Australian
National
University

Mid-Semester Examination
Semester 2, 2017

SIGNAL PROCESSING

ENGN2228

Study period: *None*
Writing period: *120 Minutes duration*
Permitted materials: *One A4 page with notes on both sides*
Calculator (memory cleared)

6 True-False Questions for a total of 6 marks,
12 Multiple-Choice Questions (choose 1 or 5), for a total of 24 marks
6 Short Answer Problems, for a total of 70 marks

Contribution to Final Assessment: 20%

- *Write your True-False and Multiple-Choice Questions answers on the answer sheet provided and place it inside the script book.*
- *Write your 6 Short Answer Problem answers in the script book provided.*
- *At the end of the exam, hand in the exam question sheets as well as the script book and the multiple-choice answers sheet.*

PART 1 — True/False Questions

CT means continuous time, and DT means discrete time; A system being LTI means the system is linear and time-invariant; The binary operator $\star$ denotes convolution for both CT and DT; $\overline{z}$ denotes the complex conjugate of z; $u[n]$ and $u(t)$ represent the unit step functions; $\delta[n]$ and $\delta(t)$ represent the impulse functions.

TF Question 1

A CT signal which is periodic with period 2π is also periodic with period 8π.

a. True

b. False

TF Question 2

A CT System with impulse response $h(t) = 2^{-t}u(t)$ is causal.

a. True

b. False

TF Question 3

A DT System with impulse response $h[n] = 2^{-n}u[n]$ is causal.

a. True

b. False

TF Question 4

The DT system

$$y[n] = 3\,x[n] + 5\,x[n+1] - x[n+2],$$

where $x[n]$ is the input and $y[n]$ is the output signal, is linear, time-invariant and causal.

a. True

b. False

TF Question 5

DT signal $x[n] = \sin(n/16)$ is not periodic.

a. True

b. False

TF Question 6

The parallel connection of two component LTI systems is equivalent to a LTI system whose impulse response is the sum of the component impulse responses.

a. True

b. False

PART 2 — Multiple Choice Questions

MC Question 1

What is the fundamental period of DT signal $x[n] = \sin(n/16)$?

a. 16

b. 16π

c. 32

d. 32π

e. It is not periodic and has no fundamental period.

Solution: The intrinsic period of sin is irrational, which implies n would need to be irrational. It needs to be an integer. □

MC Question 2

Let $\delta(t)$ be the CT unit impulse function. Which of the following is false?

a. $x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$

b. $\displaystyle\int_{-1}^{1} \delta(t)\,dt = 1$

c. $\displaystyle\int_{-\infty}^{\infty} \delta(t-1)\,dt = 1$

d. $\displaystyle\int_{-1}^{1} \delta(t-2)\,dt = 0$

e. $\displaystyle\int_{-1}^{1} x(t)\,\delta(t-2)\,dt = x(2)$

Solution: The integrand is non-zero only at $t = 2$, which is outside the range of the integration. The integral is zero. □

MC Question 3

What is the fundamental period, N, of DT signal

$$x[n] = (-j)^n + \cos(\pi\, n/3) + \sin(2\pi\, n/15)?$$

That is, the least integer N such that $x[n] = x[n + N], \quad \text{for all } n.$

a. 15

b. 30

c. 45

d. 60

e. Undefined, the signal is not periodic and, therefore, has no fundamental period.

Solution: $(-j)^n$ has period 4, $\cos(\pi\, n/3)$ has period $6 = 2 \times 3$, and $\sin(2\pi\, n/15)$ has period $15 = 3 \times 5$. The lowest common multiple is $4 \times 3 \times 5 = 60$, so $x[n + 60] = x[n]$. □

MC Question 4

Consider the DT system described by

$$y[n] = 2\,\overline{x[n]} + 3\,x[n-1]$$

where $x[n]$ is the complex input, $y[n]$ is the complex output and $\overline{\cdot}$ denotes complex conjugation. Which one of the following is true?

a. The system is linear and causal.

b. The system is linear and not causal.

$\boxed{\text{c. The system is non-linear and causal.}}$

d. The system is non-linear and not causal.

e. The system is time-varying and causal.

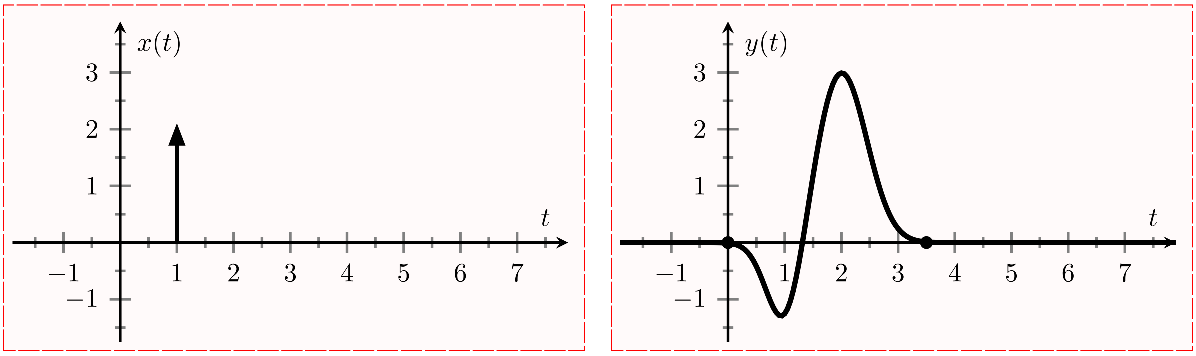
Solution: It clearly causal and time-inavriant Less obvious is the fact that it is non-linear. The input $x_1[n]$ leads to

$$y_1[n] = 2\,\overline{x_1[n]} + 3\,x_1[n-1]$$

but $x_2[n] = j\,x_1[n]$ leads to

$$\begin{aligned} y_2[n] &= -2j\,\overline{x_1[n]} + 3j\,x_1[n-1] \\ &\neq +2j\,\overline{x_1[n]} + 3j\,x_1[n-1] \;\; = j\,y_1[n]. \end{aligned}$$

So scaling with a complex number violates the scaling condition for linearity. $\square$



(a) Specific CT input signal $x(t)$

(b) CT output signal $y(t)$

Figure 1: CT signals $x(t)$ and $y(t)$ which are the input and output of an unknown LTI system.

MC Question 5

A CT LTI system with specific input $x(t) = 2\,\delta(t-1)$, shown in Figure 1(a), generates the output $y(t)$, shown in Figure 1(b).

Given that $y(t) = x(t) \star h(t)$, for some impulse response $h(t)$, which of the following is correct?

$\boxed{\text{a. } h(t) = 0.5\,y(t+1).}$

b. $h(t) = 2\,y(t+1)$.

c. $h(t) = 0.5\,y(t-1)$.

d. $h(t) = 2\,y(t-1)$.

e. The impulse response cannot be determined.

Solution: Since $x(t) = 2, \delta(t-1)$ then $\delta(t) = 0.5\, x(t+1)$. The impulse response is then $0.5\, y(t+1)$ by the LTI property. ☐

MC Question 6

Which is true of the following statements about the following DT system:

$$y[n] = 3\, x[n] + 5^{-n}\, x[n+1],$$

where $x[n]$ is the input signal and $y[n]$ is the output signal?

a. The system is linear, time-invariant and causal.

b. The system is linear, time-invariant and non-causal.

c. The system is linear, time-varying and causal.

d. The system is linear, time-varying and non-causal.

e. The system is nonlinear, time-invariant and non-causal.

Solution: This DT system is linear; the 5^{-n} makes it time-varying and $y[n]$ requires future $x[n+1]$, which make is non-causal. ☐

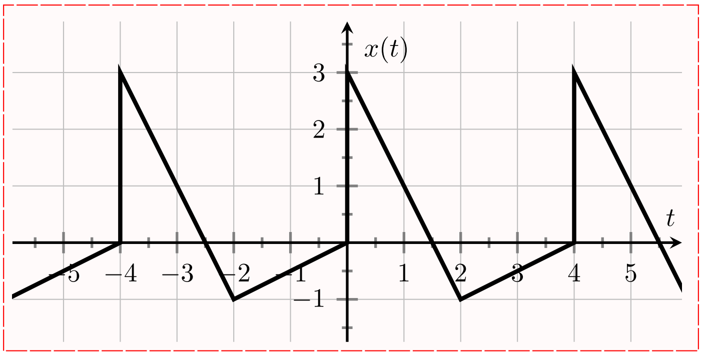


Figure 2: Piecewise linear CT periodic signal $x(t)$ with fundamental period $T = 4$.

MC Question 7

Figure 2 shows a portion of a periodic CT signal with fundamental period $T = 4$. What is the DC value:

$$\frac{1}{T}\int_0^T x(t)\,dt\,?$$

a. 1/4

b. 1/2

c. 3/4

d. 1

e. 2

Solution: The area between 0 and 1 is 2 units, between 1 and 2 is 0 units, and between 2 and 4 is -1 unit. Therefore the total area over one period is 1 unit, and the average is thereby 1/4. □

MC Question 8

The average power of the periodic signal $x(t)$ over a period T is given by

$$\frac{1}{T}\int_0^T x^2(t)\,dt.$$

What is the average power per period of the signal, $x(t)$, shown in Figure 2?

(Take care with how much time you spend on this.)

a. 0

b. 1

c. 4/3

d. 2

e. 4

Solution: $\int_0^2 (3-2t)^2\,dt = 14/3$ and $\int_2^4 (-2+t/2)^2\,dt = \int_0^2 (t/2)^2 dt = 2/3$. Therefore average power per period is $(14/3 + 2/3)/4 = 4/3$. □

MC Question 9

Which frequency (in rad/sec) corresponds to the positive fourth harmonic the signal, $x(t)$, shown in Figure 2? (That is, the frequency, in rad/sec, corresponding to $k = 4$ in the Fourier Series or four times the fundamental frequency.)

a. $\omega = \pi/2$

b. $\omega = \pi$

c. $\omega = 2\pi$

d. $\omega = 3\pi$

e. $\omega = 4\pi$

Solution: Since $T = 4$ then the fundamental frequency is $\omega = \pi/2$. So the positive fourth harmonic is 4 times the fundamental frequency which is $\omega = 2\pi$. □

MC Question 10

The equation for a LTI system with input $x[n]$ and output $y[n]$ is given by

$$y[n] = 0.5\,x[n] - 0.3\,x[n-1] + 0.1\,x[n-2].$$

Suppose the input is given by

n	-2	-1	0	1	2
$x[n]$	5	3	-2	2	6

What is output at times $n = 0,\ 1,\ 2$, that is, $y[0]$, $y[1]$ and $y[2]$?

a. $y[0] = 2.5$, $y[1] = 0$, and $y[2] = -1.4$.

b. $y[0] = -1.4$, $y[1] = 1.9$, and $y[2] = 2.2$.

c. $y[0] = 2.2$, $y[1] = 1.9$, and $y[2] = -1.4$.

d. $y[0] = 2.2$, $y[1] = -1.6$, and $y[2] = 0.6$.

e. $y[0] = -2.2$, $y[1] = +1.6$, and $y[2] = -0.6$.

Solution: Plug numbers into the difference equation, e.g., $y[1] = 0.5(2) - 0.3(-2) + 0.1(3) = 1.9$. □

MC Question 11

A DT LTI system with pulse response $h[n]$ is stable if

$$\sum_{n=-\infty}^{\infty} \big|h[n]\big| < \infty.$$

One of the following causal DT LTI systems is not stable. Which one is it?

a. $h[n] = u[n] - u[n-10]$

b. $h[n] = \dfrac{1}{n^2} u[n]$

c. $h[n] = \dfrac{1}{n} u[n]$

d. $h[n] = e^{-2n} u[n]$

e. $h[n] = \delta[n]$

Solution: The harmonic series $\sum_{n=0}^{\infty} 1/n$ diverges. All the others are finite sums or decay more rapidly than the harmonic series. □

MC Question 12

Below are 5 DT systems, where $x[n]$ denotes the input and $y[n]$ denotes the output.

System 1: $y[n] = x[n+1] + x[3-n]$

System 2: $y[n] = \big(x[n]\big)^2$

System 3: $y[n] = -0.8\,y[n-1] + x[n]$

System 4: $y[n] = \displaystyle\sum_{k=n}^{\infty} x[k]$

System 5: $y[n] = \begin{cases} x[n], & x[n] \geq 0 \\ 2\,x[n], & x[n] < 0 \end{cases}$

How many of these 5 DT systems can be fully characterized by their response to the input $x[n] = \delta[n]$ (their pulse response)?

a. 1 system

b. 2 systems

c. 3 systems

d. 4 systems

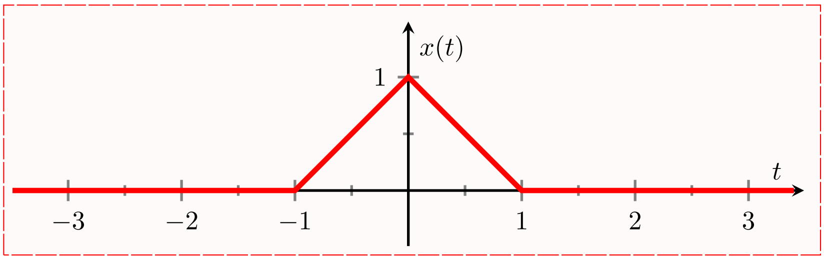
e. 5 systems

Solution: We are looking for LTI systems. System 3 is recursive but it is LTI, and System 5 is non-causal and unstable but it is LTI. System 1 and 4 are time-varying, and System 2 is non-linear. □

PART 3 — Short Answer Problems

Problem 1

A triangular pulse $x(t)$ is shown below.

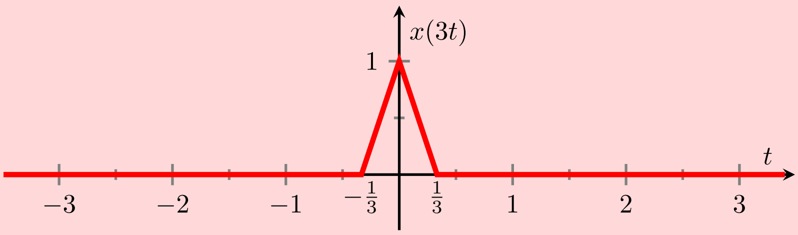


Sketch and label each of the following signals derived from the triangular pulse $x(t)$.

(a) [2 marks]

$$y_1(t) = x(3t)$$

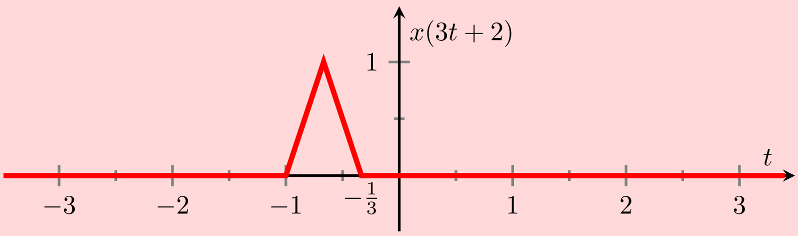
Solution:



(b) [2 marks]

$$y_2(t) = x(3t+2)$$

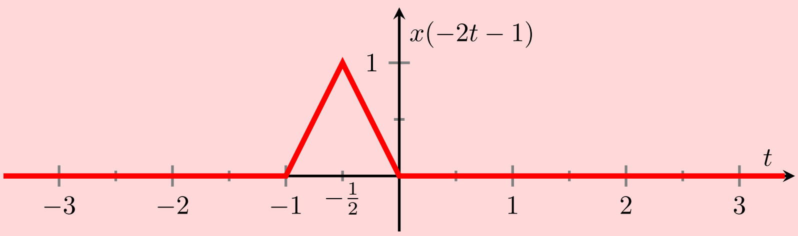
Solution:



(c) [2 marks]

$$y_3(t) = x(-2t-1)$$

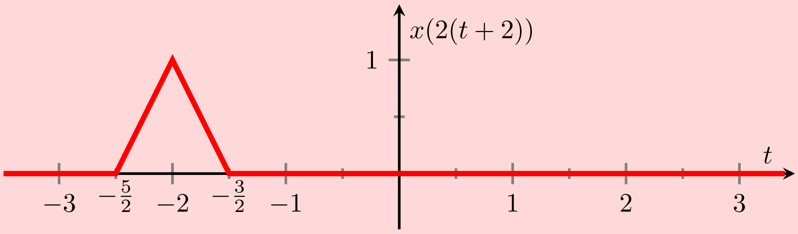
Solution:



(d) [2 marks]

$$y_4(t) = x(2(t+2))$$

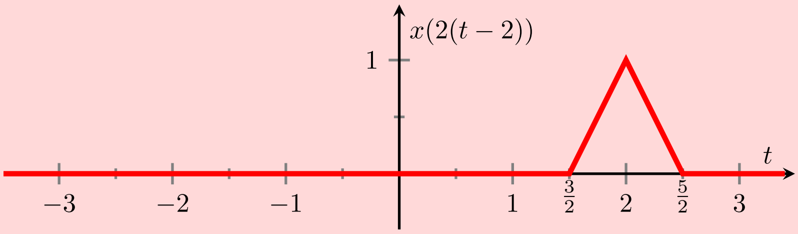
Solution:



(e) [2 marks]

$$y_5(t) = x(2(t-2))$$

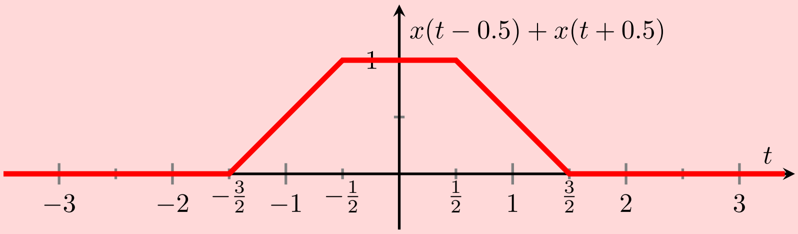
Solution:



(f) [2 marks]

$$y_6(t) = x(t-0.5) + x(t+0.5)$$

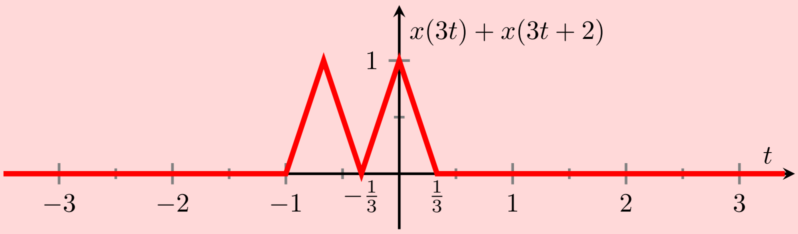
Solution:



(g) [2 marks]

$$y_7(t) = x(3t) + x(3t+2)$$

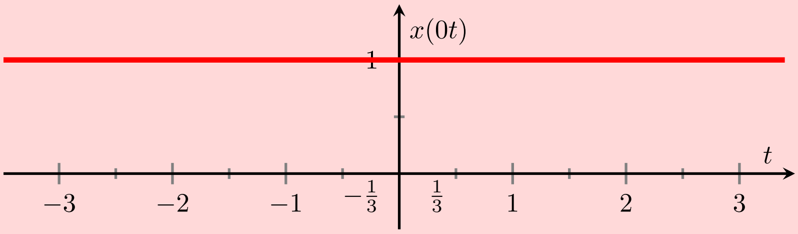
Solution:



(h) [1 mark]

$$y_8(t) = x(0t) \quad \text{which is the same as} \quad y_8(t) = x(0)$$

Solution:



Problem 2

DT signal $x[n] = e^{j\omega n}$ is not periodic for every choice of ω.

(a) [2 marks] True or false? Why?

Solution: True. To be periodic we need ωN to be a multiple of 2π for some N. □

(b) [2 marks] Is it periodic when $\omega = 4$? Why?

Solution: No, because otherwise we would need $4n$ to be a multiple of 2π which would imply π is rational , which is false ($\pi \neq 2n/k$ for integer k). □

(c) [2 marks] Is it periodic when $\omega = 19\pi$? Why?

Solution: Yes, because it has fundamental period $N_0 = 2$. $x[0] = 1$, $x[1] = -1$, $x[2] = 1$, etc. □

(d) [4 marks] How does the DT signal $x[n] = e^{j19\pi n}$ differ from $x[n] = e^{j\pi n}$? Why?

Solution: It doesn't differ. The first signal aliases to the second signal. □

Problem 3

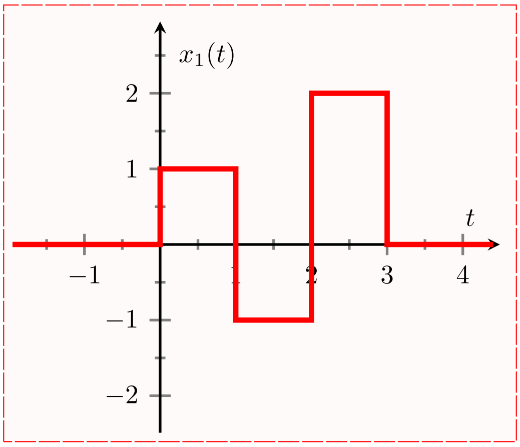
(a) [4 marks] Explain what it means in words for a system to be linear and time-invariant (LTI)?

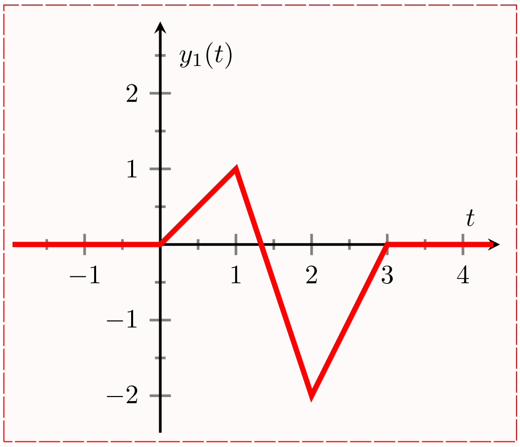
Solution: The answer should be *in words* and not equations; otherwise 0 marks.

To be LTI means the system behaves the same way irrespective of when an input is applied (time-invariance), and the output response to a linear combination of input signals leads to the same linear combination of the respective outputs (linearity or superposition).

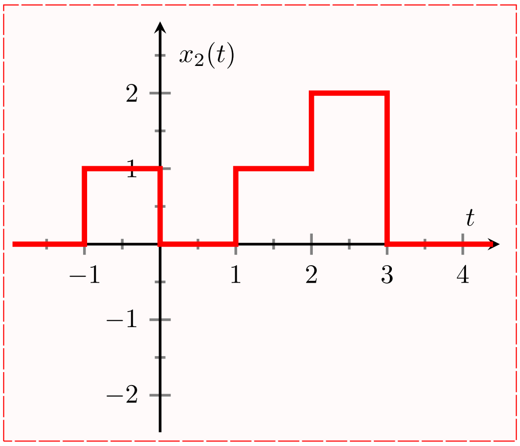
Alternatively, the LTI system is one that can be fully characterized by the response to an impulse. The output signal is given by the convolution of the impulse response and the input signal. □

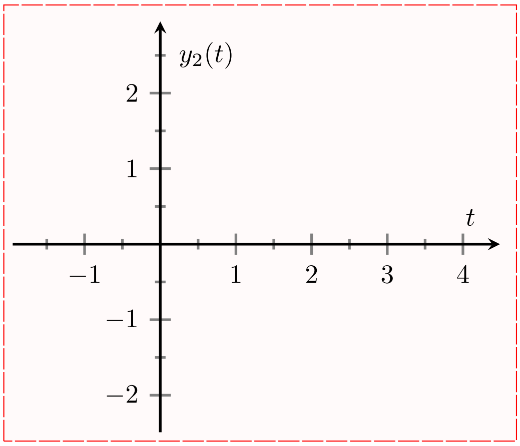
(b) [4 marks] Consider a LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ depicted below.



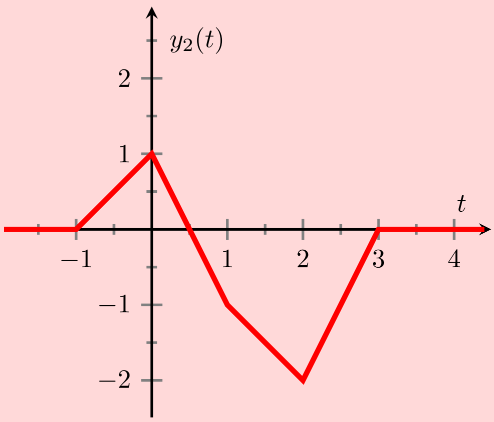


Determine and provide a labelled sketch of the response $y_2(t)$ of the system to the input $x_2(t)$ depicted below.





Solution: Since the system is a LTI system and $x_2(t) = x_1(t) + x_1(t+1)$, the output of the system is $y_2(t) = y_1(t) + y_1(t+1)$. The response $y_2(t)$ is:



(c) [2 marks] Provide concise statements to explain how you arrived at $y_2(t)$.

Solution: Since the system is a LTI system and $x_2(t) = x_1(t) + x_1(t+1)$, the output of the system is $y_2(t) = y_1(t) + y_1(t+1)$. □

Problem 4

Find the even and odd components of each of the following signals:

(a) [2 marks]

$$x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$$

Solution:

$$\begin{aligned} \text{Even:} &\quad \cos(t) \\ \text{Odd:} &\quad \sin(t)\big(1+\cos(t)\big) \end{aligned}$$

□

(b) [2 marks]

$$x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$$

Solution:

$$\begin{aligned} \text{Even:} &\quad 1 + 3t^2 + 9t^4 \\ \text{Odd:} &\quad t + 5t^3 \end{aligned}$$

□

(c) [2 marks]

$$x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$$

Solution:

$$\begin{aligned} \text{Even:} &\quad 1 + t^3\sin(t)\cos(t) \\ \text{Odd:} &\quad t\cos(t) + t^2\sin(t) \end{aligned}$$

□

(d) [2 marks]

$$x(t) = (1 + t^3)\cos^3(10t)$$

Solution:

$$\begin{aligned} \text{Even:} &\quad \cos^3(10t) \\ \text{Odd:} &\quad t^3\cos^3(10t) \end{aligned}$$

□

(e) [2 marks]

$$x(t) = u(t)$$

Solution:

$$\begin{aligned} \text{Even:} &\quad 0.5 \\ \text{Odd:} &\quad u(t) - 0.5 \end{aligned}$$

Perhaps people can argue how to treat the point at $t = 0$. Here we assume $u(0) = 0.5$. Of course, the odd part must be zero at $t = 0$ and so the even part would be $u(0)$ at $t = 0$ (for whatever definition of $u(t)$ is assumed). □

Problem 5

The equation

$$y[n] - a\,y[n-1] = x[n] \tag{1}$$

describes a DT system, with input $x[n]$ and output $y[n]$, assumed to be initially at rest, that is,

$$x[n] = 0 \text{ and } y[n] = 0 \quad \text{for all } n < 0.$$

(a) [3 marks] Show that the impulse response $h[n]$ for this system is

$$h[n] = a^n u[n],$$

where $y[n] = x[n] \star h[n]$.

Solution: We want to show that when the input is an impulse, $x[n]$ set to $\delta[n]$, then the response $y[n]$ is given by $h[n]$, that is, the following is satisfied

$$h[n] - a\,h[n-1] = \delta[n]$$

with $h[n] = a^n u[n]$. Substituting we have

$$\begin{aligned} a^n u[n] - a\,a^{n-1} u[n-1] &= a^n \big(u[n] - u[n-1]\big) \\ &= a^n\,\delta[n] && \text{since } \delta[n] = u[n] - u[n-1] \\ &= a^0\,\delta[n] = \delta[n] && \text{since } g[n]\delta[n] = g[0]\delta[n], \text{ for any } g[n] \end{aligned}$$

which shows $h[n] = a^n u[n]$. □

(b) Is this system (provide reasoning for each of your answer)

i) [2 marks] linear?

Solution: Yes, it is evident by inspection since the system output is given by a linear combination of the present input and a past output. If you are still not convinced you can plug in, for example, $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ and see that the only possible solution will involve $y[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$. □

ii) [2 marks] time-invariant?

Solution: Yes, given it is linear, then this is also clear by inspection since the coefficients of $x[n-k]$ or $y[n-k]$ are not a function of time n, for any k. □

iii) [2 marks] memoryless?

Solution: No, output $y[n]$ depends on the old input $y[n-1]$, which only depends on old inputs $x[n-1], x[n-2], x[n-3], \ldots$. □

iv) [2 marks] causal?

Solution: Yes, output $y[n]$ depends on the current into $x[n]$ and old inputs $x[n-1], x[n-2], \ldots$, but not future inputs $x[n+1], x[n+2], \ldots$. Alternatively, $u[n] = 0$ for all $n < 0$ and so $h[n] = 0$ for all $n < 0$, which is necessary and sufficient condition for causality. □

v) [2 marks] stable?

Solution: The answer depends on the value of a. If $|a| < 1$

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

and it is stable. Otherwise, if $|a| \geq 1$, it is unstable. □

(c) [2 marks] Draw a block diagram (such as Direct-I form) for the filter implementation of the system given in (1).

Solution: Not much to sketch here. □

Problem 6

Continuous time (CT) convolution of signals $x(t)$ and $h(t)$ is defined as

$$x(t) \star h(t) = \int_{-\infty}^{+\infty} x(t-\tau)\, h(\tau)\, d\tau = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

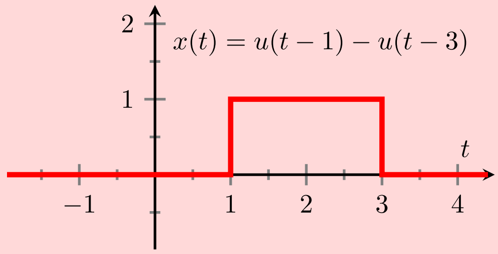
This question concerns convolution of rectangular signals (which can be expressed in terms of the unit step function $u(t)$) and delta functions $\delta(t)$.

(a) [2 marks] Draw and label the signal

$$x(t) = u(t-1) - u(t-3)$$

as a function of time t.

Solution:



□

(b) [2 marks] For $x(t)$ in part (a), determine the convolution

$$z(t) = x(t+4) \star \delta(t-4).$$

Solution: Convolution with $\delta(t-4)$ is the same as a delay of 4 or shifting in time to the right by 4. The signal $x(t+4)$ corresponds to shifting $x(t)$ to the left by 4. These two shifts cancel and so

$$z(t) = x(t+4-4) = x(t) = u(t-1) - u(t-3)$$

□

(c) [2 marks] For $x(t)$ in part (a), determine the convolution

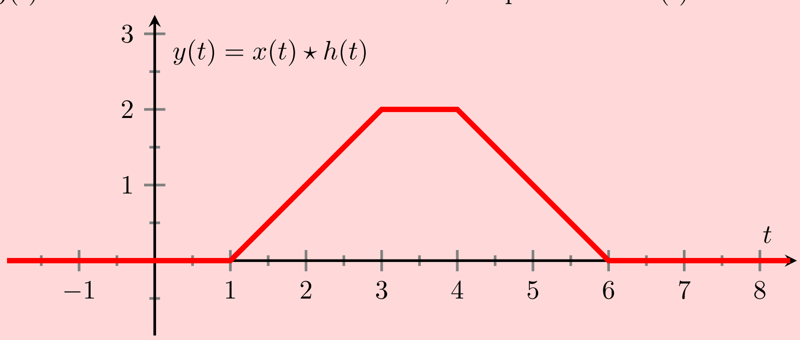
$$y(t) = x(t) \star h(t)$$

when

$$h(t) = u(t) - u(t-2).$$

(Suggestion: There are other ways to determine this convolution than applying the definition of convolution.)

Solution: $x(t)$ is rectangular of width 3 and $h(t)$ is rectangular of width 2. The rectangles overlap occurs for a time-range of 5. There is complete overlap for a time range of 1; the area when there is complete overlap is 2. If $h(t)$ as an impulse response causal with no delay; so the output $y(t)$ becomes non-zero from time $t = 1$, the point when $x(t)$ becomes non-zero.



□

(d) [4 marks] Both of the rectangular signals $x(t)$ and $h(t)$, in the parts above, have time-width of 2.

Suppose two rectangular signals, $x_1(t)$ and $x_2(t)$ have time-widths w_1 and w_2, which are generally different ($w_1 \neq w_2$). What, if anything, can be said about the width and shape of the convolution $x_1(t) \star x_2(t)$?

Solution: If you think of rectangles sliding over each other then the shape and width can be quickly determined.

When the time-widths are equal, that is, $w_1 = w_2$, then the convolution is a symmetric triangular shape of total width $2w_1$. (The question doesn't ask about the peak value but it is not too hard to determine.)

When the time-widths are unequal, that is, $w_1 \neq w_2$, then the convolution is a **symmetric flat top (plateau) triangular shape**. The plateau is of **width** $|w_1 - w_2|$ (the differences in the widths). The **width** of the base is determined by when the two rectangles just touch (on each side), and is given by $w_1 - w_2$. These observations are consistent with the $w_1 = w_2$ case. □

(end of exam)

Question	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	2	
8	2	
9	2	
10	2	
11	2	
12	2	
13	2	
14	2	
15	2	
16	2	
17	2	
18	2	
19	15	
20	10	
21	10	
22	10	
23	15	
24	10	
Total:	100	