DEEP LEARNING IN COMPUTER VISION - EXERCISE 1

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Exercise 2.5
Let $x_1, \dots, x_L$ be random samples drawn from a uniform distribution on the interval $[0, \theta]$.

(a) $\overline{x} = \sum_{l=1}^{L} x_l$ is a biased estimator of θ.
(b) $2\overline{x}$ is an unbiased estimator of θ.
(c) The standard error for the estimator $2\overline{x}$ is given by $\sqrt{\frac{4}{3L}\theta^2 + \frac{L-1}{L}\theta^2 - \theta^2}$, which tends to zero for $L \to \infty$.
(d) Another biased estimator for θ is given by $\hat{x} := \max\{x_1, \dots, x_L\}$. Its standard error is given by $\sqrt{\frac{L}{L+2}\theta^2 - \left(\frac{L}{L+1}\theta\right)^2}$ which tends to zero as $L \to \infty$.

Proof.

(a) Using

$$P(x_l < x) = \begin{cases} 1, & x \geq \theta, \\ \frac{x}{\theta}, & 0 \leq x \leq \theta, \\ 0, & x < 0, \end{cases}$$

together with the linearity of the expectation, we see that

$$\mathbb{E}[\overline{x}] = \frac{1}{L}\sum_{l=1}^{L} \mathbb{E}[x_l] = \frac{1}{L}\sum_{l=1}^{L} \int_0^\theta \frac{x}{\theta}\, dx = \frac{1}{L}\sum_{l=1}^{L} \left[\frac{x^2}{2\theta}\right]_0^\theta = \frac{\theta}{2} \neq \theta,$$

whence $\overline{x}$ is a biased estimator of θ.
(b) Obviously, since $\mathbb{E}[2\overline{x}] = \theta$ by (a) and the linearity of the expectation, the estimator $2\overline{x} = \frac{2}{L}\sum_{l=1}^{L} x_l$ is an unbiased estimator for θ.

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(c) We calculate the variance $\mathrm{Var}(2\overline{x})$. Using that x_k and x_l are independent for $k \neq l$, we obtain

$$\begin{aligned}
\mathrm{Var}(2\overline{x}) &= \mathbb{E}\left[(2\overline{x} - \mathbb{E}[2\overline{x}])^2\right] \\
&= \mathbb{E}\left[(2\overline{x} - \theta)^2\right] \\
&= \mathbb{E}\left[4\overline{x}^2 4\overline{x}\theta + \theta^2\right] \\
&= 4\mathbb{E}[\overline{x}^2] - 2\theta^2 + \theta^2 \\
&= \frac{4}{L^2}\left(\sum_{l=1}^{L}\mathbb{E}[x_l^2] + \sum_{k\neq l}\mathbb{E}[x_k x_l]\right) - \theta^2 \\
&= \frac{4}{L^2}\left(\sum_{l=1}^{L}\mathbb{E}[x_l^2] + \sum_{k\neq l}\mathbb{E}[x_k]\mathbb{E}[x_l]\right) - \theta^2 \\
&= \frac{4}{L^2}\left(\sum_{l=1}^{L}\int_0^\theta \frac{x^2}{\theta}\,dx + \sum_{k\neq l}\frac{\theta^2}{4}\right) - \theta^2 \\
&= \frac{4\theta^2}{3L} + \frac{L(L-1)}{L^2}\theta^2 - \theta^2 \\
&= \frac{4\theta^2}{3L} + \frac{(L-1)}{L}\theta^2 - \theta^2.
\end{aligned}$$

Here, we have used that $\sum_{k\neq l} 1 = 2 \cdot \frac{L(L-1)}{2} = L(L-1)$. Letting $L \to \infty$ we get

$$\mathrm{Var}(2\overline{x}) = \frac{4\theta^2}{3L} + \frac{L(L-1)}{L}\theta^2 - \theta^2 \to 0 + \theta^2 - \theta^2 = 0$$

and the assertion follows.

(d) For $\hat{x} = \max\{x_1, \dots, x_L\}$ it holds that

$$P(\hat{x} < x) = P(x_l < x, l = 1, \dots, L) = \begin{cases} 1, & x \geq \theta \\ \left(\frac{x}{\theta}\right)^L, & 0 \leq x \leq \theta, \\ 0, & x \leq 0. \end{cases}$$

Therefore, the density function of $\hat{x}$ is given by

$$f_{\hat{x}}(x) = \begin{cases} \frac{L}{\theta^L}x^{L-1}, & 0 \leq x \leq \theta, \\ 0, & x \notin [0, \theta]. \end{cases}$$

First, we show that $\hat{x}$ is an biased estimator for θ: It holds that

$$\mathbb{E}[\hat{x}] = \int_0^\theta \frac{L}{\theta^L}x^L\,dx = \frac{L}{\theta^L}\left[x^{L+1}\frac{1}{L+1}\right]_0^\theta = \frac{L}{L+1}\frac{\theta^{L+1}}{\theta^L} = \frac{L}{L+1}\theta \neq \theta,$$

whence $\hat{x}$ is biased (but the bias is getting smaller the larger L is). Again, we calculate the variance of $\hat{x}$ and we get

$$\begin{aligned}\operatorname{Var}(\hat{x}) &= \mathbb{E}\left[\left(\hat{x} - \frac{L}{L+1}\theta\right)^2\right] \\ &= \mathbb{E}[\hat{x}^2] - \left(\frac{L}{L+1}\theta\right)^2 \\ &= \int_0^\theta \frac{L}{\theta^L} x^{L+1}\,dx - \left(\frac{L}{L+1}\theta\right)^2 \\ &= \frac{L}{L+2}\theta^2 - \left(\frac{L}{L+1}\theta\right)^2.\end{aligned}$$

Taking $L \to \infty$ we obtain

$$\operatorname{Var}(\hat{x}) = \frac{L}{L+2}\theta^2 - \left(\frac{L}{L+1}\theta\right)^2 \to \theta^2 - \theta^2 = 0$$

and the assertion follows.

$\square$