Australian
National
University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 2 – SOLUTIONS

In the following: $\delta[n]$ and $u[n]$ represent the Dirac and unit step functions for discrete-time (DT). Similarly $\delta(t)$ and $u(t)$ for continuous-time (CT). Convolution of signals is written $x[n] \star h[n]$ or $x(t) \star h(t)$. Please indicate any identities or formulas used in the simplification of the results.

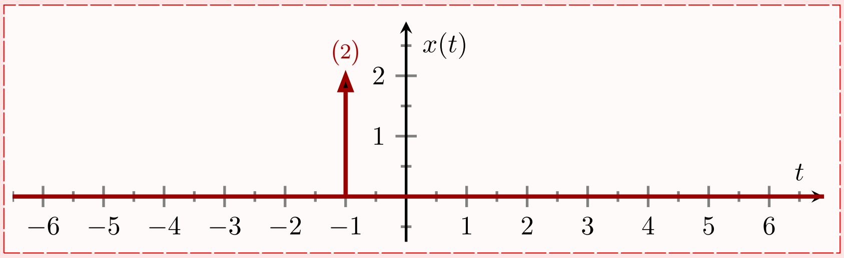
Unit Impulse and Unit Step Functions

Problem Set 2-1

Draw the following signals

(a) $x(t) = 2\,\delta(t+1)$

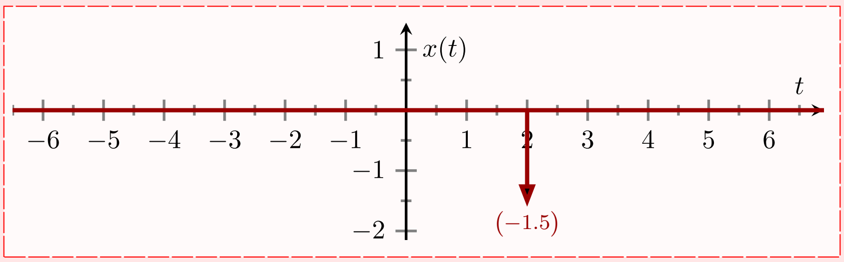
Solution:



Notice that you could draw in the fact that the function is zero everywhere except at $t = -2$. □

(b) $-1.5\,\delta(t-2)$

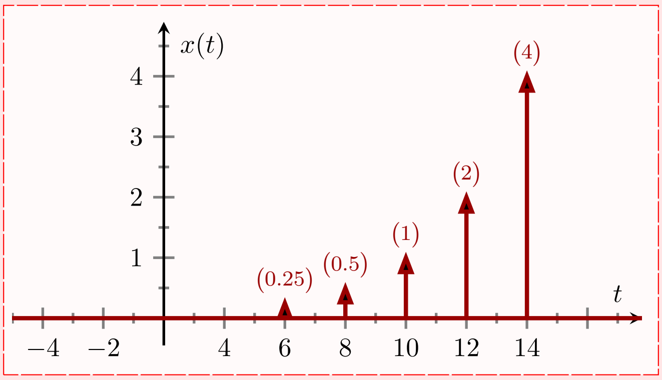
Solution:



□

(c) $x(t) = \sum_{k=3}^{7} 2^{k-5}\delta(t-2k)$

Solution:

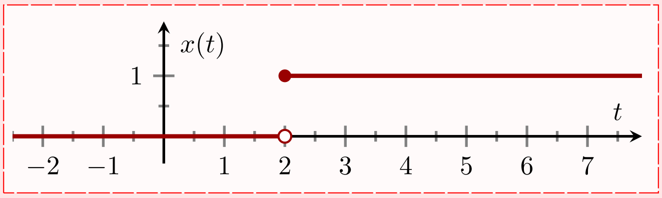


☐

(d) $x(t) = \displaystyle\int_{-\infty}^{t} \delta(\tau - 2)\, d\tau$

Solution: From textbook equation (1.71), $u(t) = \int\limits_{-\infty}^{t} \delta(\tau)d\tau$. Hence,

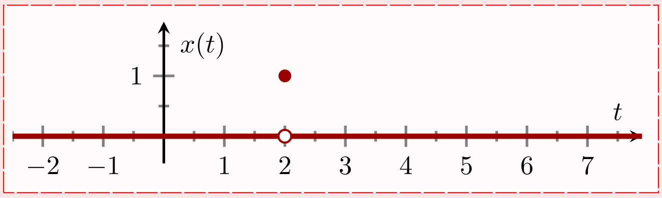
$$\int_{-\infty}^{t} \delta(\tau - 2)\, d\tau = \int_{-\infty}^{t-2} \delta(s)\, ds = u(t-2)$$



Don't worry about what the exact value is a $t = 2$. ☐

(e) $x(t) = \displaystyle\int_{-\infty}^{t} \delta(t - 2)\, d\tau$

Solution: Well this is a bit weird. Firstly, the integrand is independent of the integration variable τ. For all values $t \neq 2$ we have $\delta(t-2) = 0$ so $x(t) = 0$ for $t \neq 2$. For $t = 2$ is best to leave the $\delta(t-2)$ inside the integral. At $t = 2$ it has area 1 so $x(2) = 1$.



Not a delta function at $t = 2$ just the value of 2 (zero area). ☐

(f) $\displaystyle\int_{-t}^{t} \delta(t - 2)\, d\tau$

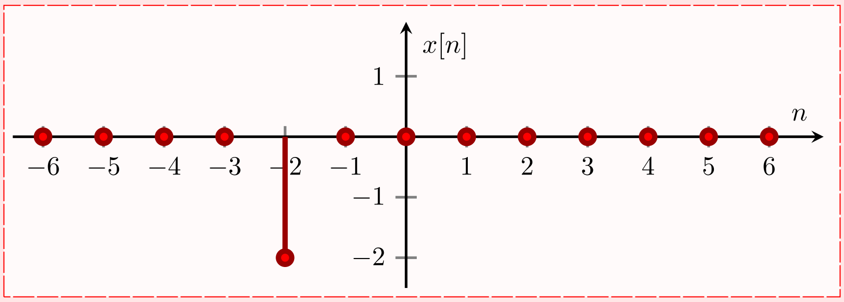
Solution: Messing with your head. Same as previous part (e). ☐

Problem Set 2-2

Draw the following signals

(a) $x[n] = -2\,\delta[n+2]$

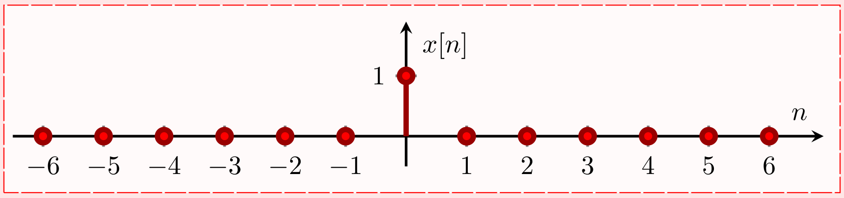
Solution:



Notice that you could draw in the fact that the function is zero everywhere except at $n = -2$. □

(b) $x[n] = u[n] - u[n-1]$

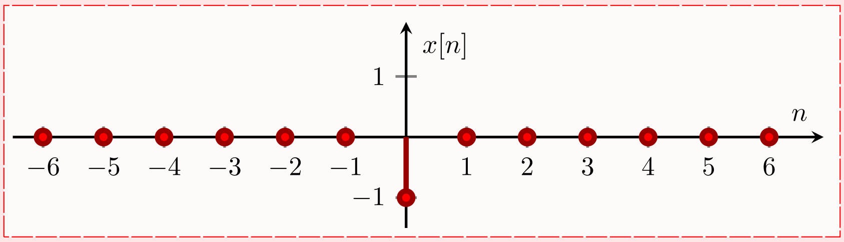
Solution:



Yep, it's the same as $\delta[n]$. □

(c) $x[n] = -u[-n] + u[-n-1]$

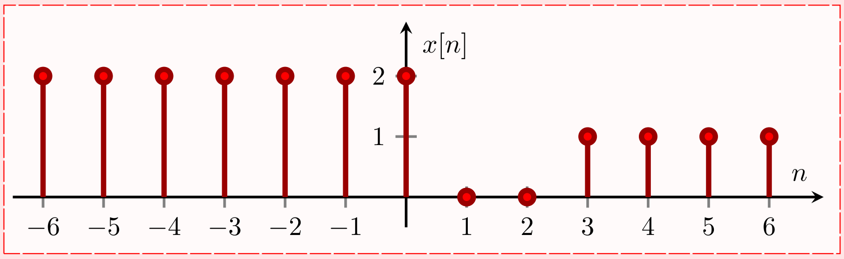
Solution:



Or $-u[-n] + u[-n-1] = -\delta[n]$. □

(d) $x[n] = 2u[-n] + u[n-3]$

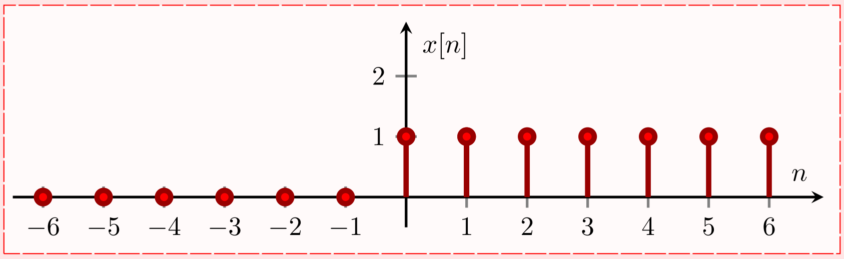
Solution:



□

(e) $x[n] = \sum_{k=-\infty}^{-1} \delta[k] + u[n]$

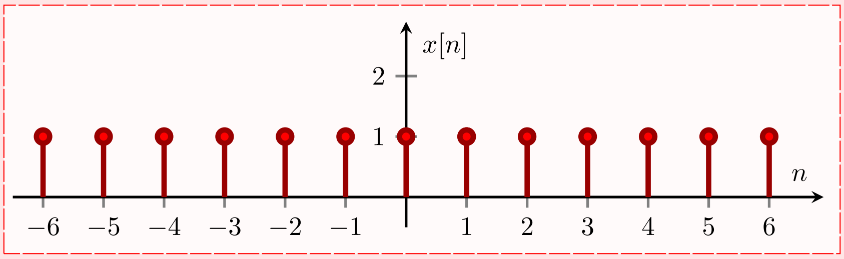
Solution:



That is, $x[n] = u[n]$. □

(f) $x[n] = \sum_{k=-\infty}^{-1} \delta[n-k] + u[n]$

Solution:



That is, $x[n] = 1$ for all n. □

Discrete-time Convolution

Problem Set 2-3

Use the graphical flip/shift method, showing intermediate working to perform the following DT convolutions. Note: solutions can be checked in Matlab.

(a) $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$ and $h[n] = 2\delta[n-4]$

Solution:
$$y[n] = 2\delta[n-4] + 4\delta[n-5] - 2\delta[n-6]$$

(b) $x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$ and $h[n] = \delta[n] + 0.5\delta[n-1]$

Solution:
$$y[n] = 2\delta[n] + 5\delta[n-1] - \delta[n-3]$$

(c) $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$ and $h[n] = 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$

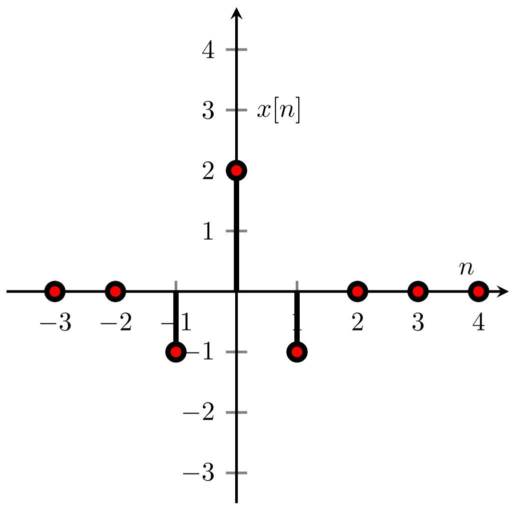
Solution:
$$y[n] = 3\delta[n-2] + 8\delta[n-3] + 2\delta[n-4] - \delta[n-6]$$

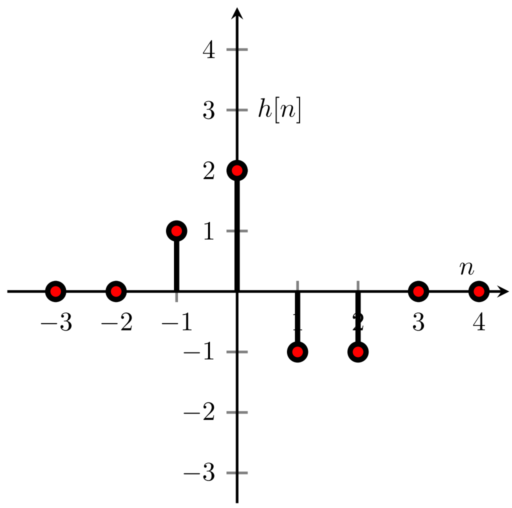
(d) $x[n] = \delta[n+1] + 2\delta[n] + 3\delta[n-1] + 4\delta[n-2]$ and $h[n] = -\delta[n+2] + 5\delta[n+1] + 3\delta[n]$

Solution:
$$y[n] = -\delta[n+3] + 3\delta[n+2] + 10\delta[n+1] + 17\delta[n] + 29\delta[n-1] + 12\delta[n-2]$$

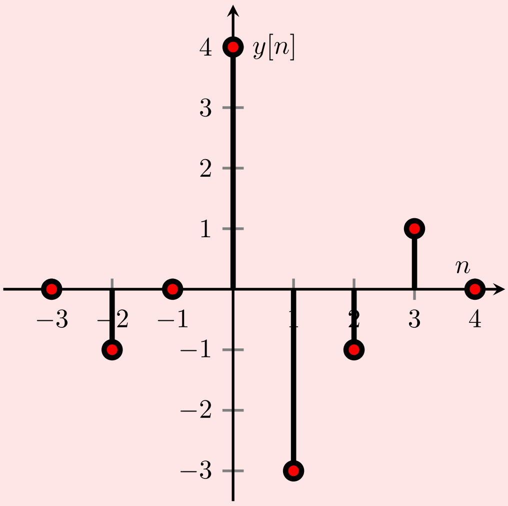
Problem Set 2-4

Compute the DT convolution of $x[n]$ and $h[n]$ as shown below





Solution: Using the graphical flip and shift method (easiest to flip $x[n]$ as $x[n]$ is even so $x[-n] = x[n]$), we obtain $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ is:



Problem Set 2-5

For a DT LTI system with impulse response

$$h[n] = u[n-1].$$

Find the output $y[n] = x[n] \star h[n]$ for input

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1].$$

Solution:

$$\begin{aligned} y[n] = x[n] \star h[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{-k} u[-k-1]u[n-k-1] \end{aligned}$$

Since $u[-k-1] = 0$ for $k > -1$,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} u[n-k-1] \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k} u[n+k-1]. \end{aligned}$$

There are two cases: $1-n \leq 1$ and $1-n > 1$ (or $n \geq 0$ and $n < 0$).

For $n \geq 0$

$$y[n] = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k} = \frac{1/3}{1-1/3} = 1/2.$$

For $n < 0$

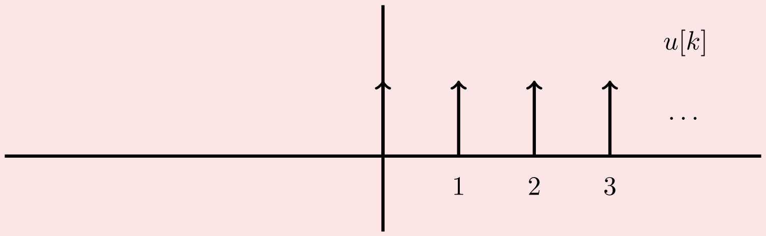
$$y[n] = \sum_{k=1-n}^{\infty} \left(\frac{1}{3}\right)^{k} = \sum_{\ell=1}^{\infty} \left(\frac{1}{3}\right)^{\ell-n} = \left(\frac{1}{3}\right)^{-n} \frac{1}{2} = \frac{3^n}{2}.$$

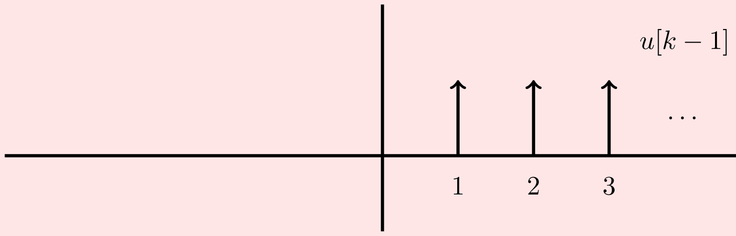
Here change of summation is used with $k+n=\ell$. □

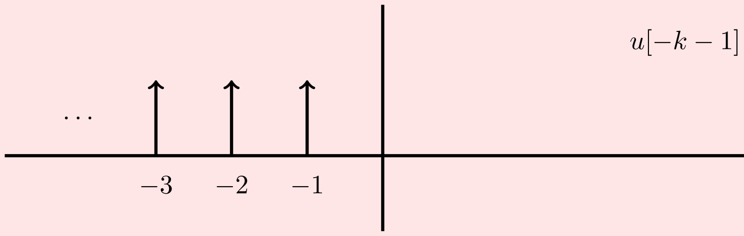
Problem Set 2-6

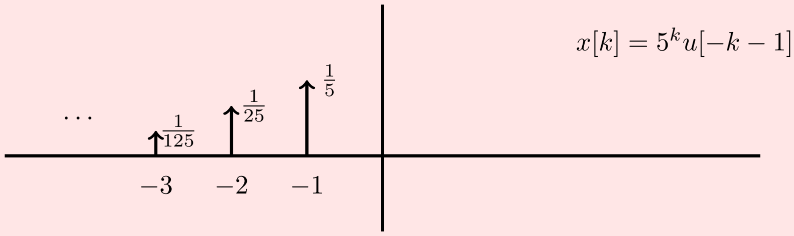
Compute the convolution $y[n] = x[n] * h[n]$ when $x[n] = 5^n\, u[-n-1]$ and $h[n] = u[n-1]$.

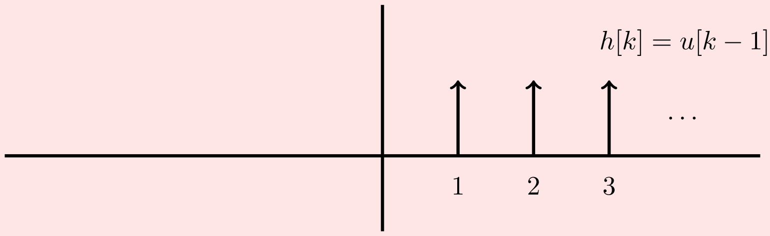
Solution:

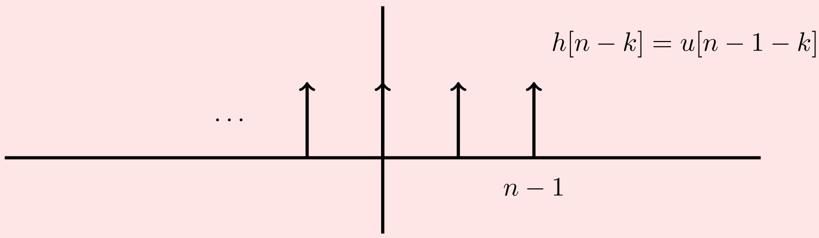












Here, we have two cases.
For $n - 1 \geq -1 \Rightarrow n \geq 0$:

$$y[n] = \sum_{k=-\infty}^{-1} x[k]h[n-k]$$
$$= \sum_{k=-\infty}^{-1} 5^k = \frac{1}{4}$$

For $n-1 < -1 \Rightarrow n < 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{n-1} 5^k = \frac{5^n}{4} \end{aligned}$$

Thus, overall we have

$$y[n] = \begin{cases} \frac{5^n}{4} & n < 0 \\ \frac{1}{4} & n \geq 0 \end{cases}$$

Discrete-time Impulse Response

Problem Set 2-7

Find the impulse response of the system, with following input and output relation where $x[n]$ denotes the input and $y[n]$ denotes the output,

$$y[n] + \frac{1}{3}y[n-2] = x[n].$$

Assume initial condition of rest, i.e., $x[n] = 0$ and $y[n] = 0$ for all $n < 0$.

Solution: For input $x[n] = \delta[n]$, the output is output $y[n] = h[n]$. Since $\delta[n] = 0$ for $n < 0$, then the initial rest condition implies that

$$h[n] = 0 \quad \text{for all } n < 0$$

which means it is causal. Next we look at $h[n]$ for $n \geq 0$.

Rearranging the system equation for the input $x[n] = \delta[n]$ and output $y[n] = h[n]$, we have

$$\begin{aligned} h[n] &= \delta[n] - \frac{1}{3}h[n-2] \\ &= \begin{cases} 0 & n < 0 & \text{(by causality, as established above)} \\ \delta[0] - \frac{1}{3}h[-2] = 1 & n = 0 & \text{(as } \delta[0] = 1 \text{ and } h[-2] = 0 \text{ by causality)} \\ -\frac{1}{3}h[n-2] & n > 0 & \text{(as } \delta[n] = 0 \text{ for all } n > 0\text{)} \end{cases} \end{aligned}$$

So the sequence $\{h[0], h[1], h[2], h[3], h[4], \ldots\}$ looks like $\{1, 0, -\frac{1}{3}, 0, \frac{1}{9}, \ldots\}$, which can be written

$$h[n] = \begin{cases} \left(-\frac{1}{3}\right)^{n/2} & n = 0, 2, 4, \ldots \\ 0 & \text{otherwise} \end{cases}.$$

□

Problem Set 2-8

Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2\,x[n-2].$$

(a) Find the response of this system to the unit pulse input $\delta[n]$ by solving the difference equation recursively or otherwise.

Solution: Let $h[n]$ denote the response of a system to the unit pulse input.
At time n, we have output $h[n] = x[n] + 2\,x[n-2] - 2\,h[n-1]$.
Since the system is initially at rest, $h[n] = 0$ for $n < 0$. The pulse response $h[n]$ at time n due to the input $x[n] = \delta[n]$ can be determined recursively as:

$$h[n] = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ -2 & n = 1, \\ (-2)^{n-2}\,6 & n \geq 2. \end{cases}$$

Just run the difference equation by hand and watch the output pattern emerge. □

(b) Find the response of this system to the input depicted in Fig. 1 by solving the difference equation recursively or otherwise.

Solution: The output $y[n]$ can be computed using the pulse response $h[n]$ computed in part (a). The input $x[n]$ can be written as

$$x[n] = \delta[n+2] + 2\,\delta[n+1] + 3\,\delta[n] + 2\,\delta[n-1] + 2\,\delta[n-2] + \delta[n-3].$$

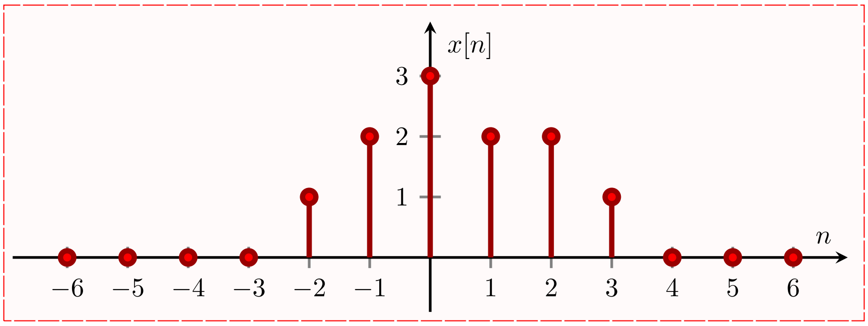


Figure 1: Input signal $x[n]$.

Using the convolution definition, we find output $y[n]$ as

$$\begin{aligned} y[n] &= x[n] \star h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= h[n+2] + 2\,h[n+1] + 3\,h[n] + 2\,h[n-1] + 2\,h[n-2] + h[n-3] \end{aligned} \tag{1}$$

Alternatively, the output can also be determined recursively using the difference equation that describes the system.

Using the result from part (a), the output $y[n]$ in (1) is given by

$$y[n] = \begin{cases} 0 & n \leq -3, \\ 1 & n = -2, \\ 0 & n = -1, \\ 5 & n = 0, \\ -4 & n = 1, \\ 16 & n = 2, \\ -27 & n = 3, \\ 58 & n = 4, \\ -(-2)^{n-5}\,114 = (-2)^{n-4}\,57 & n \geq 5. \end{cases}$$

Note that once the finite duration/support input $x[n]$ is flushed out the difference equation becomes equivalent to

$$y[n] = -2y[n-1]$$

which gives the long term $(-2)^n$ behaviour. □

Discrete-time System Properties

Problem Set 2-9

The equation

$$y[n] - a\,y[n-1] = x[n]$$

describes a DT system, with input $x[n]$ and output $y[n]$, assumed to be initially at rest, that is,

$$x[n] = 0 \text{ and } y[n] = 0 \quad \text{for all } n < 0.$$

(a) Show that the impulse response $h[n]$ for this system is

$$h[n] = a^n u[n],$$

where $y[n] = x[n] \star h[n]$.

Solution: We want to show that when the input is an impulse, $x[n]$ set to $\delta[n]$, then the response $y[n]$ is given by $h[n]$, that is, the following is satisfied

$$h[n] - a\,h[n-1] = \delta[n]$$

with $h[n] = a^n u[n]$. Substituting we have

$$\begin{aligned} a^n u[n] - a\,a^{n-1} u[n-1] &= a^n \big(u[n] - u[n-1]\big) \\ &= a^n\,\delta[n] && \text{since } \delta[n] = u[n] - u[n-1] \\ &= a^0\,\delta[n] = \delta[n] && \text{since } g[n]\delta[n] = g[0]\delta[n],\ \text{for any } g[n] \end{aligned}$$

which shows $h[n] = a^n u[n]$. □

(b) Is this system (provide reasoning for each of your answer)

i) linear?

Solution: Yes, it is evident by inspection since the system output is given by a linear combination of the present input and a past output. If you are still not convinced you can plug in, for example, $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ and see that the only possible solution will involve $y[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$. □

ii) time-invariant?

Solution: Yes, given it is linear, then this is also clear by inspection since the coefficients of $x[n-k]$ or $y[n-k]$ are not a function of time n, for any k. □

iii) memoryless?

Solution: No, output $y[n]$ depends on the old input $y[n-1]$, which only depends on old inputs $x[n-1], x[n-2], x[n-3], \ldots$. □

iv) causal?

Solution: Yes, output $y[n]$ depends on the current into $x[n]$ and old inputs $x[n-1], x[n-2], \ldots$, but not future inputs $x[n+1], x[n+2], \ldots$. Alternatively, $u[n] = 0$ for all $n < 0$ and so $h[n] = 0$ for all $n < 0$, which is necessary and sufficient condition for causality. □

v) stable?

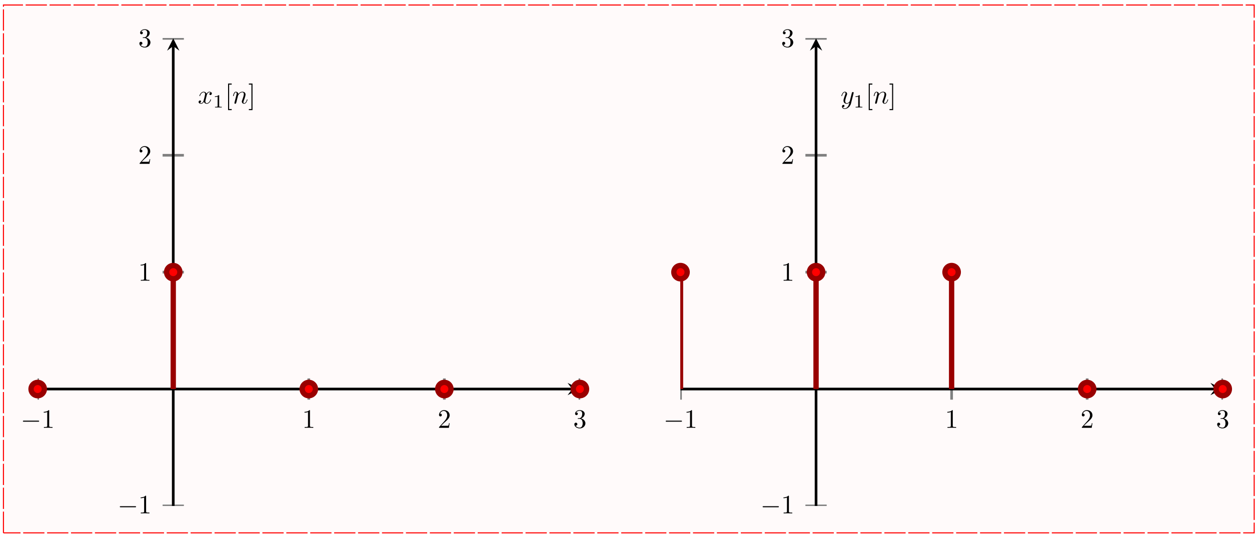
Solution: The answer depends on the value of a. If $|a| < 1$

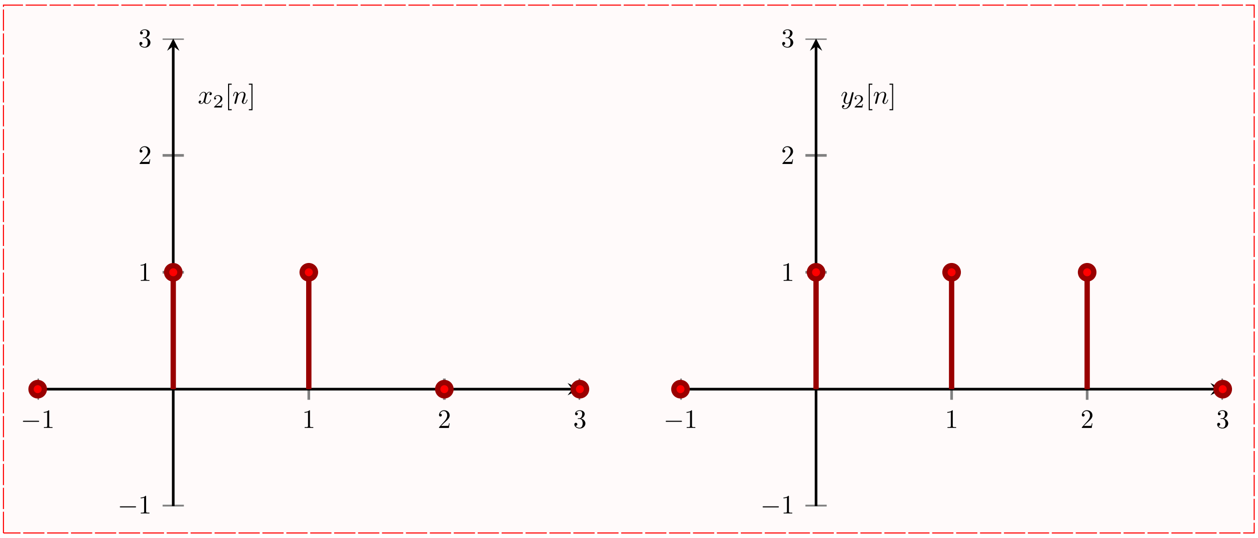
$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty$$

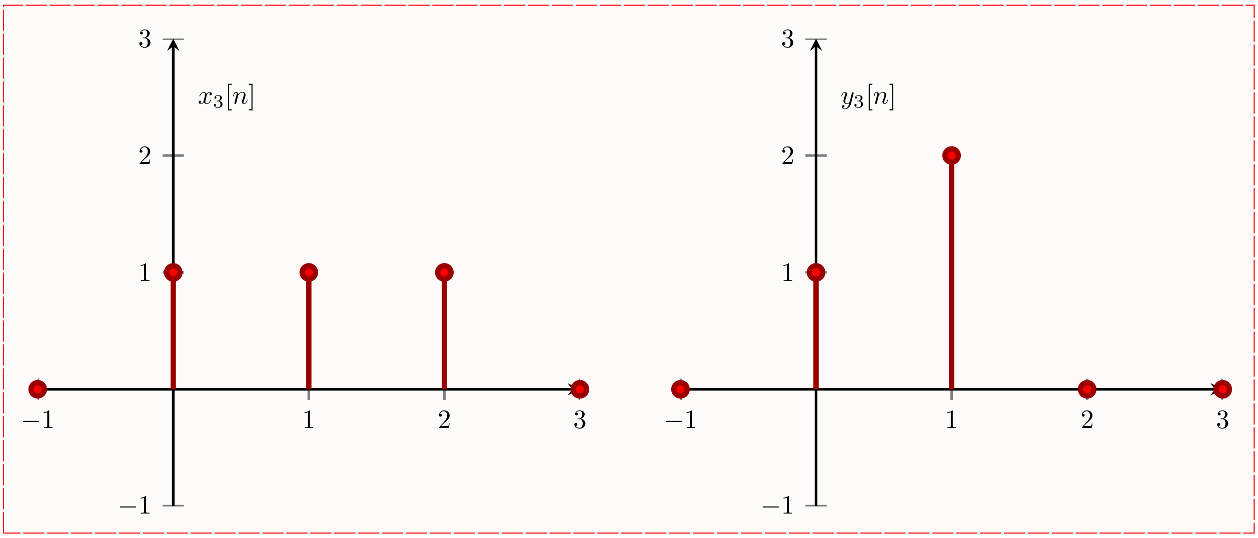
and it is stable. Otherwise, if $|a| \geq 1$, it is unstable. □

Problem Set 2-10

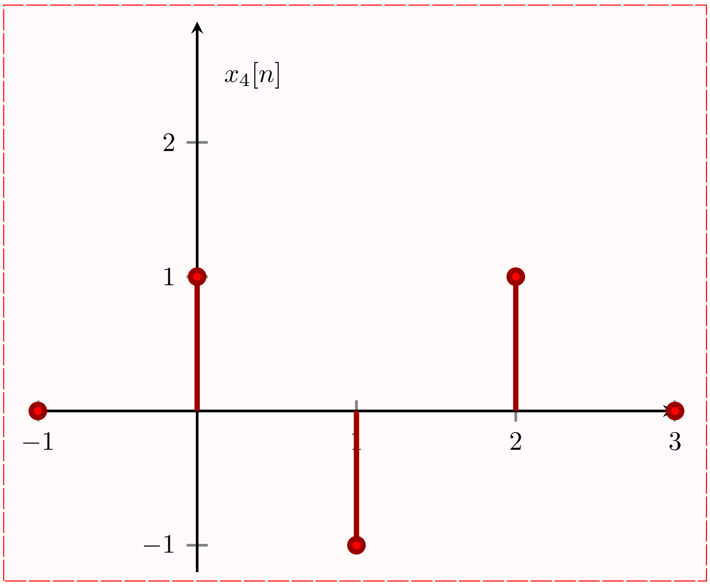
Suppose we have an unknown linear system, for which the superposition principle applies. Further, suppose we have knowledge of three outputs $y_1[n]$, $y_2[n]$, $y_3[n]$, generated by inputs $x_1[n]$, $x_2[n]$, $x_3[n]$, as shown below:



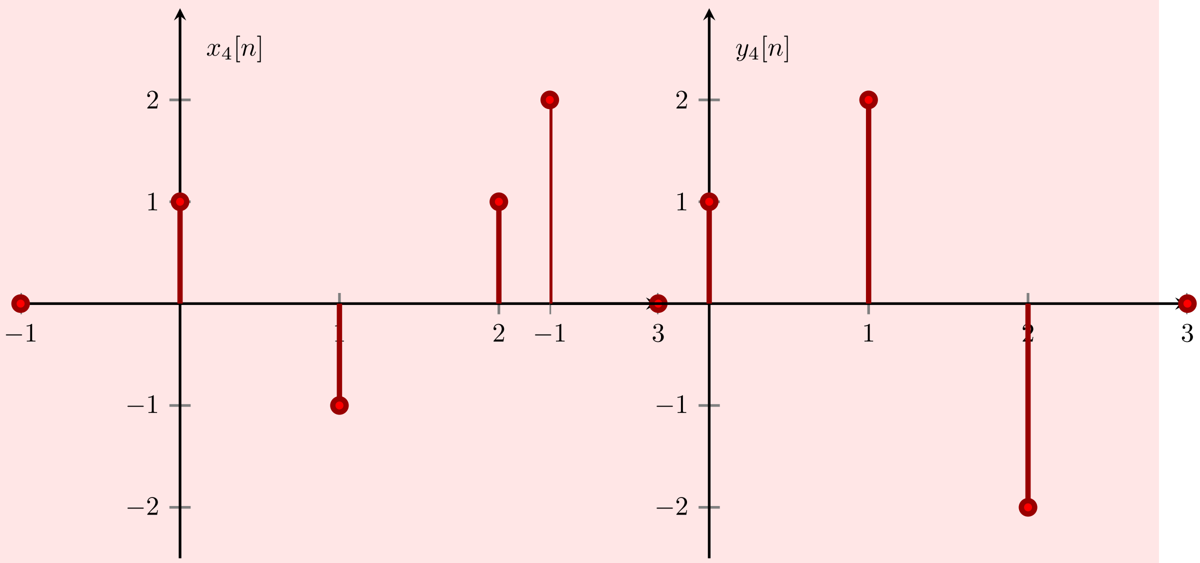




(a) Determine the response of the system $y_4[n]$ when the input is $x_4[n]$ as shown below.



Solution: A possible linear combination is $x_4[n] = 2x_1[n] - 2x_2[n] + x_3[n]$. Using superposition, the resulting output $y_4[n] = 2y_1[n] - 2y_2[n] + y_3[n]$ is shown below:



□

(b) Do we need the system to also be time-invariant?

Solution: The system is NOT time-invariant (that is, it is time-varying) because an input $x_1[n] + x_1[n-1]$ does not produce an output $y_1[n] + y_1[n-1]$. Note that while $x_1[n] + x_1[n-1] = x_2[n]$, $y_1[n] + y_1[n-1] \neq y_2[n]$. This proves the time-varying nature of the system. □

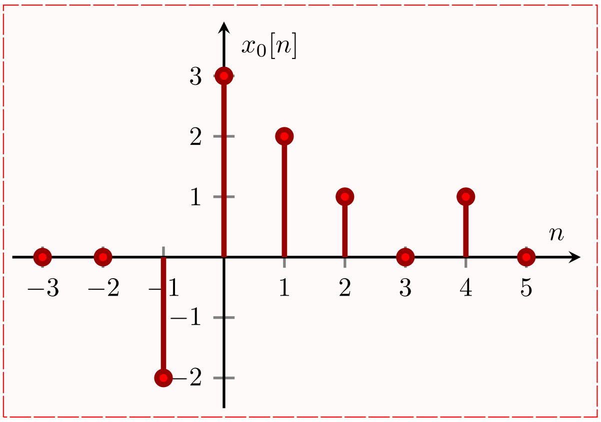
Problem Set 2-11

$$x[n] \longrightarrow \boxed{\text{DT System } S} \longrightarrow y[n] = x[3-n]$$

Consider the DT system, S, with input signal $x[n]$ and output signal given by

$$\boxed{S: \quad y[n] = x[3-n].} \tag{2}$$

(a) Write signal $x_0[n]$, shown below, in terms of linear combinations of shifted $\delta[n]$.



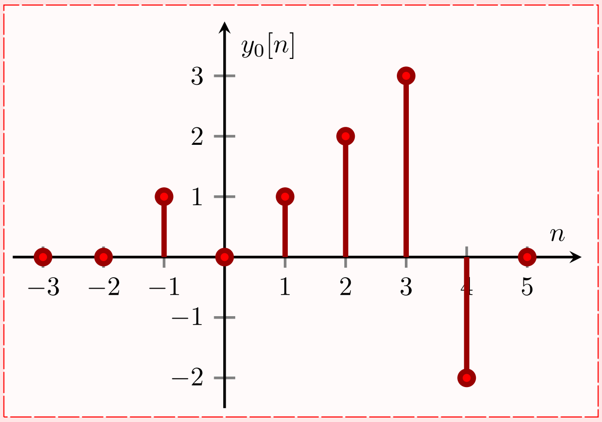
Solution:

$$x_0[n] = -2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2] + \delta[n-4]$$

□

(b) Draw the output $y_0[n]$ when the input is given by $x_0[n]$ shown above.

Solution:



□

(c) Write this signal $y_0[n]$ in terms of linear combinations of shifted $\delta[n]$.

Solution:

$$y_0[n] = \delta[n+1] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + -2\delta[n-4]$$

□

(d) Shown that the system is linear.

Solution: Suppose $x_1[n] \longrightarrow y_1[n]$ and $x_2[n] \longrightarrow y_2[n]$. Then, by (2), $y_1[n] = x_1[3-n]$ and $y_2[n] = x_2[3-n]$. Create a third input signal from the linear combination

$$x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n].$$

Then, by (2), $x_3[n] \longrightarrow y_3[n] = x_3[3-n]$. But

$$\begin{aligned} x_3[3-n] &= \alpha_1 x_1[3-n] + \alpha_2 x_2[3-n] \\ &= \alpha_1 y_1[n] + \alpha_2 y_2[n]. \end{aligned}$$

In summary

$$\boxed{\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n],}$$

and this holds for every α_1, $x_1[n]$, α_2, $x_2[n]$, meaning it is linear. □

(e) Shown that the system (2) is non-causal.

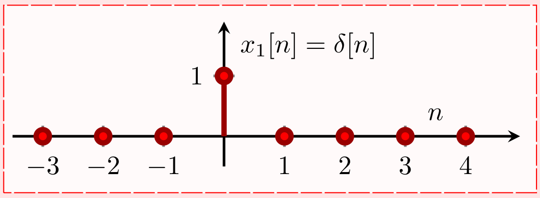
Solution: See the following solution. □

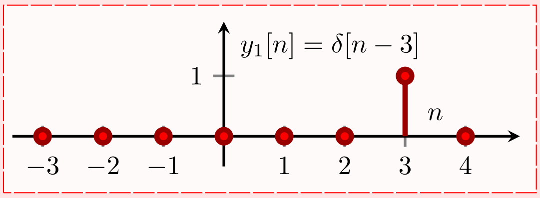
(f) Shown that the system (2) is time-varying.

Solution: We could memorise that a time reversal destroys causality and is time-varying. But it is better to provide a sound demonstration.

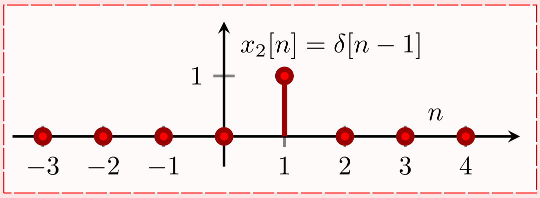
Consider the behavior of the system to three different input and corresponding output signals.

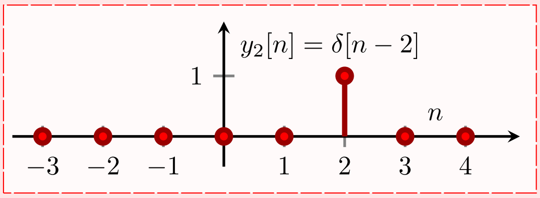
1) Input signal $x_1[n] = \delta[n]$ which yields output signal $y_1[n] = \delta[n-3]$.



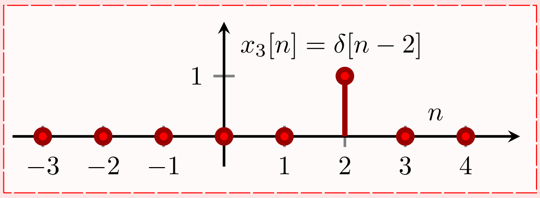


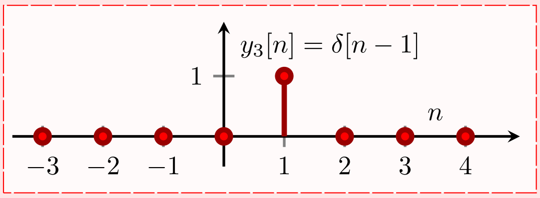
2) Input signal $x_2[n] = \delta[n-1]$ which yields output signal $y_2[n] = \delta[n-2]$.





3) Input signal $x_3[n] = \delta[n-2]$ which yields output signal $y_3[n] = \delta[n-1]$.





Non-causal: Firstly, the case $x_3[n] \longrightarrow y_3[n]$ shows it is non-causal because the output appears before the input.

Time-varying: Given $x_1[n] \longrightarrow y_1[n]$ then time shifting the input $x_1[n]$ to the right yields $x_2[n]$ (which also equals $x_1[n-1]$ or $\delta[n-1]$). However, time shifting the output $y_1[n]$ to the right, which is $y_1[n-1]$ or $\delta[n-4]$, does not equal $y_2[n]$. $\square$

(g) Suppose we have the same system but we don't know its defining relationship (2).

Let $h[n]$ be the output when $\delta[n]$ is applied. We observe $h[n] = \delta[n-3]$.

Can the system be fully characterised by this $h[n]$, that is, if we only know $h[n]$ can we determine the output for any input signal $x[n]$ for such an unknown system?

Solution: No. Firstly it is not LTI so we cannot rely only on the unit pulse response $h[n]$ to tell us everything. If it were a LTI system with unit pulse response $h[n] = \delta[n-3]$ then it must be a delay of 3 system. This is quite different from our system. $\square$