Appendix A: Proofs

In what follows we make repeated use of the notation

$$\bar{\rho} = 1 - \rho, \tag{A1}$$

$$\bar{\Lambda} = \bar{\rho}^{-1}\Lambda, \tag{A2}$$

$$\bar{\lambda} = \bar{\rho}^{-1}\lambda. \tag{A3}$$

Proof of Proposition 1

Assuming that the value function is of the posited form, we calculate the expected future value function as

$$\begin{aligned} E_t[V(x_t, f_{t+1})] = &-\frac{1}{2}x_t^\top A_{xx}x_t + x_t^\top A_{xf}(I-\Phi)f_t + \frac{1}{2}f_t^\top(I-\Phi)^\top A_{ff}(I-\Phi)f_t \\ &+\frac{1}{2}E_t(\varepsilon_{t+1}^\top A_{ff}\varepsilon_{t+1}) + A_0. \end{aligned} \tag{A4}$$

The agent maximizes the quadratic objective $-\frac{1}{2}x_t^\top J_t x_t + x_t^\top j_t + d_t$ with

$$\begin{aligned} J_t &= \gamma\Sigma + \bar{\Lambda} + A_{xx}, \\ j_t &= (B + A_{xf}(I-\Phi))f_t + \bar{\Lambda}x_{t-1}, \\ d_t &= -\frac{1}{2}x_{t-1}^\top\bar{\Lambda}x_{t-1} + \frac{1}{2}f_t^\top(I-\Phi)^\top A_{ff}(I-\Phi)f_t + \frac{1}{2}E_t(\varepsilon_{t+1}^\top A_{ff}\varepsilon_{t+1}) + A_0. \end{aligned} \tag{A5}$$

Does this prove that that the guess satisfies the Bellman equation? And then the rest of the proof is parameterizatio ns etc.

The maximum value is attained by

$$x_t = J_t^{-1}j_t, \tag{A6}$$

which is equal to $V(x_{t-1}, f_t) = \frac{1}{2}j_t^\top J_t^{-1}j_t + d_t$. Combining this fact with (6) we obtain an equation that must hold for all x_{t-1} and f_t, which implies the following restrictions on the coefficient matrices:[13]

$$-\bar{\rho}^{-1}A_{xx} = \bar{\Lambda}(\gamma\Sigma + \bar{\Lambda} + A_{xx})^{-1}\bar{\Lambda} - \bar{\Lambda}, \tag{A7}$$

$$\bar{\rho}^{-1}A_{xf} = \bar{\Lambda}(\gamma\Sigma + \bar{\Lambda} + A_{xx})^{-1}(B + A_{xf}(I-\Phi)), \tag{A8}$$

$$\begin{aligned} \bar{\rho}^{-1}A_{ff} = &(B + A_{xf}(I-\Phi))^\top(\gamma\Sigma + \bar{\Lambda} + A_{xx})^{-1}(B + A_{xf}(I-\Phi)). \\ &+(I-\Phi)^\top A_{ff}(I-\Phi). \end{aligned} \tag{A9}$$

The existence of a solution to this system of Riccati equations can be established using standard results, for example, as in Ljungqvist and Sargent (2004). In this case, however, we can derive explicit expressions as follows. We start by letting $Z = \bar{\Lambda}^{-\frac{1}{2}}A_{xx}\bar{\Lambda}^{-\frac{1}{2}}$ and $M = \bar{\Lambda}^{-\frac{1}{2}}\Sigma\bar{\Lambda}^{-\frac{1}{2}}$, and rewriting equation (A7) as

$$\bar{\rho}^{-1}Z = I - (\gamma M + I + Z)^{-1}, \tag{A10}$$

[13] Remember that A_{xx} and A_{ff} can always be chosen to be symmetric.

which is a quadratic with an explicit solution. Since all solutions Z can be written as a limit of polynomials of the matrix M, we see that Z and M commute and the quadratic can be sequentially rewritten as

$$Z^2 + Z(I + \gamma M - \bar{\rho} I) = \bar{\rho}\gamma M, \tag{A11}$$

$$\left(Z + \frac{1}{2}(\gamma M + \rho I)\right)^2 = \bar{\rho}\gamma M + \frac{1}{4}(\gamma M + \rho I)^2, \tag{A12}$$

resulting in

$$Z = \left(\bar{\rho}\gamma M + \frac{1}{4}(\rho I + \gamma M)^2\right)^{\frac{1}{2}} - \frac{1}{2}(\rho I + \gamma M), \tag{A13}$$

$$A_{xx} = \bar{\Lambda}^{\frac{1}{2}}\left[\left(\bar{\rho}\gamma M + \frac{1}{4}(\rho I + \gamma M)^2\right)^{\frac{1}{2}} - \frac{1}{2}(\rho I + \gamma M)\right]\bar{\Lambda}^{\frac{1}{2}}, \tag{A14}$$

that is,

$$\begin{aligned} A_{xx} = &\left(\bar{\rho}\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \frac{1}{4}(\rho^2\bar{\Lambda}^2 + 2\rho\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \gamma^2\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{-1}\Sigma\bar{\Lambda}^{\frac{1}{2}})\right)^{\frac{1}{2}} \\ &-\frac{1}{2}(\rho\bar{\Lambda} + \gamma\Sigma). \end{aligned} \tag{A15}$$

Note that the positive-definite choice of solution Z is the only one that results in a positive-definite matrix A_{xx}.

The other value function coefficient determining optimal trading is A_{xf}, which solves the linear equation (A8). To write the solution explicitly, we note first that, from (A7),

$$\bar{\Lambda}(\gamma\Sigma + \bar{\Lambda} + A_{xx})^{-1} = I - A_{xx}\Lambda^{-1}. \tag{A16}$$

Using the general rule that $\mathrm{vec}(XYZ) = (Z^{\top} \otimes X)\mathrm{vec}(Y)$, we rewrite (A8) in vectorized form:

$$\mathrm{vec}(A_{xf}) = \bar{\rho}\mathrm{vec}((I - A_{xx}\Lambda^{-1})B) + \bar{\rho}((I - \Phi)^{\top} \otimes (I - A_{xx}\Lambda^{-1}))\mathrm{vec}(A_{xf}), \tag{A17}$$

so that

$$\mathrm{vec}(A_{xf}) = \bar{\rho}\left(I - \bar{\rho}(I - \Phi)^{\top} \otimes (I - A_{xx}\Lambda^{-1})\right)^{-1}\mathrm{vec}((I - A_{xx}\Lambda^{-1})B). \tag{A18}$$

Finally, A_{ff} is calculated from the linear equation (A9), which is of the form

$$\bar{\rho}^{-1}A_{ff} = Q + (I - \Phi)^{\top}A_{ff}(I - \Phi) \tag{A19}$$

with

$$Q = (B + A_{xf}(I - \Phi))^{\top}(\gamma\Sigma + \bar{\Lambda} + A_{xx})^{-1}(B + A_{xf}(I - \Phi)), \tag{A20}$$

a positive-definite matrix.

The solution is easiest to write explicitly for diagonal Φ, in which case

$$A_{ff,ij} = \frac{\bar{\rho}\, Q_{ij}}{1-\bar{\rho}(1-\Phi_{ii})(1-\Phi_{jj})}. \tag{A21}$$

In general,

$$\operatorname{vec}\left(A_{ff}\right) = \bar{\rho}\left(I-\bar{\rho}(I-\Phi)^{\top}\otimes(I-\Phi)^{\top}\right)^{-1}\operatorname{vec}(Q). \tag{A22}$$

One way to see that A_{ff} is positive-definite is to iterate (A19) starting with $A^0_{ff}=0$.

We conclude that the posited value function satisfies the Bellman equation. Q.E.D.

1: Why do the authors differentiate wrt. to x_(t-1) when equation 5 is stated as max x_t and not max x_(t-1)

Proof of Proposition 2

Differentiating the Bellman equation (5) with respect to x_{t-1} gives

$$-A_{xx}x_{t-1} + A_{xf}\,f_t = \Lambda(x_t - x_{t-1}),$$

2: Even if we differentiate wrt. x_t or x_(t-1) we cannot get this result. Do they insert something extra in the Bellman equation?

which clearly implies (7) and (8).

In the case $\Lambda=\lambda\Sigma$ for some scalar $\lambda>0$, the solution to the value function coefficients is $A_{xx}=a\Sigma$, where a solves a simplified version of (A7):

$$-\bar{\rho}^{-1}a = \frac{\bar{\lambda}^2}{\gamma+\bar{\lambda}+a}-\bar{\lambda}, \tag{A23}$$

or

$$a^2+(\gamma+\bar{\lambda}\rho)a-\lambda\gamma=0, \tag{A24}$$

with solution

$$a=\frac{\sqrt{(\gamma+\bar{\lambda}\rho)^2+4\gamma\lambda}-(\gamma+\bar{\lambda}\rho)}{2}. \tag{A25}$$

It follows immediately that $\Lambda^{-1}A_{xx}=a/\lambda$.

Note that a is symmetric in $(\lambda\rho(1-\rho)^{-1},\gamma)$. Consequently, a increases in λ if and only if it increases in γ. Differentiating (A25) with respect to λ, one gets

$$2\frac{da}{d\lambda} = -\bar{\rho}^{-1}\rho+\frac{1}{2}\frac{(2(\gamma+\bar{\lambda}\rho)+4\gamma)}{\sqrt{(\gamma+\bar{\lambda}\rho)^2+4\gamma\lambda}}. \tag{A26}$$

This expression is positive if and only if

$$\bar{\rho}^{-2}\rho^2\left((\gamma+\bar{\lambda}\rho)^2+4\gamma\lambda\right)\le\left((\gamma+\bar{\lambda}\rho)\bar{\rho}^{-1}\rho+2\gamma\right)^2, \tag{A27}$$

which is verified to hold with strict inequality as long as $\bar{\rho}\gamma>0$.