Natural Deduction for Propositional Logic

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Lewis Carroll Puzzle (I)

- Derive an inescapable conclusion using all of these:
 - a. All babies are illogical
 - b. Nobody is despised who can manage a crocodile
 - c. Illogical persons are despised
 - B: it is a baby
 - L: it is logical
 - M: it can manage a crocodile
 - D: it is despised

a)
$$B \rightarrow \neg L$$

b)
$$M \rightarrow \neg D$$

c)
$$\neg L \rightarrow D$$

Natural Deduction

- Collection of proof rules, which allow to infer new formulas from existing formulas
- Given the formulas $\Phi_1 \dots \Phi_n$, we intend to infer a conclusion Ψ :

$$\Phi_1 \dots \Phi_n \vdash \Psi$$

This construct is called sequent, for example:

$$p \wedge q, r \vdash q \wedge r$$

Priority of connectives: \neg , \land , \lor , \rightarrow

Conjunction

$$\dfrac{\Phi \quad \Psi}{\Phi \wedge \Psi} \quad \wedge i$$
 and-introduction

$$egin{array}{cccc} rac{\Phi \wedge \Psi}{\Phi} & \wedge e1 \ rac{\Phi \wedge \Psi}{\Psi} & \wedge e2 \end{array}
ight]$$
 and-elimination

Prove $p \wedge q, r \vdash q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e2$ 1
4	$q \wedge r$	$\wedge i \ 3, 2$

Double Negation and Implication Elimination

$$\frac{\neg \neg \Phi}{\Phi}$$

 $\neg \neg e$

double negation-elimination

$$\frac{\Phi}{\neg\neg\Phi}$$

 $\neg \neg i$

double negation-introduction

$$\frac{\Phi \quad \Phi \to \Psi}{\Psi}$$

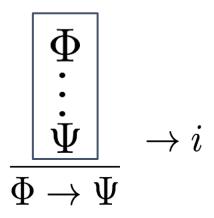
 $\rightarrow e$

implication-elimination

Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$

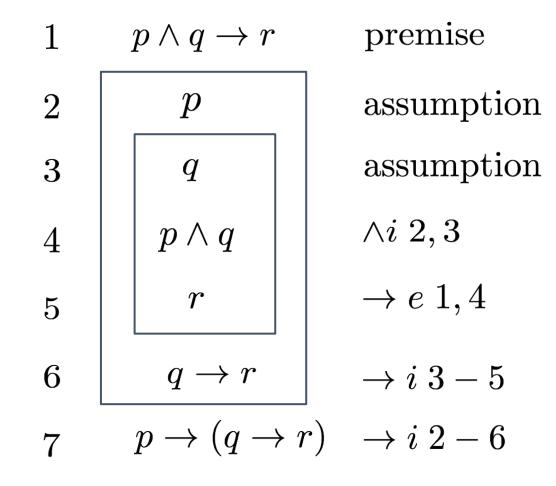
Prove
$$p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$$

Implication Introduction



In order to prove $\Phi \to \Psi$, we make the temporary assumption of Φ , and then prove Ψ . The scope of the assumption is indicated by the box

Prove $p \land q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$



Disjunction

$$\frac{\Phi}{\Phi \vee \Psi} \vee i1 \qquad \frac{\Psi}{\Phi \vee \Psi} \vee i2$$

$$egin{array}{c|c} \Phi \lor \Psi & egin{array}{c|c} \Phi & egin{array}{c} \Psi \ dots \ \dot{\chi} & \dot{\chi} \ \end{array} & \lor e \end{array}$$

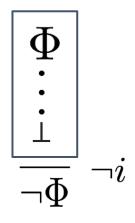
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Prove $q \rightarrow r \vdash p \lor q \rightarrow p \lor r$ $q \rightarrow r$ premise assumption $p \vee q$ 2 assumption 3 p $\forall i13$ $p \vee r$ 4 assumption 5 $\rightarrow e 1, 5$ $p \vee r$ $\vee i26$ $p \vee r$ 8 $\forall e \ 2, 3-4, 5-7$

 $p \lor q \rightarrow p \lor r$

 $\rightarrow i 2 - 8$

Negation



$$\frac{\perp}{\Phi} \perp e \quad \frac{\Phi \quad \neg \Phi}{\perp} \neg e$$

Prove $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

$$\begin{array}{cccc} 1 & p \rightarrow q & \text{premise} \\ 2 & p \rightarrow \neg q & \text{premise} \\ 3 & p & \text{assumption} \\ 3 & q & \rightarrow e \ 1, 3 \\ 5 & \neg q & \rightarrow e \ 2, 3 \\ 6 & \bot & \neg e \ 4, 5 \\ 7 & \neg p & \neg i \ 3 - 6 \end{array}$$

Prove $\neg p \lor q \vdash p \rightarrow q$ premise $\neg p \lor q$ premise $\neg p$ 2 assumption p3 $\neg e \ 3, 2$ q5 $\perp e4$ 6 $p \rightarrow q$ $\rightarrow i 3 - 5$ premise assumption pcopy 7 9 $p \rightarrow q$ $\rightarrow i 9 - 10$ $\forall e 1, 2-6, 7-10$ $p \rightarrow q$

Proof Strategy in Natural Deduction

If your goal is to prove:	Do:
$F \lor G$	Derive each of F or G; use or introduction
F o G	Assume F and derive G; use implication introduction
$F \wedge G$	Try to derive F and G as a subgoal; use and introduction

Derived Rules

$$\frac{\Phi \to \Psi \quad \neg \Psi}{\neg \Phi}$$

MT (Modus Tollens)

$$\frac{\Phi}{\neg\neg\Phi}$$

$$\neg\neg i$$



RAA ([Reductio ad Absurdum - a.k.a Proof by contradiction (PBC)]

$$\Phi \lor \neg \Phi$$

LEM (Law of excluded middle)

Summary

- To derive a conclusion, we need a set of proof rules
- Natural deduction rules we studied:
 - Conjunction
 - Implication
 - Disjunction
 - Negation
 - Derived rules
- No perfect set of rules. You can come up with your own rules!