

Geometric quantities and their Minkowski problems

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Basics and Notations

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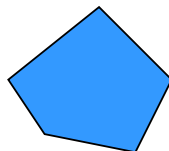
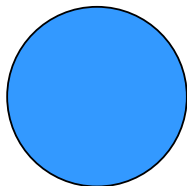
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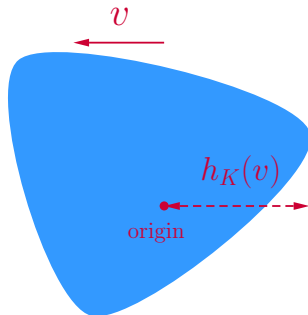


Not a convex body!

Basics and Notations

- support function $h_K : S^1 \rightarrow \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$



The Brunn-Minkowski theory

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- Steiner formula

$$A(K + t \cdot D) = \binom{2}{0} w_0(K) + \binom{2}{1} w_1(K)t + \binom{2}{2} w_2(K)t^2,$$

where D is the unit disk in \mathbb{R}^2 .

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$$A(K + t \cdot L) = \int_{\mathbb{R}^2} \chi_{K + t \cdot L}(x) dx = \int_{\mathbb{R}^2} \chi_K(x) + t \chi_L(x) dx = A(K) + t A(L)$$
$$K + t \cdot L = \{x + ty : x \in K, y \in L\}$$

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- Things get more interesting for Euclidean spaces of dimensions beyond 2. In dimension n , you get $n + 1$ quermassintegrals!

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Let's deviate to dimension n *only* for a minute!

- Steiner formula in \mathbb{R}^n

$$V(K + tB) = \sum_{i=0}^n t^i \binom{n}{i} W_i(K) \quad W_i(K) \text{ quermassintegral}$$

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in integral geometry, it is the average of projection areas:

$$W_{n-i}(K) = c \int_{G(n,i)} \mathcal{H}^i(P_\xi K) d\xi$$

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in differential geometry, integral of mean curvatures:

$$W_{n-i}(K) = c \int_{\partial K} H_{n-i-1}(K, x) dS(x)$$

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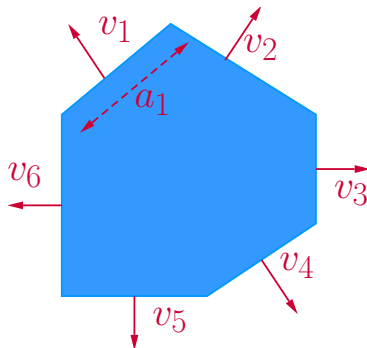
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Generates two families of geometric measures: *area measures*, *curvature measures*

They replace curvatures that are only defined for surfaces that are smooth enough.

Surface area measure for polygons

Suppose P is a polygon in \mathbb{R}^2 . S_P is the surface area measure of P .



S_P encodes the set of all outer unit normals v_1, v_2, \dots, v_N of P and attach to each v_i a positive number a_i which is equal to the length of the edge with outer normal v_i .

The Minkowski problem

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- Ans: As long as $\text{span}\{v_1, \dots, v_N\} = \mathbb{R}^2$ and

$$a_1 v_1 + \dots + a_N v_N = 0,$$

P will exist.

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- Proof is mostly just Calculus.

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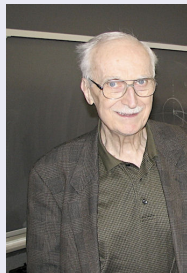
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- *Well known problem both in PDE and differential geometry.*

The Minkowski problem

Remark

E. Calabi



The importance of the Minkowski problem and its solution is to be felt both in differential geometry and elliptic partial differential equations, on either count going far beyond the impact that the literal statement may have. From the geometric point of view it is the Rosetta Stone, from which several other related problems can be solved.

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- *In PDE, the following Monge-Ampère equation:*

$$\det(h_{ij} + h\delta_{ij}) = f,$$

where unknown is h . Here (h_{ij}) is the Hessian matrix on the unit sphere, δ_{ij} is the Kronecker delta, and f is a given function.

Modern development of the BMt

The L_p Brunn-Minkowski theory

Firey, Lutwak

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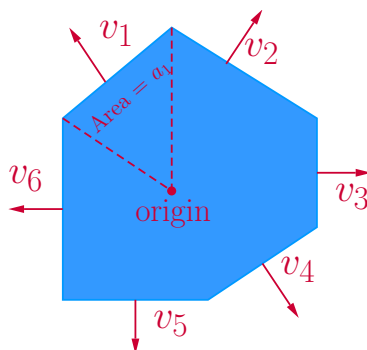
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$$V_K = \frac{1}{2} h_K \cdot S_K$$



For polygons, $V_P \rightsquigarrow v_1, \dots, v_N$ and a_1, \dots, a_N .

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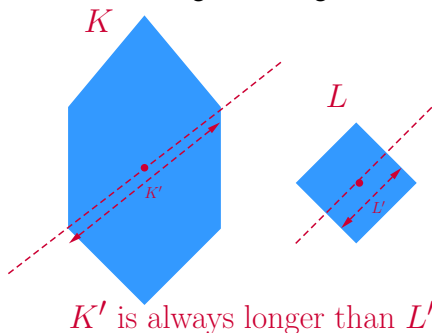
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- many “dual” concepts and results — radial summation, dual quermassintegral, **intersection body**, etc.
- Busemann-Petty problem. K, L o-symmetric, if you slice both of them using the same line through the origin



length of K' is always larger than length of L' , no matter how you slice them. Is it true K has greater area?

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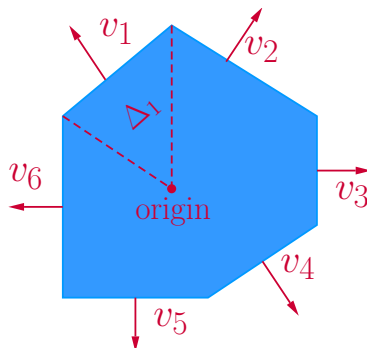
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 \rightsquigarrow **dual curvature measures** and **dual area measures**

Dual curvature measures for polygons

For each real number q , the q -th dual curvature measure $\tilde{C}_q(P, \cdot)$
 $\rightsquigarrow v_1, \dots, v_N$ the set of outer unit normals, and a_1, \dots, a_N .



Here $a_i = \frac{q}{2} \int_{\Delta_i} |x|^{q-2} dx$

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The discrete dual Minkowski problem. Let $q \in \mathbb{R}$. Given unit vectors v_1, \dots, v_N and positive numbers a_1, \dots, a_N , under what conditions is there a polygon P such that P has v_1, \dots, v_N as outer unit normals and a_1, \dots, a_N as the quantity defined on the previous slide? Moreover, if P exists, is it unique?

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- PDE:

$$\det(h_{ij}(v) + h(v)\delta_{ij}) = nh^{-1}(v)|\nabla_{S^{n-1}}h(v) + h(v)v|^{n-q}f(v).$$

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Theorem (Z., CVPDEs 2017)

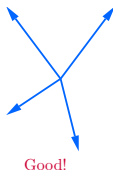
Suppose $q < 0$. There is a solution P to the discrete dual Minkowski problem if and only if v_1, \dots, v_n are not contained in any closed half plane. Moreover, P is unique (if exists).

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