Australian National University

Research School of Engineering
College of Engineering and Computer Science

ENGN2228 Signal Processing

HOMEWORK 3 – SOLUTIONS

Homework 3-1

Consider the following convolution

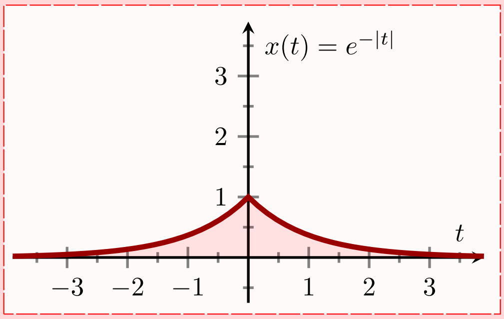
$$y(t) = x(t) \star h(t)$$

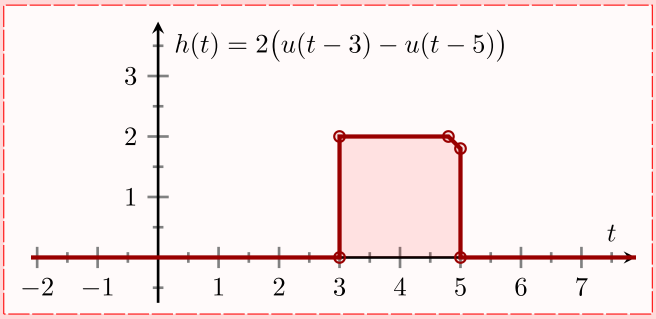
where

$$x(t) = e^{-|t|} \quad \text{and} \quad h(t) = 2\,u(t-3) - 2\,u(t-5).$$

(a) Draw $x(t)$ and $h(t)$.

Solution:



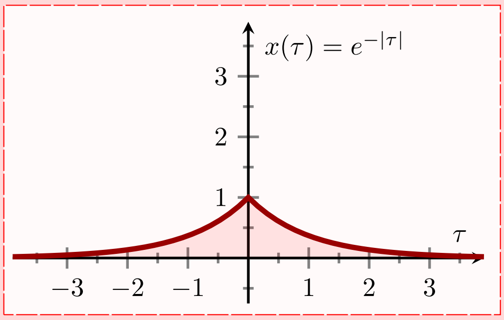


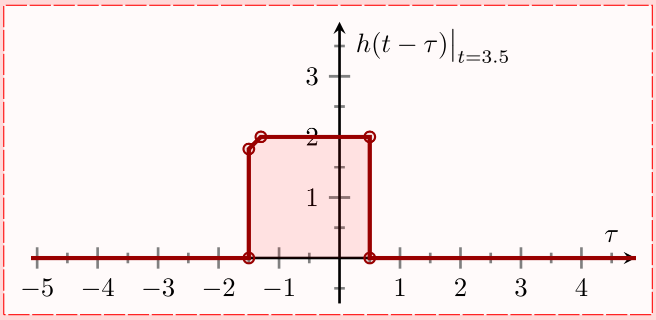
Note that $h(t)$ is drawn with a bite taken out of it, which helps interpret the convolution later. □

(b) Evaluate the integral

$$y(t) = x(t) \star h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Solution: Notice $h(t)$ has been flipped and shifted, and $h(t-\tau)$ is drawn for value $t = 3.5$. For example at $\tau = -1.5$, then $h(3.5 - (-1.5)) = h(5)$ which is the "bitten edge".





So as $h(t-\tau)$ slides around (as we vary t) then the overlap with $x(\tau)$ will consist of 3 cases:

(i) $t < 3$: here $h(t-\tau)$ overlaps with the left part of $x(\tau)$ where τ is negative

$$y(t) = 2\int_{t-5}^{t-3} e^{\tau}\,d\tau = 2\,e^{\tau}\Big|_{t-5}^{t-3} = 2(e^{t-3} - e^{t-5})$$

(ii) $3 \leq t \leq 5$: here $h(t-\tau)$ overlaps with the right part of $x(\tau)$ where τ is positive

$$y(t) = \int_{t-5}^{0} e^{-\tau}\, d\tau + \int_{0}^{t-3} e^{-\tau}\, d\tau = \cdots = 4 - 2\,e^{t-5} - 2\,e^{3-t}$$

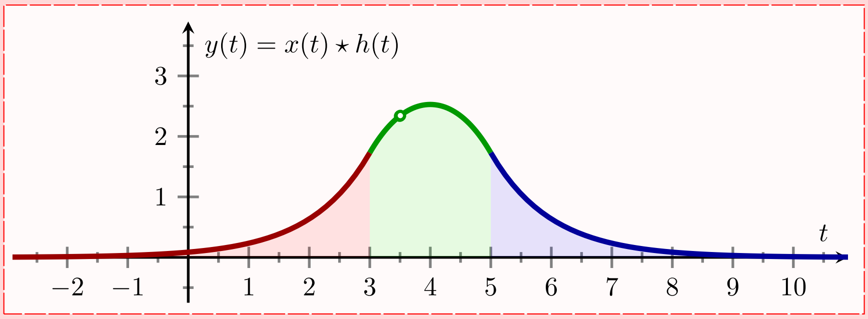
(iii) $t > 5$: here $h(t-\tau)$ overlaps across the peak of $x(\tau)$

$$y(t) = 2\int_{t-5}^{t-3} e^{-\tau}\, d\tau = -2\,e^{-\tau}\Big|_{t-5}^{t-3} = 2(e^{3-t} - e^{5-t})$$

The output $y(t)$ consists of these three parts and is plotted next. □

(c) Draw $y(t)$.

Solution: Combining the three segments we get the final answer



Hopefully you are convinced that computing the convolution from the integral expression is a pain and error prone. There are other ways of computing this convolution. Also on the figure the point for $t = 3.5$ is marked and this corresponds to the integral of the product of the two functions shown in part (b). □

Homework 3-2

In 3-1, we note that the centre of $h(t)$ is at $t = 4$. Review and understand the following:

$$\begin{aligned} h(t) &= 2\big(u(t-3) - u(t-5)\big) \\ &= 2\big(u(t+1) - u(t-1)\big) \star \delta(t-4) \end{aligned}$$

Therefore

$$\begin{aligned} y(t) &= x(t) \star h(t) \\ &= \Big(x(t) \star 2\big(u(t+1) - u(t-1)\big)\Big) \star \delta(t-4). \end{aligned}$$

So we can adjust for the delay of 4 at the end of a convolution that uses the even function

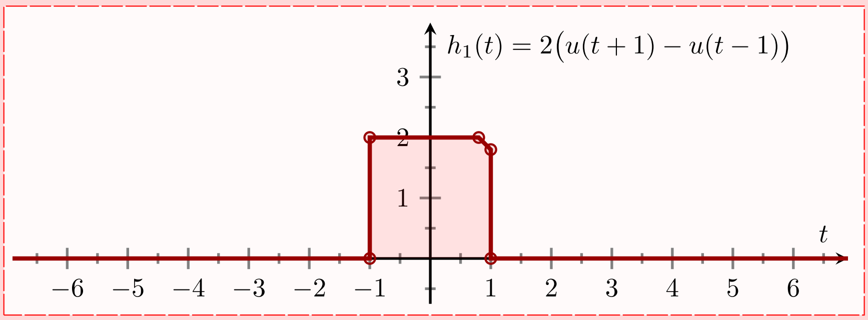
$$h_1(t) = 2\big(u(t+1) - u(t-1)\big)$$

instead of the original $h(t)$.

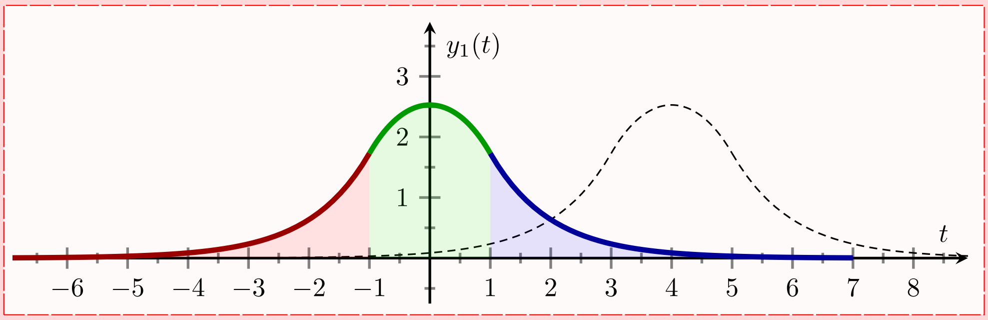
(a) Draw

$$y_1(t) = x(t) \star h_1(t) = x(t) \star 2\big(u(t+1) - u(t-1)\big).$$

Solution: First we show $h_1(t)$:



and then $y_1(t)$ is given by



Here $y(t)$ is shown dashed. $\square$

(b) Confirm $y_1(t)$'s relationship with $y(t)$.

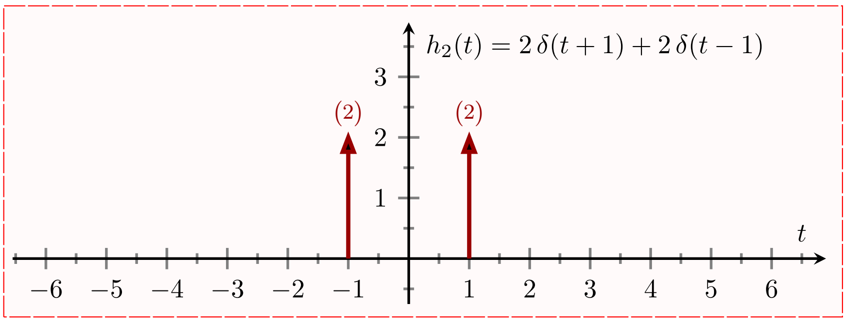
Solution: Indeed $y(t) = y_1(t-4)$. $\square$

Homework 3-3

Computing a convolution with $h_1(t)$, still requires an integral to be determined. Let's try a new "system" called $h_2(t)$ consisting of two Dirac delta functions:

$$h_2(t) = 2\,\delta(t+1) + 2\,\delta(t-1),$$

which is shown below:



The value of the delta function is presented in round brackets and that number represents the area, here (2) means the delta function has area 2 (so $h_2(t)$ has total area 4).

(a) Compute the following convolution

$$y_2(t) = x(t) \star h_2(t),$$

where $x(t) = e^{-|t|}$ as before.

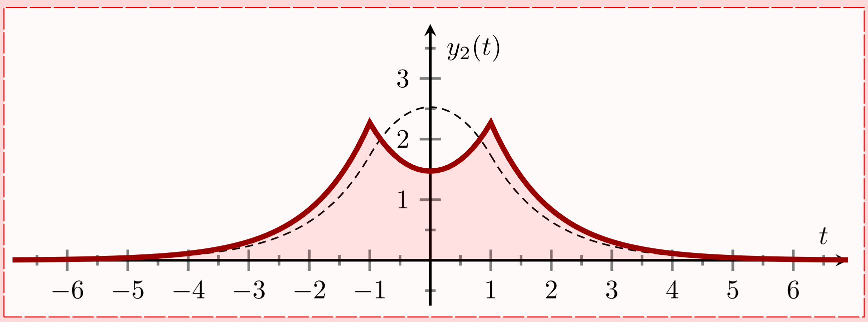
Solution:

$$\begin{aligned} y_2(t) &= e^{-|t|} \star \big(2\,\delta(t+1) + 2\,\delta(t-1)\big) \\ &= 2\,e^{-|t+1|} + 2\,e^{-|t-1|}. \end{aligned}$$

Nuf sed. $\square$

(b) Plot $y_2(t) = x(t) \star h_2(t)$ and compare with $y(t) = x(t) \star h(t)$.

Solution:

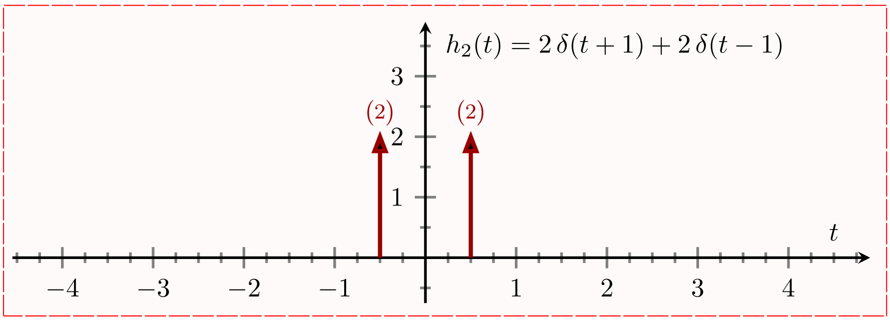


Here $y_1(t)$ is shown dashed. $\square$

(c) It is clear that $h_2(t)$ is a rough approximation to $h(t)$ and $y_2(t)$ is a (less) rough approximation to $y(t)$. Show and argue why

$$h_3(t) = 2\,\delta(t + 1/2) + 2\,\delta(t - 1/2),$$

which is shown below, is a better approximation to $h(t)$ than $h_2(t)$.

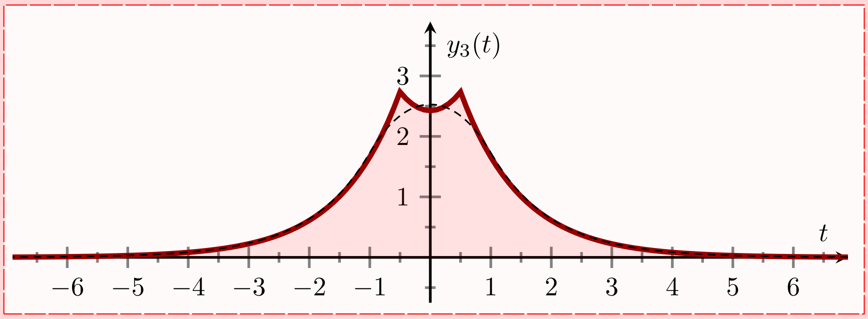


Compute and plot

$$y_3(t) = x(t) \star h_3(t),$$

where $x(t) = e^{-|t|}$ as before. Compare $y_3(t)$ with $y_2(t)$ and $y(t)$.

Solution: In a sense $h_3(t)$ is a better approximation to $h(t)$ than $h_2(t)$ with its mass more uniformly spread over the range where $h(t)$ is non-zero.

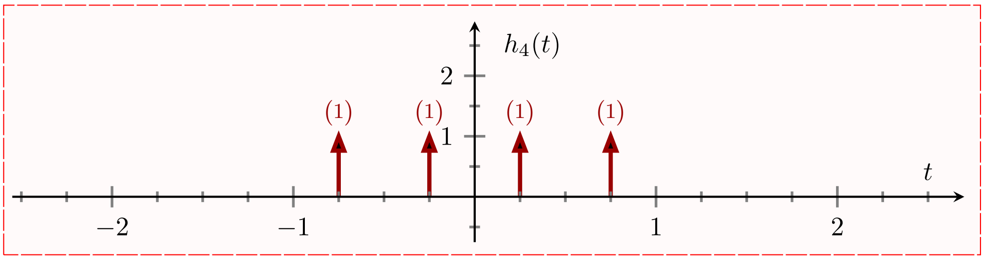


Here $y_1(t)$ is shown dashed. □

(d) Repeat (c) again for

$$h_4(t) = \delta(t + 3/4) + \delta(t + 1/4) + \delta(t - 1/4) + \delta(t - 3/4)$$

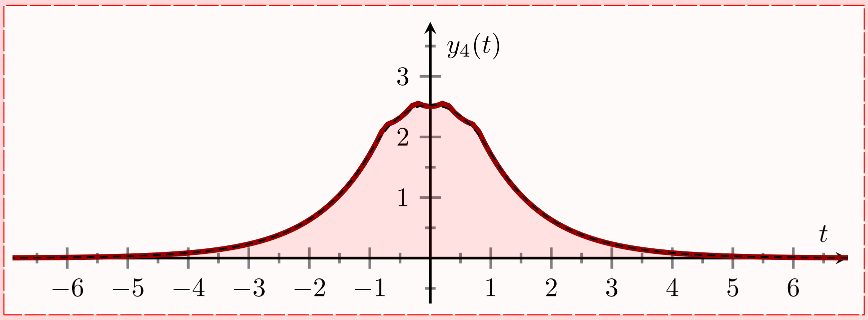
shown here



and

$$y_4(t) = x(t) \star h_4(t).$$

Solution: Here $y_4(t)$ is getting very close to $y(t)$.



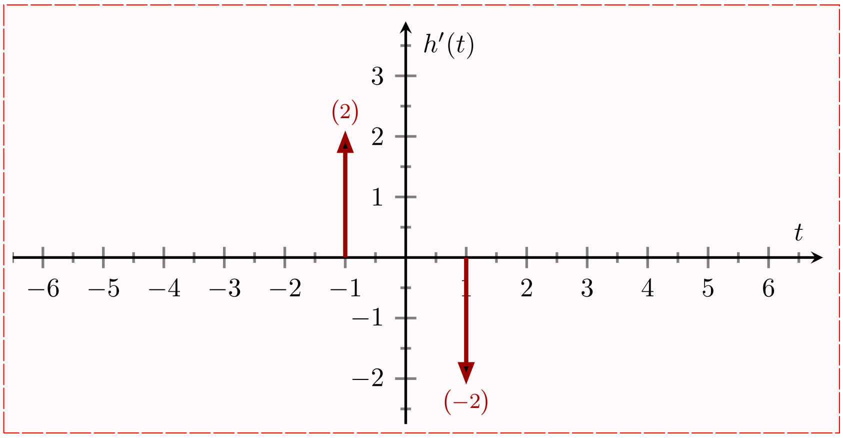
Here $y_1(t)$ is shown dashed (barely visible). □

Homework 3-4

(Hard) The derivative of $h(t)$ is given by

$$h'(t) = 2\,\delta(t+1) - 2\,\delta(t-1)$$

and is drawn below



In the lecture notes it was shown how to compute the convolution using the $u_n(t)$ functions (where $\delta(t) = u_0(t)$, $u(t) = u_{-1}(t)$, etc.).

(a) Confirm

$$x(t) \star h(t) = x(t) \star u_1(t) \star u_{-1}(t) \star h(t).$$

Solution: Convolving with $\delta(t)$ does nothing, and $u_1(t) \star u_{-1}(t) = \delta(t)$. □

(b) Use this expression to evaluate the convolution.

Solution: We have

$$\begin{aligned} x(t) \star h(t) &= x(t) \star u_1(t) \star u_{-1}(t) \star h(t) \\ &= x(t) \star u_1(t) \star h(t) \star u_{-1}(t) \\ &= x(t) \star h'(t) \star u_{-1}(t) \\ &= \int_{-\infty}^{t} x(\tau) \star h'(\tau)\, d\tau \end{aligned}$$

but

$$\begin{aligned} x(\tau) \star h'(\tau) &= x(\tau) \star \big(2\,\delta(t+1) - 2\,\delta(t-1)\big) \\ &= 2\,e^{-|t+1|} - 2\,e^{-|t-1|} \end{aligned}$$

and so

$$y(t) = 2\int_{-\infty}^{t} e^{-|\tau+1|}\, d\tau - 2\int_{-\infty}^{t} e^{-|\tau-1|}\, d\tau.$$

And give this to zombies to compute. There are better things in life to do. □

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