Signal Processing
ENGN2228

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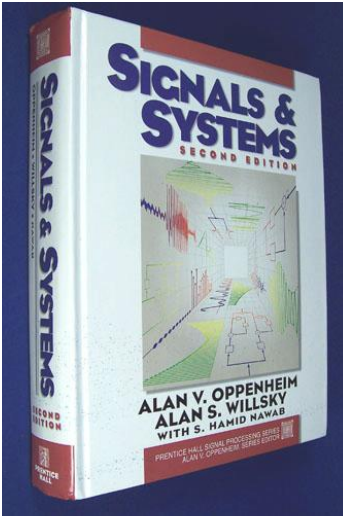
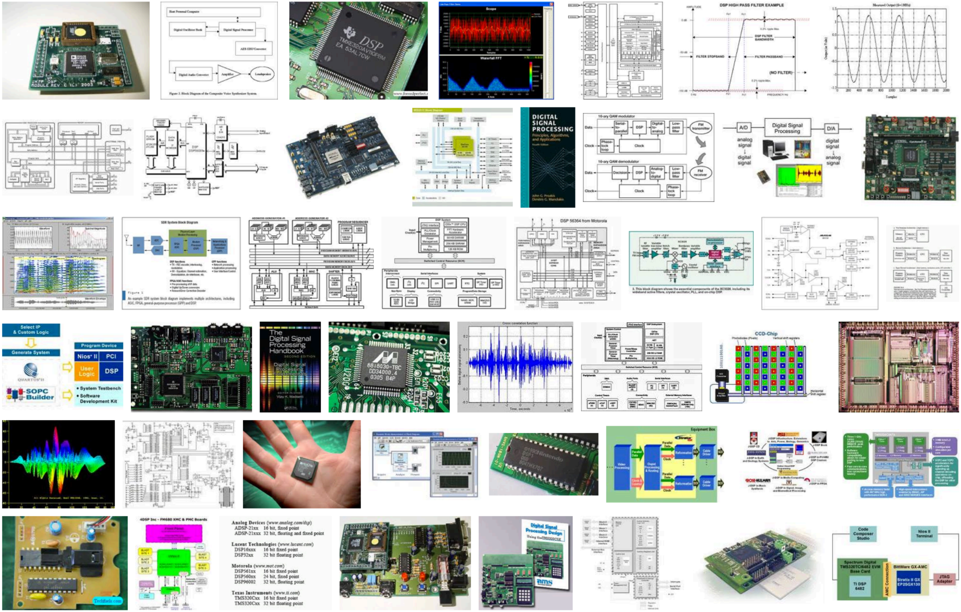
Text for the Course



Signals & Systems
Oppenheim, Willsky and Nawab
2nd Edition
Prentice Hall

What is (Digital) Signal Processing?



About this Course

Understanding:

1. what is a **signal**
2. what is a **system**
3. that signals and systems arise in **diverse application areas**: from communications, control of processes, seismology, speech processing, aeronautics, etc. Although the areas are diverse, they share a common underlying set of common features which enable us to understand and model them with

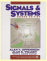
   one set of tools and one set of ideas and concepts
4. what it means for a system to be **linear**
5. the use of **frequency domain and Fourier techniques** to design and analyze signals and systems

About this Course - Uses

Essential material for:

1. any work in **physical layer communications** including **wireless** communications: IEEE 802.11g
2. understanding and defining **engineering specifications** on physical systems; spectral allocation for radio, DTV, emergency services, communications,
3. any work in the **audio and acoustic** field, digital audio, amplifiers, speaker and cross-over design, microphones; multichannel surround sound for entertainment: Dolby TrueHD, Dolby Digital Plus, DTS-HD Master Audio , Dolby Digital EX, DTS-ES Discrete 6.1, DTS 96/24, etc
4. **stability** and vibrational mode analysis; stop bridges collapsing
5. **control** system design and analysis; how to keep a plane in the sky
6. **image processing**, medical imaging, ultrasound processing; imaging the guts of people to identify disease (more reliably than a radiologist)
7. **electronics**, circuit design and analysis, filter design; what doesn't use electronics?

What are Signals?



Definition (Signals)

Signals carry information in their variations. Mathematically signals are functions of one or more independent variables.

Examples of signals are:

- speech and audio signals, output from a microphones
- electrical signals, television, radio, digital radio,
- voltages, currents
- images, stock market prices, currency exchange rates
- temperature, weather data
- biological signals
- ...

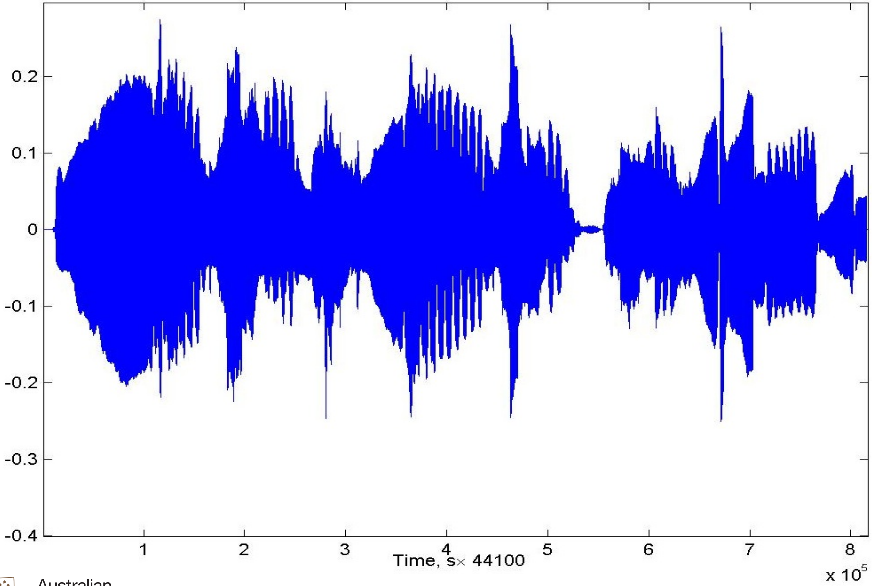
Signals are generally connected with physical quantities that vary with time or space or both.

Examples of Signals – Currency (vs time)

Examples of Signals – Share Price (vs time)

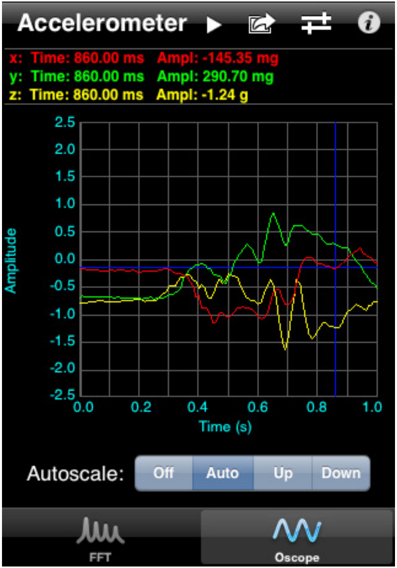
Part 1 Slide 11/many Lecturer: R. A. Kennedy

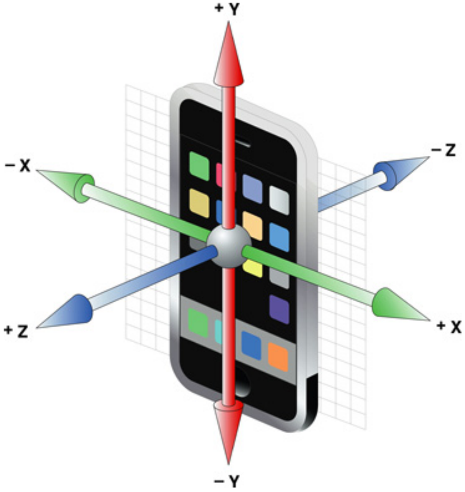
Examples of Signals – Flute Sound Waveform (vs time)



Examples of Signals – An Image (vs pixel space)

Examples of Signals – iPhone Accelerometers (vs time)





Independent Variables (types)

Signals vary with respect to the "independent variables"

- **Continuous**
  - air temperature across Australia (latitude and longitude)
  - trajectory of a bullet fired at a monkey falling from a helicopter into a volcano (in space)
- **Discrete**
  - digital image pixels (xy), 3D medical image voxels (xyz)
  - DNA base sequence: $\cdots$AATATAGACCGACCCTAATTAAGTAAAATAGACCT$\cdots$
  - digital audio (time samples)

Independent Variables – (dimension)

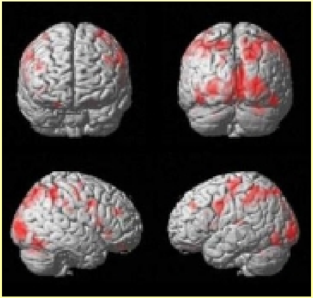
The independent variables can be one dimensional 1D, 2D, 3D, etc.

- A signal may vary with time (1D)
- An image varies with cartesian coordinates x and y in space (2D)
- The temperature can vary with position in a room horizontal x and y, and vertical z (3D)
- A movie is a 2D image that varies with time (3D)
- What dimension is a 3D movie?

Independent Variables – (dimension)

- fMRI – functional Magnetic Resonance Imaging – 3D volume of patient's brain is imaged every one or two seconds (4D, i.e., 3 space and 1 time dimensions)

Oversized pencil sharpener



fMRI

Independent Variables – (course focus on 1D)

- For this course we will focus on 1D
- That is, a single independent variable
- Most cases this is "time"

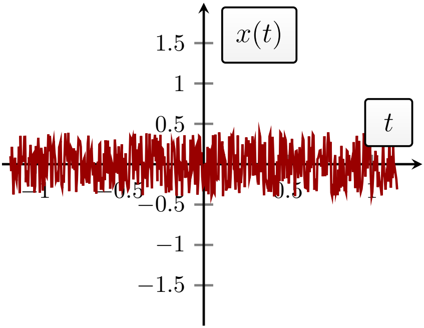
Definition (Continuous-Time, CT, Signals)

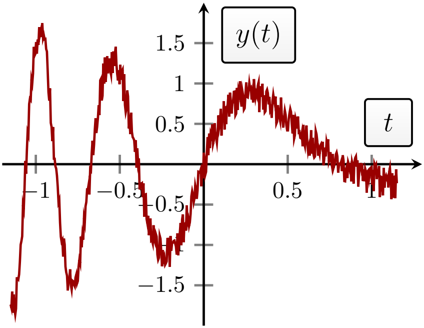
Continuous-Time Signals are signals whose independent variable is **continuous** and taken as time. That is, $x(t)$ with continuum t.

$$\boxed{x(t), \quad t \in \mathbb{R} \text{ (real numbers/time)}}$$

- Signals from the real physical world are generally CT, such as voltage, pressure, velocity, etc, as functions of time t
- Examples of real-valued CT signals (functions) are shown next: one noisy and the other some dying signal with additive noise.
- Followed by an example of a complex-valued CT signal.

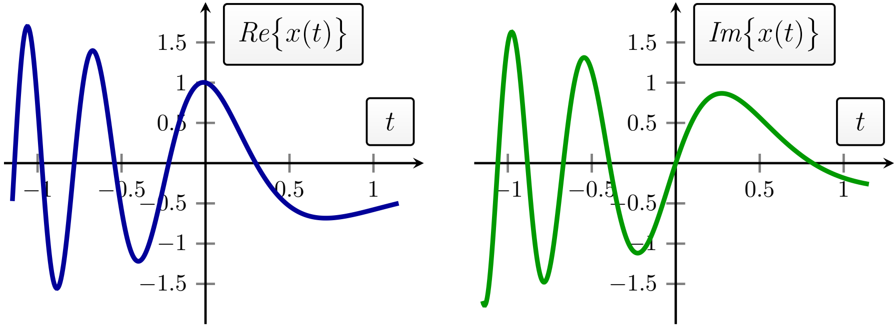
Continuous Time Signals – Examples (note (t))





Continuous Time Signals – Complex Example

$$x(t) \triangleq \exp\big(2\pi j\, t\, \exp(-0.6\, t)\big)\exp(-0.5\, t) \in \mathbb{C}$$



Note: $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$ and has period 2π

Discrete Time Signals

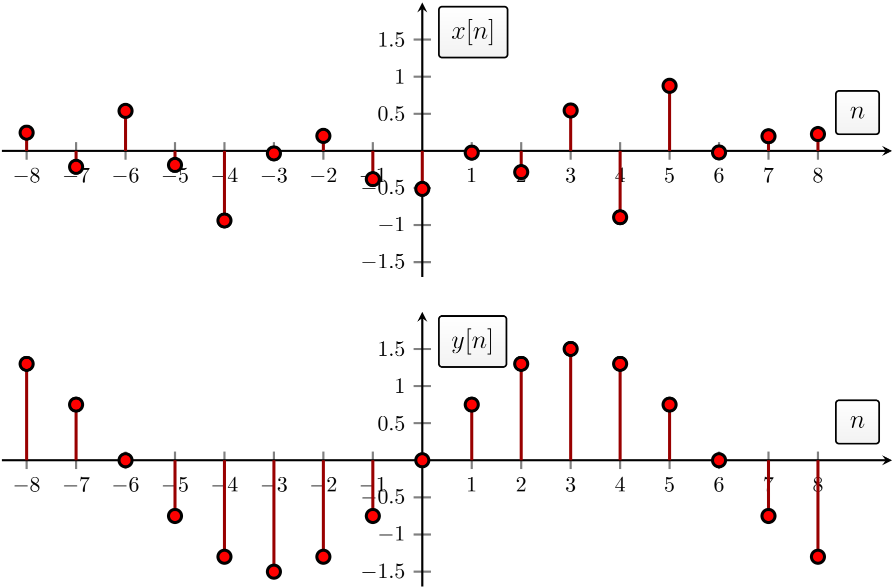
Definition (Discrete-Time (DT) Signals)

Discrete-Time Signals are signals whose independent variable takes on only a **discrete set of values** and are generally taken to be integer values. That is, $x[n]$ with discrete/integer n

$$\boxed{x[n], \quad n \in \mathbb{Z} \text{ (integers)}}$$

- Signals from the real physical world are generally not naturally DT, but most man-made and "sampled" signals are DT
- Examples of real-valued DT signals (discrete functions) are shown next slide:
  - $x[n]$ is somewhat random
  - $y[n]$ looks like a sampled sinusoid

Discrete Time Signals – Real-Valued Examples (note $[n]$)



Discrete Time Signals

Reminder:

Continuous-Time (CT) signals: $x(t)$ with t – continuous values[1]

Discrete-Time (DT) signals: $x[n]$ with n – integer values[1]

[1] Round brackets $\equiv$ continuous. Square brackets $\equiv$ discrete.

Discrete Time Signals

- Natural DT signals? Less common, e.g., DNA base sequence
- Most DT signals are man-made
- Images, digital music, stock market data, etc

DT signals are increasingly important because they are in a form that permit calculations, that is, "processing", via computers (CPUs), graphics processing units (GPU's) and digital signal processors (DSPs)

i) Intel CPU, ii) ATI RADEON GPU and iii) TI TMS320 DSP

- Notions of energy dominate our lives. In loose everyday jargon and in more rigorous scientific/engineering notions.
- There are well-defined physical notions of energy and power, and these guide developing analogous notions of energy and power for **signals**.
- We are interested in having notions of: i) finite (total) energy signals; ii) finite (average) power signals, and iii) signals where the neither energy nor power are finite (see ahead a few slides)

Definition (Power vs Energy)

Power (Watts) is energy (Joules) transferred per unit of time (seconds).

- ○ Power is the rate at which energy is delivered.
- ○ Understand the difference between energy and power.
- ○ Understand that energy and power can be defined for signals (which may have no direct physical meaning but is well-defined nonetheless).

Signal Energy and Power (physical motivation)

For a resistor R continuous time **instantaneous power** in a circuit is the product of voltage and current

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t) = R\,i^2(t)$$

The **total energy** dissipated from time t_1 to time t_2 is

$$\boxed{\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt}$$

and the **average power** over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

such that the energy delivered over the time interval $t_2 - t_1$ at the average power equals the total energy. It is the mean rate that energy is delivered.

Signal Energy and Power (signal abstraction)

Definition (Total energy of a continuous time signal $x(t)$)

Total energy of a continuous time signal $x(t)$ between real times instants t_1 and t_2 is

$$\boxed{\int_{t_1}^{t_2} \big|x(t)\big|^2 \, dt}$$

Definition (Total energy of a discrete time signal $x[n]$)

Total energy of a discrete time signal $x[n]$ between integer time instants n_1 and n_2 is

$$\boxed{\sum_{n=n_1}^{n_2} \big|x[n]\big|^2}$$

- Signals can be real or complex valued; $|\cdot|$ means "absolute" value for real and "magnitude" for complex
- Don't worry about the physical interpretation; this is just a definition

Signal Energy and Power (Infinite Time Interval Energy)

Definition (Infinite time interval total energy of a continuous time signal $x(t)$)

$$E_\infty = \lim_{T \longrightarrow \infty} \int_{-T}^{T} \left| x(t) \right|^2 dt$$

Definition (Infinite time interval total energy of a discrete time signal $x[n]$)

$$E_\infty = \lim_{N \longrightarrow \infty} \sum_{n=-N}^{N} \left| x[n] \right|^2$$

- Total energy may be finite or infinite.

Signal Energy and Power (Infinite Time Interval Time-Average Power)

Definition (Infinite time interval time-average power of a continuous time signal $x(t)$)

$$P_\infty = \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left|x(t)\right|^2 dt$$

Definition (Infinite time interval time-average power of a discrete time signal $x[n]$)

$$P_\infty = \lim_{N \longrightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2$$

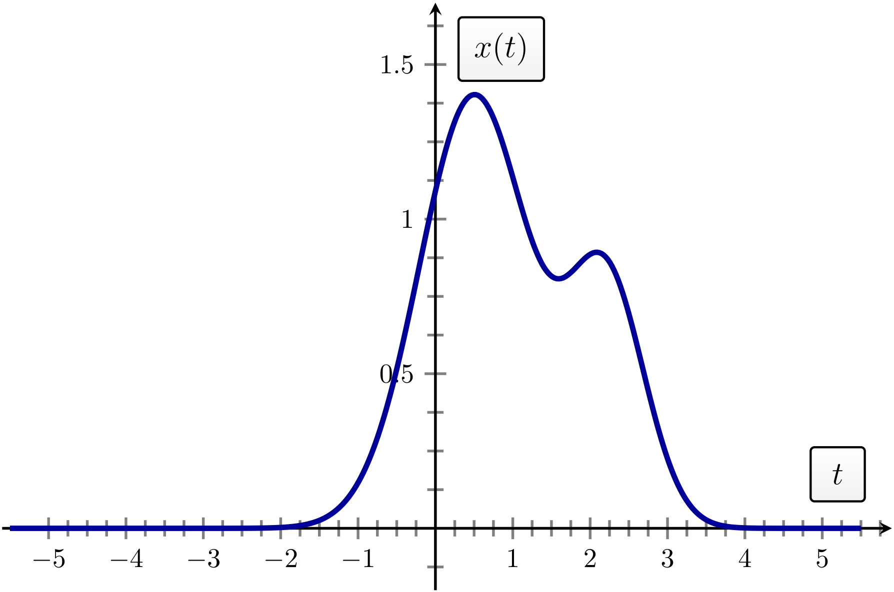
Signal Energy and Power (various cases)

- P_∞ may be finite or infinite. Natural signals are expected to be finite power.
- $E_\infty < \infty$ implies $P_\infty = 0$. Finite energy signals have zero average power over the infinite interval.
- $P_\infty > 0$ implies $E_\infty = \infty$. Finite average power signals end up delivering infinite energy over the infinite interval.
- Both $P_\infty = \infty$ and $E_\infty = \infty$ also mathematically possible, but not of much engineering interest.

Signal Transformations (independent var.)

- A signal $x(t)$ at some point transferred to another point (e.g., via communication) would generally suffer a time delay, $x(t-\Delta)$. This and other simple scenarios defines a simple but important class of **signal transformations** which are easily characterized.
- These signal transformations can be understood in terms of "affine" transformations on the independent variable
- For time as the independent variable, this just means scaling (compressing or expanding), reversing and shifting (delaying or advancing) time
- Other signal transformations not of this form are possible and treated later.

Signal Transformations (CT reference signal)



Time Shift

Independent variable change

$$t \longrightarrow t + \beta$$

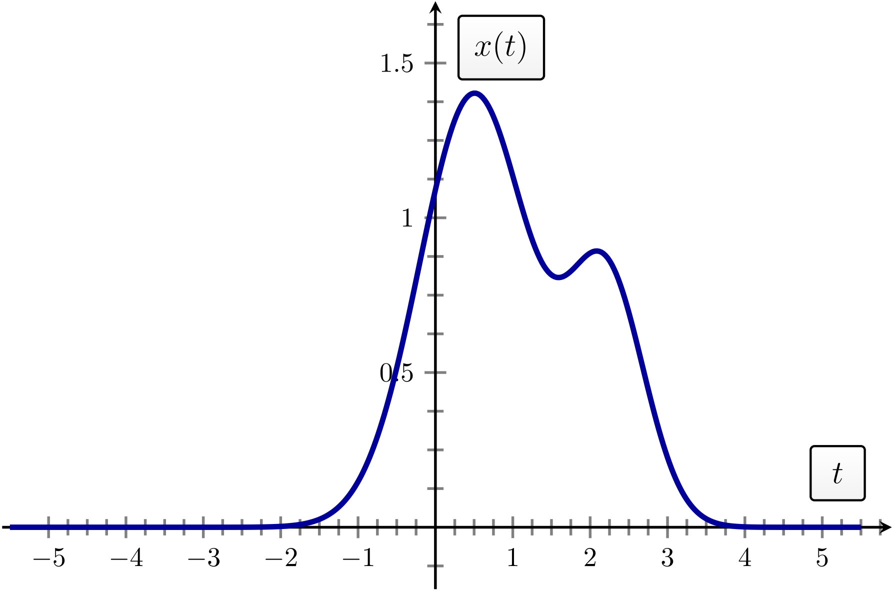
induces the signal change

$$x(t) \longrightarrow y(t) \triangleq x(t+\beta),$$

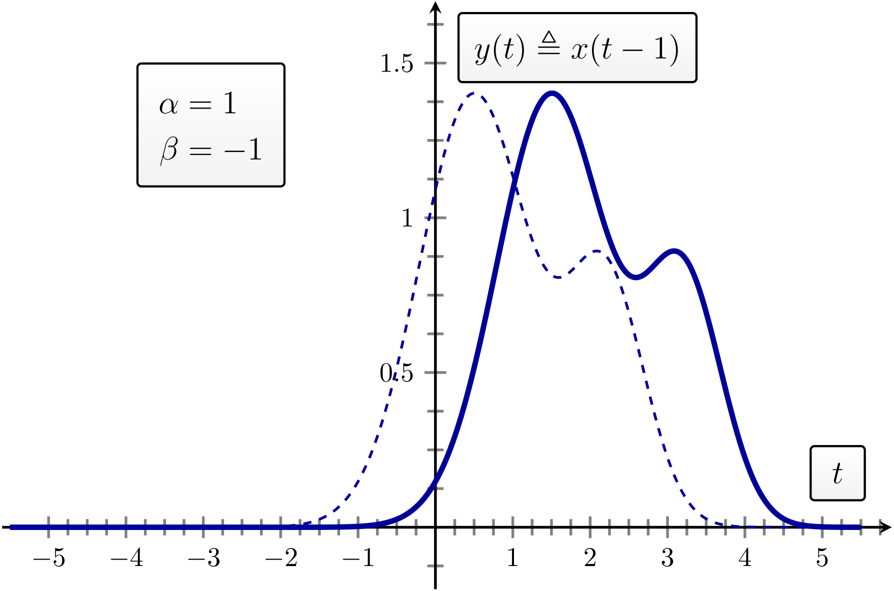
where $\beta \in \mathbb{R}$. If $\beta < 0$ then signal is delayed (shifts to the right), if $\beta = 0$ then no change, and if $\beta > 0$ then signal is advanced (shifts to the left).

(There is another parameter, α, which we'll meet shortly but for a time shift it is $\alpha = 1$.)

Signal Transformations (CT time shift)



Signal Transformations (CT time shift)



Signal Transformations (CT time reversal)

Time Reverse

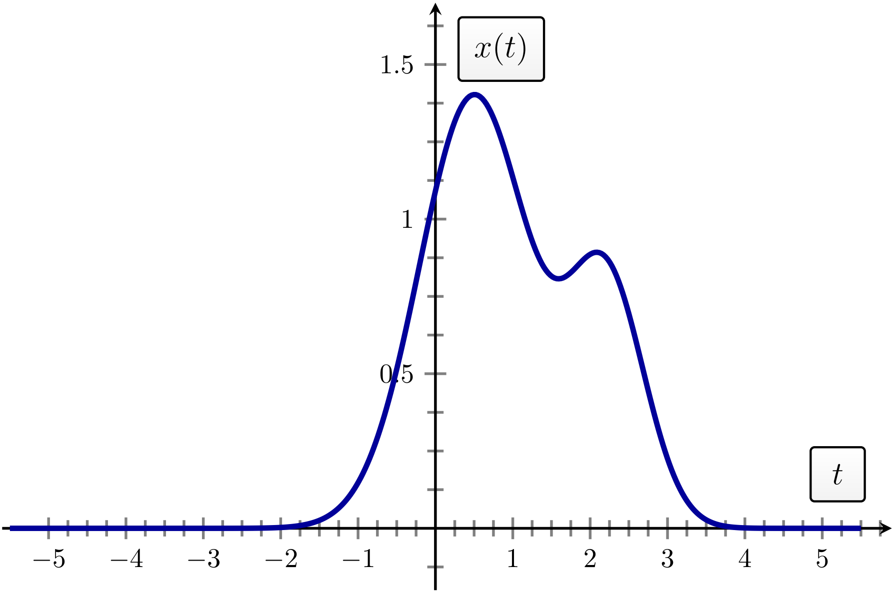
Independent variable change

$$t \longrightarrow -t$$

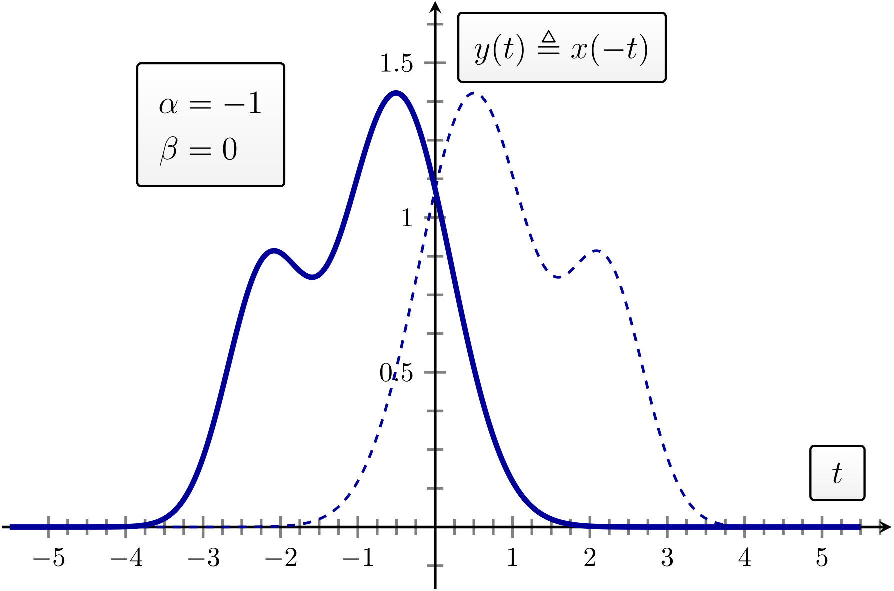
induces the signal change

$$x(t) \longrightarrow y(t) \triangleq x(-t)$$

Signal Transformations (CT time reversal)



Signal Transformations (CT time reversal)



Signal Transformations (CT time scaling)

Time Scaling

Independent variable change

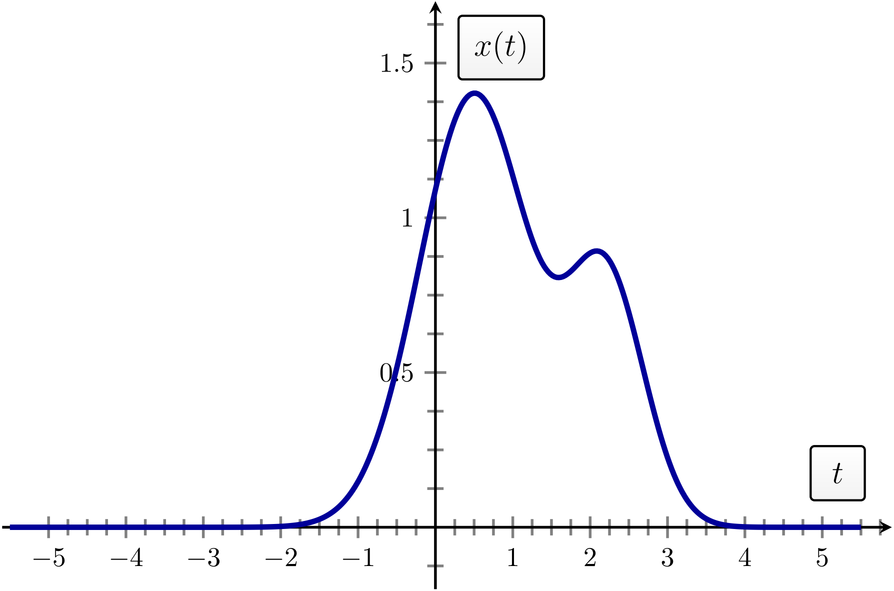
$$t \longrightarrow \alpha\, t$$

induces the signal change

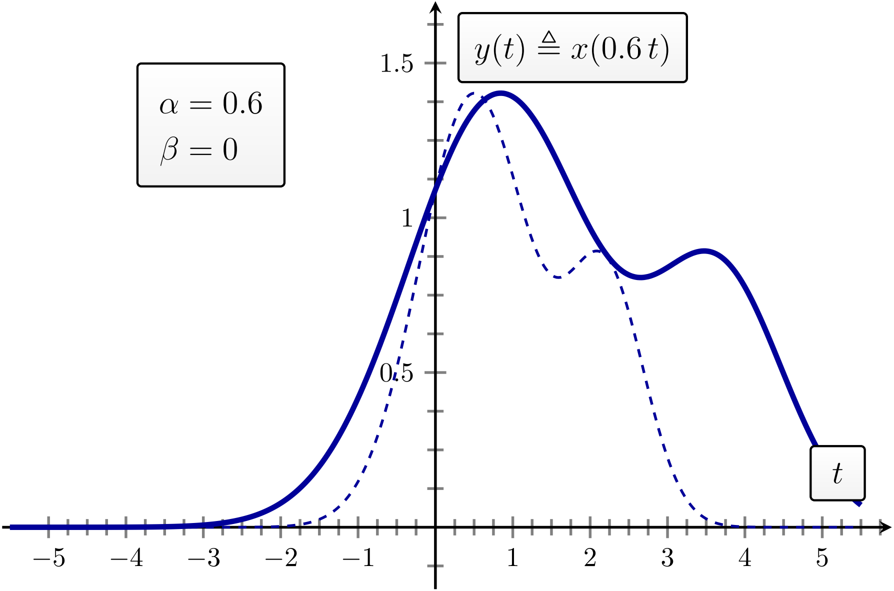
$$x(t) \;\longrightarrow\; y(t) \triangleq x(\alpha\, t),$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq 0$.

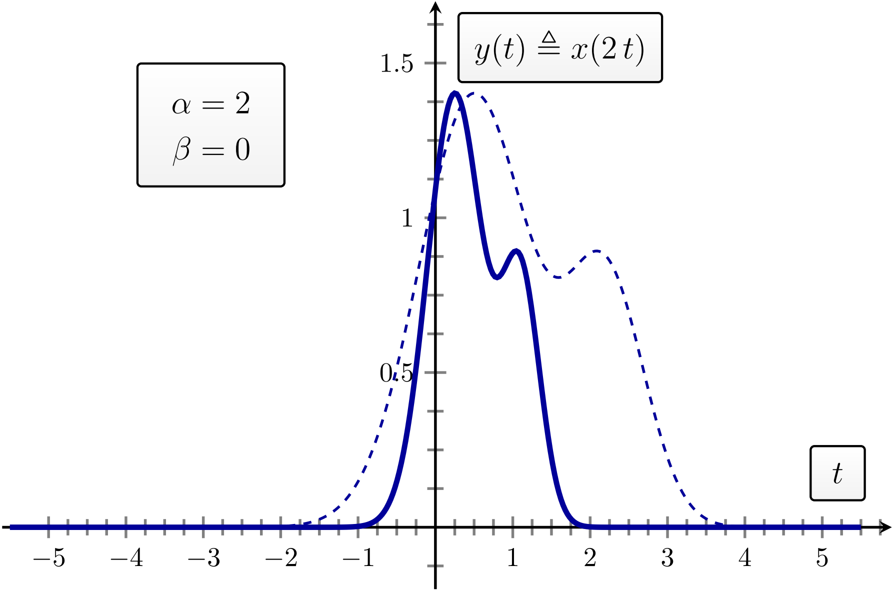
Signal Transformations (CT time scaling)



Signal Transformations (CT time scaling - expand)



Signal Transformations (CT time scaling - compress)



Signal Transformations (CT affine transformation)

Affine Transformation[2]

Independent variable change

$$t \longrightarrow \alpha\, t + \beta$$

induces the signal change

$$x(t) \longrightarrow y(t) \triangleq x(\alpha\, t + \beta)$$

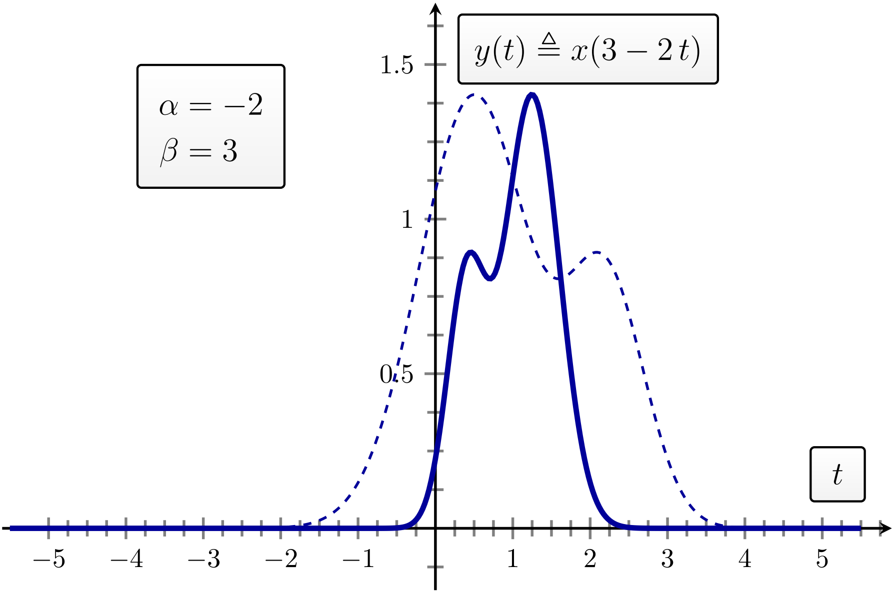
where

$\alpha \in \mathbb{R}$ **time scales** expands whenever $|\alpha| < 1$ and
compresses whenever $|\alpha| > 1$ and/or
time-reverses whenever $\alpha < 0$

$\beta \in \mathbb{R}$ **time shifts** forward in time whenever $\beta < 0$ and
backward in time whenever $\beta > 0$

[2] affine $\equiv$ linear + constant

Signal Transformations (CT affine time change)



Signal Transformations (independent variable)

- To this point we have only considered transformations

$$x(t) \longrightarrow y(t) \triangleq x(\alpha\, t + \beta)$$

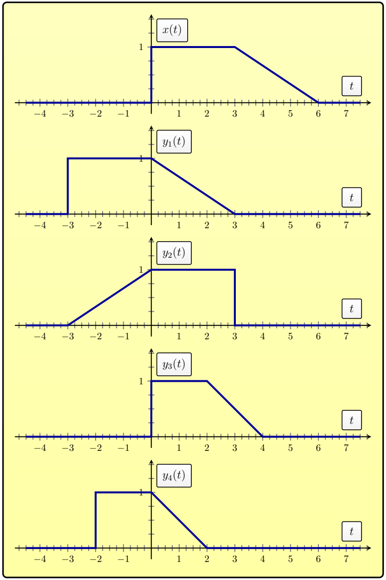
  which affinely transform the independent variable.

- But this **excludes** other simple signal transformations such as

$$\begin{aligned} x(t) &\longrightarrow y(t) \triangleq 3\, x(t) \\ x(t) &\longrightarrow y(t) \triangleq x(t) + 2.7 \\ x(t) &\longrightarrow y(t) \triangleq x(t^2) \\ x(t) &\longrightarrow y(t) \triangleq \big(x(t)\big)^5 \\ x(t) &\longrightarrow y(t) \triangleq 23 \end{aligned}$$

- Next, determine the transformation of $x(t)$ that yields the given $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$.

Signal Transformations (exercise)



- $x(t)$
- $y_1(t) = x(t+3)$, that is, $\alpha = 1 \quad \beta = 3$
- $y_2(t) = x(-t+3)$, that is, $\alpha = -1 \quad \beta = 3$
- $y_3(t) = x(1.5\,t)$, that is, $\alpha = 1.5 \quad \beta = 0$
- $y_4(t) = x(1.5\,t+3)$, that is, $\alpha = 1.5 \quad \beta = 3$

Periodic Signals (CT definition)

Definition (Periodic Continuous Time Signals)

CT Signal $x(t)$ is **periodic with period** $T > 0$ if

$$x(t) = x(t+T)$$

for all real $t \in \mathbb{R}$ (or $\forall t \in \mathbb{R}$).

- If the signal is given by

  $$x(t) = \sin(t),$$

  then $x(t) = x(t+2\pi)$, $\forall t$, and $x(t) = x(t+4\pi)$, $\forall t$, etc.
- So, $x(t) = \sin(t)$ is periodic with period $T = 4\pi$. But, of course, $T = 4\pi$ is not the smallest period. This motivates the following definition:

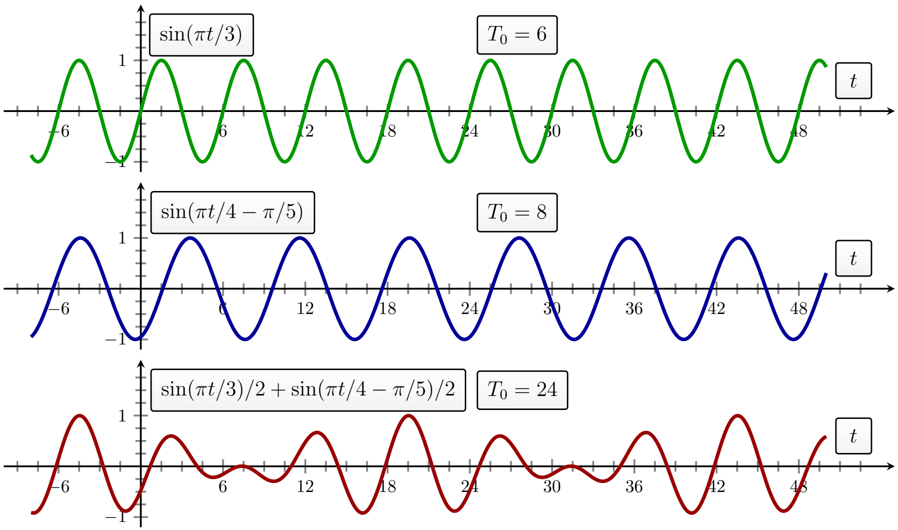
Periodic Signals (CT definition)

Definition (Fundamental Period T_0)

The fundamental period, $T_0 > 0$, is the **smallest positive period** T for which $x(t)$ is periodic (and non-constant).

- If $x(t) = \sin(t)$, then $T_0 = 2\pi$.
- Also $x(t) = \exp(jt)$ has fundamental period $T_0 = 2\pi$. So complex signals can be periodic too.
- $x(t) = \sin(4t)$ has fundamental period $T_0 = \pi/2$.
- $x(t) = \sin(3t)$ has fundamental period $T_0 = 2\pi/3$.
- $x(t) = \sin(t) + \sin(4t)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(3t) + \sin(4t)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(3t) + \sin(4t + \pi/7)$ has fundamental period $T_0 = 2\pi$.
- $x(t) = \sin(\pi\, t/3)/2 + \sin(\pi\, t/4 - \pi/5)/2$ has fundamental period $T_0 = 24$.

Examples of Signals – Example



Periodic Signals (DT definition)

Definition (Periodic Discrete Time Signals)

DT Signal $x[n]$ is **periodic with integer period** $N > 0$ if

$$x[n] = x[n + N]$$

for all integer $n \in \mathbb{Z}$.

Definition (Fundamental Period N_0)

The fundamental period, $N_0 > 0$, is the **smallest positive integer** N for which $x[n]$ is periodic.

Odd and Even Signals

Start with any signal $x(t)$, then we can write

$$\boxed{x(t) = \underbrace{\frac{1}{2}\big(x(t) + x(-t)\big)}_{x_e(t)} + \underbrace{\frac{1}{2}\big(x(t) - x(-t)\big)}_{x_o(t)}}$$

which, in a way, seems completely daft since we introduce a time reversed signal, $x(-t)$ which is cancelled.

Other notation (see O&W p.14)

$$Ev\{x(t)\} \triangleq x_e(t) = \frac{1}{2}\big(x(t) + x(-t)\big)$$

$$Od\{x(t)\} \triangleq x_o(t) = \frac{1}{2}\big(x(t) - x(-t)\big)$$

Odd and Even Signals (even signals)

We observed that

$$\begin{aligned} x_e(t) &= \frac{1}{2}\big(x(t) + x(-t)\big) \\ &= \frac{1}{2}\big(x(-t) + x(t)\big) \\ &= x_e(-t) \end{aligned}$$

Such as signal, which equals its time reverse, is called **even**.

Definition (Even Continuous Time Signals)

CT Signal $x(t)$ is **even** if

$$\boxed{x(-t) = x(t), \quad \text{for all } t}$$

Odd and Even Signals (odd signals)

We observed that

$$\begin{aligned} x_o(t) &= \frac{1}{2}\big(x(t) - x(-t)\big) \\ &= -\frac{1}{2}\big(x(-t) - x(t)\big) \\ &= -x_o(-t) \end{aligned}$$

Such a signal, which equals the negative of its time reverse, is called **odd**.

Definition (Odd Continuous Time Signals)

CT Signal $x(t)$ is **odd** if

$$\boxed{x(-t) = -x(t), \qquad \text{for all } t}$$

Odd and Even Signals (CT Signal Decomposition)

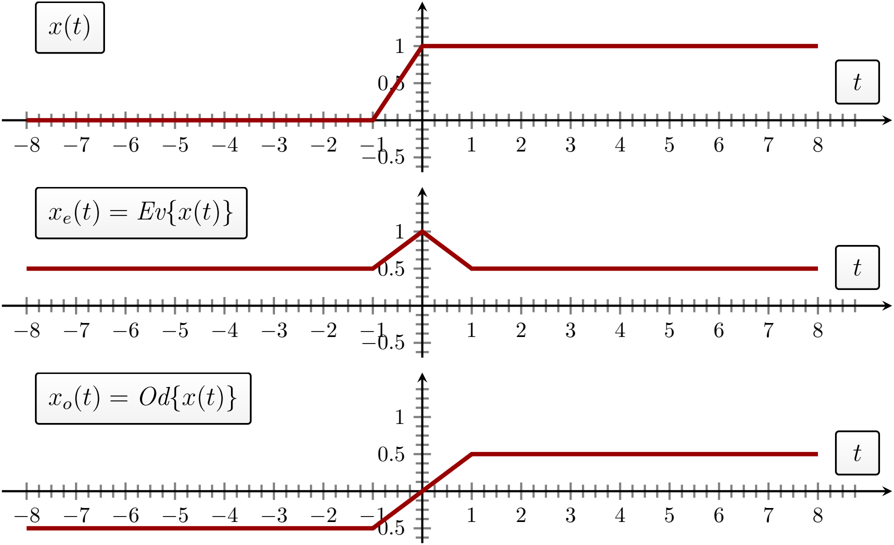
Revisiting

$$\boxed{x(t) = \underbrace{\frac{1}{2}\big(x(t) + x(-t)\big)}_{x_e(t)} + \underbrace{\frac{1}{2}\big(x(t) - x(-t)\big)}_{x_o(t)}}$$

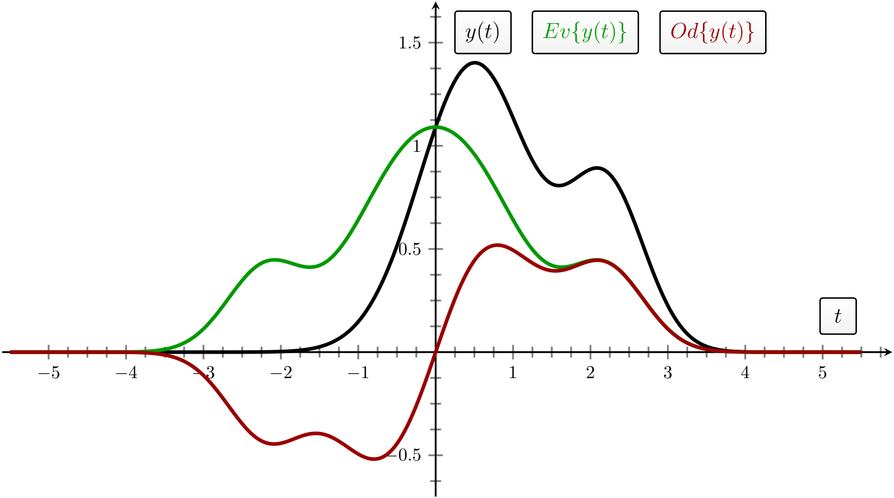
means we can always decompose any CT signal into the sum of an odd CT signal and an even CT signal.

Even signals are **symmetric** and odd signal are **anti-symmetric**. Both properties play an important role in simplifying signal processing.

Odd and Even Signals (CT Example)



Odd and Even Signals (CT 2nd Example)



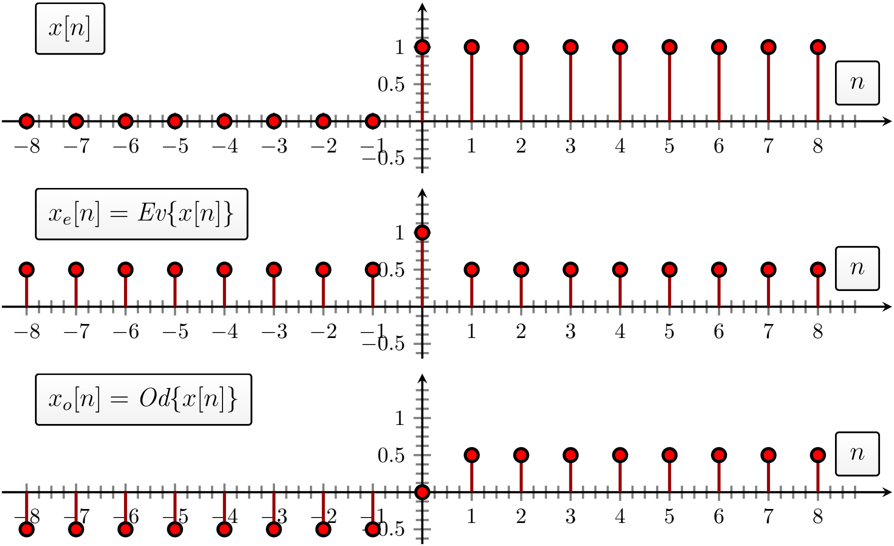
Odd and Even Signals (DT Signal Decomposition)

For DT signals

$$\boxed{x[n] = \underbrace{\frac{1}{2}\big(x[n] + x[-n]\big)}_{x_e[n]} + \underbrace{\frac{1}{2}\big(x[n] - x[-n]\big)}_{x_o[n]}}$$

means we can always decompose any DT signal into the sum of an odd DT signal and an even CT signal.

Odd and Even Signals (DT Example)



CT Exponential Signals

- A fundamental signal class is the complex exponential signals

$$x(t) = Ce^{\alpha t}$$

  with C and α complex numbers ($C, \alpha \in \mathbb{C}$).

- This is a compact way to represent: pure sinsuoids, real exponentials, exponentially growing or decaying sinusoids, etc. They are basic building blocks from which we can construct many signals of interest.

- With $C = |C|e^{j\theta}$ and $\alpha = r + j\omega_0$, we can write

$$x(t) = |C|e^{rt}e^{j(\omega_0 t + \theta)}$$

  Parameter $|C|$ is the magnitude, r is the exponential growth, ω_0 is the fundamental frequency of the oscillation in rad/sec and θ is the phase offset.

- We should be comfortable with $j \triangleq \sqrt{-1}$ such that $j^2 = -1$.

CT Exponential Signals

- Note, Euler's relation `O&W p.71`

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

which has fundamental period $T_0 = 2\pi/|\omega_0|$.

- Signal $x(t) = e^{j\omega_0 t}$ has finite, in fact unity, average power

$$\begin{aligned} P_\infty &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left|x(t)\right|^2 dt \\ &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt \\ &= 1 \end{aligned}$$

and (necessarily) infinite total energy, $E_\infty = \infty$.

CT Exponential Signals

- Many identities can be easily derived such as

$$
\begin{aligned}
A\cos(\omega_0 t+\phi) &= \frac{A}{2}e^{j\phi}e^{j\omega_0 t}+\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}\\
&= A\,\mathit{Re}\{e^{j(\omega_0 t+\phi)}\}
\end{aligned}
$$

$$
A\sin(\omega_0 t+\phi) = A\,\mathit{Im}\{e^{j(\omega_0 t+\phi)}\}
$$

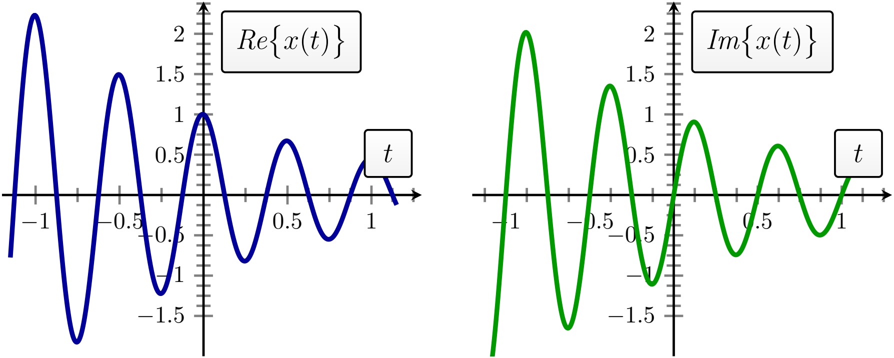
  See `O&W 1.3.1 p.17`

- Get comfortable and competent with such calculations.

CT Exponential Signals – Example

With $C = 1$ and $\alpha = -0.5 + j4\pi$ in $x(t) = Ce^{\alpha t}$ we have

$$\boxed{x(t) \triangleq \exp\left(4\pi j\, t\right) \exp(-0.5\, t) \in \mathbb{C}}$$



DT Exponential Signals

- A fundamental signal class is the class of DT complex exponential signals

$$x[n] = C\alpha^n, \quad n \in \mathbb{Z}$$

with C and α complex numbers ($C, \alpha \in \mathbb{C}$).

- Generally DT signals emulate the properties of CT signals or vice versa.
- Slightly weird things can emerge for DT signals though. You should rely on mathematical analysis to resolve any confusion rather than trying to memorize any details. However, you should make a mental note to be wary.
- First we look at the case where $|\alpha| = 1$ which we can write in the form

$$x[n] = e^{j\omega_0 n}, \quad n \in \mathbb{Z},$$

that is, $C = 1$ and $\alpha = e^{j\omega_0}$ for some $0 \le \omega_0 < 2\pi$.

Periodicity of DT Exponential Signals

- Note that CT signal $x(t) = e^{j\omega_0 t}$ is **periodic** (with fundamental period $T_0 = 2\pi/|\omega_0|$) for all values of $\omega_0 \neq 0$). This is not quite true of the analogous DT exponential signal.
- For
$$x[n] = e^{j\omega_0 n}, \quad n \in \mathbb{Z}$$
to be periodic we need $x[n] = x[n+N]$ for some period N (which may or may not be the fundamental period N_0).
- The issue is that N may not be, and in general is not likely to be, synchronized with the period implicit in ω_0 which is $2\pi/|\omega_0|$ and is generally not an integer.
- Sometimes $x[n] = e^{j\omega_0 n}$ is periodic, e.g., if $2\pi/|\omega_0| = N$ is an integer then $x[n] = e^{j\omega_0 n}$ has period N. (This is not the only case for periodicity, see next slide.)
- Generally $x[n] = e^{j\omega_0 n}$ is not periodic.

Periodicity of DT Exponential Signals

- Other possibilities for periodicity to arise are when $2\pi k/|\omega_0| = N$ for integer $k \in \mathbb{Z}$ then $x[n] = e^{j\omega_0 n}$ has (integer) period kN.
- Plonking $\omega_0 = 2\pi k/N$ into $x[n] = e^{j\omega_0 n}$ leads to the harmonic set $\phi_k[n] \triangleq e^{jk(2\pi/N)n}$ where $k \in \mathbb{Z}$ (integer values). However, only N of these are distinct

$$\phi_k[n] \triangleq e^{jk(2\pi/N)n}, \quad k = 0, 1, \ldots, N-1 \tag{1}$$

  For example, $k = 3$ and $k = N + 3$ are not distinct because $\phi_k[n] = \phi_{k+N}[n]$ for all n. This is related to "aliasing" that we will meet later.
- As bland as (1) looks, in fact this set is one of the most important sets of signals and plays a fundamental role in most signal processing systems, most modern communications systems, digital television, most scientific processing, etc.

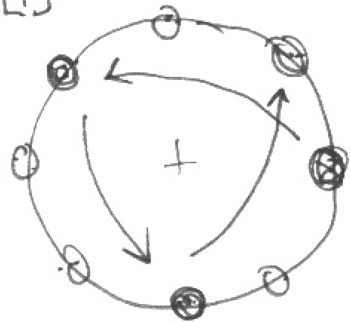
Periodicity of DT Exponential Signals – Example

$\phi_{11}[1]$

$\phi_3[1]$

$N=8$

$k=3$



$\phi_3[0]$ $\phi_{11}[0]$

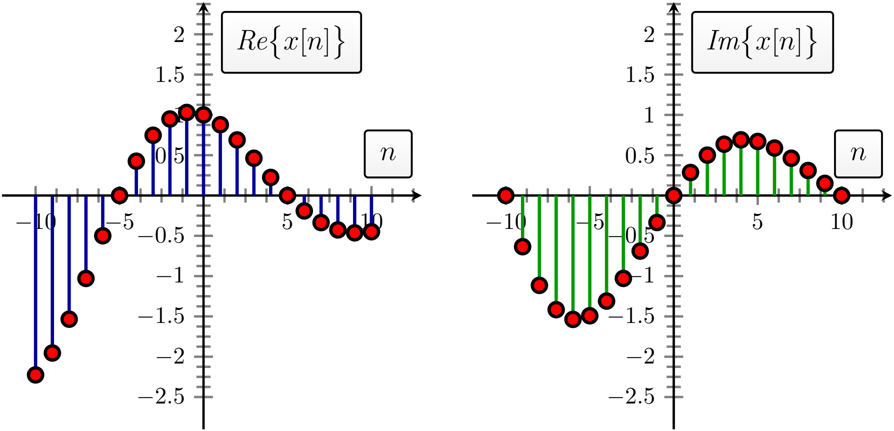
$\phi_3[2]$

$\phi_{11}[2]$ ...

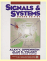
Periodicity of DT Exponential Signals – Example

With $C = 1$ and $\alpha = -0.08 + j0.1\pi$ in $x(t) = Ce^{\alpha t}$ we have

$$\boxed{x[n] \triangleq \exp\left(0.1\pi j\, n\right) \exp(-0.08\, n) \in \mathbb{C}}$$



CT and DT Systems



- A **system is a box** with an input signal $x(t)$ and an output signal $y(t)$

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$$

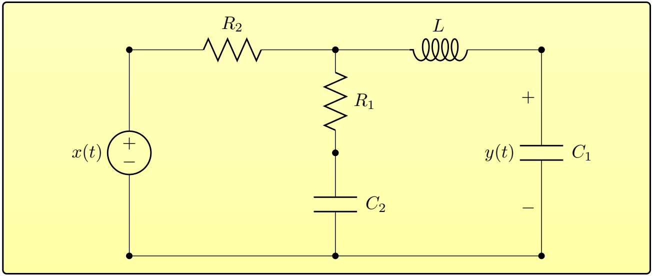
- In the discrete case with an input signal $x[n]$ and an output signal $y[n]$

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Mathematicians and physicists would say a system is an operator
- Signals are functions (containing useful information) and systems are things that transform one type of signal (function) to another signal (function)

CT and DT Systems (examples)

- An RLC circuit can be regarded as a system



where the $x(t)$ is a voltage source and $y(t)$ is the voltage across the capacitor.

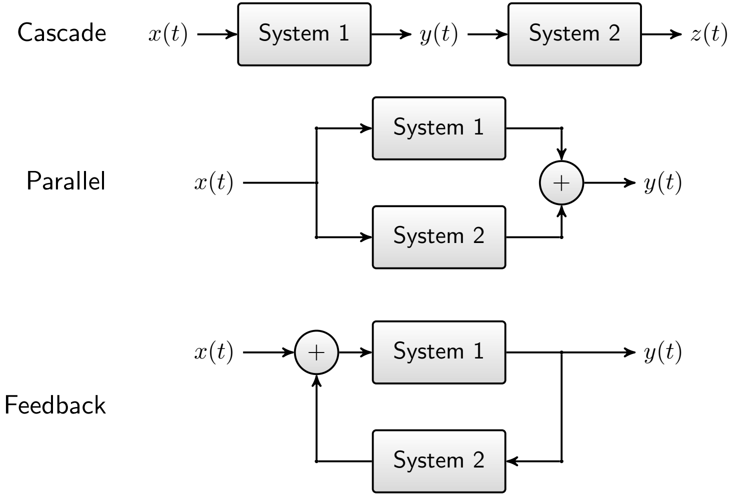
- There are a plethora of systems derived from the RLC circuit. The input $x(t)$ can be any voltage or current and output $y(t)$ can be any voltage or current.

CT and DT Systems

- Dynamics of a car in response to steering.
- An algorithm for predicting the BHP stock price.
- Medical image feature enhancement processing algorithms.
- Basically anything that has an output that responds to an input, e.g., a horse
- This course we focus on systems with one input and one output.
- This course we focus on systems that are linear (whatever that means).

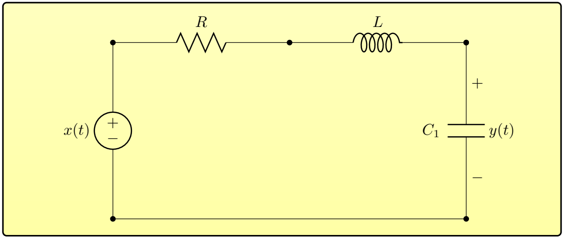
Interconnections of Systems

- More complex systems are the interconnection of simpler or component subsystems.



Interconnections of Systems

- Understanding, designing and analyzing complex interconnections of systems, in cascade (series), parallel and feedback is a core element of modern engineering.
- Complex control systems for a modern aircraft with high order dynamical flight models.
- Almost bewilderingly complex mobile communications systems (that work).



$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

System Examples – Electrical (cont'd)

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t)$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{2}$$

This is a system? (Yes)

- Signal $x(t)$ is the input. Signal $y(t)$ is the output and responds to, or depends on, this input.
- The coefficients LC and RC are constants and don't depend on time or $x(t)$ or $y(t)$.
- The "CT System" is everything in (2) except for $x(t)$ and $y(t)$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

- [Addition] The system in (2) is a second order differential equation. It is **linear**. If $y_1(t)$ is the result of setting $x(t) = x_1(t)$ and $y_2(t)$ is the result of setting $x(t) = x_2(t)$ then $y(t) = y_1(t) + y_2(t)$ is the result of setting $x(t) = x_1(t) + x_2(t)$
  - Punching you in the stomach and kicking you in the head simultaneously has the has the same effect as doing them separately (if you are linear)

- [Scaling] If $y_1(t)$ is the result of setting $x(t) = x_1(t)$ then $3y_1(t)$ is the result of setting $x(t) = 3x_1(t)$
  - Punching you in the head with three times the force has an effect three times as bad (if you are linear)

System Examples – Electrical (cont'd)

- In combination (that is, added and scaling togather), for input

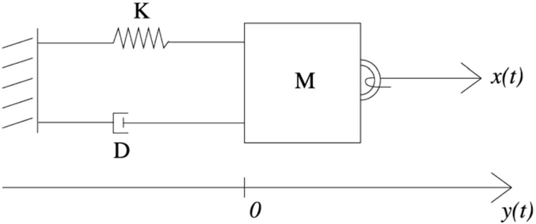
$$x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$$

  the output is

$$y(t) = \alpha_1\, y_1(t) + \alpha_2\, y_2(t)$$

  where α_1 and α_2 are (complex) scalars.

- This is linearity / superposition.



- Force Balance

$$M\frac{d^2y(t)}{dt^2} + D\frac{dy(t)}{dt} + Ky(t) = x(t)$$

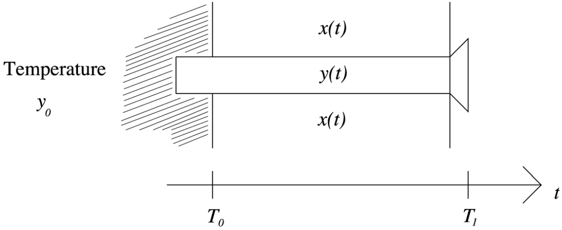
  where M is mass, K is string constant, D is damping and $x(t)$ is the applied force.

- The coefficients M, D and K are RC constants and don't depend on time or $x(t)$ or $y(t)$.

Observations:

- this can be viewed as a mechanical analog of the previous electrical example
- different physical systems / analogs may have identical or very similar mathematical descriptions
- generally you have a strong or familiar domain, say electrical, from which you can interpret other systems (e.g., resistance interpretation of damping)

System Examples – Thermal



- t – distance along the cooling fin
- $y(t)$ – fin temperature as a function of distance
- $x(t)$ – surrounding temperature along fin

System Examples – Thermal (cont'd)

$$
\begin{aligned}
\frac{d^2y(t)}{dt^2} &= k\big(y(t) - x(t)\big) \\
y(T_0) &= y_0 \\
\frac{dy}{dt}(T_1) &= 0
\end{aligned}
$$

- Here the independent variable, t, is space (not time). (OK so the notation is not the greatest.)
- Here we have boundary conditions rather than initial conditions.

System Examples – Edge Detector

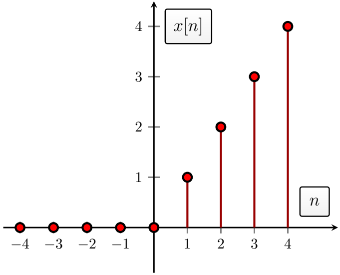
- A rough edge detector acting on a DT signal (sequence)

$$\begin{aligned} y[n] &= x[n+1] - 2x[n] + x[n-1] \\ &= \Big(x[n+1] - x[n]\Big) - \Big(x[n] - x[n-1]\Big) \end{aligned}$$

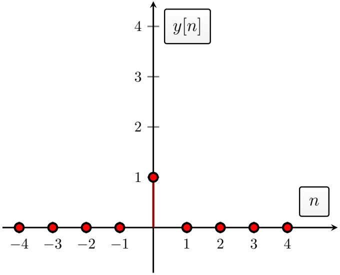
  which is a second difference. This emulates the second derivative for CT signals "change of slope".
- This is a system, $x[n]$ is the input DT signal and $y[n]$ is output detector.
- If $x[n] = n$ (a linear ramp) then $y[n] = 0$ for all n.

System Examples – Edge Detector (cont'd)

- If $x[n] = n\,u[n]$ $(= n \times u[n])$, where $u[n]$ is a step then $y[n] = 0$ for all n except $n = 0$ where $y[0] = 1$.



$x[n]$



$y[n]$

- Here $y[n]$ is a bit weird because how did it know at time $n = 0$ that the input was changing at time $n = 1$? This system is not "causal" which we treat shortly.

$$y[n] = x[n+1] - 2x[n] + x[n-1]$$

Part 3 Slide 100/many Lecturer: R. A. Kennedy

Observations:

- Differential equations and difference equations form an important class of systems.
- A system is not fully characterized by just the “dynamical” equations but also the initial conditions (or boundary conditions).

System Properties

Why study system properties?

- important practical / physical implications
- system Properties imply structure that we can exploit to analyze and understand systems more deeply

System Properties – Causality

Definition (Causality)

A system is **causal** if the output at any time depends on values of the input at only the present and past times.

- All real time-based physical systems are causal. Time flows in one direction. Effect occurs after cause.
- Non-causal systems are the play thing of science fiction. (Don't murder any of your ancestors.)
- Causality relates to time. For other independent variables, like space, there need not be such a constraint. We can approach a point in space from any direction in general without pondering the consequences of strangling an unsuspecting ancestor.

System Properties – Causality

- If you factor out a bulk delay then you can have a non-causal description of what is left.
- CT mathematical version of causality. When

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t)$$

and

$$x_1(t) = x_2(t), \quad \text{for all } t \leq t_0$$

then

$$y_1(t) = y_2(t), \quad \text{for all } t \leq t_0$$

- Only a system can be causal. It is senseless to say a signal is causal.

Terminology: causal, non-causal, anti-causal and strictly causal

- "Non-causal" means there is some output that anticipates the input for some input. For other input-output combinations the system may appear causal. (The set of numbers $\{0, -3, 7, 3, 4, 2\}$ is not positive, since at least one and not all elements are negative.)
- "Strictly causal" means the output depends on the past but not the present nor future. For example, $y[n]$ can be a function of $x[n-1]$, $x[n-2]$, ... but not a function of $x[n]$ nor $x[n+1]$, $x[n+2]$, ...
- "Anti-causal" systems always violate causality (output depends only on the future of the input). They are a type of time reversal of a strictly causal system.

System Properties – Causality

Examples: Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = \big(x(t-1)\big)^2$$

  is causal, e.g., $y(10)$ depends on $x(9)$, $y(t)$ depends strictly on past $x(t)$.

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = x(t+1)$$

  is non-causal, e.g., $y(13) = x(14)$, $y(t)$ depends on strictly future $x(t)$.

- Note a CT system is non-causal even if it is only non-causal at one time instant.

Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = x[-n]$$

is non-causal, e.g., $y[-5] = x[5]$ (but not anti-causal, as $y[5] = x[-5]$).
$y[n]$ is the time-reversal of $x[n]$.

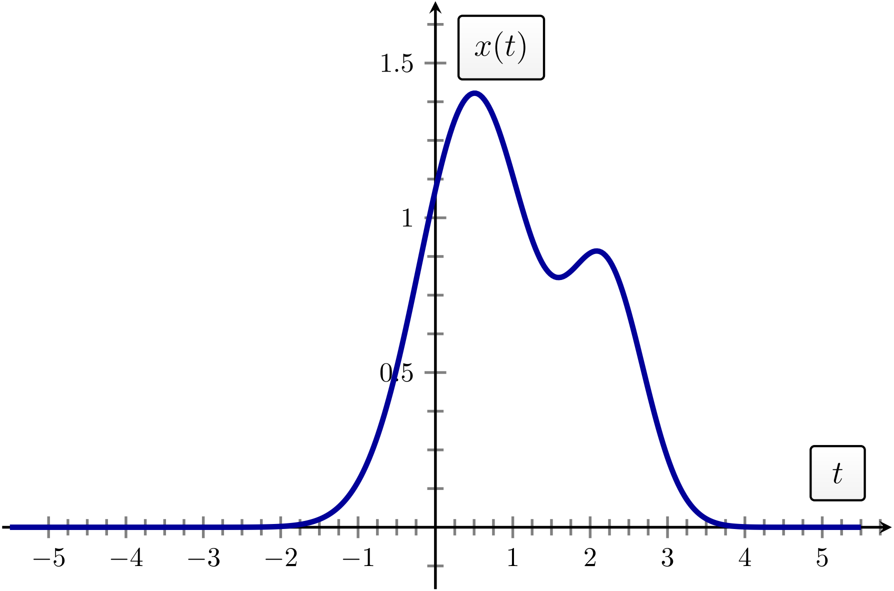
Examples (cont'd): Causal or non-causal?

- The CT system $x(t) \longrightarrow y(t)$ described by

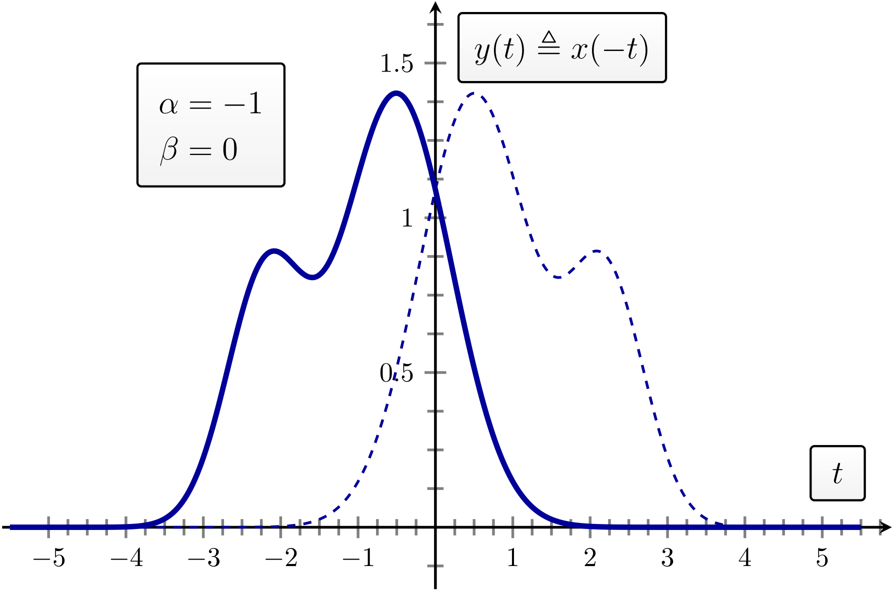
$$y(t) = x(-t)$$

  is non-causal. That is, the system that time reverses an input signal is a non-causal system.

System Properties (CT time reversal)



System Properties (CT time reversal)



Examples (cont'd): Causal or non-causal?

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \left(\frac{1}{2}\right)^{n+1} \big(x[n-1]\big)^3$$

  is causal. The weighting $(1/2)^{n+1}$ is decaying with time n increasing but this is independent of signal $x[n]$.

System Properties – Time-Invariance

Definition (DT System Time-Invariance)

A DT system is **time-invariant** if

$$x[n] \longrightarrow y[n]$$

then

$$x[n-n_0] \longrightarrow y[n-n_0]$$

for all $n_0 \in \mathbb{Z}$.

- Time-Invariance means "doesn't change with time". It is a property of a system and not of the signals input and output (which are obviously functions of time). It means that if a caveman put a signal through a TI system then the output would be the same as the same signal today.

System Properties – Time-Invariance (cont'd)

Definition (CT System Time-Invariance)

A CT system is **time-invariant** if

$$x(t) \longrightarrow y(t)$$

then

$$x(t - t_0) \longrightarrow y(t - t_0)$$

for all $t_0 \in \mathbb{R}$.

- Only a system can be time-invariant. It is senseless to say a signal is time-invariant.

Examples:

- The CT system $x(t) \longrightarrow y(t)$ described by

$$y(t) = \big(x(t+1)\big)^2$$

is time-invariant (TI).

- The DT system $x[n] \longrightarrow y[n]$ described by

$$y[n] = \left(\frac{1}{2}\right)^{n+1} \big(x[n-1]\big)^3$$

is not time-invariant.

Not time-invariant is preferably called time-varying (don't use the expression "time variant").

System Properties – Linear & Nonlinear

- Many, some say most, systems are **nonlinear**. For example, diodes, car dynamics, etc.
- In this course we focus of **linear** systems.
- Don't confuse nonlinear with time-varying linear.
- Linear models are a very important class of models because:
  - they are mathematically tractable
  - they can model small signal variations in nonlinear systems
  - they model accurately circuit elements such as resistors, capacitors, etc.
  - they can provide insights into the behavior of more complex nonlinear systems

System Properties – Linear & Nonlinear (cont'd)

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t)$$

then

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for complex scalars α_1 and α_2.

Definition (Nonlinear System)

A **nonlinear** system is a system which is not linear.

An equivalent definition:

Definition (Linear System)

A CT system is **linear** if superposition holds. If

$$x_k(t) \longrightarrow y_k(t)$$

then

$$\sum_k \alpha_k x_k(t) \longrightarrow \sum_k \alpha_k y_k(t)$$

for complex scalars α_k.

System Properties – Linear & Nonlinear (cont'd)

Linear/Nonlinear, time-invariant/time varying, causal/non-causal:

- $y(t) = \big(x(t)\big)^2 = x^2(t)$ is a square law and as a system is:
  - nonlinear (it is quadratic)
  - time-invariant ($y(t - t_0) = x^2(t - t_0)$ directly)
  - causal (current output depends only on current input)
  - $x_1(t) = t$ and $x_2(t) = t$ then $y_1(t) = t^2$ and $y_2(t) = t^2$. $x_3(t) = 2t$ implies $y_3(t) = 4t^2 \neq y_1(t) + y_2(t) = 2t^2$.
- $y(t) = x(2t)$ is a compression in time and as a system is:
  - Non-causal, since for $t > 0$ we have $2t > t$, for example, at time $t = 3$ we have $y(3) = x(6)$ which is a time advance of 3. Note for $t < 0$ we have $2t < t$, for example, at time $t = -3$ we have $y(-3) = x(-6)$ which is a delay of 3 (that is, it acts causally at time $t = -3$).
  - Linear, (think of $(x_1 + x_2)(t)$ and $\alpha x(t)$)
  - Time-varying, at time $t = 3/2$ the output is $y(3/2) = x(3)$ which is a time-advance of $3/2$, whereas at time $t = 3$ the system does $y(3) = x(6)$ which is a time-advance of 3.

System Properties – Linear & Nonlinear

Are all these combinations possible?

- Linear, time-invariant and causal?
- Linear, time-invariant and non-causal?
- Linear, time-varying and causal?
- Linear, time-varying and non-causal?
- Nonlinear, time-invariant and causal?
- Nonlinear, time-invariant and non-causal?
- Nonlinear, time-varying and causal?
- Nonlinear, time-varying and non-causal?

Yes, all combinations are possible.

Homework Problem: generate system examples for each of the 8 cases above.

Definition (Linear System)

A DT system is **linear** if superposition holds. If

$$x_k[n] \longrightarrow y_k[n]$$

then

$$\sum_k \alpha_k x_k[n] \longrightarrow \sum_k \alpha_k y_k[n]$$

for complex scalars α_k.

- For linear systems, zero input gives zero output.

Challenge Problem:

$$x(t) = 0 \text{ for } t \leq t_0 \;\longrightarrow\; y(t) = 0 \text{ for } t \leq t_0 \tag{3}$$

- A linear system is **causal** if and only if it satisfies the condition of initial rest This statement can be written in different ways (here C means "the linear system is causal"):
  - $C \iff (3)$
  - $C \implies (3)$ and $(3) \implies C$
  - $C \implies (3)$ and $\overline{C} \implies \overline{(3)}$[3]
  - $(3) \implies C$ and $\overline{(3)} \implies \overline{C}$
  - $\overline{C} \implies \overline{(3)}$ and $\overline{(3)} \implies \overline{C}$

[3] Contrapositive of $C \implies (3)$ is $\overline{(3)} \implies \overline{C}$

Summary to this Point

- Signals carry some desired information
- Signals can be CT or DT, that is, their domain is time $t \in \mathbb{R}$ for CT or time index $n \in \mathbb{Z}$ for DT.
- Properties of CT signals and DT signals are similar (and can be treated together).
- The independent variable is usually time (1D), but may be space (1D, 2D or 3D) or time (1D) or combinations, or some other parameter(s) altogether.
- Signals can be classified under various labels such as: periodic, complex exponential, etc.
- Signals are inputs and outputs of systems (systems model how signals are generated or processed or both).

Summary to this Point – (cont'd)

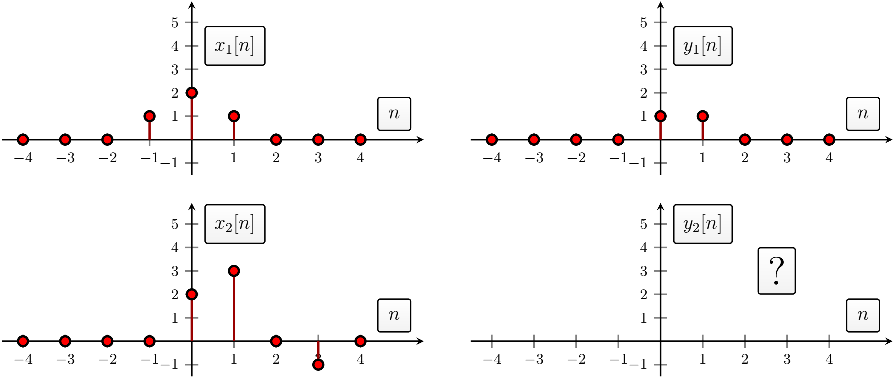
- Systems are usually described through differential equations for CT and difference equations for DT + initial conditions or boundary conditions.
- Systems descriptions for diverse physical situation (e.g., electrical, thermal, mechanical, financial, ...) may be very similar or identical.
- Systems can be linear or nonlinear, causal or non-causal, time-invariant or time-varying. This classification is not exhaustive as there are additional "interesting" properties.
- For causality, it is implicit the independent variable is time. Systems which model physical systems are causal by nature. But non-causal system descriptions are still useful.
- Systems which are linear and time-invariant (LTI) are special (easier, powerful theory and very useful). Now look at these in more detail.

LTI

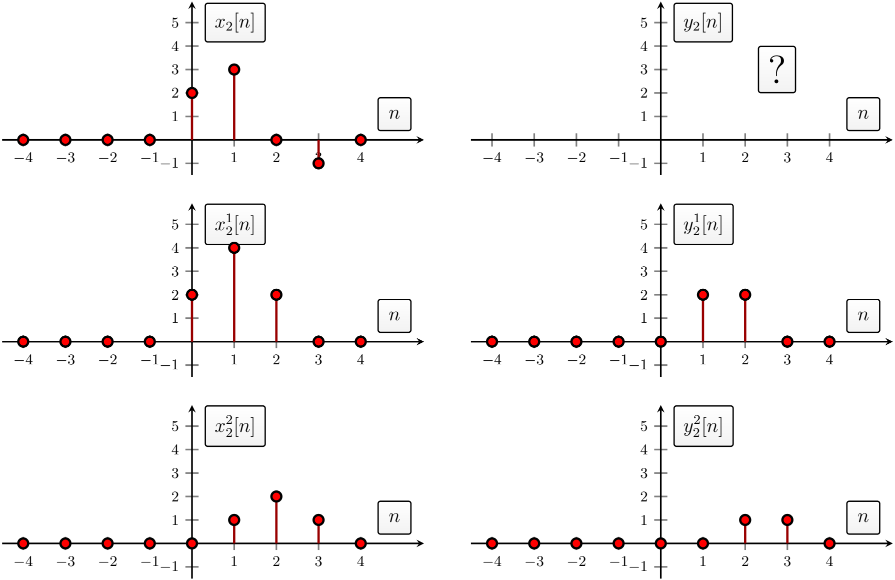
- Linear Time-Invariant (LTI) Systems is focus
- We first treat the DT case and then follow by the CT case `O&W Chapter 2`
- Practically important
- Extensive analysis tools for LTI systems
- Response of an LTI system can be understand from the response to a special type of input signal
- LTI systems can be characterized by a "signal" called a unit sample (DT) or an impulse (CT)

DT LTI Systems – Using DT LTI Property

A system is linear and time-invariant. It is unknown except that output $y_1[n]$ occurs for input $x_1[n]$, that is, $x_1[n] \longrightarrow y_1[n]$. Can we work out the output $y_2[n]$ of this system when the input is $x_2[n]$ ($x_2[n] \longrightarrow y_2[n]$)?



DT LTI Systems – Using DT LTI Property



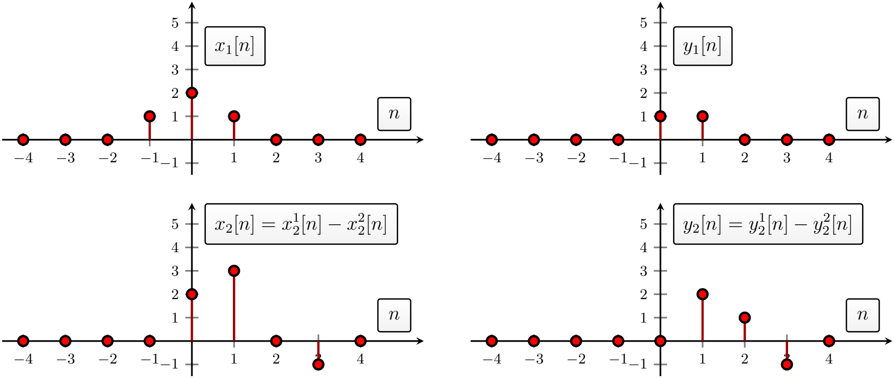
Note that $x_2^1[n]$ and $x_2^2[n]$ are shifted and scaled versions of $x_1[n]$, and further that

$$x_2[n] = x_2^1[n] - x_2^2[n]$$

So by linearity and time-invariance

$$y_2[n] = y_2^1[n] - y_2^2[n]$$

DT LTI Systems – Using DT LTI Property



DT LTI Systems – Audio Example

Enough of the maths, let's look at whether we can do something useful/real with what we have learnt so far.

- The DT system $x[n] \longrightarrow y[n]$ described by

$$\boxed{y[n] = 0.6\,x[n] + 0.4\,x[n-3000]}$$

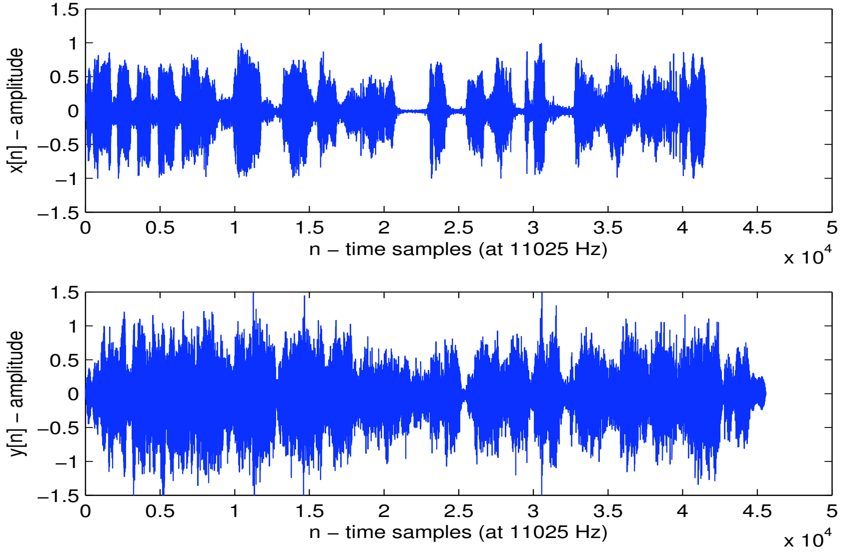
  is: a) linear, b) causal and c) time-invariant.

- Take the input $x[n]$ to be a "sound bite" with samples

$$\begin{aligned} x[0] &= -0.0781 \\ x[1] &= -0.0547 \\ x[2] &= -0.0469 \\ x[3] &= -0.0312 \\ x[4] &= 0.0000 \\ &\vdots \\ x[41580] &= 0.0859 \end{aligned}$$

- Components of sequence $x[n]$ are samples of an audio waveform taken at sampling rate, in this case, of $11,025$ Hz. $x[0]$ is the sample at time 0 seconds, $x[1]$ is the sample at time $1/11,025$ seconds, ... The total real time length is 3.77 seconds (41581 samples).
- Sample values are normalized to range $[-1,+1]$, that is, maximum is $+1$ and minimum is -1 and no signal (quiet) is 0.
- The DT output signal $y[n]$ can be regarded as an output $11,025$ Hz audio clip. What does it sound like?

DT LTI Systems – Audio Example (cont'd)



DT LTI Systems – Audio Example (cont'd)

- Output signal $y[n]$ is longer. Why?
- Output signal $y[n]$ has range outside $[-1, +1]$. Why?
- Direct path 0.6 plus 0.4 echo at $3000/11,025 \approx 0.2731$ seconds or 93 metres longer (speed sound 340m/s).
- If sounds are played back at $8,000$ Hz rate does the pitch go up or down? Does the sound play longer or shorter?
- If the original $11,025$ Hz sound bite is Daffy Duck which cartoon character does the same sound bite sound like when played back at $8,000$ Hz? **Yosemite Sam**

DT LTI Systems – Audio Example (cont'd)

```
d=wavread( 'daffyin.wav' );
fs=11025;
f8=8000;
wavwrite( d, f8, 'daffyin8.wav' );
sound( d, fs );
r(1)=0.6;
r(3001)=0.4;
dd = conv( d, r );
sound( dd, fs );
wavwrite( dd, fs, 'daffyout.wav' );
wavwrite( dd, f8, 'daffyout8.wav' );
subplot(2,1,1);
plot(d)
xlabel('n - time samples (at 11025 Hz)')
ylabel('x[n] - amplitude')
axis([0 50000 -1.5 1.5])
subplot(2,1,2);
plot(dd)
xlabel('n - time samples (at 11025 Hz)')
ylabel('y[n] - amplitude')
axis([0 50000 -1.5 1.5])
```

DT LTI Systems – Superposition + Time-Invariance

- By superposition and time-invariance

$$x_k[n] \longrightarrow y_k[n]$$

$$x[n] = \sum_k a_k x_k[n] \longrightarrow y[n] = \sum_k a_k y_k[n]$$

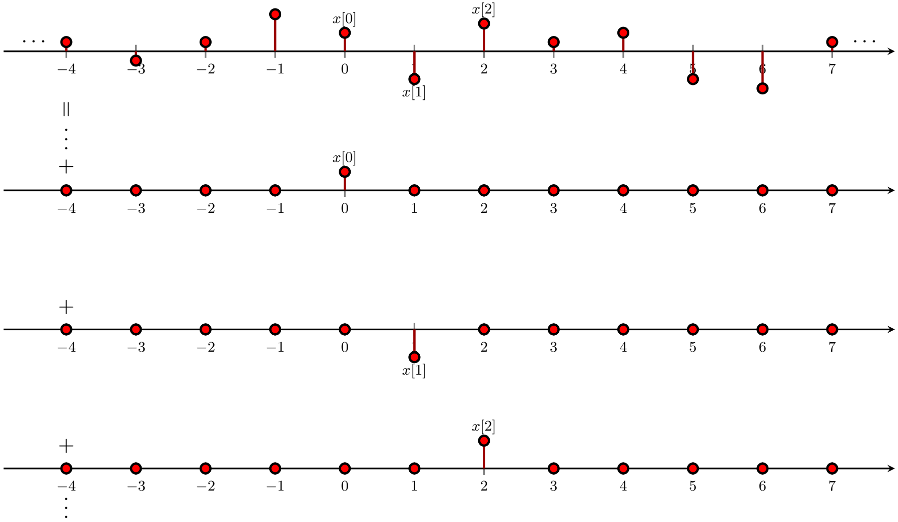
  $x_k[n]$ is an indexed set of candidate building blocks.

- Seek basic building blocks to represent any signal, that is, linear combinations of these building blocks.
- The response of LTI systems to these basic building blocks, if properly chosen, is elegant and powerful.
- In DT the natural choice of building block signals are **time shifted unit samples**.

- Define **unit sample** (this is a signal) `O&W 1.4 pages 30-32`

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

DT LTI Systems – Signal Representation (cont'd)



DT LTI Systems – Signal Representation

- Unit sample (this is a signal)

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- Expand $x[n]$ in terms of shifted unit samples

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + \\ x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

"Sifting Property" of the unit sample $\delta[n]$

DT LTI Systems – More General TV Case

- Response (output $y[n]$) of a **time-varying (TV) linear system** to $x[n]$ can be thought of in terms of superposition of unit sample responses.
- So we want to figure out the response in $y[n]$ to the $\delta[n-k]$ components in $x[n]$.
- Define $h_k[n]$ as the response to $\delta[n-k]$:

$$\delta[n-k] \longrightarrow h_k[n]$$

- By superposition, this linear (generally time-varying) system is given by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

- We need to know the response of a time-varying linear system to an **infinite** number of shifted unit samples to fully characterize it.

DT LTI Systems – More General TV Case

- If I give a linear system a kick then it has a response.
- But the response may be different depending on when I kick it. It can still be linear.
- If the response is different depending on when I kick it it is called **time-varying (TV)**. This is why we wrote

$$h_k[n]$$

  which is the result of a kick at time k, $\delta[n-k]$.
- The case where $h_k[n]$ is independent of k is what we look at in a moment.

DT LTI Systems – More General TV Case

We need all of

$$
\begin{array}{cccccccc}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h_{-2}[-3] & h_{-2}[-2] & h_{-2}[-1] & h_{-2}[0] & h_{-2}[1] & h_{-2}[2] & \cdots \\
\cdots & h_{-1}[-3] & h_{-1}[-2] & h_{-1}[-1] & h_{-1}[0] & h_{-1}[1] & h_{-1}[2] & \cdots \\
\cdots & h_{0}[-3] & h_{0}[-2] & h_{0}[-1] & h_{0}[0] & h_{0}[1] & h_{0}[2] & \cdots \\
\cdots & h_{1}[-3] & h_{1}[-2] & h_{1}[-1] & h_{1}[0] & h_{1}[1] & h_{1}[2] & \cdots \\
\cdots & h_{2}[-3] & h_{2}[-2] & h_{2}[-1] & h_{2}[0] & h_{2}[1] & h_{2}[2] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

to be able to compute the response of a **time-varying** linear system (which is a pain). This is not LTI. LTI is a special case which we consider next.

DT LTI Systems – Unit Pulse Response

- Now suppose the system is LTI, and re-visit the unit sample response(s). Define **unit sample response** $h[n]$

$$\delta[n] \longrightarrow h[n],$$

that is, $h[n]$ is the response of the LTI system to a kick at $k = 0$.

- But from time-invariance

$$\delta[n-k] \longrightarrow h[n-k],$$

that is, kick at k is the same as a kick at 0 shifted by k. Hence, $h[n]$ completely characterizes a LTI System !!!

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

DT LTI Systems – Unit Pulse Response

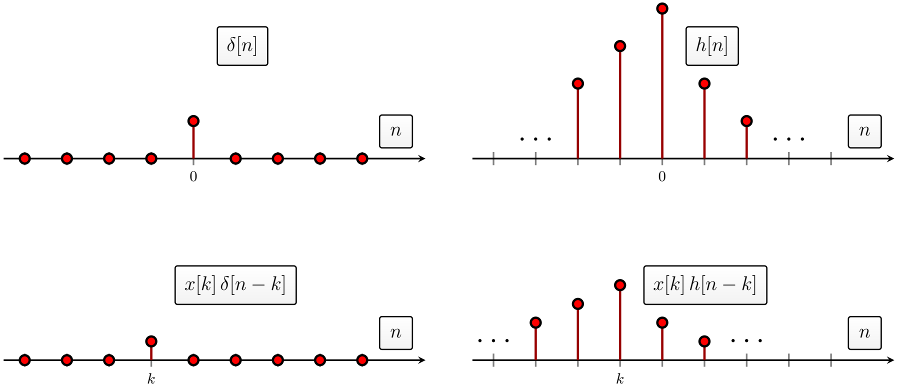
As a special case of a TV Linear System we have $h_k[n] = h[n-k]$, then:

$$\begin{array}{cccccccc}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h[-1] & h[0] & h[1] & h[2] & h[3] & h[4] & \cdots \\
\cdots & h[-2] & h[-1] & h[0] & h[1] & h[2] & h[3] & \cdots \\
\cdots & h[-3] & h[-2] & h[-1] & h[0] & h[1] & h[2] & \cdots \\
\cdots & h[-4] & h[-3] & h[-2] & h[-1] & h[0] & h[1] & \cdots \\
\cdots & h[-5] & h[-4] & h[-3] & h[-2] & h[-1] & h[0] & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We just need the unit sample response, all the $h[n]$.

DT LTI Systems – Unit Pulse Response

Interpretation:



DT LTI Systems – Unit Pulse Response

- Convolution sum, here $\star$ denotes a binary operation not multiplication

$$y[n] = x[n] \star h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- In MATLAB: `y = conv( x, h );`
- Weird but (follows from definition)

$$y[n] = x[n] \star h[n] = h[n] \star x[n]$$

- LTI System is characterized/parametrized by $h[n]$ which looks like a signal—this signal is the output of the LTI system to a unit sample input.

DT LTI Systems – Examples

Simple Reverb System: (revisit)

$$y[n] = 0.6\,x[n] + 0.4\,x[n-3000]$$

- The unit sample response is

$$h[n] = 0.6\,\delta[n] + 0.4\,\delta[n-3000]$$

- If $x[n]$ is the daffy audio signal then

$$y[n] = h[n] \star x[n]$$

  is the reverberant daffy output.
- To emulate the reverberation in the Opera House, fire a pistol $\delta[n]$ and record $h[n]$.

DT LTI Systems – Examples (cont'd)

- Define **unit step** (this is a signal) `O&W 1.4.2 pages 32-38`

$$
\begin{aligned}
u[n] &\triangleq \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \\
&= \sum_{k=-\infty}^{n} \delta[k]
\end{aligned}
$$

DT LTI Systems – Examples (cont'd)

An accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is causal LTI.

- Unit sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

  where $u[n]$ is the unit step.

- In convolution form

$$x[n] \star u[n] = \sum_{k=-\infty}^{n} x[k]$$

DT LTI Systems – Commutative Property

From the definition of convolution is can be shown

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$

- Can be used to reinterpret LTI systems. $x[n]$ as input signal to a system with unit sample response $h[n]$ is the same as $h[n]$ as input signal to a system with unit sample response $x[n]$.
- Can be used as a tool or trick:

$$s[n] = u[n] \star h[n] = h[n] \star u[n]$$

the second is the "accumulator" system. Hence

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

DT Convolution – Street Version

Dictionary

convolution

con•vo•lu•tion |ˌkänvəˈlo͞o sh ən|

noun

1 (often **convolutions**) a coil or twist, esp. one of many : *crosses adorned with elaborate convolutions.*

- a thing that is complex and difficult to follow : *the convolutions of farm policy.*
- a sinuous fold in the surface of the brain.
- the state of being coiled or twisted, or the process of becoming so : *the flexibility of the polymer chain allows extensive convolution.*

2 (also **convolution integral**) Mathematics a function derived from two given functions by integration that expresses how the shape of one is modified by the other.

- a method of determination of the sum of two random variables by integration or summation.

DERIVATIVES

con•vo•lu•tion•al |-sh ənl| |ˈkɑnvəˈluʃənl| |ˈkɑnvəˈluʃnəl| adjective

ORIGIN mid 16th cent.: from medieval Latin ***convolutio(n-)***, from ***convolvere* 'roll together'** (see CONVOLVE).

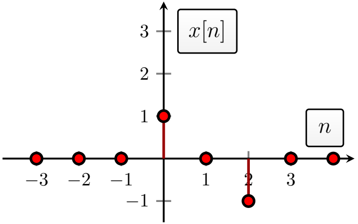
DT Convolution – Direct Version

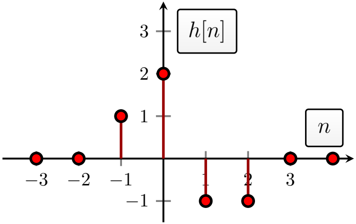
Consider the convolution of

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < 0, \text{ or } n > 2) \end{cases}$$

and (non-causal) pulse response

$$h[n] = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ -1 & n = 1 \\ -1 & n = 2 \\ 0 & \text{otherwise } (n < -1, \text{ or } n > 2) \end{cases}$$

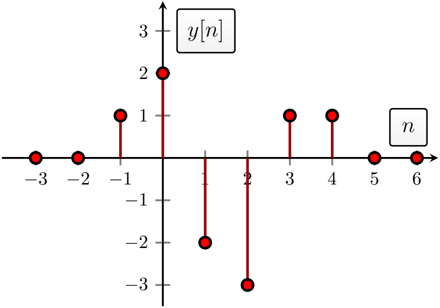




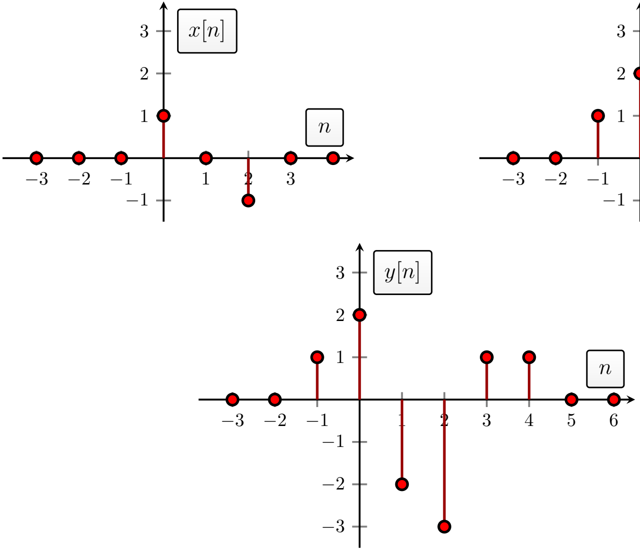
DT Convolution – Direct Version (cont'd)

The convolution is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ -1 - 1 = -2 & n = 1 \\ -1 - 2 = -3 & n = 2 \\ 1 & n = 3 \\ 1 & n = 4 \\ 0 & \text{otherwise } (n < -1, \text{ or } n > 4) \end{cases}$$



DT Convolution – Direct Version (cont'd)



DT Convolution – Direct Version (cont'd)

The calculation of DT convolution is a means to combine two vectors/sequences

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & -1 & -1 \end{bmatrix}$$

to generate a new vector/sequence

$$\begin{bmatrix} 1 & 2 & -2 & -3 & 1 & 1 \end{bmatrix}$$

and indeed this is what MATLAB command

```
y = conv( x, h );
```

does. Where else does this exact type of calculation appear?

DT Convolution – Polynomial Version

Consider the polynomial multiplication:

$$(1x^2 - 1)(x^3 + 2x^2 - x - 1) = x^5 + 2x^4 - 2x^3 - 3x^2 + x + 1$$

- $x[n]$ and $h[n]$ provide the LHS polynomials' coefficients.
- $y[n]$ provides the RHS polynomial's coefficients.
- So DT convolution appears to be related to polynomial multiplication.
- Here the non-causal shift in $h[n]$ has not been factored in (but can be).
- x is the polynomial indeterminate, it can be thought of as a unit time shift.

DT Convolution – Matrix Version

Matrix version of convolution

$$\begin{array}{rrr} -1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

- Here $h[n]$ is implemented as a 6×3 Toeplitz matrix. The rows are formed by reversing $h[n]$ and shifting.
- The orange portions above are just an aid to understand and not part of the matrix.

DT System Properties – Commutative Property

Previously we noted the **Commutative Property**:

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$$

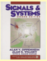
DT System Properties – Commutative Property

- This follows from (change variables to $\ell = n - k$)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$
$$= \sum_{\ell=-\infty}^{\infty} x[n-\ell]\, h[\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, x[n-\ell]$$

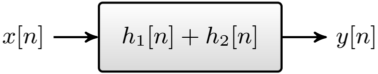
- Alternatively, if we have two polynomials say $p(x)$ and $q(x)$ then $p(x)q(x) = q(x)p(x)$. Polynomial multiplication is commutative. Convolution is commutative.

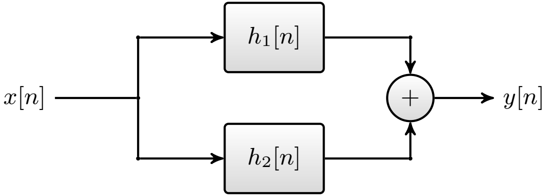
DT System Properties – Distributive Property



Consider an input signal $x[n]$ and two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **parallel**, then we have the **Distributive Property**:

$$\boxed{x[n] \star \left(h_1[n] + h_2[n]\right) = x[n] \star h_1[n] + x[n] \star h_2[n]}$$



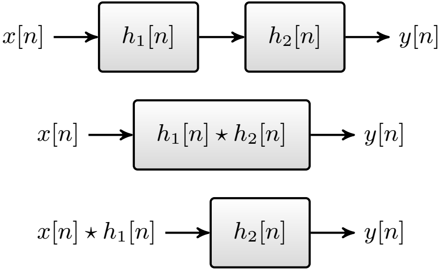


- This implies that we can combine two DT LTI systems in parallel into a single equivalent DT LTI system (by **adding** the pulse responses).

DT System Properties – Associative Property

Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

$$\boxed{x[n] \star \left(h_1[n] \star h_2[n]\right) = \left(x[n] \star h_1[n]\right) \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).

DT System Properties – Causality Property

For DT LTI Systems the **Causality Property** can be written:

Theorem (Causal DT LTI System)

A DT LTI system is **causal** *if and only if its pulse response,* $h[n]$*, satisfies*

$$h[n] = 0, \quad \textit{for all } n < 0.$$

- If $h[n] \neq 0$ for at least one $n = -n_0$ $(n_0 > 0)$ then the output at time n, $y[n]$, would contain term

  $$h[-n_0]\, x[n + n_0],$$

  for example, if $n_0 = 1$ and $h[-1] = 2$ then
  $y[n] = \cdots + h[-1]\, x[n+1] + \cdots$, and hence would not be causal.

DT System Properties – Stability Property

Definition (DT LTI System Stability)

A DT LTI system is **stable**, with pulse respond $h[n]$, if and only if

$$\sum_{n=-\infty}^{\infty} \big|h[n]\big| < \infty$$

Output $y[n]$ is bounded if and only if the input is bounded.

- $h[n] \triangleq 2^n$ is not stable
- $h[n] \triangleq 2^{-n}$ is not stable (consider $n \longrightarrow -\infty$)
- The following is stable:

$$h[n] \triangleq \begin{cases} 2^{-n} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $h[n] \triangleq e^{j2\pi n/11}$ is not stable

DT System Properties – Memoryless Property

Definition (Memoryless DT System)

A system is **memoryless** if its output at time n depends only on the input at the same time n.

The following DT Systems $x[n] \longrightarrow y[n]$ are:

- **Memoryless**
  - $y[n] = 7x[n]$
  - $y[n] = \sqrt{x[n]} + 29$
  - $y[n] = n\,x[n]$
  - $y[n] = 23$ (even though independent of $x[n]$)
- **Not memoryless** (have memory)
  - $y[n] = 7x[n-1]$

causal

DT System Properties – Review Questions

Problem:

Determine whether or not each of the following signals are: i) **time-invariant**, ii) **linear**, iii) **casual**, iv) **stable**, and v) **memoryless**.

(a) $y[n] = x[n+3] - x[1-n]$

(b) $y[n] = \begin{cases} (-1)^n x[n], & x[n] \geq 0 \\ 2x[n], & x[n] < 0 \end{cases}$

(c) $y[n] = \sum_{k=n}^{\infty} x[k]$

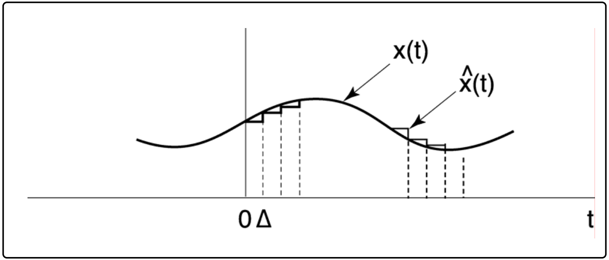
Solution:

	TI	Linear	Causal	Stable	Memoryless
(a)	no	yes	no	yes	no
(b)	no	no	yes	yes	yes
(c)	yes	yes	no	no	no

CT Signals and Systems – Preamble

- By and large, features and properties of CT signals and systems look very similar or are the same as their DT counterparts. The terminology is largely identical.
- When a CT System is LTI then it can be characterized in terms of an impulse response which itself is in the form of a signal. That is, knowing the impulse response signal of a CT LTI System completely characterizes it (apart from initial conditions).

Approximate any CT signal $x(t)$ as a sum of shifted, scaled rectangular pulses:

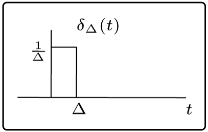


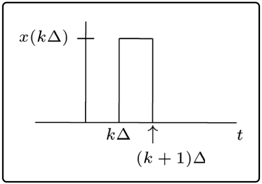
$$\widehat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$

(Here $\widehat{x}(t)$ denotes an approximation to $x(t)$.)

Define a rectangle pulse

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$$





$$= x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

Whence

$$\widehat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t - k\Delta)\Delta$$

CT Signals and Systems – Representation

In the limit as $\Delta \to 0$ we infer

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

which is the "Sifting Property" of the unit impulse:

- the integrand is zero for all $t-\tau \neq 0$, only when $\tau = t$ is the integrand non-zero,
- the unit impulse $\delta(t)$ has the area of unity; $x(\tau)\, \delta(t-\tau)$ has area of $x(t)$

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

$$\delta_\Delta \to h_\Delta(t)$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta}_{\widehat{x}(t)} \to \underbrace{\sum_{k=-\infty}^{\infty} x(k\Delta)\,h_\Delta(t-k\Delta)\Delta}_{\widehat{y}(t)}$$

$$\Downarrow \qquad \text{(as } \Delta \to 0\text{)}$$

$$\delta(t) \to h(t)$$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau}_{x(t)} \to \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}_{y(t)}$$

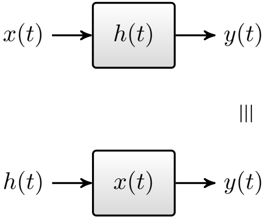
Convolution Integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Interpretation

$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau)$$
$$h(-\tau) \xrightarrow{\text{Shift}} h(t-\tau)$$
$$h(t-\tau) \xrightarrow{\text{Multiply}} x(\tau)h(t-\tau)$$
$$x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

CT System Properties – Commutativity



$$
\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= h(t) \star x(t)
\end{aligned}
$$

- This follows from (change variables to $\sigma = t - \tau$)

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} x(t-\sigma) h(\sigma)\, d\sigma
\end{aligned}
$$

CT System Properties – Sifting Property

Sifting Property:

$$x(t) \star \delta(t - t_0) = x(t - t_0)$$

That is, a system response $x(t) \xrightarrow{h(t)} y(t)$ of the form

$$h(t) = \delta(t - t_0),$$

acts as a time delay of t_0, that is,

$$y(t) = x(t - t_0).$$

CT System Properties – Sifting Property (cont'd)

Note that $x(t) \star \delta(t - t_0)$ is different from $x(t)\,\delta(t - t_0)$. The expression on the left is convolution of two signals and the expression on the right is pointwise multiplication.

- Convolution

$$\begin{aligned} x(t) \star \delta(t - t_0) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\,d\tau \\ &= x(t - t_0) \end{aligned}$$

- Pointwise multiplication

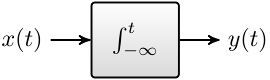
$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

which is a scaled impulse response.

CT System Properties – Integrator

Integrator Property:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \qquad (4)$$



So we need to determine the system response $h(t)$ to synthesize the integrator (4). But $y(t) = h(t)$ is output when $x(t) = \delta(t)$ is input, so substituting into (4)

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\, d\tau \\ &= u(t) \qquad \text{(step function)} \end{aligned}$$

That is, a system response of the form $h(t) = u(t)$ acts as an integrator.

CT System Properties – Step Response

Step Response: Let $x(t) = u(t)$ be a step input signal to a CT LTI System with pulse response $h(t)$, then the output signal is given by

$$\begin{aligned} s(t) &= u(t) \star h(t) = h(t) \star u(t) \\ &= \int_{-\infty}^{t} h(\tau)\, d\tau \end{aligned}$$

This is a common test signal used in diagnosing physical systems (often called "plants").

CT System Properties – Causality

Theorem (Causal CT LTI System)

A CT LTI system is **causal** *if and only if its pulse response, $h(t)$, satisfies*

$$h(t) = 0, \quad \textit{for all } t < 0.$$

- If $h(t) \neq 0$ for at least one $t = -t_0$ ($t_0 > 0$) then the output at time t, $y(t)$, would have an integrand term

$$h(-t_0)x(t + t_0)$$

and hence not be causal.

Stability Property:

Definition (CT LTI System Stability)

CT LTI system, $x(t) \xrightarrow{h(t)} y(t)$, is **stable**, if and only if

$$\int_{-\infty}^{\infty} \big|h(\tau)\big| \, d\tau < \infty$$

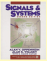
Definition (Memoryless CT System)

A CT system is **memoryless** if its output at time t depends only on the input at the same time t.

The following CT Systems $x(t) \xrightarrow{h(t)} y(t)$ are:

- **Memoryless**
  - $y(t) = 7x(t)$
  - $y(t) = \sqrt{x(t)} + 23$
  - $y(t) = t\,x(t)$
  - $y(t) = -5$ (even though independent of $x(t)$)
- **Not memoryless** (have memory)
  - $y(t) = 5\,x(t - 0.5)$
  - $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

CT System Properties – Distributivity



Signals & Systems
section 2.3.2
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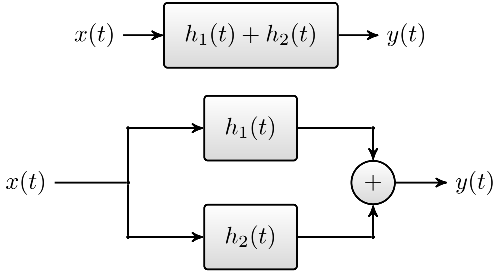
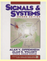
Definition (Distributivity Property of CT LTI Systems)

Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

$$\begin{aligned} y(t) &= x(t) \star \big(h_1(t) + h_2(t)\big) \\ &= x(t) \star h_1(t) + x(t) \star h_2(t) \end{aligned}$$



Two CT LTI systems in **parallel** implies we **add** their impulse responses.



Signals & Systems
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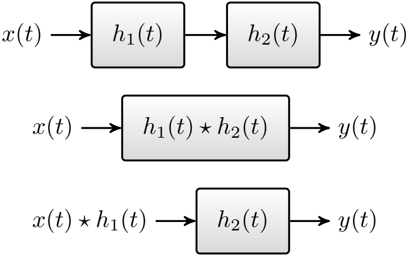
Definition (Associativity Property of CT LTI Systems)

Consider two CT LTI systems: $x(t) \xrightarrow{h_1(t)} y_1(t)$ and $x(t) \xrightarrow{h_2(t)} y_2(t)$ then

$$y(t) = \big(x(t) \star h_1(t)\big) \star h_2(t)\big) = x(t) \star \big(h_1(t) \star h_2(t)\big)$$



Two CT LTI systems in **series** implies we **convolve** their impulse responses.

Impulses and More – Unit Impulse

$\delta(t)$ – unit area (integrates to 1) pulse which is the limit of a narrow rectangular pulse with width Δ going to zero and height, $1/\Delta$, going to infinity.

- The shape of the (unit) impulse isn't important, that is, there is nothing special about the rectangular shape.
- When applied to a CT LTI System gives the output equal to the impulse response:

$$\delta(t) \star h(t) = h(t), \quad \text{for all } h(t)$$

  This is a tautology of sorts, this says "the response to an unit impulse is the impulse response". Let's look at this mathematically next.

Impulses and More – Unit Impulse (cont'd)

Start with $x(t) \star h(t) = y(t)$ which means

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

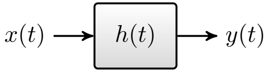


Fig: General LTI System

Then set the input an impulse, that is, set $x(t) = \delta(t)$, to yield

$$\int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau = h(t)$$

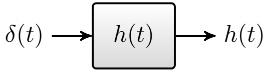
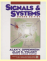


Fig: Special Input $x(t) = \delta(t)$

Here $\delta(\tau) = 0$ if $\tau \neq 0$ and it "sifts" the value of $h(t)$.



Impulses and More – Trivial System

From commutativity $x(t) \star h(t) = h(t) \star x(t) = y(t)$:

$$\begin{aligned} y(t) &= h(t) \star x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \end{aligned}$$

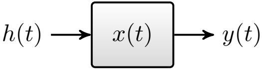


Fig: Flipped $x(t)$ and $h(t)$

With impulse response $x(t) = \delta(t)$ and input $h(t)$

$$\int_{-\infty}^{\infty} h(\tau)\, \delta(t-\tau)\, d\tau = h(t)$$

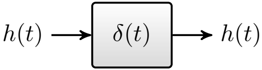
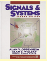


Fig: Trivial System

Here $\delta(t-\tau) = 0$ if $t \neq \tau$ and it sifts the value $h(t)$. This system, with impulse response given by the impulse, has output signal equal to the input signal, that is, just passes the input to the output — called the "Trivial System".

Impulses and More – Delay System



Now, with impulse response $h(t) = \delta(t - t_0)$ and input $x(t)$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t - t_0 - \tau)\, d\tau = x(t - t_0).$$

$$x(t) \longrightarrow \boxed{\delta(t - t_0)} \longrightarrow x(t - t_0)$$

Fig: Time Shift LTI System (Delay when $t_0 > 0$)

Here $\delta(t - t_0 - \tau) = 0$ if $t - t_0 \neq \tau$ and it "sifts" the value $h(t - t_0)$. This system, with impulse response given by the impulse with time shift, has output equal to the input with a time shift. Note that $t_0 > 0$ gives a delay and $t_0 < 0$ gives a time advance (which would be non-causal).

Note that

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which is the unit step signal.

Compare this with the fundamental theorem of calculus which asserts

$$\frac{d}{dt} \int_{-\infty}^{t} f(\tau)\, d\tau = f(t)$$

for sufficiently regular functions $f(t)$. (cont'd)

Impulses and More – Additional Results (cont'd)

With the function $\delta(t)$ as the function in the fundamental theorem of calculus, we can infer

$$\frac{d}{dt}\underbrace{\int_{-\infty}^{t}\delta(\tau)\,d\tau}_{u(t)} = \delta(t).$$

So the $\delta(t)$ "is" the derivative of the unit step $u(t)$. (cont'd)

Impulses and More – Additional Results (cont'd)

We can represent this as:

$$u(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \delta(t) = \frac{du(t)}{dt}$$

Fig: Differentiator with Unit Step Input

where the $\frac{d}{dt}$ inside the box is not the impulse response but denotes an operator. (cont'd)

Impulses and More – Additional Results (cont'd)

Taking the derivative is actually a linear time-invariant (LTI) operator. It satisfies superposition (L)

$$\frac{d}{dt}\big(\alpha_1 x_1(t) + \alpha_2 x_2(t)\big) = \alpha_1 \frac{dx_1(t)}{dt} + \alpha_2 \frac{dx_2(t)}{dt}$$

and, with the notation,

$$x'(t) = \frac{dx(t)}{dt}$$

then by the chain rule we have the time-invariance (TI)

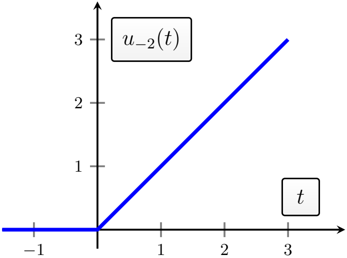
$$\frac{d}{dt} x(t - t_0) = x'(t - t_0)$$

So we conclude that the differentiator operator acts like a LTI system and so must have an **impulse response** which we'll consider shortly.

Impulses and More – Additional Results (cont'd)

Now suppose we have a linear **unit ramp**

$$u_{-2}(t) \triangleq \begin{cases} t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



or, equivalently, $u_{-2}(t) \triangleq t\,u(t) = t\,u_{-1}(t)$.

We can also write this as

$$u_{-2}(t) = u(t) \star u(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$$

(picture in your mind, convolving two unit steps $u(t)$ together).

Impulses and More – Additional Results (cont'd)

Big picture:

$$\cdots \xrightarrow{\frac{d}{dt}} u_{-2}(t) \xrightarrow{\frac{d}{dt}} u(t) \xrightarrow{\frac{d}{dt}} \delta(t) \xrightarrow{\frac{d}{dt}} \; ? \xrightarrow{\frac{d}{dt}} \cdots$$

Can we continue? What is the derivative of the impulse $\delta(t)$? (And the second derivative, etc?)

Later picture (different notation only):

$$\cdots \xrightarrow{\frac{d}{dt}} u_{-2}(t) \xrightarrow{\frac{d}{dt}} u_{-1}(t) \xrightarrow{\frac{d}{dt}} u_0(t) \xrightarrow{\frac{d}{dt}} u_1(t) \xrightarrow{\frac{d}{dt}} \cdots$$

Here, evidently, $u_0(t) = \delta(t)$ and $u_{-1}(t) = u(t)$.

Impulses and More – Additional Results (cont'd)

We give a partial explanation of some of the previous weird notation. For the trivial system, the system that just passes the input to output without modification, we found that the impulse response is just $\delta(t)$, that is,

$$h(t) \longrightarrow \boxed{\delta(t)} \longrightarrow h(t)$$

Fig: Trivial System

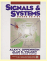
Then we can make some sense of the notation $u_0(t) = \delta(t)$

$$h(t) \longrightarrow \boxed{u_0(t)} \longrightarrow h(t)$$

Fig: Trivial System (Alternative Notation)

where 0 indicates a reference trivial (zero) system.

Impulses and More – Differentiator System



Consider the differentiator (recall this is LTI)

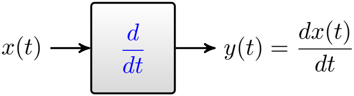


Fig: Differentiator System

What is the impulse response to describe this?

The answer is denoted $u_1(t)$, called the **unit doublet**, and has the property/definition:

$$\frac{dx(t)}{dt} = x(t) \star u_1(t)$$

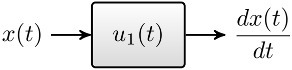


Fig: Differentiator System

We relate $u_1(t)$ to $\delta(t)$ later.

Impulses and More – Differentiator System (cont'd)

Further generalizing, for $k > 0$,

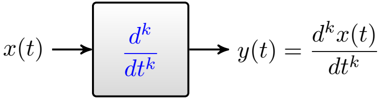


Fig: kth Differentiator System

Then, for $k > 0$,

$$\frac{d^k x(t)}{dt^k} = x(t) \star u_k(t)$$

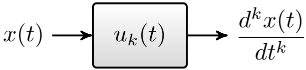


Fig: kth Differentiator System

and, by cascading differentiators we can get a nth order differentiator impulse response

$$u_n(t) \triangleq \underbrace{u_1(t) \star \cdots \star u_1(t)}_{n \text{ times}}, \quad n > 0$$

Impulses and More – Integrators

An integrator

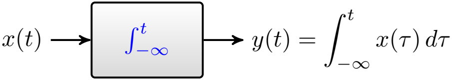
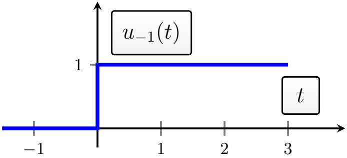


Fig: Integrator System

Impulse response is the step response:

$$u_{-1}(t) = u(t),$$



that is, the -1th derivative or simply the integral. We can write

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Further, cascading n integrators, leads to impulse response

$$u_{-n}(t) \triangleq \underbrace{u_{-1}(t) \star \cdots \star u_{-1}(t)}_{n \text{ times}}, \quad n > 0$$

Impulses and More – Integrators (cont'd)

Cascade of two integrators, with input $\delta(t)$:

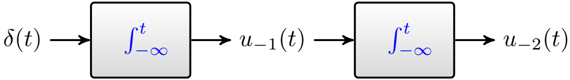


Fig: Cascade of Integrator Systems

Whence

$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)\, d\tau = \int_{-\infty}^{t} u(t)\, d\tau = t\, u(t)$$

which is the unit ramp. More generally,

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u(t), \quad n > 1$$

where $k! = k(k-1)(k-2)\cdots 2 \cdot 1$.

Further reflections on integrators. A differentiator operator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Fig: Differentiator System

"destroys" information, that is, a constant signal on input is mapped to zero (the zero signal) at the output.

The integrator

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Fig: Integrator System

would normally require some initial conditions to completely determine the output.

Impulses and More – Further Results

We have,

$$u_0(t) = \delta(t)$$

which can be viewed as:

- the impulse response for the trivial system, $y(t) = x(t)$
- the zeroth derivative operator

Then

$$u_m(t) \star u_n(t) = u_{m+n}(t), \quad m, n \in \mathbb{Z} \text{ (integers)}$$

For example,

$$u_1(t) \star u_{-1}(t) = u_0(t) \quad \Rightarrow \quad \frac{d}{dt} u(t) = \delta(t).$$

Digression: (for those interested)

- **Functions** map scalars to scalars (points to points).
- **Operators** map functions to functions
- **Functionals** map functions to scalars

Examples:

- A signal is a **function** of time. For each number we put in, say $t = 3$, we get a number out, here $x(3)$.
- Computing the energy or power of a signal is an example of a **functional** — a whole function is gobbled up and a number spat out.
- A system is an **operator**. For each input function $x(t)$ we input a function $y(t)$ is output. Also called a "filter".

Previously,

$$u_1(t) \star u_{-1}(t) = u_0(t) \quad \Rightarrow \quad \frac{d}{dt}u(t) = \delta(t).$$

Here u is an input function and δ is an output function. The operator is $D \triangleq d/dt$ and we can write

$$\frac{d}{dt}u = \delta \quad \text{or} \quad Du = \delta$$

If you want to include the independent variable t then, by convention, this is written

$$\Big(\frac{d}{dt}u\Big)(t) = \delta(t) \quad \text{or} \quad \big(Du\big)(t) = \delta(t)$$

Impulses and More – Tricks

Occasionally useful tricks:

$$\begin{aligned} x(t) \star h(t) &= x(t) \star \delta(t) \star h(t) \\ &= x(t) \star u_1(t) \star u_{-1}(t) \star h(t) \\ &= \big(x(t) \star u_1(t) \star h(t)\big) \star u_{-1}(t) \end{aligned}$$

Differentiate, then convolve then integrate. There are many other possibilities.

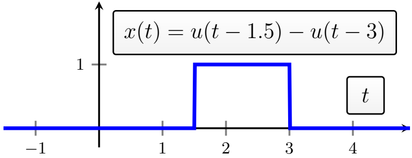
Consider unit height rectangular pulse with duration $[a, b)$ (here $a < b$ and are arbitrary, $a, b \in \mathbb{R}$):

$$x(t) = \begin{cases} 1 & a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

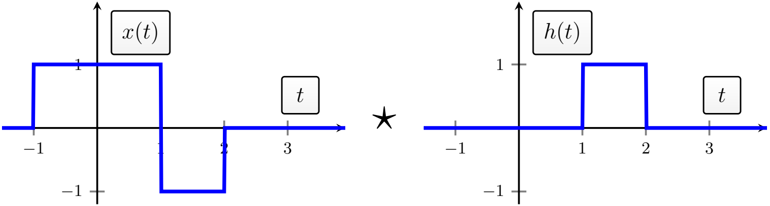
This can be written in terms of shifted $u(t)$:

$$x(t) = u(t-a) - u(t-b)$$

Example plotted on right for $a = 1.5$ and $b = 3$.



Consider the problem:



Let's use

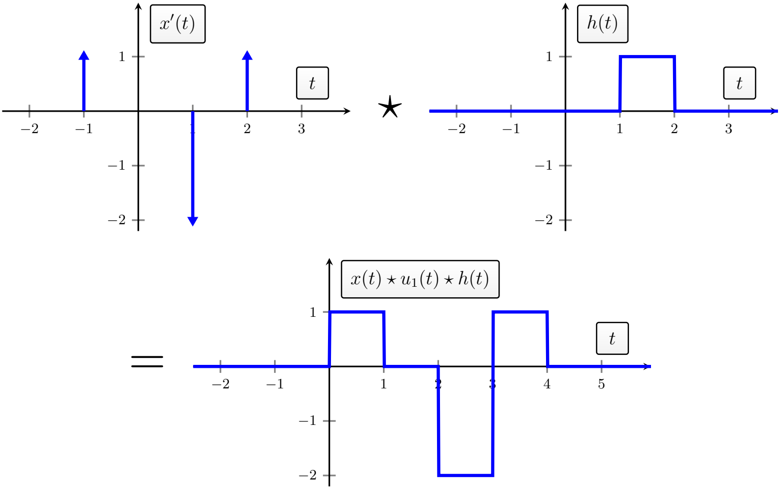
$$x(t) \star h(t) = \big(x(t) \star u_1(t) \star h(t)\big) \star u_{-1}(t)$$

to compute this convolution because it will be like having three impulses at the step transitions. First we have

$$x'(t) = x(t) \star u_1(t) = \delta(t+1) - 2\,\delta(t-1) + \delta(t-2)$$

this means, signal $x(t)$ is input to a 1st order differentiator (which is an LTI system with $u_1(t)$ as the impulse response).

Impulses and More – Tricks (cont'd)

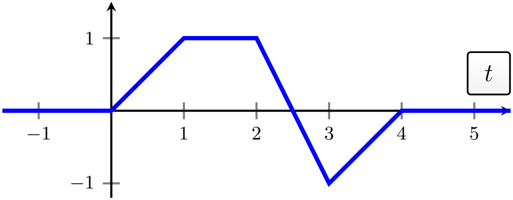


Finally,

$$x(t) \star h(t) = \int_{-\infty}^{t} \left(\frac{dx(\tau)}{d\tau} \star h(\tau) \right) d\tau$$

This can be drawn, by observation, as:

$$\boxed{\int_{-\infty}^{t} \left(\frac{dx(\tau)}{d\tau} \star h(\tau) \right) d\tau = x(t) \star u_1(t) \star h(t) \star u_{-1}(t)}$$



Scaling properties of the $\delta(t)$

- It can be shown:

$$\delta(nt) = \frac{1}{n}\,\delta(t)$$

  Start with a rectangular approximation and consider the limit as the width goes to 0. Approximation of $\delta(nt)$ is n times thinner horizontally than approximation of $\delta(t)$.

- Recall $\delta(t)$ has unit area. The above function has area $1/n$.

Revisit $u_1(t)$, called the **unit doublet**, which has the property/definition:

$$\frac{dx(t)}{dt} = x(t) \star u_1(t)$$

$$x(t) \longrightarrow \boxed{u_1(t)} \longrightarrow \frac{dx(t)}{dt}$$

Fig: Differentiator System

Signal $u_1(t)$ is the derivative of $\delta(t)$. What does this look like?

Advanced Note: This is quite mad in fact. Convolution is an integrating action and we need to significantly pre-distort the signal in the integrand to have the signal derivative come out as output.

Impulses and More – Unit Doublet Revisit (cont'd)

Consider the rectangular pulse

$$\delta_\Delta(t) = \begin{cases} 1 & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

which behaves like the impulse $\delta(t)$ as $\Delta \to 0$. Its derivative should behave like the unit doublet as $\delta(t)$ as $\Delta \to 0$. We have

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}\big(\delta(t) - \delta(t-\Delta)\big)$$

and this looks like the superposition of two delta functions of opposite sign and area $1/\Delta$. Further,

$$x(t) \star \frac{d\delta_\Delta(t)}{dt} = \frac{x(t) - x(t-\Delta)}{\Delta} \to \frac{dx(t)}{dt} \text{ as } \Delta \to \infty$$

Impulses and More – Unit Doublet Revisit (cont'd)

Two properties, we learnt earlier, of $u_0(t) = \delta(t)$ are:

$$\int_{-\infty}^{\infty} x(\tau) u_0(t-\tau)\, d\tau = x(t)$$

$$x(t)\, u_0(t) = x(0)\, \delta(t)$$

For $u_1(t)$ analogous properties (without proof) are:

$$\int_{-\infty}^{\infty} x(\tau) u_1(t-\tau)\, d\tau = -x'(t)$$

$$x(t)\, u_1(t) = x(0)\, u_1(t) - x'(0)\, \delta(t)$$

where $x'(t) \triangleq dx(t)/dt$.

Fourier Series – Where we are heading?

DT and CT LTI systems admit a simple description in terms of the impulse response. The output signal can be computed in terms of the convolution of the input and impulse response. In essence, not much else is needed.

Fourier Series, the Fourier Transform and other "frequency domain" descriptions provide an alternative (but equivalent) viewpoint with a number of key advantages:

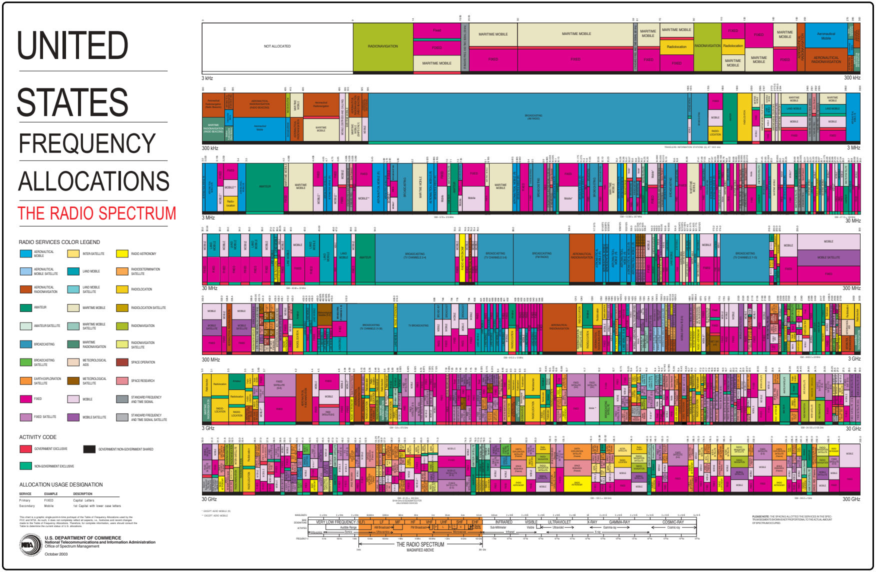
- Convolution is simplified using the Fourier representations. Convolution is "twisted and complicated" in the "time domain" but in the frequency domain it is "untwisted and simple".
- Studying the behavior of a LTI system in the frequency domain is actually natural and intuitive (after a few years).

Fourier Series – Where we are heading? (con't)

To this point we haven't said much about **design**. We want to build LTI systems that achieve certain design goals. This is best done in the frequency domain.

Design and resource allocation specifications almost always have a frequency domain formulation:

Fourier Series – Where we are heading? (con't)



Consider

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

which is a moving average of three consecutive terms. So $y[n]$ looks like a smooth version of $x[n]$.

Note that we use $1/3$ weights because if $x[n]$ is very smooth to start with then $y[n]$ should be equal or close to $x[n]$. So if $x[n] = 1$ for all n, that is, it is a constant, then $y[n] = 1/3 + 1/3 + 1/3 = 1$.

Fourier Series – Where we are heading? (con't)

So

$$y[n] = \frac{1}{5}x[n-2] + \frac{1}{5}x[n-1] + \frac{1}{5}x[n] + \frac{1}{5}x[n+1] + \frac{1}{5}x[n+2]$$

works as a smoother as well. It should smooth more.

Ditto

$$y[n] = \frac{1}{9}x[n-2] + \frac{2}{9}x[n-1] + \frac{1}{3}x[n] + \frac{2}{9}x[n+1] + \frac{1}{9}x[n+2]$$

works as a smoother too. It relies more on the current value $x[n]$ in forming $y[n]$.

Which smoother works best? What are we trying to achieve?

Fourier Series – Where we are heading? (con't)

We are lacking tools to analyze which smoother is best. We are lacking tools for design. This is one main reason to look at the description of LTI systems in the Frequency or Fourier domain.

Fourier Series – Digression

We can view the "slow" system

$$h_5[n] \triangleq \frac{1}{5}\delta[n-2] + \frac{1}{5}\delta[n-1] + \frac{1}{5}\delta[n] + \frac{1}{5}\delta[n+1] + \frac{1}{5}\delta[n+2]$$

as extracting the slow/smooth part of $x[n]$

$$x_s[n] \equiv x_5[n] \triangleq x[n] \star h_5[n]$$

$$x[n] \longrightarrow \boxed{h_5[n]} \longrightarrow x_5[n]$$

Fig: Smoother

This is called **filtering**. It is a **"low pass filter"**. It passes to the output the slowly varying parts of input $x[n]$ (mostly).

But what if I didn't want the slowly varying parts of input $x[n]$ but the opposite? Can we extract just the quickly varying parts (the parts blocked by $h_5[n]$)?

Fourier Series – Digression (con't)

Of course, the "fast" system is just the complement of the "slow" system

$$h_5^c[n] \triangleq \delta[n] - h_5[n] =$$
$$-\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]$$

and extracts the non-slow part of $x[n]$

$$x_s^c[n] \equiv x_5^c[n] \triangleq x[n] \star h_5^c[n]$$

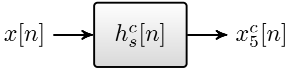


Fig: Complement of the Smoother

It is a **"high pass filter"**. It passes to the output the quickly varying parts of input $x[n]$ (mostly). Also evidently, $x[n]$ is perfectly split into the two parts $x[n] = x_5[n] + x_5^c[n]$.

Fourier Series – Digression (con't)

Finally, Goldilocks says she doesn't want the fast nor the slow but just the parts of $x[n]$ that are varying not too fast nor too slow. Can we figure this out? (Or should we just feed her to the bears?)

The complement of the five term moving average

$$
\begin{aligned}
h_5^c[n] &\triangleq \delta[n] - h_5[n] \\
&= -\frac{1}{5}\delta[n-2] - \frac{1}{5}\delta[n-1] + \frac{4}{5}\delta[n] - \frac{1}{5}\delta[n+1] - \frac{1}{5}\delta[n+2]
\end{aligned}
$$

essentially blocks the very slow parts of $x[n]$ and lets through more than the complement of the three term moving average

$$h_3[n] \triangleq \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n+1]$$

so we can get the desired action via the **"band pass filter"**

$$h_5^c[n] \star h_3[n] = \text{whatever}$$

Fourier Series – Digression (con't)

$x[n]$ can be split into slow, medium (neither fast nor slow) and fast signals

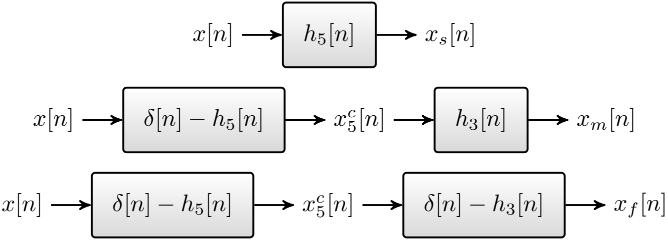


Fig: Slow, Medium and Fast Signal Processing

Here

$$x_m[n] + x_f[n] = x_5^c[n] \quad \text{and} \quad x_s[n] + x_5^c[n] = x[n]$$
$$x_s[n] + x_m[n] + x_f[n] = x[n]$$

Evidently, it seems useful to be able to characterize how a system treats fast, medium and slow inputs (then we can broadly regard it as low pass, high pass, band pass or some blurring of the three).

The most obvious thing to do is see how the system responds to **complex exponentials** (of different "frequencies"). This leads us to study this problem more completely.

Fourier Series – Response to Complex Exponent

Prior to studying the set of complex exponentials we can lay down the desirable characteristics of a set of basic signals to probe LTI systems:

- We need to be able to represent any **signal** (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or scanning in some sense).
- The response of an LTI **system** to any of these basic signals should be simple, useful and insightful.

To emphasize, it has to meet demands of both: i) signal representation and ii) systems characterization.

Fourier Series – Response to Complex Exponentials

Previous focus was unit samples and impulses.

Alternative/new focus is "eigenfunctions of LTI systems" (what the?)

Fourier Series – Response to Complex Exponentials

Start with CT LTI system convolution equation. Let the input be

$$x(t) = e^{st}$$

where $s \in \mathbb{C}$ is complex. Then, with $h(t)$ the impulse response,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau, \quad \text{with } x(t) = e^{st} \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \end{aligned}$$

(. . . yawn)

So $x(t) = e^{st}$ is a really good choice. Reflect on

$$y(t) = \underbrace{e^{st}}_{\text{the input}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\text{independent of } t}$$

The output equals the input apart from a complex multiplier

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \quad \in \mathbb{C}.$$

This is some sort of "transform" of the impulse response $h(t)$. What does this all mean?

Fourier Series – Matrix Digression

Suppose I have a $N \times N$ square matrix $\mathbf{M}$ then if there is a N-vector $\boldsymbol{\mu}$ satisfying

$$\boldsymbol{M}\boldsymbol{\mu} = \lambda\boldsymbol{\mu}, \quad \lambda \in \mathbb{C}$$

then $\boldsymbol{\mu} \in \mathbb{C}^N$ is an **eigenvector** and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

We could draw

$$\boldsymbol{\mu} \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda\boldsymbol{\mu}$$

which reveals that the output vector/signal is equal to input vector/signal apart from a complex scale factor ("gain"). The action of matrix $\boldsymbol{M}$ is simple when the input is an eigenvector.

Fourier Series – Matrix Digression (con't)

More generally, there are a set of eigenvectors

$$\boldsymbol{M}\boldsymbol{\mu}_i = \lambda_i \boldsymbol{\mu}_i, \quad i = 1, 2, \ldots, N$$

and for each we could draw

$$\boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \lambda_i \boldsymbol{\mu}_i$$

So different eigenvectors have different gains.

This is powerful because, take any $\boldsymbol{v} \in \mathbb{C}^N$ not generally an eigenvector and assume the set of N eigenvectors $\boldsymbol{\mu}_i$ span $\mathbb{C}^N$.

Then

$$\boldsymbol{v} = \sum_{i=1}^{N} a_i \boldsymbol{\mu}_i$$

and we can draw

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i$$

These eigenvectors are a special set (for this matrix $\boldsymbol{M}$).

Recall previous guidelines for a desirable signal "set"

- We need to can able to represent any **signal**/vector (or any sensible signal) in terms of such a set. The set needs to be rich enough (be sufficient or complete or spanning in some sense).
- The response of an LTI **system**/matrix to any of these basic signals should be simple, useful and insightful.

In fact it is more than a coincidence, since a system is an operator (maps functions to functions) and a matrix is a finite dimensional operator (maps vectors in $\mathbb{C}^N$ to $\mathbb{C}^N$).

In the back of our minds...

$$\sum_{i=1}^{N} a_i \boldsymbol{\mu}_i \longrightarrow \boxed{\boldsymbol{M}} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i \boldsymbol{\mu}_i, \quad \lambda_i = \lambda_i(\boldsymbol{M})$$

Eigen-behavior of LTI Systems

$$\sum_{i=1}^{N} a_i e^{s_i t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{i=1}^{N} \lambda_i a_i e^{s_i t}, \quad \lambda_i = H(s_i)$$

where

$$\boxed{H(s_i) \triangleq \int_{-\infty}^{\infty} h(\tau)\, e^{-s_i \tau}\, d\tau \in \mathbb{C}.}$$

So being able to decompose signals into complex exponentials leads to a simple characterization of an LTI system.

Fourier Series – DT Case Eigenfunctions

What can we use to in the DT case for $x[n]$ analogous to CT $x(t) = e^{st}$?

Let the input be

$$x[n] = z^n$$

where $z \in \mathbb{C}$ is complex. Then, with $h[n]$ the impulse response,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k], \quad x[n] = z^n \\ &= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \end{aligned}$$

Fourier Series – DT Case Eigenfunctions (con't)

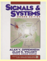
Reflect

$$y[n] = \underbrace{z^n}_{\text{the input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{\text{independent of } n}$$

The output equals the input apart from a complex multiplier

$$\boxed{H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \in \mathbb{C}}$$

This is some sort of "transform" of the impulse response $h[k]$.



Fourier Stuff – Review Key Observation

CT LTI system response to **arbitrary** input $x(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

versus

CT LTI system response to **specific, complex exponential** input $x(t) = e^{st}$, for some complex $s \in \mathbb{C}$,

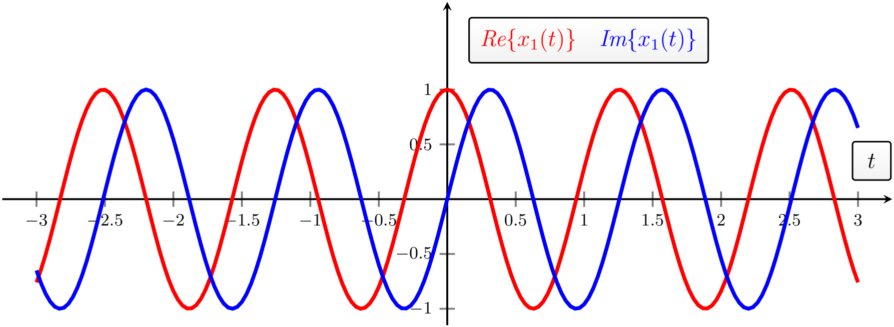
$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s) \in \mathbb{C}}$$

where $H(s)$ is just a complex number — call this the “complex gain”.

Fourier Stuff – Examples

Example 1: Let

$$x_1(t) \triangleq e^{j\,5t}$$



be input to a LTI system with impulse response $h_1(t)$ and "complex gain"

$$H_1(j5) \triangleq 1 + j\sqrt{3}.$$

Find

$$y_1(t) = x_1(t) \star h_1(t), \quad \text{where } x_1(t) = e^{j\,5t}$$

Fourier Stuff – Examples

For this input, we have

$$s = j5, \qquad \text{(recall } e^{st}\text{)}$$

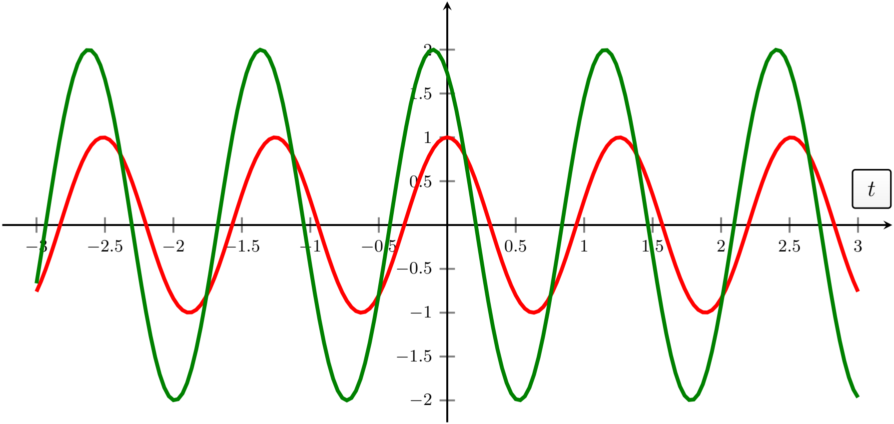
and the output is

$$\begin{aligned} y_1(t) &= e^{j\,5t} \star h_1(t) \qquad \text{(convolution)} \\ &= e^{j\,5t} H_1(j5) \qquad \text{(multiplication)} \\ &= e^{j\,5t}(1 + j\sqrt{3}) \\ &= 2\,e^{j\,5t} e^{j\,\pi/6} \\ &= 2\,e^{j\,(5t+\pi/6)} \end{aligned}$$

That is, the output is at the same frequency $\omega_1 = 5$ $(s = j\omega_1)$, is twice the amplitude of the input and is phase shifted by $\pi/6$.

Fourier Stuff – Examples

Plotting the real part of this output $y_1(t)$ (and input $x_1(t)$):



Example 2: Now let the input be only real, have a phase shift of $\pi/4$ and have magnitude 4:

$$x_1(t) \triangleq 4\cos(5t + \pi/4)$$

Let this be input to a LTI system with impulse response $h_1(t)$ which has gain

$$H_1(\pm j5) \triangleq 1 \pm j\sqrt{3}, \quad \text{(here } s = \pm j5\text{)},$$

(This notation means $H_1(j5) \triangleq 1 + j\sqrt{3}$ and $H_1(-j5) \triangleq 1 - j\sqrt{3}$ merged into one equation.) Find

$$y_1(t) = x_1(t) \star h_1(t)$$

Again we won't have to explicitly compute the convolution

$$y_1(t) = \int_{-\infty}^{\infty} 4\cos(5\tau + \pi/4)\, h_1(t-\tau)\, d\tau$$

and in fact we don't know $h_1(t)$ completely anyway.

Fourier Stuff – Examples

For this input, we have

$$\begin{aligned} x_1(t) &= 4\cos(5t + \pi/4) \\ &= 4\frac{\left(e^{j\,(5t+\pi/4)} + e^{-j(5t+\pi/4)}\right)}{2} \\ &= (\sqrt{2} + \sqrt{2}j)e^{j\,5t} + (\sqrt{2} - \sqrt{2}j)e^{-j5t} \end{aligned}$$

So we can use the principle of superposition with values $s = j5$ and $s = -j5$, that is, $s = \pm j5$,

$$(\sqrt{2} \pm j\sqrt{2})e^{\pm j5t} \longrightarrow \boxed{h_1(t)} \longrightarrow (\sqrt{2} \pm j\sqrt{2})H_1(\pm j5)e^{\pm j5t}$$

Fourier Stuff – Examples

For input

$$x_1(t) = 4\cos(5t + \pi/4)$$

the output is then

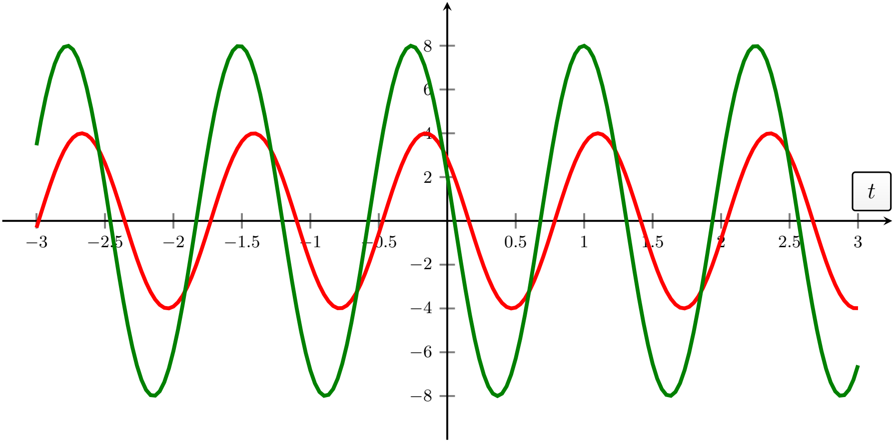
$$\begin{aligned} y_1(t) &= (\sqrt{2} + j\sqrt{2})(1 + j\sqrt{3})e^{+j5t} + (\sqrt{2} - j\sqrt{2})(1 - j\sqrt{3})e^{-j5t} \\ &= (2e^{j\,\pi/4})2e^{j\,\pi/6}e^{+j5t} + (2e^{-j\pi/4})2e^{-j\pi/6}e^{-j5t} \\ &= 8\,\frac{\left(e^{j\,(5t+5\pi/12)} + e^{-j(5t+5\pi/12)}\right)}{2} \\ &= 8\,\cos(5t + 5\pi/12) \end{aligned}$$

Or equivalently

$$\int_{-\infty}^{\infty} 4\,\cos(5\tau + \pi/4)\,h_1(t-\tau)\,d\tau = 8\,\cos(5t + 5\pi/12)$$

Fourier Stuff – Examples

Plotting input $x_1(t)$ and output $y_1(t)$:



This calculation can be considerably simplified. Note

- $\left|H_1(\pm j5)\right| = 2$ so the gain (magnitude) is just 2.
- The phase of $H_1(\pm j5)$ is $\pm\pi/4$ which is the **relative** phase shift of the output with respect to the input.
- Recall the system is time-invariant so the $\pi/4$ phase on the input is equivalent to a time shift and gets conveyed to the output.

Therefore,

$$x_1(t) = 4\cos(5t + \pi/4)$$

$$H_1(j5) = 2\, e^{j\,\pi/6} \quad \text{System}$$

$$y_1(t) = \underbrace{2 \times 4}_{8} \cos(5t + \underbrace{\pi/4 + \pi/6}_{5\pi/12})$$

Fourier Stuff – Examples

Example 3: Consider the series/cascade connection

$$x_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow y_1(t) \longrightarrow \boxed{h_2(t)} \longrightarrow y_2(t)$$

with LTI system $h_1(t)$ such that

$$H_1(j5) = 0.5\, e^{j\,\pi/4} \quad \text{and} \quad H_1(-j5) = 0.5\, e^{-j\pi/4}$$

and LTI system $h_2(t)$ such that

$$H_2(j5) = 0.3\, e^{j\,2\pi/3} \quad \text{and} \quad H_2(-j5) = 0.3\, e^{-j2\pi/3}$$

Compute

$$y_2(t) = x_1(t) \star h_1(t) \star h_2(t)$$

for input signal

$$x_1(t) \triangleq 3\cos(5t - \pi/7).$$

Fourier Stuff – Examples

First test understanding:

- $y_1(t)$ has to look like
$$\alpha \cos(5t + \beta)$$
because the input, $x_1(t)$, is of that form.
- $y_2(t)$ has to look like
$$\gamma \cos(5t + \delta)$$
because the input, $x_2(t) = y_1(t)$ is of that form.
- We only have to find two real numbers (magnitude and phase) or, equivalently, one complex number.

Overall gain is

$$3 \times \underbrace{0.5 \times 0.3}_{0.15} = 0.45$$

Overall phase

$$-\pi/7 + \underbrace{\pi/4 + 2\pi/3}_{11\pi/12} = 65\pi/84$$

Therefore,

$$y_2(t) = 0.45\, \cos(5t + 65\pi/84)$$

and we note that

$$H_1(j5)\, H_2(j5) = 0.15\, e^{j\, 11\pi/12}$$

and

$$H_1(-j5)\, H_2(-j5) = 0.15\, e^{-j\, 11\pi/12}.$$

Observation 1: This is revealing something very important. Cascading two LTI systems implies **convolving** their impulse responses. But for complex exponentials signals we only have to **multiply** complex gains. As we show much later this is the key property of frequency domain descriptions; series LTI systems lead to multiplications and parallel LTI systems lead to additions (superposition).

It seems like behavior of interconnections of LTI systems with complex exponential signals as inputs (and outputs) is relatively simple.

Fourier Stuff – Examples

Observation 2: Houston we have a problem. OK so complex exponentials are simple to work with. What about more general signals?

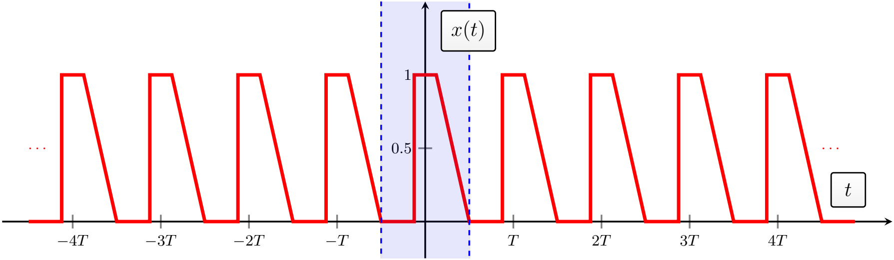
Firstly we look at **periodic signals**. Fourier series, which we now consider, show that any periodic signal can be expressed into terms of appropriate linear combinations of complex exponential signals (which are themselves periodic). Then we can appeal to superposition to characterize the response of an LTI system to general (not necessarily complex exponential) periodic signals.

CT periodic signals satisfy

$$\boxed{x(t) = x(t+T), \quad \text{for all } t}$$

where T is the **fundamental period** (smallest positive T).

An example periodic waveform with fundamental period T is shown below.



Define the **fundamental frequency** associated with fundamental period T

$$\omega_0 = \frac{2\pi}{T}$$

Prototypical periodic signal is

$$e^{j\,\omega_0 t}$$

What other related signals (to this prototypical signal) are also periodic with period T?

It is reasonably clear that ($\iff$ means if and only if)

$$e^{j\,\omega t} \text{ is periodic with period } T \iff \omega = k\omega_0, \quad k \in \mathbb{Z}$$

- For each $|k| \neq 1$, the period T is not fundamental.
- For example, with $k = -3$, $e^{-j3\omega_0 t}$ has fundamental period $T/3$ but it is still periodic with period T. It is periodic with periods: $T/3$, $2T/3$, T, $4T/3$, etc.
- Further $k = 0$, which implies $\omega = 0$, is weird, it leads to just constant function.
- $k = -1$ leads to the conjugate of $k = 1$ and has fundamental period T also.

Fourier Stuff – Fourier Series

The infinite linear combination

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk2\pi t/T}$$

- is periodic with period T because all components are periodic with period T
- $k = 0$ term is the "DC" term ("direct current")
- $k = \pm 1$ terms are the first harmonic
- $k = \pm 2$ terms are the second harmonic
- The two kth terms are called the kth harmonic ($k > 0$) and have fundamental period kT.

In

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk2\pi t/T}}$$

the $\{a_k\}$ are the **Fourier Series coefficients**.

It's clear there are an infinite number of periodic functions that can be built this way but...

Question: Can any periodic function be expressed in such a way?

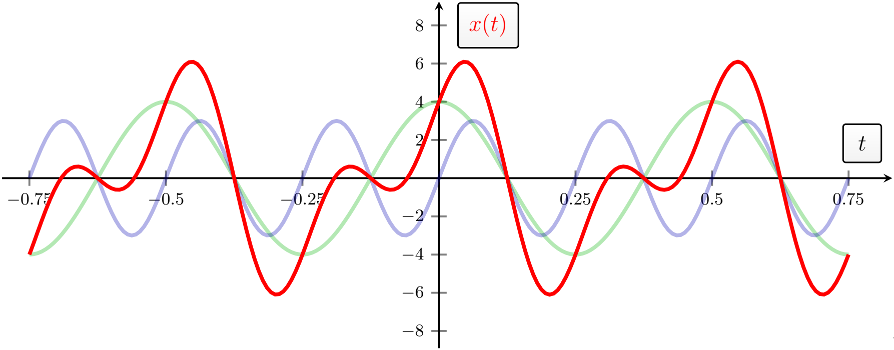
Answer: (Pretty well) yes.

Then we need to a way of determining the Fourier Series coefficients, that is:

$$\boxed{\text{Given } x(t) \text{ find } \{a_k\} \text{ for all } k \in \mathbb{Z}.}$$

(Simple) Example 1: If

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$



we find the fundamental frequency and fundamental period are:

$$\omega_0 = 4\pi, \qquad T = \frac{2\pi}{\omega_0} = \frac{1}{2},$$

Fourier Stuff – Fourier Series (cont'd)

For

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

the Fourier Series coefficients are:

$$\begin{aligned} a_1 &= 2 \\ a_{-1} &= 2 \\ a_2 &= -3j/2 \\ a_{-2} &= 3j/2 \\ a_k &= 0, \quad \text{otherwise} \end{aligned}$$

meaning

$$x(t) = 2e^{j\,4\pi t} + 2e^{-j4\pi t} - \frac{3j}{2}e^{j\,8\pi t} + \frac{3j}{2}e^{-j8\pi t}$$

For **real-valued** signals, other Fourier Series expressions are possible

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} \big(\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t)\big)$$

or

$$x(t) = \gamma_0 + \sum_{k=1}^{\infty} \big(\gamma_k \cos(k\omega_0 t + \theta_k)\big)$$

We'll stick to the complex exponential form. Then both positive and negative frequencies need to be used:

$$e^{jk\omega_0 t} \qquad e^{-jk\omega_0 t}$$

Fourier Stuff – Fourier Coefficients

For (contrived)

$$x(t) = 4\cos(4\pi t) + 3\sin(8\pi t)$$

we could read off (almost) the Fourier Series coefficients. But how do we compute them for a general function?

Given an arbitrary $x(t)$, how would we compute the coefficient a_{-37} in the expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \quad ?$$

Fourier Stuff – Fourier Coefficients (cont'd)

Consider

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \int_T \overbrace{\Big(\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \Big)}^{x(t)} e^{-jn\omega_0 t}\, dt$$
$$= \sum_{k=-\infty}^{\infty} a_k \Big(\int_T e^{j\,(k-n)\omega_0 t}\, dt \Big)$$

But

$$\int_T e^{j\,(k-n)\omega_0 t}\, dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$
$$= T\,\delta[k-n]$$

This is orthogonality. Here T is just a constant equal to the fundamental period.

Fourier Stuff – Fourier Coefficients (cont'd)

Note we have written

$$\int_T$$

rather than

$$\int_0^T$$

because the integrand is periodic with period T. As such

$$\int_0^T \equiv \int_{-T/2}^{T/2} \equiv \int_{\epsilon}^{T+\epsilon}$$

(assuming the integrand is periodic with period T).

Hence

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \Big(\int_T e^{\,j\,(k-n)\omega_0 t}\, dt \Big)$$

$$= T \sum_{k=-\infty}^{\infty} a_k\, \delta[k-n] = a_n T$$

Definition (Fourier Analysis and Synthesis)

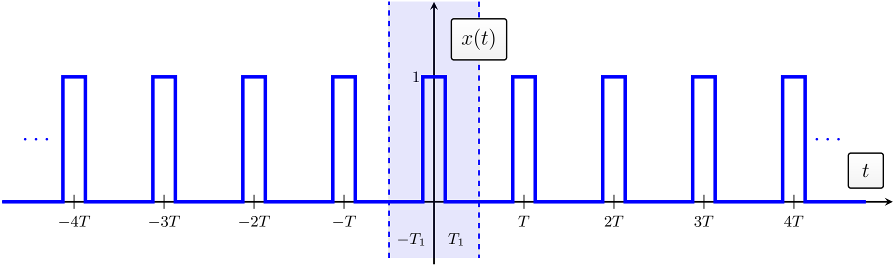
For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \le T/2$$



Fourier Stuff – Periodic Rectangular Wave (cont'd)

The Fourier coefficients can be calculated:

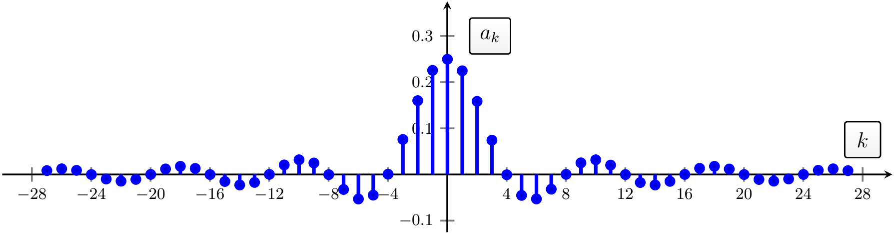
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2T_1}{T}$$

is the average or DC value, and

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt \\ &= \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad \text{recalling that} \quad \omega_0 = \frac{2\pi}{T} \end{aligned}$$

Called a "sinc" function.

With $T_1 = T/8$ the periodic rectangular wave has Fourier coefficients, $\{a_k\}$, as follows



Fourier Stuff – Periodic Rectangular Wave (cont'd)

In the following we take more and more terms in the truncated Fourier series

$$\sum_{k=-L}^{L} a_k \, e^{jk\omega_0 t}$$

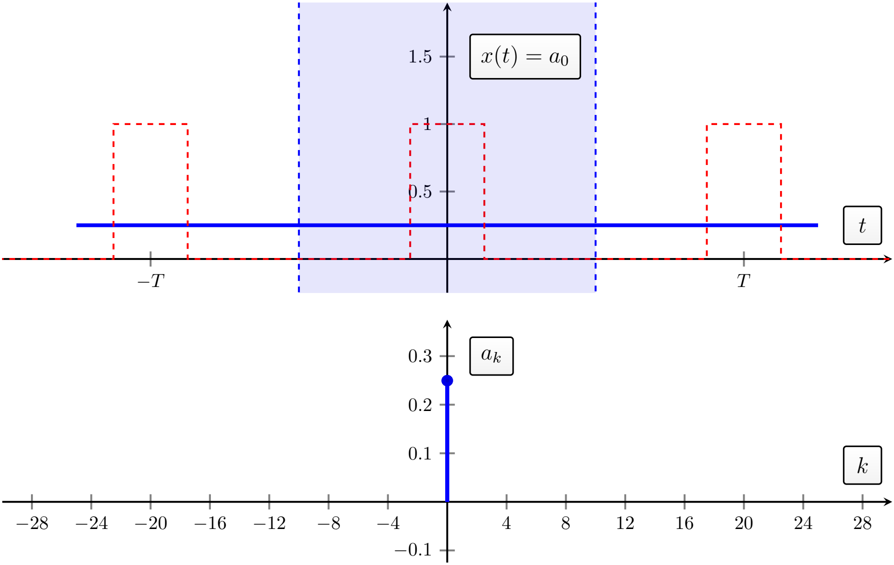
where

$$\begin{aligned} a_k &= \frac{\sin(k\,\omega_0 T_1)}{k\pi} = \frac{\sin(k\,2\pi T_1/T)}{k\pi} \\ &= \frac{\sin(k\,\pi/4)}{k\pi}, \end{aligned}$$

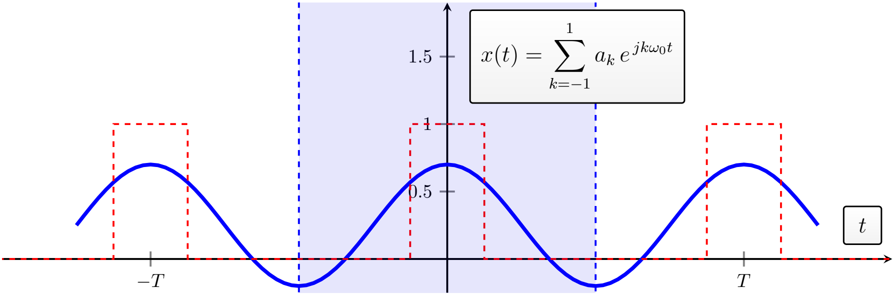
given $T_1 = T/8$, varying L as follows:

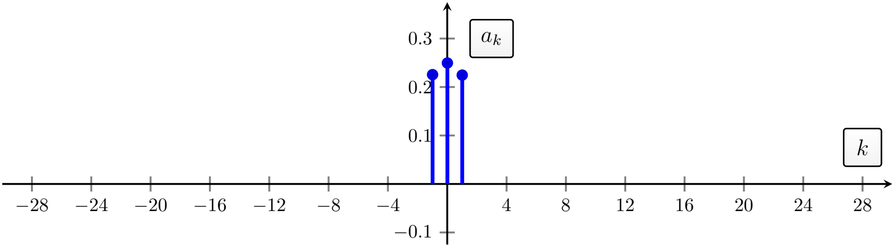
$$L = 0,\ 1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 9,\ 19,\ 27$$

Fourier Stuff – Periodic Rectangular Wave

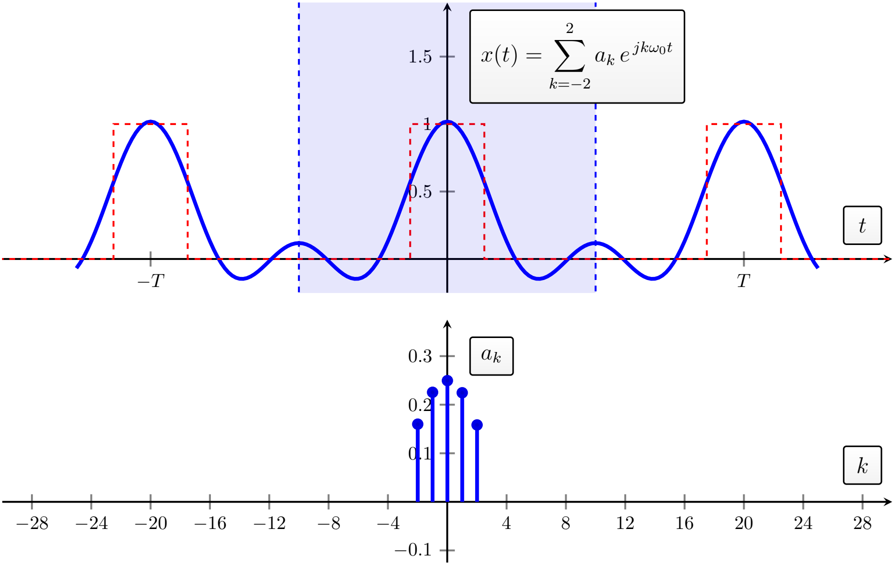


Fourier Stuff – Periodic Rectangular Wave

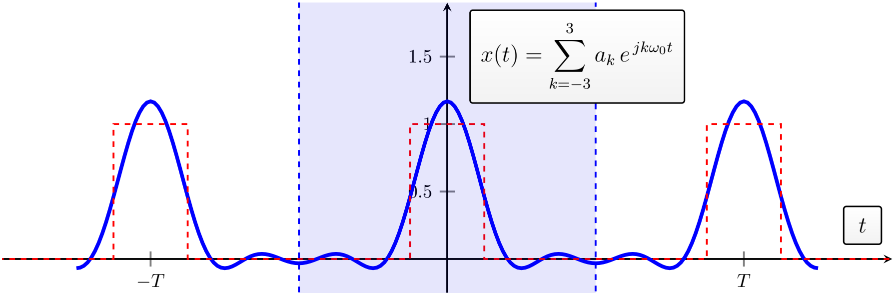


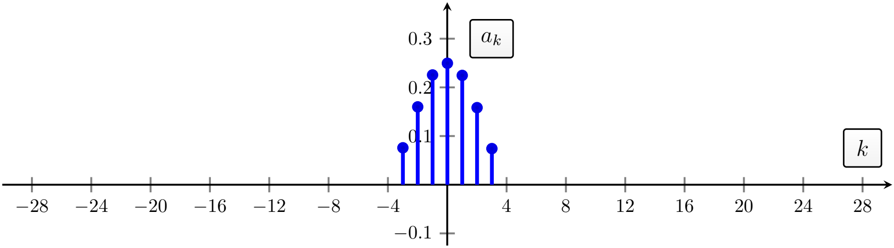


Fourier Stuff – Periodic Rectangular Wave

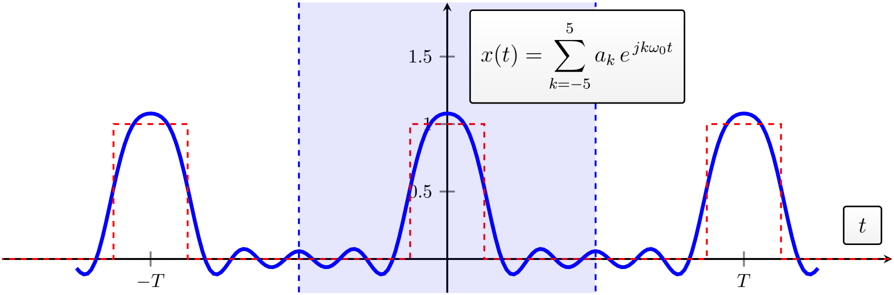


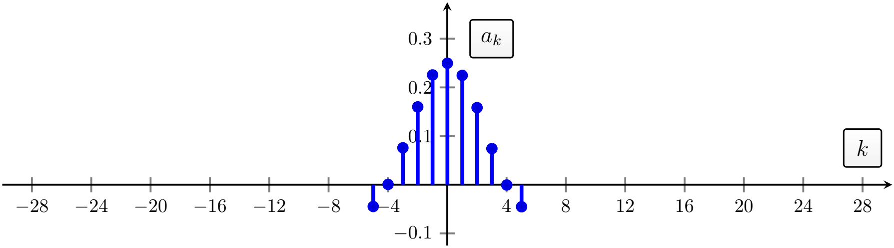
Fourier Stuff – Periodic Rectangular Wave





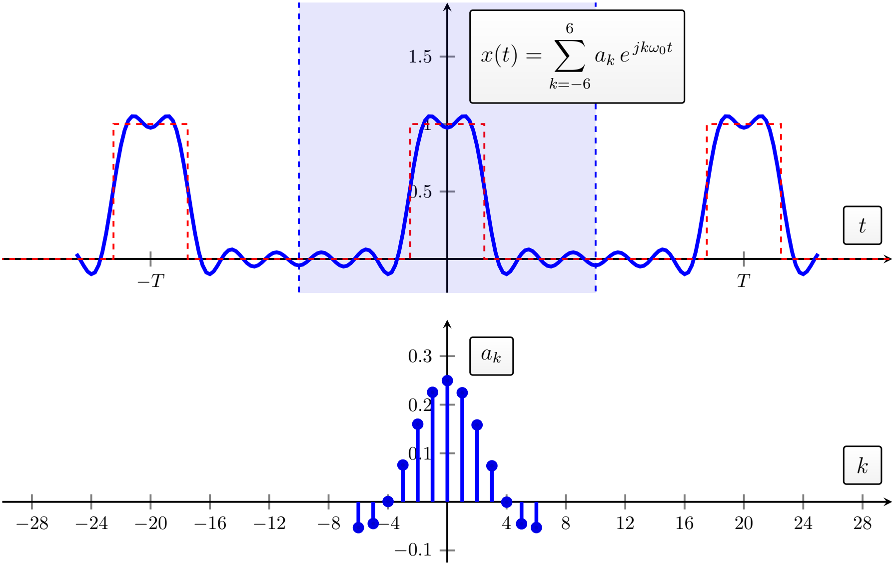
Fourier Stuff – Periodic Rectangular Wave



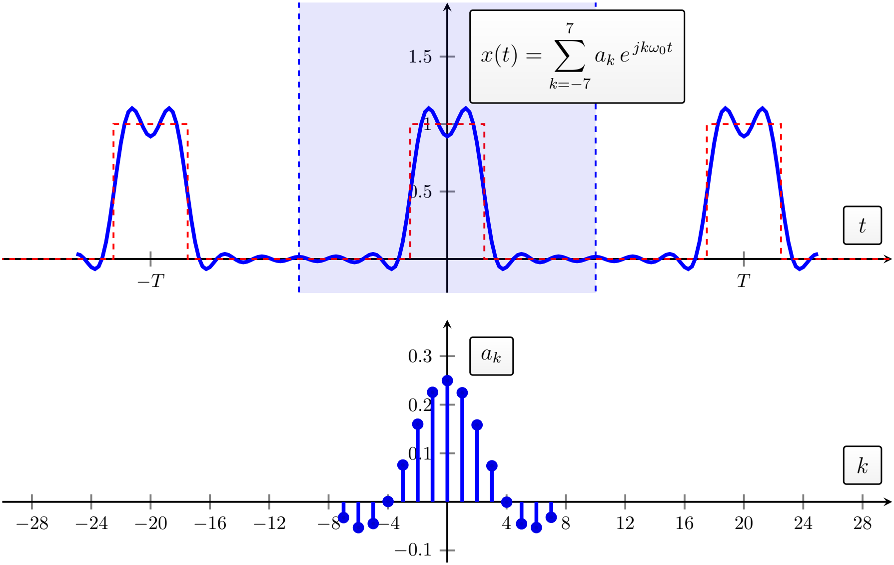


Fourier Stuff – Periodic Rectangular Wave

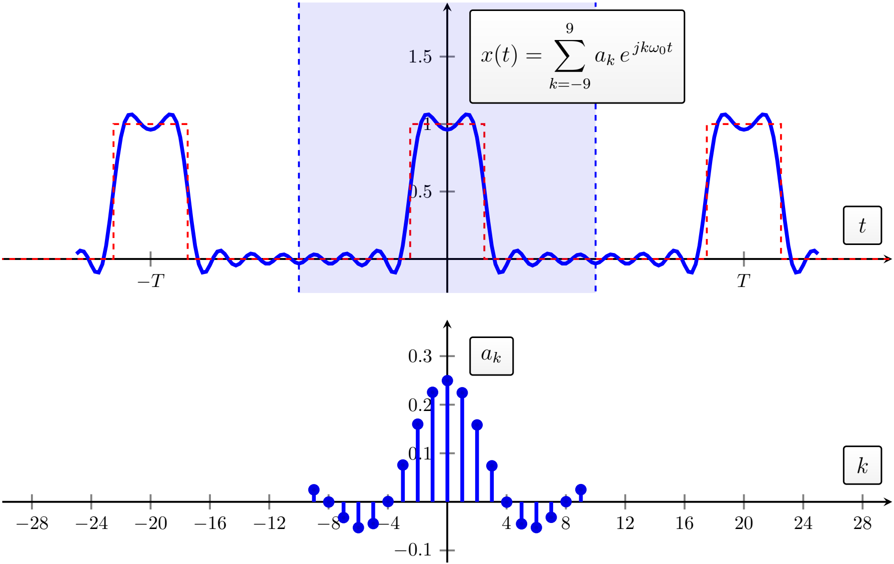
$$x(t) = \sum_{k=-6}^{6} a_k \, e^{jk\omega_0 t}$$



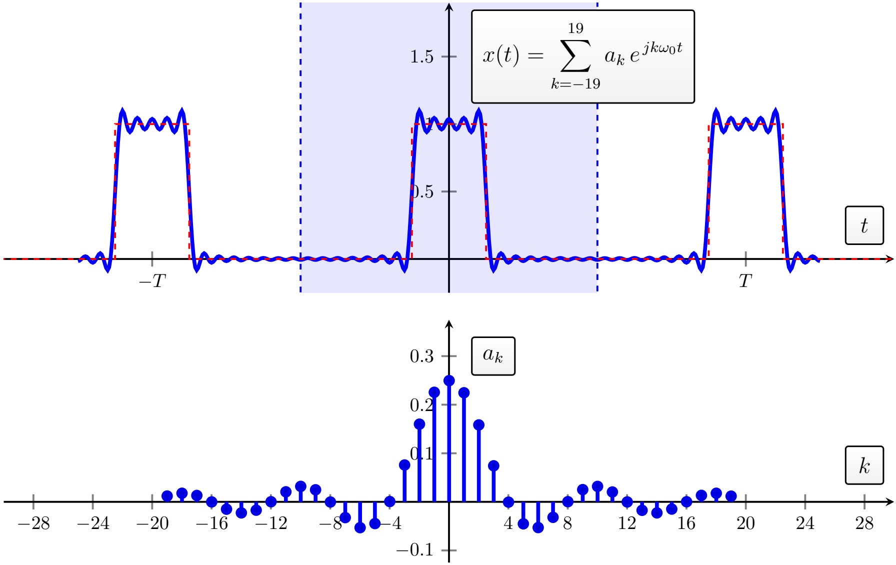
Fourier Stuff – Periodic Rectangular Wave



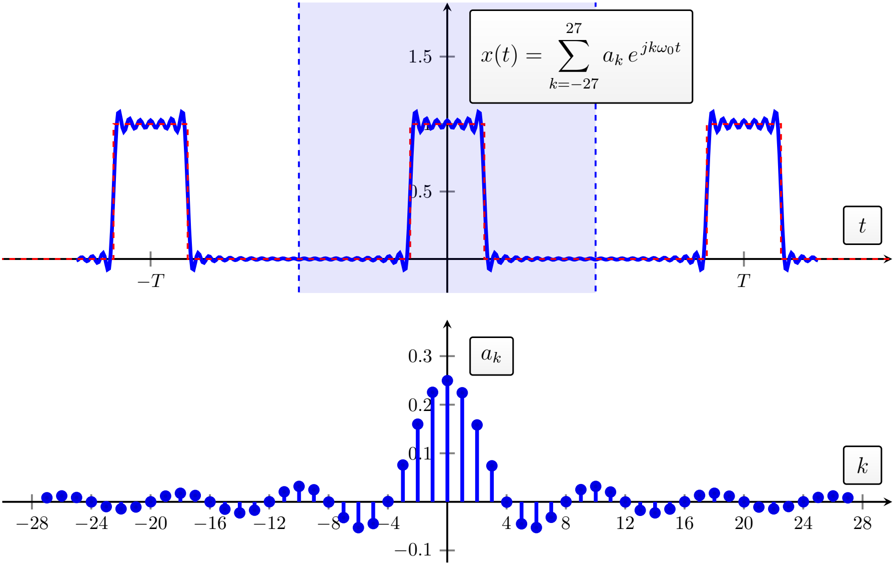
Fourier Stuff – Periodic Rectangular Wave



Fourier Stuff – Periodic Rectangular Wave



Fourier Stuff – Periodic Rectangular Wave



Australian National University

Fourier Series Properties – Notation

Previously we established:

Definition (Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad t \in \mathbb{R} \qquad (\textit{Synthesis Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\textit{Analysis Equation})$$

We adopt the shorthand

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k$$

and use this extensively.

Fourier Series Properties – Motivation

What's coming:

- We're lazy, we don't want to do work if we can avoid it. We don't want to do something complicated if there is an easy way or a trick available.
- Computing Fourier Series is not hard but can be tedious.
- With a few Fourier Series we can synthesize others. A signal derived from an original signal via a simply transformation such as time-shift, scaling, compression, etc., should have a Fourier Series some how related to the original signal.
- There are many of these, see Table 3.1 in text.

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Fourier Series Properties – Linearity

If

$$x(t) \xleftrightarrow{\mathscr{F}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\mathscr{F}} b_k$$

then

$$\boxed{\alpha\, x(t) + \beta\, y(t) \xleftrightarrow{\mathscr{F}} \alpha\, a_k + \beta\, b_k}$$

> *"Linear combinations of signals leads to identical linear combinations of the Fourier coefficients."*

For example, this can be used with parallel connection of signals.

$x(t)$ real valued $\quad \Rightarrow \quad \boxed{a_{-k} = \overline{a_k} \equiv a_k^* \quad \text{(conjugate)}}$

"Negative index Fourier coefficients are the complex conjugates of the positive index Fourier coefficients whenever the time domain signal is real valued (zero imaginary part)."

This implies for real valued signals

$$\mathit{Re}\{a_k\} \text{ is even,} \quad \mathit{Im}\{a_k\} \text{ is odd}$$

$$|a_k| \text{ is even,} \quad \angle a_k \text{ is odd}$$

Note that with any complex number written as $b\,e^{j\,\omega}$ with real $b > 0$ and angle ω, then $\angle b\,e^{j\,\omega} = \omega$, and $|b\,e^{j\,\omega}| = b$.

Fourier Series Properties – Time Shift

Given

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k$$

then

$$\boxed{x(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow} a_k\, e^{-jk\omega_0 t_0} \equiv a_k\, e^{-jk\,2\pi t_0/T}}$$

"Time shifting a signal introduces a linear phase shift $\propto t_0$ *in the Fourier coefficients."*

Fourier Series Properties – Parseval's Relation

It can be shown:

$$\boxed{\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2}$$

"Average energy is the time domain is the same as the energy in the frequency domain."

Given

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} b_k$$

then

$$\boxed{x(t)\,y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} c_k \triangleq \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell} = a_k \star b_k}$$

"The (pointwise) product of two periodic signals (of the same period, T) is the convolution of the Fourier coefficients."

For example, this can be used with serial/cascade connection of signals/systems.

Fourier Series Properties – Multiplication Property

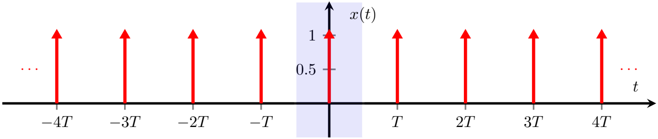
With Fourier Series pairs (time-domain and frequency-domain) convolution in one domain is multiplication in the other.

At this point we have only considered the product in time-domain leading to convolution in frequency-domain.

$$
\begin{aligned}
x(t)\,y(t) &= \sum_{\ell} a_\ell\, e^{j\,\ell\omega_0 t} \sum_m b_m\, e^{j\,m\omega_0 t} \\
&= \sum_{\ell}\sum_m a_\ell b_m\, e^{j\,(\ell+m)\omega_0 t} \\
&= \sum_k \underbrace{\Big(\sum_{\ell} a_\ell b_{k-\ell}\Big)}_{c_k} e^{jk\omega_0 t} \qquad (\ell + m = k)
\end{aligned}
$$

Periodic Impulse Train – Definition

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \quad \text{for all } k \end{aligned}$$

Periodic Impulse Train – Definition (cont'd)

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k = \frac{1}{T},$$

that is,

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

has Fourier Series

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

Periodic Impulse Train – Definition (cont'd)

$$x(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is known as the **sampling function**. As its name implies it is useful for sampling CT signals uniformly at time instants which are a multiple of T:

$$\begin{aligned} y_s(t) &\triangleq y(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} y(nT)\, \delta(t - nT) \end{aligned}$$

Periodic Impulse Train – Time Shifted PIT

Consider time shifting an **arbitrary** signal $x(t)$

$$x(t) \xleftrightarrow{\mathscr{F}} a_k$$

by the **specific** value of half the period, $t_0 = T/2$. By the "Time Shift" property, a linear phase shift is introduced relative to $x(t)$ in the Fourier coefficients

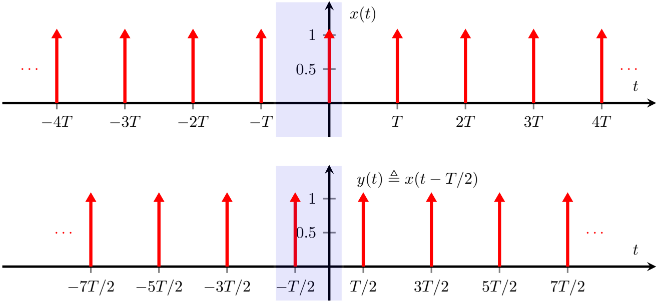
$$y(t) \triangleq x(t - T/2) \xleftrightarrow{\mathscr{F}} a_k e^{-jk\pi} \equiv (-1)^k a_k$$

from

$$e^{-jk\omega_0 T/2} = e^{-jk\pi}, \qquad e^{-j\pi} = -1$$

Next we consider the case when $x(t)$ is the impulse train.

Periodic Impulse Train – Time Shifted PIT (cont'd)



$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{T} \quad \Rightarrow \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{(-1)^k}{T}$$

Periodic Impulse Train – Time Shifted PIT

- The Fourier coefficients for both the PIT and $T/2$ time shifted PIT are purely real. Note that in the time domain both the PIT and $T/2$ time shifted PIT are even functions.
- A more general delay, e.g., $t_0 = 0.3498T$, would lead to a time shifted PIT with complex Fourier coefficients.

Periodic Convolution – Definition

By "Periodic Convolution" we mean convolution of periodic functions (of the same period T) in the time domain. But we need further qualifications.

Previously we found that multiplication in the time domain led to "discrete" convolution in the Fourier series (frequency) domain.

Doing the obvious

$$x(t) \star y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau$$

has the hazard of often being infinite. If both $x(t)$ and $y(t)$ are positive, then

$$x(t) \star y(t) = \infty$$

Periodic Convolution – Definition (cont'd)

So we define a modified periodic convolution.

Define Periodic Convolution to an integral over any one period, e.g., $-T/2$ to $T/2$. (The ends will justify the means.)

$$z(t) \triangleq \int_T x(\tau) y(t-\tau)\, d\tau$$

Let

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} a_k, \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} b_k, \quad z(t) \overset{\mathscr{F}}{\longleftrightarrow} c_k$$

Periodic Convolution – Key Result

Computing the Fourier coefficients of $z(t)$,

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T}\int_T \left(\int_{-T/2}^{T/2} x(\tau)y(t-\tau)\, d\tau\right) e^{-jk\omega_0 t}\, dt \\
&= \int_T \underbrace{\left(\frac{1}{T}\int_T y(t-\tau)\, e^{-jk\omega_0 (t-\tau)} dt\right)}_{b_k} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= b_k \int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \\
&= T\, a_k b_k
\end{aligned}$$

Multiplication in frequency (Fourier series domain).

Periodic Convolution – Key Result (cont'd)

In summary, for signals sharing a common period T,

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k, \quad y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} b_k, \quad z(t) \stackrel{\mathscr{F}}{\longleftrightarrow} c_k$$

related through Periodic Convolution

$$z(t) \triangleq \int_T x(\tau)y(t-\tau)\, d\tau$$

then

$$c_k = T\, a_k b_k$$

Alternatively

$$\left(\frac{1}{T}\int_T x(\tau)y(t-\tau)\, d\tau\right) \stackrel{\mathscr{F}}{\longleftrightarrow} a_k b_k$$

DT Periodic Signals – Definition

Now we consider **DT periodic signals** which satisfy:

$$x[n+N] = x[n] \quad \text{and} \quad \omega_0 = \frac{2\pi}{N}$$

where $N \in \mathbb{Z}$ is the fundamental period (smallest possible N).

Only complex exponentials which are periodic with period N will appear in the Fourier Series for such an $x[n]$:

$$\omega\, N = k\, 2\pi \iff \omega = k\omega_0, \quad k = 0, \pm 1, \pm 2, \cdots$$

DT Periodic Signals – Uniqueness and Aliasing

A typical N-**periodic complex** exponential signal would look like

$$e^{j\,(k\omega_0)n} \equiv e^{j\,k2\pi n/N}, \quad k, n, N \in \mathbb{Z}$$

and indeed this has the period N (consider signal at time $n+N$ and check if it is the same as the signal at time n, for all n)

$$e^{j\,k2\pi(n+N)/N} = e^{j\,k2\pi n/N} \overbrace{e^{j\,k2\pi}}^{1}$$

The way to view this, given the plethora of parameters, is that the frequency is

$$(\omega =)\ k\omega_0 \equiv k\,2\pi/N$$

and the signal can be written z^n with

$$z \triangleq e^{j\,(k\omega_0)}.$$

DT Periodic Signals – Uniqueness and Aliasing (cont'd)

For example, if $N = 16$, we could have $k = 209$ and

$$e^{j\,(209\times 2\pi)n/16} = e^{j\,(418\pi)n/16}, \quad n \in \mathbb{Z}$$

is of period 16. However, it is actually identical to

$$e^{j\,2\pi n/16}, \quad n \in \mathbb{Z}$$

since $209 = 13 \times 16 + 1$ and n takes only integer values.

DT Periodic Signals – Uniqueness and Aliasing

To visualize we give a simpler example, if $N = 4$, we could have $k = 5$ and

$$e^{j\,(5\times 2\pi)n/4} = e^{j\,10\pi n/4}$$

is a complex exponential of period 4, and is actually identical to

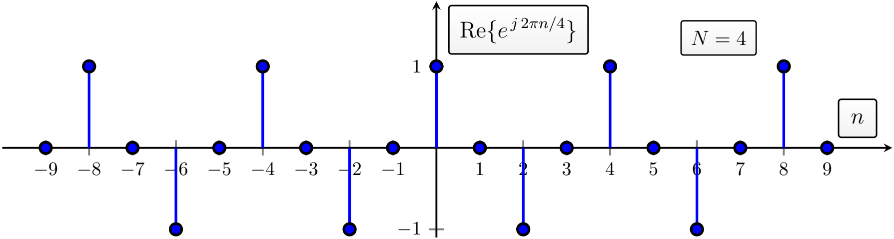
$$e^{j\,2\pi n/4}$$

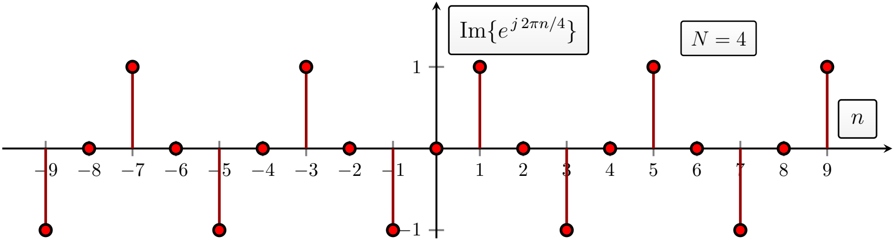
since $5 = 1 \times 4 + 1$ and n takes only integer values.

Signal $e^{j\,2\pi n/4}$ is plotted on the next few pages (top is real part and bottom is imaginary part).

DT Periodic Signals – Uniqueness and Aliasing

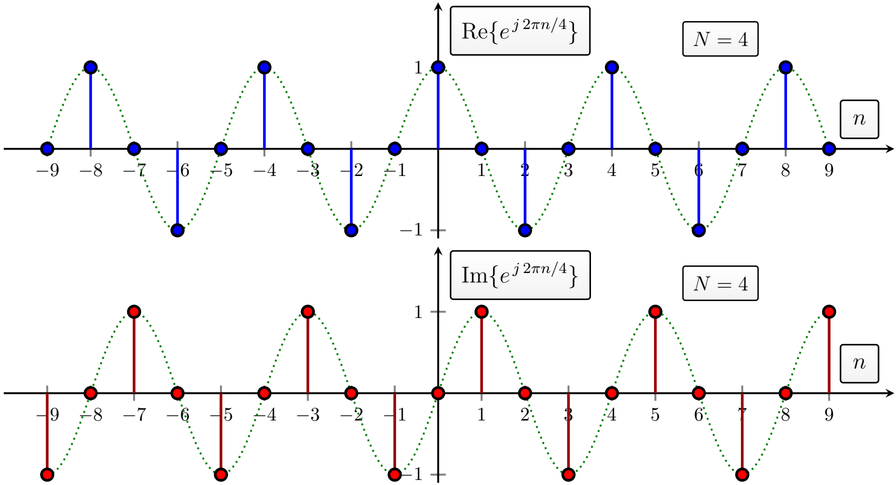
$$\ldots, j, -1, -j, 1, j, -1, -j, \ldots$$





DT Periodic Signals – Uniqueness and Aliasing

$e^{j\,2\pi n/4}$ **vs** $e^{j\,(5\times 2\pi)n/4}$



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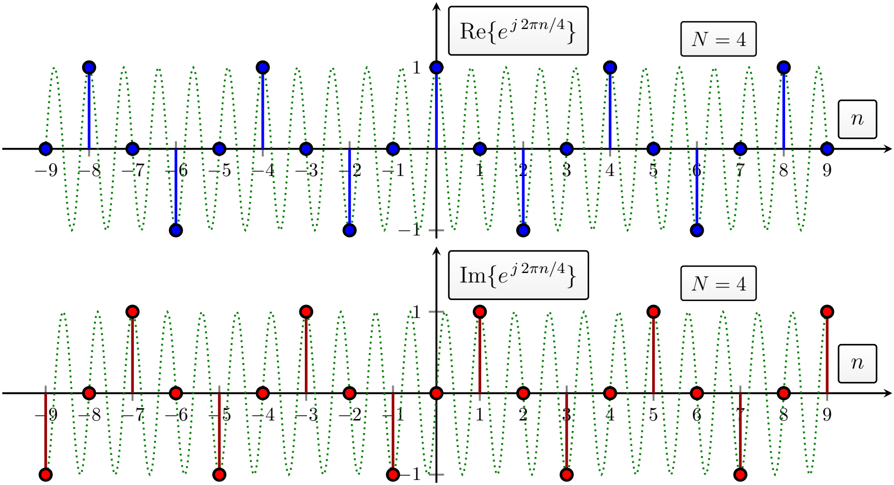


DT Periodic Signals – Uniqueness and Aliasing

Higher order frequencies are "aliased" to appear identical to lower order frequencies when sampled (movie wagon wheel effect).

We conclude there are only N distinct complex exponential signals which we may take as (other choices are fine also)

$$\underbrace{1, e^{j\omega_0 n}, e^{j2\omega_0 n}, e^{j3\omega_0 n}, \ldots, \; e^{j(N-1)\omega_0 n}}_{N \text{ signals}}, \quad \omega_0 = \frac{2\pi}{N}$$

or equivalently

$$\underbrace{1, e^{j2\pi n/N}, e^{j4\pi n/N}, e^{j6\pi n/N}, \ldots, \; e^{j(N-1)\pi n/N}}_{N \text{ signals}}$$

DT Periodic Signals – $N = 4$ Complex Exponentials

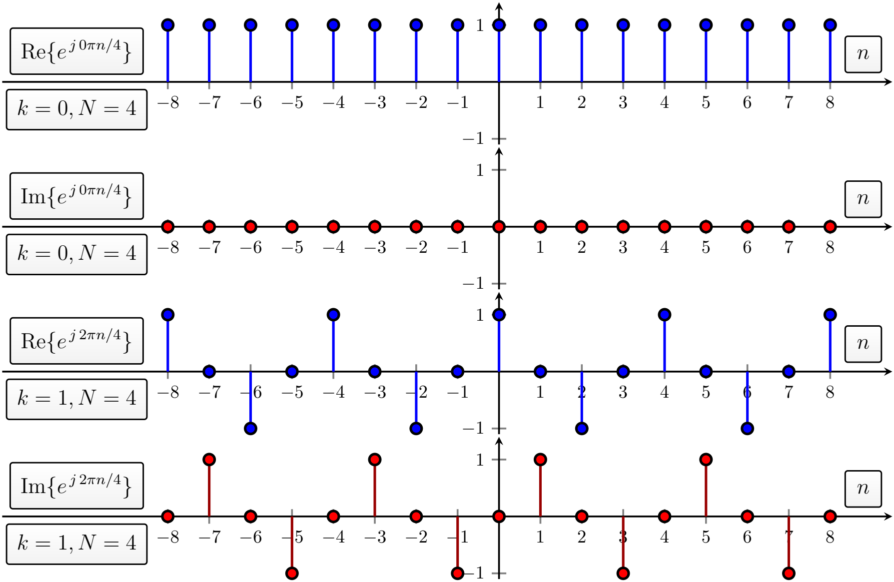
For $N = 4$, the 4 complex exponentials,

$$\boxed{e^{j2kn\pi/N}, \quad k = 0, 1, 2, 3}$$

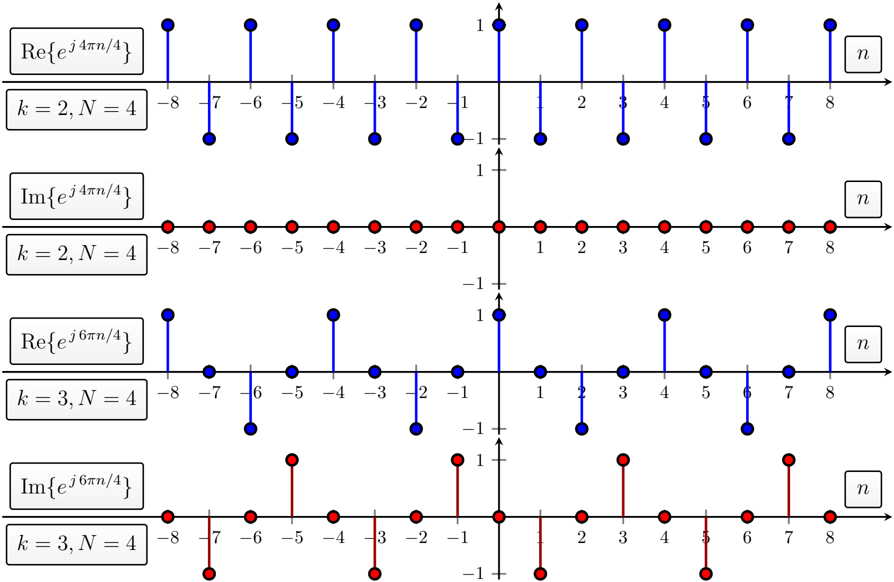
are shown in the next two slides. Remember:

- k is the index for the signal of which there are four ($k \in \{0, 1, 2, 3\}$),
- n is the discrete time index of which there are, in principle, infinity many values ($n \in \mathbb{Z}$),
- mathematically, k and n can be swapped
- we could have an infinite number of both k and n but for both, since they are swappable, we could just use 4 values; the "redundancy" in n is because the complex exponential is periodic and the "redundancy" in k is because of frequency aliasing.

DT Periodic Signals – $N = 4$ Complex Exponentials



DT Periodic Signals – $N = 4$ Complex Exponentials



Every N-period DT signal $x[n]$ has a **Fourier Series representation**

$$\boxed{x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}}$$

where the Fourier Series coefficients

$$a_k, \quad k = 0, 1, \ldots N-1$$

can be determined (the unknowns).

In fact, if we take the equations for $x[0], x[1], \ldots x[N-1]$ above then we have N linear equations in N unknowns. Just linear algebra.

Alternatively, to figure out the a_k, we have the (completeness) identity

$$\sum_{n=0}^{N-1} e^{jk\omega_0 n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$
$$= N\,\delta[n]$$

So

$$\sum_{n=0}^{N-1} x[n]\, e^{-jm\omega_0 n} = \sum_{n=0}^{N-1} \Big(\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \Big) e^{-jm\omega_0 n}$$
$$= \sum_{k=0}^{N-1} a_k \underbrace{\Big(\sum_{n=0}^{N-1} e^{j(k-m)\omega_0 n} \Big)}_{N\,\delta[k-m]}$$
$$= N\, a_m$$

DT Periodic Signals – Fourier Series (cont'd)

Definition (DT Fourier Series Pair)

For $x[n] = x[n+N]$ periodic with period N and $\omega_0 = 2\pi/N$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad (Synthesis\ Equation)$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}, \quad k \in \mathbb{Z} \qquad (Analysis\ Equation)$$

- Note that the a_k are N-periodic, $a_k = a_{k+N}$, and so only N consecutive a_k need be known/computed.
- Similarly, only N consecutive $x[n]$ are needed.

Since the discrete time signal is N periodic, the Fourier Series coefficients are N periodic and the synthesis and analysis equations are linear then the conversion between

$$N \text{ consecutive } a_k \longleftrightarrow N \text{ consecutive } x[n]$$

is describable by a $N \times N$ (invertible) matrix. Either direction would require of the order of N^2 multiplications plus additions.

There is a cleverer way to do the matrix multiplication, it is called the **Fast Fourier Transform (FFT)**. It is pervasive throughout electronic systems and communications. Your digital TV wouldn't work without it. If $N = 1024$ then the FFT is $10\times$ faster than otherwise.

DT Periodic Signals – Fourier Series (cont'd)

Recall

Definition (CT Fourier Analysis and Synthesis)

For $x(t) = x(t+T)$ periodic with period T and $\omega_0 = 2\pi/T$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\,k\omega_0 t}, \quad t \in \mathbb{R} \qquad (\mathit{Synthesis\ Equation})$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad k \in \mathbb{Z} \qquad (\mathit{Analysis\ Equation})$$

DT Periodic Signals – Recap to Here

This is getting confusing. There seems to be some patterns. What is going on?

Towards some general organizing thoughts. . .

- Signals can be either **continuous** or **discrete**.
- Signals can be periodic or not periodic.
- We have Fourier Series which is discrete in nature, it involves a summation using Fourier series coefficients. In fact, that is what a "series" means mathematically.
- The Fourier coefficient domain is also called the frequency domain.
- Sometimes the Fourier Series is periodic.
- It seems there is no continuous Fourier Series object, and if there were I wouldn't know what it meant in any case. . .

DT Periodic Signals – Recap to Here

- Signals can indeed be continuous or discrete time. (If you use delta functions, $\delta(t)$, then you can imbed the discrete time representation in a continuous time theory but there isn't too much to gain.) You never work with mixtures of these and in most practical situations the discrete time case is more important.
- The numbers $1, 0.5, 0.25, \ldots$ form a sequence and their sum $1 + 0.5 + 0.25 + \cdots$ is a **series** and equals 2. That is standard terminology. The terminology "Fourier Series" derives from this **summation** concept. How general is the Fourier Series and what signals can it represent?

DT Periodic Signals – Recap to Here

- A Fourier Series adds together terms with the same period or divisors of this period. Therefore the **signals that a Fourier Series represents must be periodic**. Less obvious but true is the fact that **virtually all periodic signals have a Fourier Series representation** (the exceptions can be considered by engineers as being pathological).
- The above statement makes no reference to whether the signal's domain is **continuous or discrete time**, and it is true regardless.
- If the Fourier Series is periodic (in addition to the signal necessarily being periodic) then it must be discrete in time. So a statement which says "a signal whose Fourier Series is periodic" contains two pieces of information and could be read "a signal which is periodic and discrete in time".

DT Periodic Signals – Recap to Here

- Later we will see that a continuous frequency domain description is possible and from our discussion cannot be for a periodic time domain signal.
- Every time we see convolution in one domain (time/frequency) there is multiplication in the other (frequency/time)

DT Periodic Signals – Examples

Example 1:

Sum of Two Sinusoids:

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

What is the period N? Look for $2\pi/N$ and the first term is slower. Period $N = 16$, which implies $\omega_0 = \pi/8$.

$$x[n] = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) + \frac{1}{2}\left(e^{j\pi/4}\, e^{j2\omega_0 n} + e^{-j\pi/4}\, e^{-j2\omega_0 n}\right)$$

Hence

$$\begin{array}{lccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & a_{-18} & a_{-17} & a_{-16} & a_{-15} & a_{-14} \\
 & a_{-2} & a_{-1} & a_0 & a_1 & a_2 \\
 & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{Coefficient Values:} & \frac{1}{2}e^{j\pi/4} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2}e^{-j\pi/4}
\end{array}$$

Example 2:

DT Square Wave:

Let $x[n]$ be periodic with period N. With N_1 such that $2N_1 + 1 \leq N$, define $x[n]$ over N-length interval

$$[-N_1, N - N_1 - 1]$$

as follows

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & N_1 + 1 \leq n < N - N_1 - 1 \end{cases}$$

For other n we can just use

$$x[n] = x[n + N]$$

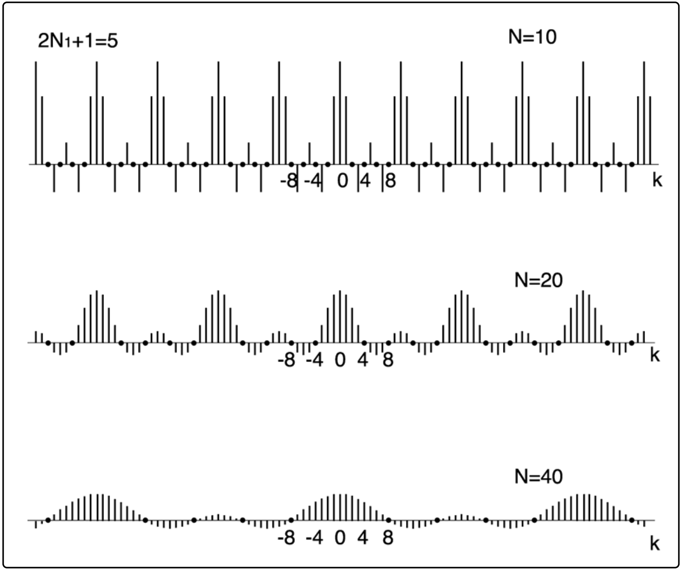
DT Periodic Signals – Examples (cont'd)

Then the (N-periodic) Fourier coefficients are

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$
$$= \frac{1}{N} \times \begin{cases} \dfrac{\sin\big(2\pi k(N_1+1/2)/N\big)}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ 2N_1 + 1, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

- This is like a "sampled" periodic sinc function.

DT Periodic Signals – Examples (cont'd)



Continuous Time

$$\delta(t) \longrightarrow \boxed{\text{CT-LTI}} \longrightarrow h(t)$$

$$\left\langle \begin{matrix} \textit{complex} \\ \textit{exponential} \end{matrix} \right\rangle x(t) \longrightarrow \boxed{h(t)} \longrightarrow \left\langle \begin{matrix} \textit{complex} \\ \textit{exponential} \end{matrix} \right\rangle y(t)$$

which leads to

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\, e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

where $s \in \mathbb{C}$ is a complex number (not necessarily of unit magnitude).

Fourier Series and LTI Systems – Eigenfunctions Revisited

Discrete Time

$$\delta[n] \longrightarrow \boxed{\text{DT-LTI}} \longrightarrow h[n]$$

$$\left\langle \begin{array}{c} \textit{complex} \\ \textit{exponential} \end{array} \right\rangle x[n] \longrightarrow \boxed{h[n]} \longrightarrow \left\langle \begin{array}{c} \textit{complex} \\ \textit{exponential} \end{array} \right\rangle y[n]$$

which leads to

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow H(z)\, z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

where $z \in \mathbb{C}$ is a complex number (not necessarily of unit magnitude).

Refresher

- CT Complex exponential signals with $|s| = 1$ are the special case of periodic signals.
- DT with $|z| = 1$ are also periodic signals but we also need $z^N = 1$ for some N. For example, $z = e^{j1}$ satisfies $|z| = 1$ but is not periodic.
- Periodic signals can be built up from such complex exponential signals.
- For periodic signals we can use Fourier Series representations.
- When we say "periodic" we mean "periodic with period $T \in \mathbb{R}$ or $N \in \mathbb{Z}$", for CT and DT, respectively.

Continuous Time

$$\left\langle \begin{array}{c}\textit{periodic}\\ \textit{signal}\end{array} \right\rangle x(t) \longrightarrow \boxed{h(t)} \longrightarrow \left\langle \begin{array}{c}\textit{periodic}\\ \textit{signal}\end{array} \right\rangle y(t)$$

Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{\,jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k \, \overbrace{H(\,jk\omega_0)}^{\text{eigenvalue}}}_{b_k} \, e^{\,jk\omega_0 t}$$

The Fourier coefficients map like

$$a_k \to b_k \triangleq H(\,jk\omega_0)\, a_k$$

Here the complex gain (eigenvalue) $H(\,jk\omega_0) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$, that is, $e^{\,jk\omega_0 t}$ (eigenfunction).

Discrete Time

$$\left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle x[n] \longrightarrow \boxed{h[n]} \longrightarrow \left\langle \begin{matrix} \textit{periodic} \\ \textit{signal} \end{matrix} \right\rangle y[n]$$

Fourier Series representation

$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y(t) = \sum_{k=0}^{N-1} \underbrace{a_k \, H(e^{jk\omega_0})}_{b_k} \, e^{jk\omega_0 n}$$

The Fourier coefficients map like

$$a_k \to b_k \triangleq H(e^{jk\omega_0}) \, a_k$$

Here complex gain $H(e^{jk\omega_0}) \in \mathbb{C}$ scales/filters the periodic signal component at frequency $k\omega_0$.

$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)\, e^{j\omega t}, \quad s = j\omega$$

Definition (CT Frequency Response)

The **CT Frequency Response** is defined by

$$H(j\omega) \triangleq \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = H(s)\Big|_{s=j\omega}$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

Frequency Response of LTI System – Discrete Time

$$e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega})\, e^{j\omega n}, \qquad z = e^{j\omega n}$$

Definition (DT Frequency Response)

The **DT Frequency Response** is defined by

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = H(z)\Big|_{z=e^{j\omega}}$$

Note that ω need not be multiples of some ω_0. ω can take any value (still have eigenfunctions).

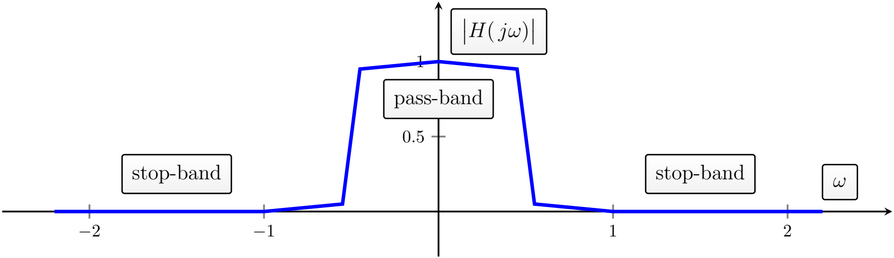
Freq Shaping and Filtering – Key Observation

We can choose/design $H(j\omega)$, or $H(e^{j\omega})$, as a function of (radial) frequency ω to determine how frequency components of the input are passed/amplified/attenuated to the output.

For example, the bass, treble and mid-range control in an audio system:

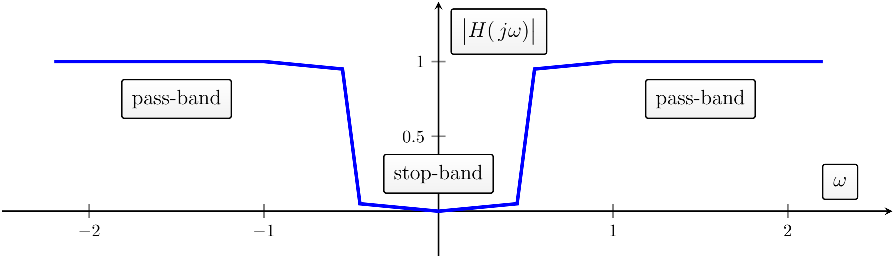
- To boost bass say at 100 Hz or $\omega = 200\pi$ then we have $|H(j200\pi)| > 1$
- To attenuate treble say at 1 kHz or $\omega = 2000\pi$ then we have $|H(j2000\pi)| < 1$

The CT "Low Pass" frequency response looks like:



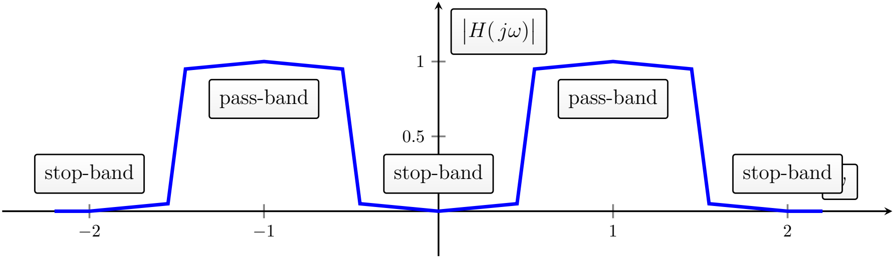
- Conventionally look at the magnitude, $|H(j\omega)|$, to characterize the type of "filter".
- Defines "passband" (low or no attenuation) and "stopband" (high attenuation).

The CT "High Pass" frequency response looks like:



- DC gain should be small or zero.
- High frequencies are passed, low frequencies are blocked. The exact shape isn't so important.

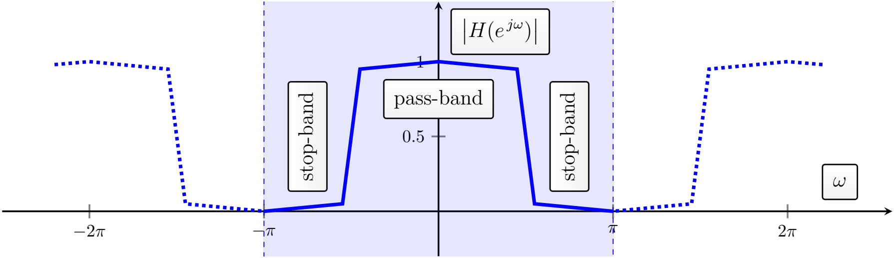
The CT “Band Pass” frequency response looks like:



- DC gain should be small or zero. The high frequency gain should be small or zero.
- High frequencies are blocked, low frequencies are blocked, mid-frequencies are passed. The exact shape isn’t so important.

DT Low Pass Filter:

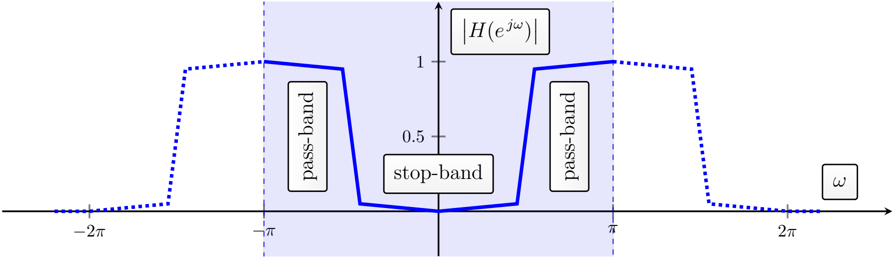
The DT "Low Pass" frequency response looks like:



- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

DT High Pass Filter:

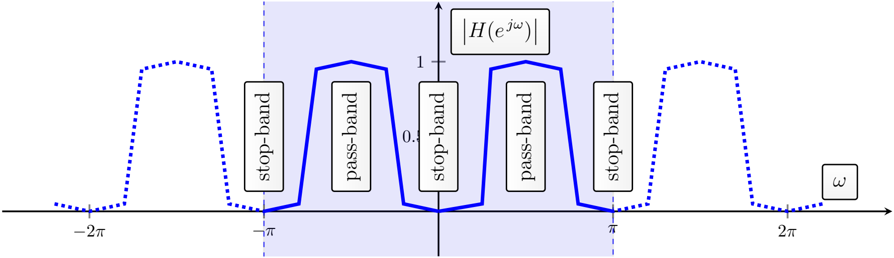
The DT "High Pass" frequency response looks like:



- Most often look at the magnitude, $|H(e^{j\omega})|$, to characterize the type of "filter".
- Note the spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π. $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

DT Band Pass Filter:

The DT "Band Pass" frequency response looks like:

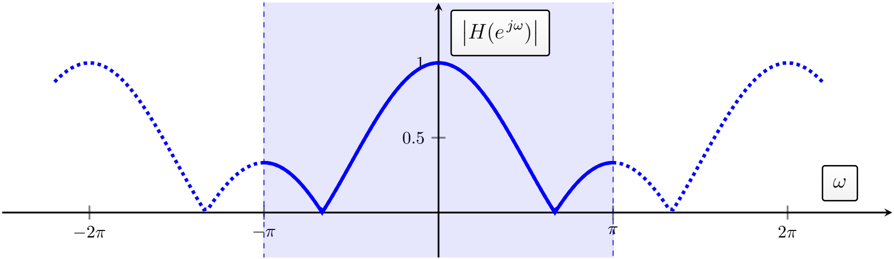


- Most often look at the magnitude, $\left|H(e^{j\omega})\right|$, to characterize the type of "filter".
- The spectrum/frequency response is periodic because $e^{j\omega}$ is periodic with period 2π.
- The frequency $\omega = \pi$ (and $\omega = -\pi$) is the "highest" frequency.

Example 1:

3 term non-causal moving average:

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$



$$\left|\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)\right|$$

Freq Shaping and Filtering – DT Band Pass Filter

From

$$y[n] = \tfrac{1}{3}\,x[n+1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n-1]$$

we deduce

$$h[n] = \begin{cases} \tfrac{1}{3}, & n = -1, 0, +1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = \tfrac{1}{3} \sum_{n=-1}^{1} e^{-j\omega n}$$

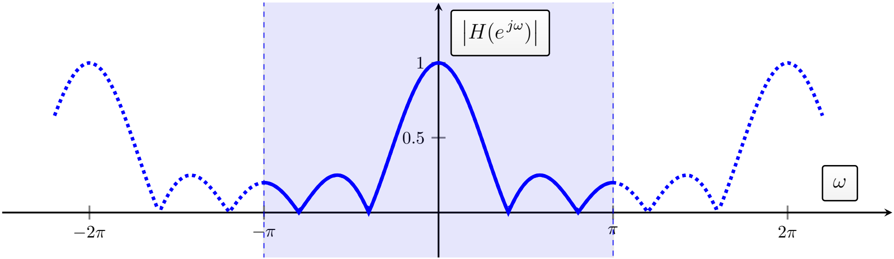
$$= \tfrac{1}{3} + \tfrac{2}{3}\,\frac{e^{j\omega} + e^{-j\omega}}{2} = \tfrac{1}{3} + \tfrac{2}{3}\cos(\omega)$$

Note that this is purely real. The phase appears to be zero. In reality it flips between 0 and π — it is π at those frequencies ω where $\tfrac{1}{3} + \tfrac{2}{3}\cos(\omega) < 0$.

Example 2:

5 term non-causal moving average:

$$y[n] = \tfrac{1}{5}\,x[n+2] + \tfrac{1}{5}\,x[n+1] + \tfrac{1}{5}\,x[n] + \tfrac{1}{5}\,x[n-1] + \tfrac{1}{5}\,x[n-2]$$



$$\left|\tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega)\right|$$

Freq Shaping and Filtering – DT Band Pass Filter

Example 3:

Causal moving average of two terms:

$$\boxed{y[n] = \tfrac{1}{2}\, x[n] + \tfrac{1}{2}\, x[n-1]}$$

which implies

$$h[n] = \begin{cases} \frac{1}{2} & n = 0, +1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} H(e^{j\omega}) &\triangleq \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = \tfrac{1}{2} \sum_{n=0}^{1} e^{-j\omega n} \\ &= \tfrac{1}{2}\left(1 + e^{-j\omega}\right) = \tfrac{1}{2} e^{-j\omega/2} 2\left(\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}\right) \\ &= e^{-j\omega/2} \cos(\omega/2) \end{aligned}$$

Note that this is complex. The phase is linear within the range $-\pi < \omega < \omega$.

Example 4:

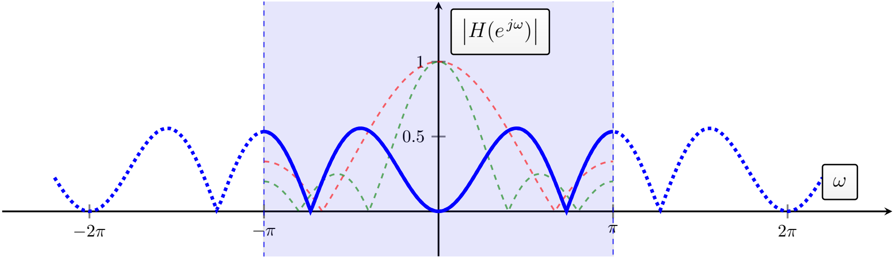
Causal moving average filters of different orders:

Order	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
1	1	1
2	$e^{-j\omega/2}\cos(\omega/2)$	$\lvert\cos(\omega/2)\rvert$
3	$e^{-j\omega}\left(\frac{1}{3}+\frac{2}{3}\cos(\omega)\right)$	$\left\lvert\frac{1}{3}+\frac{2}{3}\cos(\omega)\right\rvert$
4	$e^{-j3\omega/2}\left(\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right)$	$\left\lvert\frac{1}{2}\cos(\omega/2)+\frac{1}{2}\cos(3\omega/2)\right\rvert$
5	$e^{-j2\omega}\left(\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right)$	$\left\lvert\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right\rvert$

Example 5:

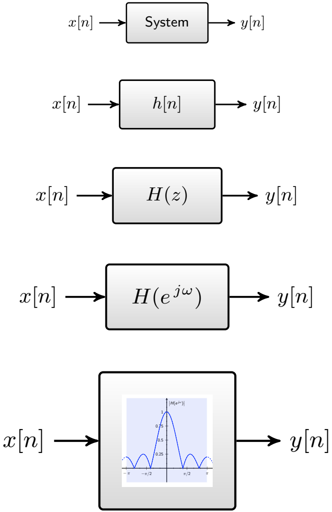
Poor Man's DT Band Pass Filter: 3/5 non-causal moving average difference:

$$\begin{aligned} y[n] = {} & \tfrac{1}{3}\,x[n-1] + \tfrac{1}{3}\,x[n] + \tfrac{1}{3}\,x[n+1] \\ & - \tfrac{1}{5}\,x[n-2] - \tfrac{1}{5}\,x[n-1] - \tfrac{1}{5}\,x[n] - \tfrac{1}{5}\,x[n+1] - \tfrac{1}{5}\,x[n+2] \end{aligned}$$



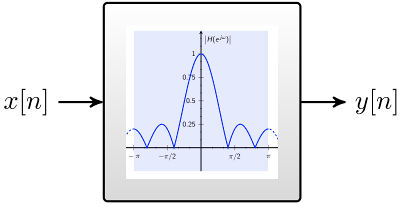
FS and LTI Systems – Summary

Evolution of System Models (DT Signals and Systems)



- System – input output functional relationship
- LTI System – pulse response picture
- Eigen-system – complex exponentials z^n
- Frequency Response – sine-wave type signals
- Filter – pass and stop various frequencies

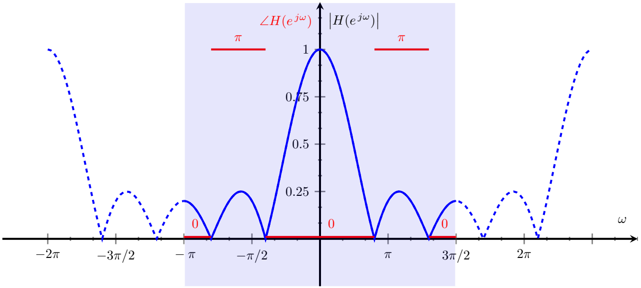
FS and LTI Systems – Summary (cont'd)



- Discrete time ("$[\cdot]$" and n is integer)
- Any frequency in the input is possible. In $H(e^{j\omega})$ there is no restriction on ω.
- But $H(e^{j\omega})$ is periodic with period 2π because $e^{j\omega}$ is periodic with period 2π.
- So DT frequency response is periodic, $\omega = 0$ is DC and maximum frequencies are really $\pm\pi$.

The 5 term non-causal moving average is:

$$y[n] = \frac{1}{5}\,x[n+2] + \frac{1}{5}\,x[n+1] + \frac{1}{5}\,x[n] + \frac{1}{5}\,x[n-1] + \frac{1}{5}\,x[n-2]$$



This has "flat" phase. This is because of the special form of the filter coefficients and a non-causal formulation.

FS and LTI Systems – Magnitude and Phase

For this example, the frequency response is:

$$\boxed{H(e^{j\omega}) = \tfrac{1}{5} + \tfrac{2}{5}\,\cos(\omega) + \tfrac{2}{5}\,\cos(2\omega) \in \mathbb{R}}$$

is purely real (by luck), but for some values of ω the frequency response takes **negative** real values. For example, when $\omega = \pi/2$ then

$$\begin{aligned} H(e^{j\pi/2}) &= \tfrac{1}{5} + \tfrac{2}{5} \times 0 + \tfrac{2}{5} \times (-1) \\ &= -\tfrac{1}{5} \equiv \tfrac{1}{5}\, e^{j\pi} \end{aligned}$$

So the magnitude of $H(e^{j\pi/2})$ is $\frac{1}{5}$ and the phase is π. This is a little annoying as the phase can show an apparent discontinuity whereas there really is none in the frequency response. Sometimes engineers do dumb things.

FS and LTI Systems – Magnitude and Phase

For this example, the **magnitude** of the frequency response is:

$$\boxed{\left|H(e^{j\omega})\right| = \left|\tfrac{1}{5} + \tfrac{2}{5}\cos(\omega) + \tfrac{2}{5}\cos(2\omega)\right|}$$

and the **phase** of the frequency response is:

$$\boxed{\angle H(e^{j\omega}) = \begin{cases} 0 & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ \pi & \text{otherwise} \end{cases}}$$

as plotted previously.

However, this phase is relatively boring and atypical. In contrast, this course is absolutely boring and typical.

FS and LTI Systems – Magnitude and Phase

- We also note that we only need to plot magnitude and phase over one period such as $\pi < \omega \leq \pi$.
- Of course, we could have taken the negating phase as $-\pi$ instead of π. This choice makes more sense because it lets us see whether the magnitude and phase are even, or odd, or neither even nor odd.
- For a **filter with real coefficients** the **magnitude is an even function** and the **phase is an odd function**.

Last lecture we saw the **causal** (implementable) 5 term moving average was:

$$\boxed{y[n] = \tfrac{1}{5}\,x[n] + \tfrac{1}{5}\,x[n-1] + \tfrac{1}{5}\,x[n-2] + \tfrac{1}{5}\,x[n-3] + \tfrac{1}{5}\,x[n-4]}$$

and had frequency response

$$\boxed{H(e^{j\omega}) = e^{-j2\omega}\big(\tfrac{1}{5} + \tfrac{2}{5}\cos(\omega) + \tfrac{2}{5}\cos(2\omega)\big), \quad \pi < \omega \le \pi}$$

The magnitude is as for the non-causal filter:

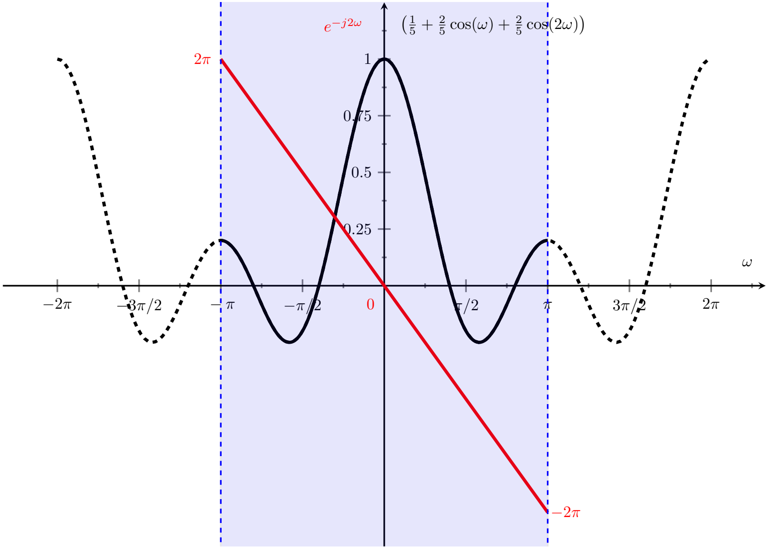
$$\boxed{\left|H(e^{j\omega})\right| = \left|\tfrac{1}{5} + \tfrac{2}{5}\ \cos(\omega) + \tfrac{2}{5}\ \cos(2\omega)\right|}$$

but the phase is linear (apart from the π discontinuities)

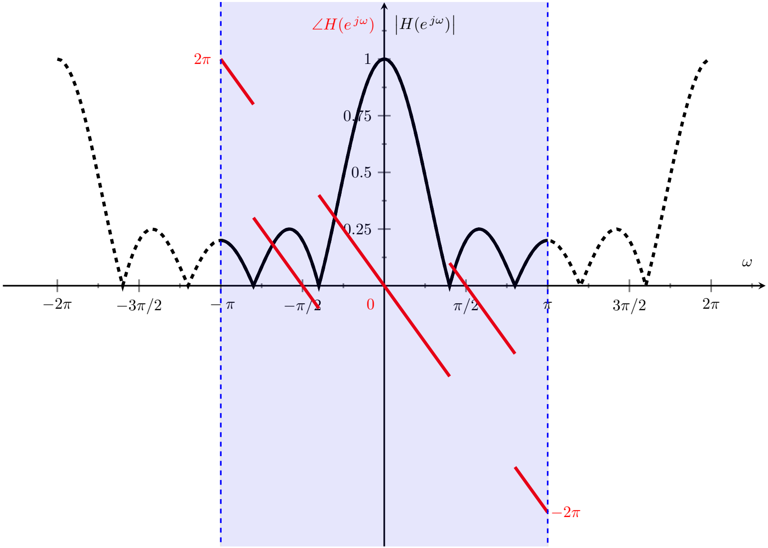
$$\boxed{\angle H(e^{j\omega}) = \begin{cases} -2j\omega & \text{if } H(e^{j\omega}) = \left|H(e^{j\omega})\right| \\ -2j\omega \pm \pi & \text{otherwise} \end{cases}}$$

Here one conventionally takes the sign of π such that $-\pi < -2j\omega \pm \pi \leq \pi$.

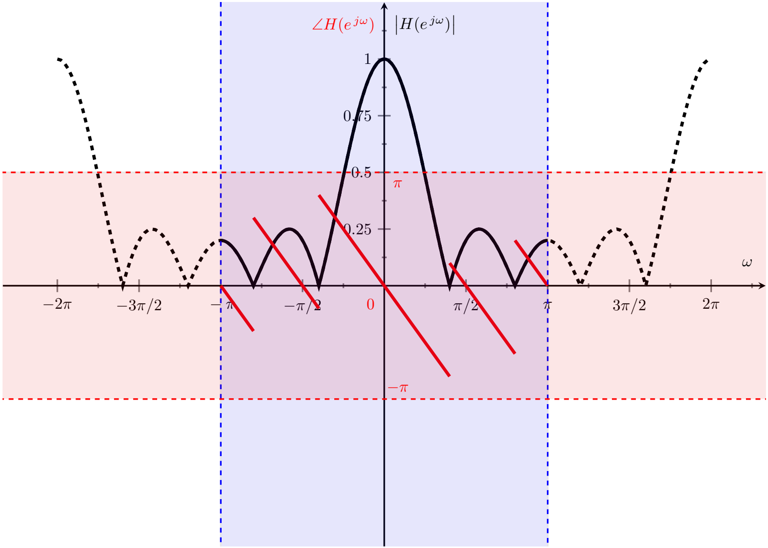
FS and LTI Systems – Magnitude and Phase



FS and LTI Systems – Magnitude and Phase



FS and LTI Systems – Magnitude and Phase



FS and LTI Systems – Magnitude and Phase

Magnitude and Phase Features:

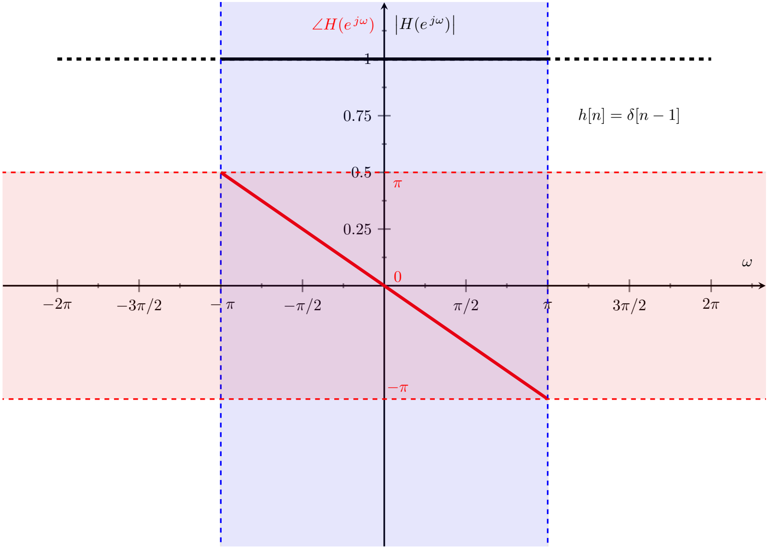
- Going from $H(e^{j\omega})$ to $\left|H(e^{j\omega})\right|$ introduces weirdness in phase such as apparent discontinuities (at frequencies ω where $H(e^{j\omega})$ gets small and passes through the origin in the complex plane).
- Can limit the phase "wind" to $-\pi < \angle H(e^{j\omega}) \leq \pi$.
- Magnitude, $\left|H(e^{j\omega})\right|$, is an **even function** of ω for real $h(t)$.
- Phase, $-\pi < \angle H(e^{j\omega}) \leq \pi$, can be made an **odd function** of ω for real $h(t)$
- We tend to define the action of a filter in terms of the magnitude of the frequency response. This can be a little deceptive.

An all-pass filter satisfies

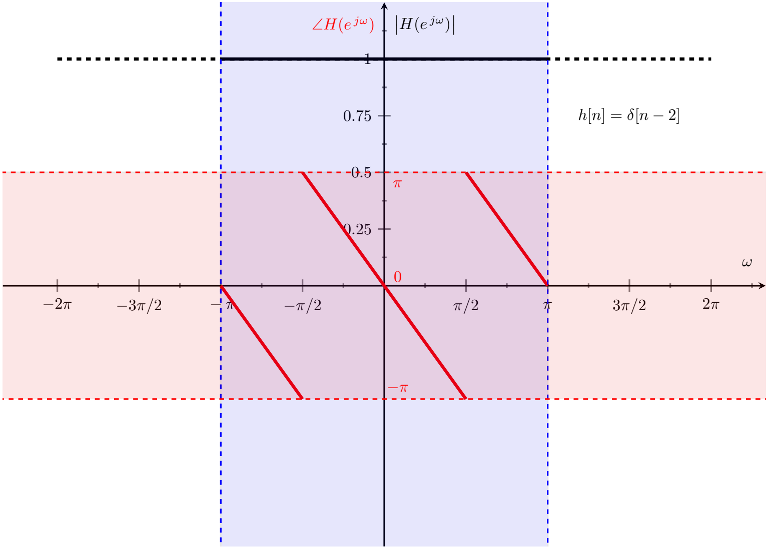
$$\boxed{\left|H(e^{j\omega})\right| = 1, \quad \text{for all } \omega}$$

- The trivial filter $h[n] = \delta[n]$ with frequency response $H(e^{j\omega}) = 1$ is **all pass**.
- A delay filter $h[n] = \delta[n - n_0]$ ($n_0 \in \mathbb{Z}$) with frequency response $H(e^{j\omega}) = e^{-jn_0\omega}$ is **all pass**.
- All frequencies at the input are passed to the output with no change in amplitude/magnitude. But the phase can be modified.
- Not every all pass filter is a delay filter.
- A delay filter has (negative) linear phase. The phase is a straight line with slope proportional to n_0. The delay is encoded in the slope.

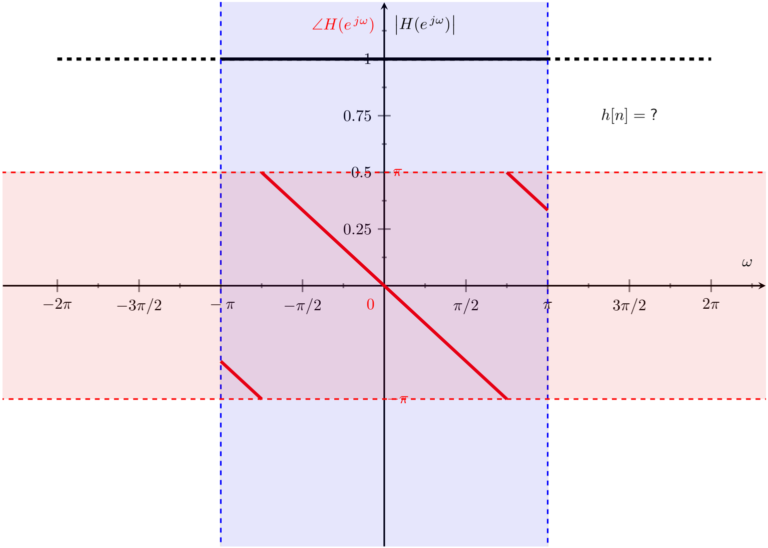
FS and LTI Systems – All Pass Filters



FS and LTI Systems – All Pass Filters



FS and LTI Systems – Challenge



$h[n] = ?$

Can I have a 1.5 (all-pass) delay filter? What does that mean?

For integer $n_0 \in \mathbb{Z}$, that is, delays of 0, 1, 2, etc (and even delays of -1, -2, etc) we have

$$h[n] = \delta[n - n_0] \quad \text{equivalently} \quad H(e^{j\omega}) = e^{-jn_0\omega}$$

Cannot have "$h[n] = \delta[n - 1.5]$" (that's nonsense).

(Can I have a 1.5 (all-pass) delay filter?) Yet

$$H(e^{j\omega}) = e^{-j1.5\omega}$$

is perfectly well defined. Indeed for any delay, $\Delta \in \mathbb{R}$,

$$H(e^{j\omega}) = e^{-j\Delta\,\omega}, \quad \Delta > 0$$

is well defined. Note here $\Delta > 0$ is a delay; $\Delta = 0$ no delay; and $\Delta < 0$ would be a non-causal time advance. For example, $H(e^{j\omega}) = e^{j3\omega}$ would be a time advance of 3.

What filter pulse response, $h[n]$, yields a delay of Δ?

CT Non-Periodic Signals – Up to this Point

CT Periodic Signals

- We met CT Periodic Signals. For example, the periodic rectangular wave. These are naturally described using Fourier Series of complex exponentials with $s = \pm j\omega$.
- Another important periodic CT signal is the impulse train. This can be thought of as an extreme example of a periodic rectangular wave.
- Periodic convolution in the time domain of two CT periodic signals is given by the product of the Fourier Series coefficients (frequency domain)

CT Non-Periodic Signals – Up to this Point

DT Periodic Signals

- We met DT Periodic Signals. N signal values (a vector) which periodically repeats itself. The natural description is to use Fourier Series of complex exponentials with $s = \pm jk\omega_0$ for $k = 0, 1, \ldots, N-1$, where $\omega_0 = 2\pi/N$.
- This description also works for finite (non-periodic) signals of N samples. The Fourier Series has N terms. So N-vector in time gives N-vector in frequency. $N \times N$ matrix to convert between domains.
- Or the Fourier Series can be repeated periodically. So DT Periodic signals yield periodic Fourier Series.

CT Non-Periodic Signals – Up to this Point

Periodic or Finite Discrete Signals

CT Periodic signals $\Longleftrightarrow$ Non-Periodic Fourier Series

DT Periodic signals $\Longleftrightarrow$ Periodic Fourier Series

(Finite DT signals $\Longleftrightarrow$ Finite Fourier Series – can use "FFT" to compute)

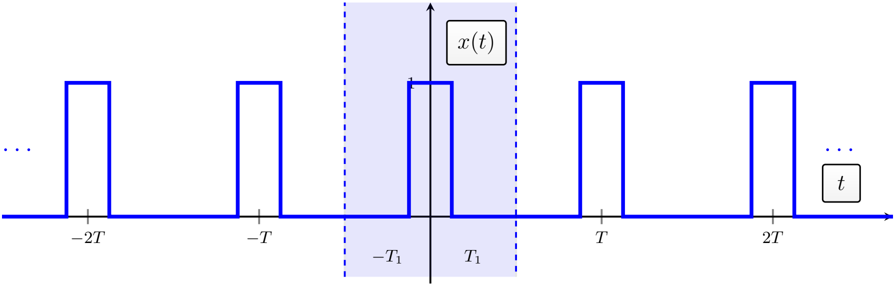
CT Non-Periodic Signals – Up to this Point

In the end, **Fourier Series** only describe/represent **periodic** time domain signals (continuous or discrete).

- But not all signals are periodic (most aren't).
- Can we generalize the Fourier Series for non-periodic time domain signals?
- Leads to the Fourier Transform. It's not too much different from Fourier Series.

Periodic Rectangular Wave: $x(t) = x(t+T)$,

$$x(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}, \qquad 0 < T_1 \leq T/2$$



What makes the periodic wave periodic?

CT Non-Periodic Signals – Periodic Wave

In the periodic wave we can imagine varying T_1, T and the amplitude. There are some interesting special case.

- $T_1 = T - T_1$ where it fills in and becomes a constant (pure DC)
- $T_1 \to 0$ with the amplitude scaled by $1/(2T_1)$ so it becomes the impulse train.
- What if $T \to \infty$? Or more strictly what if $T/T_1 \to \infty$?. This is covered in the text section 4.1. Could write a matlab code and play with the numbers.
- With above limits the signal stops being periodic (infinite period) and the signal tends to look like a bimp and is zero or dies away for time going to $\pm\infty$.

Definition (Fourier Analysis and Synthesis)

Provided the following integrals are finite/exist

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \quad t \in \mathbb{R} \qquad (Synthesis)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

A finite energy condition

$$\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt < \infty$$

guarantees $X(j\omega)$ is finite. But other conditions are possible and important in some cases.

Comments:

- Both time t and frequency ω are continuous. Going from periodic to non-periodic means frequency goes from discrete to continuous (as a rule).
- Synthesis and analysis equations are virtually identical in form. There is a -1 in one exponent, t and ω are interchanged and there is a leading constant term.
- Also called the Fourier Transform and Inverse Fourier Transform.

Key Questions:

- Fourier Transform: what is the frequency content of an "arbitrary" time domain signal. Who cares?
- Inverse Fourier Transform: what time domain signal corresponds to a given frequency domain description. Who cares?

Fourier Transform: What is the frequency content of an “arbitrary” time domain signal.

- An LTI system is a “filter”. It has a frequency response. A frequency response tells you how the frequencies in a signal are modified (magnitude and phase).
- But the action of an LTI filter is to convolve. The frequency content of the filter output which is the frequency content of the filter input modified by the filter frequency response must be the Fourier Transform of the convolution of input and system impulse response.

Inverse Fourier Transform: What time domain signal corresponds to a given frequency domain description.

- Engineering specifications are in the frequency domain most often. In communications and broadcast radio/TV each transmitter needs to confine the transmissions into tightly controlled frequency ranges. (Bandwidth is very expensive, want to limit interference to neighboring frequency bands, improve efficiency, etc.)
- If I know the frequency shape then the Inverse Fourier Transform tells me how to design the time domain filter for implementation.

Fourier Transforms – Examples

Example 1: Let $x(t) = \delta(t)$ then

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$
$$= 1, \quad \text{for all } \omega$$

This is the constant function equal to 1 for all ω.

That is, we have the synthesis:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

Could say that $\delta(t)$ is unusual in that it contains equal portions of all frequencies (phase aligned).

Fourier Transforms – Examples

Recap:

$$\delta(t) \xleftrightarrow{\mathscr{F}} 1$$

$$\mathscr{F}\left\{\delta(t)\right\} = 1$$

$$\mathscr{F}^{-1}\left\{1\right\} = \delta(t)$$

- Frequency domain function 1 is the *Fourier Transform* of $\delta(t)$
- Time domain function $\delta(t)$ is the *Inverse Fourier Transform* of 1

Fourier Transforms – Examples

Example 2: Let $x(t) = \delta(t - t_0)$, which as an impulse response gives an LTI system acting as a delay of t_0, then

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$
$$= e^{-j\omega t_0}$$

This is the a linear phase, with slope $-t_0$. Of course, the previous example is the special case $t_0 = 0$, no delay (as an impulse response, the trivial do-nothing system).

That is, we have the synthesis:

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t - t_0)}\, d\omega$$

Recap:

$$\delta(t - t_0) \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0}$$

$$\mathscr{F}\left\{\delta(t - t_0)\right\} = e^{-j\omega t_0}$$

$$\mathscr{F}^{-1}\left\{e^{-j\omega t_0}\right\} = \delta(t - t_0)$$

- Frequency domain function $e^{-j\omega t_0}$ is the *Fourier Transform* of $\delta(t - t_0)$
- Time domain function $\delta(t - t_0)$ is the *Inverse Fourier Transform* of $e^{-j\omega t_0}$

Fourier Transforms – Digression

Generally with Fourier Transforms:

- Well known and useful signals are tabulated. No need to compute the Fourier Transform (except in introductory courses like this).
- Can use superposition to combine Fourier Transforms of different component signals. Why? The Fourier Transform is a **linear** operator (functions to functions) since integration is linear.
- Also want to be able to stretch and shrink, multiply and other operations to be able to use the raw tabulated Fourier Transforms. We'll consider these later.

Example 3: Let $x(t) = e^{-at}\,u(t)$, for $a > 0$ and $u(t)$ is the unit step, then

$$
\begin{aligned}
X(\,j\omega) &= \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt \\
&= \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}
\end{aligned}
$$

Fourier Transforms – Examples

Recap:

$$e^{-at}\,u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

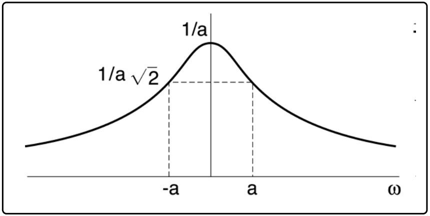
$$\mathscr{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{a+j\omega}$$

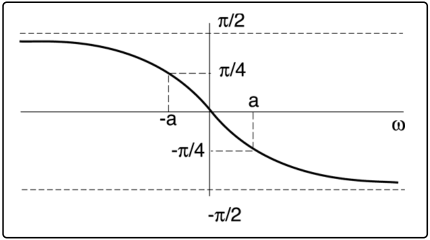
$$\mathscr{F}^{-1}\left\{\frac{1}{a+j\omega}\right\} = e^{-at}\,u(t)$$

- Frequency domain function $\frac{1}{a+j\omega}$ is the *Fourier Transform* of $e^{-at}\,u(t)$
- Time domain function $e^{-at}\,u(t)$ is the *Inverse Fourier Transform* of $\frac{1}{a+j\omega}$

Magnitude and phase are:

$$\left|X(j\omega)\right| = (a^2 + \omega^2)^{-1/2}$$
$$\angle X(j\omega) = -\tan^{-1}(\omega/a)$$





Fourier Transforms – Examples

Example 4: The Fourier Transform of a rectangular "brickwall" pulse of width $2T_1$, that is,

$$x(t) = \chi_{[-T_1,T_1]}(t) \triangleq \begin{cases} 1 & |t| \leq T_1 \\ 0 & \text{otherwise} \end{cases}$$

gives the sinc function

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \frac{2\sin(\omega T_1)}{\omega}$$

Note that if we flip time and frequency, we have a "brickwall" filter which can be realized by a sinc pulse in time. We'll look at this more closely later.

Recap:

$$\chi_{[-T_1,T_1]}(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{2\sin(\omega T_1)}{\omega}$$

$$\mathscr{F}\left\{\chi_{[-T_1,T_1]}(t)\right\} = \frac{2\sin(\omega T_1)}{\omega}$$

$$\mathscr{F}^{-1}\left\{\frac{2\sin(\omega T_1)}{\omega}\right\} = \chi_{[-T_1,T_1]}(t)$$

- Frequency domain function $\frac{2\sin(\omega T_1)}{\omega}$ is the *Fourier Transform* of $\chi_{[-T_1,T_1]}(t)$
- Time domain function $\chi_{[-T_1,T_1]}(t)$ is the *Inverse Fourier Transform* of $\frac{2\sin(\omega T_1)}{\omega}$

Example 5: Let $x(t) = e^{-at^2}$, for $a > 0$, then

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{-a[t^2 + j\omega t/a + (j\omega/2a)^2] + a(j\omega/2a)^2}\, dt \\ &= \int_{-\infty}^{\infty} e^{-a(t + j\omega/2a)^2}\, dt \cdot e^{-\omega^2/4a} \\ &= \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a} \end{aligned}$$

So the Fourier Transform of a gaussian is a compressed (or expanded) and scaled gaussian. Weird but important in applications and the “uncertainty principle”.

Recap:

$$e^{-at^2} \xleftrightarrow{\mathscr{F}} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}\left\{e^{-at^2}\right\} = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}\right\} = e^{-at^2}$$

- Frequency domain function $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$ is the *Fourier Transform* of e^{-at^2}
- Time domain function e^{-at^2} is the *Inverse Fourier Transform* of $\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$

Key Questions:

What if the CT signals are **periodic** in time?

Can the Fourier Transform sensibly handle this case or do we need to revert to the Fourier Series approach?

Answer: impulses (in the frequency domain) to the rescue.

Fourier Transforms – Periodic Signals

Begin with

$$X(j\omega) = \delta(\omega - \omega_0)$$

then

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{j\omega_0 t}\, d\omega \\
&= \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)\; d\omega \\
&= \frac{1}{2\pi} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} \delta(\omega)\; d\omega}_{1} \\
&= \frac{1}{2\pi} e^{j\omega_0 t}
\end{aligned}$$

So all the energy is concentrated at one frequency $+\omega_0$.

Recap:

$$\frac{1}{2\pi} e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} \delta(\omega - \omega_0)$$

$$\mathscr{F}\left\{\frac{1}{2\pi} e^{j\omega_0 t}\right\} = \delta(\omega - \omega_0)$$

$$\mathscr{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi} e^{j\omega_0 t}$$

- Frequency domain function $\delta(\omega - \omega_0)$ is the *Fourier Transform* of $\frac{1}{2\pi} e^{j\omega_0 t}$
- Time domain function $\frac{1}{2\pi} e^{j\omega_0 t}$ is the *Inverse Fourier Transform* of $\delta(\omega - \omega_0)$

Fourier Transforms – Periodic Signals

We have for all ω_0,

$$e^{j\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega-\omega_0),$$

therefore

$$e^{jk\omega_0 t} \xleftrightarrow{\mathscr{F}} 2\pi\,\delta(\omega-k\omega_0), \quad \forall k.$$

If $x(t)$ is **periodic** then it has Fourier Series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

where a_k are the Fourier Series coefficients. Then the Fourier Transform, by linearity, is

$$X(\,j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega-k\omega_0)$$

This has energy at only discrete frequencies $k\omega_0$ where $k \in \mathbb{Z}$.

Recap:

$$\sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0)$$

$$\mathscr{F}\left\{ \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \right\} = 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0)$$

$$\mathscr{F}^{-1}\left\{ 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0) \right\} = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

- Frequency domain function $2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0)$ is the *Fourier Transform* of $\sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$
- Time domain function $\sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$ is the *Inverse Fourier Transform* of $2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0)$

Example 6:

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

has Fourier Transform

$$X(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

The only non-zero Fourier Series coefficients are $a_{-1} = 1/2$ and $a_1 = 1/2$. In the Fourier Transform representation, $X(j\omega)$, these just get multiplied by 2π (an artifact of definitions) and becomes the weights of delta function in continuous ω frequency domain.

Recap:

$$\cos(\omega_0 t) \xleftrightarrow{\mathscr{F}} \pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$$

$$\mathscr{F}\left\{\cos(\omega_0 t)\right\} = \pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$$

$$\mathscr{F}^{-1}\left\{\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)\right\} = \cos(\omega_0 t)$$

- Frequency domain function $\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$ is the *Fourier Transform* of $\cos(\omega_0 t)$
- Time domain function $\cos(\omega_0 t)$ is the *Inverse Fourier Transform* of $\pi\,\delta(\omega-\omega_0)+\pi\,\delta(\omega+\omega_0)$

Fourier Transforms – Periodic Signals

Recall the sampling function which is periodic with (fundamental) period T:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Then $\omega_0 = 2\pi/T$ and Fourier Series coefficients are, for all integer k, given by

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jw_0 t}\, dt = \frac{1}{T}$$

Whence, from earlier (4 slides ago),

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{T}}^{a_k} \delta\Big(\omega - \overbrace{\frac{2\pi k}{T}}^{k\omega_0}\Big) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\Big(\omega - k\,\omega_0\Big)$$

which is a sampling function, periodic in ω with period ω_0 and scaled by $2\pi/T$, in the frequency domain.

Recap:

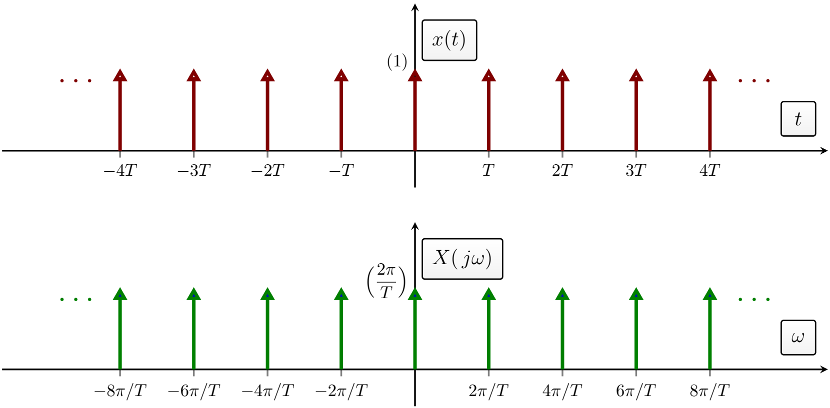
$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \xleftrightarrow{\mathscr{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\mathscr{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t-nT)\right\} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\mathscr{F}^{-1}\left\{\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)\right\} = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

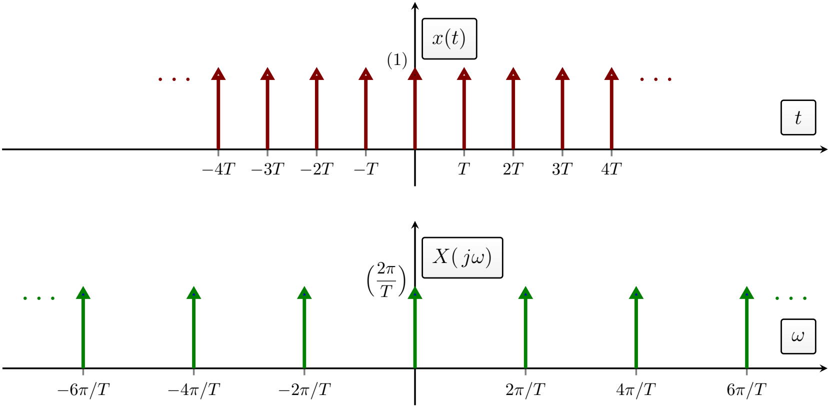
- Frequency domain function $\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ is the *Fourier Transform* of $\sum_{n=-\infty}^{\infty} \delta(t-nT)$
- Time domain function $\sum_{n=-\infty}^{\infty} \delta(t-nT)$ is the *Inverse Fourier Transform* of $\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$

Fourier Transforms – Periodic Signals



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Fourier Transforms – Periodic Signals



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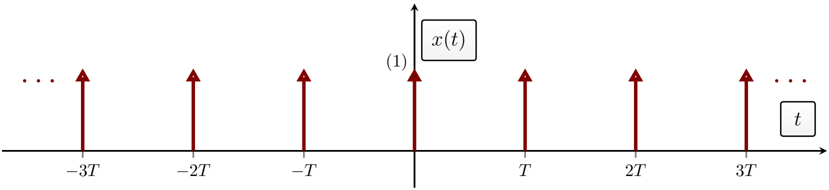
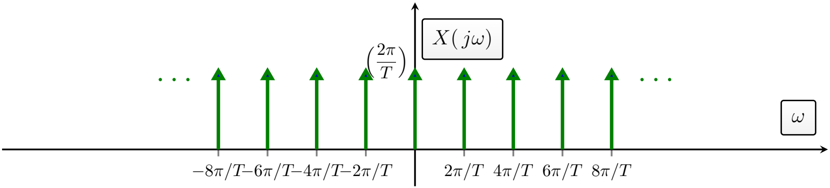




$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathscr{F}} X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Fourier Transforms – Properties

From simple manipulations of the Fourier Transform integral:

Linearity/Superposition:

$$\boxed{a\,x(t) + b\,y(t) \overset{\mathscr{F}}{\longleftrightarrow} a\,X(\,j\omega) + b\,Y(\,j\omega)}$$

Time Shift:

$$\boxed{x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0}\,X(\,j\omega)}$$

Time Scale:

$$\boxed{x(at) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}\,X(\,j\omega/a)}$$

Derivative:

$$\boxed{\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega\,X(\,j\omega)}$$

Convolution:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\boxed{y(t) = h(t) \star x(t) \xleftrightarrow{\mathscr{F}} Y(j\omega) = H(j\omega)\,X(j\omega)}$$

where

$$h(t) \xleftrightarrow{\mathscr{F}} H(j\omega)$$

This follows from the eigenfunction property of the $e^{j\omega t}$ which is central to the definition of the Fourier Transform (see over).

Terminology, $H(j\omega)$ is the **frequency response**.

Fourier Transforms – Convolution

$$
\begin{aligned}
x(t) &= \mathscr{F}^{-1}\{X(j\omega)\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega
\end{aligned}
$$

Recall the eigenfunction property:

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

$$
\begin{aligned}
y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) H(j\omega)\, e^{j\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(j\omega)\, X(j\omega)}_{Y(j\omega)}\, e^{j\omega t}\, d\omega \\
&= \mathscr{F}^{-1}\{H(j\omega)\, X(j\omega)\} \equiv \mathscr{F}^{-1}\{Y(j\omega)\}
\end{aligned}
$$

Hence

$$\boxed{y(t) = h(t) \star x(t) \iff Y(j\omega) = H(j\omega)\, X(j\omega)}$$

Fourier Transforms – Convolution

Sanity Check:

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\,\delta(\omega - \omega_0)$$

Now apply the convolution result

$$\begin{aligned} Y(j\omega) &= H(j\omega)\,X(j\omega) \\ &= H(j\omega)\,2\pi\,\delta(\omega - \omega_0) \\ &= 2\pi\,H(j\omega_0)\,\delta(\omega - \omega_0) \end{aligned}$$

$$\begin{aligned} y(t) &= \mathscr{F}^{-1}\{2\pi\,H(j\omega_0)\,\delta(\omega - \omega_0)\} \\ &= H(j\omega_0)\,\mathscr{F}^{-1}\{2\pi\,\delta(\omega - \omega_0)\} \\ &= H(j\omega_0)\,e^{j\omega_0 t} \end{aligned}$$

Fourier Transforms – Differentiator

Differentiator: A differentiator is a linear time invariant system and, therefore, has an impulse response. Call this impulse response $h(t)$ and its Fourier Transform $H(j\omega)$.

Let $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega)$ be an signal to the differentiator. Then the output is

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathscr{F}}{\longleftrightarrow} j\omega\, X(j\omega)}$$

This is a standard Fourier Transform property (given earlier but proved later).

Consider the differentiator LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \frac{dx(t)}{dt}$$

then we can infer that

$$h(t) \stackrel{\mathscr{F}}{\longleftrightarrow} j\omega \quad \text{or} \quad H(j\omega) = j\omega.$$

Look at this in more detail (over).

Starting from

$$H(j\omega) = j\omega$$

we observe this amplifies high frequencies, and kills DC because the magnitude and phase are:

$$\big|H(j\omega)\big| = |\omega| \quad \text{and} \quad \angle H(j\omega) = \overbrace{\mathsf{sgn}(\omega)}^{\pm 1}\,\pi/2$$

noting that $j = e^{j\pi/2}$.

Is this reasonable? Yes, consider

$$\begin{aligned}\frac{d}{dt}\sin(\omega_c t) &= \omega_c\,\cos(\omega_c t)\\ &= \omega_c\,\sin(\omega_c t + \pi/2)\end{aligned}$$

where ω_c is a multipler (gain) and there is a $\pi/2$ phase shift.

Fourier Transforms – Ideal Low Pass Filter

Ideal LPF: Passes only frequencies between $[-\omega_c, +\omega_c]$, where ω_c is the cut-off frequency.

The frequency domain specification is

$$\boxed{H(j\omega) = \chi_{[-\omega_c,+\omega_c]}(\omega)}$$

(RHS is the characteristic function, just some shorthand) where the phase is zero for all ω (our choice here). Then

$$\begin{aligned} h(t) = \mathscr{F}^{-1}\{H(j\omega)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-\omega_c,+\omega_c]}(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega \\ &= \frac{\sin \omega_c t}{\pi t} \\ &= \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \end{aligned}$$

where the "sinc function" is defined as

$$\operatorname{sinc}(\theta) \triangleq \frac{\sin \pi\theta}{\pi\theta}$$

Definition (Ideal Low Pass Filter)

The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin \omega_c t}{\pi t} \xleftrightarrow{\mathscr{F}} \chi_{[-\omega_c,+\omega_c]}(\omega)}$$

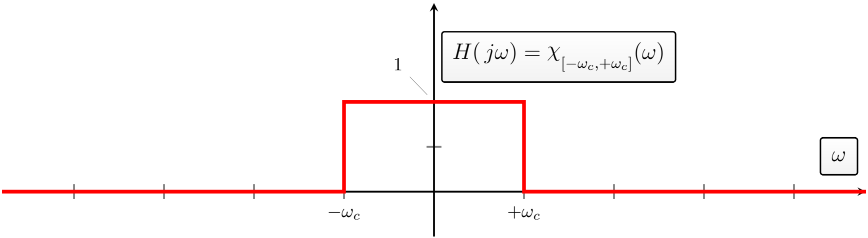
or

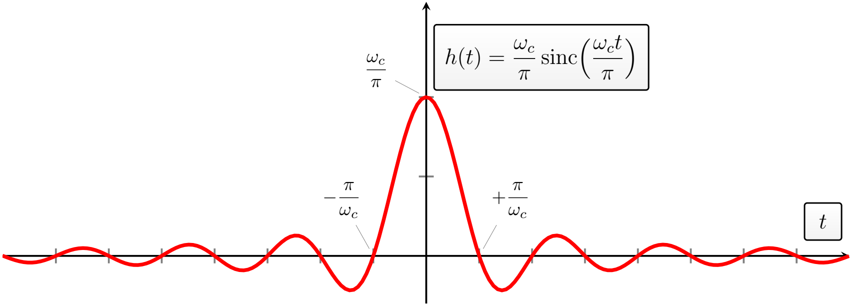
Definition (Ideal Low Pass Filter)

The LTI system that passes only frequencies with gain 1 in the range $[-\omega_c, +\omega_c]$ has impulse response and frequency response pair:

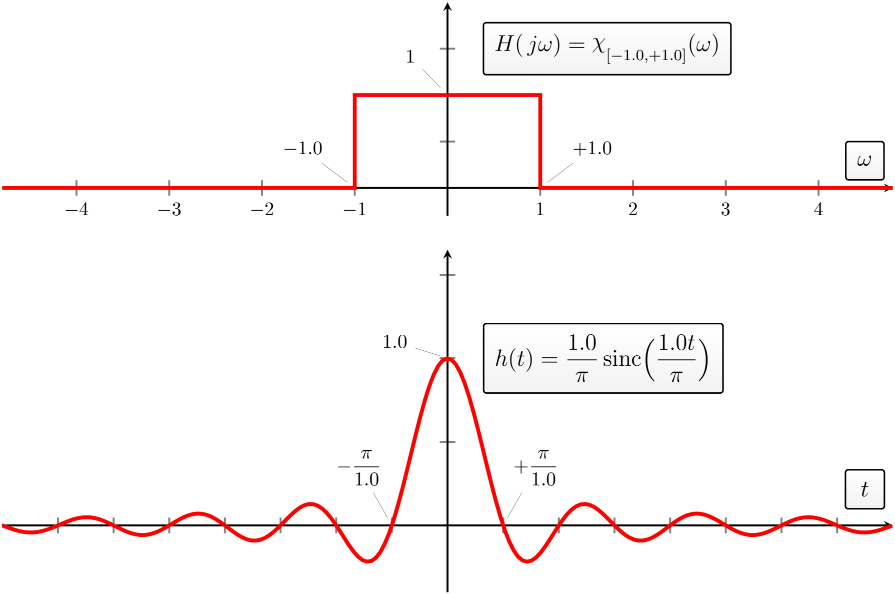
$$\boxed{\frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \xleftrightarrow{\mathscr{F}} \chi_{[-\omega_c,+\omega_c]}(\omega)}$$

Fourier Transforms – Ideal Low Pass Filter





Fourier Transforms – Ideal LPF and Sinc



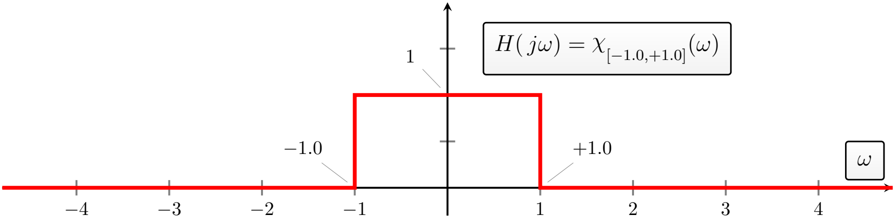
Cutoff Variations: The following frames show the effects of varying the cutoff frequency, increasing the bandwidth of an ideal LPF which contracts the sinc function in time. This can be explained in terms of

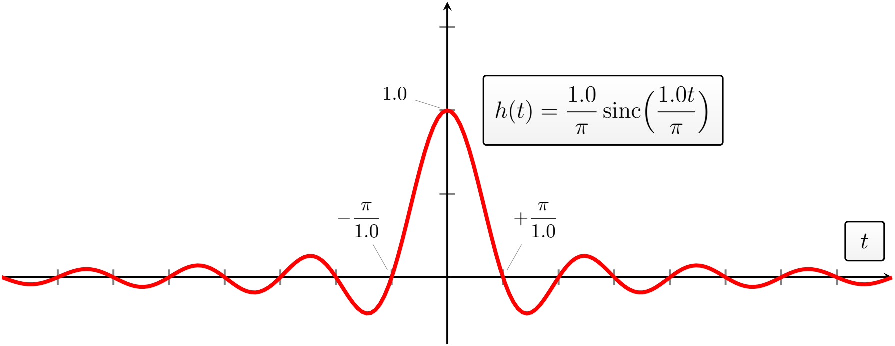
$$x(a\,t) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|}\,X\Big(\frac{j\omega}{a}\Big)$$

The plots show a sequence with $a = 1.0, 1.1, 1.2, 1.3, 1.4$ and 1.5

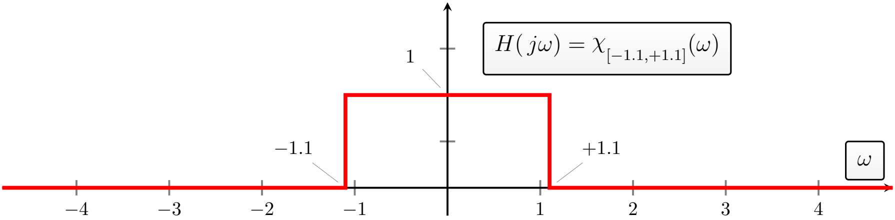
Followed by $a = 1.0, 0.9, 0.5$ and 0.25

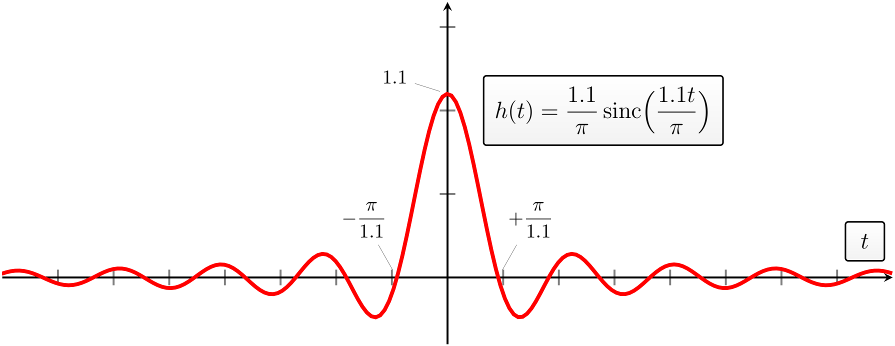
Fourier Transforms – Ideal LPF and Sinc



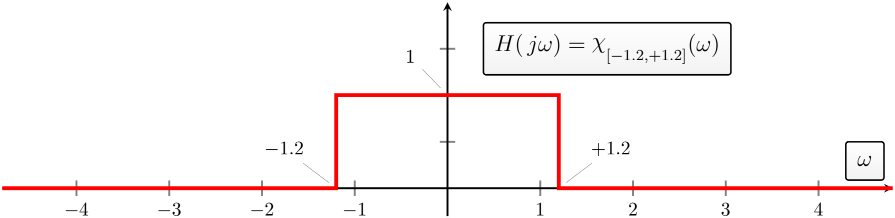


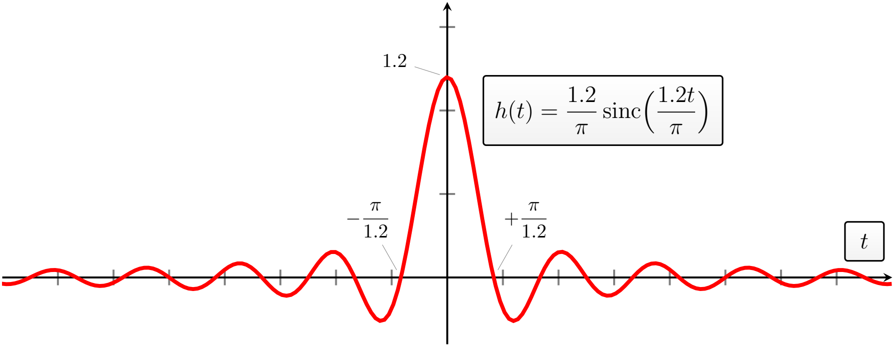
Fourier Transforms – Ideal LPF and Sinc



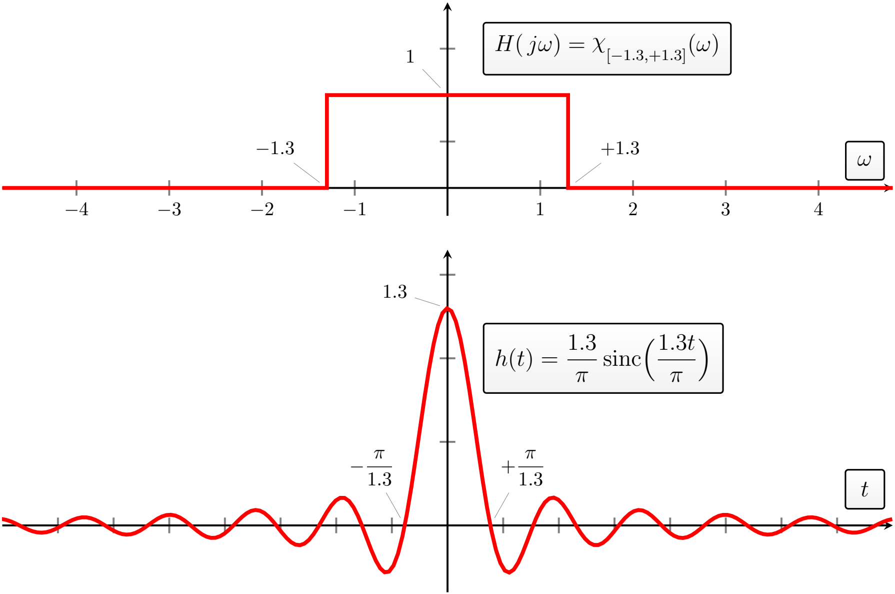


Fourier Transforms – Ideal LPF and Sinc

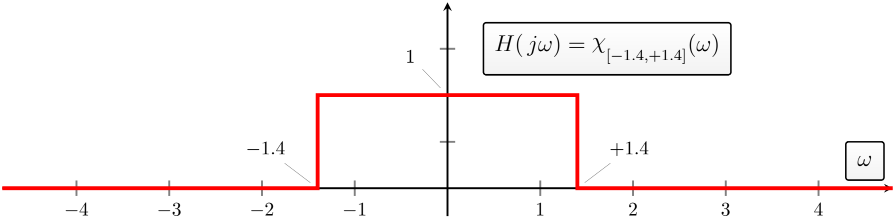


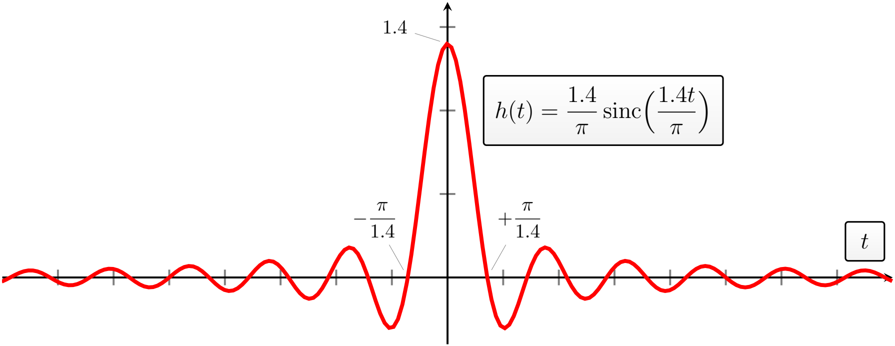


Fourier Transforms – Ideal LPF and Sinc

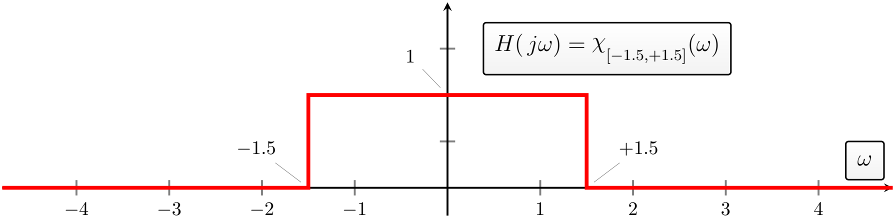


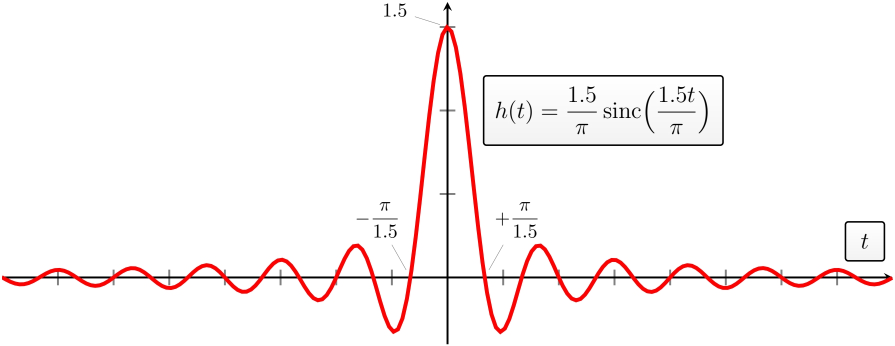
Fourier Transforms – Ideal LPF and Sinc



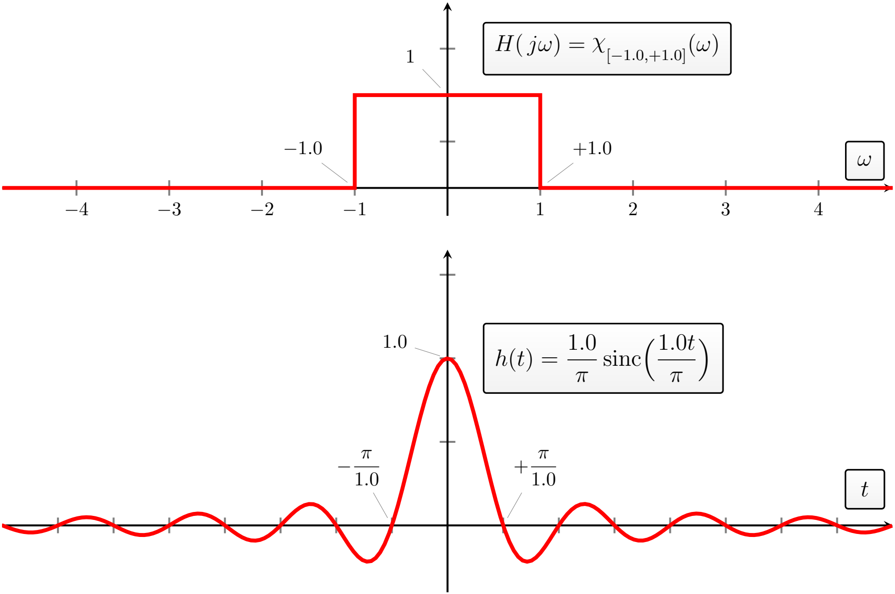


Fourier Transforms – Ideal LPF and Sinc

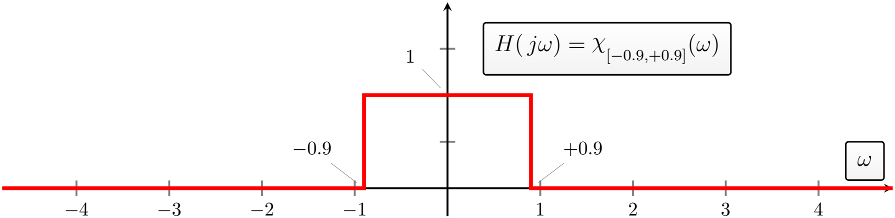


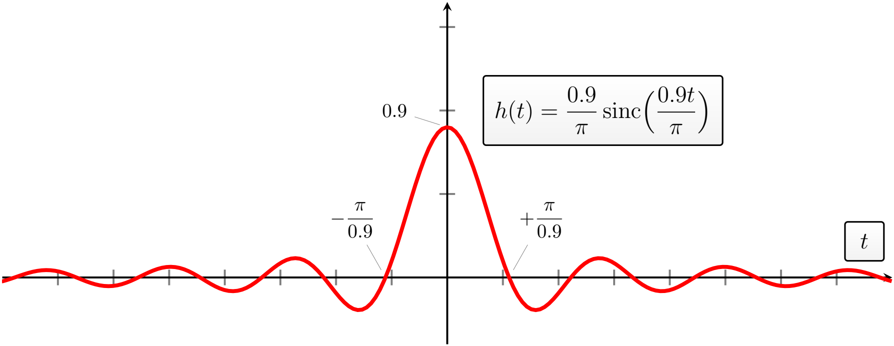


Fourier Transforms – Ideal LPF and Sinc

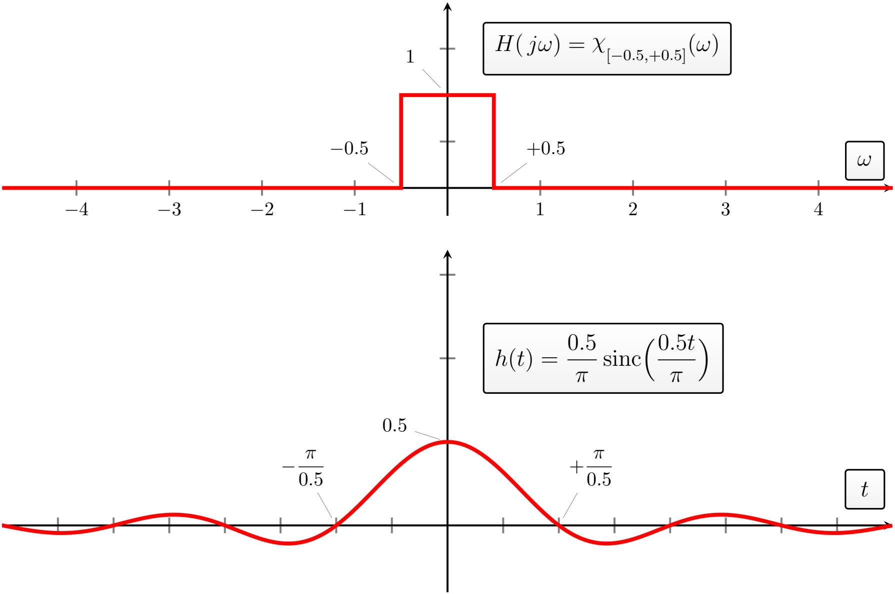


Fourier Transforms – Ideal LPF and Sinc

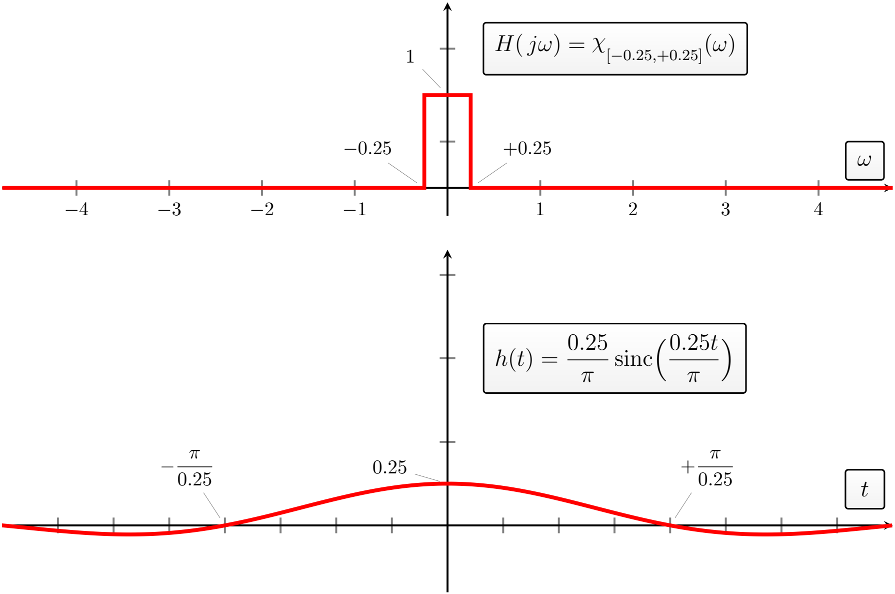




Fourier Transforms – Ideal LPF and Sinc

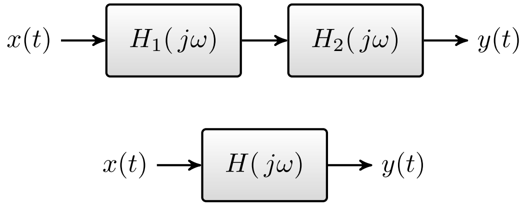


Fourier Transforms – Ideal LPF and Sinc



Fourier Transforms – Filter Cascade

Example 1: Cascading filters



Then the convolution theorem gives

$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

Note that if $H_1(j\omega) = H_2(j\omega)$ then $H_1^2(j\omega) \equiv \big(H_1(j\omega)\big)^2$ tends to have a sharper frequency selectivity/cutoff, for example, as some frequency, attenuation by 10% becomes attenuation by almost 20%, $0.9^2 = 0.81$.

Example 2: Cascading ideal LPF filters. What is

$$\underbrace{\frac{\sin(4\pi t)}{\pi t}}_{x(t)} \star \underbrace{\frac{\sin(8\pi t)}{\pi t}}_{h(t)} = ?$$

Think about $x(t)$ as the input into LTI system with impulse response $h(t)$.

Note that: $x(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 4\pi$; and $h(t)$ is the impulse response of an ideal low pass filter with cut-off $\omega_c = 8\pi$

Fourier Transforms – Filter Cascade

Answer:

$$\frac{\sin(4\pi t)}{\pi t} \star \frac{\sin(8\pi t)}{\pi t} = \frac{\sin(4\pi t)}{\pi t}$$

That's weird. Imagine the pain to calculate:

$$\int_{-\infty}^{\infty} \frac{\sin(4\pi\tau)}{\pi\tau} \frac{\sin(8\pi(t-\tau))}{\pi(t-\tau)}\, d\tau$$

But the result is clear if we consider the frequency domain.

Useful result: If we cascade (ideal) LPFs the effect is the same as just applying the (ideal) LPF of the least bandwidth (least cut-off ω_c).

Fourier Transforms – Filter Cascade

$$\frac{\sin(3.4\pi t)}{\pi t} \star \frac{\sin(2.7\pi t)}{\pi t} \star \frac{\sin(4.1\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$
$$\frac{\sin(3.7\pi t)}{\pi t} \star \frac{\sin(5.7\pi t)}{\pi t} \star \frac{\sin(5.1\pi t)}{\pi t} \star \frac{\sin(4.0\pi t)}{\pi t} \star$$
$$\frac{\sin(3.3\pi t)}{\pi t} \star \frac{\sin(2.8\pi t)}{\pi t} \star \frac{\sin(4.9\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} \star$$
$$\frac{\sin(7.4\pi t)}{\pi t} \star \frac{\sin(8.7\pi t)}{\pi t} \star \frac{\sin(5.3\pi t)}{\pi t} \star \frac{\sin(6.1\pi t)}{\pi t} \star$$
$$\frac{\sin(5.2\pi t)}{\pi t} \star \frac{\sin(8.8\pi t)}{\pi t} \star \frac{\sin(9.1\pi t)}{\pi t} \star \frac{\sin(8.2\pi t)}{\pi t} \star$$
$$\frac{\sin(4.5\pi t)}{\pi t} \star \frac{\sin(3.5\pi t)}{\pi t} \star \frac{\sin(4.8\pi t)}{\pi t} = \frac{\sin(2.7\pi t)}{\pi t}$$

Example 3: Consider

$$h(t) = e^{-t}u(t) \quad \text{and} \quad x(t) = e^{-2t}u(t)$$

What is $y(t) = h(t) \star x(t)$?

Obviously could grind through a convolution integral. But generally going via the Fourier Transform is quicker and easier.

First we work through this example and then review the general approach.

Fourier Transforms – Partial Fractions

$$
\begin{aligned}
Y(j\omega) &= H(j\omega)\,X(j\omega) \\
&= \frac{1}{(1+j\omega)}\,\frac{1}{(2+j\omega)} = \frac{1}{(1+j\omega)(2+j\omega)}
\end{aligned}
$$

Do we know what the Inverse Fourier Transform of this is? Not really. However, we can do a "partial fraction expansion" (trick):

$$\boxed{Y(j\omega) = \frac{1}{(1+j\omega)} - \frac{1}{(2+j\omega)}}$$

Whence

$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

that is, $y(t) = h(t) \star x(t)$ looks like:

$$\left(e^{-t} - e^{-2t}\right)u(t) = e^{-t}u(t) \star e^{-2t}u(t)$$

Fourier Transforms – Transform Method

Convolution via transform techniques:

- Do Fourier Transform to convert problem to Fourier/frequency domain.
- Do convolution via multiplication of Fourier Transforms.
- Do algebraic manipulations, e.g., partial fractions.
- Do Inverse Fourier Transform to get back time domain signal.

Fourier Transforms – Differential Equations

Consider the Fourier Transform of the derivative, $x'(t)$ of a signal $x(t)$:

$$\mathscr{F}\left\{x'(t)\right\} = \mathscr{F}\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{+\infty} \frac{dx(t)}{dt}\, e^{-j\omega t} dt = \int_{-\infty}^{+\infty} e^{-j\omega t}\, dx(t)$$

the RHS of which looks like "integration by parts"

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

with $u \equiv e^{-j\omega t}$ and $v \equiv x(t)$; noting $du = -j\omega\, e^{-j\omega t}\, dt$ and $dv = x'(t)\, dt$.

So

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathscr{F}} j\omega\, X(j\omega)}$$

- $uv\big|_{-\infty}^{+\infty} = 0$ given that $\left|x(t)\, e^{-j\omega t}\right| = \left|x(t)\right| \to 0$ as $t \to \pm\infty$ because $X(j\omega)$ exists (is finite);
- $-\int_{-\infty}^{+\infty} v\, du = j\omega\, X(j\omega)$.

Fourier Transforms – Differential Equations

Further, the Fourier Transform of the kth derivative, $x^{(k)}(t)$ of a signal $x(t)$:

$$\boxed{\mathscr{F}\left\{x^{(k)}(t)\right\} = \mathscr{F}\left\{\frac{d^k x(t)}{dt^k}\right\} = (\,j\omega)^k\, X(\,j\omega)}$$

In notation, $x^{(0)}(t) \equiv x(t)$, $x^{(1)}(t) \equiv x'(t)$, $x^{(2)}(t) \equiv x''(t)$, etc.

One way to interpret this is to consider the derivative system as a linear time-invariant system with frequency response $j\omega$. Then the result follows from the cascade or series connection of k such derivative systems – the impulse responses convolve but the frequency responses, each jw, just multiply.

In summary:

$$\boxed{\frac{d^k x(t)}{dt^k} \stackrel{\mathscr{F}}{\longleftrightarrow} (\,j\omega)^k\, X(\,j\omega)}$$

Fourier Transforms – Differential Equations

LCC Differential Equations: `O&W 4.7 pp.330-333`

Now solve the linear, constant coefficient differential equation:

$$\boxed{\sum_{k=0}^{K} a_k \, \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \, \frac{d^m x(t)}{dt^m}}$$

where we can interpret this as describing a system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fourier Transform both side of the differential equation to yield (see next slide)

$$\boxed{\sum_{k=0}^{K} a_k \, (j\omega)^k Y(j\omega) = \sum_{m=0}^{M} b_m \, (j\omega)^m X(j\omega)}$$

That is, take Fourier Transforms of both sides of

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

to yield

$$\begin{aligned}
\mathscr{F}\left\{\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k}\right\} &= \mathscr{F}\left\{\sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}\right\} \\
\sum_{k=0}^{K} a_k \,\mathscr{F}\left\{\frac{d^k y(t)}{dt^k}\right\} &= \sum_{m=0}^{M} b_m \,\mathscr{F}\left\{\frac{d^m x(t)}{dt^m}\right\} \\
\sum_{k=0}^{K} a_k \,(j\omega)^k Y(j\omega) &= \sum_{m=0}^{M} b_m \,(j\omega)^m X(j\omega)
\end{aligned}$$

$$Y(j\omega) = \left(\frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k} \right) X(j\omega)$$

So

$$\boxed{H(j\omega) = \frac{\sum_{m=0}^{M} b_m \, (j\omega)^m}{\sum_{k=0}^{K} a_k \, (j\omega)^k}}$$

is the frequency response of the linear, constant coefficient differential equation system. Amazing.

Called a "rational function of $j\omega$", ratio of polynomials in $j\omega$ or ω.

Filter Design: Can get different shaped $H(j\omega)$ by choosing different values for the a_k and b_m. That is, we can **design** different filters and we can implement them in practice by implementing, somehow, the differential equation. Not these days because we now use, with care, discrete time/digital techniques. This course is not about design, however.

In the CT non-periodic (in general) signal case:

Definition (Parseval's Relation)

The total energy in time domain equals total energy in frequency domain:

$$\boxed{\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 d\omega}$$

The term

$$\boxed{\frac{1}{2\pi}\left|X(j\omega)\right|^2}$$

is called the spectral density (energy per unit frequency).

Fourier Transforms – Multiplication Property

Convolution theory states

$$\boxed{x(t) \star y(t) \xleftrightarrow{\mathscr{F}} X(j\omega)\, Y(j\omega)}$$

then by "symmetry" (and a bit of book-keeping)

$$\boxed{x(t) \cdot y(t) \xleftrightarrow{\mathscr{F}} \frac{1}{2\pi} X(j\omega) \star Y(j\omega)}$$

where, convolution in frequency,

$$\frac{1}{2\pi} X(j\omega) \star Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\zeta)\, Y\big(j(\omega - \zeta)\big)\, d\zeta$$

Example 4: Amplitude modulation (AM):

> *AM or Amplitude Modulation is a method of radio broadcasting where the frequency is modulated or varied by its changing amplitude. Radio frequencies for AM broadcasts are expressed in kilohertz (kHz).*

ABC Canberra 666 means the center frequency is 666 kHz. To express in terms of ω (radians per second):

$$\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots \quad \text{radians per second}$$

What is the Fourier Transform theory behind 666 Canberra?

Fourier Transforms – Multiplication Property

First we take an audio signal, albeit a rather boring audio signal, $s(t)$. The Fourier Transform yields:

$$s(t) \overset{\mathscr{F}}{\longleftrightarrow} S(j\omega)$$

Since $s(t)$ is audio its frequencies are limited to what people can hear. The human range, for teenagers and younger is roughly 1 Hz to 20 kHz. For various reasons AM audio is further limited to 10 kHz, so $s(t)$ is low pass and occupies the frequency range

$$|\omega| \leq 10 \text{ kHz}$$

Fourier Transforms – Multiplication Property

Imagine, we could transmit such a signal directly over radio (called baseband). There would be a number of problems: interference from other baseband transmitters, ridiculously huge antennas, etc.

So ABC Canberra 666 really means $666,000 \pm 10,000$ Hz. Different stations are centered on different frequencies so that they don't interfere.

So how do we translate a baseband audio signal, $S(j\omega)$, to be centered on 666 kHz?

Fourier Transforms – Multiplication Property

Use time-domain multiplication `O&W 4.5 pp.323-324`

$$r(t) = s(t) \cdot p(t) \overset{\mathscr{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi} S(j\omega) \star P(j\omega)$$

with "modulator"

$$p(t) = \cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} P(j\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

where $\omega_0 = 2\pi \times 666,000 = 4,184,601.41\ldots$. This implies for any audio signal $s(t)$ that

$$R(j\omega) = \frac{1}{2} S\big(j(\omega - \omega_0)\big) + \frac{1}{2} S\big(j(\omega + \omega_0)\big)$$

Fourier Transforms – Multiplication Property

The frequency range for the AM modulated signal $r(t)$ is non-zero in

$$\big||\omega| - 4,184,601.41\ldots\big| \leq 10 \text{ kHz}$$

which is in RF.

At the receiver, you can multiply again by $\cos(\omega_0 t)$ to move the signal back to baseband and low pass filter to move components that go to $2\omega_0$ `O&W 4.5 pp.323–324`

$$\cos^2(\omega_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)$$

This being the combination of the modulator (transmitter) and demodulator (receiver).

DT Fourier Transforms – Recap

Periodic Signals:

$$\text{CT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Non-Periodic Fourier Series}$$

$$\text{DT Periodic signals} \overset{\mathscr{F}}{\longleftrightarrow} \text{Periodic Fourier Series}$$

DT Fourier Transforms – Recap

Finite Discrete Signals:

$$\text{Finite DT signals} \xleftrightarrow{\mathscr{F}} \text{Finite Fourier Series – FFT}$$

is very close to (mathematically equivalent to)

$$\text{DT Periodic signals} \xleftrightarrow{\mathscr{F}} \text{Periodic Fourier Series}$$

DT Fourier Transforms – Recap

CT Non-Periodic Signals:

$$\text{CT (Non-Periodic) Signals} \stackrel{\mathscr{F}}{\longleftrightarrow} \text{“Continuous” Fourier Transform}$$

DT Fourier Transforms – Recap

CT Periodic Signals (revisited):

$$\text{CT Periodic signals} \xleftrightarrow{\mathscr{F}} \text{“Impulse Sequence” Fourier Transform}$$

- Sometimes called a discrete spectrum.
- Technically the “Impulse Sequence” needs to be uniformly spaced, with delta functions lying at multiples of some ω_0.

Discrete Time Fourier Transform: `O&W 5.1.1 pp.359-362`

Here we have DT but, generally, non-periodic signals. We can have a continuum of frequencies and to synthesis the time domain signal we need to integrate over that continuum of frequencies. In the frequency domain we have 2π periodicity.

Definition (DT Fourier Analysis and Synthesis)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \qquad (Synthesis)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \qquad (Analysis)$$

DT Non-Periodic Signals:

$$\text{DT Non-Periodic signals} \xleftrightarrow{\mathscr{F}} \text{Continuous Periodic Fourier Transform}$$

Again, DT implies periodicity of the Fourier Transform and given the signal is not periodic (aperiodic) then the spectrum is continuous.

DT Fourier Transforms – Discrete Time FT

Four cases:

$$\text{continuous in time} \overset{\mathscr{F}}{\longleftrightarrow} \text{continuous in frequency}$$

$$\text{continuous in time} \overset{\mathscr{F}}{\longleftrightarrow} \text{discrete in frequency}$$

$$\text{discrete in time} \overset{\mathscr{F}}{\longleftrightarrow} \text{continuous in frequency}$$

$$\text{discrete in time} \overset{\mathscr{F}}{\longleftrightarrow} \text{discrete in frequency}$$

Technically "discrete in time" means discrete and uniformly spaced in time, and similarly "discrete in frequency" means discrete and uniformly spaced in frequency.

DT Fourier Transforms – Examples

Example 1: `O&W 5.1.3 p.367`

Unit sample signal

$$\boxed{x[n] = \delta[n]}$$

is not periodic and has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n}$$

That is,

$$\boxed{X(e^{j\omega}) = 1, \quad \text{for all } \omega}$$

- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = 0$, for all ω.

Example 2: `O&W 5.4.1 p.383`

Shifted unit sample signal, where delay $n_0 \in \mathbb{Z}$,

$$\boxed{x[n] = \delta[n - n_0]}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\, e^{-j\omega n}$$

That is,

$$\boxed{X(e^{j\omega}) = e^{-j\omega n_0}, \quad n_0 \in \mathbb{Z}}$$

- Amplitude $\left|X(e^{j\omega})\right| = 1$ and phase $\angle X(e^{j\omega}) = -j\omega\, n_0$, for all ω.
- Has linear phase (phase proportional to ω), and the slope (derivative wrt ω) gives the time shift.
- Here, slope is $-n_0$ implying a delay of n_0.

DT Fourier Transforms – Examples

Example 3: `O&W 5.1.2 pp.362-363`

Causal, exponentially decaying function

$$\boxed{x[n] = a^n\, u[n], \quad |a| < 1}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n\, u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n$$

$$= \frac{1}{1 - a\, e^{-j\omega}}, \quad \text{if } |a| < 1$$

DT Fourier Transforms – Examples

We can write this $X(e^{j\omega})$ as

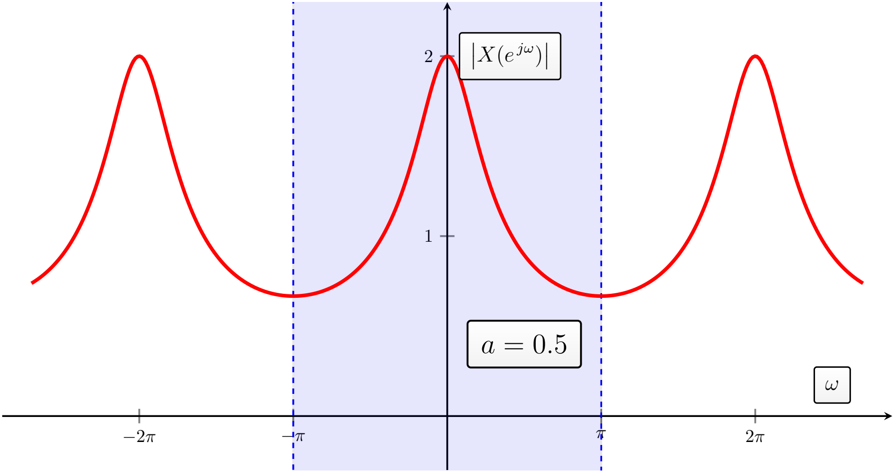
$$\boxed{X(e^{j\omega}) = \frac{1}{(1 - a\,\cos\omega) + ja\sin\omega}, \quad |a| < 1}$$

- This is complex (real plus imaginary) and a can be complex also.
- For real $a = 0.5$, $0.4, \ldots, -0.5$ we plot

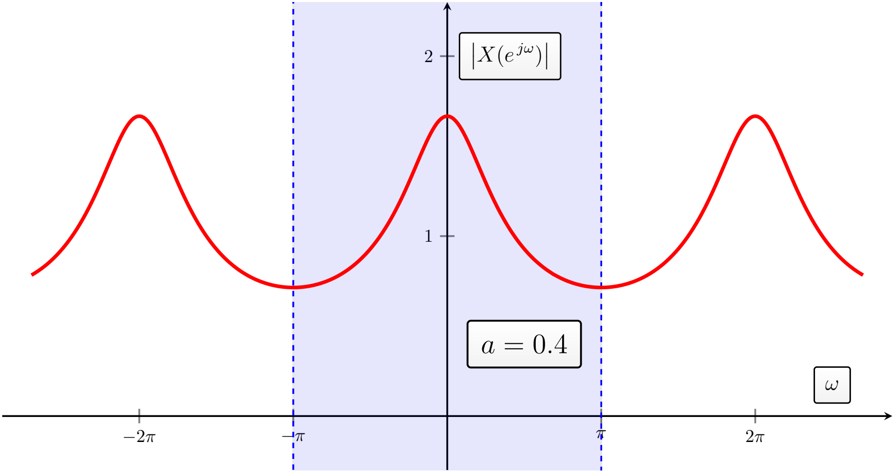
$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos(\omega) + a^2}}$$

In this way we can infer whether is acts like a low-pass filter, high-pass filter or otherwise.

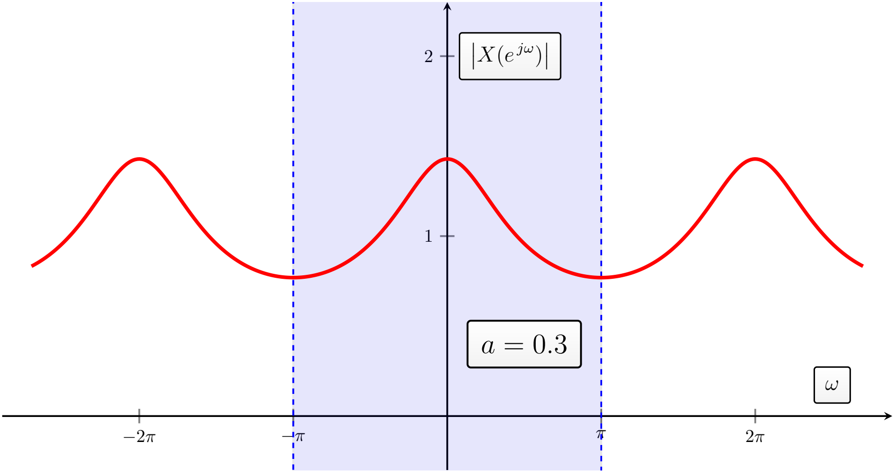
DT Fourier Transforms – Examples



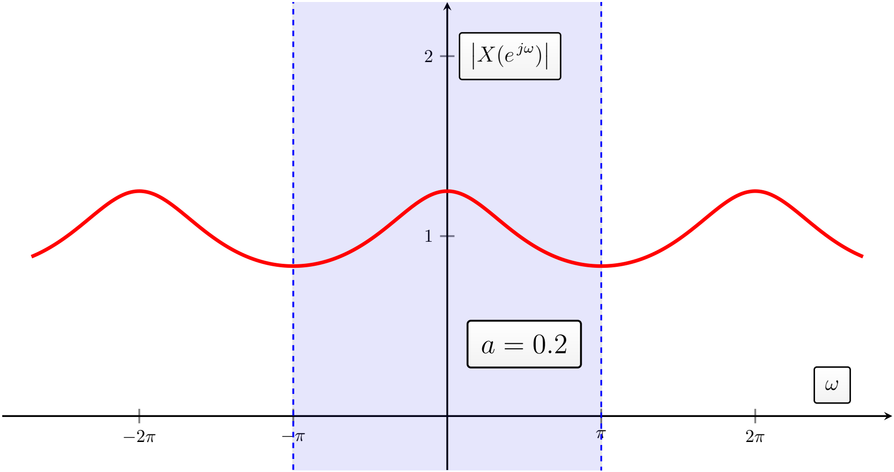
DT Fourier Transforms – Examples



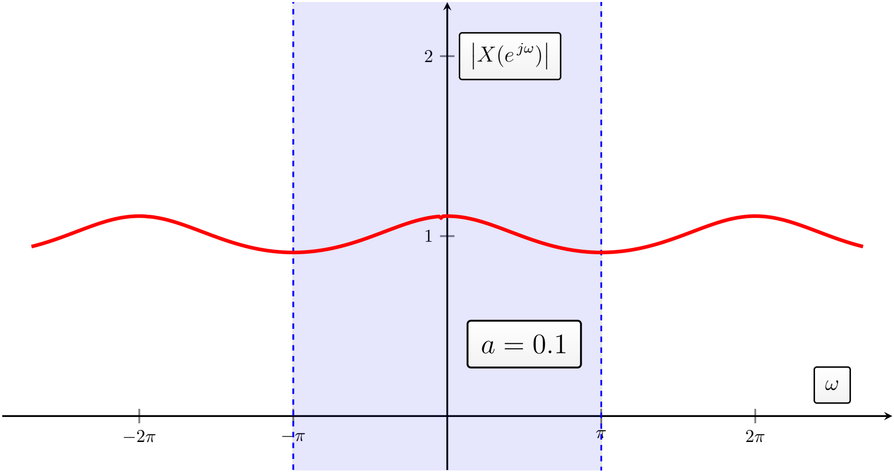
DT Fourier Transforms – Examples



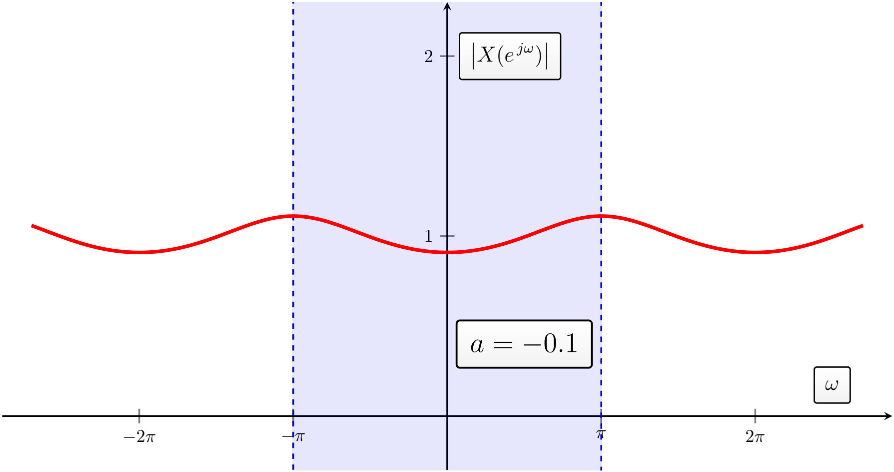
DT Fourier Transforms – Examples



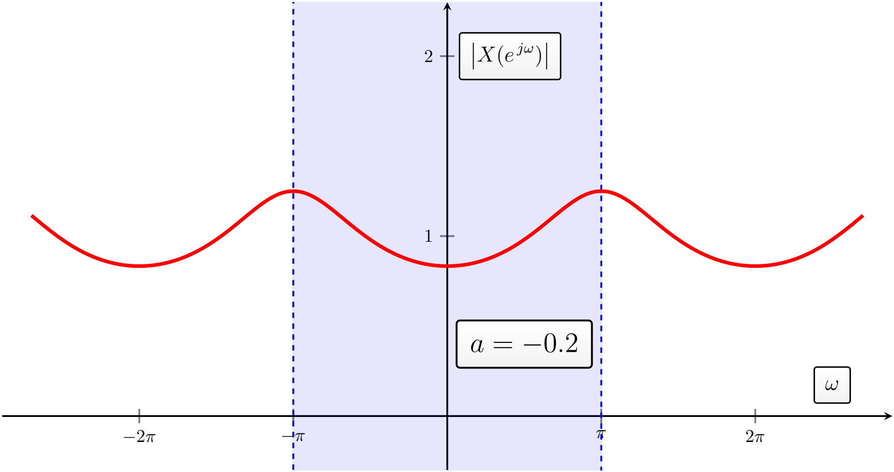
DT Fourier Transforms – Examples



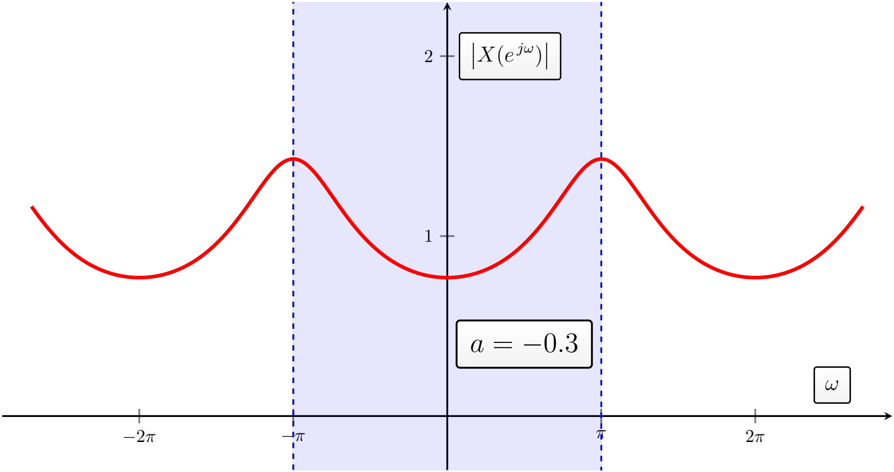
DT Fourier Transforms – Examples



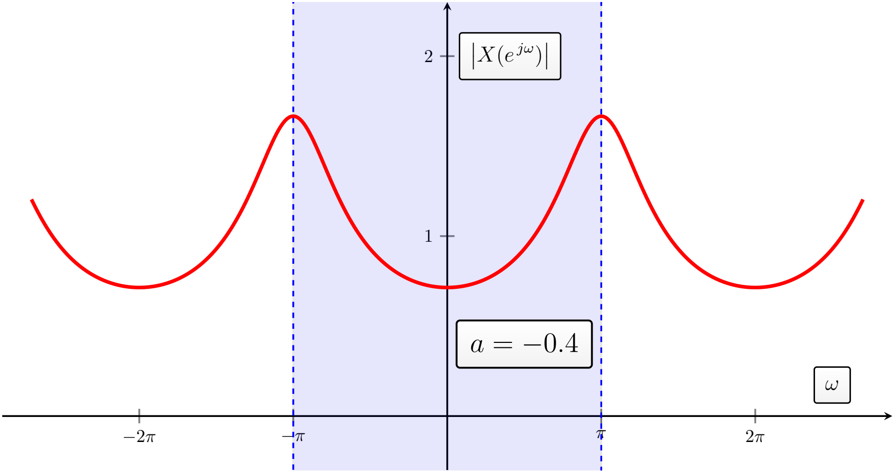
DT Fourier Transforms – Examples



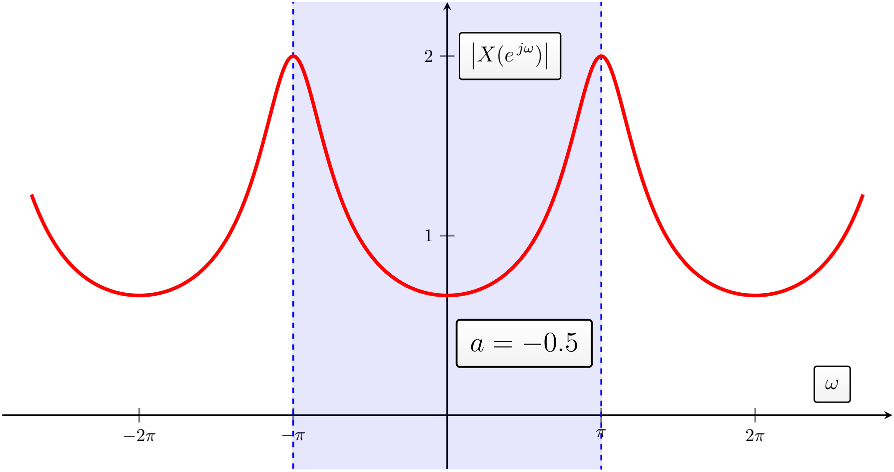
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Notes:

- With $a \to 0$ ("$a = 0$") we have $a^0 = 1$ and $a^n = 0$ for $n \neq 0$; then

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

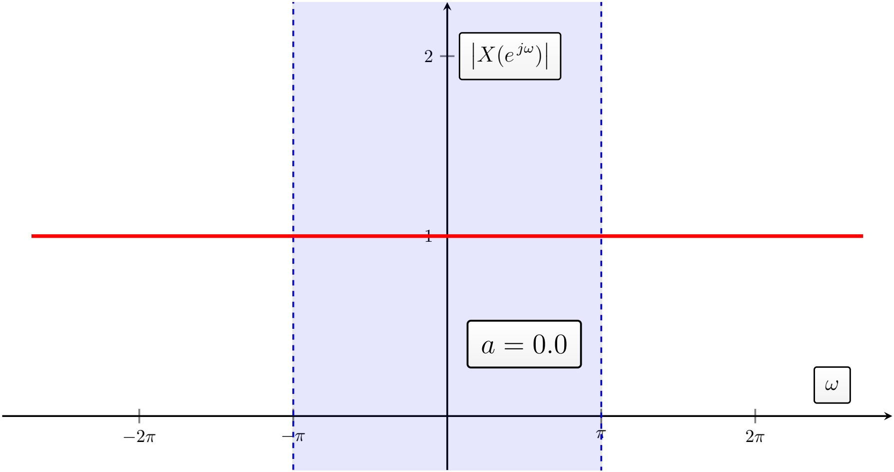
  that is,

$$x[n] = \delta[n]$$

  and

$$X(e^{j\omega}) = 1.$$

DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Example 4: `O&W 5.1.2 pp.365-366`

DT Rectangular Pulse function

$$\boxed{x[n] = \chi_{[-N_1,N_1]}[n], \quad n,\, N_1 \in \mathbb{Z}}$$

has Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \chi_{[-N_1,N_1]}[n]\, e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

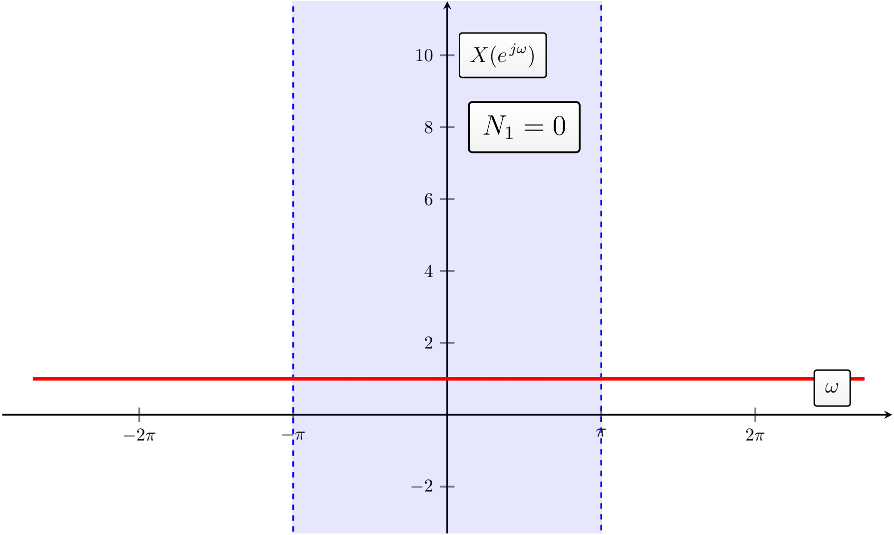
$$= \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}$$

That is,

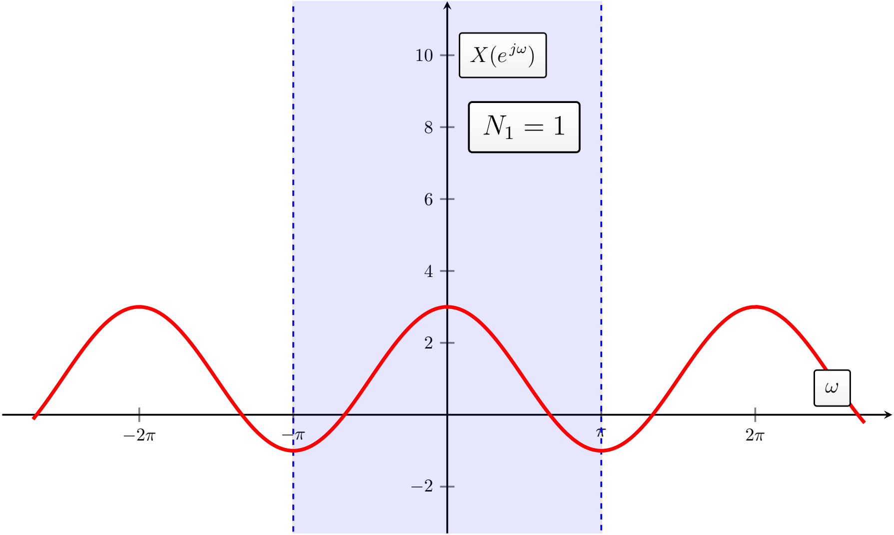
$$\boxed{X(e^{j\omega}) = \frac{\sin \omega(N_1 + 0.5)}{\sin(\omega/2)}, \quad N_1 \in \mathbb{Z}}$$

- This is purely real and we plot $X(e^{j\omega})$ rather than $\left|X(e^{j\omega})\right|$.

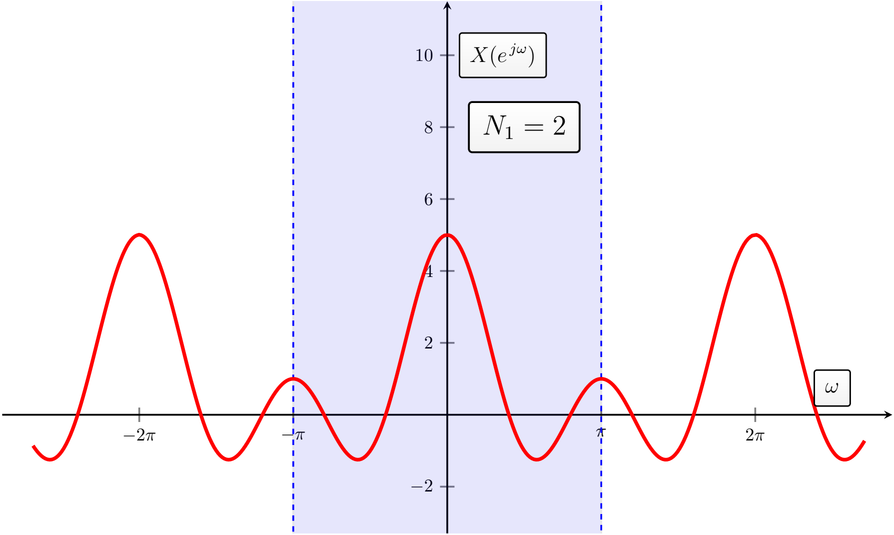
DT Fourier Transforms – Examples



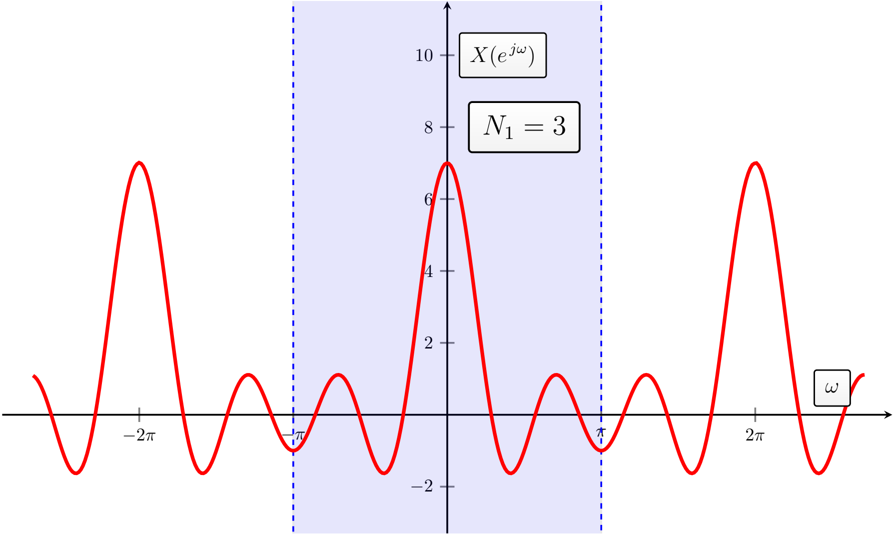
DT Fourier Transforms – Examples



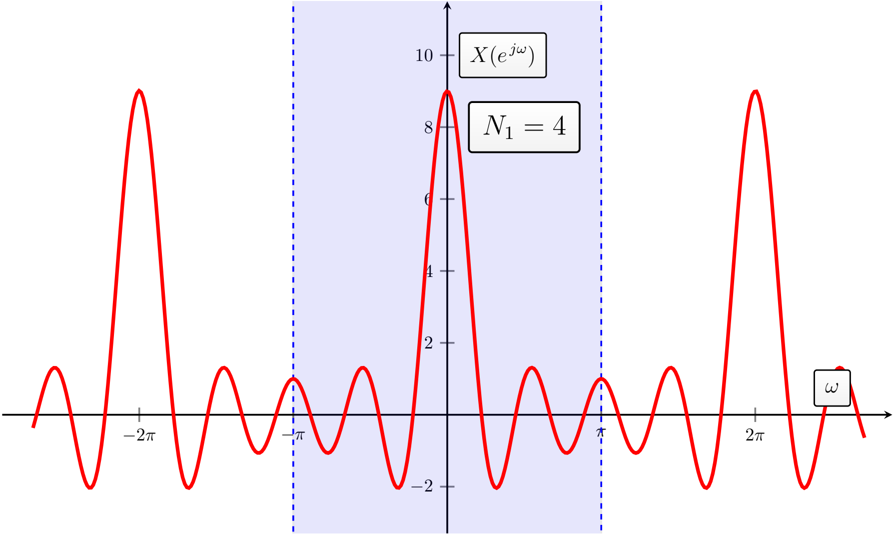
DT Fourier Transforms – Examples



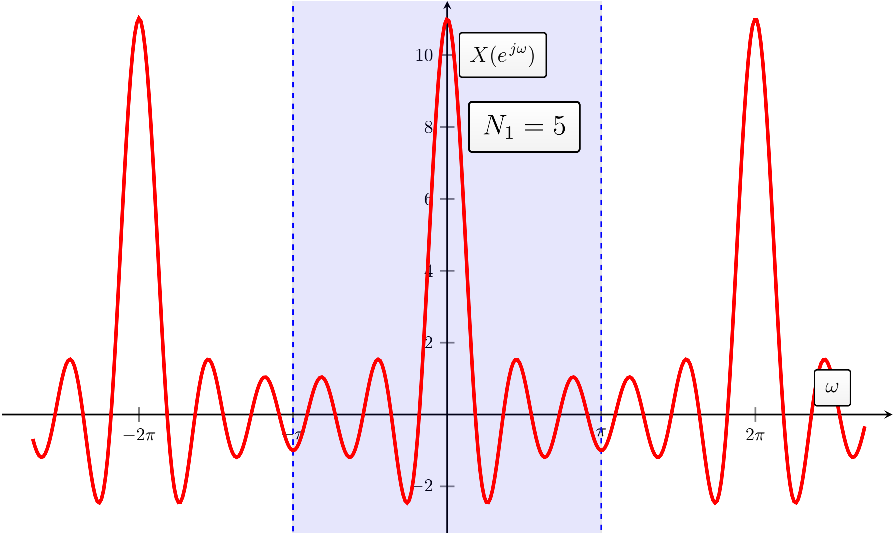
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

Notes:

- From these figures we can see that in the frequency domain the Fourier Transform of a time-domain discrete rectangular pulse looks (and is) the convolution of a sinc function with a periodic train of frequency domain delta functions. That is, the superposition of a sinc function with shifted versions of itself.

Example 5: `O&W 5.4.1 pp.383-384`

DT Ideal Low-Pass Filter, bandwidth $0 \leq W \leq \pi$, passes frequencies $-W \leq \omega \leq W$,

$$\boxed{X(e^{j\omega}) = \chi_{[-W,+W]}(\omega), \quad -\pi \leq \omega \leq \pi, \quad 0 \leq W \leq \pi}$$

and is periodic with period 2π. It has time domain sampled sinc response

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[-W,+W]}(\omega)\, e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega n}\, d\omega \\ &= \frac{\sin Wn}{\pi n}, \quad n \in \mathbb{Z} \end{aligned}$$

Definition (DT Ideal Low-Pass Filter)

The DT LTI system that passes only frequencies with gain 1 in the range $[-W, +W] \in [-\pi, +\pi]$ has impulse response and frequency response pair:

$$\boxed{\frac{\sin Wn}{\pi n} \xleftrightarrow{\mathscr{F}} \chi_{[-W,+W]}(\omega), \quad -\pi \le \omega \le \pi, \quad 0 \le W \le \pi}$$

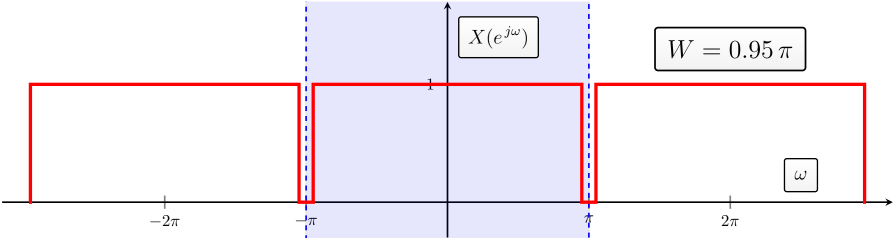
and the frequency response is periodic in ω with period 2π; or

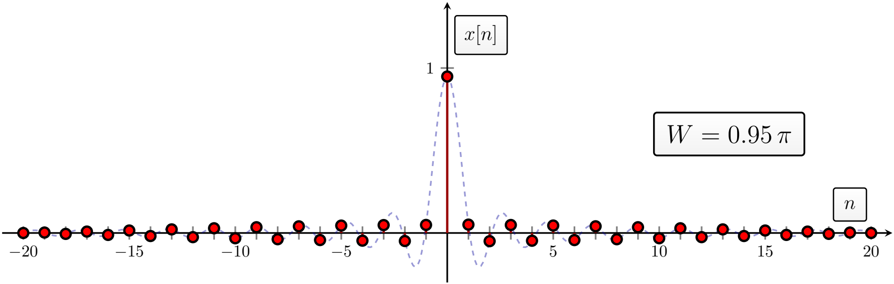
$$\boxed{\frac{W}{\pi}\,\mathrm{sinc}\Big(\frac{Wn}{\pi}\Big) \xleftrightarrow{\mathscr{F}} \chi_{[-W,+W]}(\omega), \quad -\pi \le \omega \le \pi, \quad 0 \le W \le \pi}$$

and the frequency response is periodic in ω with period 2π.

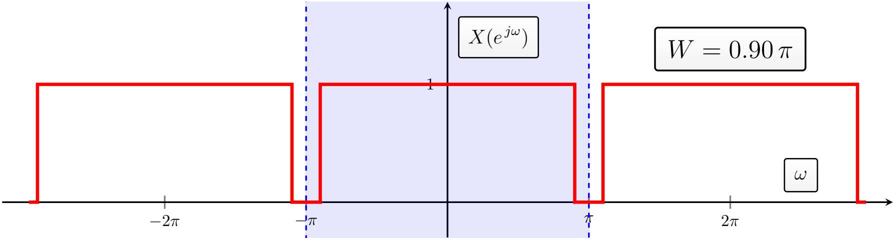
- We plot $X(e^{j\omega})$ and $x[n]$ for a range of $0 < W < \pi$.

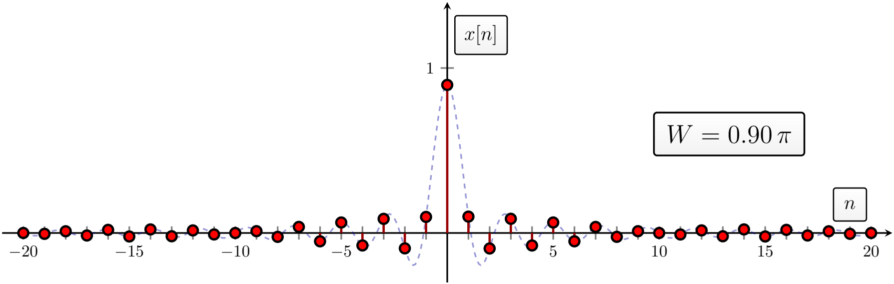
DT Fourier Transforms – Examples



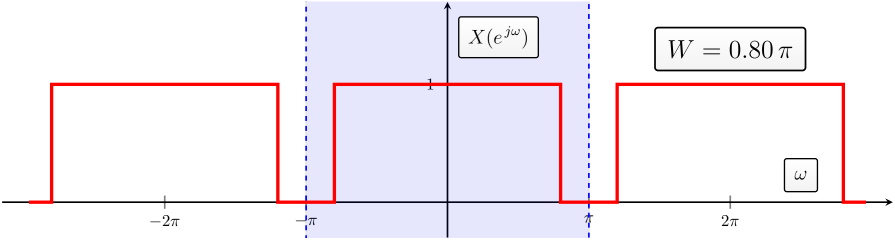


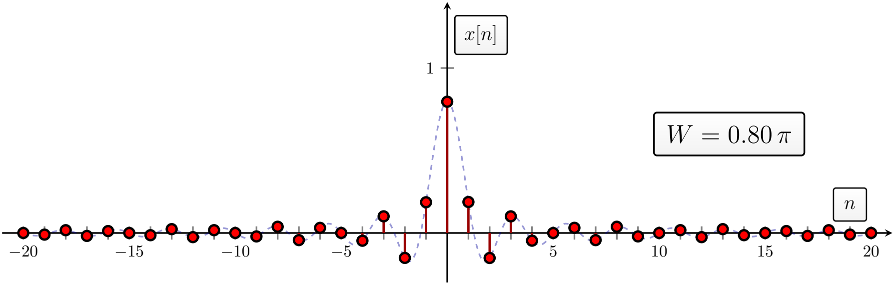
DT Fourier Transforms – Examples



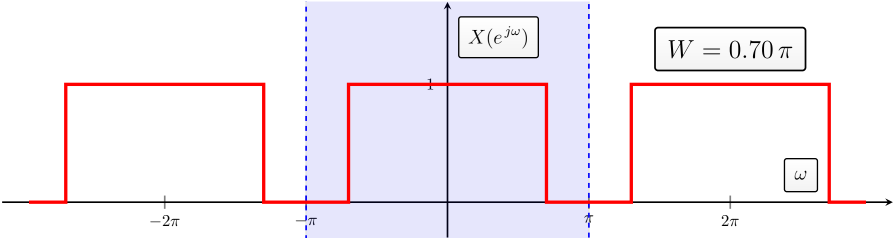


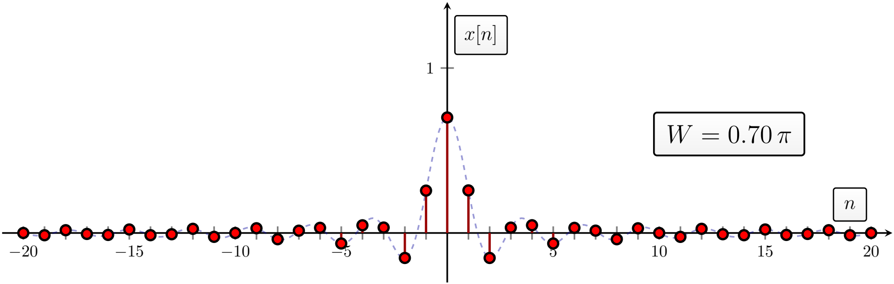
DT Fourier Transforms – Examples



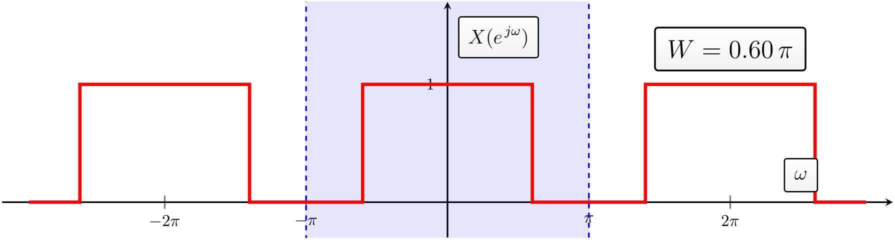


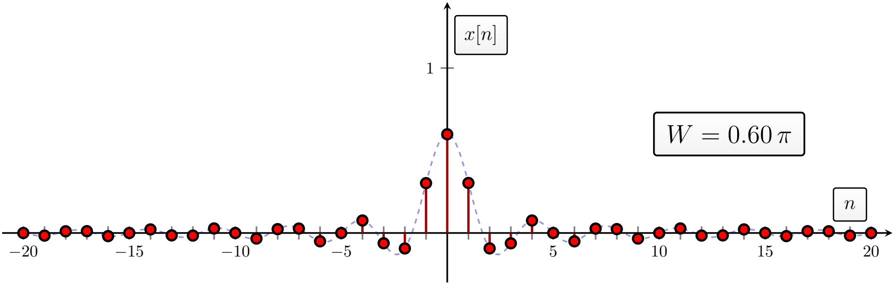
DT Fourier Transforms – Examples



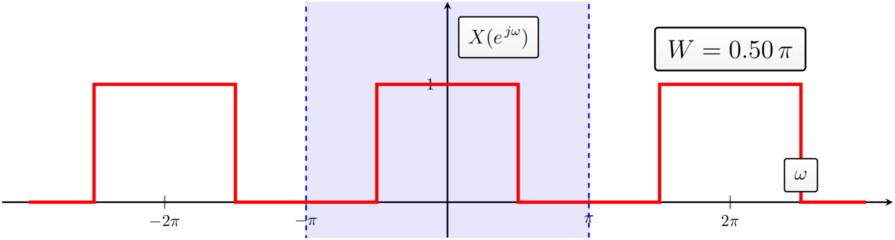


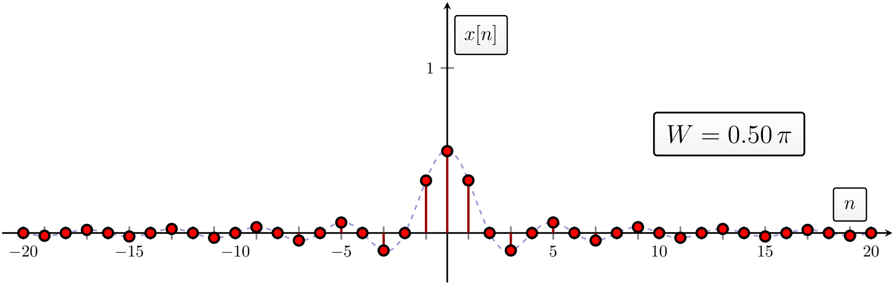
DT Fourier Transforms – Examples



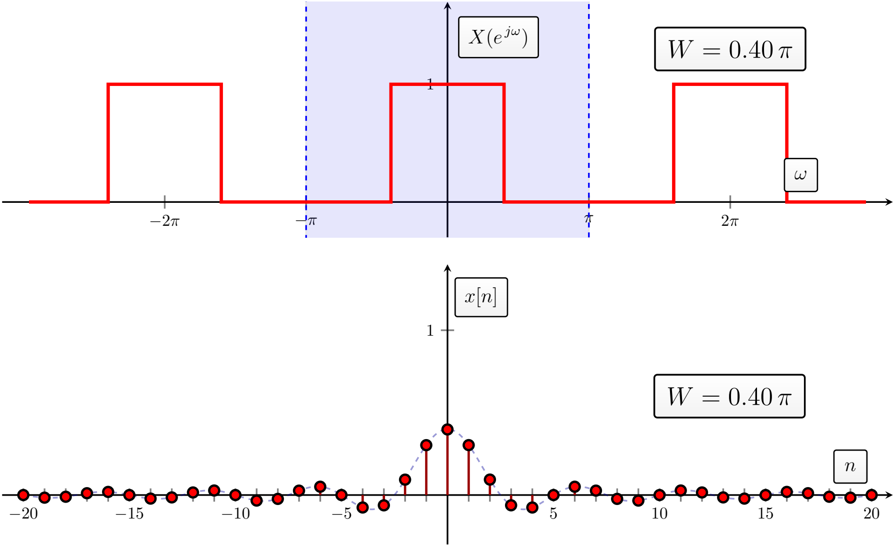


DT Fourier Transforms – Examples

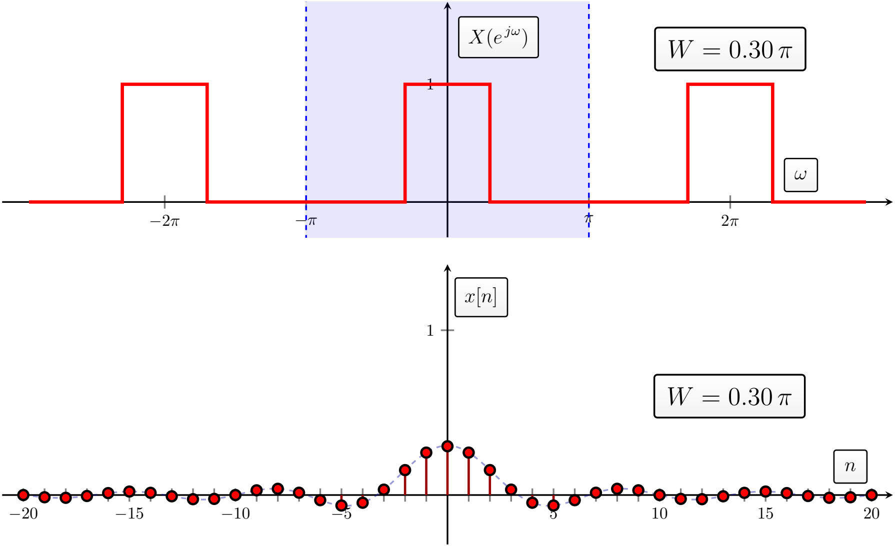




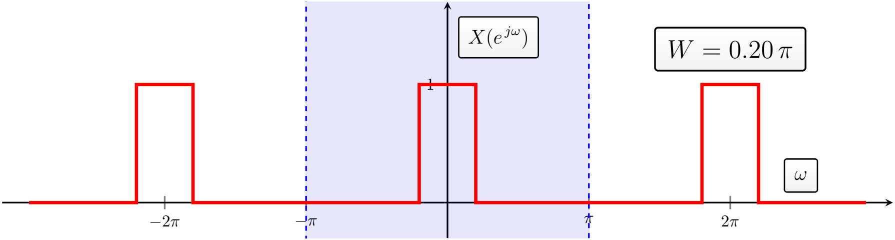
DT Fourier Transforms – Examples

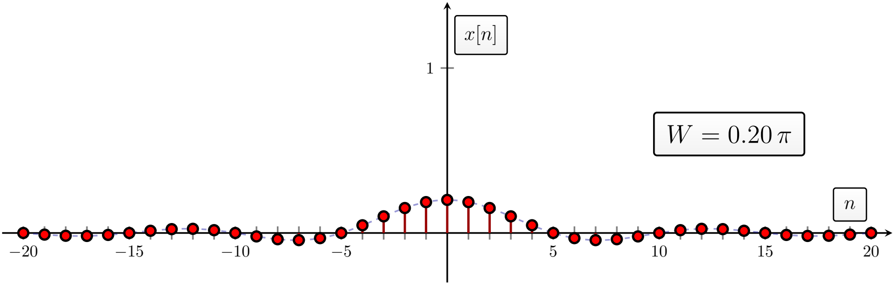


DT Fourier Transforms – Examples

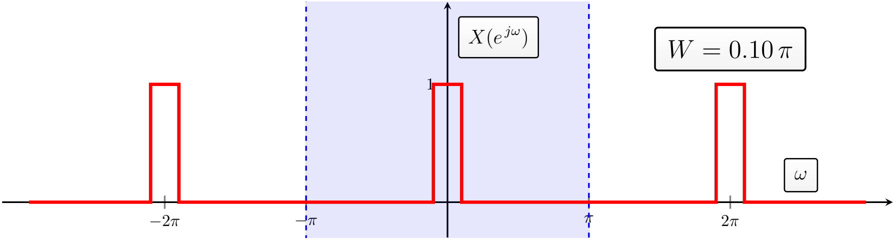


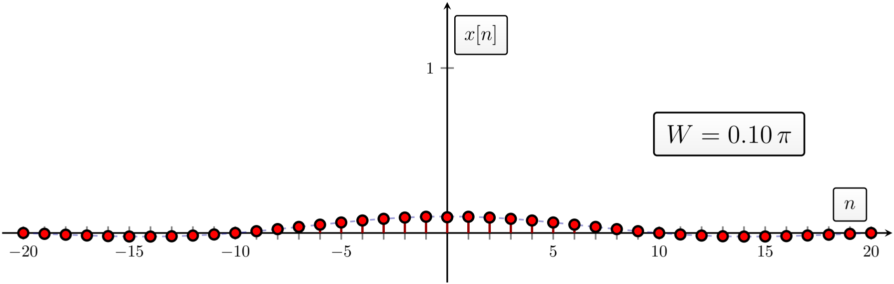
DT Fourier Transforms – Examples





DT Fourier Transforms – Examples





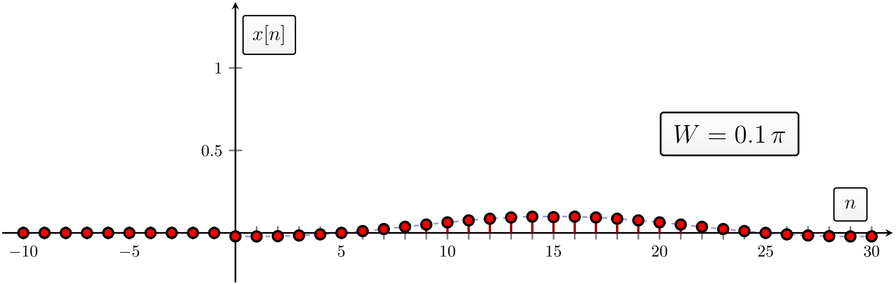
Example 6:

Implementable causal near-ideal low-pass filter (with delay of 15)

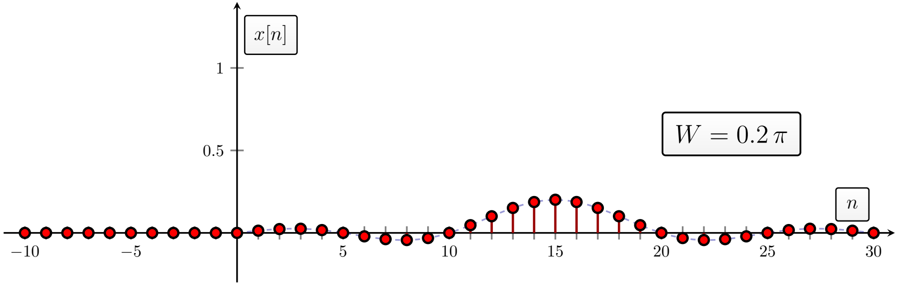
- Causal means the impulse response values are zero for all $n < 0$.
- The greater the delay the closer the magnitude response can get to an ideal low-pass filter.
- The bandwidth is varied in the following figures from $W = 0.1\,\pi$ to $W = \pi$ (all-pass; here a pure delay of 15).

$$x[n] = \begin{cases} \dfrac{\sin W(n-15)}{\pi(n-15)} & n \geq 0 \\ 0 & n < 0 \quad \text{(forced to zero by causality)} \end{cases}$$

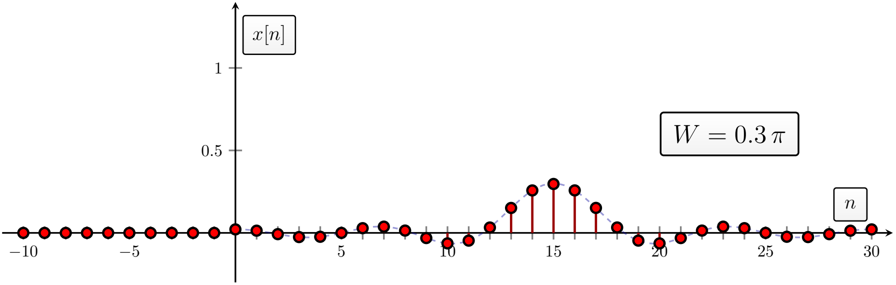
DT Fourier Transforms – Examples



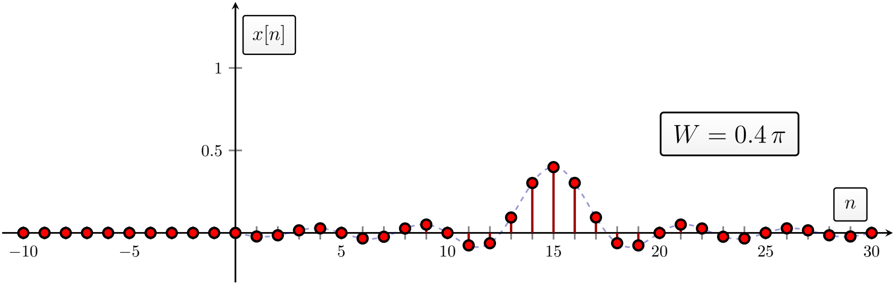
DT Fourier Transforms – Examples



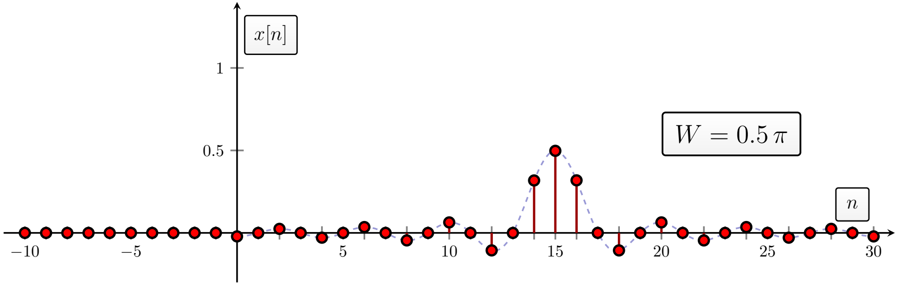
DT Fourier Transforms – Examples



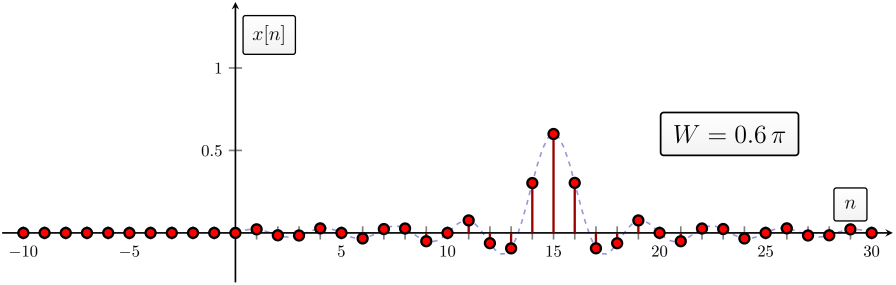
DT Fourier Transforms – Examples



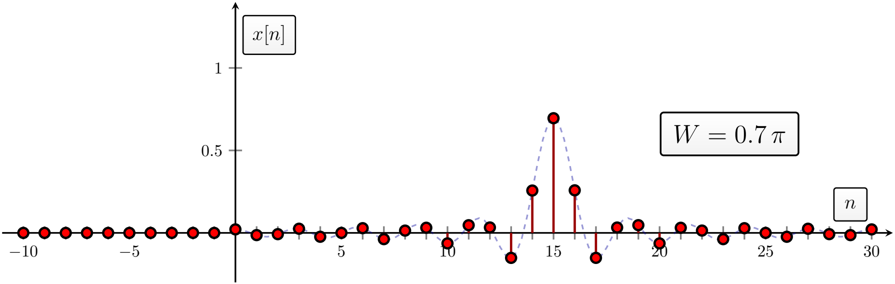
DT Fourier Transforms – Examples



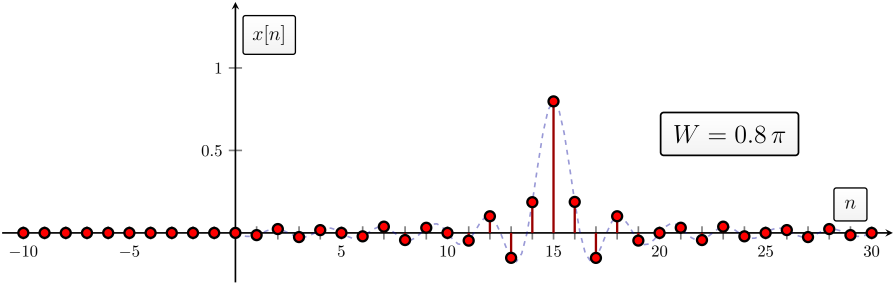
DT Fourier Transforms – Examples



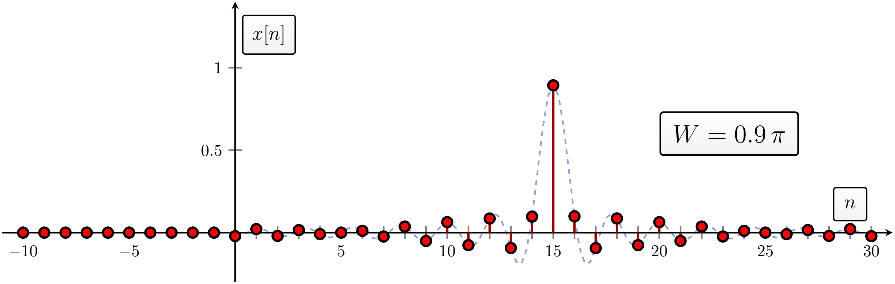
DT Fourier Transforms – Examples



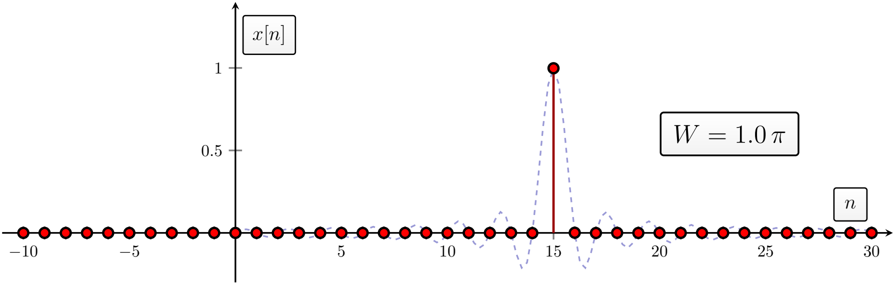
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples

That is, to get a causal version we shift the impulse response to more positive values and set any remaining non-causal values to zero. Setting these coefficients to zero, of course introduces artifacts:

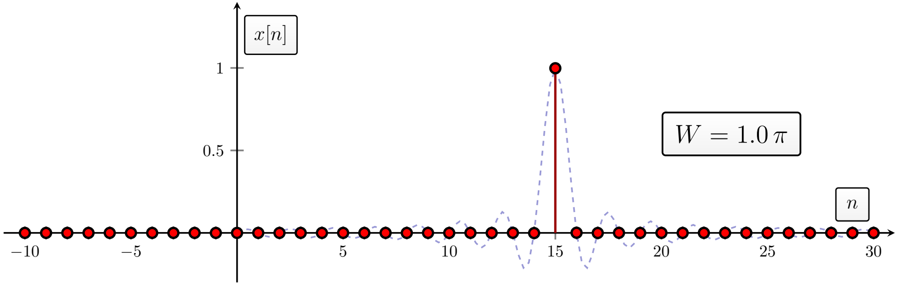
- There is a time delay introduced.
- The truncation implies that the result is not exactly the same as a delayed ideal impulse response but in engineering terms it is close.
- The closeness to an ideal impulse response can anyways improved by introducing a long delay. Then only much smaller weights which occur before zero are set to zero.

Example 7:

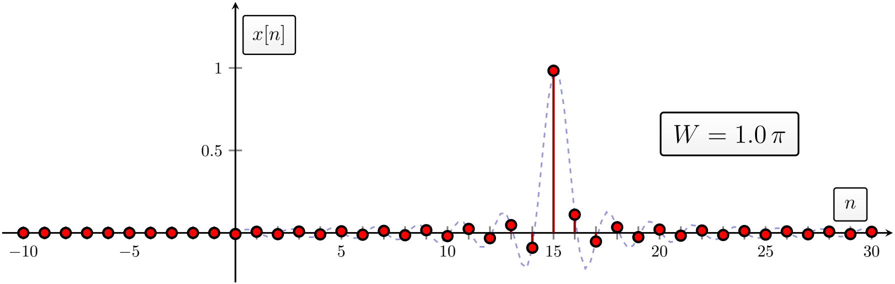
Mystery filters. . .

Previously we held the delay at 15 and varied the bandwidth W. That is, previously the sinc was centered at 15 and its width varied. Now we try a different variation where we fix the bandwidth to $W = \pi$ which means all-pass.

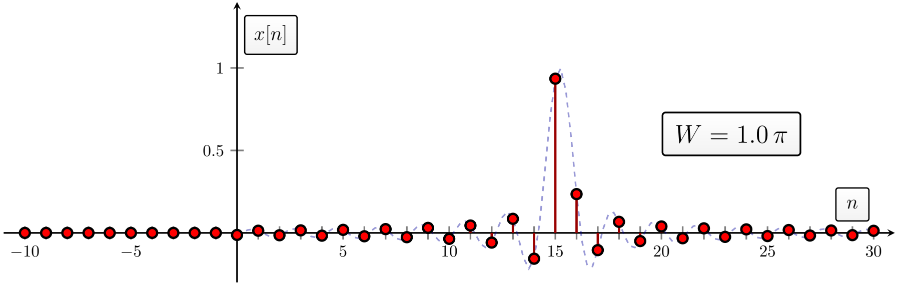
DT Fourier Transforms – Examples



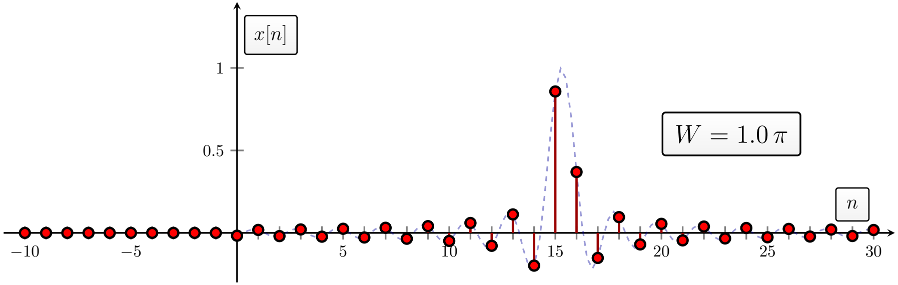
DT Fourier Transforms – Examples



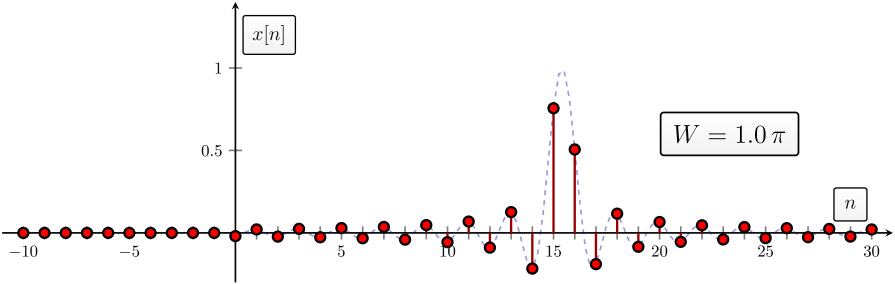
DT Fourier Transforms – Examples



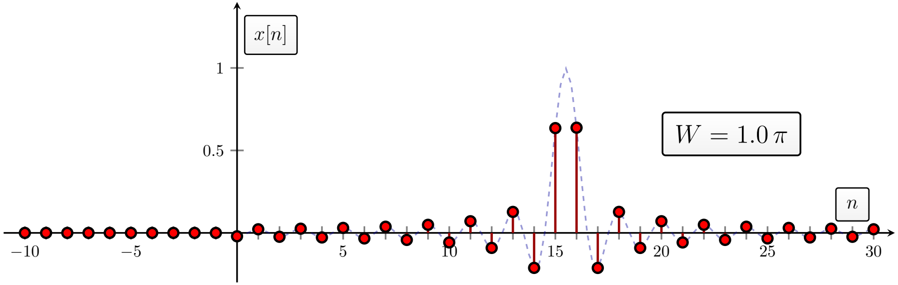
DT Fourier Transforms – Examples



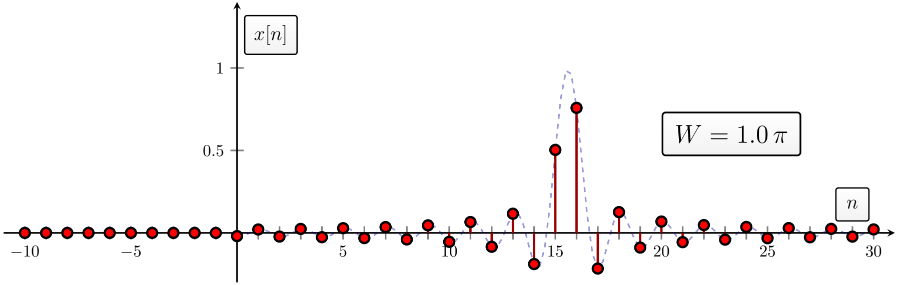
DT Fourier Transforms – Examples



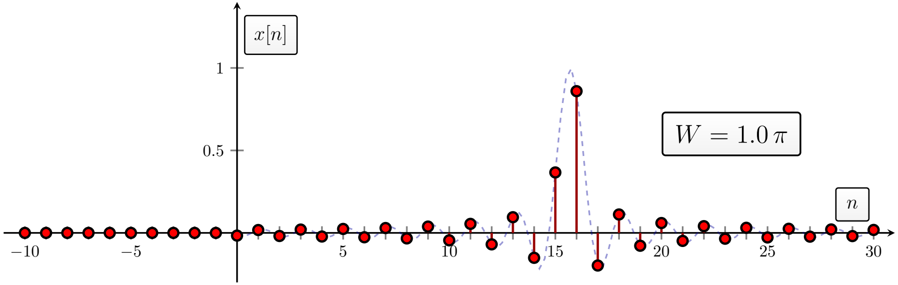
DT Fourier Transforms – Examples



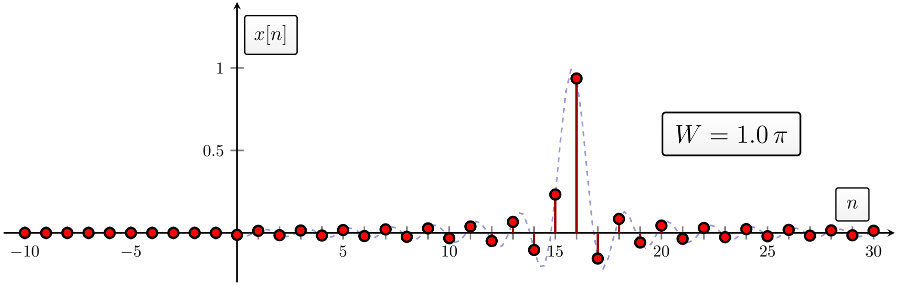
DT Fourier Transforms – Examples



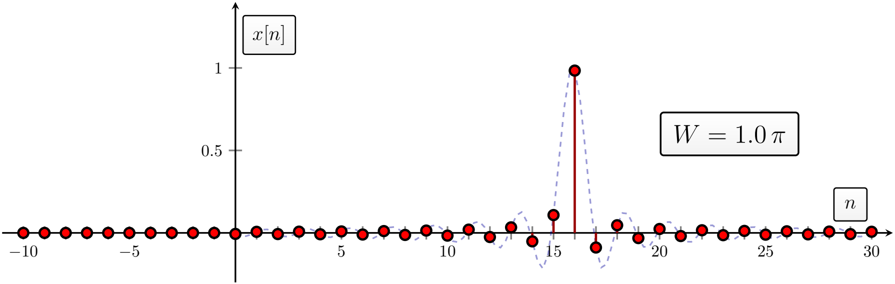
DT Fourier Transforms – Examples



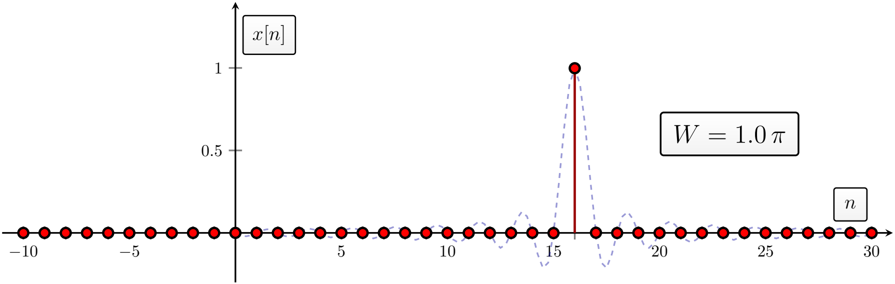
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT fractional delay filters!:

The green continuous function in the plots is a sinc function which is subject to a fractional delay, in this sequence, delays 15.0, 15.1, $\ldots$, 15.9, 16.0.

(Actually what we plotted was the causalized DT fractional delay filters since we had the impulse response values for $n < 0$ set to zero.)

Mathematically, we use the synthesis equation:

$$\boxed{h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z}}$$

and specify a linear phase all-pass filter with linear phase in the interval $[-\pi, \pi]$ with slope wrt ω equal to the delay.

So we can infer the impulse response of the fractional delay filter $h[n]$ by computing the inverse Fourier Transform of

$$\boxed{H(e^{j\omega}) = e^{-j\omega d}, \quad -\pi < \omega < \pi, \quad d \in \mathbb{R}}$$

where d is the delay and need not be integer.

$$\boxed{h[n] = \frac{\sin \pi(n-d)}{\pi(n-d)}, \quad d \in \mathbb{R}}$$

We make some observations:

DT Fourier Transforms – Examples

- With integer $d = n_0$ then

$$h[n] = \delta[n - n_0]$$

- $H(e^{j\omega}) = e^{-j\omega n_0}$ is naturally periodic with period 2π for integer $n_0 \neq 0$ ($n_0 = 0$ is not a problem). Further $H(e^{j\pi}) = H(e^{-j\pi})$.
- In contrast, $H(e^{j\omega}) = e^{-j\omega d}$ is not periodic with period 2π for non-integer d. For this reason the behaviour at $\omega = -\pi$ and $\omega = +\pi$ is not consistent and the behaviour is singular at that frequency:

$$\boxed{H(e^{j\omega}) = e^{-j\omega d}, \quad -\pi < \omega < \pi, \quad d \in \mathbb{R}}$$

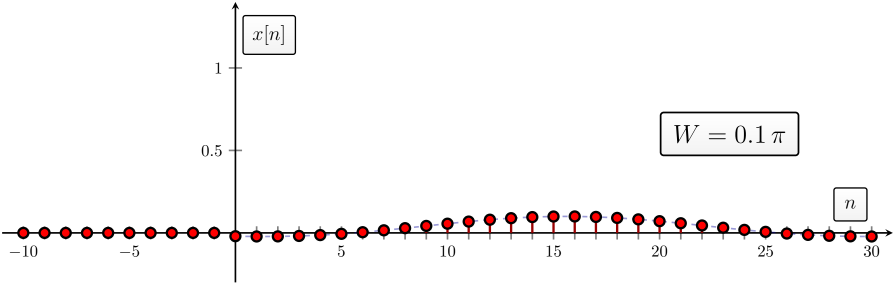
and is periodic for all ω outside $-\pi < \omega < \pi$.

DT fractional delay low-pass filter:

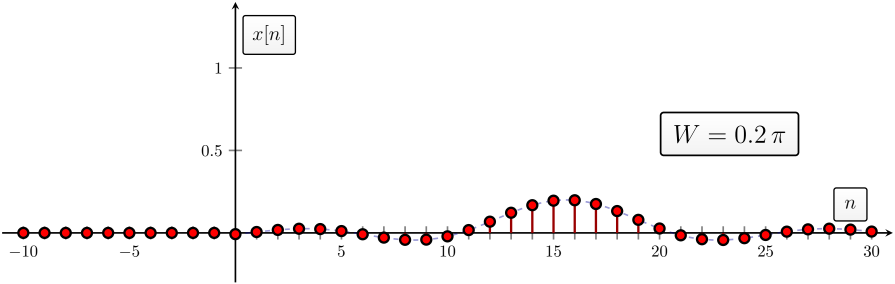
Can also have a fractional delay, here $d = 15.6$, and vary the bandwidth $W = 0.1\,\pi$ to $W = \pi$:

$$\boxed{h[n] = \frac{\sin W(n-d)}{\pi(n-d)}, \quad d \in \mathbb{R}}$$

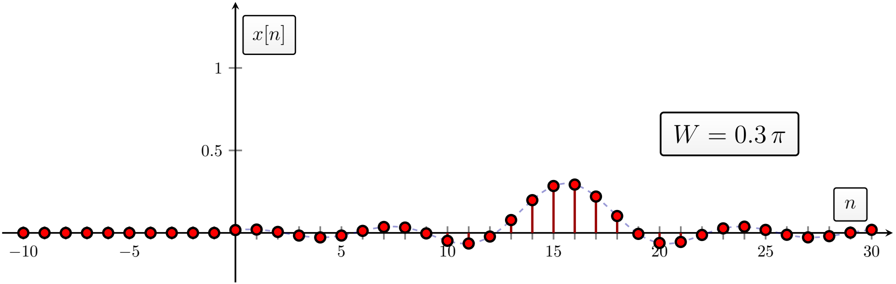
DT Fourier Transforms – Examples



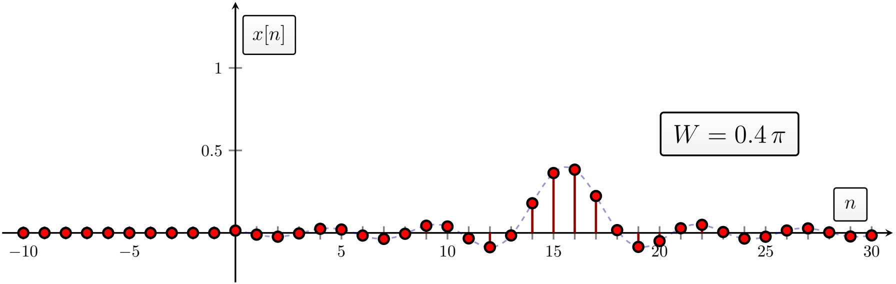
DT Fourier Transforms – Examples



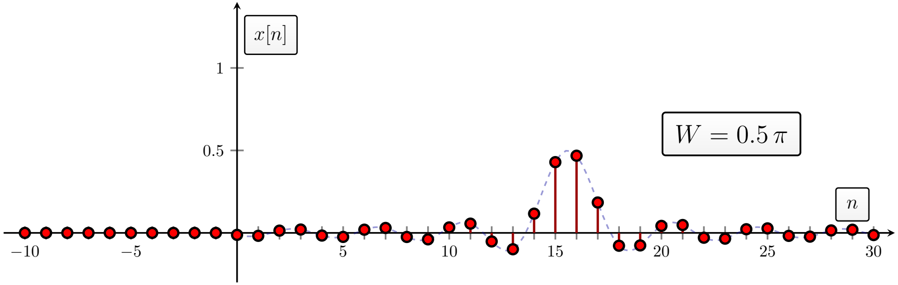
DT Fourier Transforms – Examples



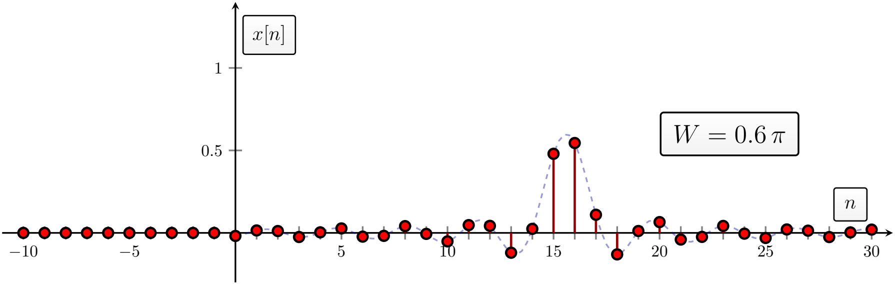
DT Fourier Transforms – Examples



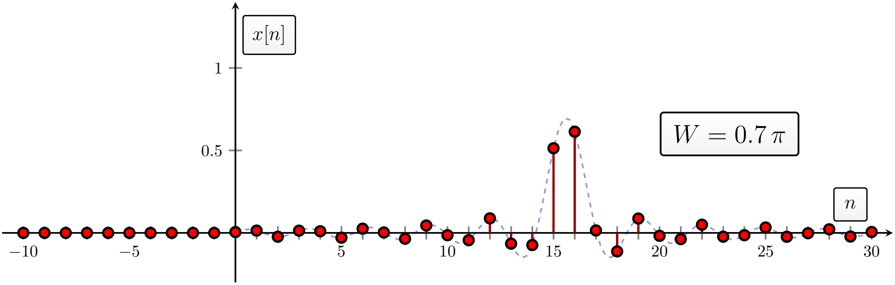
DT Fourier Transforms – Examples



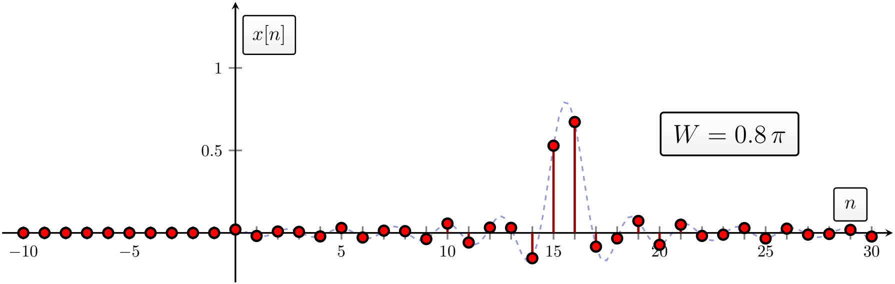
DT Fourier Transforms – Examples



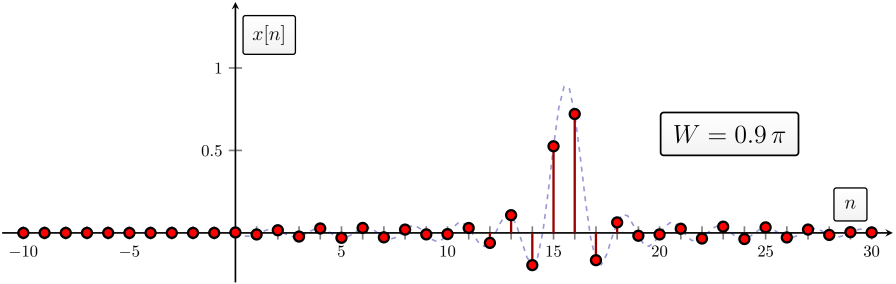
DT Fourier Transforms – Examples



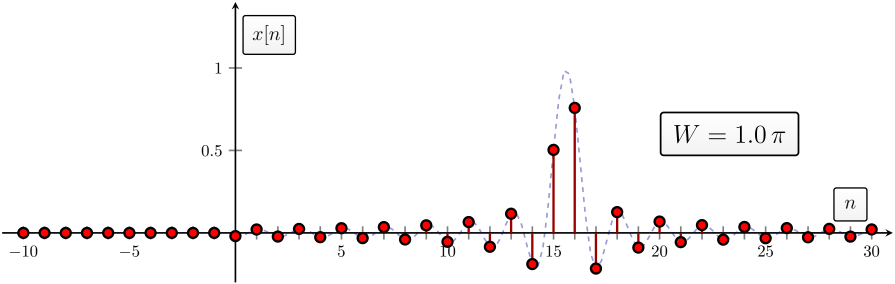
DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier Transforms – Examples



DT Fourier – Properties

Properties for DT Fourier Transform can be inferred from the Synthesis and Analysis equations. Properties that mimic the CT counterparts are:

- **Reference:**

$$\boxed{x[n] \xleftrightarrow{\mathscr{F}} X(e^{j\omega})}$$

- **Linearity/Superposition:** `O&W 5.3.2 p.373`

$$\boxed{a\,x_1[n] + b\,x_2[n] \xleftrightarrow{\mathscr{F}} a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})}$$

- **Time Shifting:** `O&W 5.3.3 p.373`

$$\boxed{x[n-n_0] \xleftrightarrow{\mathscr{F}} e^{-j\omega n_0} X(e^{j\omega}), \quad n_0 \in \mathbb{Z}}$$

DT Fourier – Properties

- **Frequency Shifting:** `O&W 5.3.3 p.373`

$$\boxed{e^{-j\omega_0 n}x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})}$$

- **Time Reversal:** `O&W 5.3.6 p.376`

$$\boxed{x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega})}$$

- **Conjugate Symmetry:** `O&W 5.3.4 p.375`

$$\boxed{x[n] \text{ real} \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega}) = X^*(e^{-j\omega})}$$

DT Fourier – Properties

Properties that more DT specific and/or tricky are:

- **Periodicity:** `O&W 5.3.1 p.373`

$$\boxed{X(e^{j\omega}) \text{ is periodic with period } 2\pi}$$

- **Differentiation in Frequency:** `O&W 5.3.8 p.380`

$$\boxed{n\,x[n] \xleftrightarrow{\mathscr{F}} j\frac{d}{d\omega}X(e^{j\omega})}$$

is multiplication by n in the time domain.

- **Parseval's Relation:** `O&W 5.3.9 p.380`

$$\boxed{\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega}$$

DT Fourier – Properties

- **Convolution:** `O&W 5.4 pp.382-388`

$$\boxed{y[n] = h[n] \star x[n] \xleftrightarrow{\mathscr{F}} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})}$$

Frequency Response $H(e^{j\omega})$ is the DTFT of the unit sample response.

- **Time Expansion:** `O&W 5.3.7 p.377-380` (how to fry your brain)

Compress by an integer factor $k \in \mathbb{Z}$ in the frequency domain:

$$\boxed{x_{(k)}[n] \xleftrightarrow{\mathscr{F}} X_{(k)}(e^{j\omega}) \triangleq X(e^{jk\omega})}$$

requires zero padding (expansion) in the time domain (DT)

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is an integer multiple of } k \\ 0 & \text{otherwise} \end{cases}$$

Finite Impulse Response (FIR): `O&W 2.4.2 p.122`

A DT system with a finite length impulse response (total duration) is called a Finite Impulse Response (FIR) filter. Just need a finite number of parameters to describe it. Ideally suited to digital signal processing implementation.

Three examples:

$$y_1[n] = x[n]$$
$$y_2[n] = x[n+1] - 3x[n] + 7x[n-89]$$
$$y_3[n] = \sum_{k=0}^{100} 0.5^k x[n-k]$$

They have total durations 1, 91 and 101 (which are clearly finite), respectively.

Infinite Impulse Response (IIR): `O&W 2.4.2 p.123`

If a DT system is not FIR then it is Infinite Impulse Response (IIR).

For example,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} a^{n-k} x[k], \quad |a| < 1, \quad n \in \mathbb{Z} \\ &= \sum_{k=0}^{\infty} a^{k} x[n-k], \quad |a| < 1, \quad n \in \mathbb{Z} \end{aligned}$$

However, IIR doesn't necessarily mean infinite complexity. In the next few slides we see why.

Consider some Low Pass Filters (LPFs). First a moving 5-average is given by

$$y[n] = 0.2\,x[n] + 0.2\,x[n-1] + \cdots + 0.2\,x[n-4]$$

and is FIR. (It is a FIR-LPF.) Possibly you want significantly more smoothing

$$y[n] = 0.001\,x[n] + 0.001\,x[n-1] + \cdots + 0.001\,x[n-999]$$

which is a moving 1000-average. Now the complexity is much higher (although there is a more efficient way to compute this).

Backfill Notes – FIR and IIR

But long memory can be achieved by the **recursive** system `O&W 3.11.1 pp.244-245`

$$\boxed{y[n] = a\,y[n-1] + x[n], \quad |a| < 1}$$

which has frequency response

$$\boxed{H(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}}}$$

by taking FT of

$$y[n] - a\,y[n-1] = x[n], \quad |a| < 1$$

and rearranging. In other words, this system is the same as

$$\boxed{y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k], \quad |a| < 1, \quad n \in \mathbb{Z}}$$

Given there are an infinite number of terms then this is IIR.

The equivalence can be seen directly from:

$$\begin{aligned} y[n] &= a\,y[n-1] + x[n], \quad |a| < 1 \\ &= a(a\,y[n-2] + x[n-1]) + x[n], \quad |a| < 1 \\ &= a^2\,y[n-2] + a\,x[n-1] + x[n], \quad |a| < 1 \\ &\;\;\vdots \end{aligned}$$

- This has roughly the same complexity as $y[n] = x[n] + a\,x[n-1]$.
- So IIR need not be infinitely complex. Here we can effectively have long memory, by making a close to 1.

Backfill Notes – FIR and IIR

Note to make a unity DC gain recursive IIR-LPF we slightly modify the previous dvelepment to:

$$\boxed{y[n] = a\,y[n-1] + (1-a)\,x[n], \quad |a| < 1}$$

This is like saying $(100\,a)$% of the last output plus $100\,(1-a)$% of the new input. Then

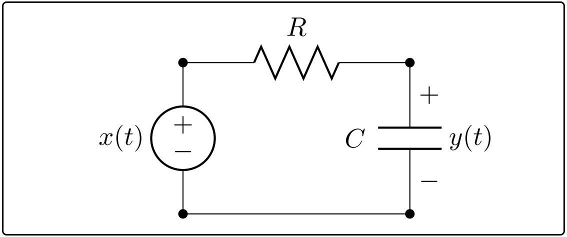
$$\boxed{H(e^{j\omega}) = \frac{1-a}{1-a\,e^{-j\omega}}}$$

from which we see $H(e^{j0}) = H(1) = 1$ at $\omega = 0$ (DC).

A digital filter (DT system) in practice is often either 1) FIR, or 2) IIR but formed by a mix of a limited amount of moving-average and a limited amount of recursive (or auto-regressive).

Backfill Notes – Circuits

In CT the RC circuit `O&W 3.10.1 pp.240-241`



acts as a system between input voltage $x(t)$ and output voltage $y(t)$. We can infer the frequency response:

$$H(e^{j\omega}) = (1/j\omega C)/(R + 1/j\omega C)$$
$$= \frac{1}{1 + j\omega RC}$$

This acts as a low pass filter, $y(t) = x(t)$ at $\omega = 0$, and $y(t) \to 0$ as $\omega \to \infty$.

LCC Difference Equations: `O&W 5.8 pp.396-399`

Linear constant coefficient difference equation (LCCDEs)

$$\boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]}$$

where we can interpret this as describing a DT-LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

for some impulse response $h[n]$. (Here $x[n]$ is the input and $y[n]$ is the output.) We can infer this system response; well at least its frequency response. That is, we can find the frequency response $H(e^{j\omega})$ corresponding to the LCCDE.

DT Fourier Stuff – Miscellanea

The LCCDE can be simplified by using

$$\boxed{x[n-k] \xleftrightarrow{\mathscr{F}} e^{-j\omega k} X(e^{j\omega}), \quad k \in \mathbb{Z}}$$

By taking the Fourier Transform of both sides of:

$$\boxed{\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]}$$

we get:

$$\boxed{\sum_{k=0}^{N} a_k e^{-jk\omega} \, Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} \, X(e^{j\omega})}$$

This type of transformation is directly analogous of what we did for differential equations.

DT Fourier Stuff – Miscellanea

$$Y(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{K} a_k e^{-jk\omega}} X(e^{j\omega})$$

So

$$\boxed{H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{K} a_k e^{-jk\omega}}}$$

is the frequency response of the linear, constant coefficient difference equation system.

$H(e^{j\omega})$ is a rational function of $e^{-jk\omega}$ amenable to the use of partial fraction expansions.

DT Fourier Stuff – Miscellanea

Multiplication Property: `O&W 5.5 pp.388-390`

$$\boxed{z[n] = x[n]\,y[n] \overset{\mathscr{F}}{\longleftrightarrow} Z(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\zeta})Y(e^{j(\omega-\zeta)})\,d\zeta}$$

Periodic convolution in the frequency domain.

Suppose we want the Fourier Transform of:

$$y[n] = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2$$

Then $y[n] = x[n]\,x[n]$ (the multiplication) where

$$x[n] = \frac{\sin(\pi n/4)}{\pi n}$$

which is the impulse response of an ideal low pass filter with cut-off $W = \pi/4$.

So

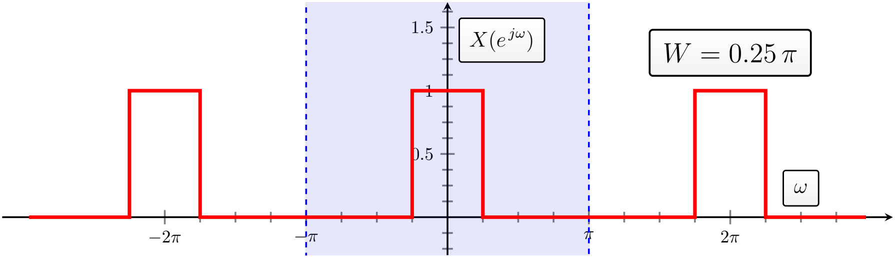
$$Y(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \star X(e^{j\omega})$$

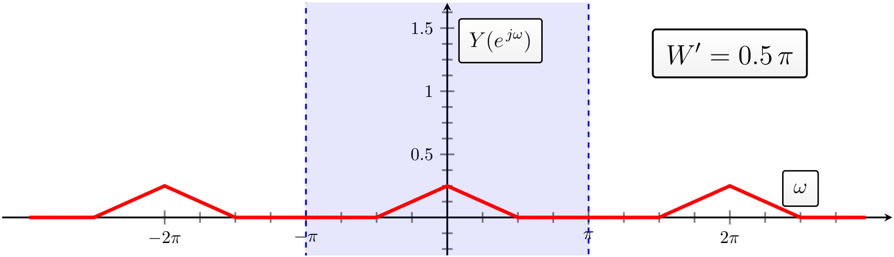
and we note, from the multiplication property,

$$\begin{aligned} Y(e^{j\omega})\Big|_{\omega=0} = Y(1) &= \frac{1}{2\pi} \int_{2\pi} \left(X(e^{j\zeta})\right)^2 d\zeta \\ &= \frac{1}{2\pi} \int_{2\pi} \left(\chi_{[-\pi/4,+\pi/4]}(\zeta)\right)^2 d\zeta \\ &= \frac{1}{2\pi} \int_{2\pi} \chi_{[-\pi/4,+\pi/4]}(\zeta)\, d\zeta \\ &= \frac{1}{2\pi} \cdot \frac{\pi}{2} = 0.25 \end{aligned}$$

Note here $W = \pi/4$. Let us denote the bandwidth of $y[n]$ as W' which in this case we'll see is given by $W' = \pi/2$, that is, twice W.

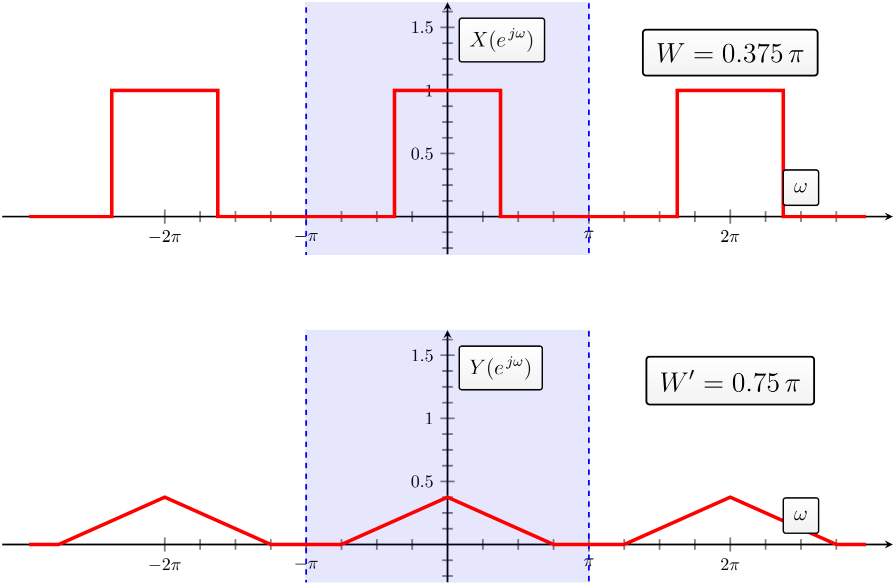
DT Fourier Stuff – Miscellanea



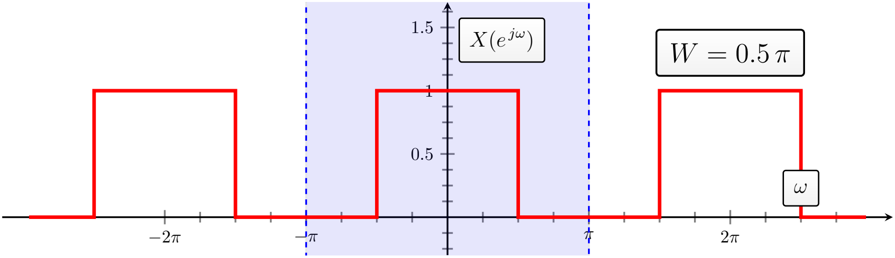


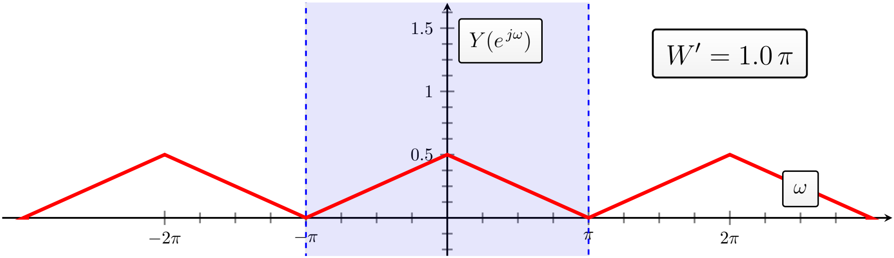
- Note, this is just a square law, which is non-linear. It doubles the bandwidth of the input provided the bandwidth of the output is no greater than π.
- Repeat for $W = 3\pi/8$ and $W = \pi/2$ which have $Y(e^{j0}) = 3/8$ and $Y(e^{j0}) = 1/2$, respectively.
- More generally, if $W = r\pi$ and $0 < r < \pi/2$ then $Y(e^{j0}) = r$.

DT Fourier Stuff – Miscellanea



DT Fourier Stuff – Miscellanea





- So if a square law doubles the bandwidth; what does a cube law do to the bandwidth, etc?
- A restriction in the previous point is that the output bandwidth is no greater than π, that is, provided the input bandwidth is small enough. For example, with a square law: $W = r\pi$ and $0 < r < \pi/2$. This is a DT issue only.

DT Fourier Stuff – Duality

Duality from Fourier Symmetry: `O&W 4.3.6; 5.7.1 and 5.7.2`

Summarizing table.

Case	Duality	Type	O&W Ref
1	CT Fourier Transform	Self Duality	`4.3.6 pp.309-311`
2	DT Fourier Series	Self Duality	`5.7.1 pp.391-394`
3	CT F. Series – DT F. Trans.	Cross Duality	`5.7.2 pp.395-396`

Roughly this means, in Case 3 for example, that the maths of DT Fourier Transforms is essentially the same as the maths of CT Fourier Series. Just need to flip around t or n (time) and k or ω (frequency) with some negative signs and 2π thrown in.

Not the most interesting set of properties...

DT Fourier Stuff – Duality

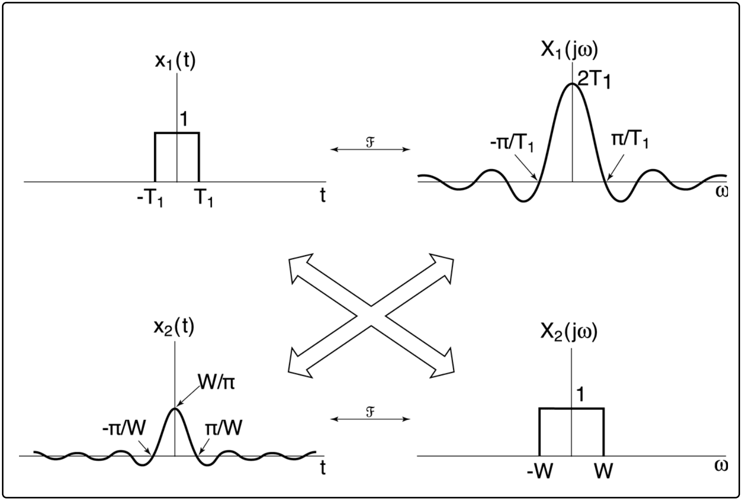
For Case 1: `O&W 4.3.6 pp.309-311`

$$\boxed{g(t) \overset{\mathscr{F}}{\longleftrightarrow} f(\omega)}$$

$$\boxed{f(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, g(-\omega)}$$

- Here CT non-periodic functions to continuous frequency non-periodic functions.
- In words: If $f(\omega)$ is the Fourier Transform of $g(t)$ then the Fourier Transform of $f(\omega)$ is $2\pi\, g(-\omega)$.

DT Fourier Stuff – Duality



DT Fourier Stuff – Duality

For Case 2: `O&W 5.7.1 pp.391-394`

Periodic and discrete in time yields periodic and discrete in frequency. Technically periodicity is the dual of discreteness.

For Case 3: `O&W 5.7.2 pp.395-396`

Here we have a mix. Interesting if you are disturbed, and disturbing if you are interested.

Duality means we can intelligently reuse results with minor scaling and flips.

Magnitude of FT: `O&W 6.2 p.427`

In CT:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \left|X(j\omega)\right| e^{j\angle X(j\omega)}$$

Parseval Relation:

$$\boxed{\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 d\omega}$$

So *magnitude* (scaled and squared) captures the **energy density** in ω.

DT Fourier Stuff – Magnitude of FT

In DT:

$$x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})}$$

Parseval Relation:

$$\boxed{\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega}$$

Ditto *magnitude* (scaled and squared) captures the **energy density** in ω.

DT Fourier Stuff – Magnitude of FT

So in summary for the magnitude.

- The magnitude defines the type of filter by the frequencies it passes and blocks.
- The magnitude is all we need in the Parseval Relation to determine the energy at various frequencies.

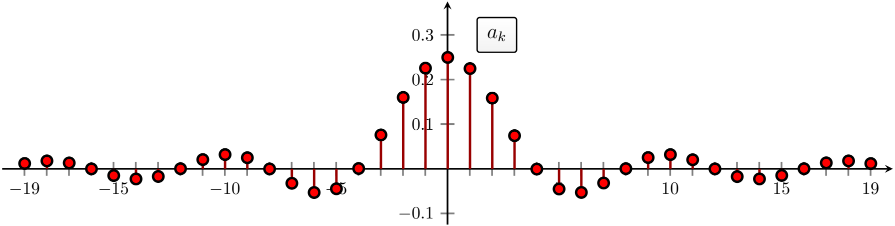
DT Fourier Stuff – Phase of FT

Phase of FT: `O&W 6.2 p.427`

- Phase is as much an issue as magnitude.
- Can have a significant effect on the signal shape and character.
- Contributes to constructive and destructive interference.
- The effect depends on the signal

DT Fourier Stuff – Phase of FT (con't)

With $T_1 = T/8$, the periodic rectangular wave has Fourier coefficients, $\{a_k\}$, as follows:



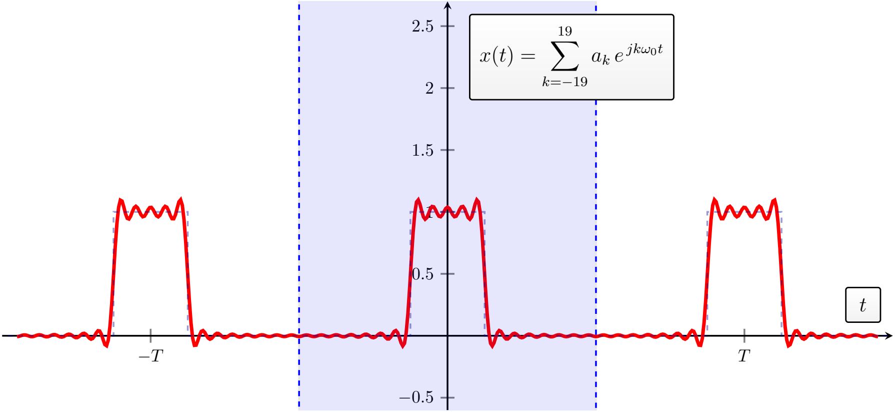
Now lets flip the sign (a π phase shift) of a subset of the Fourier coefficients. Clearly the magnitude (and Parseval's Relation) is the same.

Actually the sign is flipped the k components in the set:

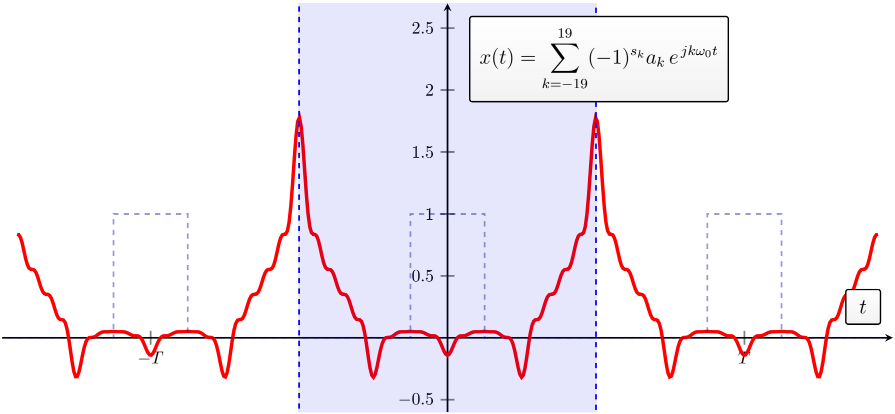
$$\mathscr{S} \triangleq \{\pm 1, \pm 3, \pm 6, \pm 9, \pm 11, \pm 14, \pm 17, \pm 19\}$$

The variable s_k, used in a later figure, is 1 for $k \in \mathscr{S}$ and 0 otherwise.

DT Fourier Stuff – Phase of FT



DT Fourier Stuff – Phase of FT



- Clearly there is a big difference in the appearance of the signal in the time domain. Phase is very important to the time domain shape.
- Of course, the energy at each frequency is not affected as this depends only on the magnitude.
- Potentially if you listened to both signals played back over loudspeakers then you might not hear a difference. Might be worth an experiment.
- Notice how the second signal has bigger peaks. This is a major problem in communication system design called "Peak to Average Ratio Reduction (in OFDM)". OFDM – Orthogonal frequency-division multiplexing – just uses the FFT to carry communication information on each regularly spaced frequency.

Linear and Nonlinear Phase `O&W 6.2.1 p.428-430`

We have seen previously that a pure delay system has linear phase

$$H(j\omega) = e^{-j\alpha\omega}$$

here α is the delay ($-\alpha$ for $\alpha > 0$ implies a delay). A pure delay is also an all-pass filter, that is, $\left|H(j\omega)\right| = 1$.

However, it would be a mistake to think that an all-pass is important here. Consider

$$H(j\omega) = \frac{\alpha - j\omega}{\alpha + j\omega}$$

which is all-pass, $\left|H(j\omega)\right| = 1$, but the phase is not a linear function of frequency (we say "the phase is nonlinear").

Linear phase means the delay as a function of frequency is appropriate to align in phase the contributions at different frequencies (constructive interference). So any remaining distortion is purely due to the magnitude.

Group Delay `O&W 6.2.2 p.430-436`

When the phase is non-linear how should we think about the signal delay? We note that the linear phase condition is a constant slope condition. Group delay is associated with the derivative of the phase, so that

$$\boxed{-\frac{d}{d\omega}\angle H(\,j\omega)}$$

reflects the local time delay at (or in the vicinity of) ω. Then for ω in the vicinity of ω_0 we have

$$\angle H(\,j\omega) \approx \angle H(\,j\omega_0) + (\omega - \omega_0)\,\left.\frac{d\angle H(\,j\omega)}{d\omega}\right|_{\omega=\omega_0}$$

Sampling – Background

CT signals (and systems) describe the physical world. We are interested in converting them to DT signals. Why?

- why not?
- we what to process them with a computer or digital signal processor (DSP)
- it may reduce the complexity of the signal
- we can use MATLAB to play with them
- they are amenable for storage (hard disk)

We have the results from earlier lectures to fully understand the process of converting a CT signal to a meaningful DT signal through “sampling”. What is sampling?

Sampling is taking snapshots of some signal $x(t)$ every T seconds. Here T is the sampling period. For example, the standard CD audio sampling rate is 44.1 kHz which means $T = 1/44100 = 0.00002267\cdots$ seconds.

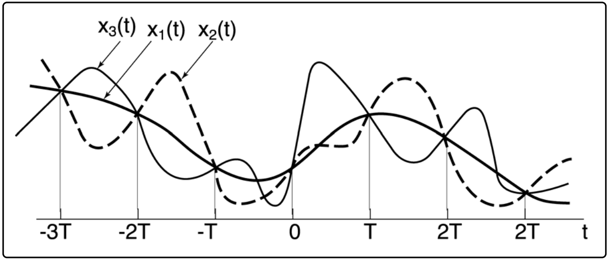
44.1 kHz is a weird number. It was chosen to: 1) be high enough and 2) to make it hard to convert from 48 kHz professional audio by designers who probably would have failed this course. And then there is 96 kHz audio which is a load of nonsense to make it appeal to the same fools who think oxygen free speaker cable makes audio sound better.

With T the sampling period, one way to create a DT signal, $x[n]$, from a given CT signal, $x(t)$, is via

$$x[n] = x(nT), \quad n \in \mathbb{Z}$$

noting $x(nT)$ means $x(t)$ evaluated or sampled at time $t = nT$.

We note that lots of CT signals have the same samples



and, therefore, information is thrown away when we sample. We lose the behavior of the signal between the samples. Or do we? Do we *always* lose information? Under what conditions can we recover or reconstruct the original CT signal $x(t)$ from its samples $x[n] = x(nT)$?

This is a non-trivial and somewhat remarkable property. If true then we can do digital signal processing (signal processing of sampled signals) without compromising the signal we are interested in.

What we expect is:

- The signals might need to have some special property (like smoothness).
- The sampling period T is short enough, or equivalently the sampling rate $1/T$ is high enough, relative to the variations we see in the signal.
- The complete CT signal $x(t)$ must be mathematically explicitly expressible in terms of its samples $x[n] = x(nT)$.

Sampling – Impulse Sampling

We need a convenient mathematical description of the sampling process. On the way through the course the signals and operations required to do this have been introduced.

Multiply the signal $x(t)$ by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

which yields

$$x_p(t) = x(t)\, p(t) = \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT)$$

By "sifting" property of the delta function:

$$\boxed{x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)}$$

$x_p(t)$ is a continuous time sampled signal. We only need to know $x(t)$ at $t = nT$ for integer $n \in \mathbb{Z}$, these are the samples.

For all other values of t (namely all t taking reals values not equal to the integers), the information in $x(t)$ is ignored or discarded. That is, the vast majority of $x(t)$ is discarded.

So surely hoping that $x_p(t)$ still contains all the information in $x(t)$ seems delusional. But the remarkable thing is, under the right conditions, we can figure out what $x(t)$ is from the continuous time sampled signal $x_p(t)$. Let's understand this further.

Time domain operation

$$x_p(t) = x(t)\,p(t)$$

implies, by the multiplication property, in the frequency domain

$$X_p(\,j\omega) = \frac{1}{2\pi} X(\,j\omega) \star P(\,j\omega)$$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\,j\omega), \quad x_p(t) \overset{\mathscr{F}}{\longleftrightarrow} X_p(\,j\omega), \quad p(t) \overset{\mathscr{F}}{\longleftrightarrow} P(\,j\omega)$$

where

$$\boxed{P(\,j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,\omega_s)}$$

and $\omega_s \triangleq 2\pi/T$ is the sampling frequency (in rad/sec).

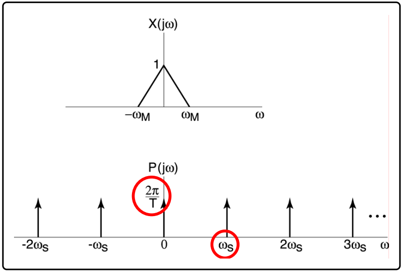
So we have

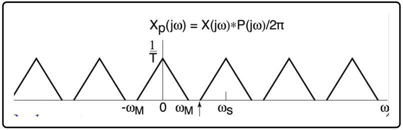
$$
\begin{aligned}
X_p(\,j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\,j\omega) \star \delta(\omega - k\,\omega_s) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(\,j(\omega - k\,\omega_s)\big)
\end{aligned}
$$

Recall $x(t) \xleftrightarrow{\mathscr{F}} X(\,j\omega)$ and $x_p(t) \xleftrightarrow{\mathscr{F}} X_p(\,j\omega)$.

So the frequency content of $x_p(t)$ is a periodic copy, with the frequency shifts given by $k\,\omega_s$ for integer $k \in \mathbb{Z}$, of the frequency content of $x(t)$ (with some scaling given by $1/T$).

Sampling – Illustration





More than an illustration but an important special case.

Here, have a band-limited signal such that $X(j\omega) = 0$ for $|\omega| > \omega_M$ and illustrated for

$$\omega_s - \omega_M > \omega_M$$

or

$$\omega_s > 2\,\omega_M$$

Sampling – Reconstruction

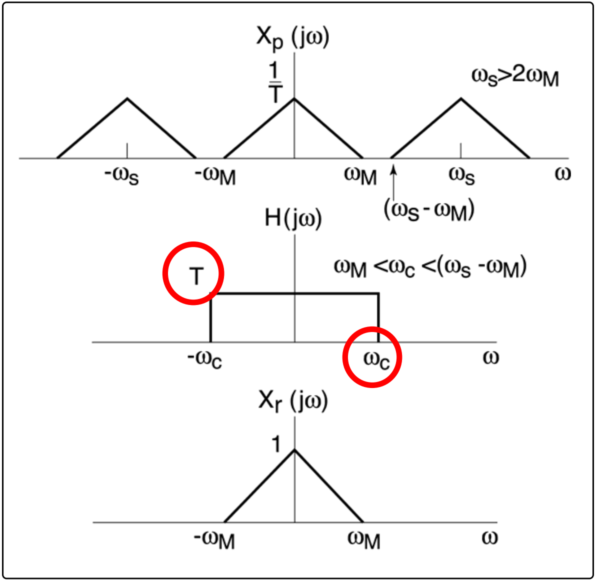
If the frequency content of the signal is appropriately low pass then there is no overlap in the periodic repetition of the spectrum. Hence we can use an ideal low pass filter to *perfectly* recover the signal.

A suitable bandwidth of the low pass filter is ω_c where

$$\omega_M < \omega_c < (\omega_s - \omega_M)$$

This condition can be "graphically" determined.

Sampling – Reconstruction



Theorem (Sampling Theorem)

Suppose $x(t)$ is band-limited so that

$$X(\,j\omega) = 0 \quad \textit{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$ if

$$\omega_s \triangleq \frac{2\pi}{T} > 2\,\omega_M \equiv \textit{“Nyquist Rate”}$$

The theorem provides sufficient conditions. You could generate a whole bunch of similar results for bandpass signals or signals with nicely spaced frequency holes, etc.

Example
Generally the top end of human hearing is given as 20 kHz (which declines with age and the number of night-clubs you frequent). The Nyquist rate or frequency is twice the highest frequency content, so the sample rate in Hz needs to be at least 40 kHz (or $2\,\omega_M = 80,000\pi$ in radians/sec).

(Continued)

Note that the CD audio rate is at least 40 kHz since it is 44.1 kHz. Why is it not 40 kHz?

- If it were 40 kHz then we would need an ideal low pass filter with cut-off at 20 kHz exactly. An ideal low pass filter has a long sinc shaped impulse response which is not nice (expensive and complicated to implement).
- Since there is a gap between 40 kHz and 44.1 kHz, the reconstruction filter needs only to satisfy

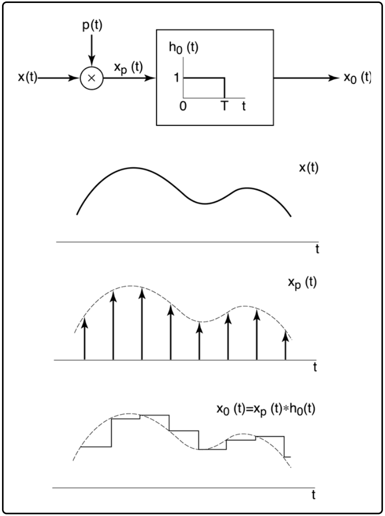
$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 2\pi \times 20,000 \\ 0 & \text{for } |\omega| > 2\pi \times 22,050 \\ \text{don't care} & \text{otherwise} \end{cases}$$

So we can design a cheaper filter.

Sampling with impulses is an idealization. An infinitely narrow time portion of a signal would have zero energy in the limit (which we can't amplify by infinity like we do mathematically).

We could add or integrate the signal around the sampling time instant to gather enough energy to get a meaningful reading. This is a distortion but not a great one. This leads to the concept of a "Zero-Order Hold" which has a convenient mathematical or system model (shown next). It is important to have such a model since it can reveal how close to ideal impulse like sampling we can get.

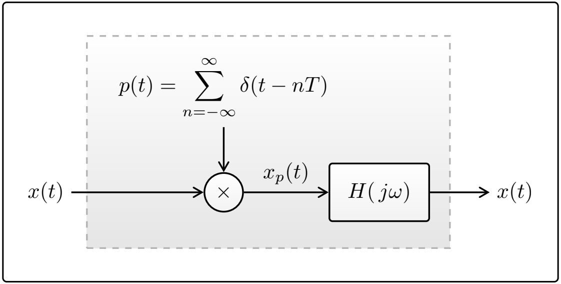
Sampling – Zero-Order Hold



Sampling – Sampling is Time-Varying

Multiplying a signal $x(t)$ by a time-varying (non-constant) function $p(t)$ implies sampling is a time-varying operation. However, sometimes combinations of such sampling and filtering reduces to something simple. For

$$\boxed{x(t) \text{ s.t. } X(j\omega) = 0 \text{ for } |\omega| > 2\pi/T}$$



with $H(j\omega)$ an ideal low pass filter. Overall this acts like the identity.

Sampling – Time Domain Reconstruction

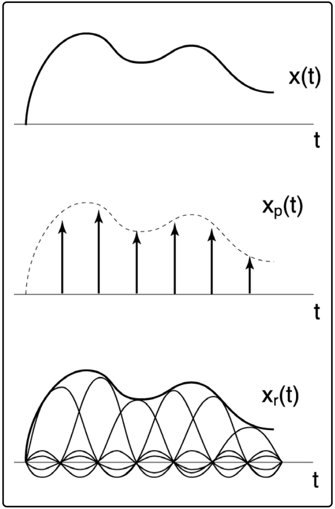
The Sampling Theorem is obvious in the frequency domain but it is useful to see what the equivalent time domain operation is (for implementation). Recall ω_c is the cut-off of the filter which we take to be an ideal LPF.

$$\begin{aligned} x_r(t) &= x_p(t) \star h(t), \qquad h(t) \triangleq \frac{T \sin \omega_c t}{\pi t} \\ &= \Big(\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \Big) \star h(t) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT) \end{aligned}$$

That is,

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \frac{T\, \sin\big(\omega_c (t - nT)\big)}{\pi (t - nT)}}$$

Sampling – Time Domain Reconstruction



Note that in going from $x(nT)$ to $x(t)$ we are filling in the values in the time gaps, which is just interpolation. So the Sampling Theorem reconstruction can be viewed as a special/optimal interpolation. We can then enumerate a few different types of interpolation

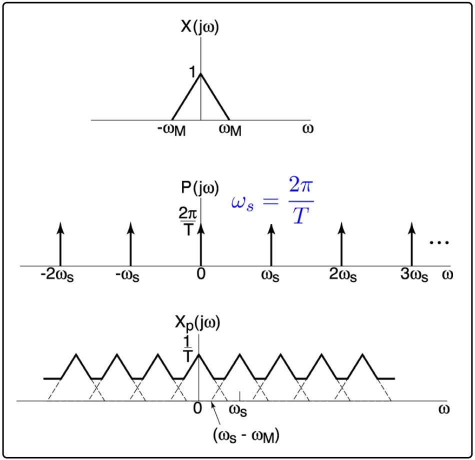
- Band-limited Interpolation — primarily the sinc style interpolation
- Zero-Order Hold — reconstructs a piecewise constant signal through the sample points
- First-Order Hold — reconstructs a piecewise linear interpolation joining the sample points (non-causal)

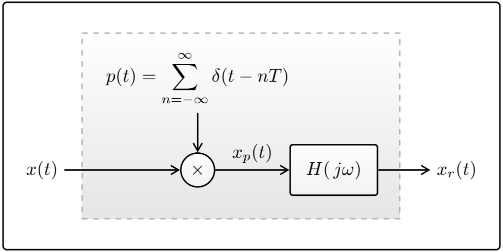
Sampling – Undersampling and Aliasing

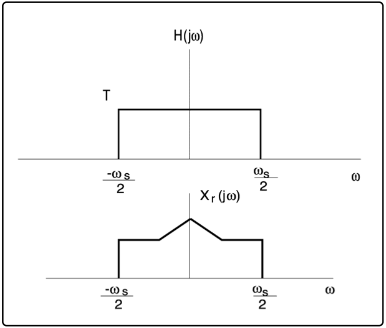
For the band-limited signal case, when the sampling rate is insufficient, that is,

$$\omega_s \leq 2\omega_M$$

there will be frequency overlapping.







Higher frequencies of $x(t)$ are aliased to lower frequencies.

Example

With

$$x(t) = \cos(\omega_0 t), \quad \text{where } \omega_M = \omega_0.$$

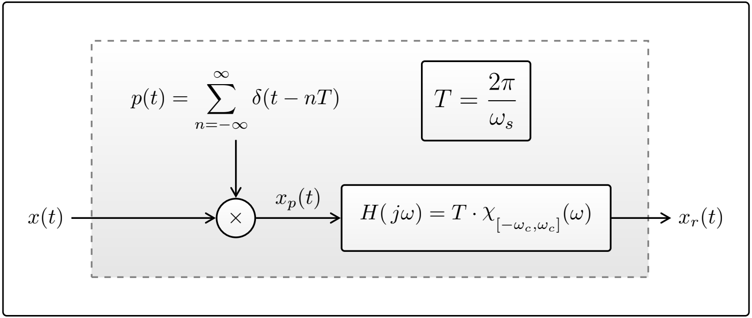
If $\omega_s > 2\omega_0$ then there is perfect reconstruction

$$x_r(t) = \cos(\omega_0 t)$$

If $\omega_s < 2\omega_0$ then there is aliasing

$$x_r(t) = \cos\big((\omega_s - \omega_0)t\big)$$

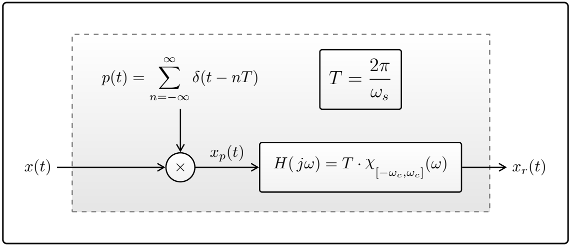
CT Processing via DT – Sampling Theorem and Reconstruction



If $X(j\omega) = 0$ for $|\omega| > \omega_M$ and $\omega_s = 2\pi/T > 2\,\omega_M$ (the **Nyquist rate**) then, assuming filter cut-off ω_c satisfies $\omega_M < \omega_c < \omega_s - \omega_M$, we have perfect recovery

$$x_r(t) = x(t)$$

CT Processing via DT – Sampling Theorem and Reconstruction



In words:

If signal $x(t)$ is low pass band-limited to maximum frequency ω_M and sampled at frequency $\omega_s = 2\pi/T$ which is at least twice ω_M, then $x(t)$ can be perfectly recovered from "the samples" (the signal $x_p(t)$ which depends only on $x(nT)$) with a scaled ideal low pass filter with cutoff frequency anywhere between the maximum frequency ω_M and $\omega_s - \omega_M$.

In the above, it doesn't need to be an ideal low pass filter but just have gain 1 in $|\omega| \leq \omega_M$ and gain 0 in $|\omega| > \omega_s - \omega_M$. In between, the gain doesn't matter since there is no signal energy beyond ω_M (here note that $\omega_s - \omega_M \geq \omega_M$).

CT Processing via DT – Digital Processing

The Importance:

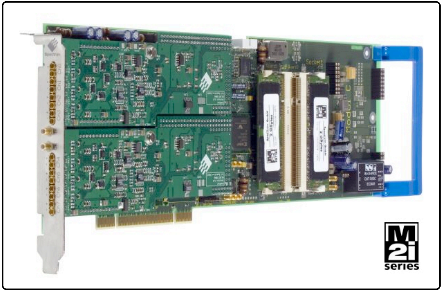
We can convert band-limited CT signals to DT signals **without any loss of information**. We want to measure a signal and process it. This can be done in the DT domain making it amenable to digital processing, that is, processing in a computer rather than building a circuit or other approach.

One can think of time being sampled or being made discrete. One also samples or **discretizes the values** (like the voltage levels) and this is really what we mean by "digital" — discrete in time **and** discrete in amplitude.

CT Processing via DT – Digital Processing

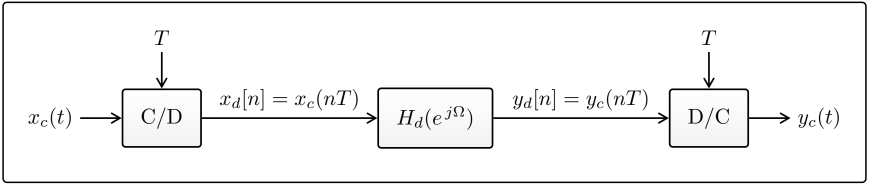
Digital Processing Means:

- MATLAB
- "PC" with PCI (Multi-Channel) 8/12/16-bit A/D
- DSP such as Texas Instruments TMS320 series
- FPGA – Field Programmable Gate Array
- ASIC – Application-specific integrated circuit



CT Processing via DT – C/D D/C System

Suppose we want to take a CT signal $x_c(t)$ (measurement) convert it to a DT form, $x_d[n]$, process it to generate a new DT signal, $y_d[n]$, and finally convert that back to a CT signal $y_c(t)$:



This circuitous type of processing has advantages:

- can be much cheaper
- extremely versatile
- can be made “adaptive”
- allows fancy more complex (higher order) processing
- can have better noise performance

CT Processing via DT – C/D D/C System

How do we analyze this hybrid system?

- The frequency domain can be thought of in both CT and DT
  - ω – CT frequency variable
  - Ω – DT frequency variable where

$$\boxed{\Omega = \omega T}$$

- Previous figure had C/D and D/C (here C–Continuous and D–Discrete). Generally you see A/D and D/A, or ADC and DAC, etc., where A–Analog and D–Digital. C/D is appropriate given time is discretized but we are *not* discretizing the values/amplitude. (We don't want to get too pedantic nor too hung-up on terminology which does vary.)

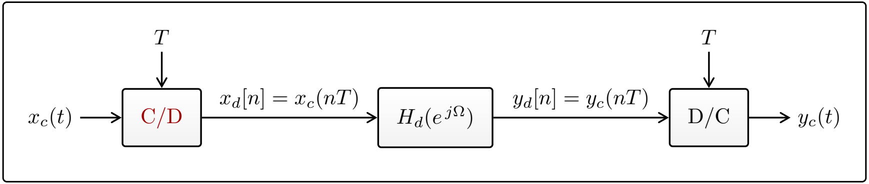
CT Processing via DT – C/D D/C System

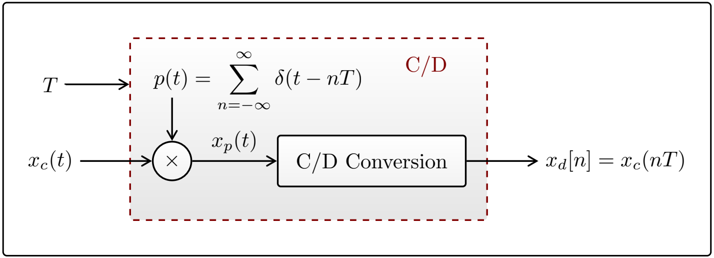
How do we analyze this hybrid system (con't)?

- In the CT representation of a "sampled signal" we have equally spaced impulses of amplitudes equal to the values of the signals at the sample time instants.
- In the DT representation of a "sampled signal" we have integer spaced values equal to the values of the signals at the sample time instants.

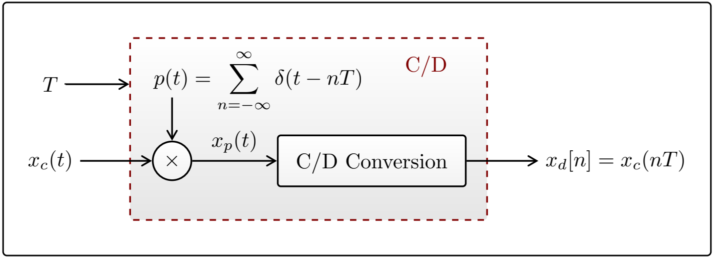
CT Processing via DT – C/D D/C System

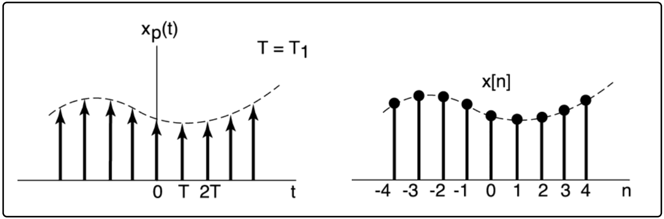
C/D Conversion:





CT Processing via DT – C/D D/C System





CT Processing via DT – C/D D/C System

Frequency Domain Interpretation of C/D Conversion:

Consider the sampled version of CT signal $x_c(t) \overset{\mathscr{F}}{\longleftrightarrow} X_c(j\omega)$:

$$\boxed{x_p(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)}$$

1) CT, in frequency variable ω, periodic with period $\omega_s = 2\pi/T$:

$$X_p(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\big(j(\omega - k\,\omega_s)\big)$$

but straight from $x_p(t)$ above, $\delta(t - nT) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega nT}$, also we have

$$\boxed{X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega nT}}$$

CT Processing via DT – C/D D/C System

2) DT, in frequency variable Ω, periodic with period 2π ($\Omega = \omega T$):

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\Omega n}$$

which can also be written

$$\boxed{X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega n}}$$

3) Comparing the two expressions (CT and DT), we infer

$$X_d(e^{j\Omega}) = X_p\big(j(\underbrace{\Omega/T}_{\omega})\big)$$

CT Processing via DT – C/D D/C System

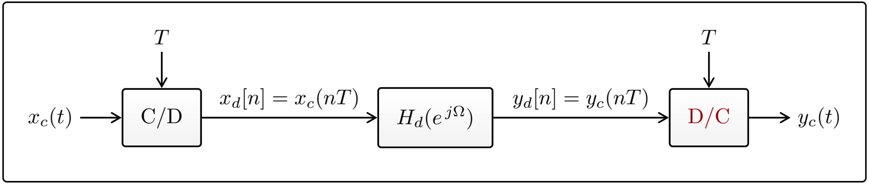
More carefully,

$$\boxed{X_d(e^{j\Omega}) = X_p\big(j(\Omega/T)\big), \quad -\pi < \Omega < \pi}$$

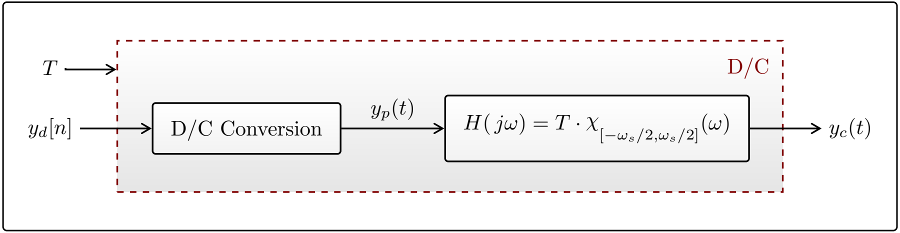
- This is the key equation defining the DT filtering specification that we need to implement. It is based on the CT filtering specification over a limited range of operating frequencies and a given sampling range T.
- As with most things this is easier to understand when we look at an example in a moment.

CT Processing via DT – C/D D/C System

D/C Conversion:



In this case we want to convert the DT signal $y_d[n]$ to a CT signal $y_c(t)$ via intermediate impulse train signal $y_p(t)$ which is ideal low pass filtered:



CT Processing via DT – C/D D/C System

Summary:

- In the DT world, signals are sequences of values, and in the continuous world signals are functions of continuous time.
- The DT sequences can be mapped to an impulse train CT Signal and vice versa. This is the most important point.
- There is a bunch of "book-keeping", mostly wrt to scaling, to account for the conversions between frequency domain descriptions and time domain sampling.
- By doing the processing in the DT world there is untold powerful filters, etc., that can be implemented. (Higher order filters, differential equations, would be very hard to implement any other way.)

Summary (con't):

- Non-obvious/non-trivial point is if the sampling theorem is satisfied then we can implement an overall CT LTI system.
- If the sampling theorem is not satisfied the overall system is time-varying. This means the input signal has some high frequency content which gets aliased to lower frequencies.
- If you look at the C/D step, the first operation is sampling, then to avoid the "damage" due to the high frequency content in the input the only option is to add a CT low pass filter before the sampling to filter out any high frequencies which would get aliased. This is called an **anti-aliasing filter**.

CT Processing via DT – Digital Differentiator

Example:

Digital Differentiator: The goal in this example is to build a differentiator using the C/D-D/C method. This illustrates there is a simple methodology to do the design which takes into account the "book-keeping" factors.

This goal doesn't make sense unless the frequency content is constrained, so actually what we need to do is specify a CT **band-limited** differentiator:

$$\boxed{H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}}$$

- Here ω_c is the cut-off frequency. Our differentiator is only going to work if the signal has frequencies in the band $|\omega| < \omega_c$.
- A differentator has a simple looking frequency response. If we want any other filter, operating in the range $|\omega| < \omega_c$, when just swap it in. If you understand this example then you understand how to design for any filter, operating in the range $|\omega| < \omega_c$.

CT Processing via DT – Digital Differentiator

Set the sampling fast enough to be twice the maximum frequency our CT band-limited differentiator can handle:

$$\boxed{\omega_s = 2\,\omega_c}$$

and this implies the maximum (slowest) sampling interval needs to be

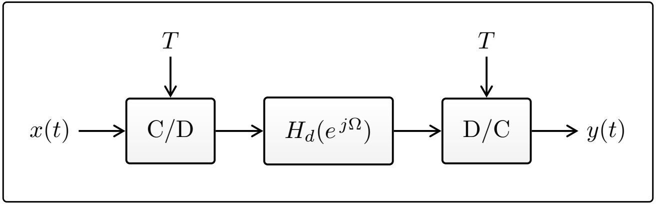
$$\boxed{T = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_c}}$$

Of course, any signal applied should have maximum frequency content

$$\boxed{\omega_M < \omega_c}$$

by the Nyquist condition.

CT Processing via DT – Digital Differentiator



We need to determine $H_d(e^{j\Omega})$ from our design specification $H_c(j\omega)$:

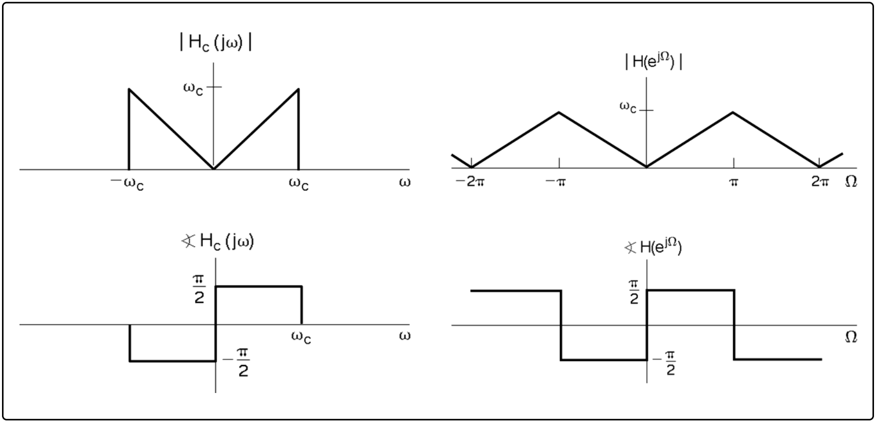
$$H_d(e^{j\Omega}) = \begin{cases} H_c(j\Omega/T), & |\Omega| < \pi \\ \text{periodic}, & |\Omega| > \pi \end{cases}$$

So implement a DT filter with the following DT frequency response

$$\boxed{H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right) = j\,\omega_c\left(\frac{\Omega}{\pi}\right), \quad |\Omega| < \pi}$$

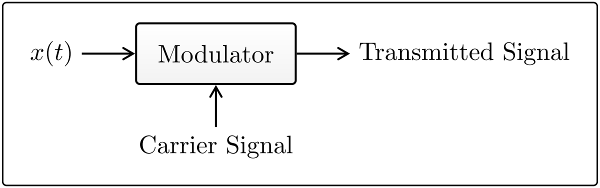
and it is periodic with period 2π for Ω outside this range.

CT Processing via DT – Digital Differentiator



Communications Applications – Modulation

The concept of modulation (met briefly earlier with “AM”)



Why modulate?

- Wireless transmission can be done at radio frequencies (RF) and done very efficiently.
- Transmit multiple signals through the same medium using different “carriers”.
- Transmit through “channels” with different characteristics primarily different pass-bands.

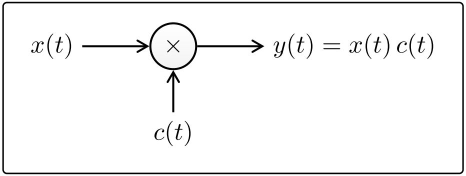
Communications Applications – Modulation

Why modulate (con't)?

- Modulation to lift and separate. "Lift" to higher frequencies which are more suitable for transmission. "Separate" because more than one user will want to do the same so we design it such that there is no interference.

How to modulate? (Many methods)

- Best to start with (simplified) Amplitude Modulation (AM).



where the multiplication is pointwise in time t.

- Actual AM, like 666 Canberra ABC, is slightly more complicated (we'll see this later).

Complex Exponential AM – Modulation

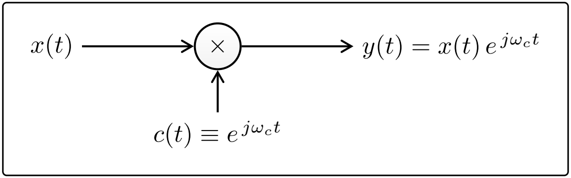
First we choose the carrier signal, $c(t)$. We make the idealistic initial choice

$$c(t) = e^{j\omega_c t}$$

where ω_c is the carrier frequency (think RF, such as $\omega_c = 2\pi \cdot 666,000$ kHz). Then the modulated signal is

$$\boxed{y(t) = x(t)\, e^{j\omega_c t}}$$

drawn as the multiplication (pointwise in time t)



Note that modulation need not be a multiplication (but is in this case).

Complex Exponential AM – Modulation

In the frequency domain this, of course, is a frequency shift

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi} X(j\omega) \star C(j\omega) \\
&= \frac{1}{2\pi} X(j\omega) \star 2\pi\, \delta(\omega - \omega_c) \\
&= X\big(j(\omega - \omega_c)\big)
\end{aligned}
$$

where $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega)$ and $c(t) \overset{\mathscr{F}}{\longleftrightarrow} C(j\omega)$.

Complex Exponential AM – Modulation

In practical application we are thinking to choose ω_c such that

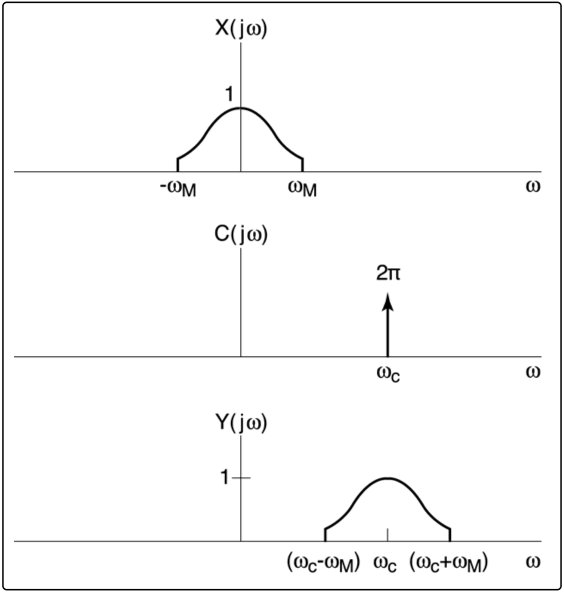
$$\boxed{\omega_c \gg \omega_M}$$

where

$$\boxed{\left|X(j\omega)\right| \leq \omega_M}$$

- ω_c is the carrier frequency usually in the MHz to GHz range (Hz $\times 2\pi$ for this in rad/sec)
- ω_M is the message/signal/information bandwidth could be 10kHz for audio or 7MHz for Digital Video Broadcasting Terrestrial (DVB-T)
- ω_c might be 666 kHz for Audio and 540 MHz for DVB-T

Complex Exponential AM – Modulation



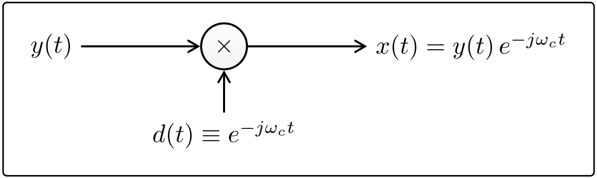
Complex Exponential AM – Demodulation

There are different views as to how we can get back the original signal:

- Since we multiplied by $c(t)$ then we can multiply by $d(t) \triangleq 1/c(t)$ to demodulate the transmitted signal.
- Looking at the frequency domain, the transmitted signal needs to be frequency shifted back to "base-band" and this frequency shift is readily attained by

$$\boxed{y(t)\, e^{-j\omega_c t} = x(t)\, e^{j\omega_c t}\, e^{-j\omega_c t} = x(t)}$$

- This is the demodulator:



where $d(t)$ is the multiplicative demodulating local oscillator.

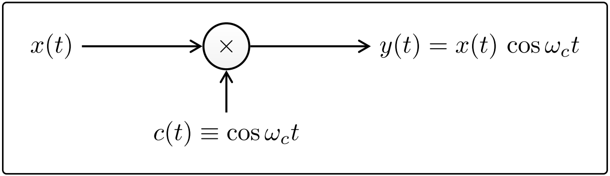
Complex Exponential AM – Demodulation

One problem with Complex Exponential AM is being complex rather than real. So mathematically we are fine but what does it mean in practice?

$$\boxed{e^{j\omega_c t} = \cos\omega_c t + j\,\sin\omega_c t}$$

which can be viewed as a superposition of two signals. We say that there are two separate modulation channels, also called quadratures, with carriers 90° out of phase. There are two carriers which are orthogonal (non-interfering); could carry two messages in the same bandwidth. Meet in other courses.

Sinusoidal AM – Modulation

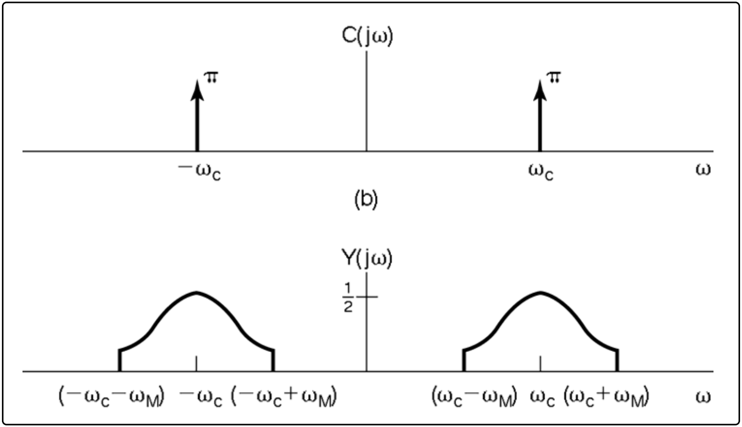


$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi} X(j\omega) \star \pi\Big(\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\Big) \\
&= \frac{1}{2} X\big(j(\omega-\omega_c)\big) + \frac{1}{2} X\big(j(\omega+\omega_c)\big)
\end{aligned}
$$

Recall from $e^{j\omega_c t} = \cos\omega_c t + j\ \sin\omega_c t$

$$\boxed{\cos\omega_c t = \frac{1}{2} e^{-j\omega_c t} + \frac{1}{2} e^{j\omega_c t}}$$

$$\boxed{Y(j\omega) = \frac{1}{2} X\big(j(\omega-\omega_c)\big) + \frac{1}{2} X\big(j(\omega+\omega_c)\big)}$$

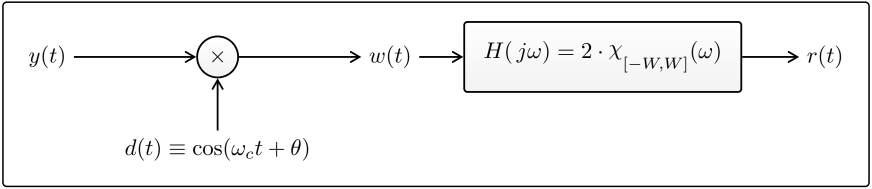


Here $\omega_c \gg \omega_M$ and, in the diagram,

$$\boxed{c(t) \equiv \cos\omega_c t \stackrel{\mathscr{F}}{\longleftrightarrow} C(\,j\omega) \equiv \pi\big(\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\big)}$$

Sinusoidal AM – Demodulation

(Potentially imperfect) demodulation in the "receiver":



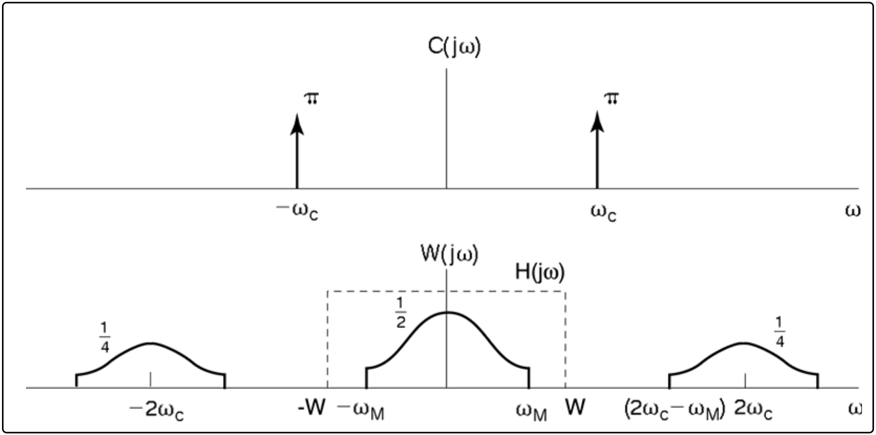
where

$$\boxed{d(t) \equiv \cos(\omega_c t + \theta)}$$

is the local oscillator (in the transmitter) with phase error θ relative to the phase of the carrier (in the receiver).

Sinusoidal AM – Demodulation

If phase error $\theta = 0$, then local oscillator is in phase with the carrier:



Sinusoidal AM – Demodulation

That is, with phase error $\theta = 0$,

$$\begin{aligned} w(t) &= y(t)\cos\omega_c t \\ &= x(t)\cos^2\omega_c t \\ &= \frac{1}{2}\,x(t) + \frac{1}{2}\,x(t)\cos 2\omega_c t \end{aligned}$$

and

$$r(t) = \frac{1}{2}\,x(t)$$

after lowpass filtering with filter cut-off W satisfying

$$\boxed{\omega_M < W < 2\omega_c - \omega_M}$$

(where $\omega_c \gg \omega_M$)

- Of course here we should amplify $r(t)$ by 2 to get $x(t)$.
- Exactly half the energy goes to "baseband" and the other half appear in two parts (two quarters): centred at $-2\omega_c$ and $2\omega_c$ which are filtered out.

Sinusoidal AM – Demodulation

If there is a phase difference, $\theta \neq 0$, then let's see what happens:

$$\begin{aligned} w(t) &= y(t)\cos(\omega_c t + \theta) \\ &= x(t)\cos(\omega_c t)\cos(\omega_c t + \theta) \\ &= \frac{1}{2}x(t)\cos\theta + \frac{1}{2}x(t)\big(\cos(2\omega_c t + \theta)\big) \end{aligned}$$

and

$$\boxed{r(t) = \frac{1}{2}\cos\theta\, x(t)}$$

after lowpass filtering with $W > \omega_M$.

- Therefore, a phase error hurts the recovery of the signal by attenuating by (an additional factor) $\cos\theta$
- can also negate depending on θ (since $-1 \leq \cos\theta \leq +1$).
- The problem cases are $\theta = \pm\pi/2$ or θ near those values.

Sinusoidal AM – Demodulation

Local oscillator analysis

$$\cos(\omega_c t + \theta) = \frac{1}{2} e^{-j\theta} e^{-j\omega_c t} + \frac{1}{2} e^{j\theta} e^{j\omega_c t}$$

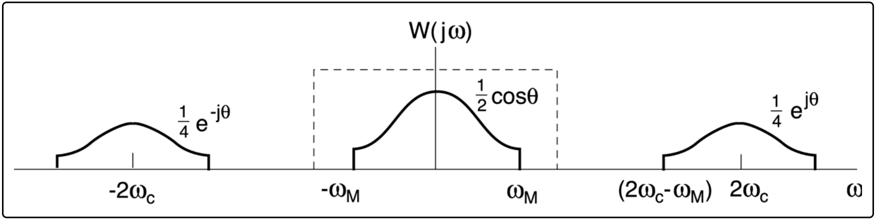
has Fourier transform

$$\pi e^{-j\theta} \delta(\omega - \omega_c) + \pi e^{j\theta} \delta(\omega + \omega_c)$$

$$\boxed{\cos(\omega_c t + \theta) \xleftrightarrow{\mathscr{F}} \pi e^{-j\theta} \delta(\omega - \omega_c) + \pi e^{j\theta} \delta(\omega + \omega_c)}$$

Sinusoidal AM – Demodulation

With phase error θ there is attenuation of the desired baseband signal:



- When $\cos\theta$ is small then we lose the desired signal. θ is the phase (offset).
- Having $\theta = 0$ (which is what we want) means the local oscillator in the receiver has to be perfectly phase synchronized with the transmitter. That is no so easy to do.
- Equally bad or worse if the local oscillator is even a tiny bit out that is, the receiver uses $\omega_c + \Delta\omega$ then that is like a time varying $\theta \equiv \Delta\omega\, t$

$$d(t) \equiv \cos\big((\omega_c + \Delta\omega)t\big) = \cos(\omega_c t + \Delta\omega\, t)$$

so the signal gets modulated (fading in and out at frequency $\Delta\omega$).

Sinusoidal AM – Demodulation

- If you do want to "track" things in the receiver so that you have $\theta = 0$ then that can be done with a thing called a Phase-Locked Loop (PLL). But there are still problems. . .
- If the signal $x(t) = 0$ goes quiet (the DJ shuts up) then with the above type of AM the transmitter transmits nothing and any tracking of the phase cannot work until the DJ starts speaking again. So a smart thing to do is constantly transmit a signal at the carrier frequency ω_c which is called "the carrier" or a "pilot".
- In the digital world you can building quite complex tracking functionality and so you could get away with using a low power pilot and use a multiplicative type demodulator (what we have been going through).
- There is a more practical solution to all this which we look at next. We in effect add a very powerful pilot and abandon multiplicative type demodulators in favour of something that can be built for a few cents (and doesn't even need power; in fact people metal tooth fillings have been able to pick up AM quite well – not to their wishes though).

Practical AM – Background

The name amplitude modulation is not captured by a process such as $y(t) = x(t)\, e^{j\omega_c t}$. Naively we might think about modulating the "amplitude" $\big|x(t)\big|$ but that would be a disaster.

Further, engineers when trying to figure out how to transmit audio didn't rush to a table of Fourier transforms. They probably couldn't spell Fourier and definitely couldn't pronounce it (no one can, not even Fourier).

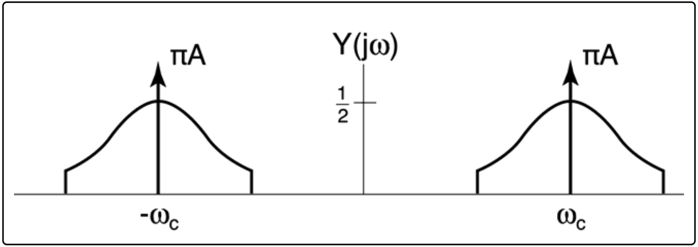
The important message here is; practical AM is designed to make demodulation easy. Think about it. The broadcast transmitter can be expensive and fancy but you want the receiver to be cheap and simple; at least 80 or more years ago that was the case.

Practical AM – Modulation

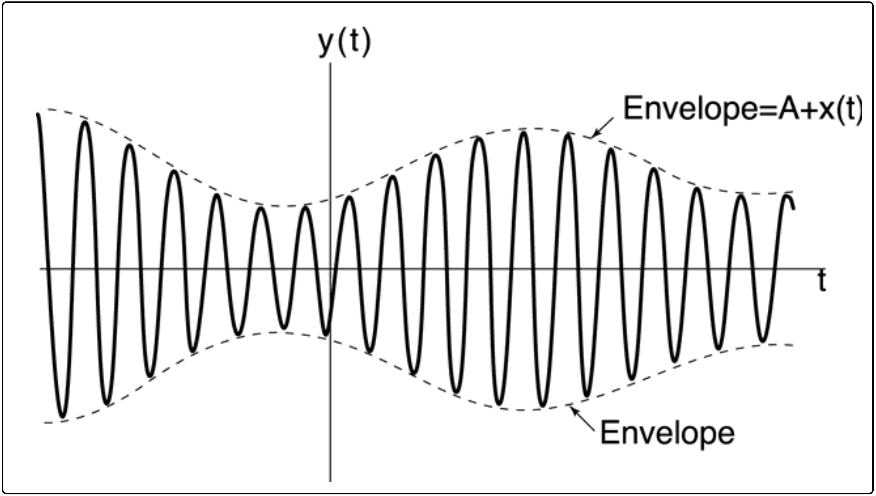
AM is Sinusoidal AM to which we add pure carrier:

$$\begin{aligned} y(t) &= \big(A + x(t)\big)\cos(\omega_c t) \\ &= A\cos(\omega_c t) + x(t)\cos(\omega_c t) \end{aligned}$$

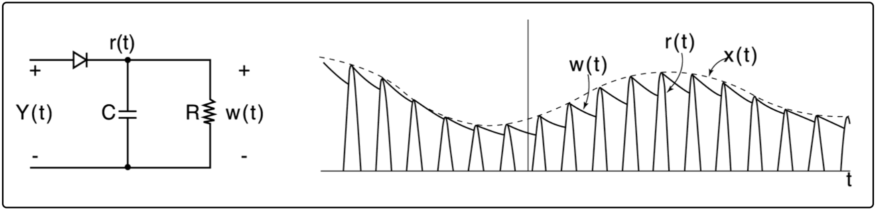
where $A > \max_t x(t)$.



Practical AM – Demodulation



Practical AM – Demodulation



(or a tooth filling)

Recovery is possible from the "envelope" if $A > \max_t x(t)$. Recovery is very simple and cheap. Weakness, a lot of power goes into the carrier; in fact it has to be more than the maximum of the time-varying power that goes into the information signs $x(t)$.