Signal Processing
ENGN2228

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Second Semester

Course Survey

Available at:

https://goo.gl/forms/zEGgsyxCuBZzsdUf2

Only takes 5 mins, your feedback is important.

Review of Module Two

At the end of the second module:

- Fourier series
  - For continous-time periodic signals
- Properties of Fourier series

Now we are going to look at:

- Discrete-time Fourier series

  - For DT periodic signals

Part 1 Outline

1. **DT Periodic Signals**
   - Definition
   - Fourier Series
   - Recap to Here
   - Inspection Method to Calculate DTFS Coefficients
   - Using Synthesis Equation
   - Using Analysis Equation to Calculate DTFS Coefficients
   - Properties DTFT

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DT Periodic Signals – Definition

Now we consider **DT periodic signals** which satisfy:

$$x[n+N] = x[n] \quad \text{and} \quad \boxed{N = \frac{2\pi}{\omega_0} m}$$

where $N_0 \in \mathbb{Z}$ is the fundamental period (smallest possible period N).

$m = 1, 2 \ldots$ such that N is an integer.

Periodicity DT and CT Complex Exponentials

Continuous-time exponentials $\boxed{e^{j\omega_0 t}:}$

- Periodic all ω_0
- $T = \frac{2\pi}{\omega_0}$
- Distinct functions for each ω_0
- Increasing ω_0 get a unique signal with faster oscillations

Discrete-time exponentials $\boxed{e^{j\omega_0 n}:}$

- Periodic only if ω_0 is a rational multiple of 2π
- $N = \frac{2\pi}{\omega_0} m$
- Functions not distinct for each ω_0
- Identical signals for values of ω_0 separated by multiples of 2π
- The rate of oscillation doesn't depend on $|\omega_0|$
- Fastest oscillation at $\omega_0 = \pi$, slowest $\omega_0 = 0$ and $\omega_0 = 2\pi$

$$x[n] = e^{j\pi/8} \leftarrow N = \frac{2\cancel{\pi}}{\cancel{\pi}/8} m = 16m$$

$$N_0 = 16 \ (m=1)$$

$$x[n] = e^{j17\pi/8} \leftarrow N = \frac{2\cancel{\pi}}{17\cancel{\pi}/8} m = \frac{16}{17} m$$

$$N_0 = 16 \ (m=17)$$

$$\frac{\pi}{8} + 2\pi = 17\pi/8$$

$$x[n] = e^{j2\pi n}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$= \left(e^{j2\pi}\right)^n = \left(\underbrace{\cancel{\cos 2\pi}}_{1} + j\sin 2\pi\right)^n$$

$$= (1)^n = 1$$

$$x[n] = e^{j\pi n} = \left(e^{j\pi}\right)^n = \left(\cos\pi + j\sin\pi\right)^n$$

$$= (-1 + 0)^n = (-1)^n$$

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$x(t)$ or $x[n]$

Four different Fourier transforms will meet in this course:

Time-domain properties	Periodic	Non-periodic	
Continuous	Fourier series (FS)	Fourier transform (FT)	**Non-periodic**
Discrete	Discrete-time Fourier series (DTFS)	Discrete-time Fourier transform (DTFT)	**Periodic**
	Discrete	**Continuous**	**Frequency-domain Properties**

a_k

Definition (DT Fourier Series Pair)

For $x[n] = x[n+N]$ periodic with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad \textit{(Synthesis Equation)}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad \textit{(Analysis Equation)}$$

- Note that the a_k are N-periodic, $a_k = a_{k+N}$, and so only N consecutive a_k need be known/computed.
- Similarly, only N consecutive $x[n]$ are needed.

DT Periodic Signals – Recap to Here (cont'd)

Definition (DT Fourier Series Pair)

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

Definition (CT Fourier Analysis and Synthesis)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

DT Periodic Signals – Proof of Analysis Equation

To figure out the a_k, we have the (completeness) identity

$$\sum_{n=0}^{N-1} e^{jk\omega_0 n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

RHS analysis equation

$$= N\,\delta[n]$$

synthesis equation

So

$\times N$

$$\sum_{n=0}^{N-1} x[n]\, e^{-jm\omega_0 n} = \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n}$$

$$= \sum_{k=0}^{N-1} a_k \underbrace{\left(\sum_{n=0}^{N-1} e^{j(k-m)\omega_0 n} \right)}_{N\,\delta[k-m]}$$

$$= N\, a_m$$

LHS analysis equation $\times N$

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Inspection Method to Calculate DTFS Coefficients

- If $x[n]$ is composed of real or complex sinusoids, then it is better to determine a_k using the inspection method (rather than using the DTFS Analysis Equation).

- This involves expanding the sinusoids in terms of complex exponentials using Euler's formula and comparing each term with the terms in the DTFS Synthesis Equation.

$\omega_0 = \frac{2\pi}{5}$, $N = \frac{2\pi}{2\pi/5} m = 5m$, $N_0 = 5$ $(m=1)$

Example 1:

$$x[n] = \sin \frac{2\pi}{5} n$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$x[n] = \frac{e^{j\frac{2\pi}{5}n} - e^{-j2\pi/5n}}{2j} = \frac{1}{2j} e^{j2\pi/5n} - \frac{1}{2j} e^{-j2\pi/5n} \quad (1)$$

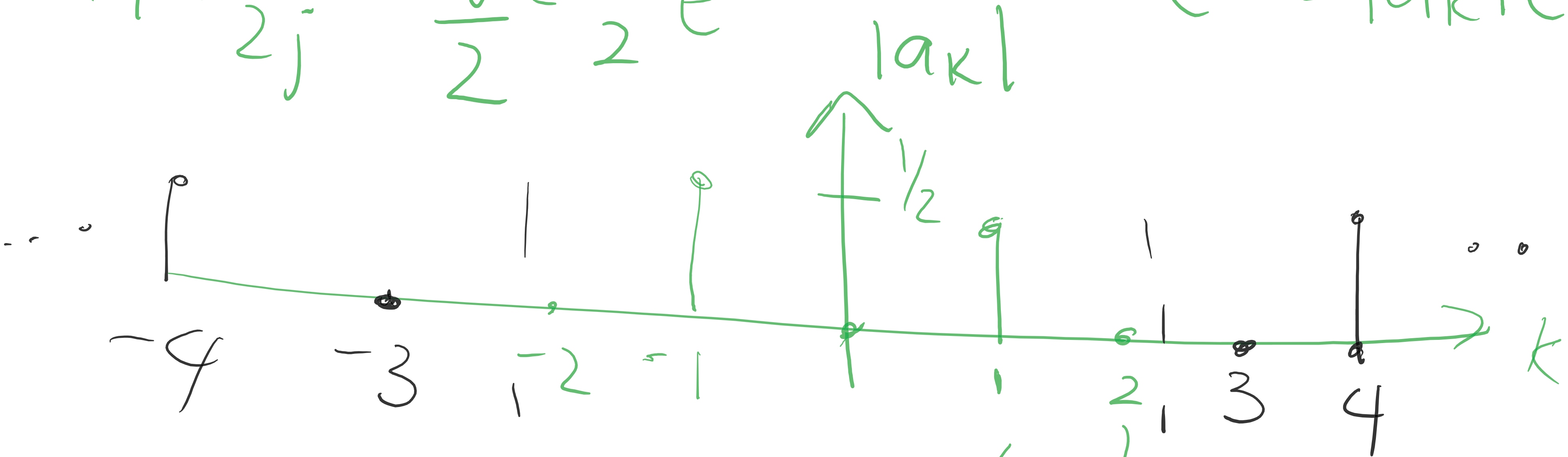
$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=-2}^{2} a_k e^{jk\frac{2\pi}{5}n}$$

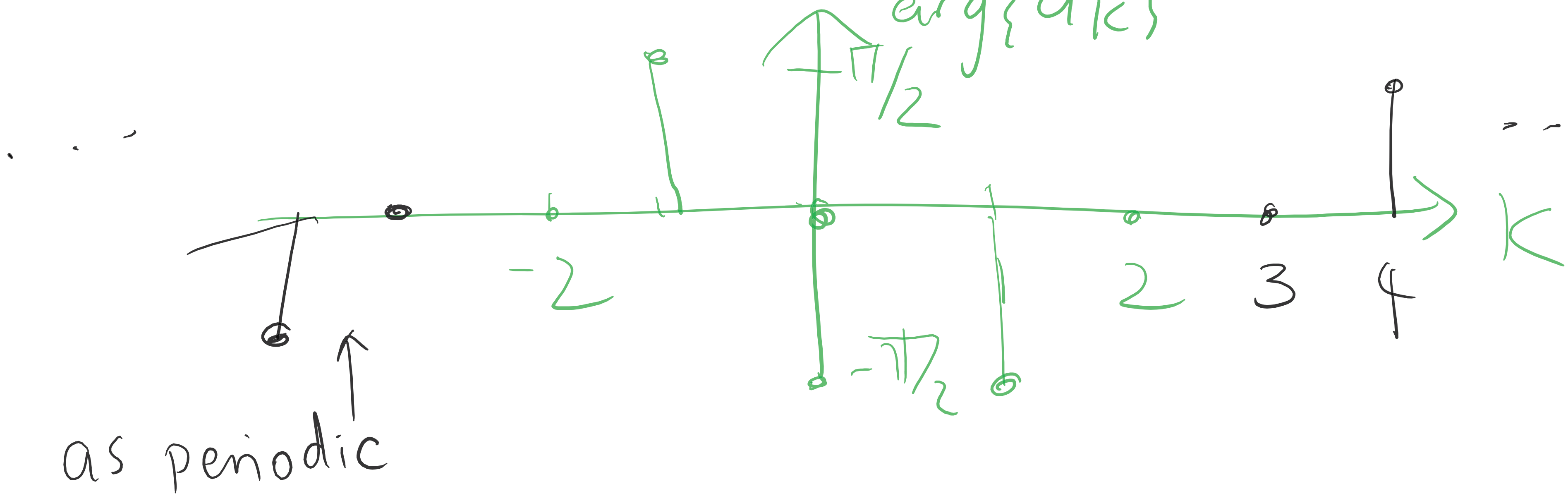
$$= a_{-2} e^{-j\frac{4\pi}{5}n} + a_{-1} e^{-j2\pi/5n} + a_0 + a_1 e^{j2\pi/5n} + a_2 e^{j4\pi/5n} \quad (2)$$

$a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$, $a_0 = a_2 = a_{-2} = 0$

$$a_1 = \frac{1}{2j} = -\frac{j}{2} = \frac{1}{2}e^{-j\pi/2} \qquad \left(\frac{1}{j} = -j\right)$$

$$a_{-1} = -\frac{1}{2j} = \frac{j}{2} = \frac{1}{2}e^{j\pi/2} \qquad re^{j\theta} = |a_k|e^{j\arg\{a_k\}}$$





Inspection Method to Calculate DTFS Coefficients

$N_0 = 16,\quad N = \frac{2\cancel{\pi}}{\cancel{\pi}/8} m = 16m \qquad N = \frac{2\cancel{\pi}}{\cancel{\pi}/4} m = 8m$

Example 3: $\omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{16} = \frac{\pi}{8}$

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$x[n] = \frac{e^{j\pi n/8} + e^{-j\pi n/8}}{2} + \frac{e^{j(\pi n/4 + \pi/4)} + e^{-j(\pi n/4 + \pi/4)}}{2}$$

$$= \frac{1}{2} e^{j\pi n/8} + \frac{1}{2} e^{-j\pi n/8} + \frac{1}{2} e^{j\pi/4} e^{j\pi n/4} + \frac{1}{2} e^{-j\pi/4} e^{-j\pi n/4} \quad (1)$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=-7}^{8} a_k e^{jk\pi/8\, n} \quad (2)$$

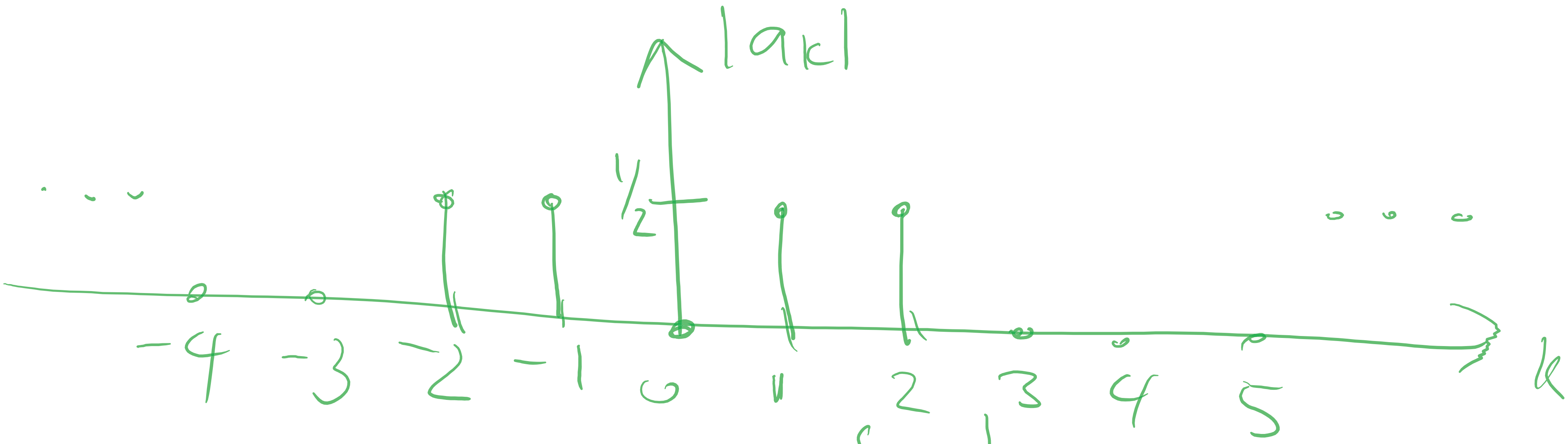
$a_1 = 1/2$

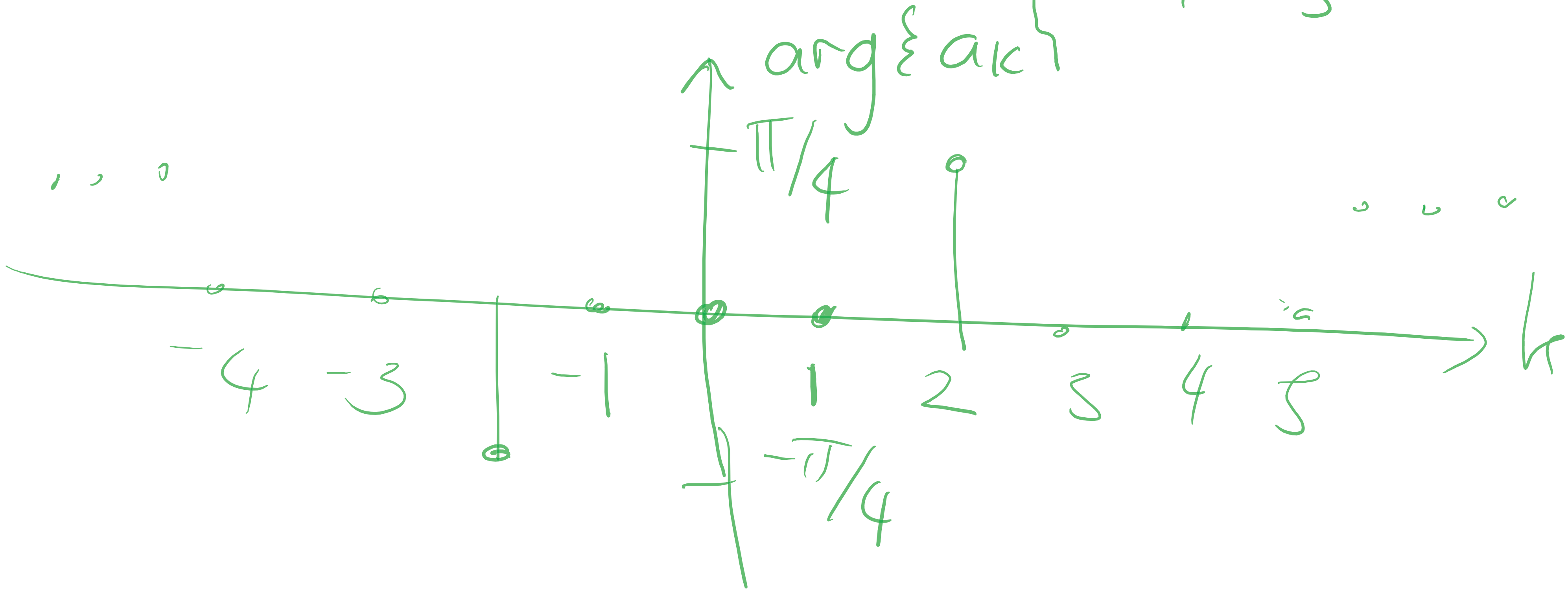
$a_{-1} = 1/2$

$a_2 = \frac{1}{2} e^{j\pi/4}$

$a_{-2} = \frac{1}{2} e^{-j\pi/4}$

$a_1 = \frac{1}{2} \quad a_{-1} = 1/2 \quad a_2 = \frac{1}{2} e^{j\pi/4} \quad a_{-2} = \frac{1}{2} e^{-j\pi/4}$





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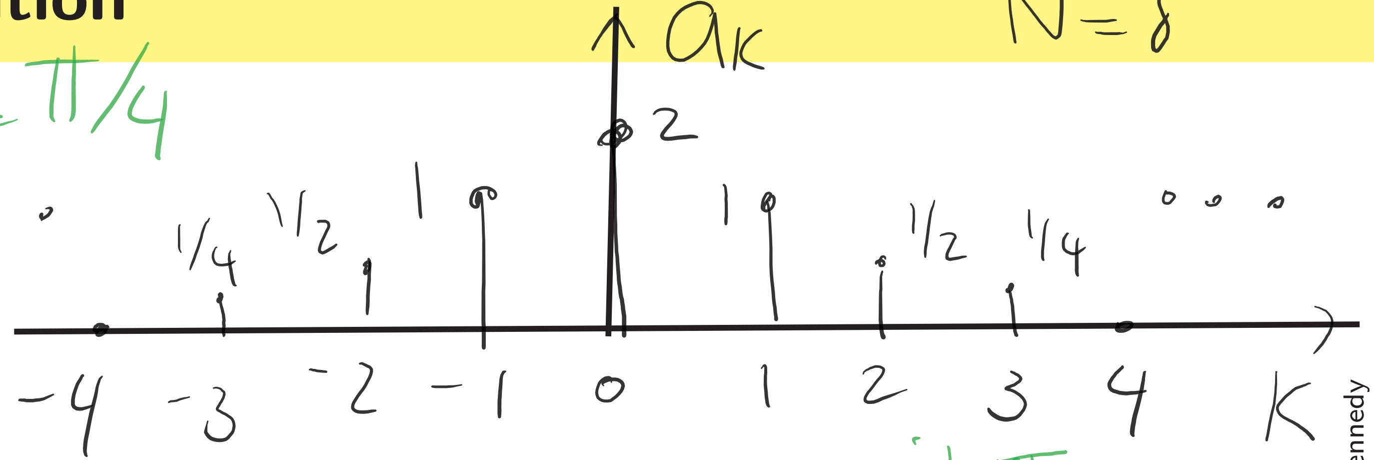
Using Synthesis Equation

$N=8$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \pi/4$$



- Given a_k find $x[n]$
- Example 1:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=-3}^{4} a_k e^{j\frac{k\pi n}{4}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \frac{1}{4} e^{-j\frac{3\pi n}{4}} + \frac{1}{2} e^{-j\frac{2\pi n}{4}} + e^{-j\frac{\pi n}{4}} + 2 + e^{j\frac{\pi n}{4}} + \frac{1}{2} e^{j\frac{2\pi n}{4}} + \frac{1}{4} e^{j3\pi n/4} + 0$$

$$= 2 + \frac{1}{4}\left(e^{j\frac{3\pi n}{4}} + e^{-j\frac{3\pi n}{4}}\right) + \frac{1}{2}\left(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right) + e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}} = 2 + \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + 2\cos\left(\frac{\pi n}{4}\right)$$

Using Synthesis Equation

$N=8$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \pi/4$$

- Given a_k find $x[n]$
- Example 2:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=-3}^{4} a_k e^{jk\frac{\pi n}{4}}$$

$$= e^{-j\frac{3\pi n}{4}} + 0 + e^{-j\frac{\pi n}{4}} + 1 + e^{j\frac{\pi n}{4}} + 0 + e^{j\frac{3\pi n}{4}} + 0 = e^{j\frac{3\pi n}{4}} + e^{-j\frac{3\pi n}{4}} + e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}} + 1 = 2\cos\left(\frac{3\pi n}{4}\right) + 2\cos\left(\frac{\pi n}{4}\right) + 1$$

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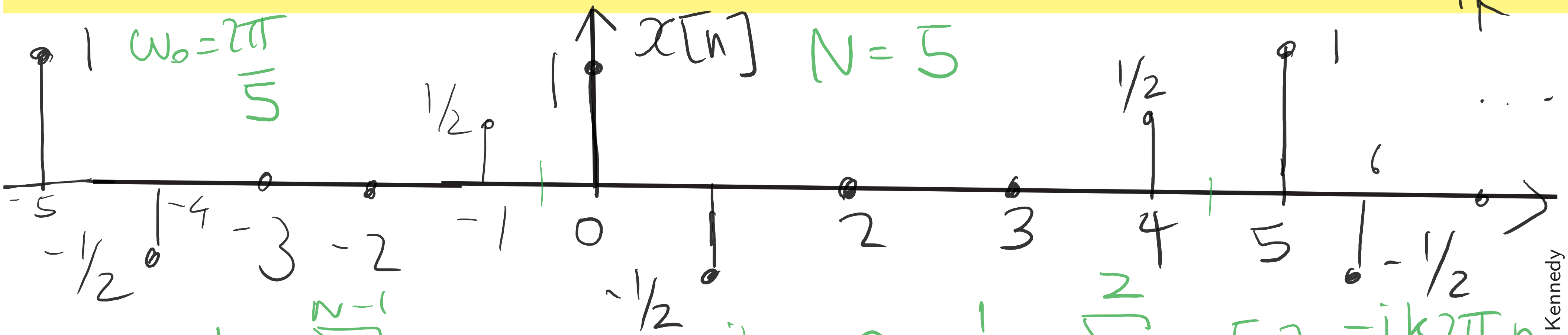
Using Analysis Equation to Calculate DTFS Coefficients

Analysis Equation:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z}$$

Using Analysis Equation to Calculate DTFS Coefficients a_k



$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} = \frac{1}{5}\sum_{n=-2}^{2} x[n]\, e^{-jk\frac{2\pi}{5}n}$$

$$= \frac{1}{5}\left[0 + \frac{1}{2}e^{j\frac{2\pi}{5}k} + 1 - \frac{1}{2}e^{-j\frac{2\pi}{5}k} + 0\right]$$

$$= \frac{1}{5}\left(1 + \frac{j}{2}\left(e^{j\frac{2\pi k}{5}} - e^{-j\frac{2\pi k}{5}}\right)\right)$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{5}\left(1 + j\sin\left(\frac{2\pi}{5}k\right)\right)$$

Using Analysis Equation to Calculate DTFS Coefficients

Instead consider $n = 0 \ldots 4$:

$$a_k = \frac{1}{5}\sum_{n=0}^{4} x[n] e^{-jk2\pi/5\, n}$$

$$= \frac{1}{5}\left(1 - \frac{1}{2}e^{-j2\pi/5\, k} + 0 + 0 + \frac{1}{2}e^{-j\frac{8\pi}{5}k}\right)$$

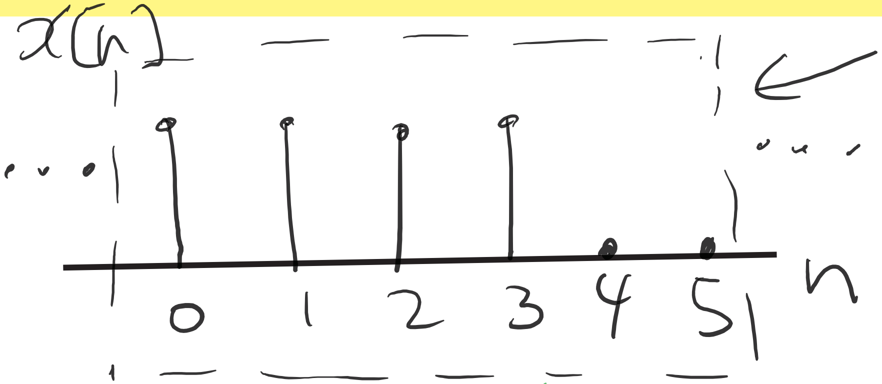
$$= \frac{1}{5}\left(1 + \frac{1}{2}\left(e^{-j8\pi/5\, k} - e^{-j2\pi/5\, k}\right)\right)$$

$$e^{-j8\pi/5\, k} = e^{-jk2\pi} e^{jk2\pi/5} = e^{jk\frac{2\pi}{5}}$$

$$a_k = \frac{1}{5}\left(1 + \frac{1}{2}\left(e^{j\frac{2\pi}{5}k} - e^{-j\frac{2\pi}{5}k}\right)\right)$$

$$= \frac{1}{5}\left(1 + j\sin\left(\frac{2\pi}{5}k\right)\right)$$

Using Analysis Equation to Calculate DTFS Coefficients



periodic part

$N=6$ $\quad \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{6} = \pi/3$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n} = \frac{1}{6}\sum_{n=0}^{5} x[n]e^{-jk\pi/3\, n}$$

$$= \frac{1}{6}\sum_{n=0}^{3} e^{-j\frac{k\pi n}{3}}$$

$$= \frac{1}{6}\sum_{n=0}^{3}\left(e^{-jk\frac{\pi}{3}}\right)^n$$

$$\left(\sum_{n=0}^{N-1}\alpha^n = \frac{1-\alpha^N}{1-\alpha}\right)$$

$$= \frac{1}{6}\left(\frac{1-\left(e^{-jk\frac{\pi}{3}}\right)^4}{1-e^{-jk\pi/3}}\right)$$

$$a_k = \frac{1}{6}\left(\frac{1-e^{-j\frac{k4\pi}{3}}}{1-e^{-j\frac{k\pi}{3}}}\right)$$

$$= \frac{1}{6}\,\frac{e^{-j\frac{k2\pi}{3}}\left(e^{j\frac{k2\pi}{3}} - e^{-j\frac{k2\pi}{3}}\right)}{e^{-j\frac{k\pi}{6}}\left(e^{j\frac{k\pi}{6}} - e^{-j\frac{k\pi}{6}}\right)}\;\frac{2j}{2j}$$

$$= \frac{1}{6}\,\frac{e^{-j\frac{k2\pi}{3}}\,\sin\left(\frac{k2\pi}{3}\right)}{e^{-j\frac{k\pi}{6}}\,\sin\left(\frac{k\pi}{6}\right)}$$

$$-j\frac{k2\pi}{3} + j\frac{k\pi}{6}$$

$$= -j\frac{k3\pi}{6}$$

$$= -j\frac{k\pi}{2}$$

$$= \frac{1}{6}\,e^{-j\frac{k\pi}{2}}\,\frac{\sin\left(k\frac{2\pi}{3}\right)}{\sin\left(\frac{k\pi}{6}\right)}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients

DT Square Wave:

Let $x[n]$ be periodic with period N. With N_1 such that $2N_1 + 1 \leq N$, define $x[n]$ over N-length interval

$$[-N_1, N - N_1 - 1]$$

as follows

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & N_1 + 1 \leq n < N - N_1 - 1 \end{cases}$$

For other n we can just use

$$x[n] = x[n + N]$$

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients (cont'd)

Example 3.12 of text book page 218. Make sure can get this answer:

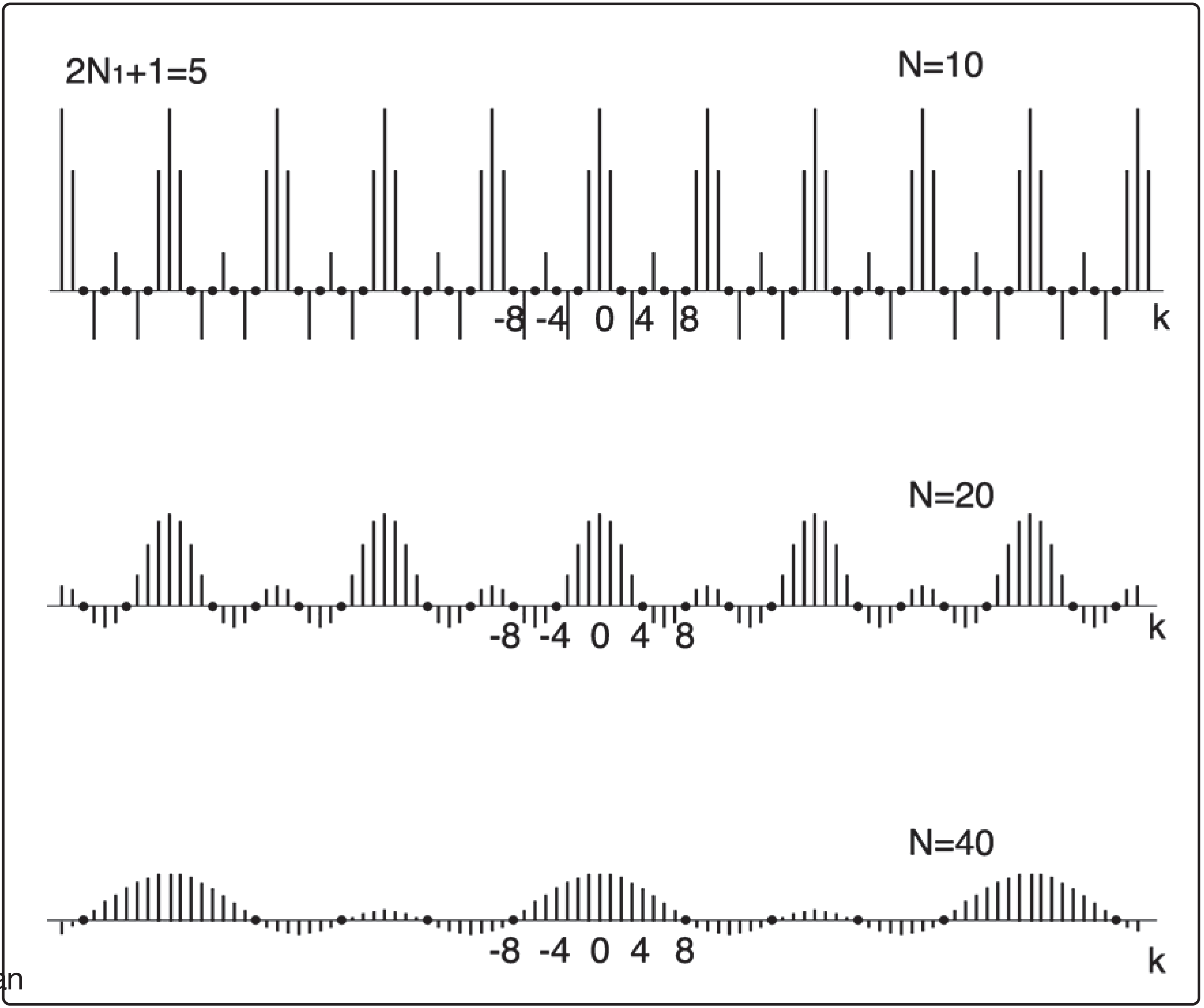
The (N-periodic) Fourier coefficients are

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \times \begin{cases} \dfrac{\sin\left(2\pi k(N_1 + 1/2)/N\right)}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ 2N_1 + 1, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

- This is like a "sampled" periodic sinc function.

DT Periodic Signals – Using Analysis Equation to Calculate DTFS Coefficients (cont'd)



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DT Periodic Signals – Properties DTFT

Given in exam:

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & \text{[x[n] real]} \\ x_o[n] = \mathcal{O}d\{x[n]\} & \text{[x[n] real]} \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

different to CT property