10. Multi-objective Problem

Bi-criterion problems

multi-objective with two objectives

decision variable x lies in decision space X.
feasible region F is a subset of X.
objective space Y.

feasible region in objective space:
$$\{ (z_1(x), z_2(x), \cdots, z_n(x)) : x \in X \} \subseteq Y.$$

Non-inferior set.

$$\max_{x \in F} \{ z_1(x), z_2(x) \}$$

$z_1(x_1) > z_1(x_2)$, $z_2(x_1) \geq z_2(x_2)$
or
$z_1(x_1) \geq z_1(x_2)$, $z_2(x_1) > z_2(x_2)$

x_1 give a strictly better result than x_2 in at least one variable.

Then x_1 dominates x_2
x_2 is inferior to x_1

NIS (Non-inferior set): a set of point in the feasible region that are not inferior to any other point in F

points in set "Pareto-optimal"

Ideal and nadir solution

Suppose a multi-objective maximisation problem with feasible region $F \subseteq X$ and NIS $N \subseteq F$.

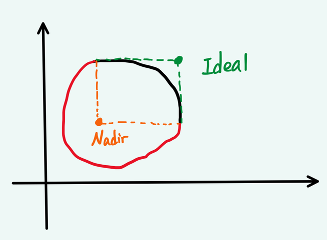
Ideal solution $z_i^* = \max\{ z_i(x) : x \in F \}$

ideal solution maximises each component over the given feasible region, regardless of feasibility.

$\hat{z} \leq z \leq z^*$

Nadir Solution $\hat{z}_i = \min\{ z_i(x) : x \in N \}$

nadir solution provides component-wise lower bound on the NIS.



Solution Method.

The Weighting method

- Quantify the relative importance of different objectives.

$z(\gamma) = (1-\gamma) z_1 + \gamma z_2$ with weighting variable $\gamma \in [0,1]$

For $z = (1-\gamma) z_1 + \gamma z_2 = \text{Const}$ is a line in objective space. (decision maker set γ)

determines the slope. $\frac{\gamma-1}{\gamma}$ $\gamma \in [0,1]$

For $\gamma = 0$, $z(0) = z_1$

increase γ, B remains optimal until C.

因为z_1比重减少, proportion of z_2 get greater

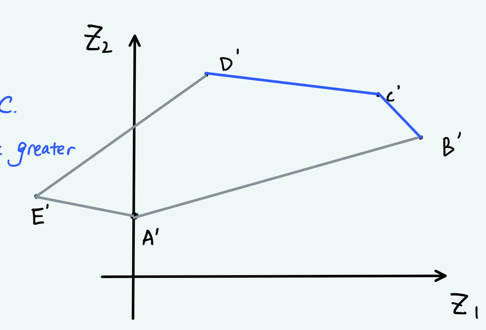
$14(1-\gamma) + 7\gamma = 12(1-\gamma) + 9\gamma$

$\gamma = \frac{1}{2}$. B.C are optimal

$12(1-\gamma) + 9\gamma = 4(1-\gamma) + 11\gamma$

$\gamma = \frac{4}{5}$, C.D are optimal

optimal solution: $\begin{cases} B, & \gamma \in [0, \frac{1}{2}] \\ C, & \gamma \in [\frac{1}{2}, \frac{4}{5}] \\ D, & \gamma \in [\frac{4}{5}, 1] \end{cases}$



Generalize,

multi-objective with objective $z_1, z_2, \cdots, z_k$ and weights $\gamma_1, \cdots, \gamma_k, \in [0,1]^k$.

$$z = \sum_{i=1}^{k} \gamma_i z_i.$$
$$\text{s.t. } \sum_{i=1}^{k} \gamma_i = 1$$

γ总是 k-1 维的.
$\gamma_1, \gamma_2, 1-\gamma_1-\gamma_2$

ε-constraints

for one or more of the objectives, there is a minimum acceptable value.

Consider a multi-objective problem:

$$\max_{x \in F} \{ z_1(x), \cdots, z_k(x) \}$$
s.t.
$g_1(x) \leq b_1$
$\vdots$
$g_m(x) \leq b_m$
$x \geq 0$

assign one objective a lower bound ε.

decision maker set ε.

the choice of ε_i may yield an infeasible LP.

$$\max_{x \in F_1} \{ z_2(x), \cdots, z_k(x) \}$$
s.t.
$g_1(x) \leq b_2$
$\vdots$
$g_m(x) \leq b_m$
$z_1(x) \geq \varepsilon_1$
$x \geq 0$

repeat until 仅剩一个变量

$$\max_{x \in F_{k-1}} \{ z_k(x) \}$$
s.t.
$g_1(x) \leq b_1$
$\vdots$
$g_m(x) \leq b_m$
$z_1(x) \geq \varepsilon_1$
$\vdots$
$z_{k-1}(x) \geq \varepsilon_{k-1}$
$x \geq 0$

Goal programming

- Quantify the cost of an objective not meeting a particular value.

Consider a multi-objective problem:

$$\max_{x \in F} \{ z_1(x), \cdots, z_k(x) \}$$
s.t.
$g_1(x) \leq b_1$
$\vdots$
$g_m(x) \leq b_m$
$x \geq 0$

choose a target value for each z_i, called goal G_i, $z_i(x) = G_i$
add slack and surplus variable.

$z_i(x) = G_i + d_i^+ - d_i^-$

d_i^+: how much greater the goal
d_i^- (≥ 0): less than goal.

α_i^+: penalty for overshooting
α_i^-: penalty for undershooting

minimize the total penalty.

$$\min \left\{ \sum_{i=1}^{k} (\alpha_i^+ d_i^+ + \alpha_i^- d_i^-) \right\}$$
s.t.
$z_1(x) = G_1 + d_1^+ - d_1^-$
$\vdots$
$z_k(x) = G_k + d_k^+ - d_k^-$
$g_1(x) \leq b_1$
$\vdots$
$g_m(x) \leq b_m$
$x_i, d_i^+, d_i^- \geq 0$

当我们希望 $z_i \geq G_i$ 时,可以 set penalty $\alpha_i^+ = 0$