Signal Processing

ENGN2228

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Second Semester

Part 0 Slide 1/many Convenor: R. A. Kennedy

DT Fourier Transforms – Recap

Periodic Signals:

$$\text{CT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Non-Periodic Fourier Series}$$

$$\text{DT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Periodic Fourier Series}$$

Finite Discrete Signals:

$$\text{Finite DT signals} \xleftrightarrow{\mathscr{F}} \text{Finite Fourier Series – FFT}$$

is very close to (mathematically equivalent to)

$$\text{DT Periodic signals} \xleftrightarrow{\mathscr{F}} \text{Periodic Fourier Series}$$

CT Non-Periodic Signals:

$$\text{CT (Non-Periodic) Signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{``Continuous'' Fourier Transform}$$

CT Periodic Signals (revisited):

$$\text{CT Periodic signals} \xleftrightarrow{\mathscr{F}} \text{“Impulse Sequence” Fourier Transform}$$

- Sometimes called a discrete spectrum.
- Technically the “Impulse Sequence” needs to be uniformly spaced, with delta functions lying at multiples of some ω_0.

Discrete Time Fourier Transform: `O&W 5.1.1 pp.359-362`

Here we have DT but, generally, non-periodic signals. We can have a continuum of frequencies and to synthesis the time domain signal we need to integrate over that continuum of frequencies. In the frequency domain we have 2π periodicity.

Definition (DT Fourier Analysis and Synthesis)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \qquad (Synthesis)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

DT Non-Periodic Signals:

$$\text{DT Non-Periodic signals} \xleftrightarrow{\mathscr{F}} \text{Continuous Periodic Fourier Transform}$$

Again, DT implies periodicity of the Fourier Transform and given the signal is not periodic (aperiodic) then the spectrum is continuous.

DT Fourier Transforms – Discrete Time FT

Four cases:

$$\text{continuous in time} \xleftrightarrow{\mathscr{F}} \text{continuous in frequency}$$

$$\text{continuous in time} \xleftrightarrow{\mathscr{F}} \text{discrete in frequency}$$

$$\text{discrete in time} \xleftrightarrow{\mathscr{F}} \text{continuous in frequency}$$

$$\text{discrete in time} \xleftrightarrow{\mathscr{F}} \text{discrete in frequency}$$

Technically "discrete in time" means discrete and uniformly spaced in time, and similarly "discrete in frequency" means discrete and uniformly spaced in frequency.

Time vs. Frequency domain:

Time-domain Properties	Periodic	Non-periodic	
Continuous	Fourier Series (FS)	Fourier Transform (FT)	Non-periodic
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)	Periodic
	Discrete	Continuous	Frequency-domain Properties

DTFT pairs example:

$$\boxed{\delta[n] \longleftrightarrow 1}$$

DT Fourier Transforms – Examples

Example 1: `O&W 5.1.3 p.367`

Unit sample signal

$$\boxed{x[n] = \delta[n]}$$

is not periodic and has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n}$$

That is,

$$\boxed{X(e^{j\omega}) = 1, \quad \text{for all } \omega}$$

- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = 0$, for all ω.

DTFT pairs example:

$$\boxed{\delta[n - n_0] \longleftrightarrow e^{-j\omega n_0}}$$

DT Fourier Transforms – Examples

Example 2: `O&W 5.4.1 p.383`

Shifted unit sample signal, where delay $n_0 \in \mathbb{Z}$,

$$\boxed{x[n] = \delta[n - n_0]}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\, e^{-j\omega n}$$

That is,

$$\boxed{X(e^{j\omega}) = e^{-j\omega n_0}, \quad n_0 \in \mathbb{Z}}$$

- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = -\omega\, n_0$, for all ω.
- Has linear phase (phase proportional to ω), and the slope (derivative wrt ω) gives the time shift.
- Here, slope is $-n_0$ implying a delay of n_0.

DTFT pairs example:

$$\boxed{a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}}$$

Example 3: `O&W 5.1.2 pp.362-363`

Causal, exponentially decaying function

$$\boxed{x[n] = a^n\, u[n], \quad |a| < 1}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n\, u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n$$

$$= \frac{1}{1 - a\, e^{-j\omega}}, \quad \text{if } |a| < 1$$

We can write this $X(e^{j\omega})$ as

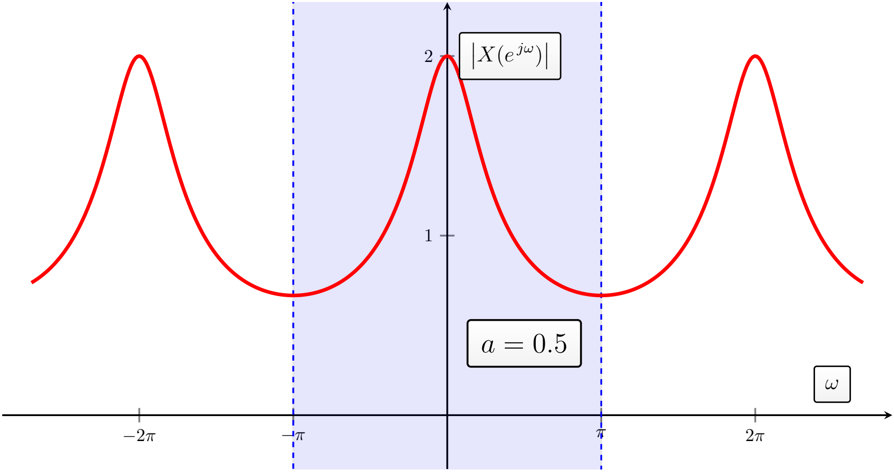
$$\boxed{X(e^{j\omega}) = \frac{1}{(1 - a\,\cos\omega) + j a \sin\omega}, \quad |a| < 1}$$

- This is complex (real plus imaginary) and a can be complex also.
- For real $a = 0.5$, $0.4, \ldots, -0.5$ we plot

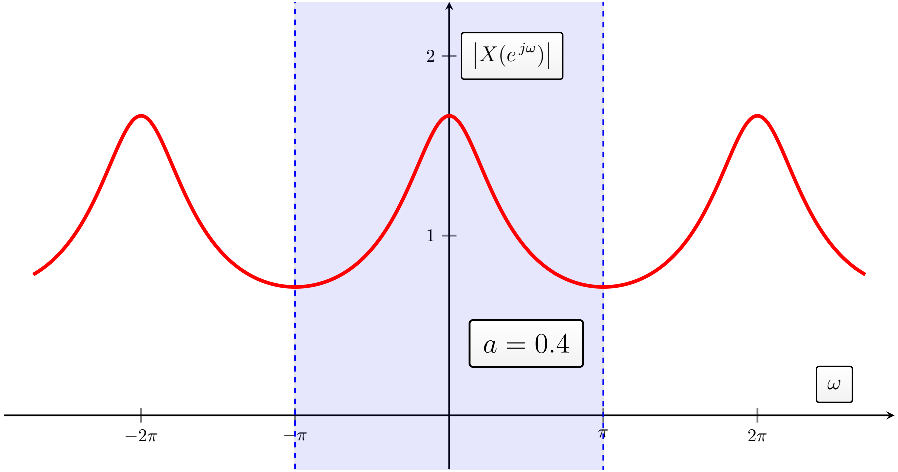
$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos(\omega) + a^2}}$$

In this way we can infer whether is acts like a low-pass filter, high-pass filter or otherwise.

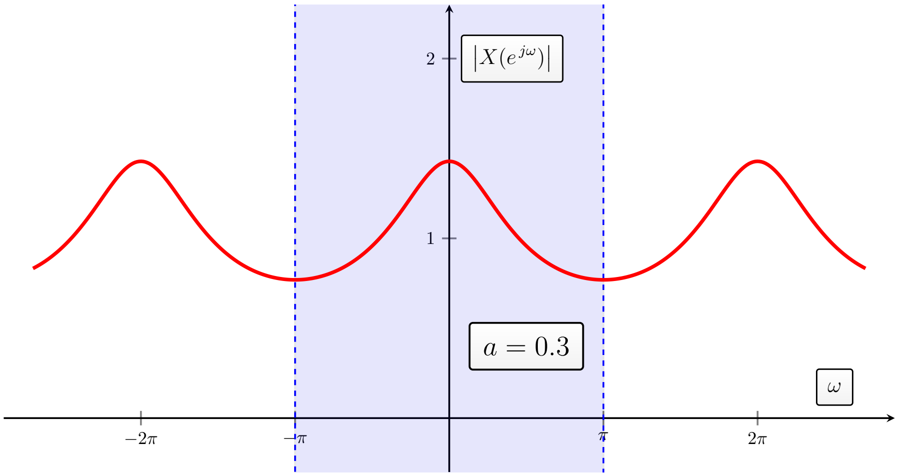
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



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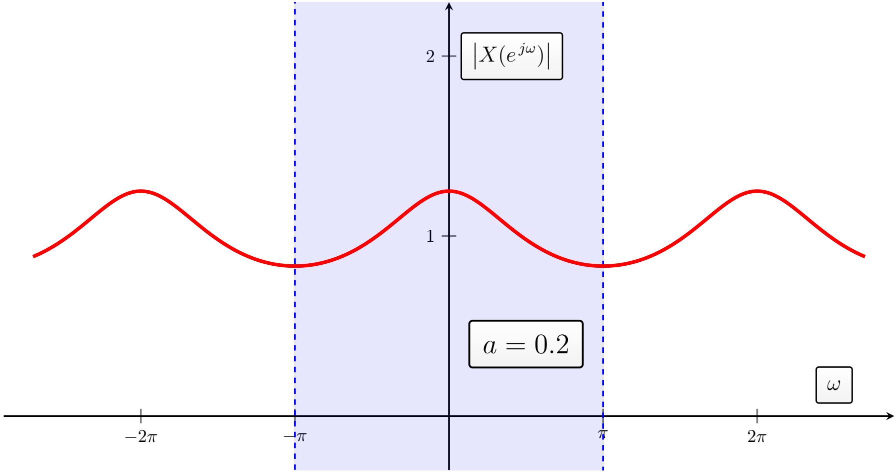

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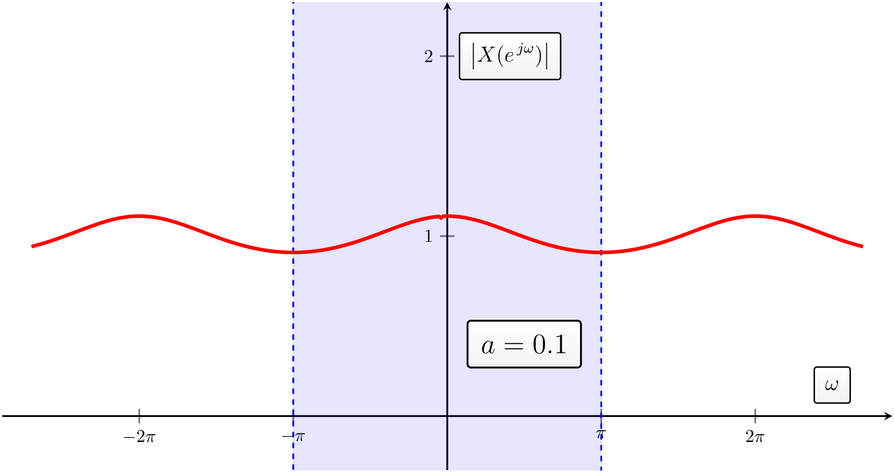

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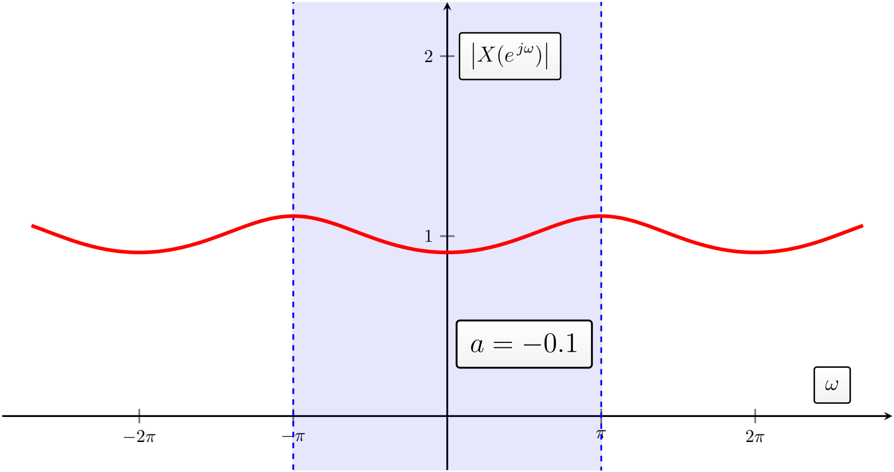

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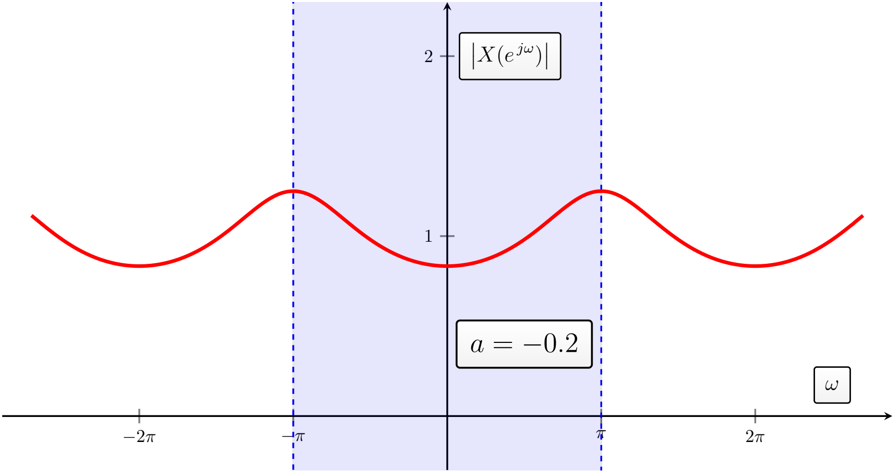

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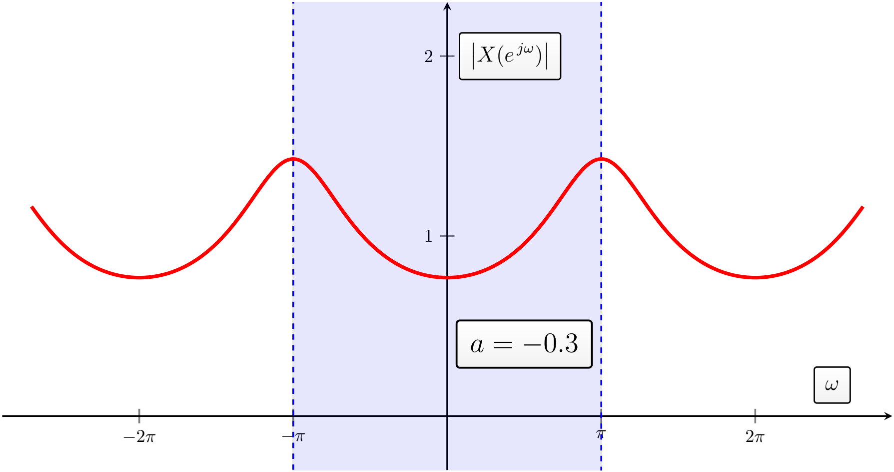

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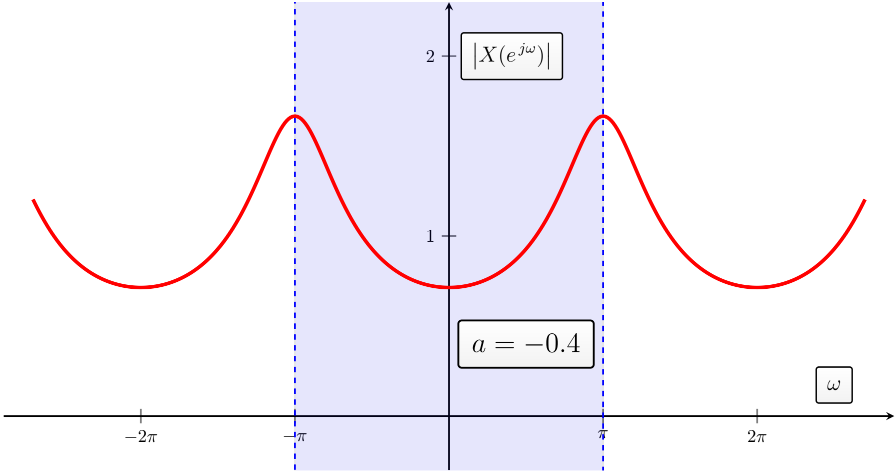
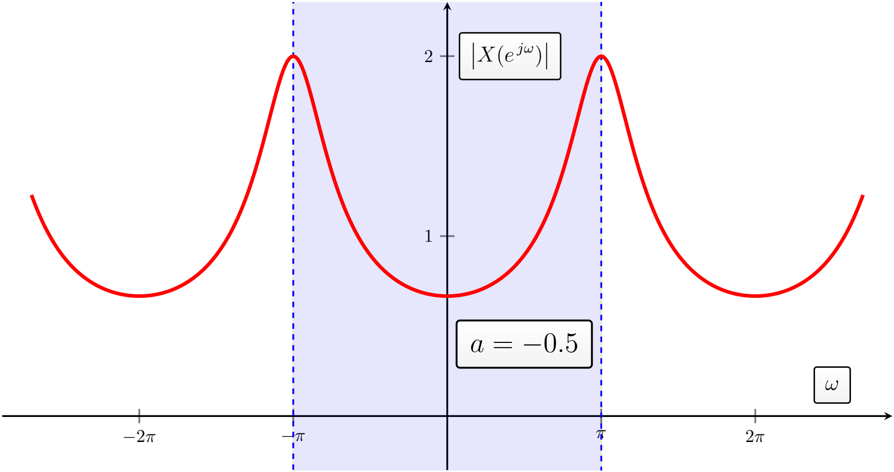


DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Notes:

- With $a \to 0$ ("$a = 0$") we have $a^0 = 1$ and $a^n = 0$ for $n \neq 0$; then

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

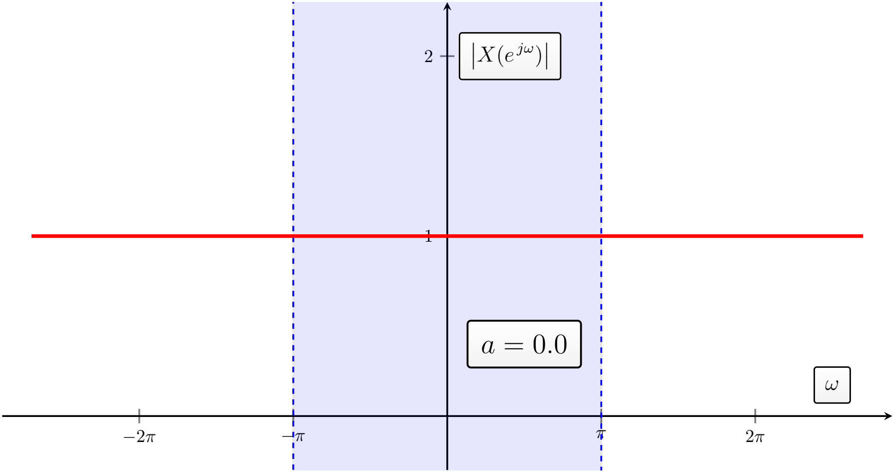
that is,

$$x[n] = \delta[n]$$

and

$$X(e^{j\omega}) = 1.$$

DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Example 4: `O&W 5.1.2 pp.365-366`

DT Rectangular Pulse function

$$\boxed{x[n] = \chi_{[-N_1,N_1]}[n], \quad n,\, N_1 \in \mathbb{Z}}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \chi_{[-N_1,N_1]}[n]\, e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}$$

That is,

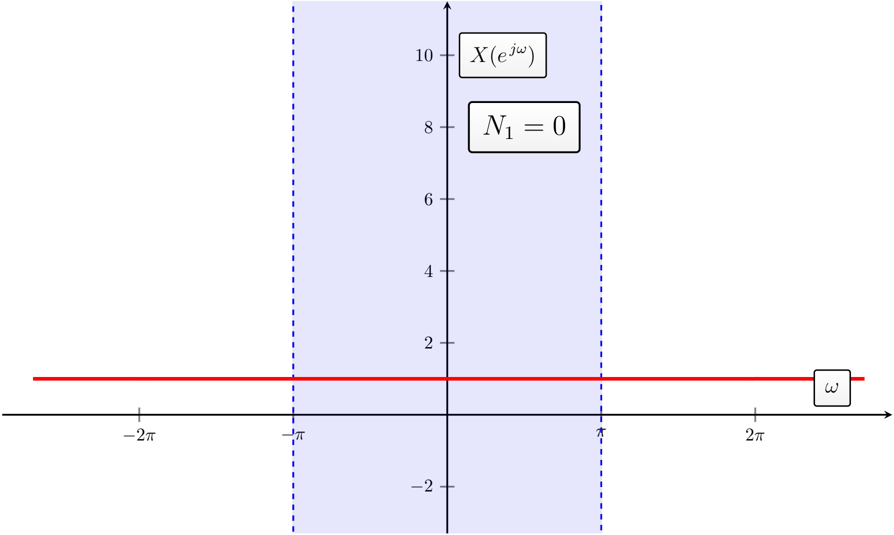
$$\boxed{X(e^{j\omega}) = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}, \quad N_1 \in \mathbb{Z}}$$

- This is purely real and we plot $X(e^{j\omega})$ rather than $\left|X(e^{j\omega})\right|$.

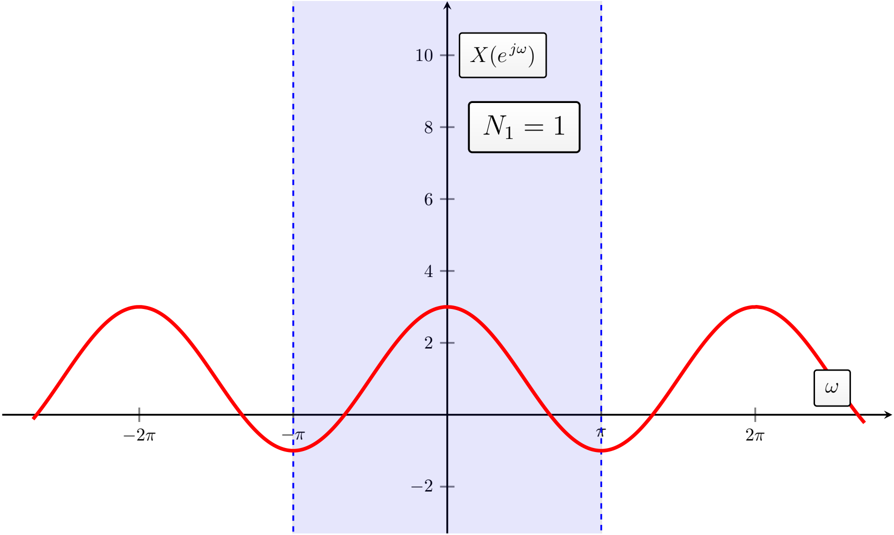
DTFT pairs example:

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \longleftrightarrow \frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



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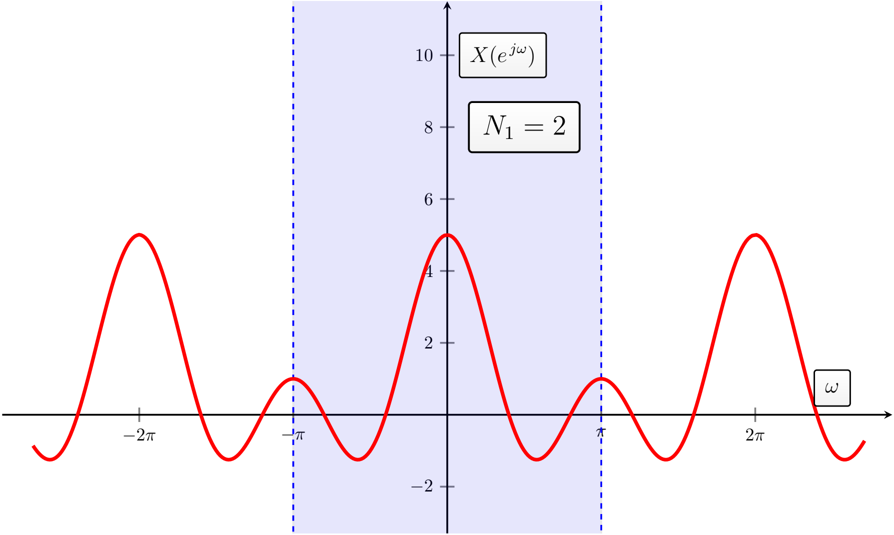

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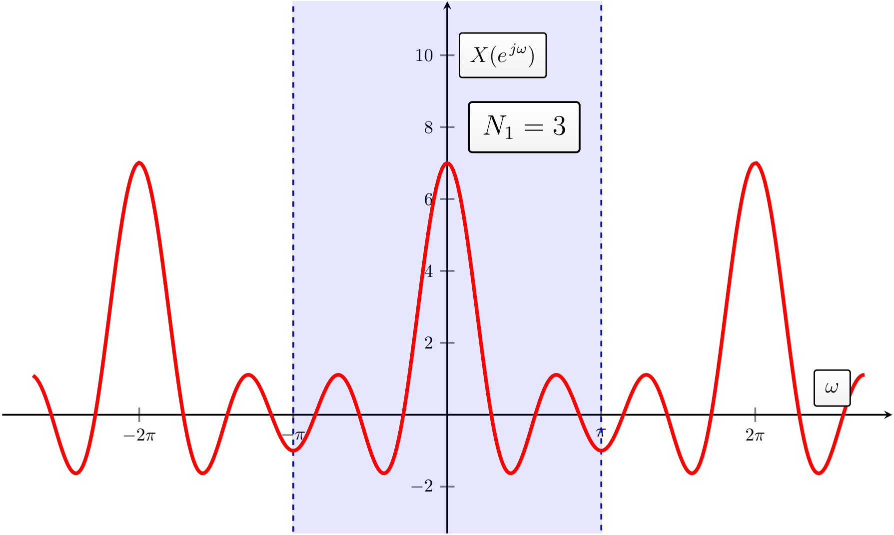

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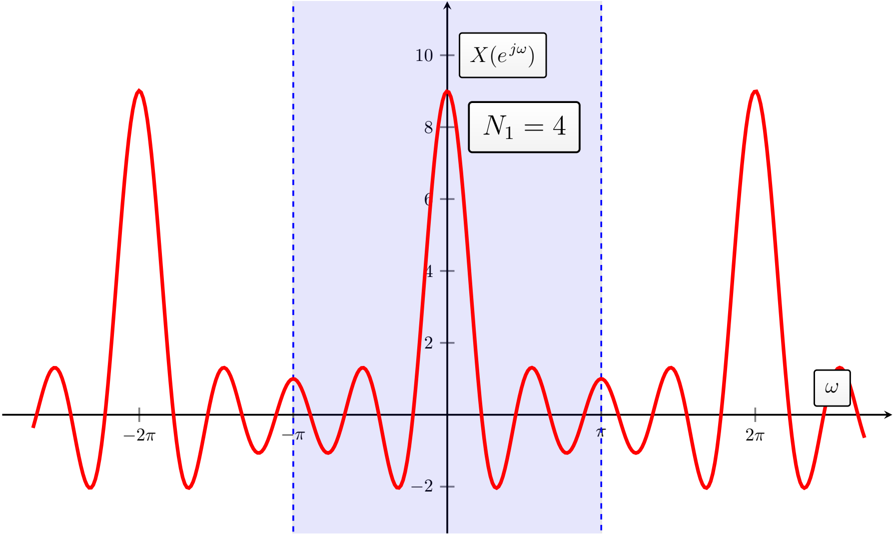

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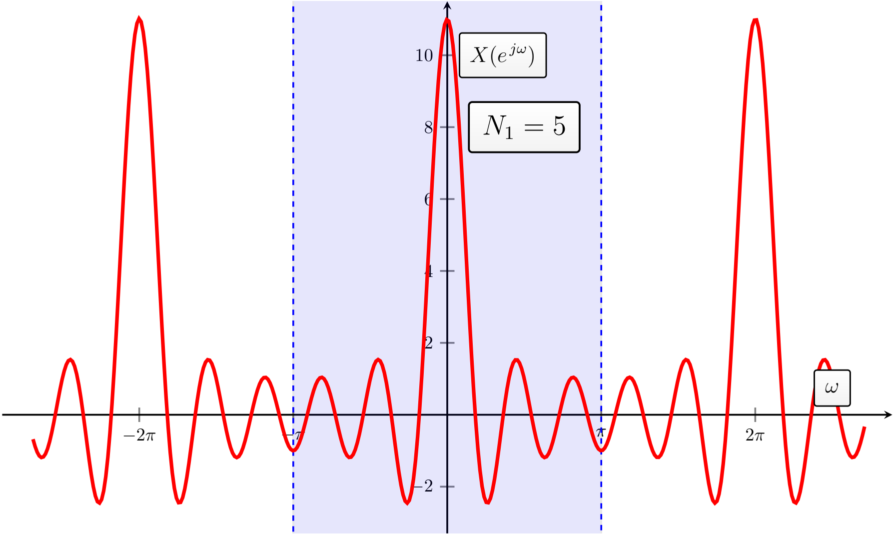


Notes:

- From these figures we can see that in the frequency domain the Fourier Transform of a time-domain discrete rectangular pulse looks (and is) the convolution of a sinc function with a periodic train of frequency domain delta functions. That is, the superposition of a sinc function with shifted versions of itself.

DT Fourier Transforms – Examples

Example 5: `O&W 5.4.1 pp.383-384`

DT Ideal Low-Pass Filter, bandwidth $0 \leq W \leq \pi$, passes frequencies $-W \leq \omega \leq W$,

$$\boxed{X(e^{j\omega}) = \chi_{[-W,+W]}(\omega), \quad -\pi \leq \omega \leq \pi, \quad 0 \leq W \leq \pi}$$

and is periodic with period 2π. It has time domain sampled sinc response

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[-W,+W]}(\omega)\, e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega n}\, d\omega \\ &= \frac{\sin Wn}{\pi n}, \quad n \in \mathbb{Z} \end{aligned}$$

DT Fourier Transforms – Examples

Definition (DT Ideal Low-Pass Filter)

The DT LTI system that passes only frequencies with gain 1 in the range $[-W, +W] \in [-\pi, +\pi]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin Wn}{\pi n} \xleftrightarrow{\mathscr{F}} \chi_{[-W,+W]}(\omega), \quad -\pi \le \omega \le \pi, \quad 0 \le W \le \pi}$$

and the frequency response is periodic in ω with period 2π; or

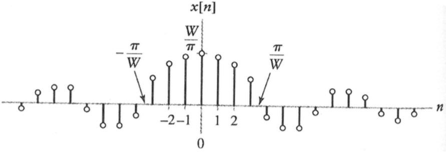
$$\boxed{\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right) \xleftrightarrow{\mathscr{F}} \chi_{[-W,+W]}(\omega), \quad -\pi \le \omega \le \pi, \quad 0 \le W \le \pi}$$

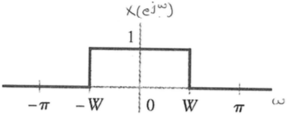
and the frequency response is periodic in ω with period 2π.

- We plot $X(e^{j\omega})$ and $x[n]$ for a range of $0 < W < \pi$.

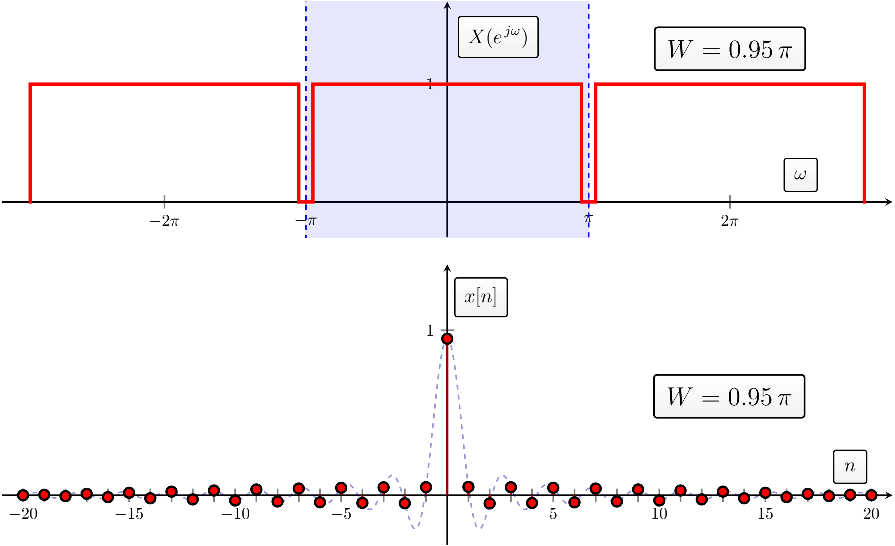
DTFT pairs example:

$$\boxed{\frac{\sin(Wn)}{\pi n} \longleftrightarrow X_{[-W,W]}(\omega) = \begin{cases} 1 & 0 \leq |\omega| \leq W \\ 0 & W \leq |\omega| \leq \pi \end{cases}}$$

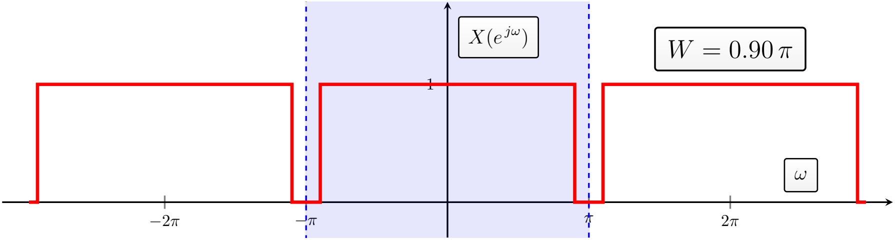


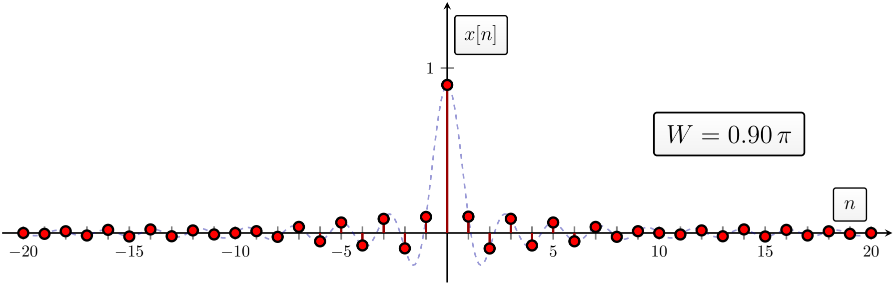


DT Fourier Transforms – Examples



DT Fourier Transforms – Examples





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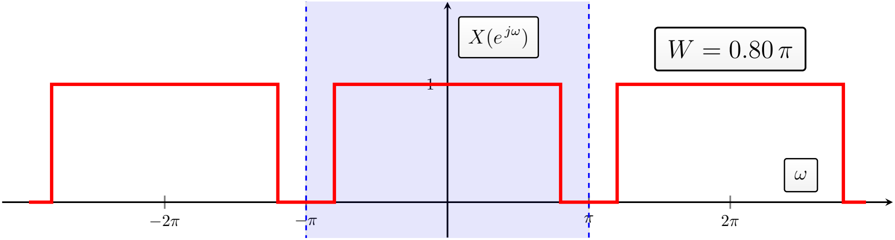

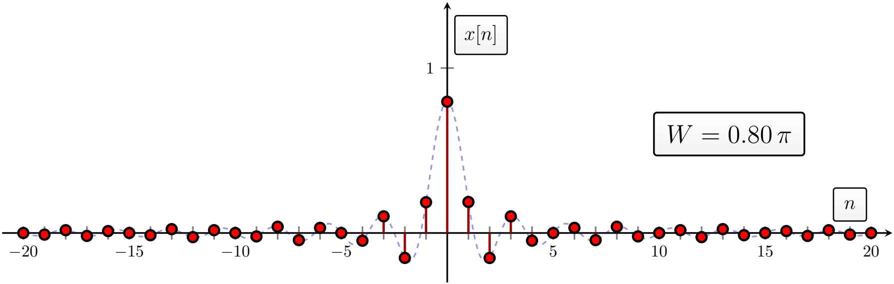

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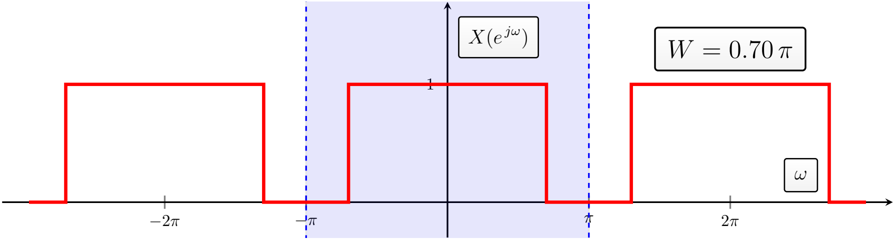

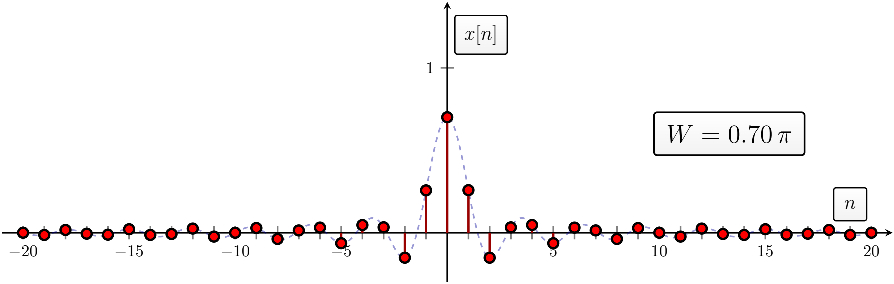

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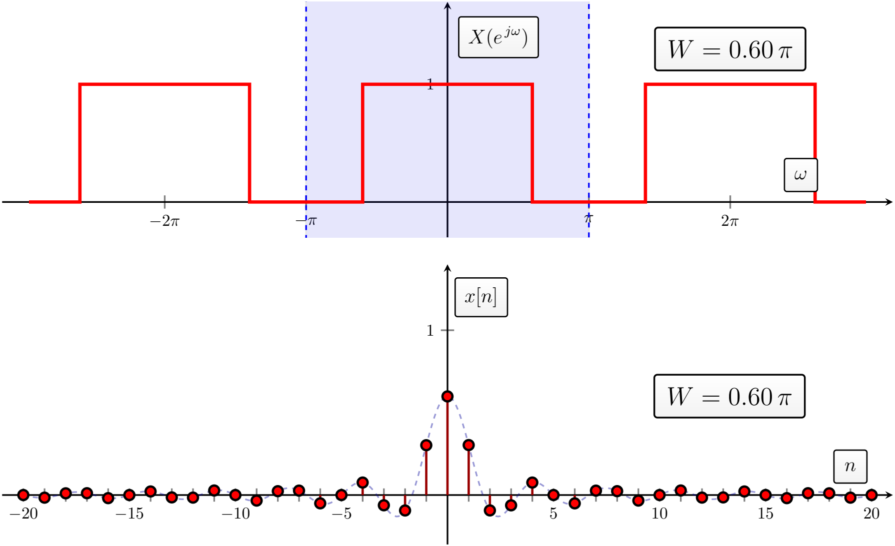

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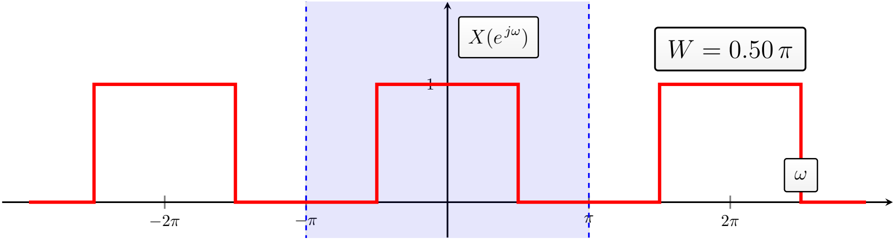

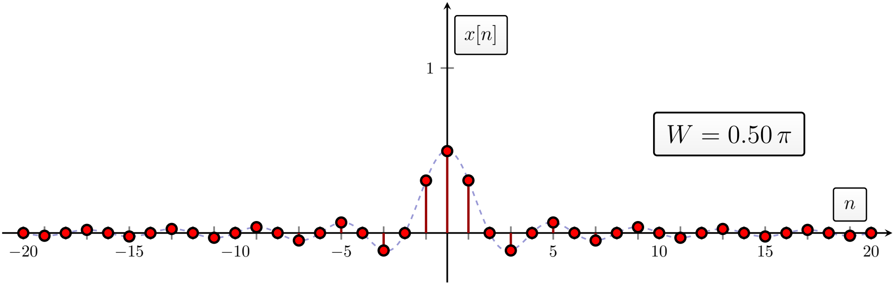

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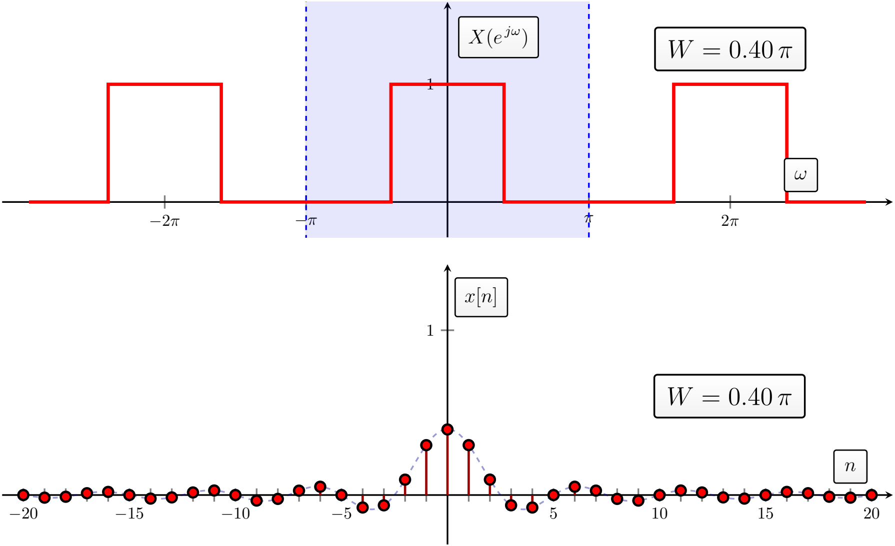

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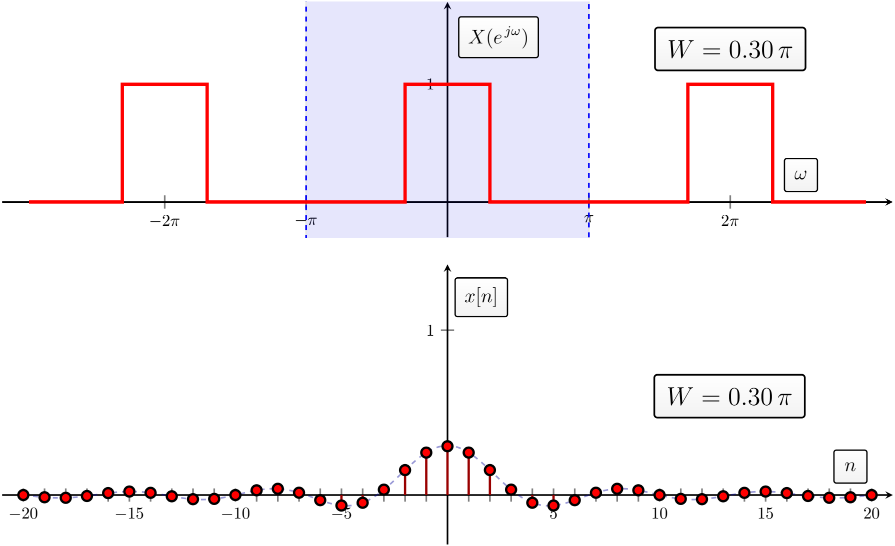
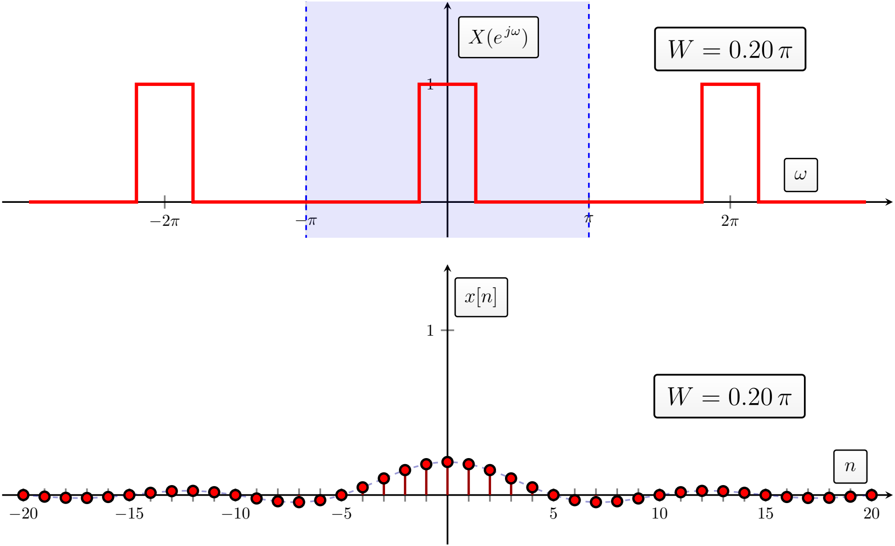


DT Fourier Transforms – Examples



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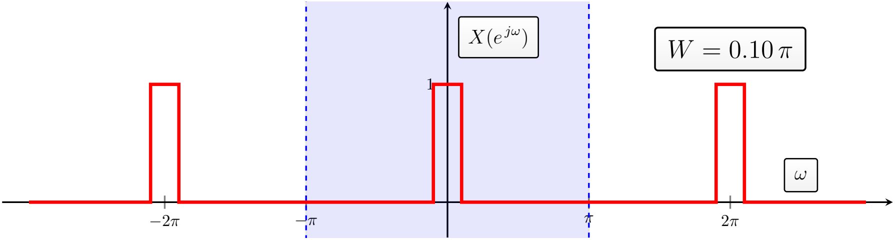
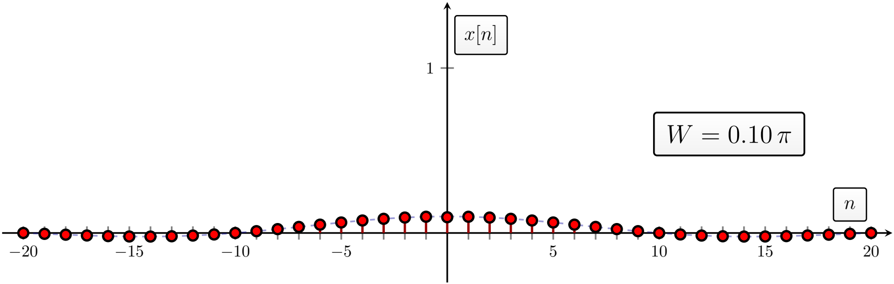




DTFT pairs example:

$$\boxed{e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2\pi l\right)}$$

DTFT pairs example:

$$\cos(\omega_0 n) \longleftrightarrow \pi \sum_{l=-\infty}^{\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$$

DTFT pairs example:

$$\boxed{\sin(\omega_0 n) \longleftrightarrow \frac{\pi}{j} \sum_{l=-\infty}^{\infty} \left\{ \delta\left(\omega - \omega_0 - 2\pi l\right) - \delta\left(\omega + \omega_0 - 2\pi l\right) \right\}}$$

DT Fourier – Properties

Properties for DT Fourier Transform can be inferred from the Synthesis and Analysis equations. Properties that mimic the CT counterparts are:

- **Reference:**

$$\boxed{x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})}$$

- **Linearity/Superposition:** `O&W 5.3.2 p.373`

$$\boxed{a\,x_1[n] + b\,x_2[n] \overset{\mathscr{F}}{\longleftrightarrow} a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})}$$

- **Time Shifting:** `O&W 5.3.3 p.373`

$$\boxed{x[n-n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}), \quad n_0 \in \mathbb{Z}}$$

DT Fourier – Properties

- **Frequency Shifting:** `O&W 5.3.3 p.373`

$$\boxed{e^{-j\omega_0 n}x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})}$$

- **Time Reversal:** `O&W 5.3.6 p.376`

$$\boxed{x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega})}$$

- **Conjugate Symmetry:** `O&W 5.3.4 p.375`

$$\boxed{x[n] \text{ real} \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega}) = X^*(e^{-j\omega})}$$

DT Fourier – Properties

Properties that more DT specific and/or tricky are:

- **Periodicity:** `O&W 5.3.1 p.373`

$$\boxed{X(e^{j\omega}) \text{ is periodic with period } 2\pi}$$

- **Differentiation in Frequency:** `O&W 5.3.8 p.380`

$$\boxed{n\,x[n] \xleftrightarrow{\mathscr{F}} j\frac{d}{d\omega}X(e^{j\omega})}$$

is multiplication by n in the time domain.

- **Parseval's Relation:** `O&W 5.3.9 p.380`

$$\boxed{\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega}$$

DTFT Tables:

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

	Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
1	$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ (2π/N)	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
2	$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
3	$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
4	$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
5	$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
6	Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
7	$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
8	$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
9	$x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
10	$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(e^{j\omega})$ $X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
11	$\delta[n]$	1	—
12	$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
13	$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
14	$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
15	$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$X(e^{j\omega})$, $Y(e^{j\omega})$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$$

Examples:

$$x[n] = \delta[n+2] - \delta[n-2] \qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$x[n] = \begin{cases} n & -3 \leq n \leq 3 \\ 0 & \text{otherwise} \end{cases} \qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$x[n] = \begin{cases} 2^n & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases} \qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$x[n] = \left(-\frac{1}{5}\right)^n u[n] - 6\left(-\frac{1}{5}\right)^{n-2} u[n-2] \qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$x[n] = \sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) \qquad\qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$x[n] = \sin\left(n\frac{\pi}{2}\right) + \cos(n) \qquad X\left(e^{j\omega}\right) = ?$$

Examples:

$$X\left(e^{j\omega}\right)=\begin{cases}1 & \frac{\pi}{4}\le|\omega|\le\frac{3\pi}{4}\\ 0 & \text{otherwise.}\end{cases}\quad -\pi<\omega<\pi \quad x[n]=?$$

Examples:

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} \left\{2\pi\delta(\omega - 2\pi l) + \pi\delta(\omega - \frac{\pi}{2} - 2\pi l) + \pi\delta(\omega + \frac{\pi}{2} - 2\pi l)\right\} \quad -\infty < \omega < \infty \quad x[n] = ?$$

Examples:

$$X\left(e^{j\omega}\right) = \frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2} \quad x[n] = ?$$

DTFT Properties:

- **Convolution:** `O&W 5.4 pp.382-388`

$$\boxed{y[n] = h[n] \star x[n] \xleftrightarrow{\mathscr{F}} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})}$$

Frequency Response $H(e^{j\omega})$ is the DTFT of the unit sample response.

Convolution using DTFT:

Find the convolution $y[n] = x[n] * h[n]$ where

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$$

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

Example:

Determine the convolution $y[n] = x[n] * h[n]$ if

$x[n] = \left(\frac{1}{4}\right)^n u[n]$

$h[n] = \frac{1}{6}[3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n]u[n]$

DT LTI System Representation:

DT LTI systems can be described via:

- Direct-form I implementation (block diagram)
- Difference equation
- Frequency response $H\left(e^{j\omega}\right)$
- Impulse response $h[n]$
- Output $y[n]$ and input $x[n]$ are given

Note: (1) using $h[n]$ we can check stability and causality.
(2) using $h[n]$ we can determine if the DT system is IIR or FIR.

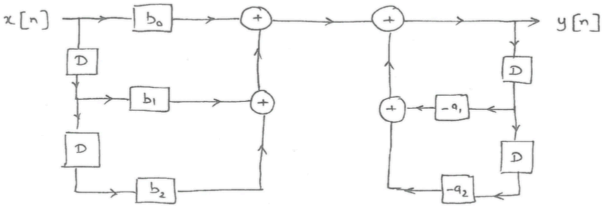
Direct-Form I implementation:

2nd order Difference Equations

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

Rearranging

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$



DT System Properties – Causality Property

For DT LTI Systems the **Causality Property** can be written:

Theorem (Causal DT LTI System)

A DT LTI system is **causal** *if and only if its pulse response, $h[n]$, satisfies*

$$h[n] = 0, \quad \textit{for all } n < 0.$$

- If $h[n] \neq 0$ for at least one $n = -n_0$ $(n_0 > 0)$ then the output at time n, $y[n]$, would contain term

$$h[-n_0]\, x[n + n_0],$$

for example, if $n_0 = 1$ and $h[-1] = 2$ then
$y[n] = \cdots + h[-1]\, x[n + 1] + \cdots$, and hence would not be causal.

DT System Properties – Stability Property

Stability: a bounded input $x[n]$ produces a bounded output $y[n]$.

Definition (DT LTI System Stability)

A DT LTI system is **stable**, with pulse respond $h[n]$, if and only if

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

is bounded if and only if the input is bounded.

Finite Impulse Response (FIR): `O&W 2.4.2 p.122`

A DT system with a finite length impulse response (total duration) is called a Finite Impulse Response (FIR) filter. Just need a finite number of parameters to describe it. Ideally suited to digital signal processing implementation.

Three examples:

$$y_1[n] = x[n]$$
$$y_2[n] = x[n+1] - 3x[n] + 7x[n-89]$$
$$y_3[n] = \sum_{k=0}^{100} 0.5^k x[n-k]$$

They have total durations 1, 91 and 101 (which are clearly finite), respectively.

Infinite Impulse Response (IIR): `O&W 2.4.2 p.123`

If a DT system is not FIR then it is Infinite Impulse Response (IIR).

For example,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} a^{n-k} x[k], \quad |a| < 1, \quad n \in \mathbb{Z} \\ &= \sum_{k=0}^{\infty} a^{k} x[n-k], \quad |a| < 1, \quad n \in \mathbb{Z} \end{aligned}$$

However, IIR doesn't necessarily mean infinite complexity. In the next few slides we see why.

LCC Difference Equations: `O&W 5.8 pp.396-399`

Linear constant coefficient difference equation (LCCDEs)

$$\boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]}$$

where we can interpret this as describing a DT-LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

for some impulse response $h[n]$. (Here $x[n]$ is the input and $y[n]$ is the output.) We can infer this system response; well at least its frequency response. That is, we can find the frequency response $H(e^{j\omega})$ corresponding to the LCCDE.

DT Fourier Stuff – Miscellanea

The LCCDE can be simplified by using

$$\boxed{x[n-k] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega k} X(e^{j\omega}), \quad k \in \mathbb{Z}}$$

By taking the Fourier Transform of both sides of:

$$\boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]}$$

we get:

$$\boxed{\sum_{k=0}^{N} a_k e^{-jk\omega}\, Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega}\, X(e^{j\omega})}$$

This type of transformation is directly analogous of what we did for differential equations.

DT Fourier Stuff – Miscellanea

$$Y(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{K} a_k e^{-jk\omega}} X(e^{j\omega})$$

So

$$\boxed{H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{K} a_k e^{-jk\omega}}}$$

is the frequency response of the linear, constant coefficient difference equation system.

$H(e^{j\omega})$ is a rational function of $e^{-jk\omega}$ amenable to the use of partial fraction expansions.

Example 1:

$$y[n-2] + 5y[n-1] + 6y[n] = 18x[n] + 8x[n-1]$$

Example 2:

A DTI system has impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{4}\right)^n u[n]$

Find the difference equation.

Sampling – Background

CT signals (and systems) describe the physical world. We are interested in converting them to DT signals. Why?

- why not?
- we what to process them with a computer or digital signal processor (DSP)
- it may reduce the complexity of the signal
- we can use MATLAB to play with them
- they are amenable for storage (hard disk)

We have the results from earlier lectures to fully understand the process of converting a CT signal to a meaningful DT signal through "sampling". What is sampling?

Sampling – Background

Sampling is taking snapshots of some signal $x(t)$ every T seconds. Here T is the sampling period. For example, the standard CD audio sampling rate is 44.1 kHz which means $T = 1/44100 = 0.00002267\cdots$ seconds.

44.1 kHz is a weird number. It was chosen to: 1) be high enough and 2) to make it hard to convert from 48 kHz professional audio by designers who probably would have failed this course. And then there is 96 kHz audio which is a load of nonsense to make it appeal to the same fools who think oxygen free speaker cable makes audio sound better.

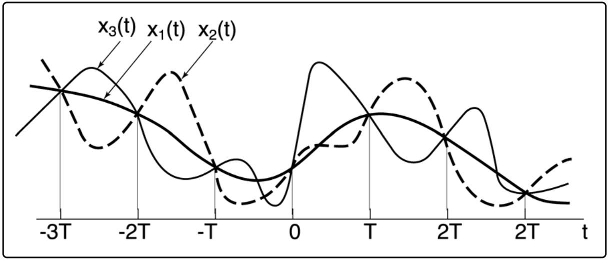
With T the sampling period, one way to create a DT signal, $x[n]$, from a given CT signal, $x(t)$, is via

$$x[n] = x(nT), \quad n \in \mathbb{Z}$$

noting $x(nT)$ means $x(t)$ evaluated or sampled at time $t = nT$.

Sampling – Background

We note that lots of CT signals have the same samples



and, therefore, information is thrown away when we sample. We lose the behavior of the signal between the samples. Or do we? Do we *always* lose information? Under what conditions can we recover or reconstruct the original CT signal $x(t)$ from its samples $x[n] = x(nT)$?

Sampling – Background

This is a non-trivial and somewhat remarkable property. If true then we can do digital signal processing (signal processing of sampled signals) without compromising the signal we are interested in.

What we expect is:

- The signals might need to have some special property (like smoothness).
- The sampling period T is short enough, or equivalently the sampling rate $1/T$ is high enough, relative to the variations we see in the signal.
- The complete CT signal $x(t)$ must be mathematically explicitly expressible in terms of its samples $x[n] = x(nT)$.

Sampling – Impulse Sampling

We need a convenient mathematical description of the sampling process. On the way through the course the signals and operations required to do this have been introduced.

Multiply the signal $x(t)$ by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

which yields

$$x_p(t) = x(t)\, p(t) = \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT)$$

By "sifting" property of the delta function:

$$\boxed{x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)}$$

$x_p(t)$ is a continuous time sampled signal. We only need to know $x(t)$ at $t = nT$ for integer $n \in \mathbb{Z}$, these are the samples.

For all other values of t (namely all t taking reals values not equal to the integers), the information in $x(t)$ is ignored or discarded. That is, the vast majority of $x(t)$ is discarded.

So surely hoping that $x_p(t)$ still contains all the information in $x(t)$ seems delusional. But the remarkable thing is, under the right conditions, we can figure out what $x(t)$ is from the continuous time sampled signal $x_p(t)$. Let's understand this further.

Sampling – Frequency Domain Analysis

Time domain operation

$$x_p(t) = x(t)\,p(t)$$

implies, by the multiplication property, in the frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \star P(j\omega)$$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega), \quad x_p(t) \overset{\mathscr{F}}{\longleftrightarrow} X_p(j\omega), \quad p(t) \overset{\mathscr{F}}{\longleftrightarrow} P(j\omega)$$

where

$$\boxed{P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,\omega_s)}$$

and $\omega_s \triangleq 2\pi/T$ is the sampling frequency (in rad/sec).

Sampling – Frequency Domain Analysis

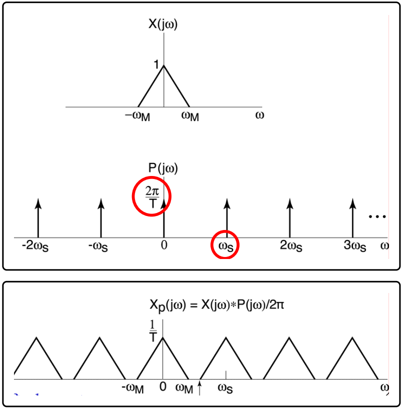
So we have

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \star \delta(\omega - k\,\omega_s)$$
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\big(\, j(\omega - k\,\omega_s)\big)$$

Recall $x(t) \xleftrightarrow{\mathscr{F}} X(j\omega)$ and $x_p(t) \xleftrightarrow{\mathscr{F}} X_p(j\omega)$.

So the frequency content of $x_p(t)$ is a periodic copy, with the frequency shifts given by $k\,\omega_s$ for integer $k \in \mathbb{Z}$, of the frequency content of $x(t)$ (with some scaling given by $1/T$).

Sampling – Illustration



More than an illustration but an important special case.

Here, have a band-limited signal such that $X(j\omega) = 0$ for $|\omega| > \omega_M$ and illustrated for

$$\omega_s - \omega_M > \omega_M$$

or

$$\omega_s > 2\,\omega_M$$

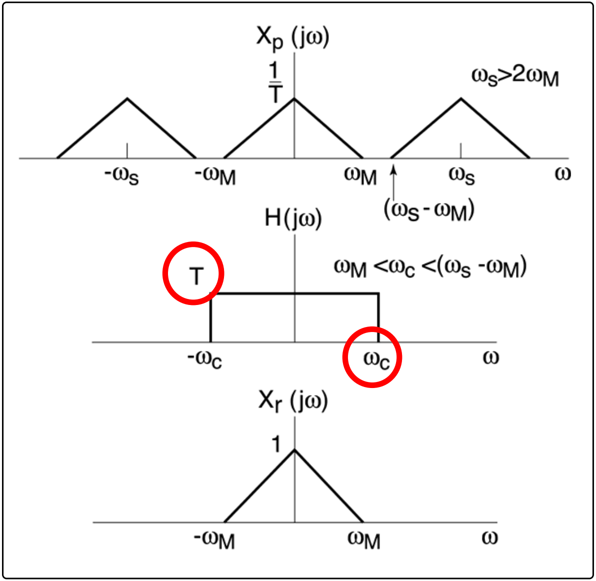
If the frequency content of the signal is appropriately low pass then there is no overlap in the periodic repetition of the spectrum. Hence we can use an ideal low pass filter to *perfectly* recover the signal.

A suitable bandwidth of the low pass filter is ω_c where

$$\omega_M < \omega_c < (\omega_s - \omega_M)$$

This condition can be “graphically” determined.

Sampling – Reconstruction



Theorem (Sampling Theorem)

Suppose $x(t)$ is band-limited so that

$$X(j\omega) = 0 \quad \textit{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$ if

$$\omega_s \triangleq \frac{2\pi}{T} > 2\,\omega_M \equiv \textit{“Nyquist Rate”}$$

The theorem provides sufficient conditions. You could generate a whole bunch of similar results for bandpass signals or signals with nicely spaced frequency holes, etc.

Example

Generally the top end of human hearing is given as 20 kHz (which declines with age and the number of night-clubs you frequent). The Nyquist rate or frequency is twice the highest frequency content, so the sample rate in Hz needs to be at least 40 kHz (or $2\,\omega_M = 80,000\pi$ in radians/sec).

(Continued)

Note that the CD audio rate is at least 40 kHz since it is 44.1 kHz. Why is it not 40 kHz?

- If it were 40 kHz then we would need an ideal low pass filter with cut-off at 20 kHz exactly. An ideal low pass filter has a long sinc shaped impulse response which is not nice (expensive and complicated to implement).
- Since there is a gap between 40 kHz and 44.1 kHz, the reconstruction filter needs only to satisfy

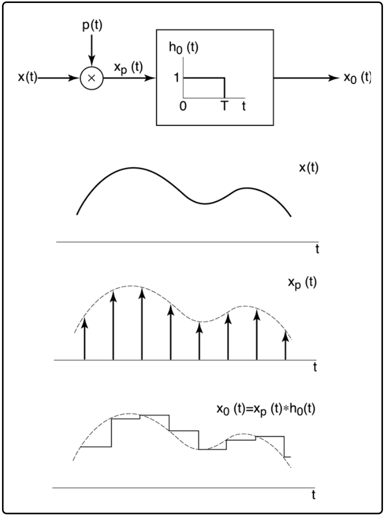
$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 2\pi \times 20,000 \\ 0 & \text{for } |\omega| > 2\pi \times 22,050 \\ \text{don't care} & \text{otherwise} \end{cases}$$

  So we can design a cheaper filter.

Sampling with impulses is an idealization. An infinitely narrow time portion of a signal would have zero energy in the limit (which we can't amplify by infinity like we do mathematically).

We could add or integrate the signal around the sampling time instant to gather enough energy to get a meaningful reading. This is a distortion but not a great one. This leads to the concept of a "Zero-Order Hold" which has a convenient mathematical or system model (shown next). It is important to have such a model since it can reveal how close to ideal impulse like sampling we can get.

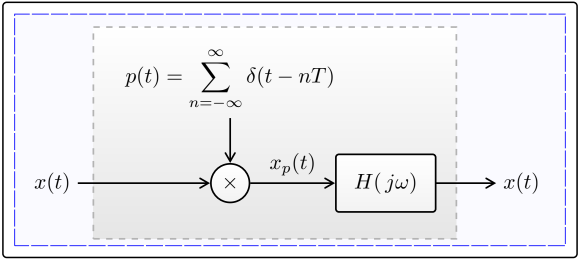
Sampling – Zero-Order Hold



Sampling – Sampling is Time-Varying

Multiplying a signal $x(t)$ by a time-varying (non-constant) function $p(t)$ implies sampling is a time-varying operation. However, sometimes combinations of such sampling and filtering reduces to something simple. For

$$\boxed{x(t) \text{ s.t. } X(j\omega) = 0 \text{ for } |\omega| > 2\pi/T}$$



with $H(j\omega)$ an ideal low pass filter. Overall this acts like the identity.

Sampling – Time Domain Reconstruction

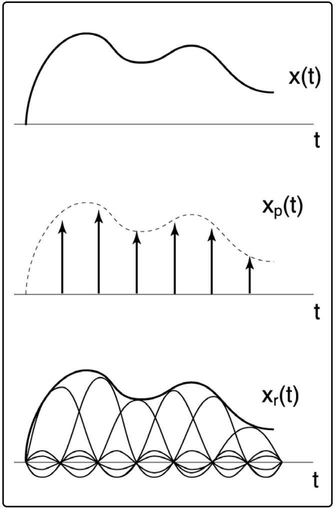
The Sampling Theorem is obvious in the frequency domain but it is useful to see what the equivalent time domain operation is (for implementation). Recall ω_c is the cut-off of the filter which we take to be an ideal LPF.

$$
\begin{aligned}
x_r(t) &= x_p(t) \star h(t), \qquad h(t) \triangleq \frac{T \sin \omega_c t}{\pi t} \\
&= \Big(\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT) \Big) \star h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT)
\end{aligned}
$$

That is,

$$
\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \frac{T\, \sin\big(\omega_c (t-nT)\big)}{\pi (t-nT)}}
$$

Sampling – Time Domain Reconstruction



Sampling – Interpolation Methods

Note that in going from $x(nT)$ to $x(t)$ we are filling in the values in the time gaps, which is just interpolation. So the Sampling Theorem reconstruction can be viewed as a special/optimal interpolation. We can then enumerate a few different types of interpolation

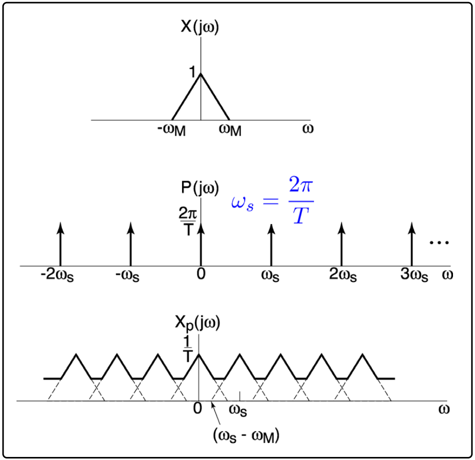
- Band-limited Interpolation — primarily the sinc style interpolation
- Zero-Order Hold — reconstructs a piecewise constant signal through the sample points
- First-Order Hold — reconstructs a piecewise linear interpolation joining the sample points (non-causal)

Sampling – Undersampling and Aliasing

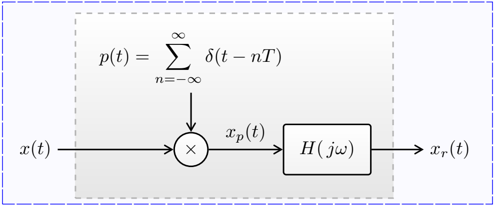
For the band-limited signal case, when the sampling rate is insufficient, that is,

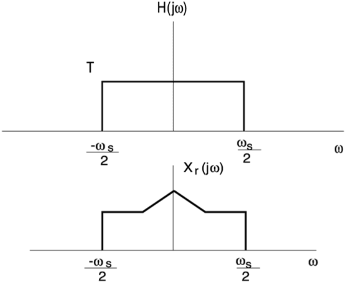
$$\omega_s \leq 2\omega_M$$

there will be frequency overlapping.



Sampling – Undersampling and Aliasing





Higher frequencies of $x(t)$ are aliased to lower frequencies.

Example

With

$$x(t) = \cos(\omega_0 t), \quad \text{where } \omega_M = \omega_0.$$

If $\omega_s > 2\omega_0$ then there is perfect reconstruction

$$x_r(t) = \cos(\omega_0 t)$$

If $\omega_s < 2\omega_0$ then there is aliasing

$$x_r(t) = \cos\big((\omega_s - \omega_0)t\big)$$