DERIVED CATEGORIES

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1. Introduction

We are going to introduce the Derived Category.

We assume knowledge of some vocabulary:

- additive category, additive functor
- abelian category
- k-linear category*
- category of complexes
- natural isomorphism
- null-homotopy
- quasi-isomorphism

We'll also need some new tools, which we'll define, explain and motivate:

- triangles (triangulated category)
- cohomological functors
- Ore localization
- right denominator set
- localization of a triangulated category

That will be the bulk of the paper. We will finish with application of the derived category to representations of quivers (?).

Our setting is the category of complexes **C(M)** of an additive category **M**. We arrive in the homotopy category of complexes **K(M)** by noting that C(M) is a linear category and the null-homotopic morphisms form a 2-sided ideal, and quotienting by that ideal. Now is when we mention **triangles** because K(M) is a **triangulated category**. It turns out that we get the Derived Category **D(M)** as the image (?) of a triangulated functor from the homotopy category; as well, it is triangulated. We construct the Derived Category as the **localization** of the homotopy category about the quasi-isomorphisms, which are a **right denominator set**. The key is we can learn about exact sequences in M from distinguished triangles in D(M).

2. Triangulated Categories

A **T-additive category** is an additive category **K** with an additive automorphsim **T**, called the "translation" or "shift" or "suspension". A **T-additive functor** is an additive functor $\mathbf{F} : K \to L$ with a natural isomorphism $\xi : F \circ T_K \to T_L \circ F$. It follows that a **morphism of T-additive functors** is a natural transformation $\eta : (F, \xi) \to (G, \nu)$.

A **triangle** is a diagram $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$, and a morphism of triangles is a triple of maps (ϕ, ψ, χ) with commutative diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & T(L) \arrow[d, "T \circ \phi"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & T(L')
\end{tikzcd}$$

A **triangulated category** is a T-additive category with a set of **distinguished triangles**, and a **triangulated functor** is a T-additive functor that sends distinguished triangles to distinguished triangles.

A set of distinguished triangles must satisfy the following axioms:

(TR1) If A is isomorphic to B and B is a distinguished triangle, so is A. For every $\alpha : L \to M$, there is a distinguished triangle $L \xrightarrow{\alpha} M \to N \to T(L)$. For every $M \in K$, $M \xrightarrow{1_M} M \to 0 \to T(M)$ is distinguished.

(TR2) The triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ is distinguished if and only if $M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L) \xrightarrow{-T(\alpha)} T(M)$ is distinguished.

(TR3) If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ and $L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} T(L')$ are distinguished triangles and $\phi : L \to L', \psi : M \to M'$ are such that $\psi \circ \alpha = \alpha' \circ \phi$, there exists a $\chi : N \to N'$ such that we get a morphism of triangles:

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, dashed, "\chi"'] & T(L) \arrow[d, "T \circ \phi"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & T(L')
\end{tikzcd}$$

(TR4) If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N' \to T(L), M \xrightarrow{\gamma} N \xrightarrow{\lambda} L' \to T(M)$ and $L \xrightarrow{\gamma\circ\alpha} N \xrightarrow{\epsilon} M' \to T(L)$ are distinguished triangles, there exists a distinguished triangle $N' \xrightarrow{\phi} M' \xrightarrow{\psi} L' \to T(N')$ such that the following diagram commutes:

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "1"'] & M \arrow[r, "\beta"] \arrow[d, "\gamma"'] & N' \arrow[r] \arrow[d, "\phi"'] & T(L) \arrow[d, "1"'] \\
L \arrow[r, "\gamma \circ \alpha"] \arrow[d, "\alpha"'] & N \arrow[r, "\epsilon"] \arrow[d, "1"'] & M' \arrow[r] \arrow[d, "\psi"'] & T(L) \arrow[d, "T(\alpha)"'] \\
M \arrow[r, "\gamma"] \arrow[d, "\beta"'] & N \arrow[r, "\delta"] \arrow[d, "\epsilon"'] & L' \arrow[r] \arrow[d, "1"'] & T(M) \arrow[d, "T(\beta)"'] \\
N' \arrow[r, dashed, "\phi"] & M' \arrow[r, dashed, "\psi"] & L' \arrow[r, dashed] & T(N')
\end{tikzcd}$$

The object N of (T1) is called the cone on $\alpha : L \to M$. An important note regarding (T3) is that the morphism χ on cones induced by the commutative square is not unique, which is a reflection that cones are not functorial.

2.1. **Getting K(M) is a triangulated category.**

We begin by defining the standard triangle associated to α when $\alpha : L \to M$ is a morphism of chain complexes in C(M). Let $N[i] = L[i+1] \oplus M[i]$ and define the boundary operator $d_N^i : N^i \to N^{i+1}$ by

$$\begin{bmatrix} -d_L^{i+1} & 0 \\ \alpha^{i+1} & d_M^i \end{bmatrix}$$

.

We call **N** the **mapping cone** of α and denote it by cone(α). Let $\beta : M \to N$ be given by

$$\begin{bmatrix} 0 \\ 1_M \end{bmatrix}$$

and $\gamma : N \to L[1]$ be given by

$$\begin{bmatrix} 1_{L[1]} & 0 \end{bmatrix}$$

.

We get a triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$. Passing to K(M), we get the **standard triangle** associated to α $L \xrightarrow{\overline{\alpha}} M \xrightarrow{\overline{\beta}} N \xrightarrow{\overline{\gamma}} L[1]$. While the standard triangle associated to α is functorial in C(M), it is not in K(M). The category K(M) becomes a triangulated category with set of distinguished triangles those isomorphic in K(M) to standard triangles in K(M).

One should now check that this set of distinguished triangles satisfies the necessary axioms (T1), (T2), (T3), (T4).

3. Cohomological Functors

Now that we have triangulated categories, we can talk about special additive functors from triangulated categories to abelian categories.

A **cohomological functor** is an additive functor $F : K \to M$ from a triangulated category to an abelian category such that every distinguished triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ in K gets mapped to an exact sequence $F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$.

We have the following useful propositions:

Proposition 1. *$F : K \to M$ cohomological, $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ distinguished in K. Then get LES $\ldots \to F(L[i]) \xrightarrow{F(\alpha[i])} F(M[i]) \xrightarrow{F(\beta[i[)} F(N[i]) \xrightarrow{F(\gamma[i])} F(L[i+1]) \xrightarrow{F(\alpha[i+1])} F(M[i+1]) \to \ldots$.*

Proposition 2. *K, triangulated. $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$, distinguished. Then $\beta \circ \alpha = 0$. Also for $P \in Ob(K)$, Hom(-, P): $K^{op} \to Ab$ and Hom(P,-): $K \to Ab$ are cohomological.*

EXPLAIN AND MOTIVATE. INCLUDE H^0 COHOMOLOGICAL?

4. Localization

A **weak localization of A with respect to S** is a pair **($\mathbf{A}_S$, Q)** consisting of a category A_S and a functor $Q : A \to A_S$ such that (i) for all $s \in S, Q(s) \in A_S$ is invertible and (ii) if $F : A \to B$ is a functor such that $F(s)$ is an isomorphism for $s \in S$, there exists a pair (F_S, η) with $F_S : A_S \to B$ and $\eta : F \to F_S \circ Q$ an isomorphism of functors. The pair (F_s, η) is unique up to unique isomorphism and the pair (A_s, Q) is unique up to unique equivalence (? CHECK, AND UNDERSTAND IMPORTANCE).

A **strict localization of A with respect to S** is a pair (A_S, Q) such that (i) for all $s \in S, Q(s) \in A_S$ is invertible, (ii) Ob(A_S) = Ob(A) and Q is the identity on objects, and (iii) if $F : A \to B$ is such that $F(s)$ is invertible for all $s \in S$, then there exists $F_S : A_S \to B$ with $F_S \circ Q = F$ as functors. The pair (A_S, Q) is unique up to unique isomorphism of categories. (? UNDERSTAND DIFFERENCE W WEAK, EXPLAIN CONNECTION.)

A **right Ore localization of A with respect to S** is a pair (A_S, Q) satisfying

(L1) Ob(A_S) = Ob(A), Q identity on objects

(L2) for all $s \in S, Q(s) \in A_S$ invertible

(L3) for all $q \in A_S$, there exists an $a \in A$ and an $s \in S$ such that $q = Q(a) \circ Q(s)^{-1}$

(L4) for all $a, b \in A$ such that $Q(a) = Q(b)$, there exists an $s \in S$ such that $a \circ s = b \circ s$

One can show that an Ore localization is a strict localization, which in turn is a weak localization. The important connection for us is that Ore localization about a subset S exists if and only if S is multiplicatively closed and a right denominator set. We define these now.

Let A be a category and $M, N \in Ob(A)$ and consider $A(M, N) = Hom_A(M, N)$. A **multiplicatively closed set S in A** is a subset $S(M, N) \subset A(M, N)$ that contains $1_M \in S(M, M)$ and for all $s \in S(L, M), t \in S(M, N)$ we have $t \circ s \in S(L, N)$.

A **right denominator set** is a multiplicatively closed subset satisfying

(D1) (Right Ore condition) for all $a \in A, s \in S$, there exist $b \in A, t \in S$ such that $a \circ t = s \circ b$

$$\begin{tikzcd}
 & K \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "b"] & \\
M \arrow[dr, "a"'] & & N \arrow[dl, "s"] \\
 & L &
\end{tikzcd}$$

(D2) (Right cancellation condition) for all $a, b \in A, s \in S$ s.t. $s \circ a = s \circ b$, there exists a $t \in S$ such that $a \circ t = b \circ t$

$$\begin{tikzcd}
K \arrow[d, dashed, "\text{t}"'] \\
M \arrow[d, bend right, "\text{a}"'] \arrow[d, bend left, "\text{b}"] \\
N \arrow[d, "\text{s}"'] \\
L
\end{tikzcd}$$

Theorem 4.1. *For $S \subset A$ a multiplicatively closed subset, right Ore localization (A_S, Q) exists $\iff$ S is a right denominator set.*

Proof. We begin with proving that the existence of the right Ore localization (A_S, Q) implies that S is a right denominator set; this is the easy/shorter direction (rather, it doesn't involve roofs!). First we'll show (D1) is satisfied. Take $a \in A, s \in S$ and consider $q = Q(s)^{-1} \circ Q(a)$. By (L3) there exists a $b \in A$ and $t \in S$ such that $q = Q(b) \circ Q(t)^{-1}$. Since $Q(s)^{-1} \circ Q(a) = Q(b) \circ Q(t)^{-1}$, we get $Q(s) \circ Q(b) = Q(a) \circ Q(t)$ so $Q(s \circ b) = Q(a \circ t)$. By (L4), there exists a $u \in S$ such that $s \circ b \circ u = a \circ t \circ u$. That is, $s \circ (b \circ u) = a \circ (t \circ u)$, where $t \circ u \in S$. Thus we have (D1). Now we'll show (D2) is satisfied. Let $a, b \in A$ and $s \in S$ such that $s \circ a = s \circ b$. Then $Q(s \circ a) = Q(s \circ b)$. Since $Q(s)$ is invertible, we get that $Q(a) = Q(b)$. By (L4), there exists a $t \in S$ such that $a \circ t = b \circ t$.

Next we work out how S being a right denominator set implies the existence of the right Ore localization (A_S, Q); this is the hard/longer direction (or really, its just fun with roofs!). We start by defining the sets $A_S(M, N)$, composition between them and identity morphisms. This is the start of what we need to show that A_S is a category; we will still need to show associativity and identity properties of composition. Next we define Q, and will still need to show that it is a functor. Finally, we verify the axioms of right Ore localization, (L1), (L2), (L3), and (L4).

Consider

$$A \times S = \textstyle\prod_{L \in Ob(A)} A(L, M) \times S(L, N)$$

and consider the relation $(a_1, s_1) \sim (a_2, s_2)$ if there exist $b_1, b_2 \in S$ from K such that $a_1 \circ b_1 = a_2 \circ b_2$ and $s_1 \circ b_1 = s_2 \circ b_2$.

We have the following lemma.

Lemma 4.2. *If the right Ore condition is satisfied (D1), then $\sim$ is an equivalence relation on A_S.*

To see this, take $K = L$ and $b_i = 1_L ::\to L$. Then $(a_1, s_1) \sim (a_1, s_1)$. Hence we have reflexivity. Symmetry is clear. Now we use the right Ore condition (D1) to get transitivity. Suppose $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$.

$$\begin{tikzcd}
 & & I \arrow[dll, dashed, "t"'] \arrow[drr, dashed, "b"] & & \\
K \arrow[d, "b_1"'] \arrow[drr, "b_2"] & & & & J \arrow[dll, "c_2"'] \arrow[d, "c_3"] \\
L1 \arrow[d, "s_1"'] \arrow[drrrr, "a_1"' near end] & & L2 \arrow[dll, "s_2"' near start] \arrow[drr, "a_2" near start] & & L3 \arrow[d, "a_3"] \arrow[dllll, "s_3" near end] \\
M & & & & N
\end{tikzcd}$$

$$\begin{tikzcd}
 & H \arrow[d, dashed, "u"] & \\
 & I \arrow[dl, "t"'] \arrow[dr, "b"] & \\
K \arrow[dr, "b_2"'] & & J \arrow[dl, "c_2"] \\
 & L2 &
\end{tikzcd}$$

So we get $(a_1, s_1) \sim (a_3, s_3)$. With this transitivity, we have that $\sim$ is an equivalence relation. We can then quotient $A \times S$ by $\sim$ to define the set A_S. $\square$

We define composition of two morphisms $q_1 \in A_S(M_0, M_1), q_2 \in A_S(M_1, M_2)$ by choosing representatives $(a_i, s_i) \in A \times S(M_{i-1}, M_i)$ such that $q_i = \overline{(a_i, s_i)}$.

By (D1), given a_1, s_2 we have c, u from K such that $a_1 \circ u = s_2 \circ c$:

$$\begin{tikzcd}
 & & K \arrow[dl, dashed, "u"'] \arrow[dr, dashed, "c"] & & \\
 & L \arrow[dl, "s_1"'] \arrow[dr, "a_1"] & & L' \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & \\
M0 & & M1 & & M2
\end{tikzcd}$$

Let $q_2 \circ q_1 = \overline{(a_2 \circ c, s_1 \circ u)}$. We now to check this is well-defined:

Suppose $q_i' = \overline{(a_i', s_i')}$ and choose u', c'. We must show that $\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a_2' \circ c', s_1' \circ u')}$.

Since $(a_i, s_i) \sim (a_i', s_i')$, there exist b_i, b_i' from Ji such that:

$$
\begin{tikzcd}
J1 \arrow[d, dashed, "b_1"'] \arrow[dr, dashed, "b_1'"] & & & & J2 \arrow[dl, dashed, "b_2"'] \arrow[d, dashed, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}
$$

By (D1) we get the compositions through K, K' commute to $M1$:

$$
\begin{tikzcd}
J1 \arrow[d, "b_1"'] \arrow[dr, "b_1'"] & K \arrow[dl, dashed, "c"'] \arrow[drr, dashed, "u"] & & K' \arrow[dll, dashed, "c'"'] \arrow[dr, dashed, "u'"] & J2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}
$$

By (D1) again, we get the diagram commutes above $L1$ and through $L2'$ to $M1$:

$$
\begin{tikzcd}
 & I1 \arrow[dl, dashed, "v_1"'] \arrow[d, dashed, "d_1"] & & \tilde{I2} \arrow[d, dashed, "\tilde{v_2}"'] \arrow[dr, dashed, "\tilde{d}_2"] & \\
J1 \arrow[d, "b_1"'] \arrow[dr, "b_1'"] & K \arrow[dl, "c"'] \arrow[drr, "u"] & & K' \arrow[dll, "c'"'] \arrow[dr, "u'"] & J2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"'] & L1' \arrow[dl, "s_1'"'] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"'] & L2' \arrow[dll, "s_2'"'] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}
$$

By (D2), we get commutation above $L2'$:

$$
\begin{tikzcd}
 & I2 \arrow[d, dashed, "\tilde{v}"'] & \\
 & \tilde{I2} \arrow[dl, "\tilde{v_2}"'] \arrow[dr, "\tilde{d}_2"] & \\
K' \arrow[dr, "c'"'] & & J2 \arrow[dl, "b_2'"] \\
 & L2' \arrow[d, "s_2'"'] & \\
 & M1 &
\end{tikzcd}
$$

At this point, we've constructed the following diagram so that is is all commutative.

$$\begin{tikzcd}
 & I1 \arrow[dl, dashed, "v_1"'] \arrow[d, dashed, "d_1"] & & I2 \arrow[d, dashed, "v_2"'] \arrow[dr, dashed, "d_2"] & \\
J1 \arrow[d, "b_1"'] \arrow[dr, "c"] & K \arrow[dl, "b_1'"] \arrow[drr, "u"] & & K' \arrow[dll, "c'"'] \arrow[dr, "u'"] & J2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}$$

By (D1), we can get commutation into $M0$ from $I1$ and $I2$ by considering $w \in A$ and $e \in S$ from H:

$$\begin{tikzcd}
 & & H \arrow[dl, dashed, "w"'] \arrow[dr, dashed, "e"] & & \\
 & I1 \arrow[dl, "v_1"'] \arrow[d, "d_1"] & & I2 \arrow[d, "v_2"'] \arrow[dr, "d_2"] & \\
J1 \arrow[d, "b_1"'] \arrow[dr, "c"] & K \arrow[dl, "b_1'"] \arrow[drr, "u"] & & K' \arrow[dll, "c'"'] \arrow[dr, "u'"] & J2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}$$

While $H \to I_2 \to M_0 \in S$, we could fail to have commuting paths $H \to L_1'$ and $H \to L2$.

$$\begin{tikzcd}
 & & H \arrow[dl, dashed, "w"'] \arrow[dr, dashed, "e"] & & \\
 & I1 \arrow[dl, "v_1"'] \arrow[d, "d_1"] & & I2 \arrow[d, "v_2"'] \arrow[dr, "d_2"] & \\
J1 \arrow[dr, "b_1'"'] & K \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] & J2 \arrow[dl, "b_2"] \\
 & L1' & & L2 &
\end{tikzcd}$$

Noting that upon composing with s_1' into M_0, our paths from $H \to L_1'$ become equal, so by (D2) we get $w' \in S$ from H' such that we now have commutation above L_1'.

For the same reasons with composition with s_2, we get commutation above L_2 when mapping from H''. (CHECK, DOESN'T MAKE SENSE TO MET YET.)

Now all paths $H'' \to M_2$ are equal and all paths $H'' \to M_0$ are equal and in S:

$$\begin{tikzcd}
 & & H'' \arrow[d, dashed, "w''"'] & & \\
 & & H' \arrow[d, dashed, "w'"'] & & \\
 & & H \arrow[dl, dashed, "w"'] \arrow[dr, dashed, "e"] & & \\
 & I1 \arrow[dl, "v_1"'] \arrow[d, "d_1"] & & I2 \arrow[d, "v_2"'] \arrow[dr, "d_2"] & \\
J1 \arrow[d, "b_1"'] \arrow[dr, "b_1'"] & K \arrow[dl, "c"'] \arrow[drr, "u"] & & K' \arrow[dll, "c'"'] \arrow[dr, "u'"] & J2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L1' \arrow[dl, "s_1'"'] \arrow[dr, "a_1'"] & & L2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L2' \arrow[dll, "s_2'"'] \arrow[d, "a_2'"] \\
M0 & & M1 & & M2
\end{tikzcd}$$

Thus we have

$$\begin{tikzcd}
 & & H'' \arrow[dl, "w \circ w' \circ w''"'] \arrow[dr, "e \circ w' \circ w''"] & & \\
 & I1 \arrow[d, "d_1"'] & & I2 \arrow[d, "v_2"] & \\
 & K \arrow[dl, "u"'] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"] & \\
L1 \arrow[d, "s_1"'] & L1' \arrow[dl, "s_1'"] & & L2 \arrow[dr, "a_2"'] & L2' \arrow[d, "a_2'"] \\
M0 & & & & M2
\end{tikzcd}$$

and so $\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a_2' \circ c', s_1' \circ u')}$!

So we are done with proving well-defined composition. Take the identity morphism to be $\overline{(1_M, 1_M)}$. One now must check the identity and associativity properties of this definition of composition. Define $Q : A \to A_s$ to be $Q(M) = M, Q(a) = \overline{(a, 1_M)}$ for $a : M \to N$. One must also chech that Q is indeed a functor.

We now verify that the axioms of right Ore localization are satisfied. We have (L1) satisfied by our definition of Q. We can easily check that the inverse of Q(s) is $\overline{(1, s)}$ to get (L2) and $\overline{(a, s)} = \overline{(a, 1)} \circ \overline{(1, s)}$ to get (L3). Lastly, we have $Q(a_1) = Q(a_2)$ means that $(a_1, 1) \sim (a_2, 1)$, so there exist $b_1, b_2 \in A$ such that $a_1 \circ b_1 = a_2 \circ b_2$ and $1 \circ b_1 = 1 \circ b_2 \in S$. Let $s = b_1 \in S$, get $a_1 \circ S = a_2 \circ S$.

□

5. Cohomological Functors Revisited, briefly

Proposition 3. *If H: K →M is a cohomological functor from a triangulated category K to an abelian category M and we consider* $S = \{s \in K | H(T^i(s))$ *is invertible for all* $i \in \mathbb{Z}\}$. *Then S is a left and right denominator set.*

6. Localization of Triangulated categories:

Theorem 6.1. *If S is the right denominator set associated to a cohomological functor on K, and (K_S, Q) is the right Ore localization, then K_S has a unique triangulated structure such that (i) Q is a triangulated functor (ii) if $F : K \to E$ is another triangulated functor such that $F(s)$ is invertible for all $s \in S$, then the localization $F_S : K_S \to E$ is triangulated.*

7. The Big Reveal...

The set S(M) of quasi-isomorphisms in K(M) is equivalent to the set $\{s \in K(M) | H^i(s)$ is an isomorphism for all $i\}$, and the functor $H^0 : K(M) \to M$ is cohomological. Therefore, we can define the **Derived Category** of M as the k-linear triangulated category that results from localization of K(M) about S(M); that is, **D(M)** $= K(M)_{S(M)}$.