Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

PROBLEM SET 4

Fourier Analysis and Synthesis of Periodic Continuous Time Signals

Problem Set 4-1

Using the inspection method, determine the Fourier Series coefficients a_k of the signal

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t)$$

Problem Set 4-2

Find the Fourier coefficients for each of the following signals if $\omega_0 = 2\pi$:

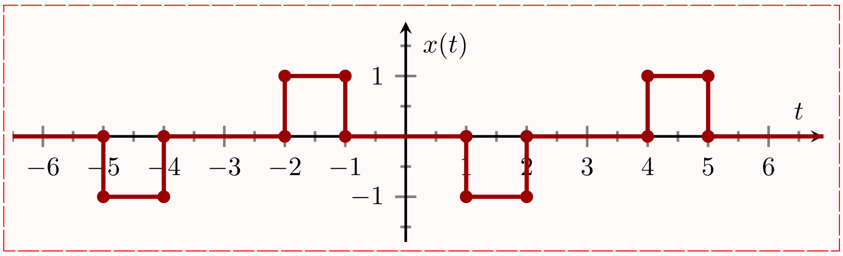
(a) $x(t) = 1 + \cos(2\pi t)$

(b) $y(t) = \sin(10\pi t + \pi/6)$

(c) $z(t) = \big(1 + \cos 2\pi t\big)\sin(10\pi t + \pi/6)$

Problem Set 4-3

Determine the Fourier series of following signal $x(t)$ by



(a) using analysis equation

(b) combinations of derivatives, impulse trains, linearity, hallucinogenic drugs, etc.

Problem Set 4-4

The Fourier series coefficient of a continuous time signal with period $T = 4$ seconds is specified below.

$$a_k = \begin{cases} 0, & k = 0 \\ (-1)^k \dfrac{\sin(k\pi/4)}{k\pi}, & \text{otherwise} \end{cases}$$

(a) Determine and sketch the signal $x(t)$ using the properties of Fourier series (Module 2: slide 398) and the result of Example 3.5 in the textbook (this periodic rectangular wave example was also solved in Module 2: slide 379,380). Hint: $(-1)^k = e^{j\pi k}$.

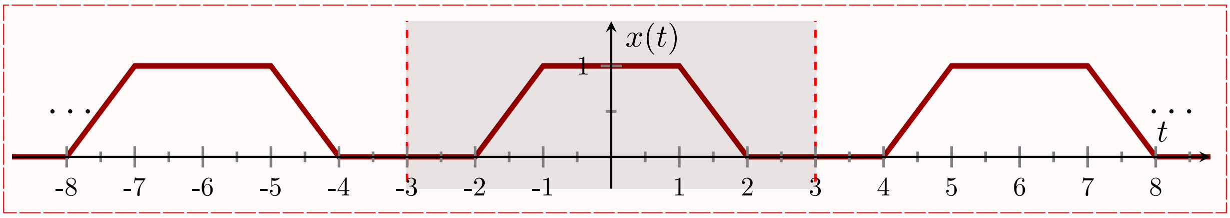


Figure 1: Periodic signal $x(t)$ for Problem 1.

Problem Set 4-5

Suppose $x(t)$ is a periodic signal as given in Fig. 1 below with period $T = 6$ seconds.

Here,

$$x(t) = \begin{cases} 0, & -3 \le t \le -2 \\ t+2, & -2 \le t \le -1 \\ 1, & -1 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & 2 \le t \le 3 \end{cases}$$

(a) Find the value of a_0, that is,

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

Write a sentence to intuitively explain your answer.

(b) Determine the Fourier series coefficients for this signal, that is,

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Show that

$$a_k = \frac{6}{k^2 \pi^2} \sin\left(\frac{\pi k}{2}\right) \sin\left(\frac{\pi k}{6}\right)$$

You must show your intermediate steps. Do not substitute the value of $\omega_0 = \frac{2\pi}{T}$ until the final step. You may wish to use all or some of the following results to help with your derivation:

$$\int t e^{-jk\omega_0 t} dt = \frac{e^{-jkt\omega_0}(1 + jkt\omega_0)}{k^2 \omega_0^2}$$

$$\int e^{-jk\omega_0 t} dt = \frac{j e^{-jkt\omega_0}}{k\omega_0}$$

$$\int_a^b e^{-jk\omega_0 t} dt = -\frac{j(e^{-jak\omega_0} - e^{-jbk\omega_0})}{k\omega_0}$$

$$\int_a^b (t+c) e^{-jk\omega_0 t} dt = \frac{e^{-ik\omega_0(a+b)} \left(e^{iak\omega_0}(1 + ik\omega_0(b+c)) - ie^{ibk\omega_0}(k\omega_0(a+c) - i)\right)}{k^2 \omega_0^2}$$

$$\int_a^b (-t+c) e^{-jk\omega_0 t} dt = \frac{e^{-ik\omega_0(a+b)} \left(e^{ibk\omega_0}(1 + ik\omega_0(a-c)) + e^{iak\omega_0}(-1 - ik\omega_0(b-c))\right)}{k^2 \omega_0^2}$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$$

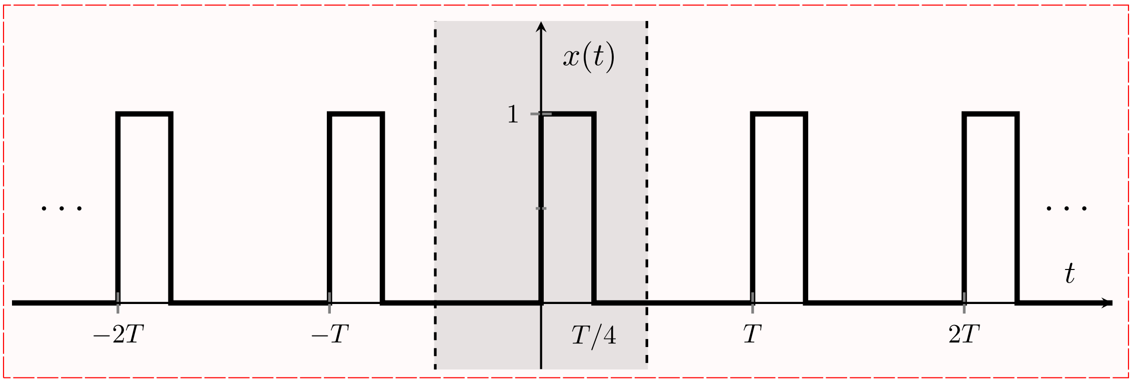


Figure 2: Periodic signal $x(t)$.

Fourier Series Properties of CT Periodic Signals

Problem Set 4-6

Suppose $x(t)$ is a periodic signal given as in Fig. 2 below with period T.

(a) Determine the Fourier series coefficients for this signal, that is,

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

using two different strategies. For example, you can use the direct integration as the first strategy and transform the periodic rectangular waveform from the lectures Part 9 as the second strategy.

(b) Consider the periodic convolution of $x(t)$ with itself, that is,

$$y(t) = x(t) \star x(t)$$

Determine the signal $y(t)$, plot and compare with signal $x(t)$. Part 10 of the lectures should be useful here.

(c) For the signal $y(t)$ in part (b) determine its Fourier series.

Problem Set 4-7

Suppose we are given following information about a signal $x(t)$

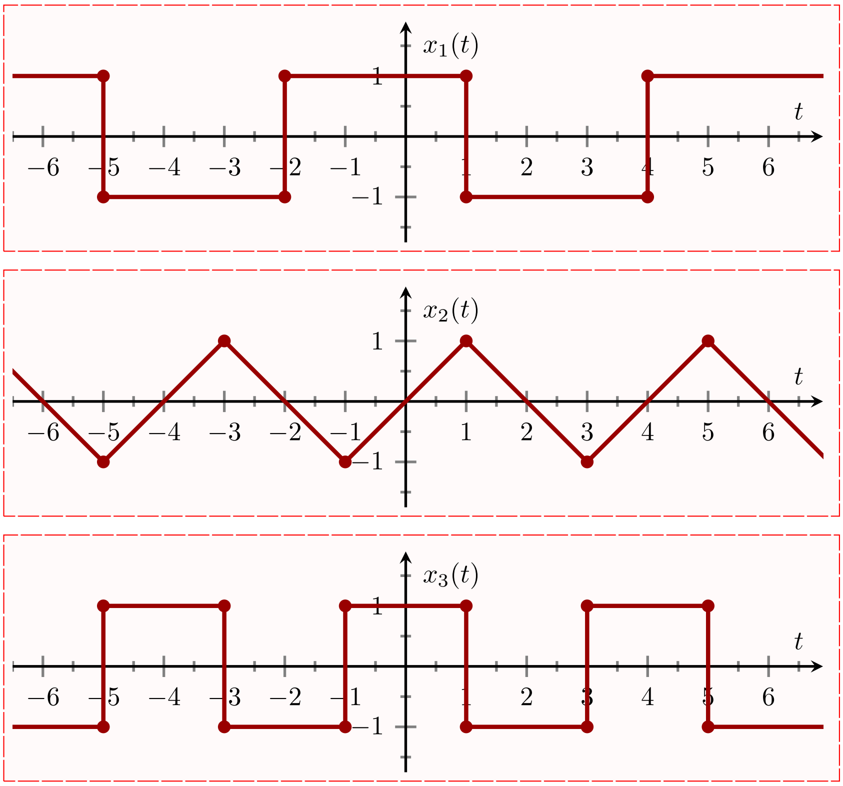
1. $x(t)$ is real and odd
2. $x(t)$ is periodic with period $T = 2$
3. The Fourier coefficients are a_k, such that $a_k = 0$ for $k > 1$
4. $\frac{1}{2} \int_0^2 \left|x(t)\right|^2 dt = 1$

Specify two different signals that satisfy these conditions.

Problem Set 4-8

Without evaluating the Fourier series coefficients, find which of the following periodic signals have Fourier coefficients with the following properties:

1. Only odd harmonics
2. Only real harmonics
3. Only imaginary harmonics

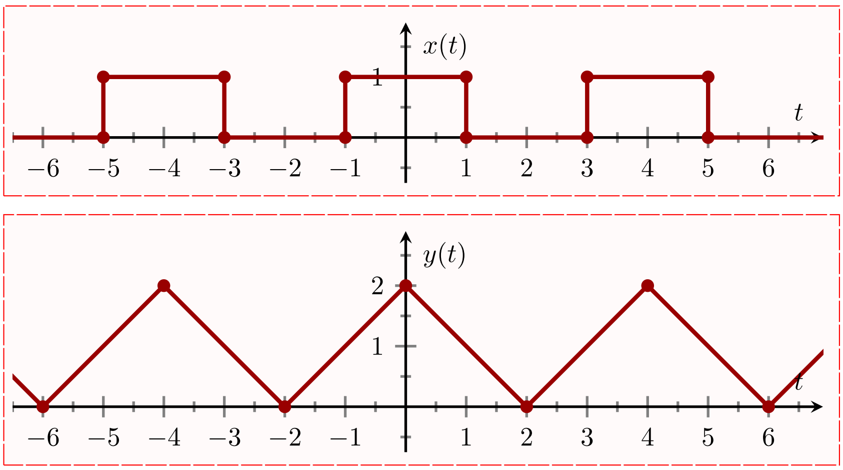


Problem Set 4-9

In the figure below $x(t)$ is a periodic rectangular wave with period $T = 4$ and has the Fourier series coefficients

$$a_0 = \frac{1}{2}, \qquad a_k = \frac{\sin(k\pi/2)}{k\pi}.$$

Using these Fourier series coefficients of $x(t)$, find the Fourier series coefficients, b_k, of the triangular wave with period 4, $y(t)$, as shown in the figure.



Problem Set 4-10

Let $x(t)$ be a periodic signal with fundamental frequency ω_0 and Fourier coefficients a_k, that is,

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{j\omega_0 t}.$$

Similarly for periodic

$$y(t) = \sum_{k=\infty}^{\infty} b_k e^{j\omega_0 t},$$

where the coefficients are b_k.

Find the Fourier coefficients b_k in terms of the Fourier coefficients a_k for the following signals.

(a) $y(t) = -2\,x(t) + j\,x(t)$

(b) $y(t) = x(t-1)$

(c) $y(t) = x'(t) = \frac{d}{dt}x(t)$

(d) $y(t) = x(1-t)$

(e) $y(t) = x^2(t)$

Problem Set 4-11

A normal mains voltage waveform versus time is shown as $x(t)$ in the figure below.

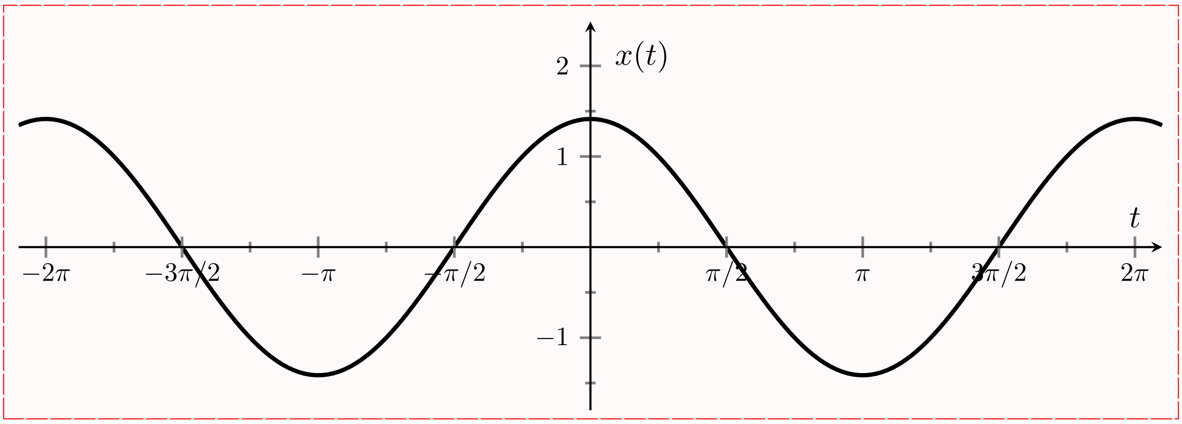


Figure 3: Normal mains voltage waveform (normalized).

Normally the voltage is 230 volts which is an RMS measure the peak voltage is thereby $230\sqrt{2}$ and the frequency of oscillation is 50 Hz or $\omega_0 = 100\pi$ rad/sec. For simplicity for this question the peak value is taken as $\sqrt{2}$, the fundamental period is $T_0 = 2\pi$ and fundamental frequency $\omega_0 = 1$.

(a) With $x(t) = \sqrt{2}\cos(t)$, show the average power per period of $x(t)$ is 1.

(b) Modern light dimmers work by gating (or chopping up) the main voltage waveform $x(t)$. Normally these are called trailing-edge and leading edge dimmers. For this problem we simplify the action of the dimmer to be like a combination of both trailing and leading edge dimmers, generating the periodic signal $y_\alpha(t)$ as shown in Fig. 4.

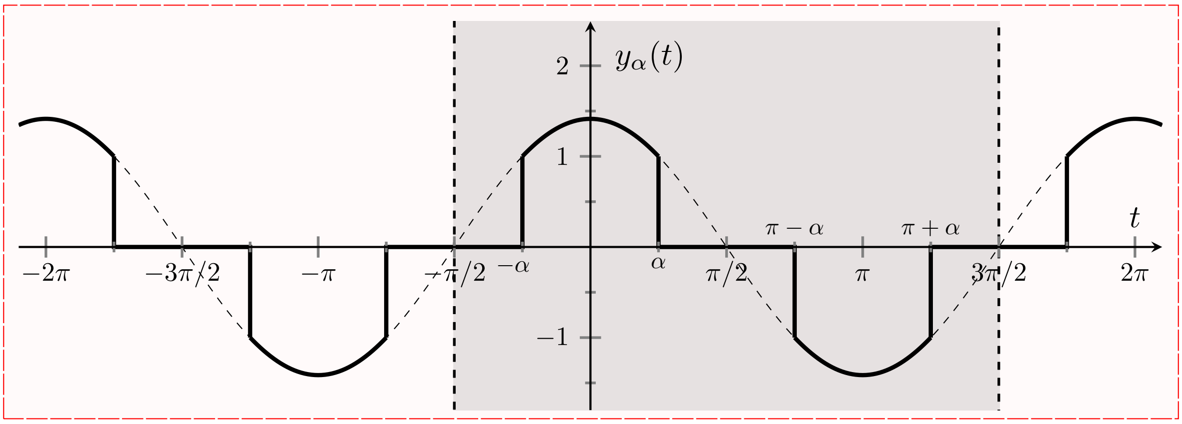


Figure 4: Gated mains voltage waveform for dimming, where $\alpha \in [0, \pi/2]$ adjusts the dimming.

Mathematically we can define $y_\alpha(t)$ over one period, and it is convenient to take the interval as $[-\pi/2, 3\pi/2]$ (shown shaded in Fig. 4):

$$y_\alpha(t) = \begin{cases} x(t) & t \in [-\alpha, \alpha] \cup [\pi-\alpha, \pi+\alpha] \\ 0 & \text{otherwise} \end{cases}, \quad t \in [-\pi/2, 3\pi/2]$$

and $y_\alpha(t+2\pi) = y_\alpha(t)$. A valid range of values for the parameter α is

$$0 \le \alpha \le \pi/2 \quad \text{or} \quad \alpha \in [0, \pi/2]$$

and corresponds to the dimmer dial setting.

Find as a function of α the average power per period

$$P(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \left|y_\alpha(t)\right|^2 dt$$

and confirm that $P(0) = 0$ and $P(\pi/2) = 1$.

(c) Both $x(t) = \sqrt{2}\cos(t)$ and $y_\alpha(t)$ are periodic with the same fundamental frequency $\omega_0 = 1$ and both have zero DC component. The *total harmonic distortion* (THD) is the ratio of the power per period of the harmonics $|k| > 1$ divided by the power per period in the first harmonic components $k = \pm 1$ (or $|k| = 1$).

Compute the total harmonic distortion (THD) as a function of α of $y_\alpha(t)$, that is,

$$\mathrm{THD}(\alpha) = \frac{\sum_{k=-\infty}^{-2} \left|b_k(\alpha)\right|^2 + \sum_{k=2}^{\infty} \left|b_k(\alpha)\right|^2}{\left|b_{-1}(\alpha)\right|^2 + \left|b_1(\alpha)\right|^2}$$

where the Fourier Series coefficients are given by

$$b_k(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} y_\alpha(t)\, e^{-jk\omega_0 t}\, dt.$$

where $\omega_0 = 1$ and $b_0(\alpha) = 0$.

(You should probably want to use Parseval's Relation, as given in Module 2: slide 398, unless you are a glutton for punishment. Also note that both $x(t)$ and $y_\alpha(t)$ are even real-valued functions.)

Problem Set 4-12

Find the output $y(t)$ of a causal LTI system for the periodic input $x(t) = \cos(2\pi t)$, where

$$\frac{d}{dt} y(t) + 4y(t) = x(t)$$

Fourier Analysis and Synthesis for Discrete-time Periodic Signals

Problem Set 4-13

Suppose $x[n]$ is a periodic signal as given in Fig. 5 below with period $N = 10$ seconds.

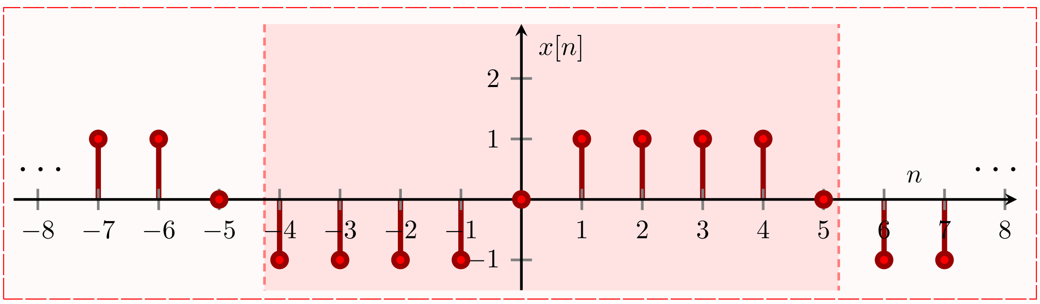


Figure 5: Signal $x[n]$ with period $N = 10$ for Problem 3.

(a) Show that

$$a_k = -\frac{j}{5} \sin\left(\frac{\pi k}{2}\right) \frac{\sin\left(\frac{2\pi k}{5}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$

Hint: Introduce a +ve and a -ve unit sample at $n = 0$ so series sums can be calculated using the series sum formulas.

Problem Set 4-14

Determine the DTFS coefficients of the following periodic signals using inspection method:

(a)

$$x[n] = 1 + \sin\left(\frac{n\pi}{12} + \frac{3\pi}{8}\right)$$

(b)

$$x[n] = 1 + \cos\left(\frac{n\pi}{30}\right) + 2\sin\left(\frac{n\pi}{90}\right)$$

Problem Set 4-15

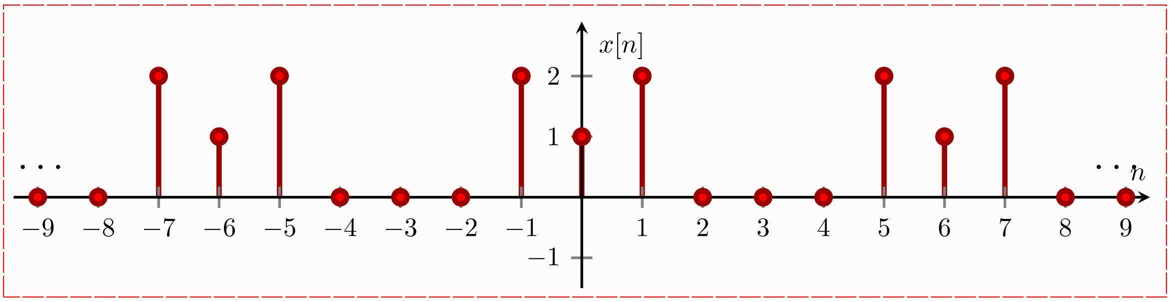
Using Matlab, find the time domain signal $x[n]$ corresponding to the DTFS coefficients

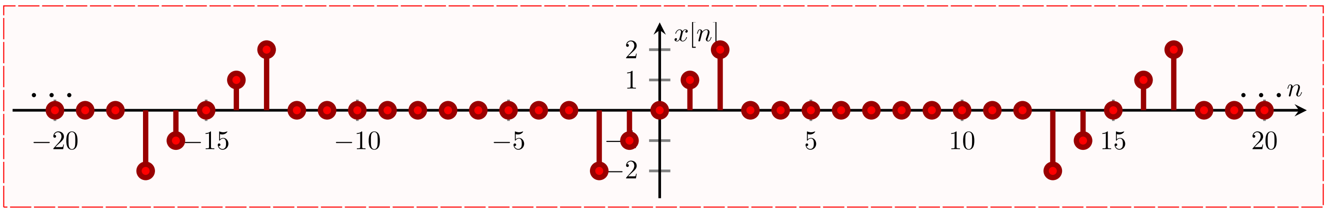
$$a_k = \cos\left(\frac{k4\pi}{11}\right) + 2j\sin\left(\frac{k6\pi}{11}\right)$$

- This problem is just too cumbersome to solve by hand.
- Hints: You have to find N first. Show that a_k is periodic with period $N = 11$. Use the DTFS synthesis equation, summing from $k = -5$ to 5. Evaluate for each value of n.
- For some values of n, due to finite machine precision, Matlab may give an answer which is very very small (1e-15 or 1e-16) which means the value is 0.

Problem Set 4-16

Determine the DTFS coefficients of the periodic signals depicted in the figures below using the DTFS analysis equation, determine N for each plot first.





Fourier Series Properties of DT Periodic Signals

Problem Set 4-17

Find the output $y[n]$ of a causal LTI system for the periodic input $x[n] = \cos\frac{n\pi}{6}$, where

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Frequency Response of Discrete-time Filters

The following problems involve the $N = 4$ DT periodic signal $x[n]$ and DT pulse response $h[n]$ of some LTI system, shown in the figures below. For $x[n]$ the values in the shaded region covers one period and are repeated indefinitely for both positive and negative n.

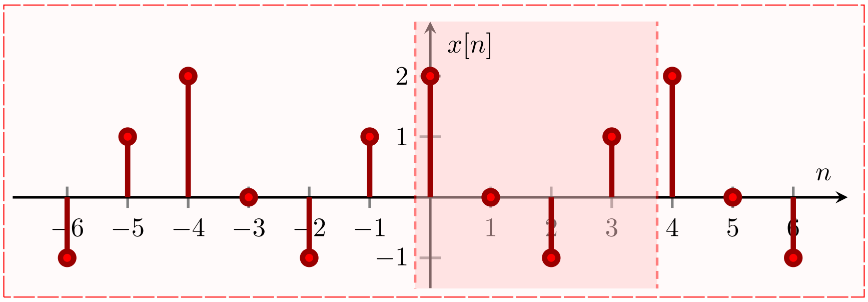


Figure 6: Signal $x[n]$ with period $N = 4$.

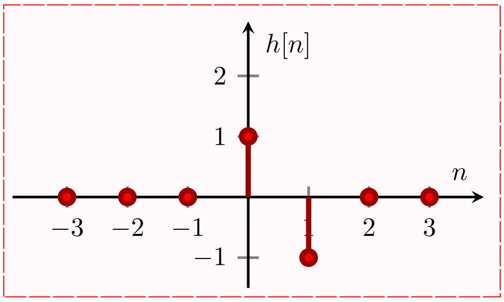


Figure 7: Pulse response $h[n]$ of DT LTI system.

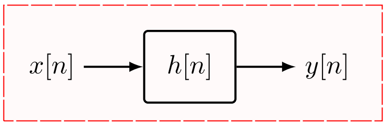


Figure 8: DT LTI system with pulse response $h[n]$, input $x[n]$ and output $y[n]$.

Problem Set 4-18

Questions on Expressing the Signals Algebraically:

(a) Express $h[n]$ in terms of a superposition of time-shifted unit pulse signals $\delta[n]$.

(b) Express $x[n]$ in terms of a superposition of shifted unit pulse signals $\delta[n]$.

Problem Set 4-19

Questions on the Convolution Output:

(a) Compute the DT convolution of $x[n]$ and $h[n]$

$$\begin{aligned} y[n] &= x[n] \star h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \end{aligned}$$

and give the answer in terms of a superposition of shifted unit pulse signals $\delta[n]$.

(b) Plot $y[n]$ using the template shown in Figure 9 (in a manner similar to Figure 6)

Problem Set 4-20

Questions on the DC Gain of the System:

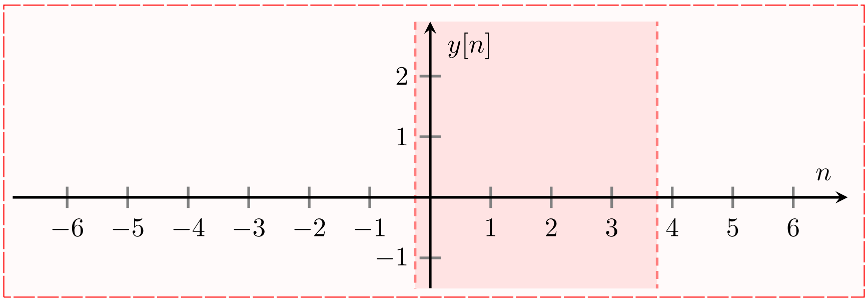


Figure 9: (Template) Signal $y[n]$ with period $N = 4$.

(a) What is the DC (zero frequency response) value of $x[n]$?

(b) What can you say about the DC value of $y[n]$ and how does it relate to the DC gain of $h[n]$?

Problem Set 4-21

Questions on DT Fourier Series:

(a) Four distinct complex exponentials that have period $N = 4$ are given by

$$\phi_k[n] = e^{j\pi kn/2}, \quad k = 0, 1, 2, 3,$$

and the Fourier series synthesis equation for $x[n]$ is then given by

$$x[n] = \sum_{k=0}^{3} a_k \, e^{j\pi kn/2}.$$

Determine the Fourier series coefficients a_0, a_1, a_2 and a_3 corresponding to $x[n]$ in Figure 6.

(b) In the above, the $N = 4$ periodic signal $x[n]$ is characterized by 4 numbers, which by convention are taken as the four values shown in the shaded portion of Figure 6, and can be written as a 4-vector

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Similarly the Fourier series coefficients can be written as a 4-vector

$$\boldsymbol{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Determine the 16 entries, $\phi_{i,j}$, in the following 4×4 (analysis equation) matrix, $\boldsymbol{\Phi}$, that relates these two 4-vectors through the matrix equation

$$\boldsymbol{a} = \frac{1}{4} \boldsymbol{\Phi} \, \boldsymbol{x},$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \phi_{0,2} & \phi_{0,3} \\ \phi_{1,0} & \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,0} & \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,0} & \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \phi_{0,2} & \phi_{0,3} \\ \phi_{1,0} & \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,0} & \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,0} & \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(c) Confirm the values you got for the Fourier coefficients of $x[n]$ in the previous part by using the new matrix calculation. That is, compute $\frac{1}{4} \boldsymbol{\Phi} \, \boldsymbol{x}$.

(d) One of the grim realities of life is trying to make sense of poorly documented material or material that uses different notations and conventions. Review the following documentation:

`http://www.mathworks.com.au/help/matlab/ref/fft.html` and determine how the analysis equation calculations

$$\boldsymbol{a} = \frac{1}{4}\boldsymbol{\Phi}\,\boldsymbol{x},$$

performed above, are related to the MATLAB functions `Y=fft(x)` and/or `y=ifft(X)`.

Problem Set 4-22

Questions on DT Frequency Response:

(a) For the pulse response $h[n]$ in Figure 7 determine its frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n},$$

and simplify the expression in the form of a complex exponential times a real function of ω.

(b) Determine $\left|H(e^{j\omega})\right|$ and plot it in the range $\omega \in [-2\pi, 2\pi]$ using the template shown in Figure 10.

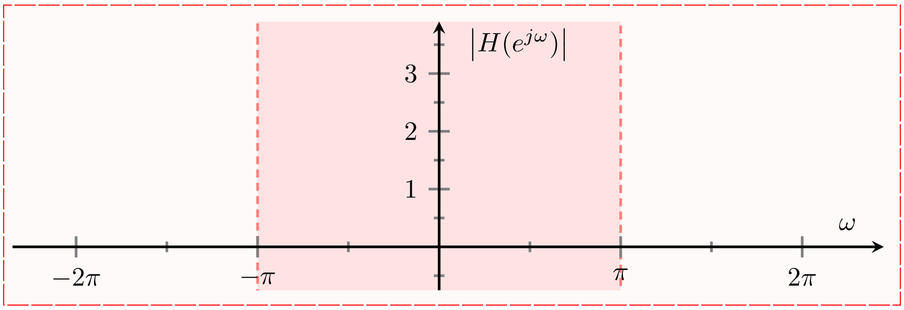


Figure 10: (Template) Frequency response $\left|H(e^{j\omega})\right|$ over range -2π to 2π.

(c) What type of filter is it and why? (low-pass, band-pass, high-pass, all-pass)

Problem Set 4-23

Questions on System, Input and Output:

(a) Redraw the frequency response plot for $H(e^{j\omega})$ from the previous problem. Now add arrows to that plot to indicate the frequencies (within the frequency range $[-\pi, \pi]$) present in the period-4 signal $x[n]$ given in Figure 6. Such a plot should show which frequencies are input to the system defined by $h[n]$.

(Figure 11 gives an example of how to indicate these frequencies in the case where the frequencies ω equal $-4\pi/5$ and $+4\pi/5$, and a nonsense $\left|H(e^{j\omega})\right| = 1.5 + 0.5\cos(3\pi\omega)$.)

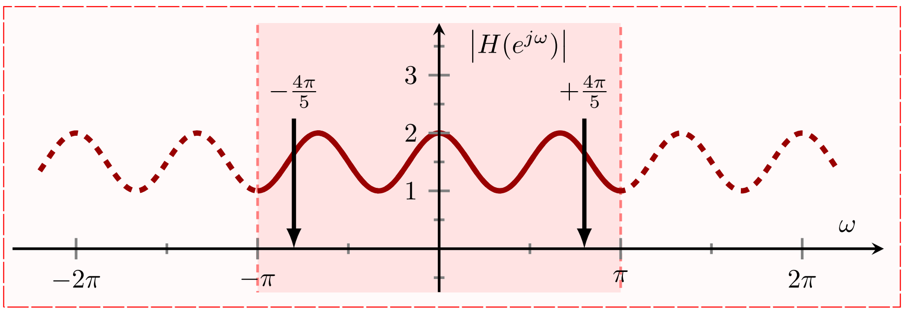


Figure 11: (Template) Example frequency response $\left|H(e^{j\omega})\right| = 1.5 + 0.5\cos(3\pi\omega)$ (which is not the right answer) over range -2π to 2π. Example of how to show the input frequencies of $x[n]$, in this case two frequencies at ω equal $-4\pi/5$ and $+4\pi/5$ (which is not the right answer).

(b) For the period $N = 4$ signal $y[n]$ find its Fourier coefficients b_0, b_1, b_2 and b_3.

Problem Set 4-24

Questions on Basic Filter Design:

(a) Design or find a new causal LTI system, $g[n]$ that produces zero output for the period-4 input $x[n]$ input shown in Figure 6. That is, find a non-trivial (non-zero) $g[n]$ such that

$$x[n] \star g[n] = 0.$$

(b) Plot the frequency response $\left|G(e^{j\omega})\right|$ and explain why your design works.

(c) [Difficult] It is likely that your design for $g[n]$ above filters out all period-4 signals and not just $x[n]$. Design a new filter, $p[n]$ that filters out $x[n]$ but has a non-zero output for other (more general) period-4 signals.

Problem Set 4-25

Consider the following pairs of signal $x[n]$ and $y[n]$, which are the input and output of a system shown in Fig. 12. For each pair, determine whether there is a discrete-time LTI system for which $y[n]$ is the output when the corresponding $x[n]$ is the input. If such a system exists, determine whether the system is unique (i.e., whether there is more than one LTI system with the given input-output pair). Also determine the frequency response of an LTI system with the desired behaviour. If no such LTI system exists for a given $x[n]$, $y[n]$ pair, explain why.

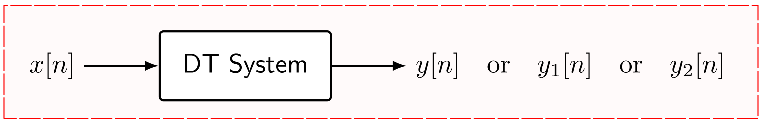


Figure 12: System with input $x[n]$ and output $y[n]$ or $y_1[n]$ (in part 2(h)) or $y_2[n]$ (in part 2(i)).

(a) $x[n] = (0.5)^n$ and $y[n] = (0.25)^n$

(b) $x[n] = (0.5)^n u[n]$ and $y[n] = (0.25)^n u[n]$

(c) $x[n] = 0.5^n u[n]$ and $y[n] = 4^n u[-n]$

(d) $x[n] = e^{jn/8}$ and $y[n] = 2\,e^{jn/8}$

(e) $x[n] = e^{jn/8} u[n]$ and $y[n] = 2\,e^{jn/8} u[n]$

(f) $x[n] = j^n$ and $y[n] = 2\,j^n(1-j)$

(g) $x[n] = \cos(\pi n/3)$ and $y[n] = \cos(\pi n/3) + \sqrt{3}\sin(\pi n/3)$

(h) $x[n]$ and $y_1[n]$ shown in Fig. 13.

(i) $x[n]$ and $y_2[n]$ shown in Fig. 13.

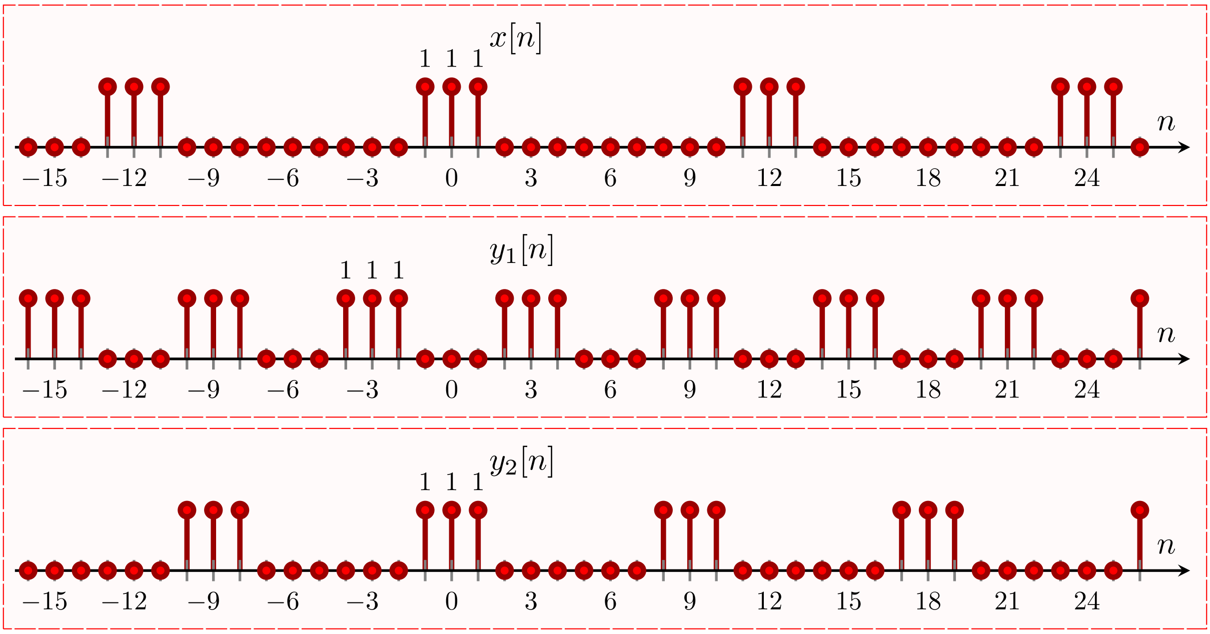


Figure 13: Periodic input signal $x[n]$ and two periodic output signals $y_1[n]$ and $y_2[n]$.