Dual curvature measures and the dual Minkowski problem

Yiming Zhao

Department of Mathematics
Tandon School of Engineering, New York University

Analysis & Probability seminar, Case Western Reserve University

Outline

- Introduce *dual curvature measures*.
  — background, what are they, why are they called "dual" curvature measures

Outline

- Introduce *dual curvature measures*.
  — background, what are they, why are they called "dual" curvature measures
- Introduce *the dual Minkowski problem*.

Outline

- Introduce *dual curvature measures*.
  — background, what are they, why are they called "dual" curvature measures
- Introduce *the dual Minkowski problem*.
- Solve *the dual Minkowski problem* — in some cases.

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

can extend h_K to $\mathbb{R}^n$ by making it homogeneous of degree 1.

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

  can extend h_K to $\mathbb{R}^n$ by making it homogeneous of degree 1.
- For K with $o \in \operatorname{int} K$, radial function $\rho_K : S^{n-1} \to \mathbb{R}$

$$\rho_K(u) = \max\{r > 0 : ru \in K\}.$$

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

  can extend h_K to $\mathbb{R}^n$ by making it homogeneous of degree 1.
- For K with $o \in \operatorname{int} K$, radial function $\rho_K : S^{n-1} \to \mathbb{R}$

$$\rho_K(u) = \max\{r > 0 : ru \in K\}.$$

Basic concepts and notations

- convex bodies K — compact convex subset of $\mathbb{R}^n$ with non-empty interior.
- support function $h_K : S^{n-1} \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$

  can extend h_K to $\mathbb{R}^n$ by making it homogeneous of degree 1.
- For K with $o \in \operatorname{int} K$, radial function $\rho_K : S^{n-1} \to \mathbb{R}$

$$\rho_K(u) = \max\{r > 0 : ru \in K\}.$$

  can extend ρ_K to $\mathbb{R}^n \setminus o$ by making it homogeneous of degree -1.

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\qquad$ $K + t \cdot L = \{x + ty : x \in K, y \in L\}$

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\qquad$ $K + t \cdot L = \{x + ty : x \in K, y \in L\}$
- Steiner formula

$$V(K + tB) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K) \qquad W_i(K) \text{ quermassintegral}$$

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\quad K + t \cdot L = \{x + ty : x \in K, y \in L\}$
- Steiner formula

$$V(K + tB) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K) \qquad W_i(K) \text{ quermassintegral}$$

- $(n-i)$-th quermassintegral $W_{n-i}(K)$

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\qquad K + t \cdot L = \{x + ty : x \in K, y \in L\}$
- Steiner formula

$$V(K + tB) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K) \qquad W_i(K) \text{ quermassintegral}$$

- $(n-i)$-th quermassintegral $W_{n-i}(K)$

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\qquad K + t \cdot L = \{x + ty : x \in K, y \in L\}$
- Steiner formula

$$V(K + tB) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K) \qquad W_i(K) \text{ quermassintegral}$$

- $(n-i)$-th quermassintegral $W_{n-i}(K)$
  in integral geometry, it is the average of projection areas:

$$W_{n-i}(K) = c \int_{G(n,i)} \mathcal{H}^i(P_\xi K) d\xi$$

The Brunn-Minkowski theory

The BM theory studies convex bodies and geometric invariants, measures associated with convex bodies.

- *volume* and *Minkowski combination*
  $V(K)$ $\qquad K + t \cdot L = \{x + ty : x \in K, y \in L\}$
- Steiner formula

$$V(K + tB) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K) \qquad W_i(K) \text{ quermassintegral}$$

- $(n-i)$-th quermassintegral $W_{n-i}(K)$
  in integral geometry, it is the average of projection areas:

$$W_{n-i}(K) = c \int_{G(n,i)} \mathcal{H}^i(P_\xi K) d\xi$$

  in differential geometry, integral of mean curvatures:

$$W_{n-i}(K) = c \int_{\partial K} H_{n-i-1}(K, x) dS(x)$$

The Brunn-Minkowski theory

- localized Steiner formula

The Brunn-Minkowski theory

- localized Steiner formula

The Brunn-Minkowski theory

- localized Steiner formula
  For $x \notin K$

The Brunn-Minkowski theory

- localized Steiner formula
  For $x \notin K$
  $p_K(x)$ — closest point to x in K

The Brunn-Minkowski theory

- localized Steiner formula
  For $x \notin K$
  $p_K(x)$ — closest point to x in K
  $d(K, x)$ — distance from x to K

The Brunn-Minkowski theory

- localized Steiner formula
  For $x \notin K$
  $p_K(x)$ — closest point to x in K
  $d(K, x)$ — distance from x to K
  $\nu_K(x) = \frac{x - p_K(x)}{|x - p_K(x)|}$ — a unit normal of K at $p_K(x)$

The Brunn-Minkowski theory

- localized Steiner formula
  For $x \notin K$
  $p_K(x)$ — closest point to x in K
  $d(K, x)$ — distance from x to K
  $\nu_K(x) = \frac{x - p_K(x)}{|x - p_K(x)|}$ — a unit normal of K at $p_K(x)$
  ω, η Borel subsets of S^{n-1}. Define

$$A_t(K, \omega) = \{x \in \mathbb{R}^n : 0 < d(K, x) \leq t \text{ and } \frac{p_K(x)}{|p_K(x)|} \in \omega\}$$
$$B_t(K, \eta) = \{x \in \mathbb{R}^n : 0 < d(K, x) \leq t \text{ and } \nu_K(x) \in \eta\}$$

The Brunn-Minkowski theory

- localized Steiner formula

  For $x \notin K$

  $p_K(x)$ — closest point to x in K

  $d(K, x)$ — distance from x to K

  $\nu_K(x) = \frac{x - p_K(x)}{|x - p_K(x)|}$ — a unit normal of K at $p_K(x)$

  ω, η Borel subsets of S^{n-1}. Define

  $$A_t(K, \omega) = \{x \in \mathbb{R}^n : 0 < d(K, x) \leq t \text{ and } \frac{p_K(x)}{|p_K(x)|} \in \omega\}$$

  $$B_t(K, \eta) = \{x \in \mathbb{R}^n : 0 < d(K, x) \leq t \text{ and } \nu_K(x) \in \eta\}$$

  Localized Steiner Formula [Schneider, 2014]:

  $$V(A_t(K, \omega)) = \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} t^{n-i} C_i(K, \omega) \qquad \text{— curvature measures}$$

  $$V(B_t(K, \eta)) = \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} t^{n-i} S_i(K, \eta) \qquad \text{— area measures}$$

The Brunn-Minkowski theory

In particular,
$C_0(K,\cdot)$ — Aleksandrov's integral curvature
$S_{n-1}(K,\cdot)$ — surface area measure

The Brunn-Minkowski theory

In particular,
$C_0(K,\cdot)$ — Aleksandrov's integral curvature
$S_{n-1}(K,\cdot)$ — surface area measure

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that*

$$\mu = S_{n-1}(K,\cdot)$$

In particular,
$C_0(K,\cdot)$ — Aleksandrov's integral curvature
$S_{n-1}(K,\cdot)$ — surface area measure

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that*

$$\mu = S_{n-1}(K,\cdot)$$

Ans: μ is not concentrated in any closed hemisphere and

$$\int_{S^{n-1}} v d\mu(v) = o.$$

In particular,
$C_0(K,\cdot)$ — Aleksandrov's integral curvature
$S_{n-1}(K,\cdot)$ — surface area measure

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that*

$$\mu = S_{n-1}(K,\cdot)$$

Ans: μ is not concentrated in any closed hemisphere and

$$\int_{S^{n-1}} v\,d\mu(v) = o.$$

Minkowski, Aleksandrov, Fenchel–Jessen, Cheng–Yau

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that $\mu = S_{n-1}(K, \cdot)$.*

The Brunn-Minkowski theory

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that $\mu = S_{n-1}(K, \cdot)$.*

Remark

- *when K is smooth, the density of $S_{n-1}(K, \cdot)$ is the reciporical Gauss curvature. In this case, the Minkowski problem is the problem of prescribing Gauss curvature.*

The Brunn-Minkowski theory

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that $\mu = S_{n-1}(K, \cdot)$.*

Remark

- *when K is smooth, the density of $S_{n-1}(K, \cdot)$ is the reciporical Gauss curvature. In this case, the Minkowski problem is the problem of prescribing Gauss curvature.*
- *also in the smooth case, when μ has a density f, the MP becomes the Monge-Ampère equation*

$$\det(h_{ij} + h\delta_{ij}) = f.$$

The Brunn-Minkowski theory

Problem

The Minkowski problem. *Given a non-zero finite Borel measure μ on S^{n-1}. Under what condition is there a K such that $\mu = S_{n-1}(K, \cdot)$.*

Remark

- *when K is smooth, the density of $S_{n-1}(K, \cdot)$ is the reciporical Gauss curvature. In this case, the Minkowski problem is the problem of prescribing Gauss curvature.*
- *also in the smooth case, when μ has a density f, the MP becomes the Monge-Ampère equation*

$$\det(h_{ij} + h\delta_{ij}) = f.$$

- *other Minkowski problems: Aleksandrov problem, L_p Minkowski problem — in particular, the logarithmic Minkowski problem [Böröczky-LYZ, JAMS '12]*

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s.

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s.

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, ’14], [Gardner, ’06].

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, ’14], [Gardner, ’06].
- many “dual” concepts and results — radial summation, dual mixed volume, intersection body, etc.

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric, $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$?

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric, $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$?

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric, $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$? — true when $n = 3, 4$, false when $n \geq 5$.

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric,
  $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$? — true when $n = 3, 4$, false when $n \geq 5$.
- no measure, no PDE

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric, $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$? — true when $n = 3, 4$, false when $n \geq 5$.
- no measure, no PDE

The dual Brunn-Minkowski theory

- introduced by Lutwak in 1970s. [Schneider, '14], [Gardner, '06].
- many "dual" concepts and results — radial summation, dual mixed volume, intersection body, etc.
- Busemann-Petty problem. K, L o-symmetric, $\mathcal{H}^{n-1}(K \cap H) \leq \mathcal{H}^{n-1}(L \cap H)$ for every hyperplane H that crosses the origin
  Is it true that $V(K) \leq V(L)$? — true when $n = 3, 4$, false when $n \geq 5$.
- no measure, no PDE — until [Huang-LYZ, ACTA 2016]

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$

  $\rho_{K \tilde{+} t \tilde{\cdot} L} = \rho_K + t\rho_L$

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$
  $\rho_{K \tilde{+} t \tilde{\cdot} L} = \rho_K + t\rho_L$
- dual Steiner formula

$$V(K \tilde{+} t \tilde{\cdot} B) = \sum_{i=0}^{n} t^{n-i} \binom{n}{i} \tilde{W}_{n-i}(K)$$

— dual quermassintegral

The dual Brunn-Minkowski theory

- radial combination $K\tilde{+}t\tilde{\cdot}L$
  $\rho_{K\tilde{+}t\tilde{\cdot}L} = \rho_K + t\rho_L$
- dual Steiner formula

$$V(K\tilde{+}t\tilde{\cdot}B) = \sum_{i=0}^{n} t^{n-i}\binom{n}{i}\tilde{W}_{n-i}(K)$$ — dual quermassintegral

- dual quermassintegral $\tilde{W}_{n-i}(K)$

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$

  $\rho_{K \tilde{+} t \tilde{\cdot} L} = \rho_K + t\rho_L$
- dual Steiner formula

$$V(K \tilde{+} t \tilde{\cdot} B) = \sum_{i=0}^{n} t^{n-i} \binom{n}{i} \tilde{W}_{n-i}(K) \qquad \text{— dual quermassintegral}$$

- dual quermassintegral $\tilde{W}_{n-i}(K)$

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$
  $\rho_{K \tilde{+} t \tilde{\cdot} L} = \rho_K + t\rho_L$
- dual Steiner formula

$$V(K \tilde{+} t \tilde{\cdot} B) = \sum_{i=0}^{n} t^{n-i} \binom{n}{i} \tilde{W}_{n-i}(K)$$

— dual quermassintegral

- dual quermassintegral $\tilde{W}_{n-i}(K)$
  in integral geometry, mean of intersection areas:
  $\tilde{W}_{n-i}(K) = c \int_{G(n,i)} \mathcal{H}^i(K \cap \xi) d\xi$.

The dual Brunn-Minkowski theory

- radial combination $K \tilde{+} t \tilde{\cdot} L$
  $\rho_{K \tilde{+} t \tilde{\cdot} L} = \rho_K + t\rho_L$
- dual Steiner formula

$$V(K \tilde{+} t \tilde{\cdot} B) = \sum_{i=0}^{n} t^{n-i} \binom{n}{i} \tilde{W}_{n-i}(K) \qquad \text{— dual quermassintegral}$$

- dual quermassintegral $\tilde{W}_{n-i}(K)$
  in integral geometry, mean of intersection areas:
  $\tilde{W}_{n-i}(K) = c \int_{G(n,i)} \mathcal{H}^i(K \cap \xi) d\xi$.
  has integral form: $\tilde{W}_{n-i}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^i(u) du = \frac{i}{n} \int_K |x|^{i-n} dx$.

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]
  For $x \in \mathbb{R}^n \setminus o$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]
  For $x \in \mathbb{R}^n \setminus o$
  $\tilde{p}_K(x) = \rho_K(x)x \in \partial K$ — the point on ∂K that shares the same direction as x.

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]
  For $x \in \mathbb{R}^n \setminus o$
  $\tilde{p}_K(x) = \rho_K(x)x \in \partial K$ — the point on ∂K that shares the same direction as x.
  $\tilde{d}(K, x) = |x - \tilde{p}_K(x)|$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]
  For $x \in \mathbb{R}^n \setminus o$
  $\tilde{p}_K(x) = \rho_K(x)x \in \partial K$ — the point on ∂K that shares the same direction as x.
  $\tilde{d}(K, x) = |x - \tilde{p}_K(x)|$
  $\tilde{d}(K, x) = 0$ if $x \in K$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]
  For $x \in \mathbb{R}^n \setminus o$
  $\tilde{p}_K(x) = \rho_K(x)x \in \partial K$ — the point on ∂K that shares the same direction as x.
  $\tilde{d}(K, x) = |x - \tilde{p}_K(x)|$
  $\tilde{d}(K, x) = 0$ if $x \in K$
  ω, η Borel subsets of S^{n-1}. Define

$$\tilde{A}_t(K, \eta) = \{x \in \mathbb{R}^n : 0 \leq \tilde{d}(K, x) \leq t \text{ and } K \text{ at } \tilde{p}_K(x) \text{ has outer normal in } \eta\}$$

$$\tilde{B}_t(K, \omega) = \{x \in \mathbb{R}^n : 0 \leq \tilde{d}(K, x) \leq t \text{ and } \frac{x}{|x|} \in \omega\}$$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]

$$V(\tilde{A}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{C}_i(K,\eta) \qquad \text{— dual curvature measure}$$

$$V(\tilde{B}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{S}_i(K,\omega) \qquad \text{— dual area measure}$$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]

$$V(\tilde{A}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{C}_i(K,\eta) \qquad \text{— dual curvature measure}$$

$$V(\tilde{B}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{S}_i(K,\omega) \qquad \text{— dual area measure}$$

- $\tilde{C}_0(K,\cdot)$ is “the same as” $C_0(K^*,\cdot)$

The dual Brunn-Minkowski theory

- localized dual Steiner formula [Huang-LYZ, ACTA 2016]

$$V(\tilde{A}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{C}_i(K,\eta) \qquad \text{— dual curvature measure}$$

$$V(\tilde{B}_t(K,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \tilde{S}_i(K,\omega) \qquad \text{— dual area measure}$$

- $\tilde{C}_0(K,\cdot)$ is "the same as" $C_0(K^*,\cdot)$
- $\tilde{C}_n(K,\cdot)$ is "the same as" cone volume measure (L_p surface area measure when $p=0$)

Dual curvature measures

- integral representation:

$$\tilde{C}_i(K, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(\eta)} \rho^i_K(u) du,$$

where α^*_K is very similar to inverse Gauss map.

Dual curvature measures

- integral representation:

$$\tilde{C}_i(K,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(\eta)} \rho^i_K(u)du,$$

  where α^*_K is very similar to inverse Gauss map.
- easily extends to all real numbers. (will use $\tilde{C}_q(K,\cdot)$)

The dual Minkowski problem

Problem

The dual Minkowski problem: *Given a nonzero finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist a K such that $\mu = \tilde{C}_q(K, \cdot)$?*

Problem

The dual Minkowski problem: *Given a nonzero finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist a K such that $\mu = \tilde{C}_q(K, \cdot)$?*

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = nh^{-1}(v)|\nabla_{S^{n-1}} h(v) + h(v)v|^{n-q} f(v)$$

The dual Minkowski problem

Problem

The dual Minkowski problem: *Given a nonzero finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist a K such that $\mu = \tilde{C}_q(K, \cdot)$?*

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = nh^{-1}(v)|\nabla_{S^{n-1}} h(v) + h(v)v|^{n-q} f(v)$$

Problem

The even dual Minkowski problem: *Given a nonzero even finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$?*

The dual Minkowski problem

Problem

The dual Minkowski problem: *Given a nonzero finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist a K such that $\mu = \tilde{C}_q(K, \cdot)$?*

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = nh^{-1}(v)|\nabla_{S^{n-1}} h(v) + h(v)v|^{n-q} f(v)$$

Problem

The even dual Minkowski problem: *Given a nonzero even finite Borel measure μ on S^{n-1} and $q \in \mathbb{R}$, under what condition(s) on μ does there exist an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$?*

$q = 0$ Aleksandrov problem, $q = n$ logarithmic Minkowski problem

Results

When $q < 0$, complete solution to the DMP

Theorem (Z., CVPDEs 2017)

Suppose μ is a non-zero finite Borel measure on S^{n-1} and $q < 0$. There exists a K such that $\mu = \tilde{C}_q(K, \cdot)$ iff μ is not concentrated in any closed hemisphere. Moreover, K is unique (if exists).

Results

When $0 < q < n$, even case

Results

When $0 < q < n$, even case
q-th subspace mass inequality (I):
We say that μ satisfies the q-th subspace mass inequality, if
when $1 < q < n$

$$\frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < 1 - \frac{(n-i)(q-1)}{(n-1)q},$$

for each $i = 1, \cdots, n-1$ and each i-dim subspace ξ_i;

Results

When $0 < q < n$, even case
q-th subspace mass inequality (I):
We say that μ satisfies the q-th subspace mass inequality, if
when $1 < q < n$

$$\frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < 1 - \frac{(n-i)(q-1)}{(n-1)q},$$

for each $i = 1, \cdots, n-1$ and each i-dim subspace ξ_i;
when $0 < q \leq 1$

μ is not concentrated in any subspheres.

Results

Theorem (Huang-LYZ, ACTA 2016)

Suppose μ is a nonzero even finite Borel measure on S^{n-1} and $0 < q < n$. If μ satisfies the q-th subspace mass inequality (I), then there exists an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Results

Theorem (Huang-LYZ, ACTA 2016)

Suppose μ *is a nonzero even finite Borel measure on* S^{n-1} *and* $0 < q < n$. *If* μ *satisfies the q-th subspace mass inequality (I), then there exists an o-symm* K *such that* $\mu = \tilde{C}_q(K, \cdot)$.

Remark

- $0 < q \leq 1$, μ *not concentrated in any subsphere, obviously optimal.*

Results

Theorem (Huang-LYZ, ACTA 2016)

Suppose μ is a nonzero even finite Borel measure on S^{n-1} and $0 < q < n$. If μ satisfies the q-th subspace mass inequality (I), then there exists an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Remark

- $0 < q \leq 1$, *μ not concentrated in any subsphere, obviously optimal.*
- $1 < q < n$, *the condition is not necessary.*

Results

Theorem (Huang-LYZ, ACTA 2016)

Suppose μ is a nonzero even finite Borel measure on S^{n-1} and $0 < q < n$. If μ satisfies the q-th subspace mass inequality (I), then there exists an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Remark

- $0 < q \leq 1$, *μ not concentrated in any subsphere, obviously optimal.*
- $1 < q < n$, *the condition is not necessary.*

Results

Theorem (Huang-LYZ, ACTA 2016)

Suppose μ is a nonzero even finite Borel measure on S^{n-1} and $0 < q < n$. If μ satisfies the q-th subspace mass inequality (I), then there exists an o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Remark

- $0 < q \leq 1$, μ *not concentrated in any subsphere, obviously optimal.*
- $1 < q < n$, *the condition is not necessary.*

$$T_a = \left\{ x \in \mathbb{R}^n : x_1^2 + \cdots + x_i^2 \leq a^2 \text{ and } x_{i+1}^2 + \cdots + x_n^2 \leq 1 \right\}$$

Results

when $q \in (1, n)$

Results

when $q \in (1, n)$
q-th subspace mass inequality (II):

Results

when $q \in (1, n)$
q-th subspace mass inequality (II):
We say that μ satisfies the q-th subspace mass inequality, if

$$\frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < \begin{cases} \frac{i}{q}, & \text{if } i < q, \\ 1, & \text{if } i \geq q, \end{cases}$$

for each $i = 1, \cdots, n-1$ and each i-dim subspace ξ_i.

Results

when $q \in (1, n)$
q-th subspace mass inequality (II):
We say that μ satisfies the q-th subspace mass inequality, if

$$\frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < \begin{cases} \frac{i}{q}, & \text{if } i < q, \\ 1, & \text{if } i \geq q, \end{cases}$$

for each $i = 1, \cdots, n-1$ and each i-dim subspace ξ_i.

Theorem (Z., JDG, to appear)

Suppose μ is a non-zero even finite Borel measure on S^{n-1} and q is an integer in $(1, n)$. If μ satisfies the q-th subspace mass inequality (II), then there exists o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Results

when $q \in (1, n)$
q-th subspace mass inequality (II):
We say that μ satisfies the q-th subspace mass inequality, if

$$\frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < \begin{cases} \frac{i}{q}, & \text{if } i < q, \\ 1, & \text{if } i \geq q, \end{cases}$$

for each $i = 1, \cdots, n-1$ and each i-dim subspace ξ_i.

Theorem (Z., JDG, to appear)

Suppose μ is a non-zero even finite Borel measure on S^{n-1} and q is an integer in $(1, n)$. If μ satisfies the q-th subspace mass inequality (II), then there exists o-symm K such that $\mu = \tilde{C}_q(K, \cdot)$.

Theorem (Böröczky & Henk & Pollehn, JDG, to appear)

q-th subspace mass inequality (II) is also necessary for $q \in (1, n)$.

Variational method

1. find a maximization problem whose Euler-Lagrange equation implies that the maximizer is the solution to the DMP.

Variational method

1. find a maximization problem whose Euler-Lagrange equation implies that the maximizer is the solution to the DMP.
2. show the maximum can be obtained.

The maximization problem

Given μ even.

The maximization problem

Given μ even. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(h) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}([h]),$$

where $[h] = \{x \in \mathbb{R}^n : x \cdot v \le h(v), \forall v \in S^{n-1}\}$ is *the Wulff shape*.

The maximization problem

Given μ even. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h]),$$

where $[h] = \{x \in \mathbb{R}^n : x \cdot v \leq h(v), \forall v \in S^{n-1}\}$ is *the Wulff shape*.
Obviously Φ is homogeneous of degree 0.

The maximization problem

Given μ even. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h]),$$

where $[h] = \{x \in \mathbb{R}^n : x \cdot v \leq h(v), \forall v \in S^{n-1}\}$ is *the Wulff shape*. Obviously Φ is homogeneous of degree 0.

Problem (The maximization problem)

$$\sup\{\Phi(h) : h \in C_e^+(S^{n-1})\}$$

The maximization problem

Given μ even. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(h) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h]),$$

where $[h] = \{x \in \mathbb{R}^n : x \cdot v \le h(v), \forall v \in S^{n-1}\}$ is *the Wulff shape*. Obviously Φ is homogeneous of degree 0.

Problem (The maximization problem)

$$\sup\{\Phi(h) : h \in C_e^+(S^{n-1})\}$$

Lemma

If a maximizer exists, then it must be the support function of some o-symm convex body.

The maximization problem

Given μ even. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(h) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h(v)d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}([h]),$$

where $[h] = \{x \in \mathbb{R}^n : x \cdot v \leq h(v), \forall v \in S^{n-1}\}$ is *the Wulff shape*. Obviously Φ is homogeneous of degree 0.

Problem (The maximization problem)

$$\sup\{\Phi(h) : h \in C_e^+(S^{n-1})\}$$

Lemma

If a maximizer exists, then it must be the support function of some o-symm convex body. Moreover, the maximizer (up to a dilation) is a solution to the DMP.

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\Big|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\bigg|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h])$.

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\Big|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h])$.

Proof to the lemma.

Suppose h_K is the maximizer.

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\bigg|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h])$.

Proof to the lemma.

Suppose h_K is the maximizer. Because Φ is homogeneous of degree 0, we may dilate K so that $\tilde{W}_{n-q}(K) = |\mu|$.

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\Big|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v)\, d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h(v)\, d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}([h])$.

Proof to the lemma.

Suppose h_K is the maximizer. Because Φ is homogeneous of degree 0, we may dilate K so that $\tilde{W}_{n-q}(K) = |\mu|$.
For arbitrary even cont. f, consider $h_t = h_K e^{tf}$.

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\Big|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}([h])$.

Proof to the lemma.

Suppose h_K is the maximizer. Because Φ is homogeneous of degree 0, we may dilate K so that $\tilde{W}_{n-q}(K) = |\mu|$.
For arbitrary even cont. f, consider $h_t = h_K e^{tf}$.

$$0 = \frac{\partial}{\partial t}\Big|_{t=0} \Phi(h_t)$$

The maximization problem

Variational formula [Huang-LYZ, ACTA 2016]: convex body K, continuous function f on the sphere

$$\frac{\partial}{\partial t}\bigg|_{t=0} \log \tilde{W}_{n-q}([h_K e^{tf}]) = \frac{q}{\tilde{W}_{n-q}(K)} \int_{S^{n-1}} f(v) d\tilde{C}_q(K, v).$$

Recall $\Phi(h) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h(v) d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}([h])$.

Proof to the lemma.

Suppose h_K is the maximizer. Because Φ is homogeneous of degree 0, we may dilate K so that $\tilde{W}_{n-q}(K) = |\mu|$.
For arbitrary even cont. f, consider $h_t = h_K e^{tf}$.

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\bigg|_{t=0} \Phi(h_t) \\ &= -\frac{1}{|\mu|}\int_{S^{n-1}} f(v) d\mu(v) + \frac{1}{\tilde{W}_{n-q}(K)}\int_{S^{n-1}} f(v) d\tilde{C}_q(K, v). \end{aligned}$$

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K)$.

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K)$.

sketch of the proof.

Suppose o-symm Q_l is a maximizing sequence.

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K).$

sketch of the proof.

Suppose o-symm Q_l is a maximizing sequence.Since Φ is homogeneous of degree 0, assume diameter of each Q_l is 1.

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K)$.

sketch of the proof.

Suppose o-symm Q_l is a maximizing sequence.Since Φ is homogeneous of degree 0, assume diameter of each Q_l is 1.
By Blaschke's selection theorem, (taking a subsequence) we may assume $Q_l \to Q_0$.

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K)$.

sketch of the proof.

Suppose o-symm Q_l is a maximizing sequence.Since Φ is homogeneous of degree 0, assume diameter of each Q_l is 1.
By Blaschke's selection theorem, (taking a subsequence) we may assume $Q_l \to Q_0$.
In general Q_0 is compact, convex, o-symm.

Existence of maximizer

Problem (The maximization problem)

$$\sup\{\Phi(h_K) : K \in \mathcal{K}_e^n\},$$

where $\Phi(h_K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K d\mu + \frac{1}{q}\log \tilde{W}_{n-q}(K)$.

sketch of the proof.

Suppose o-symm Q_l is a maximizing sequence.Since Φ is homogeneous of degree 0, assume diameter of each Q_l is 1.
By Blaschke's selection theorem, (taking a subsequence) we may assume $Q_l \to Q_0$.
In general Q_0 is compact, convex, o-symm.
If we can further show $o \in \operatorname{int} Q_0$, or $Q_0 \in \mathcal{K}_e^n$, then done (by continuity of Φ). □

Existence of maximizer

- Show $o \in \operatorname{int} Q_0$

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise.

Existence of maximizer

- Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace.

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

John ellipsoid E_l of Q_l, $E_l \subset Q_l \subset \sqrt{n} E_l$.

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

John ellipsoid E_l of Q_l, $E_l \subset Q_l \subset \sqrt{n}E_l$.

Say

$$E_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1\},$$

where $0 < a_{1l} \leq a_{2l} \leq \cdots \leq a_{nl} \leq 1$.

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

John ellipsoid E_l of Q_l, $E_l \subset Q_l \subset \sqrt{n}E_l$.

Say

$$E_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1\},$$

where $0 < a_{1l} \leq a_{2l} \leq \cdots \leq a_{nl} \leq 1$.

Assume $a_{1l}, \cdots, a_{kl} \to 0$ and $a_{k+1,l}, \cdots, a_{nl} \to> 0$

Existence of maximizer

• Show $o \in \text{int}\, Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

John ellipsoid E_l of Q_l, $E_l \subset Q_l \subset \sqrt{n} E_l$.

Say

$$E_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1\},$$

where $0 < a_{1l} \leq a_{2l} \leq \cdots \leq a_{nl} \leq 1$.

Assume $a_{1l}, \cdots, a_{kl} \to 0$ and $a_{k+1,l}, \cdots, a_{nl} \to > 0$

Let

$$T_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{kl}|^2}{a_{kl}^2} \leq 1$$
$$\text{and } |x \cdot e_{k+1,l}|^2 + \cdots + |x \cdot e_{nl}|^2 \leq 1\}.$$

Existence of maximizer

• Show $o \in \operatorname{int} Q_0$

Assume otherwise. Then Q_0 is contained in some lower dimensional subspace. At the end of the day, show $\Phi(h_{Q_l}) \to -\infty$.

John ellipsoid E_l of Q_l, $E_l \subset Q_l \subset \sqrt{n}E_l$.

Say

$$E_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1\},$$

where $0 < a_{1l} \leq a_{2l} \leq \cdots \leq a_{nl} \leq 1$.

Assume $a_{1l}, \cdots, a_{kl} \to 0$ and $a_{k+1,l}, \cdots, a_{nl} \to > 0$

Let

$$T_l = \{x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{kl}|^2}{a_{kl}^2} \leq 1$$
$$\text{and } |x \cdot e_{k+1,l}|^2 + \cdots + |x \cdot e_{nl}|^2 \leq 1\}.$$

Obviously $E_l \subset T_l$.

Hence $E_l \subset Q_l \subset \sqrt{n} T_l$.

Hence $E_l \subset Q_l \subset \sqrt{n} T_l$. This implies

$$\Phi(h_{Q_l}) \leq -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{E_l}(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}(T_l) + c.$$

Hence $E_l \subset Q_l \subset \sqrt{n} T_l$. This implies

$$\Phi(h_{Q_l}) \leq -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{E_l}(v) d\mu(v) + \frac{1}{q} \log \tilde{W}_{n-q}(T_l) + c.$$

Estimate the first term by "partition of the sphere" and "subspace mass inequality"

Hence $E_l \subset Q_l \subset \sqrt{n}T_l$. This implies

$$\Phi(h_{Q_l}) \leq -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{E_l}(v)d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}(T_l) + c.$$

Estimate the first term by “partition of the sphere” and “subspace mass inequality”

$$\text{first term} \leq -\left(\frac{1}{q} - \varepsilon_0\right)\log(a_{1l}\cdots a_{kl}) + c$$

Hence $E_l \subset Q_l \subset \sqrt{n}T_l$. This implies

$$\Phi(h_{Q_l}) \leq -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{E_l}(v)d\mu(v) + \frac{1}{q}\log \tilde{W}_{n-q}(T_l) + c.$$

Estimate the first term by "partition of the sphere" and "subspace mass inequality"

$$\text{first term} \leq -\left(\frac{1}{q} - \varepsilon_0\right)\log(a_{1l}\cdots a_{kl}) + c$$

Estimate the second term by using "spherical coordinates"

$$\text{second term} \leq \frac{1}{q}\log(a_{1l}\cdots a_{kl}) + c$$

Remaining problems

- What about $q > n$?

Remaining problems

- What about $q > n$?
- What about non-even case?

Remaining problems

- What about $q > n$?
- What about non-even case?
- Uniqueness when $q > 0$?

Remaining problems

- What about $q > n$?
- What about non-even case?
- Uniqueness when $q > 0$?
- Regularity?

Remaining problems

- What about $q > n$?
- What about non-even case?
- Uniqueness when $q > 0$?
- Regularity?
- Everything that can be done with surface area measure.