Probability Basics

Part 1

Mehrdad Zakershahrak
Guest Speaker
Arizona State University

Ira A. Fulton Schools of Engineering
Arizona State University

Why Study Probability?

- **Measure the uncertainty**
- **“Mathematics is the logic of certainty; probability is the logic of uncertainty”**
  - Introduction to probability by Joseph K Blitzstein and Jessica Hwang

Probability Applications

- **Biology**
  - Gene inheritance
- **Medicine**
  - Medical treatment success rate
- **Computer Science**
  - AI, Machine Learning
- **Physics**
  - Quantum Physics
- **Finance**
  - Risk in investment

Definitions

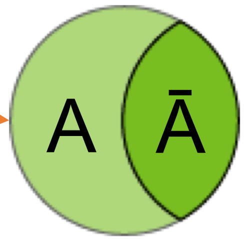
- A **sample space** S is the set of all possible outcomes of an experiment.
- An **event** $A \subseteq S$ is a subset of a sample space.
- Assuming that all outcomes are equally likely and the space is finite:

$$P(A) = \frac{\#\text{ favorable outcomes}}{\#\text{ possible outcomes}}$$

is the **probability** that A occurs.

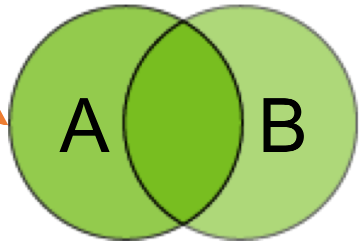
Definitions

$$P(\bar{A}) = 1 - P(A)$$



- If $A \subseteq B$, then $P(A) \leq P(B)$

$$P(A) \cup P(B) = P(A) + P(B) - P(A \cap B)$$



- Events A and B are independent iff

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap B) = P(A, B)$$

- Probability that at least one of the A_i occurs:

$$P(A_1 \cup ... \cup A_n)$$

- Probability that none of the A_i occurs:

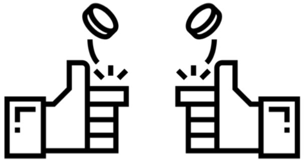
$$1 - P(A_1 \cup ... \cup A_n)$$

- Probability that all of the A_i occurs:

$$P(A_1 \cap ... \cap A_n) = 1 - P(\bar{A}_1 \cup ... \cup \bar{A}_n)$$

Example

- Suppose we toss two coins. What is the probability that we see *two heads*? [Each coin has two possible outcomes: H (heads) and T (tails).]

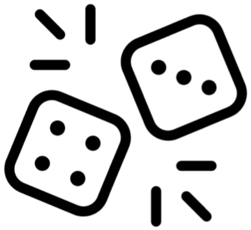


$\{\boxed{(h,h)}, (t,t), (h,t), (t,h)\}$ $\quad 1/4$

- What is the probability that we see at least one tail?

$3/4$

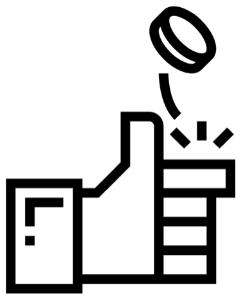
- We have two dice that we roll them at the same time. What is the probability that we do *not* see any 6s?



$1-\left(\frac{1}{6} * \frac{1}{6}\right)$

Discrete Random Variables

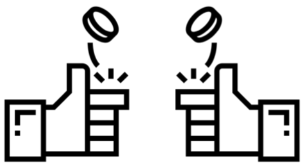
- Assuming the probability of head is 0.5, how many heads would you get if you toss a coin:
  - 4 times $\{0, \dots, 4\}$
  - 20 times $\{0, \dots, 20\}$
  - 100 times $\{0, \dots, 100\}$
- X is a **random variable (RV)** with **arity** of k if it can get exactly one value out of $\{x_1, \dots x_k\}$



Probability of Discrete RV

- We toss a fair coin 2 times…

$P(0 \text{ heads}) = \frac{1}{4}$
$p(2 \text{ heads}) = \frac{1}{4}$
$p(1 \text{ head}) = \frac{1}{2}$



- Probability mass function: $P(X = x_i)$

$$\sum_i P(X = x_i) = 1$$

- $P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$ if $i \neq j$
- $P(X = x_i \cap X = x_j) = 0$ if $i \neq j$

$$P(X = x_1 \cup \ldots \cup X = x_k) = 1$$

Probability Function

- We need a **mathematical function** that provides the **probabilities of occurrence** of different possible outcomes
- How different events will be **distributed** throughout the sample space

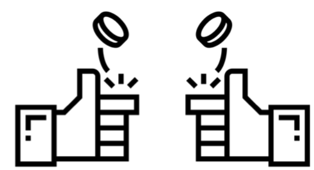
Common Distributions

- **Uniform** $X \sim U[1, \ldots, N]$
  - X takes values $1,2,\ldots,N$

    $P(X = i) = 1/N$ $\frac{1}{5}$

- **Binomial** $X \sim Bin(n, p)$
  - X takes values $0, 1,\ldots,N$

    $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$



- **Joint: Given two discrete RVs X and Y, their joint distribution is the distribution of X and Y together**

$$\sum_x \sum_y P(X = x \cap Y = y) = 1$$

Conditional Probability

- $P(X = x|Y = y)$ **is the probability of** $X = x$, **given the occurrence of** $Y = y$
  - You get A in Artificial Intelligence, given that you have studied the materials for the exam

$$P(X = x|Y = y) = \frac{P(X = x \cap Y = y)}{\boxed{P(Y = y)}} ?$$

Marginalization

- Probability of a subset of a collection of random variables
- Probability distribution of variables contained in the subset

- If X and Y are two discrete RVs:

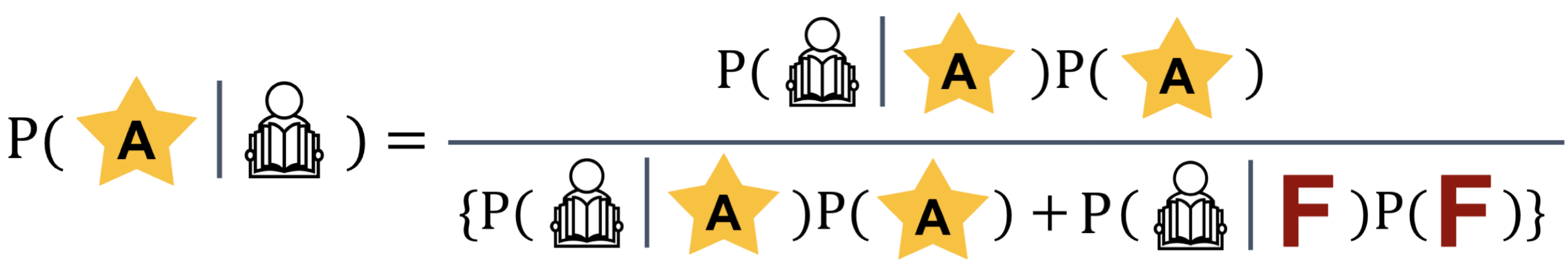
$$P(X = x_i) = \sum_i P(X = x_i \cap Y = y_j)$$

$$= \sum_j P(X = x_i | Y = y_j) P(Y = y_j)$$

Bayes' Rule

$$P(X = x | Y = y) = \frac{P(X = x \cap Y = y)}{P(Y = y)}$$

$$P(X = x_i | Y = y_j) = \frac{P(Y = y_j | X = x_j) P(X = x_i)}{\sum_k P(Y = y_j | X = x_k) P(X = x_k)}$$



Summary

- **Probability definition**
- **Discrete random variables and their probability**
- **Distributions**
  - Uniform
  - Binomial
  - Joint
- **Marginalization**
- **Bayes' Rule**