Geometric quantities and their Minkowski problems

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Basics and Notations

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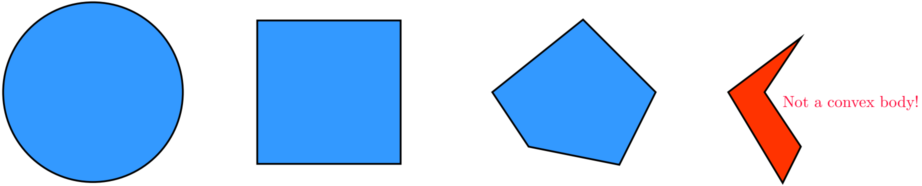
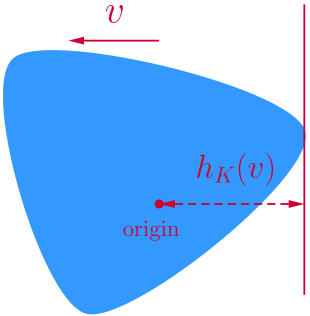


Basics and Notations

- support function $h_K : S^1 \to \mathbb{R}$

$$h_K(v) = \max\{x \cdot v : x \in K\}$$



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- Things get more interesting for Euclidean spaces of dimensions beyond 2. In dimension n, you get $n+1$ quermassintegals!

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Let's deviate to dimension *n only* for a minute!

- Steiner formula in $\mathbb{R}^n$

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$$W_{n-i}(K)=c\int_{G(n,i)} \mathcal{H}^i(P_\xi K)d\xi$$

  in differential geometry, integral of mean curvatures:

$$W_{n-i}(K)=c\int_{\partial K} H_{n-i-1}(K,x)dS(x)$$

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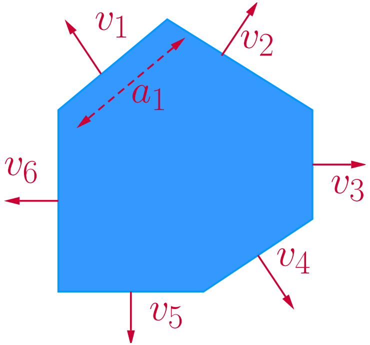
Generates two families of geometric measures: *area measures*, *curvature measures*

They replace curvatures that are only defined for surfaces that are smooth enough.

Surface area measure for polygons

Suppose P is a polygon in $\mathbb{R}^2$. S_P is the surface area measure of P.



S_P encodes the set of all outer unit normals $v_1, v_2, \ldots, v_N$ of P and attach to each v_i a positive number a_i which is equal to the length of the edge with outer normal v_i.

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The discrete Minkowski problem. *Given unit vectors $v_1, \ldots, v_N$ and positive numbers $a_1, \ldots, a_N$,*

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- *Ans: As long as* $\text{span}\{v_1, \dots, v_N\} = \mathbb{R}^2$ *and*

$$a_1 v_1 + \cdots + a_N v_N = o,$$

P will exist.

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- *Proof is mostly just Calculus.*

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- *Well known problem both in PDE and differential geometry.*

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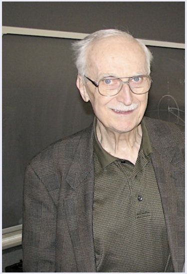
E. Calabi



The importance of the Minkowski problem and its solution is to be felt both in differential geometry and elliptic partial differential equations, on either count going far beyond the impact that the literal statement may have. From the geometric point of view it is the Rosetta Stone, from which several other related problems can be solved.

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- *In PDE, the following Monge-Ampère equation:*

$$det(h_{ij} + h\delta_{ij}) = f,$$

where unkown is h. Here (h_{ij}) is the Hessian matrix on the unit sphere, δ_{ij} is the Kronecker delta, and f is a given function.

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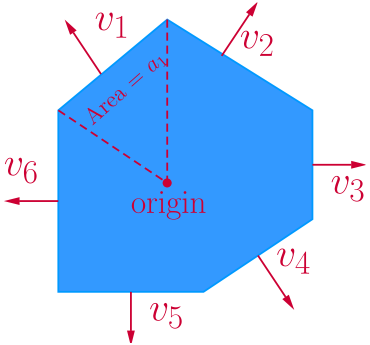

For polygons, $V_P \leftrightsquigarrow v_1, \ldots, v_N$ and $a_1, \ldots a_N$.

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- *When the given data is even, or in another word, when $\pm v_1, \dots, \pm v_N$, solved in [Böröczky-Lutwak-Yang-Zhang; JAMS 2012] for arbitrary dimension.*

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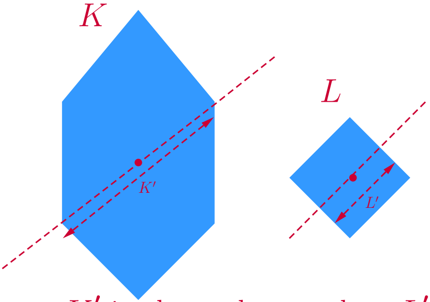
- Busemann-Petty problem. K, L o-symmetric, if you slice both of them using the same line through the origin



K' is always longer than L'

length of K' is always larger than length of L', no matter how you slice them. Is it true K has greater area?

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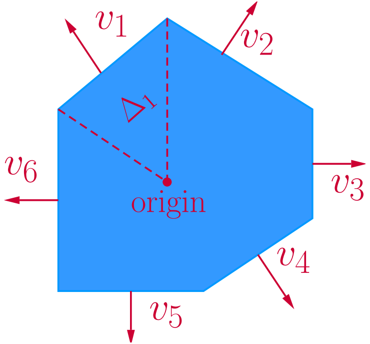
  $\rightsquigarrow$ dual curvature measures and dual area measures

Dual curvature measures for polygons

For each real number q, the q-th dual curvature measure $\widetilde{C}_q(P, \cdot)$
$\leftrightsquigarrow$ $v_1, \cdots, v_N$ the set of outer unit normals, and $a_1, \cdots, a_N$.



Here $a_i = \frac{q}{2} \int_{\Delta_i} |x|^{q-2} dx$

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- *PDE:*

$$det(h_{ij}(v) + h(v)\delta_{ij}) = nh^{-1}(v)|\nabla_{S^{n-1}} h(v) + h(v)v|^{n-q} f(v).$$

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Theorem (Z., CVPDEs 2017)

Suppose $q < 0$. There is a solution P to the discrete dual Minkowski problem if and only if $v_1, \ldots, v_n$ are not contained in any half plane. Moreover, P is unique (if exists).

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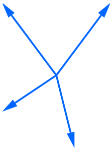
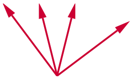


Good!



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