UNIVERSITY OF COPENHAGEN
FACULTY OF SOCIAL SCIENCES

Thesis

Dynamic Trading with a GARCH volatility model

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Keystrokes: , normal pages:

Abstract

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Division of Labor

Casper Nordblom Eneqvist (xfj248):
Section
Niels Eriksen (dvp957):
Section
Mutual Work:
Section

Contents

1 Introduction

1.1 Literature review

Introduce the fields of Modern Portfolio Theory, dynamic trading strategies and GARCH volatility modelling

I suggest: start with the problem formulation, then explain the literature that is relevant – no need for a history of GARCH papers!

1. Markowitz
2. Engle
3. Pedersen 2012

motivate why this is an interesting problem

1.2 Problem formulation

What is the optimal dynamic portfolio trading strategy for an investor with a long horizon where volatility is modelled as a GARCH process?

Sub questions: Why should volatility be modelled with a multivariate GARCH model? What is the difference between the optimal Markowitz portfolio and the optimal MGARCH portfolio in a static setting?

Questions for Stefan:

We have decided to move away from using one risk-free asset and one or more risky assets, and instead use the excess return $\tilde{r}_t = r_t - r_f$ of risky assets instead.

1. Can it be true that the mean, $\tilde{\mu}$, does not matter for standard mean-variance maximization with standard restrictions ($v_t'\mathbf{1} = 1$)? If so, is there any intuition behind this result? (see page 6 and equation (3) and page 18-19 and equation (23)+(24))

   (a) we found a paper [Levy and Markowitz, 1979] that states that mean-variance maximization and minimizing variance almost always gives the same result (weights). Is it thus smart to minimize the variance rather than maximizing the utility as the minimizing the variance is easier than maximizing the utility?

2. Why doesn't Heje 2012 have any constraints, for example that that the weights sum to 1? (see their paper equation (4)-(6))

   If you look at excess returns, there is no constraint. The reason is simply that you can think of excess returns as borrowing at the risk-free rate and investing everything in the risky asset. The only relevant question is now: How much should you borrow? As long as you are not faced to capital constraints nothing speaks against borrowing massively. By introducing a restriction you implicity assume that the bank only allows you to leverage for example 100% of your wealth.

3. Is our assumption that $\mathbb{E}(v_{t+1} - v_t) = 0$ reasonable? (see page 20-21)

   (a) and by extension does the result with trading costs included (equation 27) make sense?

   $\mu'\Sigma\iota = (\mu'\Sigma\iota)' = \iota'\Sigma'\mu = \iota'\Sigma\mu$

   holds because $\mu'\Sigma\iota$ is a scalar and $\Sigma' = \Sigma$

4. A small note: We used your slide set from AEF (specifically slides 1, page 33) to help us derive the efficient portfolio for static models, and we think we found an error when you define $D = \mu'\Sigma\iota$ and $D = \iota'\Sigma\mu$ but $\mu'\Sigma\iota \neq \iota'\Sigma\mu$. However, this does not change the result for w_{eff} (see equation (4) on page 8 and derivations above)

2 Modern portfolio theory

- Justification for ignoring expected returns and factor models in general. A central part of all portfolio optimization problems is the expected return μ. Forecasting the expected return is notoriously difficult and prone to high statistical uncertainty. Furthermore, predicting tomorrows returns based on information available today is also theoretically impossible according to EMH. However, this is what the factor models attempt to do, where one regresses so called factors on past asset returns.

I agree if you literally think of daily data. However, returns can be predictable at the long run by macroeconomic variables that proxy for the marginal rate of substitution. This does not directly invalidate the EMH because it may be impossible to hedge against a market downturn. In your setup with long-term perspectives that may play a role but I suggest you do not incorporate additional complexity!

- Discuss the actual objective function.
- Rule out intermediate consumption. But at the same time be clear that we care about intermediate returns and volatility.

2.1 Characterization of the investor and the asset markets

The market in this paper is characterized by many risky assets, S, and one risk-less asset, B always paying the risk-less rate r_f. The risky assets may be stocks, EFTs or other derivative and the risk-less assets may be thought of as a government bond like US treasuries with almost no risk. Additionally, investors pays no taxes and markets are perfectly liquid. Initially, the investor faces no trading cost, though this assumption will be relaxed.

Finally, investors care only about the return and variance of their portfolio returns. This is in line with the vast majority of investors being risk averse. This type of preference is called mean-variance utility which can be given by the following utility function

$$U_t(\tilde{r}_t) = \mathbb{E}[\tilde{r}_t] - \frac{\lambda}{2}\mathbb{V}[\tilde{r}_t] \tag{1}$$

where λ is the level of risk aversion and $\tilde{r}_t = r_t - r_f$ is excess returns over the return on the risk-free asset r_f. We implicitly include r_f in the utility function by defining the return in terms of excess returns to r_f. The daily excess portfolio return B_t is thus

normalized by what?

$$B_t = v'_{t-1}\tilde{r}_t$$

Sure B_t is useful if B is the risk-free asset?

with normalized weights v_{t-1} determined in the previous period. This portfolio will have mean excess return and variance

$$\mathbb{E}[B_t] = v'_{t-1}(\mathbb{E}[r_t] - r_f) = v'_{t-1}(\mu - r_f) = v'_{t-1}\tilde{\mu} \qquad \text{and} \qquad \mathbb{V}[B_t] = v'_{t-1}\Sigma v_{t-1}$$

This utility function essentially captures the trade-off of the risk-averse investor between maximizing returns $\mathbb{E}[B_t]$ and minimizing the variance $\frac{\lambda}{2}\mathbb{V}[B_t]$. Note that for $\lambda = 0$, $\frac{\lambda}{2}\mathbb{V}[B_t]$ disappears and thus the investor is risk neutral and will only care about maximizing his expected return regardless of the risk it may pose. This is a simple setup, which we will expand on later.

You can safely ignore risk-neutrality

discuss why this approach was taken. State other approaches/Utility functions

What do you mean?

2.2 Mean-Variance Approach

Start ud med [Markowitz, 1952] og redegør kort for hans idéer og lav en overgang til simplest case. Og samtidig giver kilden til problemet (Markowitz 1952)

2.2.1 Simplest case with two risky assets

Consider two risky assets, $S_{1,t}$ and $S_{2,t}$, the excess returns of which, $\tilde{r}_{1,t}$ and $\tilde{r}_{2,t}$, are assumed to be given by a bivariate Gaussian $N_2(\tilde{\mu}, \Sigma)$ distribution known to the investor. With the constraint that the allocation must sum to 1 which is automatically fulfilled in this simple case as the allocation into the other asset is given as $1 - v_t$.

The optimization problem for an investor maximizing his expected utility in period $t+1$

$$\max_{v_t}\{\mathbb{E}[U_{t+1}(B_t)]\} = \max_{v_t}\{\mathbb{E}[B_t] - \frac{\lambda}{2}\mathbb{V}[B_t]\} \tag{2}$$

with the portfolio excess return and variance given as

$$\mathbb{E}[B_t] = v_1\tilde{\mu}_1 + (1-v_1)\tilde{\mu}_2 \qquad \text{and} \qquad \mathbb{V}[B_t] = v_1^2\sigma_1^2 + (1-v_1)^2\sigma_2^2 + 2v_1(1-v_1)\sigma_1\sigma_2\rho$$

Setting up the Lagrangian

$$\mathcal{L}(v_1) = v_1\tilde{\mu}_1 + (1-v_1)\tilde{\mu}_2 - \frac{\lambda}{2}\big(v_1^2\sigma_1^2 + (1-v_1)^2\sigma_2^2 + 2v_1(1-v_1)\sigma_1\sigma_2\rho\big)$$

Taking first order conditions with respect to the weight v_1

$$\frac{\partial\mathcal{L}}{\partial v_1} = \tilde{\mu}_1 - \tilde{\mu}_2 - \frac{\lambda}{2}\big(2v_1\sigma_1^2 - 2(1-v_1)\sigma_2^2 + (2-4v_1)\sigma_1\sigma_2\rho\big) = 0 \Leftrightarrow$$

Solving for v_1 by isolating v_1 on one side and factorizing

$$\tilde{\mu}_1 - \tilde{\mu}_2 + \lambda\sigma_2^2 - \lambda\sigma_1\sigma_2\rho = \lambda v_1\big(\sigma_2^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\big) \Leftrightarrow v_1^* = \frac{\tilde{\mu}_1 - \tilde{\mu}_2 + \lambda\sigma_2^2 - \lambda\sigma_1\sigma_2\rho}{\lambda\big(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\big)}$$

Though this problem is trivial to solve, the intuition behind the choice of the investor is clearer than with p assets. An investor with mean-variance preferences choosing between two assets faces two trade-offs.

Firstly, the investor weighs the expected returns of both assets against each other, $\tilde{\mu}_1 - \tilde{\mu}_2$. Simply put, when the expected return of asset 1 increases, the investor increases his exposure to asset 1 and decreases his exposure to asset 2.

Secondly, the investors weighs the variance of asset 2, $\lambda\sigma_2^2$ and covariance, $\lambda\sigma_1\sigma_2\rho$ against the combined portfolio variance $\lambda\big(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\big)$. So, as the variance of asset 2 increases it becomes relatively more risky compared to asset 1 and thus the investor will increase the weight in asset 1. This trade-off logic is built in to all the following optimal solutions.

No rational investor would only consider a portfolio of two assets as the investor would be ignoring obvious possible diversification benefits. Thus, we now consider a problem with p assets for the investor to choose from.

2.2.2 Multiple Assets Problem

Now, the market is defined by many risky assets, the excess returns of which is given by a multivariate Gaussian distribution $N_p(\tilde{\mu}, \Sigma)$. $\tilde{\mu}$ is a $p \times 1$ vector of mean excess returns and Σ is the $p \times p$ covariance matrix of the p assets. v_t is now a $p \times 1$ vector of weights instead of a scalar. The optimization problem for an investor maximizing his expected utility in period $t+1$ is:

$$\max_{v_t}\{E[U_{t+1}(\tilde{r}_{t+1})]\} = \max_{v_t}\{E(\tilde{r}_{t+1}) - \frac{\lambda}{2}\mathbb{V}[\tilde{r}_{t+1}]\} =$$

$$\max_{v_t}\{v_t'\tilde{\mu} - \frac{\lambda}{2}v_t\Sigma v_t'\} \qquad \text{s.t.} \quad v_t'\mathbf{1} = 1$$

are you sure this is correct?

[Levy and Markowitz, 1979] states that this problem is equivalent to minimizing the variance of the portfolio when the investors utility is quadratic, which an investor with mean-variance utility has. This is a nice feature as the minimizing the variance is easier than maximizing the expected utility

$$\min_{v_t}\{\frac{1}{2}v_t\Sigma v_t'\} \qquad \text{s.t.} \quad v_t'\mathbf{1} = 1$$

To solve the problem, we set up the Lagrangian where the investor is limited by the normalized weights need to sum to 1 which is the constraint $v_t'\mathbf{1} = 1$ with a lagrangian multiplier κ_1

$$\mathcal{L}(v_t) = \frac{1}{2}v_t\Sigma v_t' - \kappa_1(v_t'\mathbf{1} - 1)$$

Taking first order conditions with respect to the weight, v_t

$$\frac{\partial \mathcal{L}}{\partial v_t} = \Sigma v_t - \kappa_1\mathbf{1} = 0$$

Solving for v_t yields

$$\kappa_1\mathbf{1} = \Sigma v_t \Leftrightarrow v_t = \Sigma^{-1}\mathbf{1}\kappa_1$$

The constraint requires that $v_t'\mathbf{1} = 1$, which can be used to solve for the Lagrangian multiplier κ

$$\begin{aligned} 1 &= v_t'\mathbf{1} = \mathbf{1}'v_t \\ 1 &= \mathbf{1}'\Sigma^{-1}\mathbf{1}\kappa_1 \\ \kappa_1 &= \frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \end{aligned}$$

Now, we insert the expression into the weights v_t

$$v_t = \Sigma^{-1}\mathbf{1}\kappa_1 = \Sigma^{-1}\mathbf{1}\frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$$

μ is irrelevant because it is not part of the objective

$$v_t = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = v_t^{\text{MVP}} \tag{3}$$

The Minimum Variance Portfolio (MVP) is the portfolio that has the lowest variance of all possible portfolios. To visualize this consider figure 1, where the Minimum Variance Portfolio (the green dot) has the lowest annualized volatility of all portfolios along the efficient frontier.

While reducing risk is beneficial to the investor, the low risk comes with lower returns. Thus, only a highly risk-averse investor will pick the MVP as they can achieve a higher Sharpe ratio and thus, higher utility by picking a slightly riskier portfolio with higher expected return. Consider the efficient portfolio (the red dot) in figure 1, where the investor earns a much higher annualized return for only a small increase in annualized volatility.

An investor who is seeking a higher return than the MVP offers can set a expected excess return target, $\tilde{\mu}^*$, that the investor's portfolio must achieve in expectation. This problem is in many ways similar to the MVP problem, but with an extra constraint, $v_t'\tilde{\mu} \geq \tilde{\mu}^*$.

$$\max_{v_t}\{v_t'\tilde{\mu} - \frac{\lambda}{2}v_t\Sigma v_t'\} \qquad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \text{and} \quad v_t'\tilde{\mu} \geq \tilde{\mu}^*$$

This problem is equivalent to minimizing the portfolio variance when the investors utility is quadratics as ours is. Thus, the investor essentially finds the portfolio that minimizes the variance i.e. risk given his return target, $\tilde{\mu}^*$.

$$\min_{v_t}\{\frac{1}{2}v_t\Sigma v_t'\} \qquad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \text{and} \quad v_t'\tilde{\mu} \geq \tilde{\mu}^*$$

We set up the Lagrangian for the problem with the two constraints and an extra Lagrangian multiplier κ_2.

$$\mathcal{L}(v_t) = \frac{1}{2}v_t\Sigma v_t' - \kappa_1(v_t'\mathbf{1} - 1) - \kappa_2(v_t'\tilde{\mu} - \tilde{\mu}^*)$$

Taking first order conditions with respect to the weight v_t

$$\frac{\partial \mathcal{L}}{\partial v_t} = \Sigma v_t - \kappa_1\mathbf{1} - \kappa_2\tilde{\mu} = 0$$

Solving for v_t yields

$$\kappa_1\mathbf{1} + \kappa_2\tilde{\mu} = \Sigma v_t \Leftrightarrow v_t = \Sigma^{-1}(\kappa_1\mathbf{1} + \kappa_2\tilde{\mu})$$

To ease notation going forward, we will quickly define the following:

$$A \equiv \mathbf{1}'\Sigma^{-1}\mathbf{1}, \quad B \equiv \mathbf{1}'\Sigma^{-1}\tilde{\mu}, \quad C \equiv \tilde{\mu}'\Sigma^{-1}\mathbf{1}, \quad D \equiv \tilde{\mu}'\Sigma^{-1}\tilde{\mu}$$

Finding κ_1 and κ_2 starting with their constraints

$$\begin{aligned}
1 &= v_t'\mathbf{1} \\
1 &= \mathbf{1}'v_t \\
1 &= \mathbf{1}'\Sigma^{-1}(\kappa_1\mathbf{1} + \kappa_2\tilde{\mu}) \\
1 &= \kappa_1\mathbf{1}'\Sigma^{-1}\mathbf{1} + \kappa_2\mathbf{1}'\Sigma^{-1}\tilde{\mu} \\
\kappa_1\mathbf{1}'\Sigma^{-1}\mathbf{1} &= 1 - \kappa_2\mathbf{1}'\Sigma^{-1}\tilde{\mu} \\
\kappa_1 &= \big(1 - \kappa_2\mathbf{1}'\Sigma^{-1}\tilde{\mu}\big)(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1} \\
\kappa_1 &= \frac{1 - \kappa_2 B}{A}
\end{aligned}$$

now for κ_2

$$\begin{aligned}
\tilde{\mu}^* =& v_t'\tilde{\mu} \\
\tilde{\mu}^* =& \tilde{\mu}'v_t \\
\tilde{\mu}^* =& \tilde{\mu}'\Sigma^{-1}(\kappa_1\mathbf{1} + \kappa_2\tilde{\mu}) \\
\tilde{\mu}^* =& \kappa_1\tilde{\mu}'\Sigma^{-1}\mathbf{1} + \kappa_2\tilde{\mu}'\Sigma^{-1}\tilde{\mu}) \\
\tilde{\mu}^* =& \kappa_1 C + \kappa_2 D
\end{aligned}$$

Insert expression for κ_1

$$\begin{aligned}
\tilde{\mu}^* =& \frac{1-\kappa_2 B}{A}C + \kappa_2 D \\
\tilde{\mu}^* - \frac{C}{A} =& \frac{-\kappa_2 BC}{A} + \kappa_2 D \\
\tilde{\mu}^* - \frac{C}{A} =& \kappa_2\Big(- \frac{BC}{A} + D\Big) \\
\kappa_2 =& \frac{\tilde{\mu}^* - C/A}{D - BC/A} = \frac{C - A\tilde{\mu}^*}{BC - AD}
\end{aligned}$$

Then the optimal weights are given as

$$\begin{aligned}
v_t^* &= \Sigma^{-1}\Big(\frac{1-\kappa_2 B}{A}\mathbf{1} + \kappa_2\tilde{\mu}\Big) \\
v_t^* &= \Sigma^{-1}\Big(\frac{1-\kappa_2\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\mathbf{1} + \kappa_2\tilde{\mu}\Big) \\
v_t^* &= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} + \Sigma^{-1}\Big(\kappa_2\tilde{\mu} - \kappa_2\frac{\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\mathbf{1}\Big) \\
v_t^* &= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} + \kappa_2\Big(\Sigma^{-1}\tilde{\mu} - \frac{\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\Big) \\
v_t^* &= v_t^{\text{MVP}} + \kappa_2\Big(\Sigma^{-1}\tilde{\mu} - \frac{\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\Big) \\
v_t^* &= v_t^{\text{MVP}} + \kappa_2\Big(\Sigma^{-1}\tilde{\mu} - \mathbf{1}'\Sigma^{-1}\tilde{\mu}\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Big)
\end{aligned}$$

thus resulting in

$$v_t^* = v_t^{\text{MVP}} + \kappa_2\Big(\Sigma^{-1}\tilde{\mu} - \mathbf{1}'\Sigma^{-1}\tilde{\mu}v_t^{\text{MVP}}\Big) \tag{4}$$

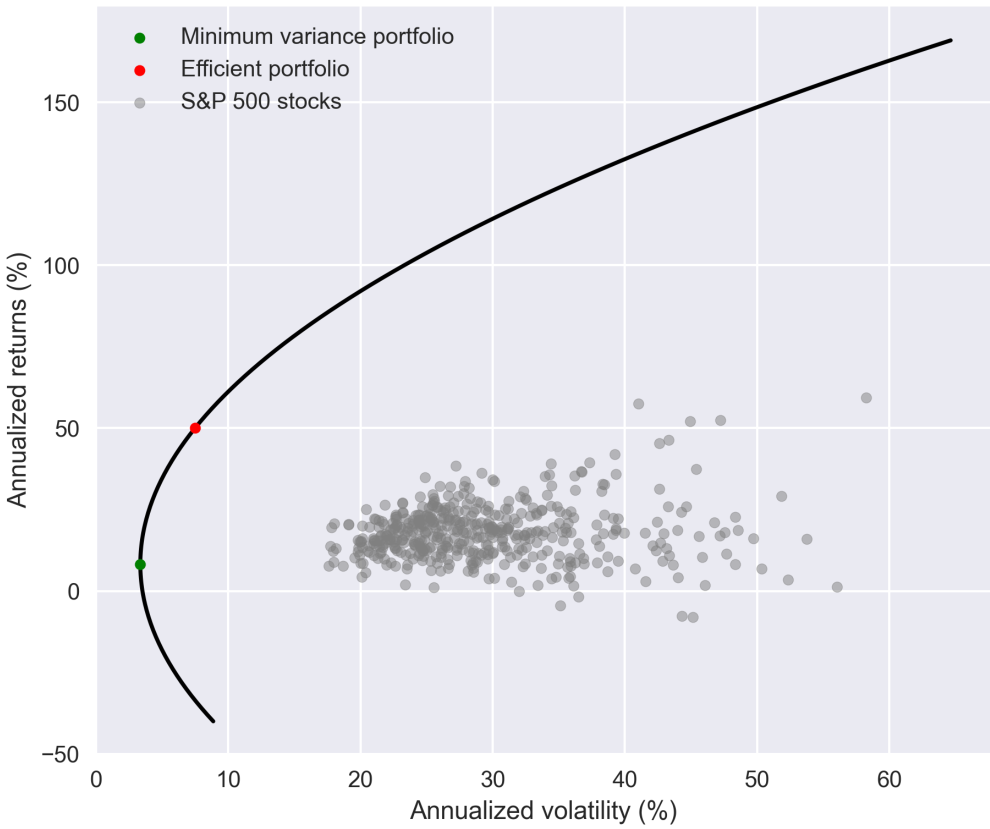
Thus, the efficient portfolio consists of the minimum variance portfolio v_t^{MVP} and a self-financing portfolio $\Sigma^{-1}\tilde{\mu} - \mathbf{1}'\Sigma^{-1}\tilde{\mu}v_t^{\text{MVP}}$. We see that the latter term is self-financing as

$$\mathbf{1}'(\Sigma^{-1}\tilde{\mu} - \mathbf{1}'\Sigma^{-1}\tilde{\mu}v_t^{\text{MVP}}) = \mathbf{1}'\Sigma^{-1}\tilde{\mu} - \mathbf{1}'\Sigma^{-1}\tilde{\mu}\mathbf{1}'\Sigma^{-1}\mathbf{1}(\mathbf{1}'\Sigma^{-1}\mathbf{1})' = 0$$

such that the normalized weight of the self-financing portfolio is 0, so the efficient portfolio v_t^* has normalized weight equal to 1. Additionally, the efficient portfolio will have expected return equal to the return target $\tilde{\mu}^*$ as

$$\tilde{\mu}'v_t^* = \tilde{\mu}'\Big(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} + \kappa_2\Big(\Sigma^{-1}\tilde{\mu} - \frac{\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\Big)\Big)$$

Figure 1: Efficient frontier with minimum variance portfolio and efficient portfolio



Source: Yahoo Finance package in python. Data from 1^{st} January 2013 to 1^{st} September 2021

be more precise. Daily? Monthly?

Insert definitions of κ_2 and for A, B, C and D

$$= \frac{C}{A} + \frac{\tilde{\mu}^* - C/A}{D - BC/A}(D - BC/A) = \frac{C}{A} - \tilde{\mu}^* - \frac{C}{A}$$

$$\tilde{\mu}' v_t^* = \tilde{\mu}^*$$

An investor seeking an excess return of $\tilde{\mu}^*$ will invest all available funds into the minimum variance portfolio and create a self-financing portfolio that increases the expected excess return of the investor's portfolio to $\tilde{\mu}^*$ and increase the risk of the portfolio too. In terms of figure 1, the investor moves upwards from the MVP along the efficient frontier as the investor increases the expected excess return goal, $\tilde{\mu}^*$ of his portfolio.

The Markowitz inspired approach presented in two asset and multi asset problem seem to easily solve the investor's problem but there are multiple problems with the approach.

Firstly, we do not know the true mean of excess returns, $\tilde{\mu}$, and the true covariance matrix of excess returns, Σ. At best, we can estimate them with their empirical counterparts, $\hat{\tilde{\mu}}$ and $\hat{\Sigma}$ which have considerable estimation uncertainties. These uncertainties introduce instability in the optimal portfolio weight, v^*, which in the real world with trading costs adds cost to the investor without adding any benefits as the changes to the estimated optimal weights may arise from uncertainty rather than changes in the market.

Secondly, returns do not follow a Gaussian distribution empirically, but the simple approach relies on this assumption. make clear where this assumption is required!

How can we mitigate these problems? We do this by changing the assumption about the data generating process to a more realistic process in the following section.

3 GARCH processes

Modelling returns as Gaussian is mathematically convenient and preserves the major point of intuition behind the central results of the mean-variance approach, but it is not empirically correct. This section explores stylized facts about financial time-series and the GARCH type models that captures these features.

3.1 Stylized facts about returns

Numerous empirical studies of returns of financial time-series have revealed some stylized facts about returns. When modelling return it is important that the model of choice mimics these stylized facts.

1. *The distribution of return is non Gaussian*

The distribution of returns does not follow a Gaussian distribution as it has a higher kurtosis and fatter tails than a Gaussian distribution allows for. Thus, returns are more likely to be centered around the mean than the Gaussian distribution and more likely to be extreme events of either sign than the Gaussian distribution.

Better fits to the empirical distribution of returns includes the Generalized Error Distribution (GED) or the Student's t-distribution, both offering higher kurtosis and fatter tails than the Gaussian distribution.[1]

2. *There is almost no correlation between returns for different days*

Consider the sample autocorrelation function between period period t and $t+\tau$ for $\tau > 0$

$$\hat{\rho}_{t,\tau} = \sum_{t=1}^{T-\tau}(r_t - \mathbb{E}[r_t])(r_{t+\tau} - \mathbb{E}[r_t]) \bigg/ \sum_{t=1}^{n}(r_t - \mathbb{E}[r_t])^2 \tag{5}$$

which measure the autocorrelation i.e. the average correlation between the values of a time series in different point in time. For almost all financial timeseries, the empirical autocorrelation, $\hat{\rho}_{t,\tau}$, is insignificant and thus close to zero.[2] Thus, an autoregressive (AR) model for returns would likely be a poor fit for financial time series and past returns gives little to no information about future returns.

[1][Taylor, 2011],*Asset price dynamics, volatility, and prediction*,p. 70-76

[2][Taylor, 2011],*Asset price dynamics, volatility, and prediction*,p. 76+77

3. *There is positive dependency squared (or absolute) returns of nearby days*

Previously, we consider autocorrelation of returns where we found no correlation, thus, one might be tempted to think that returns are *identical and independently distributed* or IID. However, this is not correct as many transformation of returns feature strong dependency across time. The most common of which is squared returns, the autocorrelation of which is rarely insignificant.[3] This also implies *volatility clustering* of returns which means that period of higher or low volatility tend to be clustered together across time or as [Mandelbrot, 1967] simply puts it:

> *"...large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes."*

This latter part implies that returns are not homoskedastic i.e. they do not have a constant variance across time.

To summarize, it would be inaccurate to model returns as IID or Gaussian. Thus, we want to built a model in which returns are not Gaussian and where returns are not modelled as independent of each other across time. There are several approach to achieve such a model, but one of the most widely used is Autoregressive Conditional Heteroskedacity or ARCH model with a non normal distribution.

3.2 Univariate GARCH models

The precursor to the GARCH model was developed by [Engle, 1982] to model and forecast variances more accurately. The resulting Autoregressive Conditional Heteroskedacity model, or ARCH for short, models the conditional variance as depending on past shocks to the time series being modelled. Consider a simple case with the return of an asset with constant mean

$$r_t = \mu + \epsilon_t \tag{6}$$

Now consider the the variance of this process is heteorskedastic and depends on the lagged shocks to the process, $\sum_{i=0}^{t-1} \epsilon_i$. The example below being the simple ARCH(1) where the conditional variance only depends on last periods shock, ϵ_{t-1}.

$$\epsilon_t = \sigma_t z_t \qquad z_t \sim D(0,1) \tag{7}$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 \qquad \omega > 0, \alpha \geq 0 \tag{8}$$

with initial values taken as given and $t = 0, 1, ...T$. The restrictions on the parameters is needed to ensure strictly positive variance for all t. $D(0,1)$ is some distribution with mean zero and unit variance. This is often a Gaussian distribution for easy of computation, but other choices like a Student's t-distribution or a Generalized Error Distribution (GED) can be used with better fits to financial time series.

[3][Taylor, 2011],*Asset price dynamics, volatility, and prediction*,p. 82-86

ω is a constant ensuring strictly positive variance. The parameter α is the effect of past shocks on the system and can also be interpreted as short run persistence of the variance.

The ARCH model had one empirical weakness in that the conditional variances from the model converged back the unconditional variance after a large shock much quicker than empirical estimates indicate it should even for large lag lengths. [Bollerslev, 1986] solved this problem with Generalized Autoregressive Conditional Heteroskedacity (GARCH) model which added persistence between the individual measures of the conditional variance across time by adding $\beta\sigma_{t-1}^2$ to the equation for the conditional variance in equation (8).

the statement is correct but it would be nice to illustrate this. Maybe in form of an IRF plot of the vola process in response to an epsilon shock for different alphas and betas?

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad \omega > 0, \alpha, \beta \geq 0 \tag{9}$$

This reduces the speed which the variance decreased after a big shock as the parameter β is how persistent the conditional variance is and can be interpreted as long run persistence.

While the standard GARCH model does an admirable job of explaining the conditional variance of financial time series, econometricians have continuously worked to improved upon the work done by Engle and Bollerslev, like allowing for asymmetric effects of shocks which [Glosten et al., 1993] did with the GJR-GARCH where the conditional variance is given by

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \kappa\epsilon_{t-1}^2 I_{\{\epsilon_{t-1}<0\}} \qquad \omega > 0, \alpha, \beta, \kappa \geq 0 \tag{10}$$

such that the effect on the conditional variance is $\alpha + \kappa$ if the shock is negative which is well documented to be true and named the "leveraged effect".

Other approach exists like the Exponential GARCH (EGARCH) developed by [Nelson, 1991] which likewise captures the leverage effect with the conditional variance given as

$$\log\sigma_t^2 = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]) + \gamma z_{t-1} + \beta\log(\sigma_{t-1}^2) \qquad \omega > 0, \alpha, \beta, \kappa \geq 0 \tag{11}$$

For $\gamma < 0$ negative shocks will have a bigger impact on future volatility than positive shocks of the same magnitude.

All fine but elaborate on this only if used later on

3.2.1 Univariate GARCH models in Portfolio theory

It should be clear by now that univariate GARCH models does far better at capturing key empirical features of financial time series than drawing IID from some distribution. An obvious question is, thus, where to apply these models?

The better fit to the empirical features should result in better predictions of the variance of the time series. Thus, instead of using the empirical estimates of the variance, $\sigma_{t+1}^2 = \hat{\sigma}^2$, we could use the GARCH model from equation (9) and set $\sigma_{t+1}^2 = \omega + \alpha\epsilon_t^2 + \beta\sigma_t^2$ as the variance of the assets and use these rather than $\hat{\sigma}^2$

Consider again the investor's problem with two asset from section 2.2.1. Now, the two asset are given by a univariate GARCH model given by equation (6), (7) and (9). The investor faces the problem in equation (2). The portfolio excess return and variance given are now given by

$$\mathbb{E}[B_t] = v_1\tilde{\mu}_1 + (1-v_1)\tilde{\mu}_2 \qquad \mathbb{V}[B_t] = v_1^2\sigma_{1,t+1}^2 + (1-v_1)^2\sigma_{2,t+1}^2 + 2v_1(1-v_1)\sigma_{1,t+1}\sigma_{2,t+1}\rho$$

where $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ are given by equation (9) or any other GARCH type model. Setting up the Lagrangian for the problem

$$\mathcal{L}(v_1) = v_1\tilde{\mu}_1 + (1-v_1)\tilde{\mu}_2 - \frac{\lambda}{2}\big(v_1^2\sigma^2_{1,t+1} + (1-v_1)^2\sigma^2_{2,t+1} + 2v_1(1-v_1)\sigma_{1,t+1}\sigma_{2,t+1}\rho\big)$$

Taking first order conditions with respect to the weight v_1

$$\frac{\partial\mathcal{L}}{\partial v_1} = \tilde{\mu}_1 - \tilde{\mu}_2 - \frac{\lambda}{2}\big(2v_1\sigma^2_{1,t+1} - 2(1-v_1)\sigma^2_{2,t+1} + (2-4v_1)\sigma_{1,t+1}\sigma_{2,t+1}\rho\big) = 0 \Leftrightarrow$$

Solving for v_1 by isolating v_1 on one side and factorizing

$$\tilde{\mu}_1 - \tilde{\mu}_2 + \lambda\sigma^2_{2,t+1} - \lambda\sigma_{1,t+1}\sigma_{2,t+1}\rho = \lambda v_1\big(\sigma^2_{2,t+1} + \sigma^2_{2,t+1} - 2\sigma_{1,t+1}\sigma_{2,t+1}\rho\big) \Leftrightarrow$$

$$v_1^* = \frac{\tilde{\mu}_1 - \tilde{\mu}_2 + \lambda\sigma^2_{2,t+1} - \lambda\sigma_{1,t+1}\sigma_{2,t+1}\rho}{\lambda\big(\sigma^2_{1,t+1} + \sigma^2_{2,t+1} - 2\sigma_{1,t+1}\sigma_{2,t+1}\rho\big)}$$

At first glance, the addition of the univariate GARCH model for asset 1 and 2 does not seem to change the result expect that the variances,$\sigma^2_{i,t+1}$, and by extension the standard deviations,$\sigma_{i,t+1}$, are now time varying and can be written as variables that are in $\mathcal{I}_t$ by the GARCH equation in (9). Thus, the weight of asset 1, v_1, would now be time-varying if this problem was multi-period.

The univariate class of GARCH model is useful when modelling a single or a few assets. However, it is often more relevant to model many assets and more importantly their covariances. A natural development to the univariate GARCH models is to consider multivariate GARCH models or MGARCH.

3.3 Multivariate GARCH models

In the univariate case, we consider one assets and the volatility of this single assets. Now, consider p assets given by a $p \times 1$ vector $S_t = (S_{1,t}, S_{2,t}, ..., S_{p,t})'$ with a $p \times 1$ vector returns, r_t, given by a similar constant mean model now of vectors

$$r_t = \mu + \epsilon_t \tag{12}$$

with μ being a $p \times 1$ vector of the empirical mean of the individual assets and ϵ_t being $p \times 1$ vector of error terms.

In the univariate case, ensuring positive variance was simply to restrict a few parameters. However, for multivariate GARCH models or MGARCH, this is much more complicate. To ensure that the $p \times p$ covariance matrix, Ω_t, is indeed a covariance matrix it must positive definite. The question is thus parameterizing the model ensuring that Ω_t is positive definite for all t.

BEKK MGARCH

A mathematically simple yet empirically impractical approach is the BEKK GARCH by [Engle and Kroner, 1995] where the conditional covariance matrix resembles the univariate GARCH in form as a BEKK GARCH(1,1)

$$\epsilon_t = \Omega_t^{1/2} z_t \qquad z_t \sim D(0, I_p) \tag{13}$$

$$\Omega_t(\theta) = \Omega + A\epsilon_{t-1}\epsilon'_{t-1}A' + B\Omega_{t-1}B' \tag{14}$$

with initial values taken as given and $t = 1, 2, ..., T$. Ω is positive definite and A and B are $p \times p$ dimensional matrices. θ is a vector of parameters where $\{\Omega, A, B\} \in \theta$ and D is some multidimensional distribution which could be a Gaussian distribution for easy of computation, but other choices like a Student's t-distribution or a Generalized Error Distribution (GED) fit financial time series better. Note that the individual components of the matrix of parameters are next to impossible to get a meaningful interpretation of but the overall interpretation of A is completely analog to α in the univariate case and the same for B to β.
To better illustrate the workings of the model consider an example with $p = 2$

$$\begin{aligned}\Omega_t(\theta) = &\begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix} + \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\epsilon_{1t-1}\\ \epsilon_{2t-1}\end{pmatrix}\begin{pmatrix}\epsilon_{1t-1} & \epsilon_{1t-1}\end{pmatrix}\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}\\ &+ \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\begin{pmatrix}b_{11} & b_{21}\\ b_{12} & b_{22}\end{pmatrix}\end{aligned}$$

A nice feature of the BEKK GARCH is that $\Omega_t(\theta)$ is positive definite for all t for any A and B and thus, the BEKK GARCH have an easy way around the paramterization problem. However, the number of parameters is $p(p+1)/2+2p^2$ meaning that the number of parameters to estimates explodes as the number of assets increase. For $p = 4$, it is 42 and for $p = 10$ it is 255, thus this model is only empirical practical when analyzing a small group of assets like $p < 10$. One way around this problem is to simplify the model in the Scalar BEKK(1,1) where A and B becomes scalars.

$$\Omega_t(\theta) = \Omega + \alpha\epsilon_{t-1}\epsilon'_{t-1} + \beta\Omega_{t-1} \qquad \alpha, \beta \geq 0 \tag{15}$$

This comes a great loss to generality as all (co)variance responds similarly to shocks and have similar persistence, which is a debatable assumption at best.

Similarly to univariate GARCH models, there exist multiple different Multivariate GARCH models, one of which is the Dynamical Conditional Correlation MGARCH or DCC MGARCH by [Engle, 2002], which have another way around the parameterization problem.

DCC MGARCH

Consider now a different variance process to the one presented above but an otherwise similar constant mean model for the returns. The DCC MGARCH use that the $p \times p$ covaraince matrix, Ω_t, can be decomposed into two variances matrices, Var_t, and a correlation matrix, Γ_t,

$$\Omega_t = \text{Var}_t \Gamma_t \text{Var}_t \tag{16}$$

Where

$$\text{Var}_t = \text{diag}(\Omega_t) = \begin{pmatrix} \sigma^2_{1,t} & 0 & \cdots & 0 \\ 0 & \sigma^2_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2_{p,t} \end{pmatrix} \tag{17}$$

and each diagonal element is given by a univariate GARCH process. This GARCH process can by any univariate GARCH process like GJR-GARCH etc.

$$\sigma^2_{i,t} = \omega_i + \alpha_i \epsilon^2_{i,t-1} + \beta_i \sigma^2_{i,t-1} \quad \text{for} \quad i = 1, 2, ..., p \qquad \omega_i > 0, \alpha_i, \beta_i \geq 0 \tag{18}$$

the correlation matrix is given as

$$\Gamma_t = Q^{*-1}_t Q_t Q^{*-1}_t = \begin{pmatrix} 1 & \rho_{12,t} & \cdots & \rho_{1p,t} \\ \rho_{21,t} & 1 & \cdots & \rho_{2p,t} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1,t} & \rho_{p2,t} & \cdots & 1 \end{pmatrix} \tag{19}$$

and the pseudo correlation, Q_t, follows a scalar BEKK(1,1) MGARCH process given by

$$Q_t = \bar{Q}(1 - a - b) + a\eta_{t-1}\eta'_{t-1} + bQ_{t-1} \qquad a, b \geq 0 \tag{20}$$

where $\eta_t = \text{Var}^{-1}_t \epsilon_t \sim N(0, \Gamma_t)$ is the standardized disturbances of the correlations. Note that

$$Q^{*-1}_t = \text{diag}(Q_t)^{-1} = \begin{pmatrix} \sqrt{q_{1,t}} & 0 & \cdots & 0 \\ 0 & \sqrt{q_{2,t}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{q_{p,t}} \end{pmatrix} \qquad \bar{Q} = \frac{1}{T}\sum_{t=1}^{T} \eta_t \eta'_t > 0 \tag{21}$$

The advantages of this is that it is possible to estimates this model in two stages. First, estimates p univariate GARCH model for the p assets and second, estimate a multivariate scalar BEKK for the conditional correlation. The model has with a simple GARCH(1,1) and a scalar BEKK(1,1) has $3p + (p^2 - p)/2$ parameters. In comparison to the BEKK(1,1) for $p = 4$ the DCC MGARCH(1,1) has 6 parameters and 45 for $p = 10$. However, both models number of variables grows fast and thus to truly model a large number of assets, Eigenvalue MGARCH or λ-MGARCH models are preferable.

3.3.1 Multivariate GARCH models in Portfolio theory

Consider the investors problem for p assets presented in section 2.2.2. Now the p assets are given by a constant mean DCC MGARCH(1,1) model given by equation (12) and (16)-(21) with mean $\tilde{\mu}$. The investor minimize the risk given that the normalized weight sums to 1.

$$\min_{v_t}\{\frac{1}{2}v_t \Omega_{t+1} v'_t\} \qquad \text{s.t.} \quad v'_t \mathbf{1} = 1$$

Note that the covariance matrix is now given as Ω_{t+1} which comes out of the DCC MGARCH model rather than the constant Σ. But note that Ω_{t+1} is $\mathcal{F}_t$ measurable as a result of the MGARCH model such that the investor with some accuracy can forecast Ω_{t+1} in period t. The Lagrangian is given as

$$\mathcal{L}(v_t) = \frac{1}{2} v_t \Omega_{t+1} v_t' - \kappa_1 (v_t' \mathbf{1} - 1)$$

Taking the first order conditions and solving for v_t and κ_1 as in section 2.2.2 yields the minimum variance portfolio after light algebra which can be found in appendix A.2.2

$$v_t = \frac{\Omega_{t+1}^{-1} \mathbf{1}}{\mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1}} = v_t^{\text{MVP}}$$

The result is rather predictable in that Ω_{t+1} replaces the constant Σ. Consider a case where the investor set a excess return target of $\tilde{\mu}^*$ as in section 2.2.2 and thus face the following problem

$$\min_{v_t} \{ \frac{1}{2} v_t \Omega_{t+1} v_t' \} \qquad \text{s.t.} \quad v_t' \mathbf{1} = 1 \quad \text{and} \quad v_t' \tilde{\mu} = \tilde{\mu}^*$$

The Lagrangian is given

$$\mathcal{L}(v_t) = \frac{1}{2} v_t \Omega_{t+1} v_t' - \kappa_1 (v_t' \mathbf{1} - 1) - \kappa_2 (v_t' \tilde{\mu} - \tilde{\mu}^*)$$

After some tendious algebra which is covered in full in appendix A.2.2, the efficient weights are given by

$$v_t^* = v_t^{\text{MVP}} + \kappa_2 \left(\Omega_{t+1}^{-1} \tilde{\mu} - \mathbf{1}' \Omega_{t+1}^{-1} \tilde{\mu} v_t^{\text{MVP}} \right)$$

with κ_2 given as

$$\kappa_2 = \frac{C - A\tilde{\mu}^*}{BC - AD}$$

Note that the result is analog to the result in section 2.2.2 with Ω_{t+1} replacing Σ. The result thus have the same interpretation. The main takeaway from this is that while it is easy to replace Σ with $\hat{\Sigma}$ and likewise for $\tilde{\mu}$, it is not necessarily the best option and other and more sophisticated options for $\tilde{\mu}$ and Σ exists. We have explored replacing Σ with the covariance matrix of a MGARCH model, Ω_{t+1} as they empirically fit the financial time series better. However, to truly benefit from the added complexity of GARCH models, we need to consist multi-period problem or dynamic trading strategies.

4 Dynamic Trading Strategies

Start with presenting the framework, where the investor can choose between two assets. Then extend the model and present the new solution. Compare the two and explain the intuition.

4.1 Dynamic programming by the Bellman Equation

Dynamic programming was by pioneered by [Bellman, 1966] to solve multi-period optimization problems. We have chosen to more closely follow the notation from [Gan and Lu, 2014] and [Gârleanu and Pedersen, 2013].

Consider a agent that seeks a policy or rule that defines that optimal action that the agent should take at time t in state s, such that the policy $\{x_t^*\}_{t=1}^{\infty}$ maximizes the present value of current rewards and future expected rewards, $f(x_t, s_t)$, discounted by $(1-p) \in (0,1]$ given some constraint $g(x_t, s_t)$.

$$\max_{\{x_i\}_{i=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty}(1-\rho)^t f(x_t, s_t)\right] \quad \text{s.t.} \quad g(x_t, s_t) = 0 \tag{\&}$$

The tools of dynamic programming that Bellman developed essentially decomposes the multi-period problem into of two period models, "now" and "later". Consider an agent facing the maximization problem in (&) in period $t = 0$ and period $t = 1$

$$L_0 = \max_{\{x_i\}_{i=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty}(1-\rho)^t f(x_t, s_t)\right] + \lambda[g(x_t, s_t)]$$
$$L_1 = \max_{\{x_i\}_{i=1}^{\infty}} \mathbb{E}\left[\sum_{t=1}^{\infty}(1-\rho)^t f(x_t, s_t)\right] + \lambda[g(x_t, s_t)]$$

These problems are very similar. Note that L_1 is included in L_0. Thus, we can essentially rewrite L_0 as the problem of today plus the problem of tomorrow

$$L_0 = \max_{\{x_i\}_{i=0}^{\infty}} \mathbb{E}\left[(1-\rho)^0 f(x_t, s_t) + L_1\right] + \lambda[g(x_t, s_t)] = \max_{\{x_i\}_{i=0}^{\infty}} \left[f(x_t, s_t) + \mathbb{E}[L_1]\right] + \lambda[g(x_t, s_t)]$$

Changing notation, where we drop subscripts and anything in the future is denoted with prime ex. s' and denoting L the value function, V, as a function of s.

The value function (or Bellman equation), $V(s)$ describes the maximum reward of current and expected future rewards given the process is in state s and time t.

$$V(s) = \max_{x \in X(s)} \left[\underbrace{f(x,s)}_{(i)} + \underbrace{\mathbb{E}[V(s')]}_{(ii)}\right] + \underbrace{\lambda[g(x,s)]}_{(iii)}$$

The value function captures the trade off between immediate reward, (i) future reward, (ii) under some constraint (iii). Given the value function, the optimal policies are the solutions to the optimizing problem contained within the in Bellman's equation, $V(s)$. The policies can be may different real world objects of interest for instance, the optimal interest rate, the optimal level of production for some firm or as for this paper, the optimal weight of wealth allocated into different assets.

4.2 Many risky assets and no trading cost

Consider p assets given by a $p \times 1$ vector $S_t = (S_{1,t}, S_{2,t}, ..., S_{p,t})'$ with excess returns $\tilde{r}_t = (\tilde{r}_{1,t}, \tilde{r}_{2,t}, ..., \tilde{r}_{p,t})'$ which by the investor are assumed given by a multivariate Gaussian distribution $N_p(\tilde{\mu}, \Sigma)$.

An investor with mean-variance preferences given equation (1) seeks as dynamic trading strategy $\{v_t\}_{t=0}^{\infty}$ that solves the multi-period maximization problem:

$$\max_{\{v_t\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\tilde{r}_{t+1} - \frac{\lambda}{2}v_t\Sigma v_t'\right)\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$

$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\tilde{\mu} + \frac{\lambda}{2}v_t\Sigma v_t'\right)\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \tag{22}$$

Can you elaborate more on this? Conditional on $t=0$, future weights are not constant!

This problem can be solved via dynamic programming using the Bellman equation using the method in section 4.1. Consider an investor with preference as (1) and facing the maximization problem in (22) in both period $t = 0$ and $t = 1$, denoted L_t, are remarkably similar

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\tilde{\mu} + \frac{\lambda}{2}v_t\Sigma v_t'\right)\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$
$$L_1 = \max_{\{v_t\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty}(1-\rho)^t\left(v_t\tilde{\mu} + \frac{\lambda}{2}v_t\Sigma v_t'\right)\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 1$$

As in Section 4.1, we note that L_1 is contained with L_0 and we can thus rewrite the L_0 as "two period" model with immediate reward (i) and later rewards (ii) under the constraint the normalized weights sum to 1 (iii).

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\underbrace{(1-\rho)^0\left(v_0\tilde{\mu} - \frac{\lambda}{2}v_0\Omega_1 v_0'\right)}_{(i)} + \underbrace{\mathbb{E}_t[L_1]}_{(ii)}\right] \quad \text{s.t.} \quad \underbrace{v_t'\mathbf{1} = 1}_{(iii)} \quad \forall t \geq 0$$
$$= \max_{\{v_t\}_{t=0}^{\infty}} \left[v_0\tilde{\mu} - \frac{\lambda}{2}v_0\Omega_1 v_0' + \mathbb{E}_t[L_1]\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$

Denoting L as a function of v_t and denoting it as the value function, V, yields

$$V(v_t) = \max_{v_t}\left[v_t\tilde{\mu} - \frac{\lambda}{2}v_t\Sigma v_t' + \mathbb{E}_t[V(v_{t+1})]\right] - \kappa_1(v_t'\mathbf{1} - 1)$$

The dynamic nature of problem changes what the investor maximizes with respect to from $\{v_t\}_{t=0}^{\infty}$ to v_t as the investor dynamically solves the problem one period at the time.

The value function (or Bellman equation) measures the value of the portfolio at time $t+1$ with weight of v_t of the risky asset. Additionally, the value function captures the trade-off between utility now and utility later.

Solving for the optimal weight requires solving the first order conditions. Now, the first order conditions are given by

$$\frac{\partial V(v_t)}{\partial v_t} = \tilde{\mu} - \kappa_1 \mathbf{1} - \lambda v_t \Sigma = 0 \Leftrightarrow v_t = \lambda^{-1}\Sigma^{-1}(\tilde{\mu} - \kappa_1 \mathbf{1})$$

The constraint requires that $v_t'\mathbf{1} = 1$, which can be used to solve for the Lagrangian multiplier κ_1

$$\begin{aligned}
1 =& v_t'\mathbf{1} = \mathbf{1}'v_t = \mathbf{1}'\big(\lambda^{-1}\Sigma^{-1}(\tilde{\mu} - \kappa_1\mathbf{1})\big) \\
1 =& \lambda^{-1}\mathbf{1}'\Sigma^{-1}\tilde{\mu} - \lambda^{-1}\mathbf{1}\Sigma^{-1}\mathbf{1}\kappa_1 \\
\lambda^{-1}\mathbf{1}'\Sigma^{-1}\tilde{\mu} - 1 =& \lambda^{-1}\mathbf{1}\Sigma^{-1}\mathbf{1}\kappa_1 \\
\kappa_1 =& \frac{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\tilde{\mu} - 1}{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\mathbf{1}}
\end{aligned}$$

Now, we insert the expression into the weights v_t

$$\begin{aligned}
v_t = \lambda^{-1}\Sigma^{-1}\left(\tilde{\mu} - \left[\frac{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\tilde{\mu} - 1}{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right]\mathbf{1}\right) &= \lambda^{-1}\Sigma^{-1}\left(\tilde{\mu} - \left[\frac{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\tilde{\mu}}{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\mathbf{1}} - \frac{1}{\lambda^{-1}\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right]\mathbf{1}\right) \\
= \lambda^{-1}\Sigma^{-1}\left(\tilde{\mu} - \frac{\mathbf{1}'\Sigma^{-1}\tilde{\mu}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} + \frac{\lambda\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right) &= \lambda^{-1}\Sigma^{-1}\left(\tilde{\mu} - \tilde{\mu} + \frac{\lambda\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right) \\
v_t &= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = v_t^{\text{MVP}} \qquad (23)
\end{aligned}$$

Note that given that variance, Σ, is assumed constant by the investor, the weight, v_t, are likewise constant. One could interpret this as an investor who rarely if ever updates his beliefs or information set and thus, would have no reason to update the weights of his portfolio. Thus, the optimal weight are identical to the static case in section 2.2.2 as the investor essentially solves infinitely many static allocation problems.

Now consider the case where the investor have a different belief about the data generating process where the investor believes that the excess returns $\tilde{r}_t = (\tilde{r}_{1,t}, \tilde{r}_{2,t}, ..., \tilde{r}_{p,t})'$ are given by a constant mean DCC MGARCH(1,1) model given by equation (12) and (16)-(21) with mean $\tilde{\mu}$. The investor now faces a slightly different maximization problem.

$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\tilde{\mu} + \frac{\lambda}{2}v_t\Omega_{t+1}v_t'\right)\right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \qquad (24)$$

The solution to the problem is almost identical to the case with Gaussian returns as such the full problem can be found in appendix A.5.1. The value function of the problem is

$$V(v_t) = \max_{v_t}\left[v_t\tilde{\mu} - \frac{\lambda}{2}v_t\Omega_{t+1}v_t' + \mathbb{E}_t[V(v_{t+1})]\right] - \kappa_1(v_t'\mathbf{1} - 1)$$

where the optimal weight can be found from the first order conditions of the value function wrt. the weights, v_t. After some light algebra, the optimal weights are given as

$$v_t = \frac{\Omega_{t+1}^{-1}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} = v_t^{\text{MVP}} \qquad (25)$$

Note that given that covariance matrix, Ω_t is time-varying the weight, v_t, are likewise time-varying. Now the investor updates his beliefs about the future variances and covariance i.e. the risk and as such can make change to his portfolio and continuously update the optimal weights. Note that Ω_{t+1} can be written as variables that are $\mathcal{F}_t$-measurable i.e. the investor can with some accuracy forecast the covariance matrix within the near future or at the very least the investor believes he can forecast the covariance matrix one period into the feature.

4.3 Many risky assets and trading costs

Consider p assets given by a $p \times 1$ vector $S_t = (S_{1,t}, S_{2,t}, ..., S_{p,t})'$ with returns with returns given by a constant mean DCC MGARCH(1,1) model given by equation (12) and (16)-(21). However, now the investor faces cost from trading and thus faces the following problem when seeking a dynamic investment strategy:

$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \left(v_t\tilde{\mu} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' - \frac{1}{2} \Delta v_t' \Lambda \Delta v_t \right) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \tag{26}$$

with trading cost taking the form $-\frac{1}{2}\Delta v_t' \Lambda \Delta v_t$. We solve this problem by using the method in section 4.1. Consider an investor with preference given by equation (1) and facing the maximization problem in (26) in period $t = 0$ and $t = 1$, denoted L_t

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \left(v_t\tilde{\mu} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' - \frac{1}{2} \Delta v_t' \Lambda \Delta v_t \right) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$
$$L_1 = \max_{\{v_t\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} (1-\rho)^t \left(v_t\tilde{\mu} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' - \frac{1}{2} \Delta v_t' \Lambda \Delta v_t \right) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 1$$

As in Section 2.1, we note that L_1 is contained within L_0 and we can thus rewrite the L_0 as a "two period" model with immediate reward and later rewards.

$$\begin{aligned} L_0 &= \max_{\{v_t\}_{t=0}^{\infty}} \left[(1-\rho)^0 \left(v_0\tilde{\mu} - \frac{\lambda}{2} v_0 \Omega_1 v_0' - \frac{1}{2} \Delta v_0' \Lambda \Delta v_0 \right) + \mathbb{E}_t[L_1] \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \\ &= \max_{\{v_t\}_{t=0}^{\infty}} \left[v_0\tilde{\mu} - \frac{\lambda}{2} v_0 \Omega_1 v_0' - \frac{1}{2} \Delta v_0' \Lambda \Delta v_0 + \mathbb{E}_t[L_1] \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \end{aligned}$$

Denoting L as a function of v_t and denoting it as the value function, V, yields

$$V(v_t) = \max_{v_t} \left[v_t\tilde{\mu} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' - \frac{1}{2} \Delta v_t' \Lambda \Delta v_t + \mathbb{E}_t[V(v_{t+1})] \right] - \kappa_1 (v_t'\mathbf{1} - 1)$$

Note that v_t is also within $\mathbb{E}_t[V(v_{t+1})]$ in the form of $-\frac{1}{2}(v_{t+1} - v_t)'\Lambda(v_{t+1} - v_t)$. Thus, the first order conditions are given by

$$\frac{\partial V(v_t)}{\partial v_t} = \tilde{\mu} - \lambda \Omega_{t+1} v_t - \Lambda(v_t - v_{t-1}) - \mathbb{E}_t[\Lambda(v_{t+1} - v_t)] - \kappa_1 \mathbf{1} = 0$$

By assuming that in period t, the investor do not expect to make any change to his portfolio in period $t+1$ as else the agent should make them in period t, we get

$$v_t(\lambda\Omega_{t+1} + \Lambda) = \tilde{\mu} + \Lambda v_{t-1} - \underbrace{\mathbb{E}_t[\Lambda(v_{t+1} - v_t)]}_{=\Lambda\mathbb{E}_t[v_{t+1}-v_t]=0} - \kappa_1\mathbf{1}$$

and thus the weights, v_t is given by

$$v_t = (\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1} - \kappa_1\mathbf{1})$$

The constraint requires that $v_t'\mathbf{1} = 1$, which can be used to solve for the Lagrangian multiplier κ_1

$$\begin{aligned}
1 =& v_t'\mathbf{1} = \mathbf{1}'v_t = \mathbf{1}'\big((\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1} - \kappa_1\mathbf{1})\big) \\
\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}\kappa_1 =& \mathbf{1}'\big((\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1})\big) - 1 \\
\kappa_1 =& \frac{\mathbf{1}'\big((\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1})\big) - 1}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}
\end{aligned}$$

Insert κ_1 into v_t

$$\begin{aligned}
v_t &= (\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1} - \kappa_1\mathbf{1}) \\
v_t &= (\lambda\Omega_{t+1} + \Lambda)^{-1}\left(\tilde{\mu} + \Lambda v_{t-1} - \left[\frac{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}(\tilde{\mu} + \Lambda v_{t-1}) - 1}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}\right]\mathbf{1}\right) \\
v_t &= \frac{\tilde{\mu} + \Lambda v_{t-1}}{\lambda\Omega_{t+1} + \Lambda} - (\lambda\Omega_{t+1} + \Lambda)^{-1}\left(\frac{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}(\tilde{\mu} + \Lambda v_{t-1}) - \mathbf{1}}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}\right) \\
v_t &= \frac{\tilde{\mu} + \Lambda v_{t-1}}{\lambda\Omega_{t+1} + \Lambda} - (\lambda\Omega_{t+1} + \Lambda)^{-1}\left(\frac{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}(\tilde{\mu} + \Lambda v_{t-1})}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}} - \frac{\mathbf{1}}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}\right) \\
v_t &= \frac{\tilde{\mu} + \Lambda v_{t-1}}{\lambda\Omega_{t+1} + \Lambda} - (\lambda\Omega_{t+1} + \Lambda)^{-1}\left(\tilde{\mu} + \Lambda v_{t-1} - \frac{\mathbf{1}}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}\right) \\
v_t &= \frac{\tilde{\mu} + \Lambda v_{t-1}}{\lambda\Omega_{t+1} + \Lambda} - \frac{\tilde{\mu} + \Lambda v_{t-1}}{\lambda\Omega_{t+1} + \Lambda} + \frac{(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}
\end{aligned}$$

thus resulting

$$v_t = \frac{(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}}{\mathbf{1}'(\lambda\Omega_{t+1} + \Lambda)^{-1}\mathbf{1}} \tag{27}$$

It is quite evident that this result is closely related to the MVP portfolios of equation (23) and (25). We see that the addition of trading cost, Λ, acts as some form of regularization of the covariance matrix.

5 Description of data

6 Empirical

6.1 Backtesting using historical prices

1. Calculate return and std

2. Compare Sharpe ratios

3. Compare certainty equivalents

6.2 Monte Carlo simulation

6.3 Calculate numerically optimal strategy and compare it to theoretically derived strategy

6.3.1 Fitting the model

6.3.2 How robust is the model to parameter differences between the DGP and the fitted model?

7 Discussion

- Q: Could we have included a factor model to predict expected returns? A: Yes, but empirical evidence suggets it would have been a waste of time, volatility is what drives portfolio allocations.

- Q: Could we have updated model parameters for often after initial fitting? A: Makes it unnecessarily complicated to update it over time.

8 Conclusion

References

[Bellman, 1966] Bellman, R. (1966). Dynamic programming. *Science*, 153(3731):34–37.

[Bollerslev, 1986] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of econometrics*, 31(3):307–327.

[Engle, 2002] Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, 20(3):339–350.

[Engle, 1982] Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica: Journal of the Econometric Society*, pages 987–1007.

[Engle and Kroner, 1995] Engle, R. F. and Kroner, K. F. (1995). Multivariate simultaneous generalized arch. *Econometric theory*, 11(1):122–150.

[Gan and Lu, 2014] Gan, X. and Lu, W. (2014). A general setting and solution of bellman equation in monetary theory. *Journal of Applied Mathematics*, 2014.

[Gârleanu and Pedersen, 2013] Gârleanu, N. and Pedersen, L. H. (2013). Dynamic trading with predictable returns and transaction costs. *The Journal of Finance*, 68(6):2309–2340.

[Glosten et al., 1993] Glosten, L. R., Jagannathan, R., and Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *The journal of finance*, 48(5):1779–1801.

[Levy and Markowitz, 1979] Levy, H. and Markowitz, H. M. (1979). Approximating expected utility by a function of mean and variance. *The American Economic Review*, 69(3):308–317.

[Mandelbrot, 1967] Mandelbrot, B. (1967). The variation of some other speculative prices. *The Journal of Business*, 40(4):393–413.

[Markowitz, 1952] Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.

[Nelson, 1991] Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica: Journal of the Econometric Society*, pages 347–370.

[Taylor, 2011] Taylor, S. J. (2011). *Asset price dynamics, volatility, and prediction.* Princeton university press.

A Appendix

A.1 Multi Assets Problem

A.1.1 MVP with return target

Insert the tendious algrebra from the multi asset problem

A.2 GARCH Asset Problems

A.2.1 Simplest case with two risky assets

A.2.2 Multi Asset Problem

Taking first order conditions with respect to the weight v_t

$$\frac{\partial \mathcal{L}}{\partial v_t} = \Omega_{t+1} v_t - \kappa_1 \mathbf{1} = 0$$

Solving for v_t yields

$$\kappa_1 \mathbf{1} = \Omega_{t+1} v_t \Leftrightarrow v_t = \Omega_{t+1}^{-1} \mathbf{1} \kappa_1$$

The constraint requires that $v_t' \mathbf{1} = 1$, which can be used to solve for the Lagrangian multiplier κ

$$\begin{aligned} 1 =& v_t' \mathbf{1} = \mathbf{1}' v_t \\ 1 =& \mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1} \kappa_1 \\ \kappa_1 =& \frac{1}{\mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1}} \end{aligned}$$

Now, we insert the expression into the weights v_t

$$v_t = \Omega_{t+1}^{-1} \mathbf{1} \kappa_1 = \Omega_{t+1}^{-1} \mathbf{1} \frac{1}{\mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1}}$$

$$v_t = \frac{\Omega_{t+1}^{-1} \mathbf{1}}{\mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1}} = v_t^{\text{MVP}}$$

Return target

$$\frac{\partial \mathcal{L}}{\partial v_t} = \Omega_{t+1} v_t - \kappa_1 \mathbf{1} - \kappa_2 \tilde{\mu} = 0$$

Solving for v_t yields

$$\kappa_1 \mathbf{1} + \kappa_2 \tilde{\mu} = \Omega_{t+1} v_t \Leftrightarrow v_t = \Omega_{t+1}^{-1} (\kappa_1 \mathbf{1} + \kappa_2 \tilde{\mu})$$

$$A \equiv \mathbf{1}' \Omega_{t+1}^{-1} \mathbf{1}, \quad B \equiv \mathbf{1}' \Omega_{t+1}^{-1} \tilde{\mu}, \quad C \equiv \tilde{\mu}' \Omega_{t+1}^{-1} \mathbf{1}, \quad D \equiv \tilde{\mu}' \Omega_{t+1}^{-1} \tilde{\mu}$$

Finding κ_1 and κ_2 starting with their constraints

$$
\begin{aligned}
1 =& v_t'\mathbf{1} \\
1 =& \mathbf{1}'v_t \\
1 =& \mathbf{1}'\Omega_{t+1}^{-1}(\kappa_1\mathbf{1} + \kappa_2\tilde{\mu}) \\
1 =& \kappa_1\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1} + \kappa_2\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} \\
\kappa_1\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1} =& 1 - \kappa_2\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} \\
\kappa_1 =& \big(1 - \kappa_2\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}\big)(\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1})^{-1} \\
\kappa_1 =& \frac{1 - \kappa_2 B}{A}
\end{aligned}
$$

now for κ_2

$$
\begin{aligned}
\tilde{\mu}^* =& v_t'\tilde{\mu} \\
\tilde{\mu}^* =& \tilde{\mu}'v_t \\
\tilde{\mu}^* =& \tilde{\mu}'\Omega_{t+1}^{-1}(\kappa_1\mathbf{1} + \kappa_2\tilde{\mu}) \\
\tilde{\mu}^* =& \kappa_1\tilde{\mu}'\Omega_{t+1}^{-1}\mathbf{1} + \kappa_2\tilde{\mu}'\Omega_{t+1}^{-1}\tilde{\mu}) \\
\tilde{\mu}^* =& \kappa_1 C + \kappa_2 D
\end{aligned}
$$

Insert expression for κ_1

$$
\begin{aligned}
\tilde{\mu}^* =& \frac{1 - \kappa_2 B}{A}C + \kappa_2 D \\
\tilde{\mu}^* - \frac{C}{A} =& \frac{-\kappa_2 BC}{A} + \kappa_2 D \\
\tilde{\mu}^* - \frac{C}{A} =& \kappa_2\left(-\frac{BC}{A} + D\right) \\
\kappa_2 =& \frac{\tilde{\mu}^* - C/A}{D - BC/A} = \frac{C - A\tilde{\mu}^*}{BC - AD}
\end{aligned}
$$

Then the optimal weights are given as

$$v_t^* = \Omega_{t+1}^{-1}\left(\frac{1-\kappa_2 B}{A}\mathbf{1} + \kappa_2\tilde{\mu}\right)$$
$$v_t^* = \Omega_{t+1}^{-1}\left(\frac{1-\kappa_2 \mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\mathbf{1} + \kappa_2\tilde{\mu}\right)$$
$$v_t^* = \frac{\Omega_{t+1}^{-1}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} + \Omega_{t+1}^{-1}\left(\kappa_2\tilde{\mu} - \kappa_2\frac{\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\mathbf{1}\right)$$
$$v_t^* = \frac{\Omega_{t+1}^{-1}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} + \kappa_2\left(\Omega_{t+1}^{-1}\tilde{\mu} - \frac{\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\Omega_{t+1}^{-1}\mathbf{1}\right)$$
$$v_t^* = v_t^{\mathrm{MVP}} + \kappa_2\left(\Omega_{t+1}^{-1}\tilde{\mu} - \frac{\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\Omega_{t+1}^{-1}\mathbf{1}\right)$$
$$v_t^* = v_t^{\mathrm{MVP}} + \kappa_2\left(\Omega_{t+1}^{-1}\tilde{\mu} - \mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}\frac{\Omega_{t+1}^{-1}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\right)$$
$$v_t^* = v_t^{\mathrm{MVP}} + \kappa_2\left(\Omega_{t+1}^{-1}\tilde{\mu} - \mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}v_t^{\mathrm{MVP}}\right)$$

A.3 Dymanic problems

A.4 Two asset with Gaussian returns and no trading cost

Consider an investor in the rather unlike situation having to invest in only two assets $S_{1,t}$ and $S_{2,t}$. The excess return of which, $\tilde{r}_{1,t}$ and $\tilde{r}_{1,t}$, follows a known Gaussian distribution $N_2(\tilde{\mu}, \Sigma)$ The investor's objective is to choose a dynamic trading strategy $\{v_t\}_{t=0}^{\infty}$ that maximizes the present value of all future expected excess returns

$$\max_{\{v_t\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\tilde{r}_{1,t} + (1-v_t)\tilde{r}_{2,t} - \frac{\lambda}{2}v_t^2\mathbb{V}[r_t^B]\right)\right] \tag{\#}$$

Inserting the expected mean and variance from the Gaussian distribution yields

$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2}v_t^2\sigma^2\right)\right]$$

where $v_t r_t + (1-v_t)r_f$ is utility gained from returns and $\frac{\lambda}{2}v_t^2\sigma^2$ is disutility gained from risk scaling with risk-aversion λ.

This problem can be solved via dynamic programming using the Bellman equation.

Solving the model

We solve this problem by using the method in section 2.1. Consider an investor with preference as ($) and facing the maximization problem in (#) then the period $t = 0$ and $t = 1$, denoted L_t, are remarkably similar

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty}(1-\rho)^t\left(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2}v_t^2\sigma^2\right)\right]$$
$$L_1 = \max_{\{v_t\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty}(1-\rho)^t\left(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2}v_t^2\sigma^2\right)\right]$$

As in Section 2.1, we note that L_1 is contained with L_0 and we can thus rewrite the L_0 as "two period" model with immediate reward and later rewards.

$$\begin{aligned} L_0 &= \max_{\{v_t\}_{t=0}^{\infty}} \left[(1-\rho)^0 \left(v_0\mu + (1-v_0)r_f - \frac{\lambda}{2} v_0^2 \sigma^2 \right) + \mathbb{E}[L_1] \right] \\ &= \max_{\{v_t\}_{t=0}^{\infty}} \left[v_0\mu + (1-v_0)r_f - \frac{\lambda}{2} v_0^2 \sigma^2 + \mathbb{E}[L_1] \right] \end{aligned}$$

Denoting L as a function of v_t and denoting it as the value function, V, yields

$$V(v_t) = \max_{v_t} \left[v_t\mu + (1-v_t)r_f - \frac{\lambda}{2} v_t^2 \sigma^2 + \mathbb{E}[V(v_{t+1})] \right]$$

Since

$$\mathbb{E}[V(v_{t+1})] = \mathbb{E}\left[\max_{v_{t+1}} \left[v_{t+1}\mu + (1-v_{t+1})r_f - \frac{\lambda}{2} v_{t+1}^2 \sigma^2 + \mathbb{E}[V(v_{t+2})] \right] \right]$$

The expectation of the value function in the next period $\mathbb{E}[V(v_{t+1})]$ goes away in $V(v_t)$ since it doesn't contain any v_t terms. The dynamic nature of problem changes what the investor maximizes with respect to from $\{v_t\}_{t=0}^{\infty}$ to v_t as the investor dynamically solves the problem. The value function (or Bellman equation) measures the value of the portfolio at time $t+1$ with weight of v_t of the risky asset. Solving for the optimal weight requires solving the first order conditions.

$$\frac{\partial V(v_t)}{\partial v_t} = \mu - r_f - \lambda v_t \sigma^2 = 0 \Leftrightarrow v_t = \frac{\mu - r_f}{\lambda \sigma^2}$$

Note that $\mu, r_f, \sigma^2, \lambda$ are assumed constant across time and thus the investors should hold a constant fraction in the risky asset with the remaining in the risk-less asset.

A.5 One Asset following a GARCH process and no trading cost

Now consider the case where returns are not Gaussian, which is empirically more appealing. Instead, returns are given by a constant mean univariate GARCH(1,1) model given by equation (6), (7) and (9). There is also a risk-less asset paying r_f.

An investor with mean-variance preferences as in ($) and solving the maximization problem in (#) but with r_t following the above process. Inserting these into (#) yields

$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \left(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2} v_t^2 \sigma_{t+1}^2 \right) \right] \tag{\%}$$

The key difference is that the variance σ_t^2 is now time-varying.

Solving the model

We solve this problem by using the method in section 2.1. Consider an investor with preference as ($) and facing the maximization problem in (%) then the period $t=0$ and $t=1$, denoted

L_t, are eerily similar

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \Big(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2} v_t^2 \sigma_{t+1}^2 \Big) \right]$$
$$L_1 = \max_{\{v_t\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} (1-\rho)^t \Big(v_t\mu + (1-v_t)r_f - \frac{\lambda}{2} v_t^2 \sigma_{t+1}^2 \Big) \right]$$

As in Section 2.1, we note that L_1 is contained with L_0 and we can thus rewrite the L_0 as "two period" model with immediate reward and later rewards.

$$\begin{aligned} L_0 &= \max_{\{v_t\}_{t=0}^{\infty}} \left[(1-\rho)^0 \Big(v_0\mu + (1-v_0)r_f - \frac{\lambda}{2} v_0^2 \sigma_1^2 \Big) + \mathbb{E}[L_1] \right] \\ &= \max_{\{v_t\}_{t=0}^{\infty}} \left[v_0\mu + (1-v_0)r_f - \frac{\lambda}{2} v_0^2 \sigma_1^2 + \mathbb{E}[L_1] \right] \end{aligned}$$

Denoting L as a function of v_t and denoting it as the value function, V, yields

$$V(v_t) = \max_{v_t} \left[v_t\mu + (1-v_t)r_f - \frac{\lambda}{2} v_t^2 \sigma_{t+1}^2 + \mathbb{E}[V(v_{t+1})] \right]$$

Now, the first order conditions are given by

$$\frac{\partial V(v_t)}{\partial v_t} = \mu - r_f - \lambda v_t \sigma_{t+1}^2 = 0 \Leftrightarrow v_t = \frac{\mu - r_f}{\lambda \sigma_{t+1}^2} = \frac{1}{\lambda} \frac{\mu - r_f}{\omega + \alpha\epsilon_t^2 + \beta\sigma_t^2}$$

Note that given that variance, σ_t^2 is time-varying the weight, v_t, are likewise time-varying.

A.5.1 Many GARCH Assets

Consider p assets given by a $p \times 1$ vector $S_t = (S_{1,t}, S_{2,t}, ..., S_{p,t})'$ with excess returns $\tilde{r}_t = (\tilde{r}_{1,t}, \tilde{r}_{2,t}, ..., \tilde{r}_{p,t})'$ given by a constant mean DCC MGARCH(1,1) model given by equation (12) and (16)-(21) with mean $\tilde{\mu}$.

An investor with mean-variance preferences given equation (1) seeks as dynamic trading strategy $\{v_t\}_{t=0}^{\infty}$ which comes out of a multi-period maximization problem:

$$\max_{\{v_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} (1-\rho)^t \Big(v_t \tilde{r}_{t+1} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' \Big) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$
$$\max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \Big(v_t \tilde{\mu} + \frac{\lambda}{2} v_t \Omega_{t+1} v_t' \Big) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \tag{28}$$

This problem can be solved via dynamic programming using the Bellman equation using the method in section 2.1. Consider an investor with preference as (1) and facing the maximization problem in (28) then the period $t = 0$ and $t = 1$, denoted L_t, are remarkably similar

$$L_0 = \max_{\{v_t\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} (1-\rho)^t \Big(v_t \tilde{\mu} + \frac{\lambda}{2} v_t \Omega_{t+1} v_t' \Big) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0$$
$$L_1 = \max_{\{v_t\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} (1-\rho)^t \Big(v_t \tilde{\mu} + \frac{\lambda}{2} v_t \Omega_{t+1} v_t' \Big) \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 1$$

As in Section 2.1, we note that L_1 is contained with L_0 and we can thus rewrite the L_0 as "two period" model with immediate reward and later rewards.

$$\begin{aligned} L_0 &= \max_{\{v_t\}_{t=0}^{\infty}} \left[(1-\rho)^0 \left(v_0\tilde{\mu} - \frac{\lambda}{2} v_0 \Omega_1 v_0' \right) + \mathbb{E}_t[L_1] \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \\ &= \max_{\{v_t\}_{t=0}^{\infty}} \left[v_0\tilde{\mu} - \frac{\lambda}{2} v_0 \Omega_1 v_0' + \mathbb{E}_t[L_1] \right] \quad \text{s.t.} \quad v_t'\mathbf{1} = 1 \quad \forall t \geq 0 \end{aligned}$$

Denoting L as a function of v_t and denoting it as the value function, V, yields

$$V(v_t) = \max_{v_t} \left[v_t\tilde{\mu} - \frac{\lambda}{2} v_t \Omega_{t+1} v_t' + \mathbb{E}_t[V(v_{t+1})] \right] - \kappa_1(v_t'\mathbf{1} - 1)$$

The dynamic nature of problem changes what the investor maximizes with respect to from $\{v_t\}_{t=0}^{\infty}$ to v_t as the investor dynamically solves the problem one period at the time.
The value function (or Bellman equation) measures the value of the portfolio at time $t+1$ with weight of v_t of the risky asset. Additionally, the value function captures the trade-off between utility now and utility later.
Solving for the optimal weight requires solving the first order conditions. Now, the first order conditions are given by

$$\frac{\partial V(v_t)}{\partial v_t} = \tilde{\mu} - \kappa_1\mathbf{1} - \lambda v_t \Omega_{t+1} = 0 \Leftrightarrow v_t = \lambda^{-1}\Omega_{t+1}^{-1}(\tilde{\mu} - \kappa_1\mathbf{1})$$

The constraint requires that $v_t'\mathbf{1} = 1$, which can be used to solve for the Lagrangian multiplier κ_1

$$\begin{aligned} 1 &= v_t'\mathbf{1} = \mathbf{1}'v_t = \mathbf{1}'\left(\lambda^{-1}\Omega_{t+1}^{-1}(\tilde{\mu} - \kappa_1\mathbf{1})\right) \\ 1 &= \lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} - \lambda^{-1}\mathbf{1}\Omega_{t+1}^{-1}\mathbf{1}\kappa_1 \\ \lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} - 1 &= \lambda^{-1}\mathbf{1}\Omega_{t+1}^{-1}\mathbf{1}\kappa_1 \\ \kappa_1 &= \frac{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} - 1}{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} \end{aligned}$$

Now, we insert the expression into the weights v_t

$$\begin{aligned} v_t &= \lambda^{-1}\Omega_{t+1}^{-1}\left(\tilde{\mu} - \left[\frac{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu} - 1}{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\right]\mathbf{1}\right) = \lambda^{-1}\Omega_{t+1}^{-1}\left(\tilde{\mu} - \left[\frac{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}}{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} - \frac{1}{\lambda^{-1}\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\right]\mathbf{1}\right) \\ &= \lambda^{-1}\Omega_{t+1}^{-1}\left(\tilde{\mu} - \frac{\mathbf{1}'\Omega_{t+1}^{-1}\tilde{\mu}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} + \frac{\lambda\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\right) = \lambda^{-1}\Omega_{t+1}^{-1}\left(\tilde{\mu} - \tilde{\mu} + \frac{\lambda\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}}\right) \\ v_t &= \frac{\Omega_{t+1}^{-1}\mathbf{1}}{\mathbf{1}'\Omega_{t+1}^{-1}\mathbf{1}} = v_t^{\text{MVP}} \end{aligned} \tag{29}$$

Note that given that variance, Ω_t is time-varying the weight, v_t, are likewise time-varying. Additionally, Ω_{t+1} can be written as variables that are $\mathcal{F}_t$-measurable, similar to the univariate case.