Natural Deduction for Propositional Logic

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Lewis Carroll Puzzle (I)

Derive an inescapable conclusion using all of these:

a. All babies are illogical
b. Nobody is despised who can manage a crocodile
c. Illogical persons are despised

- B: it is a baby
- L: it is logical
- M: it can manage a crocodile
- D: it is despised

$$a)\ B \rightarrow \neg L$$
$$b)\ M \rightarrow \neg D$$
$$c)\ \neg L \rightarrow D$$

Natural Deduction

- Collection of **proof rules**, which allow to infer new formulas from existing formulas
- Given the formulas $\Phi_1 \ldots \Phi_n$, we intend to infer a conclusion Ψ:

$$\Phi_1 \ldots \Phi_n \vdash \Psi$$

This construct is called **sequent**, for example:

$$p \wedge q, r \vdash q \wedge r$$

- Priority of connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$

Natural Deduction Rules

Conjunction

$$\frac{\Phi \quad \Psi}{\Phi \wedge \Psi} \; \wedge i$$

***and*-introduction**

$$\frac{\Phi \wedge \Psi}{\Phi} \; \wedge e1$$

$$\frac{\Phi \wedge \Psi}{\Psi} \; \wedge e2$$

***and*-elimination**

Natural Deduction Rules

- **Prove** $p \wedge q, r \vdash q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e2\ 1$
4	$q \wedge r$	$\wedge i\ 3, 2$

Natural Deduction Rules

Double Negation and Implication Elimination

$$\frac{\neg\neg\Phi}{\Phi} \quad \neg\neg e$$

double negation-elimination

$$\frac{\Phi}{\neg\neg\Phi} \quad \neg\neg i$$

double negation-introduction

$$\frac{\Phi \qquad \Phi \rightarrow \Psi}{\Psi} \quad \rightarrow e$$

implication-elimination

Natural Deduction Rules

- Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i$ 1
4	$q \wedge r$	$\neg\neg e$ 2
5	r	$\wedge e2$ 4
6	$\neg\neg p \wedge r$	$\wedge i$ 3, 5

Natural Deduction Rules

- Prove $p, p \to q, p \to (q \to r) \vdash r$

1	$p \to (q \to r)$	premise
2	$p \to q$	premise
3	p	premise
4	$q \to r$	$\to e$ 1, 3
5	q	$\to e$ 2, 3
6	r	$\to e$ 4, 5

Natural Deduction Rules

Implication Introduction

$$\dfrac{\boxed{\begin{array}{c}\Phi \\ \vdots \\ \Psi\end{array}}}{\Phi \to \Psi} \to i$$

In order to prove $\Phi \to \Psi$,
we make the temporary
assumption of Φ,
and then prove Ψ.
The scope of the assumption is
indicated by the box

Natural Deduction Rules

Prove $p \wedge q \to r \vdash p \to (q \to r)$

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge i$ $2, 3$
5	r	$\to e$ $1, 4$
6	$q \to r$	$\to i$ $3-5$
7	$p \to (q \to r)$	$\to i$ $2-6$

Natural Deduction Rules

Disjunction

$$\frac{\Phi}{\Phi \vee \Psi} \vee i1 \qquad \frac{\Psi}{\Phi \vee \Psi} \vee i2$$

$$\frac{\Phi \vee \Psi \quad \boxed{\begin{matrix}\Phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\Psi \\ \vdots \\ \chi\end{matrix}}}{\chi} \vee e$$

Natural Deduction Rules

- **Prove** $q \to r \vdash p \lor q \to p \lor r$

1	$q \to r$	premise
2	$p \lor q$	assumption
3	p	assumption
4	$p \lor r$	$\lor i1\ 3$
5	q	assumption
6	r	$\to e\ 1,5$
7	$p \lor r$	$\lor i2\ 6$
8	$p \lor r$	$\lor e\ 2, 3-4, 5-7$
9	$p \lor q \to p \lor r$	$\to i\ 2-8$

Natural Deduction Rules

Negation

$$\frac{\boxed{\begin{array}{c}\Phi\\ \vdots\\ \bot\end{array}}}{\neg\Phi}\ \neg i \qquad \frac{\bot}{\Phi}\ \bot e \qquad \frac{\Phi \quad \neg\Phi}{\bot}\ \neg e$$

Natural Deduction Rules

Prove $p \to q, p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to e\ 1, 3$
5	$\neg q$	$\to e\ 2, 3$
6	$\bot$	$\neg e\ 4, 5$
7	$\neg p$	$\neg i\ 3-6$

Natural Deduction Rules

- **Prove** $\neg p \lor q \vdash p \rightarrow q$

1	$\neg p \lor q$	premise
2	$\neg p$	premise
3	p	assumption
4	$\bot$	$\neg e\ 3, 2$
5	q	$\bot e 4$
6	$p \rightarrow q$	$\rightarrow i\ 3-5$
7	q	premise
8	p	assumption
9	q	copy 7
10	$p \rightarrow q$	$\rightarrow i\ 9-10$
11	$p \rightarrow q$	$\lor e\ 1, 2-6, 7-10$

Proof Strategy in Natural Deduction

If your goal is to prove:	Do:
$F \vee G$	Derive each of F or G; use or introduction
$F \rightarrow G$	Assume F and derive G; use implication introduction
$F \wedge G$	Try to derive F and G as a subgoal; use and introduction

Derived Rules

$$\frac{\Phi \to \Psi \qquad \neg\Psi}{\neg\Phi}$$

MT (Modus Tollens)

$$\frac{\Phi}{\neg\neg\Phi}$$

$\neg\neg i$

$$\frac{\boxed{\begin{array}{c}\neg\Phi \\ \vdots \\ \bot\end{array}}}{\Phi}$$

RAA ([Reductio ad Absurdum - a.k.a
Proof by contradiction (PBC)]

$$\Phi \vee \neg\Phi$$

LEM (Law of excluded middle)

Summary

- **To derive a conclusion, we need a set of proof rules**
- **Natural deduction rules we studied:**
  - Conjunction
  - Implication
  - Disjunction
  - Negation
  - Derived rules
- **No perfect set of rules. You can come up with your own rules!**