DERIVED CATEGORIES

NIKKI SANDERSON, LUCAS SIMON

1. Introduction

We are going to introduce the Derived Category.

We assume knowledge of some vocabulary:

- additive category, additive functor
- abelian category
- k-linear category*
- category of complexes
- natural isomorphism
- null-homotopy
- quasi-isomorphism

We'll also need some new tools, which we'll define, explain and motivate:

- triangles (triangulated category)
- cohomological functors
- Ore localization
- right denominator set
- localization of a triangulated category

That will be the bulk of the paper. We will finish with application of the derived category to representations of quivers (?).

Our setting is the category of complexes **C(M)** of an additive category **M**. We arrive in the homotopy category of complexes **K(M)** by noting that C(M) is a linear category and the null-homotopic morphisms form a 2-sided ideal, and quotienting by that ideal. Now is when we mention **triangles** because K(M) is a **triangulated category**. It turns out that we get the Derived Category **D(M)** as the image (?) of a triangulated functor from the homotopy category; as well, it is triangulated. We construct the Derived Category as the **localization** of the homotopy category about the quasi-isomorphisms, which are a **right denominator set**. The key is we can learn about exact sequences in M from distinguished triangles in D(M).

2. Triangulated Categories

A **T-additive category** is an additive category **K** with an additive automorphsim **T**, called the "translation" or "shift" or "suspension". A **T-additive functor** is an additive functor $\mathbf{F} : K \to L$ with a natural isomorphism $\xi : F \circ T_K \to T_L \circ F$. It follows that a **morphism of T-additive functors** is a natural transformation $\eta : (F, \xi) \to (G, \nu)$ with commutative diagram

INSERT NAT. TRANS DIAGRAM

A **triangle** is a diagram $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$, and a morphism of triangles is a triple of maps (ϕ, ψ, χ) with commutative diagram

INSERT TRIANGLE MORPH DIAGRAM

A **triangulated category** is a T-additive category with a set of **distinguished triangles***, and a **triangulated functor** is a T-additive functor that sends distinguished triangles to distinguished triangles.

A set of distinguished triangles must satisfy the following axioms:

(TR1) If A is isomorphic to B and B is a distinguished triangle, so is A. For every $\alpha : L \to M$, there is a distinguished triangle $L \xrightarrow{\alpha} M \to N \to T(L)$. For every $M \in K$, $M \xrightarrow{1_M} M \to 0 \to T(M)$ is distinguished.

(TR2) The triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ is distinguished if and only if $M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L) \xrightarrow{-T(\alpha)} T(M)$ is distinguished.

(TR3) If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ and $L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} T(L')$ are distinguished triangles and $\phi : L \to L', \psi : M \to M'$ are such that (? CHECK THAT WE NEED COMMUTATIVITY), there exists a $\chi : N \to N'$ such that (? WHOLE DIAGRAM COMMUTES). ***BIG IMPORTANT NOTE: NOT FUNCTORIAL, DEEP CONSEQUENCES!!!***

INSERT DIAGRAM

(TR4) If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N' \to T(L), M \xrightarrow{\gamma} N \xrightarrow{\lambda} L' \to T(M)$ and $L \xrightarrow{\gamma\circ\alpha} N \xrightarrow{\epsilon} M' \to T(L)$ are distinguished triangles, there exists a distinguished (? CHECK) triangle $N' \xrightarrow{\phi} M' \xrightarrow{\psi} L' \to T(N')$ such that the following diagram commutes

INSERT BIG CHAIN DIAGRAM

- - - EXPLAINATION AND MOTIVATION OF TRIANGLE AXIOMS - - - - -

2.1. Getting K(M) is a triangulated category.

We begin by defining the standard triangle associated to α when $\alpha : L \to M$ is a morphism of chain complexes in C(M). Let $N[i] = L[i+1] \oplus M[i]$ and define the boundary operator $d_N^i : N^i \to N^{i+1}$ = INSERT ARRAY. We call **N** the **mapping cone** of α and denote it by cone(α). Let $\beta : M \to N$ be given by INSERT ARRAY and $\gamma : N \to L[1]$ be given by

INSERT ARRAY. We get a triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$. Passing to K(M), we get the **standard triangle** associated to α $L \xrightarrow{\overline{\alpha}} M \xrightarrow{\overline{\beta}} N \xrightarrow{\overline{\gamma}} L[1]$. The category K(M) becomes a triangulated category with set of distinguished triangles those isomorphic in K(M) to standard triangles in K(M).

We now check that this set of distinguished triangles satisfies the necessary axioms (T1), (T2), (T3), (T4).

Thm. BIG PROOF. PROVE? DISCUSS PROOF?

3. Cohomological Functors (tangent, aside? understand use better):

Now that we have triangulated categories, we can talk about special additive functors from triangulated categories to abelian categories. A **cohomological functor** is an additive functor $F : K \to M$ from a triangulated category to an abelian category such that every distinguished triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ in K gets mapped to an exact sequence $F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$. (? CHECK LACK OF PRECISION IN MY DEF DIDNT MESS THINGS UP)

Two Props that are nice (? WHY):

1) $F : K \to M$ cohomological, $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ distinguished in K. Then get LES $\ldots \to F(L[i]) \xrightarrow{F(\alpha[i])} F(M[i]) \xrightarrow{F(\beta[i[)} F(N[i]) \xrightarrow{F(\gamma[i])} F(L[i+1]) \xrightarrow{F(\alpha[i+1])} F(M[i+1]) \to \ldots$.

2) K, triangulated. $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$, distinguished. Then $\beta \circ \alpha = 0$. Also for $P \in Ob(K)$, Hom(-, P): $K^{op} \to$ Ab and Hom(P,-): $K \to$Ab are cohomological.

EXPLAIN AND MOTIVATE. Discuss proof?

4. Localization

A **weak localization of A with respect to S** is a pair **($\mathbf{A}_S$, Q)** consisting of a category A_S and a functor $Q : A \to A_S$ such that (i) for all $s \in S, Q(s) \in A_S$ is invertible and (ii) if $F : A \to B$ is a functor such that $F(s)$ is an isomorphism for $s \in S$, there exists a pair (F_S, η) with $F_S : A_S \to B$ and $\eta : F \to F_S \circ Q$ an isomorphism of functors. The pair (F_s, η) is unique up to unique isomorphism and the pair (A_s, Q) is unique up to unique equivalence (? CHECK, AND UNDERSTAND IMPORTANCE).

A **strict localization of A with respect to S** is a pair (A_S, Q) such that (i) for all $s \in S, Q(s) \in A_S$ is invertible, (ii) Ob(A_S) = Ob(A) and Q is the identity on objects, and (iii) if $F : A \to B$ is such that $F(s)$ is invertible for all $s \in S$, then there exists $F_S : A_S \to B$ with $F_S \circ Q = F$ as functors. The pair (A_S, Q) is unique up to unique isomorphism of categories. (? UNDERSTAND DIFFERENCE W WEAK, EXPLAIN CONNECTION.)

A **right Ore localization of A with respect to S** is a pair (A_S, Q) satisfying

(L1) Ob(A$_S$) = Ob(A), Q identity on objects

(L2) for all $s \in S, Q(s) \in A_S$ invertible

(L3) for all $q \in A_S$, there exists an $a \in A$ and an $s \in S$ such that $q = Q(a) \circ Q(s)^{-1}$

(L4) for all $a, b \in A$ such that $Q(a) = Q(b)$, there exists an $s \in S$ such that $a \circ s = b \circ s$

CONNECT TO PREVIOUS WEAK/STRICT, mention left/right, AND TO NEXT RIGHT DENOMINATOR SET.

A **right denominator set** is a multiplicatively closed subset satisfying

(D1) for all $a \in A, s \in S$, there exist $b \in A, t \in S$ such that $a \circ t = s \circ b$

(D2) for all $a, b \in A, s \in S$ s.t. $s \circ a = s \circ b$, there exists a $t \in S$ such that $a \circ t = b \circ t$

SOME STORY AND EXPLANATIONS OF CONNECTING PROOF PIECES, TIE TOGETHER ORE AND DENOMINATOR SET, ROOFS, EQUIVALENCES;

Let A be a category and $M, N \in Ob(A)$ and consider $A(M, N) = Hom_A(M, N)$. A **multiplicatively closed set S in A** is a subset $S(M, N) \subset A(M, N)$ that contains $1_M \in S(M, M)$ and for all $s \in S(L, M), t \in S(M, N)$ we have $t \circ s \in S(L, N)$.

IMPORTANT THEOREM 10.2.6: For $S \subset A$ a multiplicatively closed subset, right Ore localization (A_S, Q) exists $\iff$ S is a right denominator set.

BIG PROOF. INCLUDE. (is this where well-defined based on choice of representative comes in?) (this IS where we use Lemma 10.2.7, right Ore condition implies $\sim$ is an equivalence - use to define set A$_S$)

Proof. We begin with proving the existence of the right Ore localization (A_S, Q) implies that S is a right denominator set; this is the easy/shorter direction. First we'll show (D1) is satisfied. Take $a \in A, s \in S$ and consider $q = Q(s)^{-1} \circ Q(a)$. By (L3) there exists a $b \in A$ and $t \in S$ such that $q = Q(b) \circ Q(t)^{-1}$. Since $Q(s)^{-1} \circ Q(a) = Q(b) \circ Q(t)^{-1}$, we get $Q(s) \circ Q(b) = Q(a) \circ Q(t)$ so $Q(s \circ b) = Q(a \circ t)$. By (L4), there exists a $u \in S$ such that $s \circ b \circ u = a \circ t \circ u$. That is, $s \circ (b \circ u) = a \circ (t \circ u)$, where $t \circ u \in S$. Thus we have (D1). Now we'll show (D2) is satisfied. Let $a, b \in A$ and $s \in S$ such that $s \circ a = s \circ b$. Then $Q(s \circ a) = Q(s \circ b)$. Since $Q(s)$ is invertible, we get that $Q(a) = Q(b)$. By (L4), there exists a $t \in S$ such that $a \circ t = b \circ t$.

Next we work out how S being a right denominator set implies the existence of the right Ore localization (A_S, Q); this is the hard/longer direction. We start by defining the sets $A_S(M, N)$, composition between them and identity morphisms. This is the start of what we need to show that A_S is a category; we will still need to show associativity and identity properties of composition (? CHECK - AND DO. LTR). Next we define Q and show it is a functor (ALSO LTR). Finally, we verify the axioms of right Ore localization, (L1), (L2), (L3), and (L4) (NEED TO PUT WORK INTO).

Consider $A \times S = \prod_{L \in Ob(A)} A(L,M) \times S(L,N)$ and consider the relation $(a_1, s_1) \sim (a_2, s_2)$ if there exist $b_1, b_2 \in S$ from K such that $a_1 \circ b_1 = a_2 \circ b_2$ and $s_1 \circ b_1 = s_2 \circ b_2$.

Lemma 10.2.7 - if right Ore condition is satisfied, $\sim$ is an equivalence relation on A_S.

Take $K = L$ and $b_i = 1_L ::\to L$. Then $(a_1, s_1) \sim (a_1, s_1)$. Hence we have reflexivity. Symmetry is clear. Now we use the right Ore condition (D1) to get transitivity. Suppose $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$.

INCLUDE DIAGRAM; L1, L2, L3 three triangles up from pair M, N connected by equivalences K and J. Note K,J to M so by (D1) get I.

INCLUDE DIAGRAM; by (D2) there exists a u from H to I, get all paths $H \to M$ and $H \to N$ commute (CHECK AGAIN) and all paths ending in M are in S.

So we get $(a_1, s_1) \sim (a_3, s_3)$.

So we can quotient $A \times S$ by $\sim$ to define the set A_S.

We define composition of two morphisms $q_1 \in A_S(M_0, M_1), q_2 \in A_S(M_1, M_2)$ by choosing representatives $(a_i, s_i) \in A \times S(M_{i-1}, M_i)$ such that $q_i = \overline{(a_i, s_i)}$.

INCLUDE DIAGRAM OF NEXT TO MORPHISMS; highlight a_1, s_2, fill box with c, u from K

(caption) By (D1), given a_1, s_2 we have c, u from K such that $a_1 \circ c = s_2 \circ u$.

Let $q_2 \circ q_1 = \overline{(a_2 \circ c, s_1 \circ u)}$. We now to check this is well-defined:

Suppose $q_i = \overline{(a_i', s_i')}$ and choose u', c'. We must show that $\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a_2' \circ c', s_1' \circ u')}$.

Since $(a_i, s_i) \sim (a_i', s_i')$, there exist b_i, b_i' such that

INCLUDE DIAGRAM two triangles up from each pair $(M_0, M_1), (M_1, M_2)$ with tops $L_i, , L_i'$, connect each triangle pair top vertex with (b_i, b_i') pair J_i

INCLUDE DIAGRAM include K, K' roof up between (L1,L2) and (L1',L2') by (D1); include I_1 between J_1 and K_1 by (D1)

INCLUDE DIAGRAM include $\tilde{I_2}$ roof between K' and J2 by (D1)

INCLUDE DIAGRAM by (D2) there is a $\tilde{v} \in S$ such that $I2 \xrightarrow{\tilde{v}} \tilde{I}2$ and paths $I2 \to L2$ are equal

At this point, we've constructed this diagram so that is is all commutative.

INCLUDE DIAGRAM by (D1) choose $w \in A$ and $e \in S$from H to fill in $I_2 \to M_0 \leftarrow I_2$;

INCLUDE SAME DIAGRAM, with highlighting: NOTE, $H \to I_2 \to M_0 \in S$ but could fail to commute in paths $H \to L_1'$ and $H \to L2$.

INCLUDE DIAGRAM, with highligting: look at two paths in $H \to L_1'$, note satisfy left comp. w s_1' equal, so exists $w' \in S$ from H' for right composition

INCLUDE DIAGRAM: include H'' for same reasons from $H' \to L_2$ with w''.

Now all paths $H'' \to M_2$ are equal and all paths $H'' \to M)9$ are equal and in S.

INCLUDE FINAL DIAGRAM: i.e. $(a_2 \circ c, s_1 \circ u) \sim (a_i' \circ c', s_1' \circ u')$.

We are done with proving well-defined composition. Take the identity morphism to be $\overline{(1_M, 1_M)}$.

CHECK IDENTITY PROPE-RTY AND ASSOCIATIVITY OF COMPOSITION. (Identity is easy and associativity is messy, haven't quite gotten).

Define $Q : A \to A_s$ to be $Q(M) = M, Q(a) = \overline{(a, 1_M)}$ for $a : M \to N$. CHECK Q IS FUNCTOR.

We now verify that the axioms of right Ore localization are satisfied. We have (L1) satisfied by our definition of Q. We can easily check that the inverse of Q(s) is $\overline{(1, s)}$ to get (L2) and $\overline{(a, s)} = \overline{(a, 1)} \circ \overline{(1, s)}$ to get (L3). Lastly, we have $Q(a_1) = Q(a_2)$ means that $(a_1, 1) \sim (a_2, 1)$, so there exist $b_1, b_2 \in A$ such that $a_1 \circ b_1 = a_2 \circ b_2$ and $1 \circ b_1 = 1 \circ b_2 \in S$. Let $s = b_1 \in S$, get $a_1 \circ S = a_2 \circ S$.

□

5. Cohomological Functors Revisited, briefly

KEY Prop (11.2.1?): If H: K →M is a cohomological functor from a triangulated category K to an abelian category M and we consider $S = \{s \in K | H(T^i(s)) \text{ is invertible for all } i \in \mathbb{Z}\}$. Then S is a left and right denominator set.

SHORT PROOF. INCLUDE.

6. Localization of Triangulated categories:

SEEMS LIKE TIME FOR PROOFS, TIEING THINGS TOGETHER TO GET DERIVED CATEGORY WITH TRIANGULATED STRUCTURE.

KEY Thm (11.2.2) If S is the right denominator set associated to a cohomological functor on K, and (K_S, Q) is the right Ore localization, then K_S has a unique triangulated structure such that (i) Q is a triangulated functor (ii) if $F : K \to E$ is another triangulated functor such that $F(s)$ is invertible for all $s \in S$, then the localization $F_S : K_S \to E$ is triangulated.

BIG PROOF. But a lot of steps are using well-defined* step from above (see where). INCLUDE.

Note: Need to show K_S additive to even get triangulated. Skip proof, but mention (?).
Note: Where is linear category used?

NEED TO UNDERSTAND WHY LINEAR PART IS INCLUDED BETTER (AND IF I NEED TO INCLUDE IT TOO), AND WHICH PROOFS TO INCLUDE/EXPLAIN)

7. The Big Reveal...

STATE DEF OF DERIVED CATEGORY IN TERMS WE"VE JUST BUILT UP. MAYBE STILL NEED TO SHOW H^0 IS COHOMOLOGICAL FUNCTOR.