Introduction to Complex Numbers

ENGN2228 Signal Processing
Tutorial 0

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Introduction of j

- To see a complex number we have to first see where it shows up.
- Solve both of these

$$\begin{aligned} x^2 - 81 &= 0 \\ x^2 &= 81 \\ x &= \pm 9 \end{aligned} \qquad \begin{aligned} x^2 + 81 &= 0 \\ x^2 &= -81 \end{aligned}$$

- $x = \pm\sqrt{-81}$ does not have a real solution; but it has an *imaginary* answer by creating a new variable j

$$j = \pm\sqrt{-1}$$

Definition of j

- j is an representation of $\sqrt{-1}$.
- The hint to deal with j is to treat j like a variable.

$$x = \pm\sqrt{-81} = \pm\sqrt{81}\sqrt{-1} = \pm 9j$$

- It cycles

$$j^2 = -1,\ j^3 = -j,\ j^4 = 1, \cdots$$

Every j^4 does not matter. For larger exponents, divide the exponent by 4, then use the remainder as your exponent instead.

$$j^{23} = j^3 = -j$$

Definition of complex numbers

- A complex number has two parts, a real part and an imaginary part

$$z = x + jy$$

with the real part $Re\{z\} = x$ and the imaginary part $Im\{z\} = y$.

- All real numbers are complex, $a = a + 0j$.
All imaginary numbers are complex, $bj = 0 + bj$.
- Elementary operations: *treat j as a variable*

$$\begin{aligned}
\text{Addition :}&(a+bj)+(c+dj) = (a+c)+j(b+d) \\
\text{Subtraction :}&(a+bj)-(c+dj) = (a-c)+j(b-d) \\
\text{Multiplication :}&(a+bj)\times(c+dj) = (ac-bd)+j(ad+bc) \\
\text{Division :}&\frac{a+bj}{c+dj} = \frac{(a+bj)(c-dj)}{(c+dj)(c-dj)} = \frac{ac+bd+j(bc-ad)}{c^2+d^2}
\end{aligned}$$

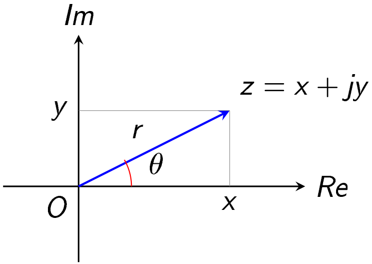
Examples

$$
\begin{aligned}
(8+3j)+(6-2j) &= (8+6)+j(3-2) = 14-j \\
(8+3j)-(6-2j) &= (8-6)+j(3-(-2)) = 2+5j \\
(8+3j)\times(6-2j) &= 48+18j-16j-6j^2 = 54+2j \\
\frac{(8+3j)}{(6-2j)} &= \frac{(8+3j)(6+2j)}{(6-2j)(6+2j)} = \frac{42+34j}{40}
\end{aligned}
$$

Addition and Subtraction: add or subtract the real parts, then add or subtract the imaginary parts.

Multiplication: Treat the j's like variables, then change any that are not to the first power.

Polar form



Cartesian form : $z = x + jy$

Magnitude : $r = |z| = \sqrt{x^2 + y^2}$

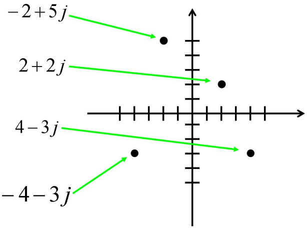
Angle : $\theta = \arg z$

$$x = r \sin \theta$$

$$y = r \cos \theta$$

Polar form : $z = re^{j\theta}$

Euler's relation : $e^{j\theta} = \cos \theta + j \sin \theta$



Examples: Polar coordinates and Cartesian coordinates

Multiplication and division in polar form

$$\begin{aligned} z_1 \times z_2 &= r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}, \;\; z^n = r^n e^{jn\theta} \\ \frac{z_1}{z_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \end{aligned}$$

$$z_1 = r_0 e^{-j\theta_0} \qquad x_1 = r_0 \cos\theta_0, \quad y_1 = -r_0 \sin\theta_0$$

$$z_2 = r_0 \qquad x_2 = r_0, \quad y_2 = 0$$

$$z_3 = r_0 e^{j(-\theta_0+\pi)} \qquad x_3 = -r_0 \cos\theta_0, \quad y_3 = r_0 \sin\theta_0$$

$$1 + j\sqrt{3} \qquad 2(1/2 + j\sqrt{3}/2) = 2e^{j\pi/3}$$

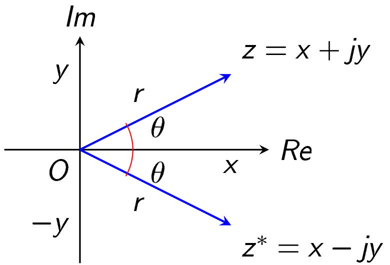
$$(\sqrt{3} + j^3)(1 - j) \qquad 2e^{-j\pi/6}\sqrt{2}e^{-j\pi/4} = 2\sqrt{2}e^{-j5\pi/12}$$

$$\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}} \qquad \frac{1/2 + j\sqrt{3}/2 - 1}{1 + j\sqrt{3}} = \frac{e^{j2\pi/3}}{2e^{j\pi/3}} = \frac{1}{2}e^{j\pi/3}$$

Complex conjugate

- The complex conjugate of a complex number $z = x + jy$ is

$$z^* = x - jy = re^{-j\theta}$$



- Geometrically, z^* is the "reflection" of z about the real axis. In particular, conjugating twice gives the original complex number, $(z^*)^* = z$.

Examples: Deriving elementary operations

$$z + z^* = x + jy + x - jy = 2x = 2Re\{z\}$$

$$|z| = |re^{j\theta}| = r = |re^{-j\theta}| = |z^*|$$

$$(z_1 + z_2)^* = ((x_1 + x_2) + j(y_1 + y_2))^* = x_1 - jy_1 + x_2 - jy_2 = z_1^* + z_2^*$$

$$(z_1 z_2)^* = (r_1 e^{j\theta_1} r_2 e^{j\theta_2})^* = r_1 e^{-j\theta_1} r_2 e^{-j\theta_2} = z_1^* z_2^*$$

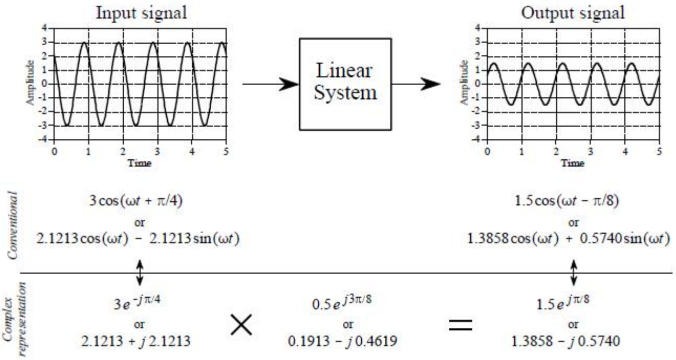
$$(e^z)^* = (e^{x+jy})^* = (e^x e^{jy})^* = e^x e^{-jy} = e^{x-jy} = e^{z^*}$$

Complex numbers in signal analysis: Compact representations

- Complex representation of Sinusoids

$$M\cos(wt) = MRe\{e^{j(wt)}\}, \; M\sin(wt) = MIm\{e^{j(wt)}\}$$

- Complex representation of Systems



Examples: Integral of exponentials

$$\int e^{at} dt = \frac{e^{at}}{a}$$

$$\int_0^4 e^{j\pi t/2} dt = \frac{e^{j\pi t/2}}{j\pi/2}\Big|_0^4 = \frac{e^{j2\pi} - 1}{j\pi/2} = 0$$

$$\int_0^\infty e^{-(1+j)t} dt = \frac{e^{-(1+j)t}}{-(1+j)t}\Big|_0^\infty = \frac{0-1}{-(1+j)} = \frac{1}{1+j} = \frac{1-j}{2}$$

$$\int_0^\infty e^{-t}\cos(t) dt = \int_0^\infty e^{-t}\frac{e^{jt} + e^{-jt}}{2} dt = \frac{1}{2}$$

Examples: Summation of exponentials

1. Finite sum formula

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \ (Hint: (1-\alpha)\sum_{n=0}^{N-1} \alpha^n) \end{cases}$$

2. Infinite sum formula

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \ |\alpha| < 1$$

3.

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{1-\alpha}, \ |\alpha| < 1$$

4.

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}, \ |\alpha| < 1$$