Semantics of Propositional Logic

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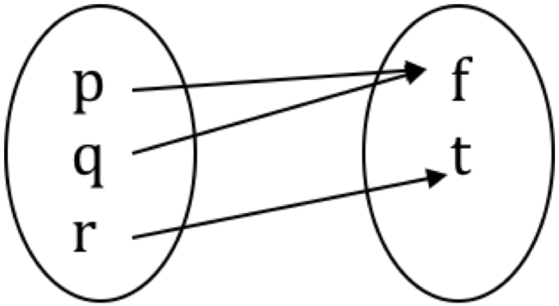
Interpretation

- An **interpretation** of a propositional signature σ is a function from σ into $\{f, t\}$
  - A propositional signature is a set of symbols called atoms, such as p, q, r
  - The symbols f and t are called truth values.
  - If σ is finite, an interpretation can be defined by a truth table



p	q	r
f	f	t

Basics

Basic valuations:

$\neg f$	t
$\neg t$	f
$f \vee t$, $t \vee f$, $f \vee t$	t
$f \vee f$	f
$t \wedge t$	t
$t \wedge f$, $f \wedge t$, $f \wedge f$	f
$t \longrightarrow f$	f
$t \longrightarrow t$, $f \longrightarrow t$, $f \longrightarrow f$	t

Valuation

- A **valuation** of a formula *F* is an assignment of each propositional atom in *F* to a truth value
  - F=(p→¬q)→(q∨¬p)
  - List all valuations of its subformulas

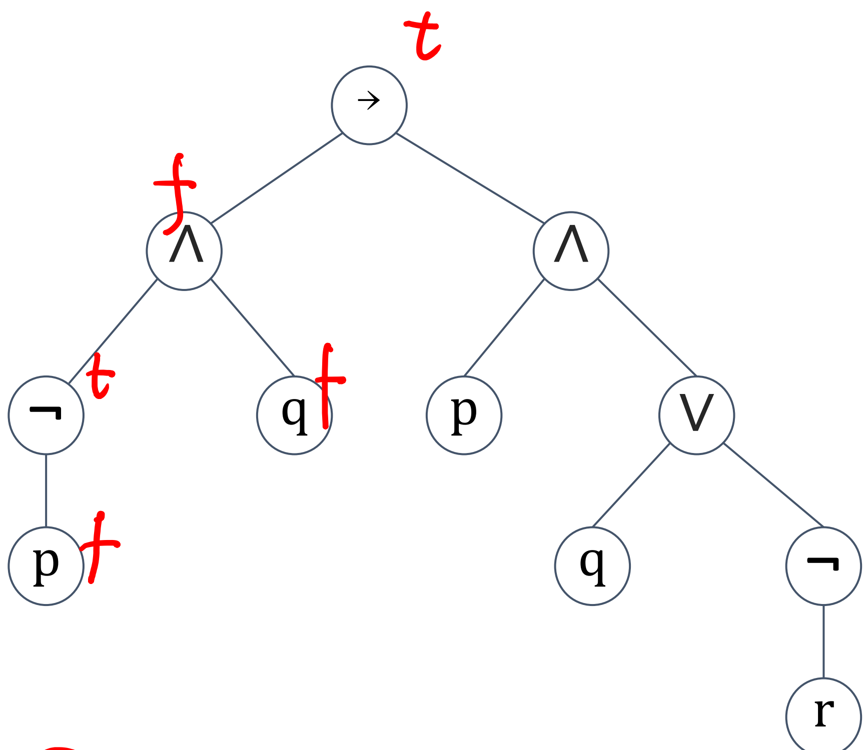
p	q	¬p	¬q	p→¬q	q∨¬p	(p→¬q)→(q∨¬p)
t	t	f	f	f	t	t
t	f	f	t	t	f	f
f	t	t	f	t	t	t
f	f	t	t	t	t	t

Valuation with Parse Tree

- **Example:** F= $\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$

I=

p	q	r
f	f	t



← parse tree

- If $F^I = t$ then we say that the interpretation I satisfies F (symbolically $I \models F$)

Valuation with computers

- **For any formula F and any interpretation I, the truth value F^I that is assigned to F by I is defined recursively, as follows:**
  - For any atom $F, F^I = I(F)$
    $\bot^I = f, \top^I = t$
    $(\neg F)^I = \neg(F^I)$
  - $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$

Entailment

- A set Γ of formulas **entails** a formula F (symbolically, $\Gamma \models F$), if every interpretation that satisfies all formulas in Γ satisfies F also.
  - c.f. Entailment uses the same symbol as satisfaction, the difference being what appears on the left of $\models$.
  - The formulas entailed by Γ are also called the logical consequences of Γ .

Example

- **Q: True or false?**
  - $\{A, A \rightarrow B\} \models B$

A	B	A→B
t	t	t
t	f	f
f	t	t
f	f	t

Tautology

- A propositional formula F is a **tautology** if every interpretation satisfies F
  - Q: Is the following formula a tautology?

$F =$ $(p \rightarrow q) \rightarrow (\neg p \vee q)$

p	q	¬p	p→q	¬p∨ q	(p→q)→(¬p∨ q)
t	t	f	t	t	t
t	f	f	f	f	t
f	t	t	t	t	t
f	f	t	t	t	t

interpretations

Satisfiability

- A propositional formula F is **satisfiable** if some interpretation satisfies F
  - Q: Is the following formula satisfiable?

$$(p \to (q \to p))$$

p	q	q⟶p	(p ⟶(q⟶p))
t	t	t	t
t	f	t	t
f	t	f	t
f	f	t	t

Equivalence

- F is **equivalent** to G (symbolically, $F \Leftrightarrow G$) if, for every interpretation I, $F^I = G^I$
  - Q: Are the following formulas equivalent?

p⟶q and ¬pV q

$p \to q \Longleftrightarrow \neg p \vee q$

p	q	¬p	p⟶q	¬pV q
t	t	f	t	t
t	f	f	f	f
f	t	t	t	t
f	f	t	t	t

Soundness and Completeness

- **When we define a logic (or any type of calculus), we want to show that it is useful.**
  - Soundness: Formulas that we derive using the calculus reflect a “real” truth.
  - Completeness: Every formula corresponding to a “real” truth can be inferred using rules of the calculus.

Soundness and Completeness

- **In the case of propositional logic, given the formulas $\Phi_1, \Phi_2, \ldots, \Phi_n$ and Ψ, we have:** natural deduction
  entails
  - Soundness: if $\Phi_1, \ldots, \Phi_n \vdash \Psi$ holds, then $\Phi_1, \ldots, \Phi_n \vDash \Psi$ holds.
    entails natural
  - Completeness: if $\Phi_1, \ldots, \Phi_n \vDash \Psi$ holds, then $\Phi_1, \ldots, \Phi_n \vdash \Psi$ holds.
- **Natural Deduction is Sound and Complete**

Summary

- **Semantics of propositional logic**
  - Interpretation
  - Valuation
- **Logic relationships based on semantics**
  - Entailment
  - Tautology
  - Satisfiability
  - Equivalence
- **Natural deduction is sound and complete**