULSE RESPONSE (FIR) system.

- If a DT LTI system, with the condition of initial rest, will have an impulse response of finite duration, then the system is called an INIFINITE IMPULSE RESPONSE (IIR) system.

Moving-Average System

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$$

- Moving-average system is a FIR system.
- Such systems are used for enhancement of some feature in a data-set, such as identifying the underlying trend in data that are fluctuating.
- The value of N determines the degree to which the system smooths the input data.

$y[n] = \frac{1}{2}\left[x[n] + x[n-1]\right] \longrightarrow h[n] = \frac{1}{2}\left[\delta[n] + \delta[n-1]\right]$