Portfolio Selection Based on Estimated Conditional Mean and Variance from Multivariate GARCH Models

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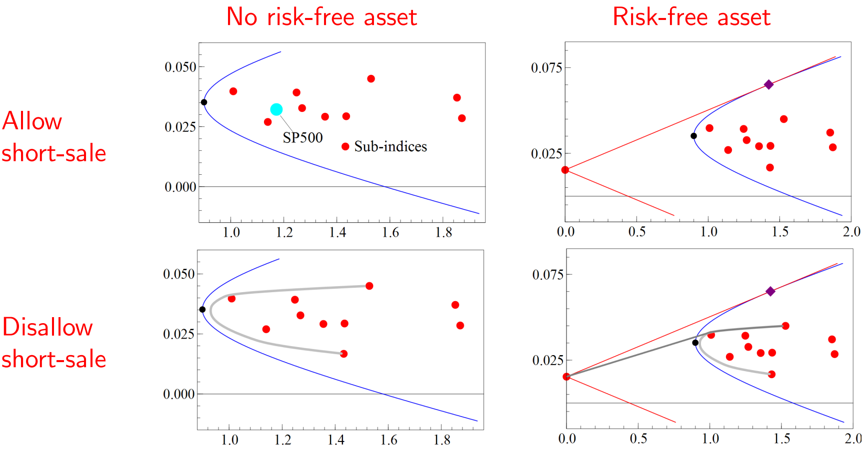
Recall the Mean-Variance Portfolio Selection

Consider returns $R_t \in \mathbb{R}^p$ with $\mu = E(R_t)$ and $\Omega = E((R-\mu)(R-\mu)')$.
The Markowitz analysis finds *(i)* opportunity set and *(ii)* optimal choice.



Dynamic Analysis Based on Multivariate GARCH

- The Markowitz analysis is a one-period optimization based on μ and Ω. To form a portfolio at some point in time T, we may use estimates of mean and variance based on historical data. To do so we need a statistical model.

- Assuming i.i.d. data:

$$R_t \text{ is i.i.d. } N(\mu, \Omega).$$

  The inputs to the portfolio selection could be the MLEs

$$\hat{\mu} = T^{-1} \textstyle\sum_{t=1}^{T} R_t \quad \text{and} \quad \hat{\Omega} = T^{-1} \textstyle\sum_{t=1}^{T} (R_t - \mu)(R_t - \mu)'.$$

- Assuming Multivariate GARCH data:

$$R_t = \mu_t + \epsilon_t, \quad \text{with} \quad \epsilon_t = \Omega_t^{1/2} z_t \quad \text{and} \quad z_t \text{ i.i.d.}(0, I_p),$$

  with some chosen specification

$$\begin{aligned} \mu_t &= E(R_t \mid \mathcal{F}_{t-1}) &&= f(\theta, R_{t-1}, R_{t-2}, ...) \\ \Omega_t &= E((R_t - \mu_t)(R_t - \mu_t)' \mid \mathcal{F}_{t-1}) &&= g(\theta; R_{t-1}, R_{t-2}, ...). \end{aligned}$$

  Now the opportunity set typically changes from time to time.

Example: BEKK Model

- Consider the $p-$dimensional return vector, R_t, $t = 1, 2, ..., T$.

- A simple BEKK-GARCH model is given by

$$\begin{aligned} R_t &= \mu + \epsilon_t \\ \epsilon_t &= \Omega_t^{1/2} z_t, \quad z_t \text{ i.i.d. } N(0, I_p) \\ \Omega_t &= LL' + A\epsilon_{t-1}\epsilon_{t-1}' A' + B\Omega_{t-1} B', \end{aligned}$$

  with $\mu \in \mathbb{R}^p$, $L \in \mathbb{R}^{p\times p}$ lower triangular, and $A, B \in \mathbb{R}^{p\times p}$ unrestricted.

- Number of parameters in $\theta = (\mu, L, A, B)$ is $p + p(p+1)/2 + 2p^2$.

- Given parameters estimates, $\hat{\theta}$, we can calculate

$$\hat{\mu}_t = \hat{\mu} \quad \text{and} \quad \hat{\Omega}_t = \hat{L}\hat{L}' + \hat{A}(R_{t-1} - \hat{\mu})(R_{t-1} - \hat{\mu})'\hat{A}' + \hat{B}\Omega_{t-1}\hat{B}',$$

  and construct portfolio selection based on $\hat{\mu}_t$ and $\hat{\Omega}_t$.

Recursive (Real-Time) Analysis

- In practice, we have to think about the information set for estimation. When forming the portfolio a time T_0, we only know data up to T_0.

- To implement a real-time analysis, we therefore
  1. Use the sample $t = 1, 2, ..., T_0$, and get estimate, $\hat{\theta}_{T_0}$.
  2. Forecast $\mu_{T_0+1|T_0}$ and $\Omega_{T_0+1|T_0}$ and form optimal portfolio weights, v_{T_0+1}.
  3. The portfolio return is given by

$$\bar{R}_{T_0+1} = v'_{T_0+1} R_{T_0+1}$$

  4. Update the sample to $t = 1, 2, ..., T_0 + 1$ and go to step 1.

- We may compare time series of portfolio returns $\bar{R}_{T_0+1}, \bar{R}_{T_0+2}, ...$ based on different models for returns and different portfolio selection strategies.

- To avoid re-estimation at each point, we may reestimate less frequently. In historical analyses it is also normal to "cheat" and use full-sample estimates $\hat{\theta}_T$ in all sub-periods.

Some Additional Open Issues

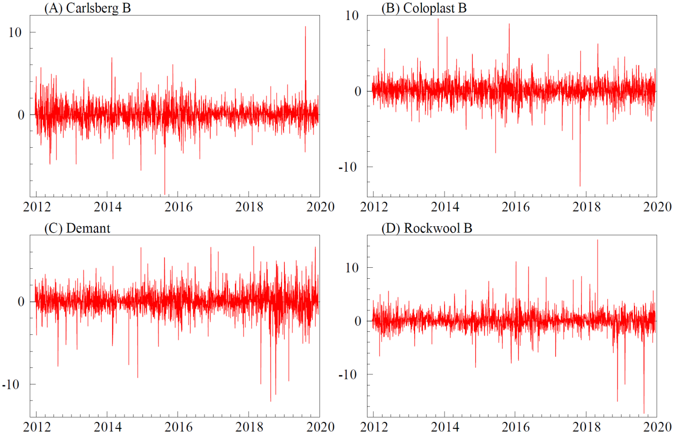
- Timing: Often we consider daily closing prices.
  We can form the portfolio weights at the end of day T_0, and look at the return the next day. This is slightly overoptimistic because it is not, in general, possible to buy at the closing price.

- Trading costs are very important in practice. The simple implementation above rebalances portfolios at each point in time, and implies a lot of trading.

- Performance Measures: There is a large literature on the measurement of portfolio performance.
  High return, low variance, Sharpe ratio, utility, ...

Empirical Example

Consider daily observations, 2012 to 2019, for Carlsberg B, Coloplast B, Demant, and Rockwool B.



Empirical Example

- Consider a BEKK model for returns, $R_t \in \mathbb{R}^4$,

$$\begin{aligned} R_t &= \mu + \epsilon_t \\ \epsilon_t &= \Omega_t^{1/2} z_t \\ \Omega_t &= LL' + A\epsilon_{t-1}\epsilon_{t-1}' A' + B\Omega_{t-1}B', \end{aligned}$$

  for $t = 1, 2, ..., T$ and with z_t i.i.d. $\mathrm{N}_4(0, I_4)$.

- We use the full sample to estimate the parameters, $\hat{\mu}$, $\hat{L}$, $\hat{A}$, and $\hat{B}$.

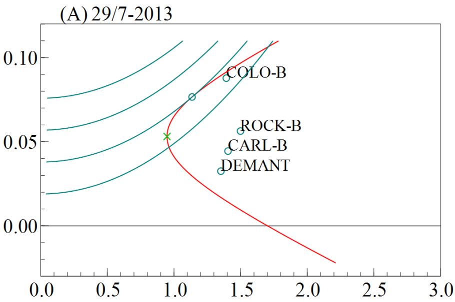
- For each point, $t = 1, 2, ..., T$, we use $\hat{\mu}$ and $\hat{\Omega}_t$ to select a portfolio. The weight, v_t, is measurable at time $t-1$ (and we trade at the closing price).

- The portfolio return is given by

$$\bar{R}_t = v_t' R_t, \quad t = 1, 2, ..., T.$$

Two Random Days

Consider the opportunity set on two random days:

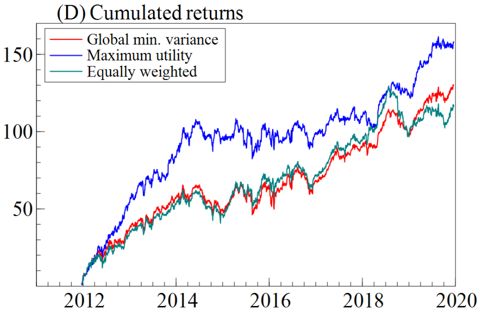


The implied GMV weights are given by

	Carlsberg B	Coloplast B	Demant	Rockwool B
29/7-2013	0.253	0.213	0.292	0.243
5/8-2019	0.476	0.375	−0.051	0.201

Compare Performance

	Global min. var.	Utility optimiz.	Equally weighted
Average return	0.065237	0.079244	0.058272
Standard deviation	0.97864	1.1717	1.0320
Skewness	−0.19563	−0.27737	−0.29419
Kurtosis	4.9364	6.8955	4.8677
Sharpe ratio	0.066661	0.067632	0.056465



Coding

The posted code is somewhat complicated:

1. Data handling.
2. MLE of preferred model (possibly real-time reestimation).
3. Storage of conditional mean and variance.
4. Calculation of optimal portfolios.
5. Analysis of time series of portfolio returns.

Analysis can be split into parts.

DCC Example

- An alternative model is the DCC

$$
\begin{aligned}
R_t &= \mu + \epsilon_t \\
\epsilon_t &= \Omega_t^{1/2} z_t \\
\Omega_t &= D_t \Gamma_t D_t,
\end{aligned}
$$

for $t = 1, 2, ..., T$ and with z_t i.i.d. $\mathrm{N}_4(0, I_4)$.

- Here

$$
D_t = \begin{pmatrix} \sigma_{1t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{4t} \end{pmatrix},
$$

with σ_{it}^2, $i = 1, 2, 3, 4$, a univariate ARCH model, e.g.

$$
\sigma_{it}^2 = \omega_i + \alpha_i \epsilon_{it-1}^2 + \beta_i \sigma_{it-1}^2.
$$

- For the correlations,

$$Q_t = Q(1-a-b) + a\eta_{t-1}\eta'_{t-1} + bQ_{t-1}.$$

This is not a correlation matrix, and we standardize to get

$$\Gamma_t = Q_t^{*-1} Q_t Q_t^{*-1} = \begin{pmatrix} 1 & \rho_{12t} & \rho_{13t} & \rho_{14t} \\ \rho_{12t} & 1 & \rho_{23t} & \rho_{24t} \\ \rho_{13t} & \rho_{23t} & 1 & \rho_{34t} \\ \rho_{14t} & \rho_{24t} & \rho_{34t} & 1 \end{pmatrix},$$

and

$$Q_t^{*-1} = \begin{pmatrix} \sqrt{q_{11t}} & 0 & 0 & 0 \\ 0 & \sqrt{q_{22t}} & 0 & 0 \\ 0 & 0 & \sqrt{q_{33t}} & 0 \\ 0 & 0 & 0 & \sqrt{q_{44t}} \end{pmatrix}.$$

- Observe that a and b are scalars.
  All correlations move with the same speed/sensitivity.

- Q may be estimated as a parameter matrix or fixed (variance targeting).

- The DCC model can be estimated in multiple steps. Consistent but not MLE.

Coding

The posted code is simpler by using the OxMetrics MGARCH package:

1. Data handling.
2. MLE of preferred model.
3. Storage of conditional mean and variance.
4. Calculation of optimal portfolio.