Kernel-Based Formulations of Spatio-Spectral Transform and Three Related Transforms on the Sphere

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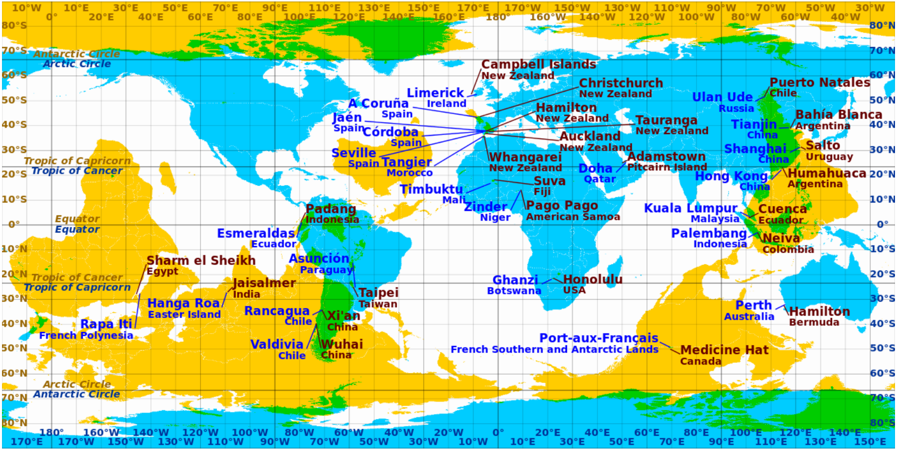
Tuesday 15 July 2014
Science on the Sphere:
Royal Society International Scientific Seminar
Chicheley Hall


Antarctic Circle
Arctic Circle
Tropic of Capricorn
Tropic of Cancer
Equator
Equator
Tropic of Cancer
Tropic of Capricorn
Arctic Circle
Antarctic Circle
Campbell Islands
New Zealand
Christchurch
New Zealand
Limerick
Ireland
A Coruña
Spain
Hamilton
New Zealand
Jaén
Spain
Tauranga
New Zealand
Córdoba
Spain
Auckland
New Zealand
Seville
Spain
Tangier
Morocco
Whangarei
New Zealand
Adamstown
Pitcairn Island
Doha
Qatar
Timbuktu
Mali
Suva
Fiji
Zinder
Niger
Pago Pago
American Samoa
Ulan Ude
Russia
Puerto Natales
Chile
Tianjin
China
Bahía Blanca
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Shanghai
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Uruguay
Hong Kong
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Humahuaca
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Palembang
Indonesia
Neiva
Colombia
Padang
Indonesia
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Ecuador
Asunción
Paraguay
Sharm el Sheikh
Egypt
Jaisalmer
India
Taipei
Taiwan
Hanga Roa
Easter Island
Rancagua
Chile
Xi'an
China
Rapa Iti
French Polynesia
Valdivia
Chile
Wuhai
China
Ghanzi
Botswana
Honolulu
USA
Perth
Australia
Hamilton
Bermuda
Port-aux-Français
French Southern and Antarctic Lands
Medicine Hat
Canada

Past work in spherical harmonics

- Radial transformation of RF feed horn patterns — radio astronomy cassegrain reflector systems (ATNF)
- Broadband and near-field arrays and beamforming — audio/acoustic applications
- Head-related transfer functions (HRTF) — hearing
- 3D-audio surround sound encoding and reproduction — multi eigen-channel rather than multi-channel
- 3D-audio microphone capture — spherical microphones
- MIMO (multi-antenna) communication channel modelling — 3D modal multipath modeling

Led to interest in **generic spherical methods** with many applications and **trawling SP and IT methods** for spherical signal applications.

- Slepian 1960s — Slepian on Sphere 1990's
- Time-frequency Methods — linear and Wigner-Ville (Cohen)
- Signals and Systems — functions and operators

$$\text{spatial in } L^2(\mathbb{S}^2) \longrightarrow \text{spectral in } \ell^{\,2}$$

For a complex-valued spatial signal on the 2-sphere, $f(\widehat{\boldsymbol{x}})$, the **Spherical Harmonic Transform (SHT)** is given by

$$(f)_\ell^m := \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}).$$

This is the natural Fourier transform on the sphere.

spatial in $L^2(\mathbb{S}^2) \longleftarrow$ spectral in ℓ^2

The **Inverse Spherical Harmonic Transform (ISHT)** is given by

$$f(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_\ell^m \, Y_\ell^m(\widehat{\boldsymbol{x}}),$$

where $(f)_\ell^m := \langle f, Y_\ell^m \rangle$. This is the inverse Fourier transform on the sphere.

spatial $\longrightarrow$ (joint) spatio-spectral on $\mathbb{S}^2 \times \mathbb{Z}$

$f(\widehat{\boldsymbol{x}}) \longrightarrow g(\widehat{\boldsymbol{x}}; \ell, m)$, using symmetric window $h(\widehat{\boldsymbol{x}})$

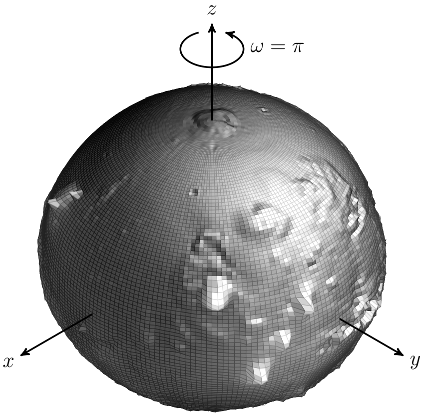
The **spatio-spectral SLSHT** [1, 2] was originally given by

$$g(\widehat{\boldsymbol{x}}; \ell, m) := \int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}}) \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}), \tag{1}$$

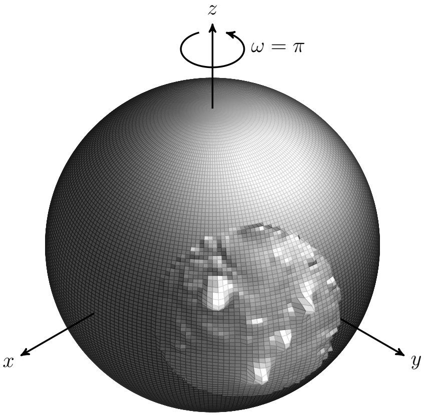
where $h(\widehat{\boldsymbol{y}})$ is an azimuthally symmetric window function satisfying

$$\langle h, Y_\ell^m \rangle = 0, \quad \forall m \neq 0. \tag{2}$$

$\big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}})$ is the window $h(\widehat{\boldsymbol{y}})$ rotated/centered about the point $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$.

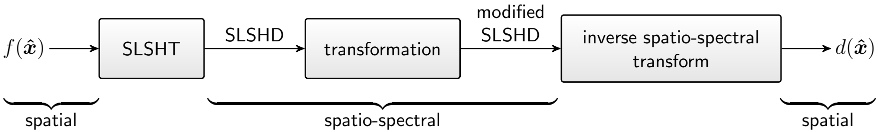


$f(\widehat{\boldsymbol{y}})$



$\big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})$

Why?


$f(\hat{\boldsymbol{x}})$
SLSHT
SLSHD
transformation
modified SLSHD
inverse spatio-spectral transform
$d(\hat{\boldsymbol{x}})$
spatial
spatio-spectral
spatial

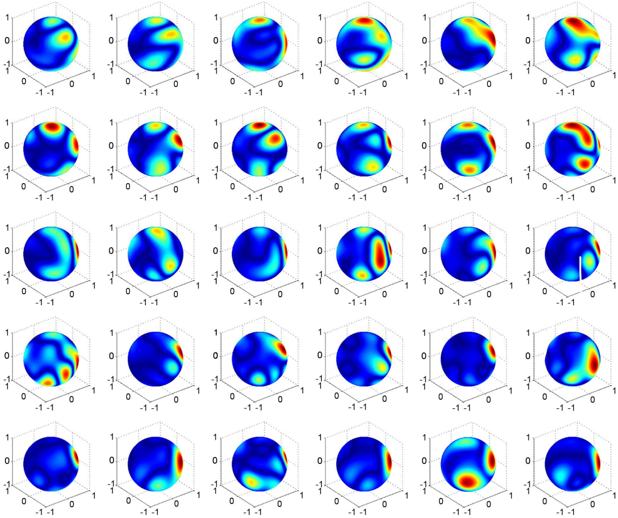
$\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})$	SHT of f
$\int_{\mathbb{S}^2} h(\widehat{\boldsymbol{y}})\, f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})$	SHT of f with window
$\int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})$	SHT of f with rotated window
$\int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}})\overline{Y_\ell{}^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}})$	SLSHT spatio-spectral arguments

Evaluate this over all rotations $\widehat{\boldsymbol{x}}$ and all spectral indices ℓ, m and this gives you a spatio-spectral domain representation of f

- parameterised by the window h
- analog of short-time Fourier Transform
- can generalise to directional windows and SO(3) rotations with "fast" computational methods [3]
- information preserving with inverse (to recover f)
- spatio-spectral domain processing generally needs pseudo-inverse [4]

$$\mathbb{S}^2 \times \mathbb{Z}$$

Insanity, really, . . .



$$\int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}})$$

is not the best notation for the SLSHT because it really means two equations

$$\boxed{g(\widehat{\boldsymbol{x}};\ell,m) := \int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}}),}$$

where $h(\widehat{\boldsymbol{y}})$ is an azimuthally symmetric window function satisfying

$$\boxed{\langle h, Y_\ell^m \rangle = 0, \quad \forall m \neq 0.}$$

- This **symmetry isn't well-exploited** in the expression for the SLSHT.
- **Two** equations.
- SLSHT processing when using this formulation [4] **isn't intuitive**.

The **spatio-spectral SLSHT** $g(\widehat{\boldsymbol{x}};\ell,m)$ has various interpretations. The window $h(\widehat{\boldsymbol{x}})$ is chosen to concentrate analysis into a local region on the sphere centered on the point $\widehat{\boldsymbol{x}}$:

- $g_{\widehat{\boldsymbol{x}}_0}(\ell,m) \coloneqq g(\widehat{\boldsymbol{x}}_0;\ell,m)$ — fixing the spatial $\widehat{\boldsymbol{x}}_0 \in \mathbb{S}^2$ and varying the spectral degree ℓ and order m we get information of **which spherical harmonics contribute most to explain that localized portion of the spatial signal $f(\widehat{x})$ within the windowed region**; and
- $g_{\ell_0,m_0}(\widehat{\boldsymbol{x}}) \coloneqq g(\widehat{\boldsymbol{x}};\ell_0,m_0)$ — fixing both the spectral degree ℓ_0 and order m_0 and varying spatially $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$ we can infer from **which parts of the sphere the signal most strongly contribute to the (global) spherical harmonic coefficient**.
- tears apart the spatial signal on the sphere $\mathbb{S}^2$ into a cartesian product domain $\mathbb{S}^2 \times \mathbb{Z}$
- but not the only way to move the signal to a cartesian domain

- Seems slightly different to short-time Fourier Transform where the cartesian product domain is $\mathbb{R} \times \mathbb{R}$ for time-frequency. Somewhat symmetric, or self-dual or something.
- For spatio-spectral SLSHT the two domains $\mathbb{S}^2$ and $\mathbb{Z}$ seem world's apart. (Here, recall, $\mathbb{Z}$ is the single countable indexing of degrees ℓ and orders m.)
- Let's fix that.

The **Spatio-Spectral SLSHT** [1] given by (1) and (2) can be written

$$\boxed{g(\widehat{\boldsymbol{x}};\ell,m) = \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}}),} \tag{3}$$

where the kernel is

$$H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}}) \coloneqq \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}}) \tag{4}$$

and coefficients $(h)_\ell^0 = \langle h, Y_\ell^0\rangle$.

- $H(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}) = H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})$ is an isotropic convolution kernel.
- Follows from the Spherical Harmonic Addition Theorem.
- (3) is self-contained and completely defines the Spatio-Spectral SLSHT.
- (4) just shows the relationship to the symmetric spatial window $h(\widehat{\boldsymbol{x}})$.
- Not a profound improvement but cleaner and simpler.

Now for something weird. . .

- Fourier Transform, Spherical Harmonic Transform, Spatio-Spectral SLSHT are **Integral Transforms**
- (Also for discrete spectra the inverse transforms being series are essentially Integral Transform-like)
- So it is strange to get, next, a non-integral transform. In fact it looks odd, simple and probably useless (or is it?).
- (It has an odd-looking name.)

The **Spatio-Spatial SLSHT** is defined by

$$g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}}) := \sum_{\ell,m} g(\widehat{\boldsymbol{x}};\ell,m) Y_\ell^m(\widehat{\boldsymbol{z}}), \quad \widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}} \in \mathbb{S}^2, \tag{5}$$

which is the ISHT in the 2nd argument of (3).

In kernel form, **Kernel-Based Spatio-Spatial SLSHT**,

$$\boxed{g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}}) := H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}}) f(\widehat{\boldsymbol{z}}), \quad \widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}} \in \mathbb{S}^2,} \tag{6}$$

- No integral. No Integral Transform.
- Just the **integrand** in the Kernel-Based Spatio-Spectral SLSHT (3)
- Here the cartesian product domain is $\mathbb{S}^2 \times \mathbb{S}^2$ — didn't we want that?
- Holds **the same information** as the original Spatio-Spectral SLSHT

$$\boxed{g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}}) := H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})f(\widehat{\boldsymbol{z}}), \quad \widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}} \in \mathbb{S}^2,}$$

- $\widehat{\boldsymbol{x}}$ is the spatial portion ("Spatio-")
- $\widehat{\boldsymbol{z}}$ is the **spatial representation of the spectral portion** ("Spatial")
- on $\mathbb{S}^2 \times \mathbb{S}^2$
- inversion example:

$$f(\widehat{\boldsymbol{x}}) = \frac{1}{H(1)} g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{x}}), \quad \text{where } H(1) \neq 0$$

- leads to fast new spatio-spectral transforms and inverses [5]
- leads to $O(L)$ improvement over existing (non-symmetric) SS-algorithms

Another transform.

The **Kernel-Based Spectro-Spatial SLSHT** of a signal $f(\cdot)$ can be written

$$\boxed{g(p,q;\widehat{\boldsymbol{z}}) := f(\widehat{\boldsymbol{z}}) \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{z}}) \overline{Y_p^q(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}), \quad \widehat{\boldsymbol{z}} \in \mathbb{S}^2,} \tag{7}$$

using the isotropic convolution kernel $H(\cdot)$, given in (4).

- p, q is the **spectral representation of the spatial portion** ("Spectro-")
- $\widehat{\boldsymbol{z}}$ is the **spatial representation of the spectral portion** ("Spatial")
- LSD-version

Yet another transform.

The **Kernel-Based Spectro-Spectral SLSHT** of a signal $f(\cdot)$ can be written

$$\boxed{g(p,q;\ell,m) := \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})\, f(\widehat{\boldsymbol{y}})\,\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,\overline{Y_p^q(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{y}})ds(\widehat{\boldsymbol{x}}),} \tag{8}$$

using the isotropic convolution kernel $H(\cdot)$, given in (4).

- p, q is the **spectral representation of the spatial portion** ("Spectro-")
- ℓ, m is the spectral portion ("Spectral")
- cartesian product domain is $\mathbb{Z}\times\mathbb{Z}$ (nice)
- naturally get here computationally but tends to be a false-friend

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
g(\widehat{\boldsymbol{x}}; \ell, m) \arrow[r, dashed, shift left, "\text{2-ISHT}"] \arrow[d, shift left, "\text{1-SHT}"] & g(\widehat{\boldsymbol{x}}; \widehat{\boldsymbol{z}}) \arrow[l, shift left, "\text{2-SHT}"] \arrow[d, shift right, "\text{1-SHT}"'] \\
g(p, q; \ell, m) \arrow[u, dashed, shift left, "\text{1-ISHT}"] \arrow[r, dashed, shift right, "\text{2-ISHT}"'] & g(p, q; \widehat{\boldsymbol{z}}) \arrow[l, shift right, "\text{2-SHT}"'] \arrow[u, dashed, shift right, "\text{1-ISHT}"']
\end{tikzcd}
$$

Figure: Transformations between the four variants of the SLSHT. In notation, 1-(I)SHT/2-(I)SHT denotes the (I)SHT on the 1st/2nd argument.

- Write the integral transform using kernel K as $\mathcal{L}_K$. With K positive definite then we can work in a "smoother and smaller" space

$$\mathcal{L}_K^{1/2}\colon f \in L^2(\mathbb{S}^2) \longrightarrow h = \mathcal{L}_K^{1/2} f$$

  This describes all the functions in the new space as the **low-pass filtered** versions of finite energy functions.
- If the filtering is strong enough (Hilbert-Schmidt) then we can manufacture RKHS's.
- Strict band-limiting to degree L is a degenerate version of this, ends up being a subspace.
- The square root operator generates of new smooth space that generally is **not** a subspace of $L^2(\mathbb{S}^2)$. The inner-product is different to incorporate the decay of the eigenvalues.

- Can generalise the concentration measures in spatial and spectral domains, without damaging the nice properties — incorporate different tapers.
- "Remarkable" dual-orthogonality of Slepian eigen-functions is a general feature of an abstract problem, not a happy accident. So there are Slepian-like alternatives, for example, which might suppress the wildness of the eigenfunctions near the edges of spatial regions.

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