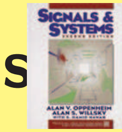
Part 2 Outline

Fourier Series and LTI Systems – Eigenfunctions Revisited



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Representations of inputs and outputs LTI system:

- Previous focus was unit samples $\delta[n]$ and impulses $\delta(t)$ - convolution
- Alternative focus is eigenfunctions of LTI systems

Eigenfunction definition:

A signal for which the system output is a (possibly complex) constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is refereed to as the systems *eigenvalue*.

$$Ax = \lambda x$$

matrix

eigenvector

eigenvalue

$$\mathcal{L} f = \lambda f$$

opperator

eigenfunction

Fourier Series and LTI Systems – Eigenfunctions Revisited

- $e^{j\omega t}$, e^{st} are eigenfunctions of CT LTI systems
- $e^{j\omega n}$, z^n are eigenfunctions of DT LTI systems

- Study of CT LTI systems using $e^{j\omega t}$:
  - Fourier series (FS) - periodic signals
  - Fourier transform (FT) - general signals

- Study of CT LTI systems using e^{st}:
  - Laplace transform (outside the scope of this course)

- Study of DT LTI systems using $e^{j\omega n}$:
  - Discrete-time Fourier series (DTFS) - periodic signals
  - Discrete-time Fourier transform (DTFT) - general signals

- Study of DT LTI systems using z^n:
  - z-transform (to be covered in ENGN4537 - DT Signal Processing)

Fourier Series and LTI Systems – Eigenfunctions Revisited

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\text{CT LTI} \quad h(t)} \longrightarrow y(t) = ?$$

$$\begin{aligned}
y(t) = x(t) * h(t) &= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\
&= \boxed{e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}
\end{aligned}$$

Therefore

$$y(t) = e^{j\omega t} \boxed{H(j\omega)} \quad \text{— independent of } t$$

where $\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}$.

Generalisation: If the input to an LTI system is expressed as a linear combination of periodic complex exponentials, the output can also be expressed in this form.

Fourier Series and LTI Systems – Eigenfunctions Revisited

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\text{DT LTI} \quad h[n]} \longrightarrow y[n] = ?$$

$$\begin{aligned} y[n] &= x[n] * h[n] = h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \end{aligned}$$

Therefore

$$y[n] = e^{j\omega n} H(e^{j\omega})$$

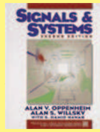
where $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

independent of n

Part 2 Outline

Part 2 Outline

Frequency Response of LTI System – Continuous Time



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$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

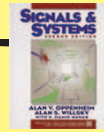
Definition (CT Frequency Response)

The **CT Frequency Response** is defined by

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = H(j\omega)$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

Part 2 Outline



Frequency Response of LTI System – Discrete T[…]e

$$e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega})\, e^{j\omega n}$$

Definition (DT Frequency Response)

The **DT Frequency Response** is defined by

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = H(e^{j\omega})$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

Part 2 Outline

Continuous Time

$$\left\langle \begin{matrix}\textit{periodic}\\ \textit{signal}\end{matrix} \right\rangle x(t) \longrightarrow \boxed{h(t)} \longrightarrow \left\langle \begin{matrix}\textit{periodic}\\ \textit{signal}\end{matrix} \right\rangle y(t)$$

Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k\, \overbrace{H(jk\omega_0)}^{\text{eigenvalue}}}_{b_k}\, e^{jk\omega_0 t}$$

The Fourier coefficients map like

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$a_k \to b_k \triangleq H(jk\omega_0)\, a_k$$

Here the complex gain (eigenvalue) $H(jk\omega_0) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$, that is, $e^{jk\omega_0 t}$ (eigenfunction).

Discrete Time

$$\left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle x[n] \longrightarrow \boxed{h[n]} \longrightarrow \left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle y[n]$$

DT

Fourier Series representation

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y(t) = \sum_{k=0}^{N-1} \underbrace{a_k\, H(e^{jk\omega_0})}_{b_k}\, e^{jk\omega_0 n}$$

The Fourier coefficients map like

$$a_k \to b_k \triangleq H(e^{jk\omega_0})\, a_k$$

Here complex gain $H(e^{jk\omega_0}) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$.

Part 2 Outline

Remember:

- $h(t)$ impulse response $\delta(t) \to h(t)$
- $H(j\omega)$ frequency response LTI system, by definition
  $\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt}$
- Example: $h(t) = e^{-t}u(t)$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-t} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-t(j\omega+1)} dt = \frac{-1}{(j\omega+1)} \left[e^{-t(j\omega+1)} \right]_0^{\infty}$$

$$= \frac{-1}{j\omega+1} \left[e^{-\infty} - e^{0} \right] = \frac{-1}{j\omega+1} [0 - 1]$$

$$= \frac{1}{j\omega+1}$$

Remember:

- $h[n]$ impulse response $\delta[n] \to h[n]$
- $H(e^{j\omega})$ frequency response LTI system, by definition
  $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$
- Example: $h[n] = \alpha^n u[n], \quad -1 < \alpha < 1$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

α in geometric series

Frequency Response of LTI System – Finding $y(t)$

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$, what is $y(t)$?

- Could find $y(t) = x(t) * h(t)$
- Use Fourier series to solve for $y(t)$ instead
- Output Fourier coefficients given by $b_k = H(jk\omega_0)a_k$
- First need $H(j\omega)$:

$\updownarrow$ b_k

$\omega = k\omega_0$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \frac{1}{RC}e^{-t/RC}u(t)e^{-j\omega t}\,dt$$

$$= \frac{1}{RC}\int_0^{\infty} e^{-t(j\omega + 1/RC)}\,dt = \frac{1}{RC} \times \frac{-1}{j\omega + \frac{1}{RC}}\left[e^{-t(j\omega+\frac{1}{RC})}\right]_0^{\infty}$$

$$= \frac{1}{\cancel{RC}} \times \frac{-\cancel{RC}}{j\omega RC + 1}\left(e^{-\infty} - e^{0}\right) = \frac{-1}{j\omega RC + 1}(0 - 1)$$

$$= \frac{1}{j\omega RC + 1}$$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$, what is $y(t)$?

- $x(t) = \cos(2\pi 1000 t)$ (1kHz) $\leftarrow$ $\omega_0 = 2\pi 1000$
- $R = \frac{1000}{2\pi}$
- C $= 10\ \mu F$
- Output Fourier coefficients given by $b_k = H(jk\omega_0) a_k$

Need to find $H(jk\omega_0)$: $H(j\omega)$, $\omega = k\omega_0$

$H(jk\omega_0) = \frac{1}{jk\omega_0 RC + 1} = \frac{1}{jk\, \cancel{2\pi} 1000 \times \frac{1000}{\cancel{2\pi}} \times 10 \times 10^{-6} + 1}$

$= \frac{1}{jk10 + 1}$

Need to find FS coefficients $x(t)$, a_k:

$x(t) = \frac{1}{2} e^{j2\pi 1000 t} + \frac{1}{2} e^{-j2\pi 1000 t}$ $a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$

Frequency Response of LTI System –Finding $y(t)$ Cont.

RC circuit with input $x(t)$. We know $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$, what is $y(t)$?

- Output Fourier coefficients given by $b_k = H(jk\omega_0)a_k$
- Finally $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$

$$b_1 = H(j\omega_0)a_1 = \frac{1}{j10+1} \times \frac{1}{2} = 0.04975e^{-j1.4711}$$

$$b_{-1} = H(-j\omega_0)a_{-1} = \frac{1}{-j10+1} \times \frac{1}{2} = 0.04975e^{j1.4711}$$

$$y(t) = \sum_{k=-1}^{1} b_k e^{jk\omega_0 t} = 0.04975e^{j1.4711}e^{-j\omega_0 t}$$

$$+ 0 + 0.04975e^{-j1.4711}e^{j\omega_0 t}$$

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Frequency Response of LTI System –Finding $y(t)$ Cont.

$$= 0.04975\left[e^{-j(\omega_0 t - 1.4711)} + e^{j(\omega_0 t - 1.4711)}\right]$$

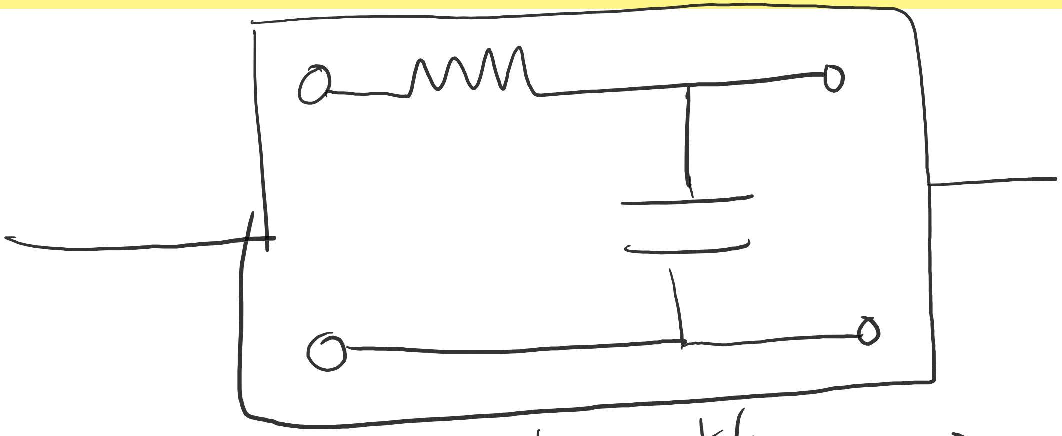
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= 2 \times 0.04975 \cos(\omega_0 t - 1.4711)$$

$$= 0.0995 \cos(2\pi 1000 t - 1.4711)$$

CT Example Summary

$x(t)$



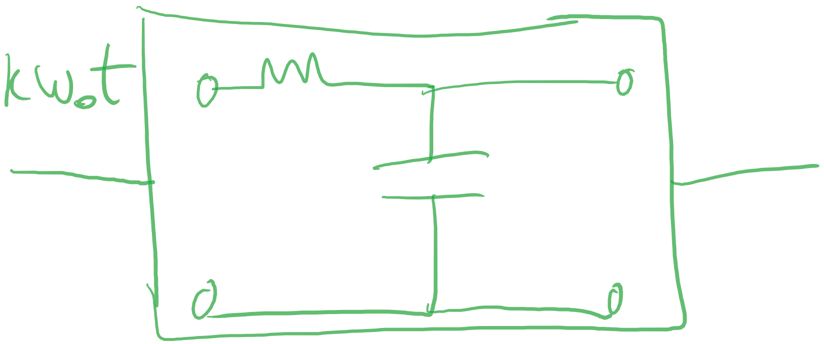
$y(t) = x(t) * h(t)$

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$x(t) \Leftrightarrow a_k$ $\qquad$ $y(t) \Leftrightarrow b_k$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$



$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k H(jk\omega_0)}_{b_k} e^{jk\omega_0 t}$$

Difference equation: $y[n] - \frac{1}{4}y[n-1] = x[n]$ — Causal

Sol. 1: Solve difference equation to find $h[n]$, then find $H(e^{j\omega})$

as system is causal

$y[n] = \frac{1}{4}y[n-1] + x[n]$

$h[n] = \frac{1}{4}h[n-1] + \delta[n]$, $h[n] = 0$, $n < 0$

$h[0] = \frac{1}{4}h[-1] + \delta[0] = 0 + 1 = 1$

$h[1] = \frac{1}{4}h[0] + \delta[1] = \frac{1}{4} + 0 = \frac{1}{4}$

$h[2] = \frac{1}{4}h[1] + \delta[2] = \left(\frac{1}{4}\right)^2 + 0$

$h[n] = \left(\frac{1}{4}\right)^n u[n]$

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n]e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{-jw}}{4}\right)^n = \frac{1}{1-\frac{1}{4}e^{-jw}}$$

Frequency Response of LTI System –Given $h[n]$ find $H(e^{j\omega})$

Let $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

Difference equation: $y[n] - \frac{1}{4}y[n-1] = x[n]$

Sol. 2: Avoids recursion

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$$H(e^{j\omega})\cancel{e^{j\omega n}} - \tfrac{1}{4}H(e^{j\omega})\cancel{e^{j\omega n}}e^{-j\omega} = \cancel{e^{j\omega n}}$$

$$H(e^{j\omega})\left(1 - \tfrac{1}{4}e^{-j\omega}\right) = 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \tfrac{1}{4}e^{-j\omega}}$$