Signal Processing
ENGN2228

Lecturer: Dr. Amin Movahed

Research School of Engineering, CECS
The Australian National University
Canberra ACT 2601 Australia

Second Semester

Lectures 28-29

DT Fourier Transforms – Recap

Periodic Signals:

$$\text{CT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Non-Periodic Fourier Series}$$

$$\text{DT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Periodic Fourier Series}$$

CT Non-Periodic Signals:

$$\text{CT (Non-Periodic) Signals} \xleftrightarrow{\;\mathscr{F}\;} \text{“Continuous” Fourier Transform}$$

CT Periodic Signals (revisited):

$$\text{CT Periodic signals} \xleftrightarrow{\mathscr{F}} \text{“Impulse Sequence” Fourier Transform}$$

- Sometimes called a discrete spectrum.
- Technically the “Impulse Sequence” needs to be uniformly spaced, with delta functions lying at multiples of some ω_0.

Recall:

FS

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

DTFS

For $x[n] = x[n+N]$ periodic with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

FT

Provided the following integrals are finite/exist

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

Discrete Time Fourier Transform: `O&W 5.1.1 pp.359-362`

Here we have DT but, generally, non-periodic signals. We can have a continuum of frequencies and to synthesis the time domain signal we need to integrate over that continuum of frequencies. In the frequency domain we have 2π periodicity.

Definition (DT Fourier Analysis and Synthesis)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \qquad (Synthesis)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

DT Fourier Transforms – Discrete Time FT

DT Non-Periodic Signals:

$$\underbrace{\text{DT}}\ \underbrace{\text{Non-Periodic}}\ \text{signals} \overset{\mathscr{F}}{\longleftrightarrow} \underbrace{\text{Continuous}}\ \underbrace{\text{Periodic}}\ \text{Fourier Transform}$$

Again, DT implies periodicity of the Fourier Transform and given the signal is not periodic (aperiodic) then the spectrum is continuous.

DT Fourier Transforms – Discrete Time FT

Four cases:

$$\text{continuous in time} \xleftrightarrow{\mathscr{F}} \text{continuous in frequency}$$

$$\text{continuous in time} \xleftrightarrow{\mathscr{F}} \text{discrete in frequency}$$

$$\text{discrete in time} \xleftrightarrow{\mathscr{F}} \text{continuous in frequency}$$

$$\text{discrete in time} \xleftrightarrow{\mathscr{F}} \text{discrete in frequency}$$

Technically "discrete in time" means discrete and uniformly spaced in time, and similarly "discrete in frequency" means discrete and uniformly spaced in frequency.

Time vs. Frequency domain:

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

DTFT Tables:

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

	Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
1	$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ $(2\pi/N)$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
* 2	$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
* 3	$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
* 4	$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$, $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
5	$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
6	Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \ne 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \dots$
7	$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
* 8	$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
* 9	$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
* 10	$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(e^{j\omega})$ $X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
* 11	$\delta[n]$	1	—
12	$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
* 13	$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
14	$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
15	$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

FT table 5.2

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \ne \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$

5.3.9 Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$$

FT Table 5.1

DTFT pairs example:

DTFT

$$8 \quad \boxed{a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}} \qquad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

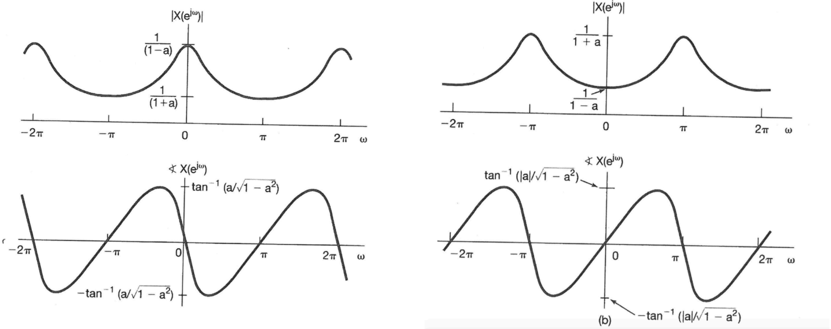
$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

$$* \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

DTFT pairs example:



Example 3: `O&W 5.1.2 pp.362-363`

Causal, exponentially decaying function

$$\boxed{x[n] = a^n \, u[n], \quad |a| < 1}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n \, u[n] \, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n \, e^{-j\omega n} = \sum_{n=0}^{\infty} (a \, e^{-j\omega})^n$$

$$= \frac{1}{1 - a \, e^{-j\omega}}, \quad \text{if } |a| < 1$$

We can write this $X(e^{j\omega})$ as

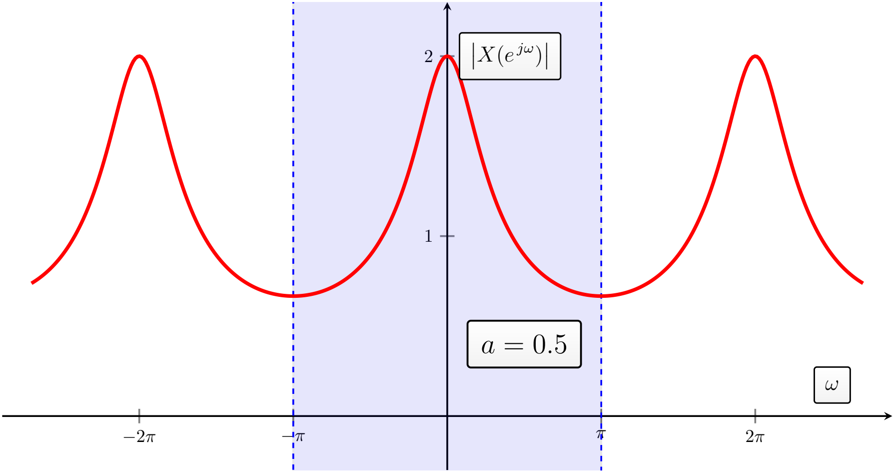
$$\boxed{X(e^{j\omega}) = \frac{1}{(1 - a\,\cos\omega) + ja\sin\omega}, \quad |a| < 1}$$

- This is complex (real plus imaginary) and a can be complex also.
- For real $a = 0.5$, $0.4, \ldots, -0.5$ we plot

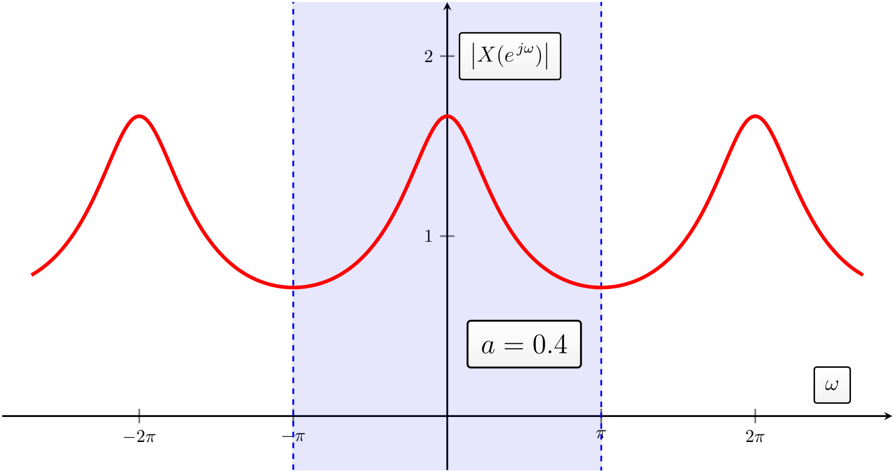
$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos(\omega) + a^2}}$$

In this way we can infer whether is acts like a low-pass filter, high-pass filter or otherwise.

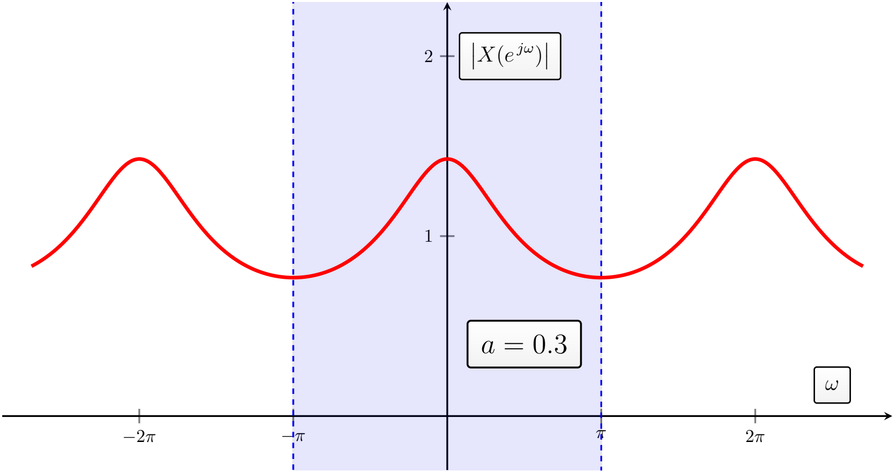
DT Fourier Transforms – Examples



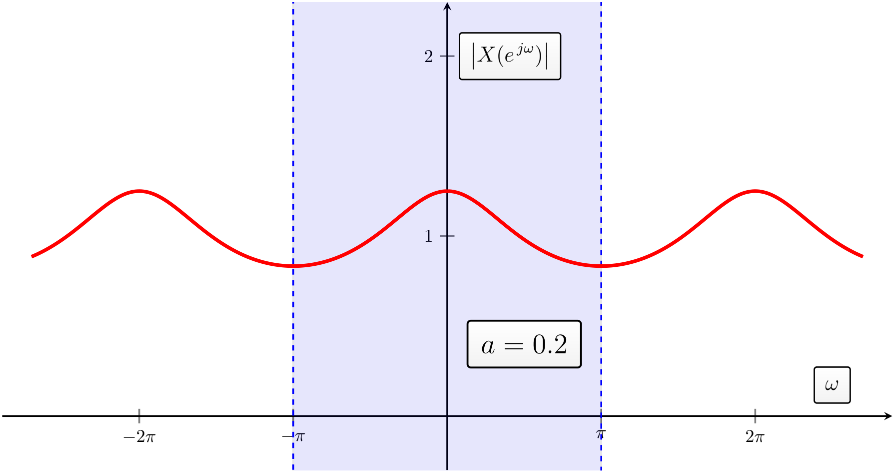
DT Fourier Transforms – Examples



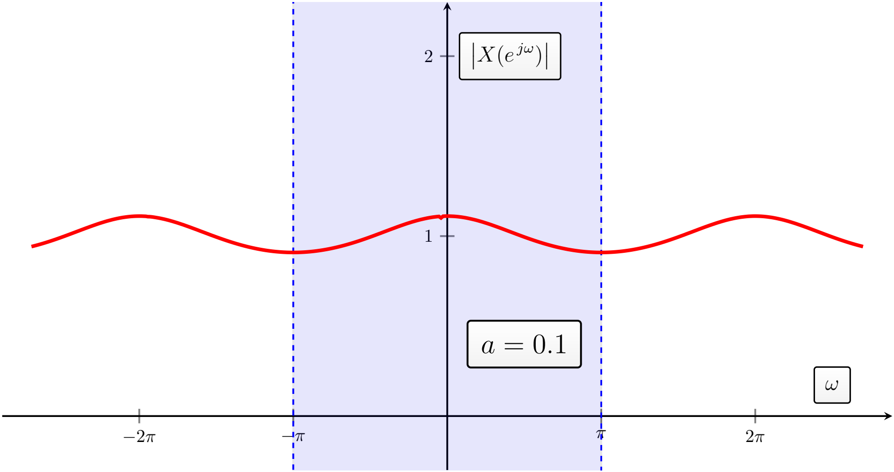
DT Fourier Transforms – Examples



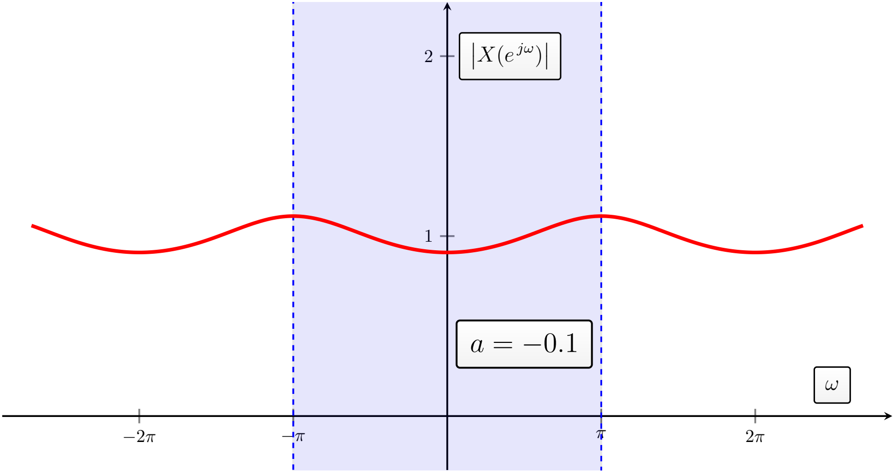
DT Fourier Transforms – Examples



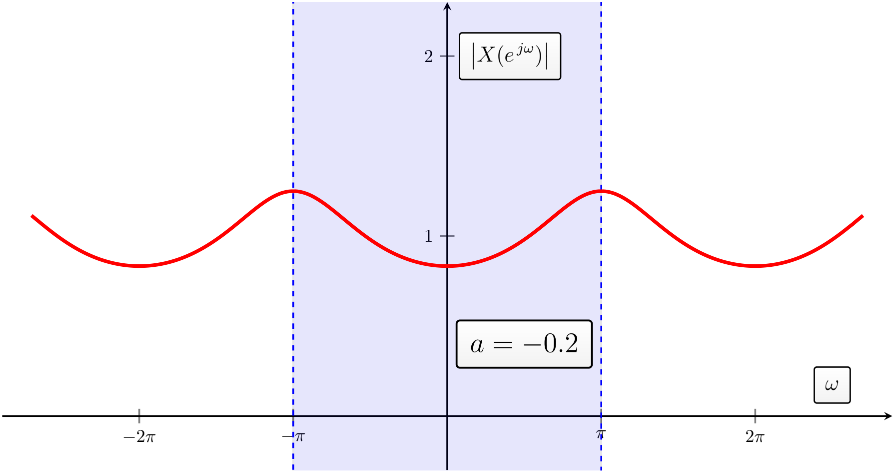
DT Fourier Transforms – Examples



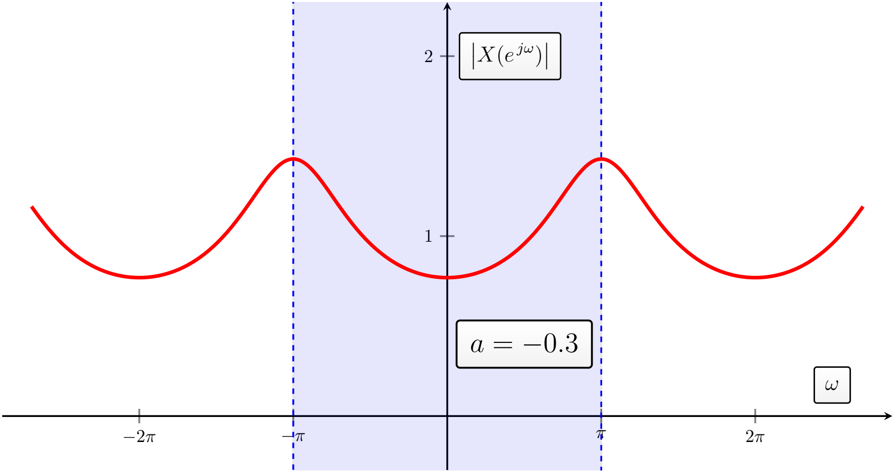
DT Fourier Transforms – Examples



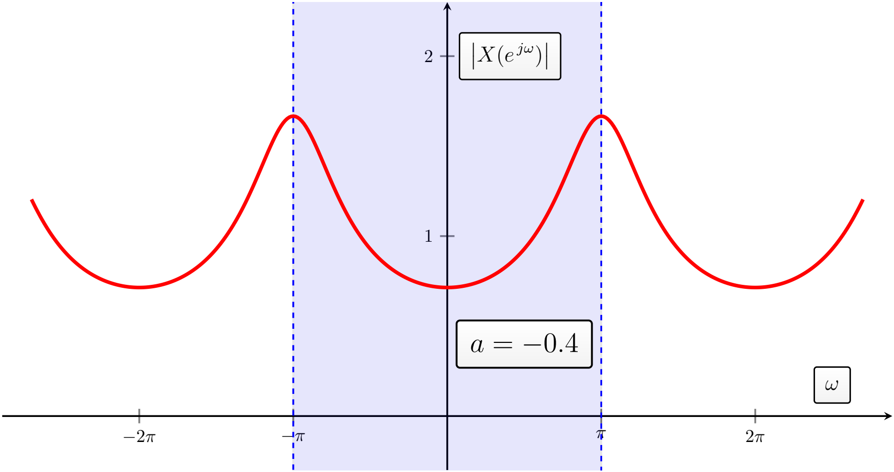
DT Fourier Transforms – Examples



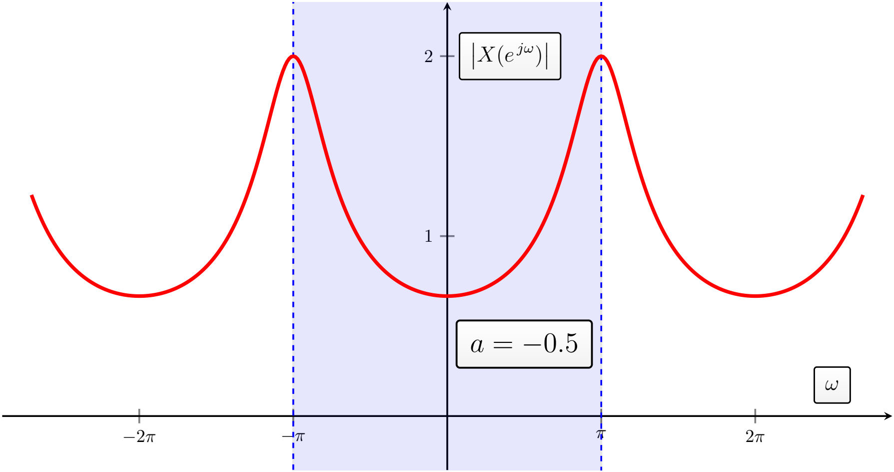
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Notes:

- With $a \to 0$ ("$a = 0$") we have $a^0 = 1$ and $a^n = 0$ for $n \neq 0$; then

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

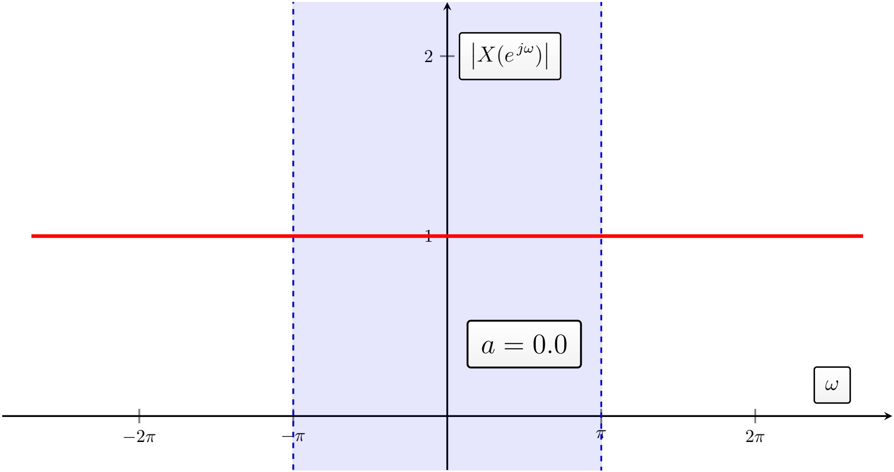
that is,

$$x[n] = \delta[n]$$

and

$$X(e^{j\omega}) = 1.$$

DT Fourier Transforms – Examples



DTFT pairs example:

14

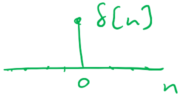
$$\boxed{(n+1)a^n u[n], \ \ |a|<1 \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}}$$

15

$$\boxed{\frac{(n+r-1)!}{n!\,(r-1)!}a^n u[n], \ \ |a|<1 \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^r}}$$

* proofs are not assessable.

DTFT pairs example:



11 $\boxed{\delta[n] \longleftrightarrow 1}$

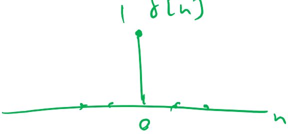
$\delta(t) \longleftrightarrow 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

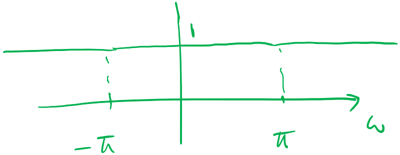
$$= \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n}$$

$$= e^{-j\omega(0)}$$

$$= 1 \checkmark$$



$X(e^{j\omega})$



DT Fourier Transforms – Examples

Example 1: `O&W 5.1.3 p.367`

Unit sample signal

$$\boxed{x[n] = \delta[n]}$$

is not periodic and has Fourier Transform

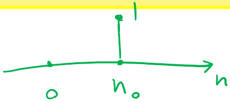
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n}$$

That is,

$$\boxed{X(e^{j\omega}) = 1, \quad \text{for all } \omega}$$

- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = 0$, for all ω.

DTFT pairs example:



13 $\boxed{\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}}$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-jwn}$$

$$= e^{-jwn_0}$$

DT Fourier Transforms – Examples

Example 2: `O&W 5.4.1 p.383`

Shifted unit sample signal, where delay $n_0 \in \mathbb{Z}$,

$$\boxed{x[n] = \delta[n - n_0]}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\, e^{-j\omega n}$$

That is,

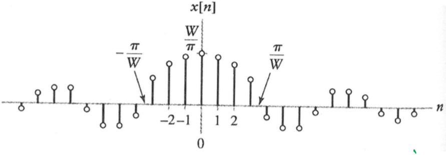
$$\boxed{X(e^{j\omega}) = e^{-j\omega n_0}, \quad n_0 \in \mathbb{Z}}$$

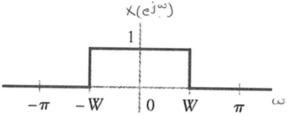
- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = -\omega\, n_0$, for all ω.
- Has linear phase (phase proportional to ω), and the slope (derivative wrt ω) gives the time shift.
- Here, slope is $-n_0$ implying a delay of n_0.

DTFT pairs example:

10

$$\frac{\sin(Wn)}{\pi n} \longleftrightarrow X_{[-W,W]}(\omega) = \begin{cases}1 & 0 \le |\omega| \le W \\ 0 & W \le |\omega| \le \pi\end{cases}$$





$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W}(1)\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left(\frac{1}{jn}\right)\left. e^{j\omega n}\right|_{-W}^{W}$$

$$= \frac{1}{\pi n}\,\frac{1}{2j}\left[e^{jWn} - e^{-jWn}\right] = \frac{\sin(Wn)}{\pi n} \checkmark$$

DT Fourier Transforms – Examples

Example 5: `O&W 5.4.1 pp.383-384`

DT Ideal Low-Pass Filter, bandwidth $0 \le W \le \pi$, passes frequencies $-W \le \omega \le W$,

$$\boxed{X(e^{j\omega}) = \chi_{[-W,+W]}(\omega), \quad -\pi \le \omega \le \pi, \quad 0 \le W \le \pi}$$

and is periodic with period 2π. It has time domain sampled sinc response

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[-W,+W]}(\omega)\, e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega n}\, d\omega \\ &= \frac{\sin Wn}{\pi n}, \quad n \in \mathbb{Z} \end{aligned}$$

DT Fourier Transforms – Examples

Definition (DT Ideal Low-Pass Filter)

The DT LTI system that passes only frequencies with gain 1 in the range $[-W, +W] \in [-\pi, +\pi]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin Wn}{\pi n} \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-W,+W]}(\omega), \quad -\pi \leq \omega \leq \pi, \quad 0 \leq W \leq \pi}$$

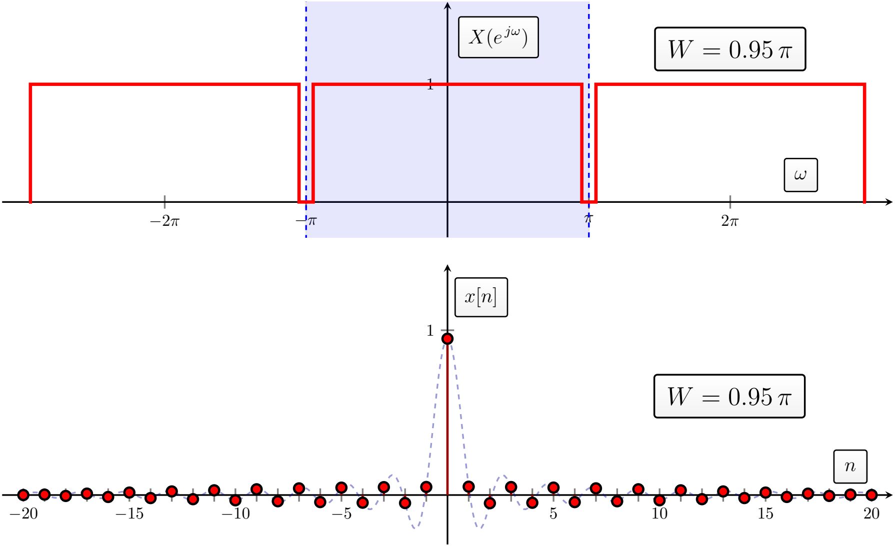
and the frequency response is periodic in ω with period 2π; or

$$\boxed{\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right) \overset{\mathscr{F}}{\longleftrightarrow} \chi_{[-W,+W]}(\omega), \quad -\pi \leq \omega \leq \pi, \quad 0 \leq W \leq \pi}$$

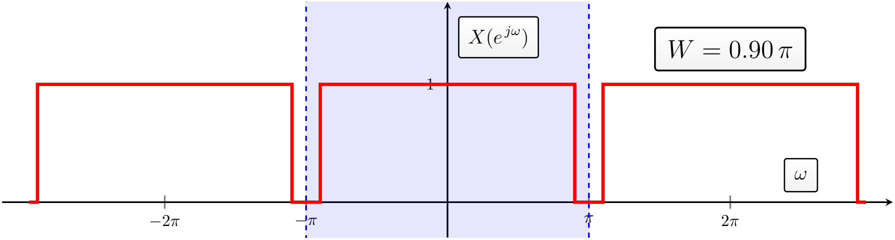
and the frequency response is periodic in ω with period 2π.

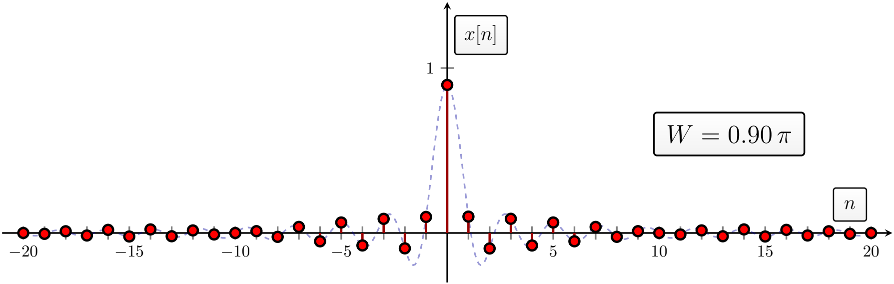
- We plot $X(e^{j\omega})$ and $x[n]$ for a range of $0 < W < \pi$.

DT Fourier Transforms – Examples

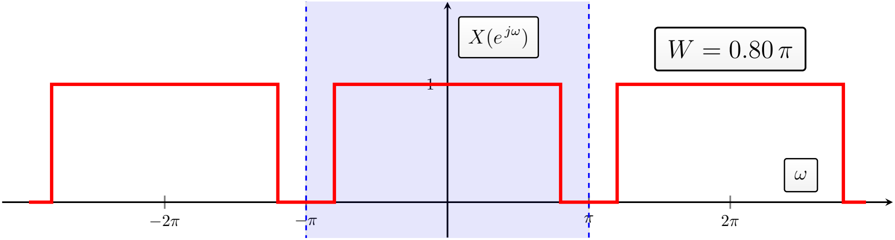


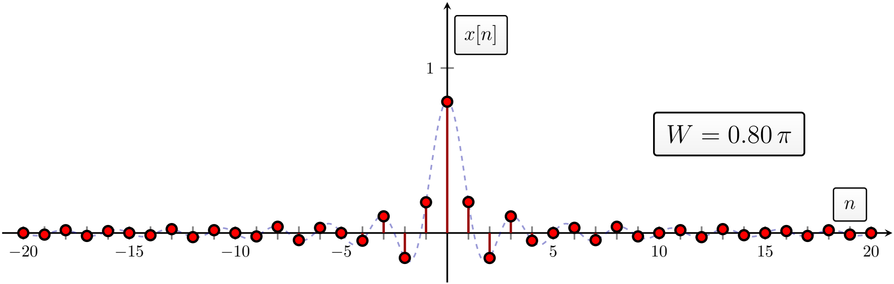
DT Fourier Transforms – Examples



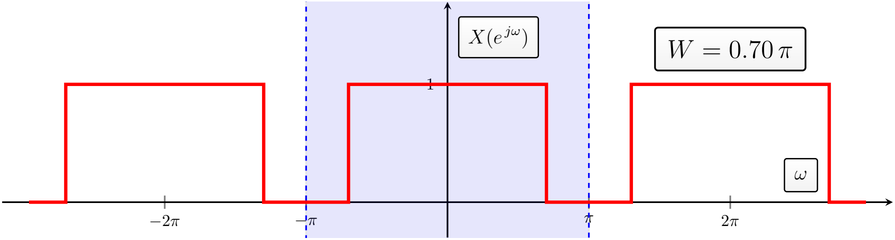


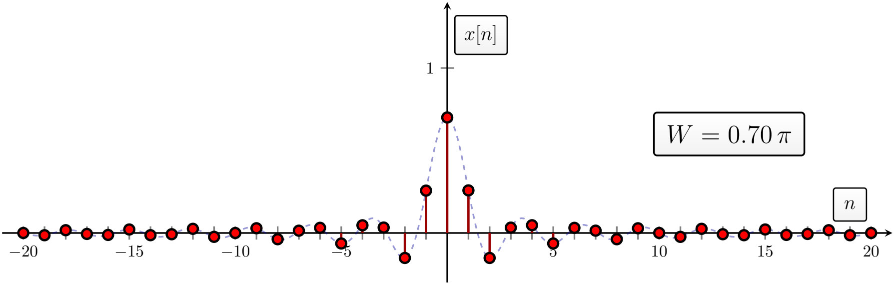
DT Fourier Transforms – Examples



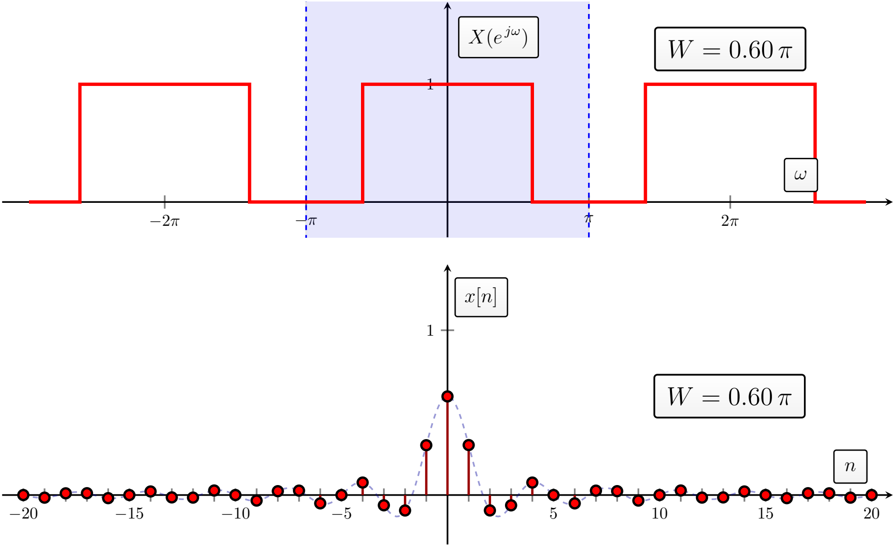


DT Fourier Transforms – Examples

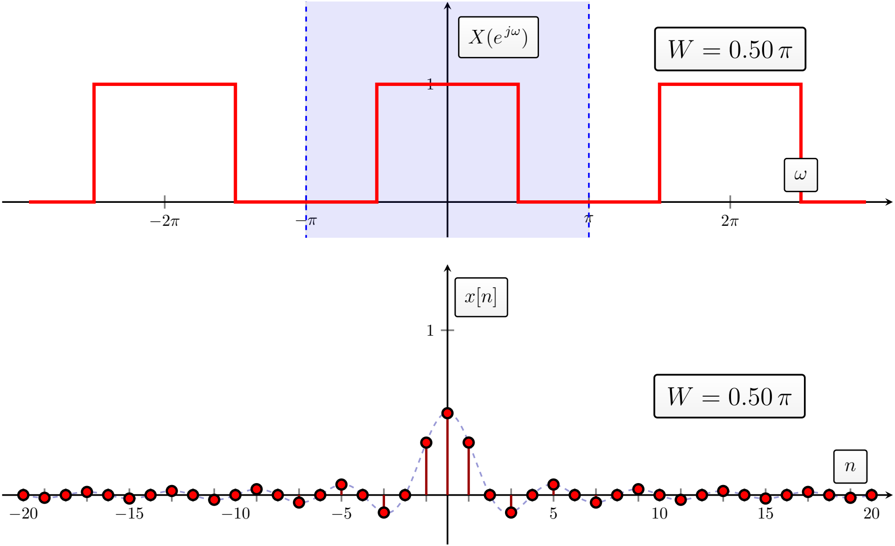




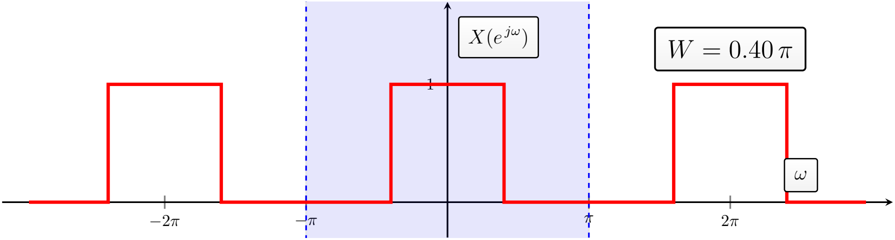
DT Fourier Transforms – Examples

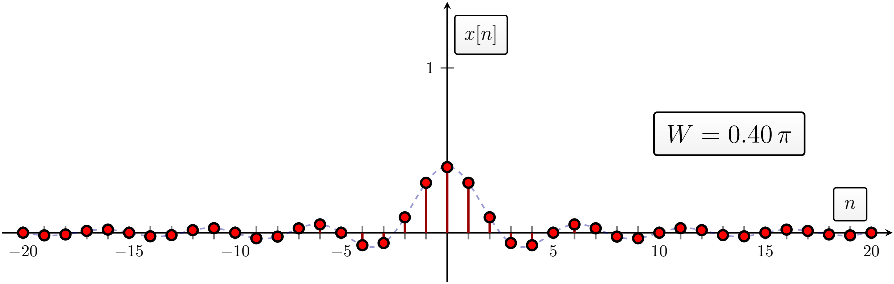


DT Fourier Transforms – Examples

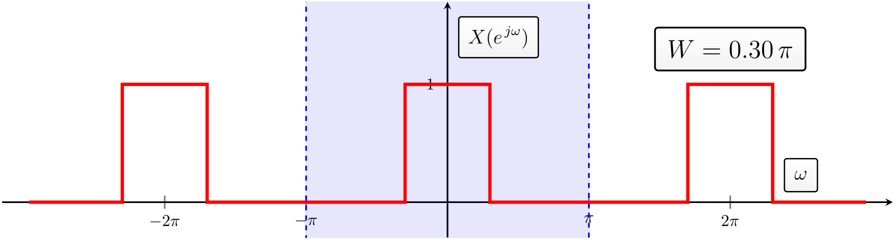


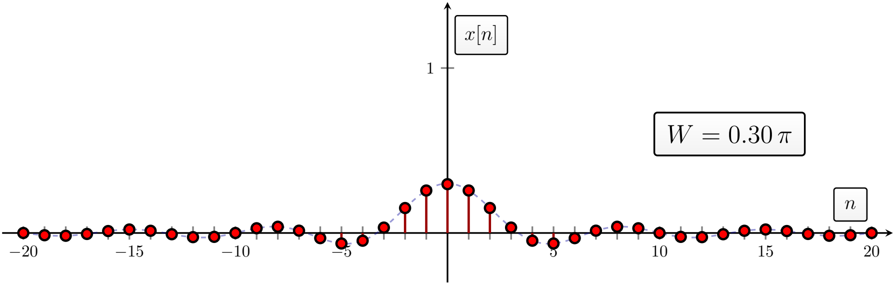
DT Fourier Transforms – Examples



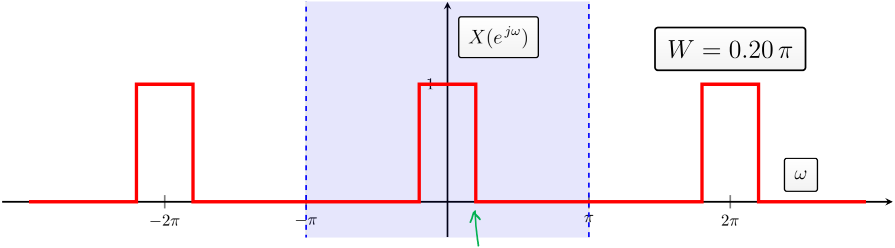


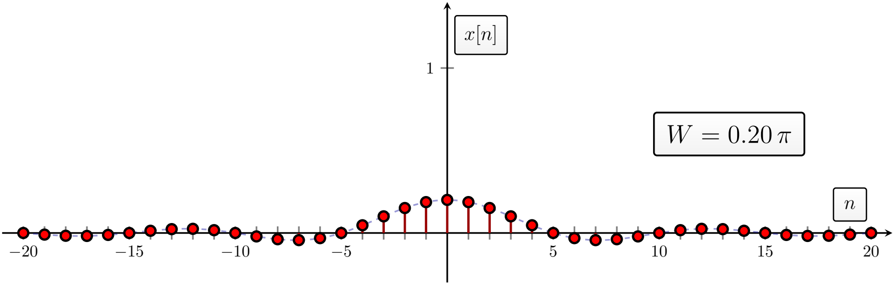
DT Fourier Transforms – Examples



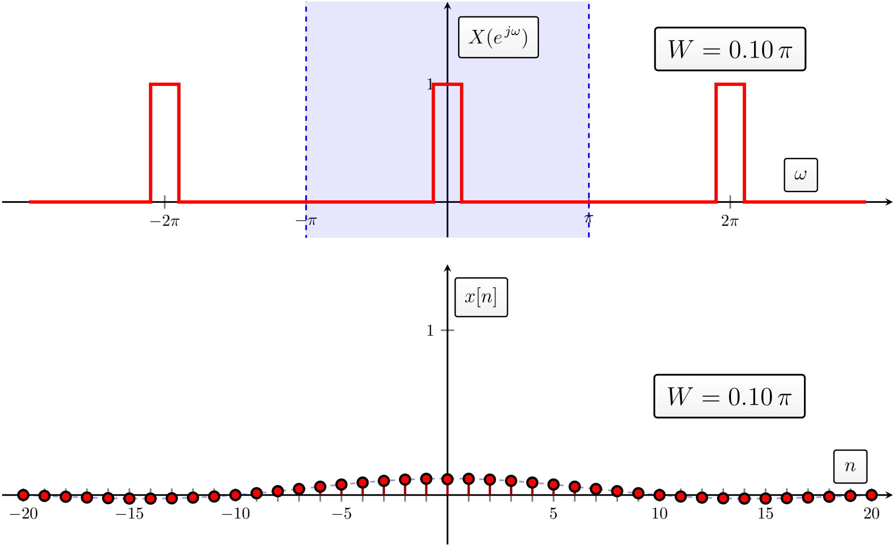


DT Fourier Transforms – Examples





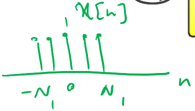
DT Fourier Transforms – Examples



DTFT pairs example:

9

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \longleftrightarrow \frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$* \; r = n + N_1 \longrightarrow$

$r - N_1 = n$

$$= \sum_{r=0}^{2N_1} e^{-j\omega(r-N_1)}$$

$$= e^{j\omega N_1} \sum_{r=0}^{2N_1} e^{-j\omega r}$$

$$* \sum_{n=0}^{M-1} \alpha^n = \begin{cases} \frac{1-\alpha^M}{1-\alpha} & \alpha \neq 1 \\ M & \alpha = 1 \end{cases}$$

$$X(e^{j\omega}) = \begin{cases} e^{j\omega N_1}\left[\frac{1-(e^{-j\omega})^{2N_1+1}}{1-e^{-j\omega}}\right] & \omega \neq 0 \\ e^{j\omega N_1}(2N_1+1) & \omega = 0, \pm 2\pi, \dots \end{cases}$$

$$e^{jwN_1}\frac{\left(1-e^{-jw(2N_1+1)}\right)}{1-e^{-jw}} = e^{jwN_1}\cdot\frac{e^{-jw/2(2N_1+1)}\left[e^{jw/2(2N_1+1)}-e^{-jw/2(2N_1+1)}\right]}{e^{-jw/2}\left(e^{jw/2}-e^{-jw/2}\right)}$$

$$= \underbrace{e^{jwN_1}\cdot e^{-jw/2(2N_1+1)}\cdot e^{jw/2}}_{1}\cdot\frac{\left(\frac{e^{jw/2(2N_1+1)}-e^{-jw/2(2N_1+1)}}{2j}\right)}{\left(\frac{e^{jw/2}-e^{-jw/2}}{2j}\right)}$$

$$= \frac{\sin\left(w/2\,(2N_1+1)\right)}{\sin(w/2)} \checkmark$$

for $w \to 0$ $\quad \lim\limits_{w\to 0}\dfrac{\sin\left(w/2\,(2N_1+1)\right)}{\sin(w/2)} = e^{jwN_1}(2N_1+1)$

DT Fourier Transforms – Examples

Example 4: `O&W 5.1.2 pp.365-366`

DT Rectangular Pulse function

$$\boxed{x[n] = \chi_{[-N_1, N_1]}[n], \quad n,\, N_1 \in \mathbb{Z}}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \chi_{[-N_1,N_1]}[n]\, e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}$$

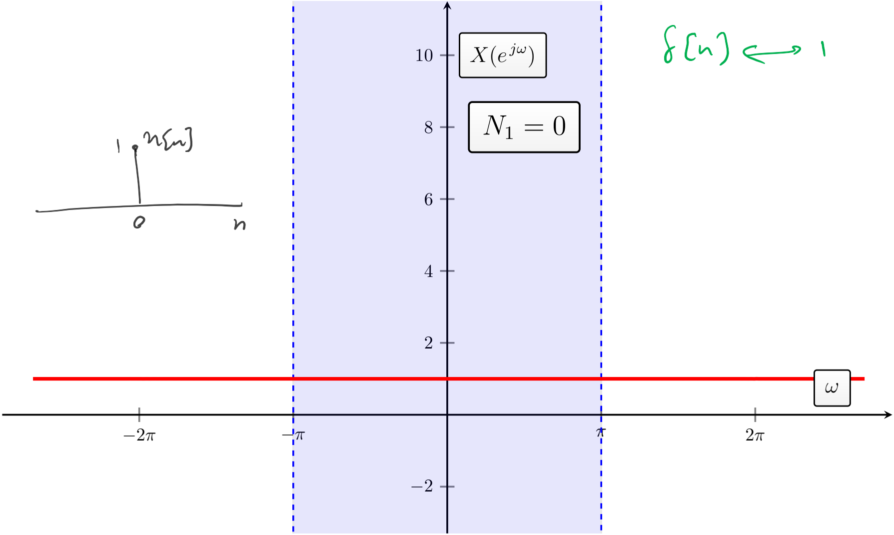
That is,

$$\boxed{X(e^{j\omega}) = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}, \quad N_1 \in \mathbb{Z}}$$

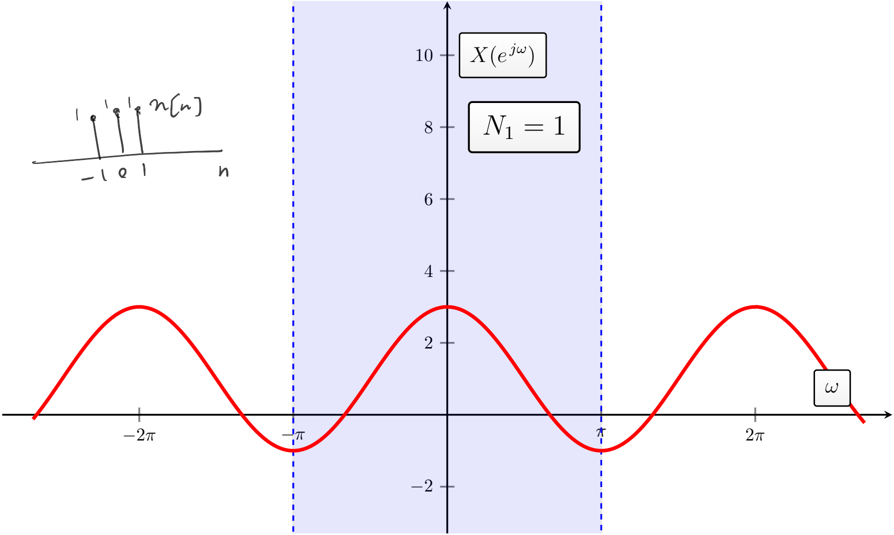
- This is purely real and we plot $X(e^{j\omega})$ rather than $\left|X(e^{j\omega})\right|$.

DT Fourier Transforms – Examples

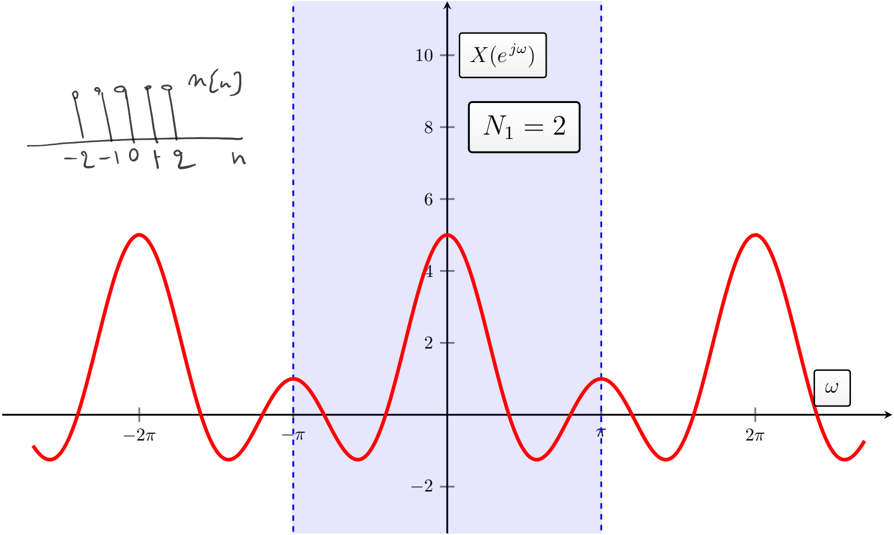
$\delta[n] \longleftrightarrow 1$



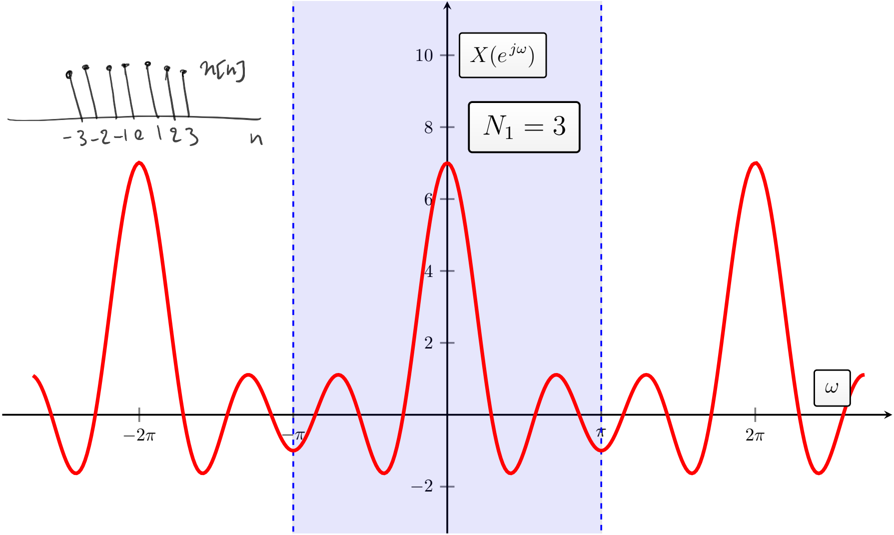
DT Fourier Transforms – Examples



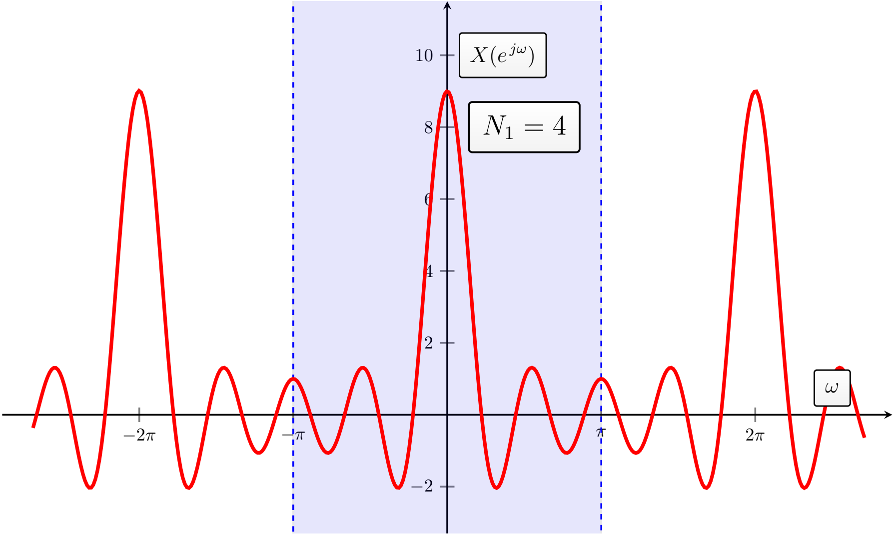
DT Fourier Transforms – Examples



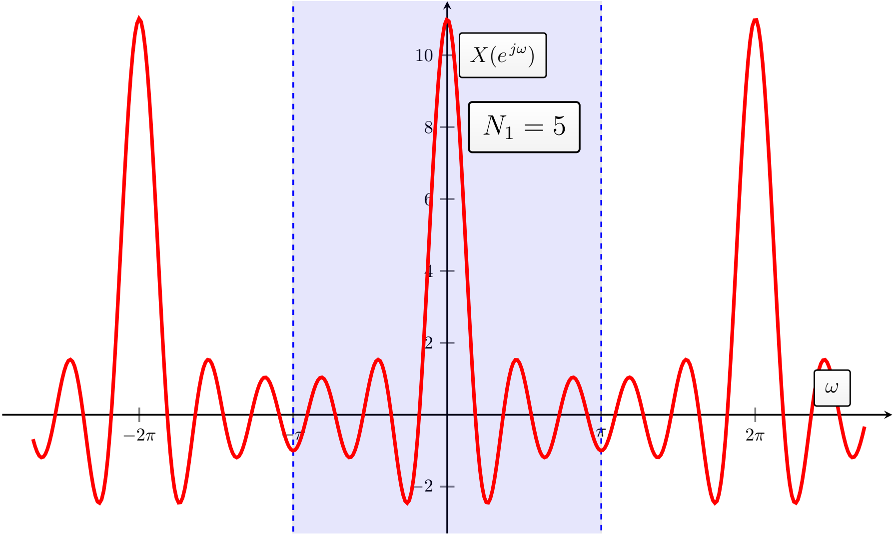
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



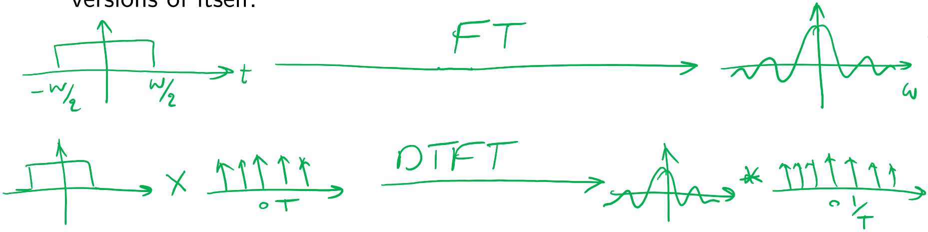
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Notes:

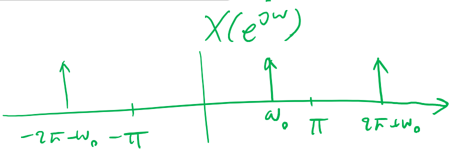
- From these figures we can see that in the frequency domain the Fourier Transform of a time-domain discrete rectangular pulse looks (and is) the convolution of a sinc function with a periodic train of frequency domain delta functions. That is, the superposition of a sinc function with shifted versions of itself.



DTFT pairs example:

2 $\quad \boxed{e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)}$ $\quad -\infty < \omega < +\infty$

$\cos \omega_0 n + j \sin \omega_0 n$



$$X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \qquad -\infty < \omega < \infty$$

OR

$$X(e^{j\omega}) = 2\pi\, \delta(\omega - \omega_0) \qquad -\pi < \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\,\delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega$$

$$= e^{j\omega_0 n} \quad \checkmark$$

DTFT pairs example:

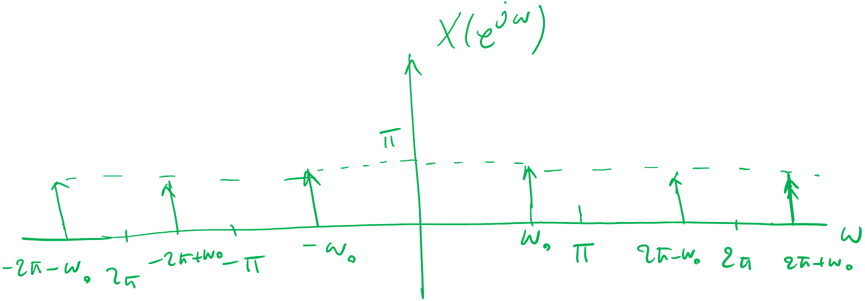
3 $$\cos(\omega_0 n) \longleftrightarrow \pi \sum_{l=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$$

$$\cos(w_0 n) = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2}$$

Using linearity property of DTFT

$$F(\cos w_0 n) = \frac{F(e^{jw_0 n}) + F(e^{-jw_0 n})}{2}$$

$$= \pi \sum_{l=-\infty}^{\infty} \delta(w - w_0 - 2\pi l) + \pi \sum_{l=-\infty}^{\infty} \delta(w + w_0 - 2\pi l) \checkmark$$



X(e^{jw})
π
ω
-2π-ω₀
2π
-2π+ω₀
-π
-ω₀
ω₀
π
2π-ω₀
2π
2π+ω₀

DTFT pairs example:

4 $$\sin(\omega_0 n) \longleftrightarrow \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$$

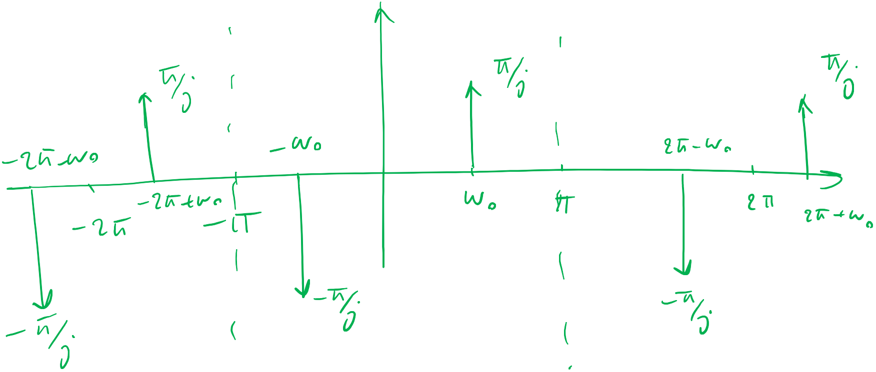
$$\sin(w_0 n) = \frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}$$

Using linearity property of DTFT

from 2

$$F(\sin(w_0 n)) = \frac{F(e^{jw_0 n}) - F(e^{-jw_0 n})}{2j}$$

$$= \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \{\delta(w - w_0 - 2\pi l) - \delta(w + w_0 - 2\pi l)\}$$



π/j
π/j
π/j
−2π−w₀
−w₀
2π−w₀
−2π+w₀
w₀
π
2π
2π+w₀
−2π
−π
−π/j
−π/j
−π/j

DTFT pairs example:

1 $$\boxed{\sum_{k=0}^{N-1} a_k e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)}$$ $\omega_0 = \frac{2\pi}{N}$

* the proof is not assessable

In pairs 1, 6, 7, $X(e^{j\omega})$ is expressed in form valid for $-\pi < \omega < \pi$

In pairs 2, 3, 4, 5, $X(e^{j\omega})$ is expressed in form valid for $-\infty < \omega < \infty$

DT Fourier – Properties

Properties for DT Fourier Transform can be inferred from the Synthesis and Analysis equations. Properties that mimic the CT counterparts are:

- **Reference:**

$$\boxed{x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})}$$

- **Linearity/Superposition:** `O&W 5.3.2 p.373`

$$\boxed{a\, x_1[n] + b\, x_2[n] \overset{\mathscr{F}}{\longleftrightarrow} a\, X_1(e^{j\omega}) + b\, X_2(e^{j\omega})}$$

- **Time Shifting:** `O&W 5.3.3 p.373`

$$\boxed{x[n - n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}), \quad n_0 \in \mathbb{Z}}$$

DT Fourier – Properties

- **Frequency Shifting:** `O&W 5.3.3 p.373`

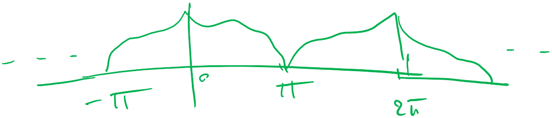
$$\boxed{e^{-j\omega_0 n}x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})}$$

- **Time Reversal:** `O&W 5.3.6 p.376`

$$\boxed{x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega})}$$

- **Conjugate Symmetry:** `O&W 5.3.4 p.375`

$$\boxed{x[n] \text{ real} \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega}) = X^*(e^{-j\omega})}$$



DT Fourier – Properties

Properties that more DT specific and/or tricky are:

- **Periodicity:** `O&W 5.3.1 p.373`

$$\boxed{X(e^{j\omega}) \text{ is periodic with period } 2\pi}$$

- **Differentiation in Frequency:** `O&W 5.3.8 p.380`

$$\boxed{n\,x[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(e^{j\omega})}$$

is multiplication by n in the time domain.

- **Parseval's Relation:** `O&W 5.3.9 p.380`

$$\boxed{\sum_{n=-\infty}^{\infty} \big|x[n]\big|^2 = \frac{1}{2\pi}\int_{2\pi} \big|X(e^{j\omega})\big|^2\, d\omega}$$

DTFT Tables:

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

	Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
1	$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ $(2\pi/N)$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
2	$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
3	$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
4	$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
5	$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
6	Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
7	$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
8	$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
9	$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
10	$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(e^{j\omega})$ $X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
11	$\delta[n]$	1	—
12	$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
13	$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
14	$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
15	$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$X(e^{j\omega})$, $Y(e^{j\omega})$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \ne \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega - \theta)}) d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega}) X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$$