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Florian Herzog
ETH Zurich

Hans P. Geering
ETH Zurich

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STRATEGIC ASSET ALLOCATION WITH FACTOR MODELS FOR RETURNS AND GARCH MODELS FOR VOLATILITIES

Florian Herzog
Measurement and Control Laboratory
Swiss Federal Institute of Technology (ETH)
CH-8092 Zurich, Switzerland
herzog@imrt.mavt.ethz.ch
+41-1-6320861

Gabriel Dondi
Measurement and Control Laboratory
Swiss Federal Institute of Technology (ETH)
CH-8092 Zurich, Switzerland
dondi@imrt.mavt.ethz.ch

Hans P. Geering
Measurement and Control Laboratory
Swiss Federal Institute of Technology (ETH)
CH-8092 Zurich, Switzerland
geering@imrt.mavt.ethz.ch

ABSTRACT
This paper develops a discrete time portfolio optimization for multi asset and long-term investment objectives. The expected returns of the risky assets are modelled using a factor model based on stochastic Gaussian processes. A multivariable Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model describes volatilities and correlations of the risky assets. A mean-variance objective function is maximized over terminal wealth κ steps into the future resulting in the decision rules for the optimal portfolio choice. In order to solve this multi-period constrained optimization problem, a model predictive control approach is used to solve the optimal control problem.

KEY WORDS
Stochastic Optimization, Optimal Control, Statistical and Probabilistic Modelling, Portfolio Management, and Finance.

1 Introduction

“Strategic asset allocation” was introduced in [1] as an expression describing a portfolio optimization problem with time-varying returns and objectives typical for long-term investments. In this paper, we approach the strategic asset allocation problem based on a dynamic model for both the expected returns and the expected risks. The risky asset model consists of two parts: the expected conditional return model and the conditional covariance model. The expected returns are modelled as an affine function of the factor levels, which in turn are modelled as stochastic linear (Gaussian) processes. The conditional covariance matrix is modelled by a dynamic correlation model and a dynamic volatility model for the individual volatilities of the assets. The portfolio optimization is carried out with respect to a mean-variance objective function, where the mean and variance of the portfolio are computed κ steps into the future. By using a model predictive control approach, the optimal control problem for the portfolio optimization is solved. Furthermore, we show that the problem is a true multi-period asset allocation problem rather than a multiple single-period decision problem. In a case study the usefulness of the approach is demonstrated.

2 Asset Returns

The returns of assets (or asset classes), in which we are able to invest, are described by

$$r_t = \mu_t + \epsilon_t^r \,, \tag{1}$$

where $r_t \in \mathbb{R}^n$ is the vector of asset returns, ϵ_t^r is a white noise process with $\mathrm{E}[\epsilon_t^r] = 0$ and $\mathrm{E}_t[\epsilon_t^r \epsilon_t^{rT}] = H_t$, μ_t is the expected return, and H_t is the conditional covariance matrix. We assume that the conditional expectation and the conditional covariance are time-varying and stochastic.

2.1 Conditional Expected Returns

The expected returns of the assets μ_t are modelled by

$$\mu_t = Gx_{t-1} + g \,, \tag{2}$$

where $G \in \mathbb{R}^{n\times m}$ is the factor loading matrix, g is a constant, and $x_t \in \mathbb{R}^m$ is the vector of factor levels. The factor process x_t allows us to model variables of either macroeconomic or industry specific nature that effect the mean returns of the assets and vary over time. We assume that the factors are driven by linear stochastic processes, described by

$$x_t = Ax_{t-1} + b + \epsilon_t^x \,, \tag{3}$$

where $A \in \mathbb{R}^{m\times m}$ and $b \in \mathbb{R}^m$. The white noise process can be written as $\epsilon_t^x = \nu\xi_t^x$, with $\nu \in \mathbb{R}^{m\times m}$ and where the standard residuals are characterized by $\mathrm{E}[\xi_t^x] = 0$ and $\mathrm{E}[\xi_t^x \xi_t^{xT}] = I$.

2.2 Conditional Covariances

Realistic models for asset returns should reflect observed stylized facts of financial time series, such as the following: return data is not identically and independently distributed (i.i.d.) and possesses low serial correlation; squared returns have high serial correlation; the standard deviation of return data, also called volatility, is observed to be time-varying and stochastic; volatility has the tendency to appear in clusters (persistence) and has the tendency to increase more when equity returns are falling (leverage effect). For this reasons, we choose to model the covariances by a multivariate GARCH model, which captures these stylized facts. The dynamics of the conditional covariance matrix $H_t = \psi_t \psi_t^T$ are described by a multi-variable GARCH process. The structure of the random process is given by

$$\epsilon_t^r = \psi_t \xi_t^r \,, \tag{4}$$

where we assume that $\mathrm{E}[\xi_t] = 0$ and $\mathrm{E}[\xi_t^r \xi_t^{rT}] = I \in \mathbb{R}^{n \times n}$. In this work we make the assumption that ξ_t^r is drawn from an elliptical symmetric distribution, such as the normal or the student t distribution. The model of the covariance matrix is given by

$$H_t = \Sigma_t \Lambda_t \Sigma_t \,, \tag{5}$$

where

$$\Sigma_t = \begin{pmatrix} \sigma_{1t} & 0 & \dots & 0 \\ 0 & \sigma_{2t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{nt} \end{pmatrix} . \tag{6}$$

The conditional covariance matrix H_t must be a positive definite symmetric matrix and therefore $\sigma_{it} > 0$ and $\Lambda_t > 0$. The GARCH structure is determined by the model we choose for the individual σ_{it} and the conditional correlation matrix Λ_t. For the individual conditional variances we choose the threshold GARCH (TARCH) model given by

$$\begin{aligned} \sigma_{it}^2 \quad = \quad & \omega_i + \sum_{j=1}^{q_i} \alpha_{ij} \epsilon_{i(t-j)}^{r2} + \sum_{j=1}^{p_i} \beta_{ij} \sigma_{i(t-j)}^2 \\ & + \sum_{j=1}^{o_i} \gamma_{ij} \chi_{i(t-j)} \epsilon_{i(t-j)}^{r2} \,, \end{aligned} \tag{7}$$

where $w_i > 0$, $\alpha_{ij} > 0$, $\beta_{ij} > 0$, and $\gamma_{ij} > 0$ and $\chi_{i(t-j)} = 1$ if $\epsilon_{i(t-j)}^r < 0$ and zero otherwise. The parameters (o_i, p_i, q_i) give the number of respective lags used for the three terms of the model. The TARCH(o,p,q) model correctly describes the three main observed stylized facts. The volatility increases when large absolute returns occur, volatility shows persistence (clusters), and volatility rises strongly when negative returns occur. A discussion of this volatility model can be found in [2] and [3]. In the case when $\gamma_{ij} = 0$, the TARCH model is reduced to the well known GARCH(p,q) model.
The conditional correlation, which describes the dependence structure of the asset returns, is modelled with the so-called dynamic conditional correlation (DCC) model. It was derived to improve the constant correlation model, since empirical data analysis of stock and bond prices suggest time variation of correlations. The so-called DCC(k,l) model is obtained by

$$\begin{aligned} Q_t = & (1 - \sum_{j=1}^{k} \delta_j - \sum_{i=1}^{l} \eta_i) \overline{Q} \\ & + \sum_{j=1}^{k} \delta_j (\zeta_{t-j} \zeta_{t-j}^T) + \sum_{j=1}^{l} \eta_j Q_{t-j} \end{aligned} \tag{8}$$

$$\Lambda_t = Q_t^{*-1} Q_t Q_t^{*-1} \tag{9}$$

$$Q_t^* = \begin{pmatrix} \sqrt{q_{11}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sqrt{q_{nn}} \end{pmatrix} , \tag{10}$$

where $\overline{Q} \in \mathbb{R}^{n \times n}$ denotes the unconditional correlation matrix of standardized residuals $\zeta_t = \Sigma_t^{-1} \epsilon_t^r$, $\delta_j > 0$, and $\eta_i > 0$ in order to make sure that Λ_t is a positive definite matrix. The DCC model allows us to follow changes in the dependence of assets and helps us to optimally design a diversification strategy. With the constant correlation assumption, we miss the important fact that correlations change over time and thus do not hedge accordingly. The properties of the dynamic conditional correlation model are discussed in [4] and [5].

2.3 Portfolio Dynamics

Based on the return models of the risky assets, we derive an approximation to the portfolio (wealth) dynamics in discrete time. We assume that an investor can chose among n risky assets and a risk-free bank account. The risky assets returns r_t and the interest rate of the bank account r_t^B are given by

$$r_t \quad = \quad G x_{t-1} + g + \epsilon_t^r \tag{11}$$

$$r_t^B \quad = \quad F_B x_{t-1} + h_B \,, \tag{12}$$

where the dynamics of the risky asset returns are as described in Section 2.1 and 2.2 and the bank account interest rate is an affine function of the factor levels x_t. The investments into the bank account, denoted by u_{0t-1} and the investment into the risky assets, denoted by u_{t-1}, must sum up to 1, i.e., $u_{0t-1} + u_t^T \overline{o} = 1$, where $\overline{o} = (1, 1, \dots, 1)^T$. The portfolio return at time t is given by $R_t = u_{0t-1} r_t^B + u_{t-1}^T r_t$. When solving for $u_{0t-1} = 1 - u_t^T \overline{o}$, we can express the portfolio return as

$$\begin{aligned}R_t &= r_t^B + u_{t-1}^T(r_t - \bar{o} r_t^B) \\ &= F_B x_{t-1} + h_B + u_{t-1}^T(F x_{t-1} + h) \\ &\quad + u_{t-1}^T \epsilon_t^r \,, \end{aligned} \tag{13}$$

where $u_t = (u_{1t}, \dots, u_{nt})^T$, $F = G - \bar{o}F_B$, $h = g - \bar{o}h_B$, and u_{jt} denotes the fraction of portfolio value (wealth) invested into the j-th risky investment. The wealth dynamics are given by

$$\begin{aligned}W_t &= (1 + R_t)W_{t-1} \\ w_t &= \ln(1 + R_t) + w_{t-1} \,, \end{aligned} \tag{14}$$

where $W_t \in \mathbb{R}$ denotes the portfolio value at time t and $w_t = \ln W_t$. The nonlinear dynamics of (14) are replaced by the following Taylor series approximation (around the mean of R_t)

$$\ln(1 + R_t) \approx R_t - \frac{1}{2}R_t^2 \,. \tag{15}$$

Also, we replace R_t^2 by its conditional expectation, i.e., Var$[R_t]$. The same approximation is also used in [6]. The error term of the approximation has a magnitude of order of R_t^3. This corresponds to the magnitude of order of the third moment. In the case of weekly S&P 500 data, the mean error is of the size 10^{-3}%.
The wealth dynamics are obtained by

$$\begin{aligned}w_t &= w_{t-1} + F_B x_{t-1} + h_B + u_{t-1}^T(F x_{t-1} + h) \\ &\quad -\frac{1}{2}u_{t-1}^T H_t u_{t-1} + u_{t-1}^T \epsilon_t^r \,, \end{aligned} \tag{16}$$

where Var$[R_t] = u_{t-1}^T H_t u_{t-1}$ and $w_t \in \mathbb{R}$. For the portfolio dynamics, we derive the conditional expectation and variance which we use for the portfolio optimization. The portfolio dynamics are fully described by

$$\begin{aligned}w_{t+1} &= w_t + F_B x_t + h_B + u_t^T(F x_t + h) \\ &\quad -\frac{1}{2}u_t^T H_{t+1} u_t + u_t \epsilon_{t+1}^r \\ x_{t+1} &= A x_t + b + \epsilon_{t+1}^x \,, \end{aligned} \tag{17}$$

where we have shifted the time index by one. Additionally, we assume that standard residuals of the two withe noise processes are correlated, i.e., $\mathrm{E}[\xi_{t+1}^r \xi_{t+1}^x] = \rho$.

3 Portfolio Optimization

For the portfolio optimization we assume that the investor pursues a long-term goal, such as to optimize the portfolio risk-adjusted return κ periods ahead. Since we assume that returns are not i.i.d, the optimization of a gaol κ periods ahead is not simply κ times the single period optimization. This is a crucial difference to the standard Markovitz portfolio optimization, where returns are assumed to be i.i.d., see [7, Chapter 6]. The investor is able to change the composition of his portfolio at every period. Thus, the underlying problem is a multi-period asset allocation problem.
We assume that the objective function of the investor is a mean-variance objective of the portfolio wealth κ steps into the future. It can be shown, that maximizing a power utility is equivalent to a mean-variance objective function for the portfolio returns, as derived in [8].

3.1 Portfolio Mean and Variance

The portfolio equations (17) can be written as

$$\begin{aligned}\underbrace{\begin{pmatrix} w_{t+1} \\ x_{t+1} \end{pmatrix}}_{y_{t+1}} &= \underbrace{\begin{bmatrix} 1 & F_B + u_t^T F \\ 0 & A \end{bmatrix}}_{A_y} \underbrace{\begin{pmatrix} w_t \\ x_t \end{pmatrix}}_{y_t} \\ &+ \underbrace{\begin{bmatrix} h_B + u_t^T h - \frac{1}{2}u_t^T H_{t+1} u_t \\ b \end{bmatrix}}_{b_y} + \underbrace{\begin{pmatrix} u_t \epsilon_{t+1}^r \\ \epsilon_{t+1}^x \end{pmatrix}}_{\epsilon_{t+1}}, \end{aligned} \tag{18}$$

or even more concisely :

$$y_{t+1} = A_y y_t + b_y + \epsilon_{t+1} \,. \tag{19}$$

Note that

$$\begin{aligned}\mathrm{E}_t[\epsilon_{t+1}\epsilon_{t+1}^T] &= \Omega_{t+1} \\ &= \begin{bmatrix} u_t^T H_{t+1} u_t & u_t^T \psi_{t+1} \rho \nu \\ \nu^T \rho^T \psi_{t+1}^T u_t & \nu \nu^T \end{bmatrix} . \end{aligned}$$

The dynamic evolution of the conditional covariance matrix is obtained by

$$V_{\tau+1|t} = A_y V_{\tau|t} A_y^T + \mathrm{E}_t[\Omega_{\tau+1}] \,, \tag{20}$$

where $\tau > t$ is a future time as described in [8, Appendix] and $V_{\tau|t} = \mathrm{E}_t[(y_\tau - \mathrm{E}_t[y_\tau])(y_\tau - \mathrm{E}_t[y_\tau])^T]$. The covariance matrix dynamics are computed conditioned on the information at time t. The variance of the portfolio system can be decomposed into

$$V_{\tau|t} = \begin{bmatrix} V_{\tau|t}^{(w)} & V_{\tau|t}^{(wx)} \\ V_{\tau|t}^{(wx)T} & V_{\tau|t}^{(x)} \end{bmatrix},$$

where the superscripted index in brackets indicate the covariance between w and x, or rather the covariance matrix of x and the variance of w. We expand equation (20) by plugging in A_y, $V_{\tau|t}$, and $\mathrm{E}_t[\Omega_{\tau+1}]$. This yields

$$\begin{aligned}V_{\tau+1|t}^{(w)} &= V_{\tau|t}^{(w)} + (F_B + u_\tau^T F)(V_{\tau|t}^{(wx)})^T \\ &+ V_{\tau|t}^{(wx)}(F_B^T + F^T u_\tau) \\ &+ (F_B + u_\tau^T F)V_{\tau|t}^{(x)}(F_B^T + F^T u_\tau) \\ &+ u_\tau^T \mathrm{E}_t[H_{\tau+1}]u_\tau \qquad (21) \\ V_{\tau+1|t}^{(wx)} &= V_{\tau|t}^{(wx)} A^T + (F_B + u_\tau^T F)V_{\tau|t}^{(x)} A^T \\ &+ u_\tau^T \mathrm{E}_t[\psi_{\tau+1}]\rho\nu \qquad (22) \\ V_{\tau+1|t}^{(x)} &= A V_{\tau|t}^{(x)} A^T + \nu\nu^T \,. \qquad (23) \end{aligned}$$

We can calculate the solutions of the difference equation by induction and get

$$V^{(x)}_{t+\kappa|t} = \sum_{i=0}^{\kappa-1} A^i \nu\nu^T (A^T)^i \tag{24}$$

$$V^{(wx)}_{t+\kappa|t} = \sum_{i=0}^{\kappa-1} (F_B + u^T_{t+i}F) V^{(x)}_{t+i|t} A^{\kappa-i} + \sum_{i=0}^{\kappa-1} u^T_{t+i} \mathrm{E}_t[\psi_{t+1+i}]\rho\nu A^{\kappa-1-i} \tag{25}$$

$$V^{(w)}_{t+\kappa|t} = \sum_{i=0}^{\kappa-1} \Big\{ (F_B + u^T_{t+i}F)(V^{(wx)}_{t+i|t})^T \tag{26}$$

$$+(F_B + u^T_{t+i}F) V^{(x)}_{t+i|t} (F_B^T + F^T u_{t+i}) + u^T_{t+i}\mathrm{E}_t[H_{t+1+i}]u_{t+i} \Big\}, \tag{27}$$

where we used the initial value $V_{t|t} = 0$. In order to compute the variance of the wealth equation, we insert $V^{(wx)}_{t+\kappa|t}$ and $V^{(x)}_{t+\kappa|t}$ into $V^{(w)}_{t+\kappa|t}$, and write

$$\begin{aligned} V^{(w)}_{t+\kappa|t} = \sum_{i=0}^{\kappa-1} \Big[& 2(F_B + u^T_{t+i}F) \\ & \cdot\Big(\sum_{j=0}^{i-1} \Big\{ A^{i-1-j}\nu^T\rho^T \overline{\psi}^T_{t+1+j} u_{t+j} \\ & + A^{i-j} \sum_{l=0}^{j-1} (A^l \nu\nu^T (A^T)^l) \\ & \cdot (F_B^T + F^T u_{t+j}) \Big\} \Big) \\ & + (F_B + u^T_{t+i}F) \sum_{j=0}^{i-1} (A^j \nu\nu^T (A^T)^j) \\ & \cdot (F_B^T + F^T u_{t+i}) \\ & + u^T_{t+i} \overline{H}_{t+1+i} u_{t+i} \Big], \end{aligned} \tag{28}$$

where $\overline{\psi}_{t+1+i} = \mathrm{E}_t[\psi_{t+1+i}]$ and $\overline{H}_{t+1+i} = \mathrm{E}_t[H_{t+1+i}]$. In order to compute the variance matrix of the portfolio we need to compute the expectation of the covariance matrix κ steps into the future, i.e., $\mathrm{E}_t[H_{t+\kappa}]$. One may notice that covariance depends on multiplications of u_{t+j} and u_{t+i}. This fact links the decision variables from one period with the decision variables from another period and thus makes this problem a true multi-period decision problem. Therefore, the portfolio allocation is a strategic rather than a tactical asset allocation problem. The factor linking the different periods are the correlation of returns and factors. The conditional mean for a time τ can be computed as $m_{\tau+1|t} = A_y m_{\tau|t} + b_y$ and is obtained by

$$\begin{aligned} m^{(w)}_{\tau+1|t} &= m^{(w)}_{\tau|t} + F_B m^{(x)}_{\tau|t} + h_B \\ &\quad + u^T_\tau \big(F m^{(x)}_{\tau|t} + h\big) - \frac{1}{2} u^T_\tau \overline{H}_{\tau+1} u_\tau \\ m^{(x)}_{\tau+1|t} &= A m^{(x)}_{\tau|t} + b\,, \end{aligned} \tag{29}$$

where $m^{(w)}_{\tau|t} = \mathrm{E}_t[w_\tau]$ and $m^{(x)}_{\tau|t} = \mathrm{E}_t[x_\tau]$. By iterating (29) over $\kappa - 1$ time steps we obtain

$$\begin{aligned} m^{(w)}_{t+\kappa} = \; & w_t + h_B\kappa \\ & + \sum_{i=0}^{\kappa-1} \Big\{ (F_B + u^T_{t+i}F) \big[A^i x_t \\ & + \sum_{j=0}^{i-1} A^{i-1-j} b \big] + u^T_{t+i} h \\ & - u^T_{t+i} \overline{H}_{t+i+1} u_{t+i} \Big\}, \end{aligned} \tag{30}$$

where the fact that $m^{(x)}_{t|t} = x_t$ and $m^{(w)}_{t|t} = w_t$ is already used.

3.2 Covariance Matrix Prediction

In this section we briefly outline the prediction of the DCC and the TARCH models based on the TARCH(1,1,1) and DCC(1,1) examples. The general case is beyond the scope of this paper and can be found in [8]. Using (7) and setting $p_i = q_i = o_i = 1$, we compute the expectation of σ_{t+2} based on the conditional information until time t and obtain

$$\begin{aligned} \mathrm{E}_t[\sigma^2_{it+2}] = \; & \omega_i + \alpha_i \mathrm{E}_t[\epsilon^2_{it+1}] + \beta_i \mathrm{E}_t[\sigma^2_{it+1}] \\ & + \gamma_i \mathrm{E}_t[\chi_{it+1}\epsilon^2_{it+1}]\,. \end{aligned} \tag{31}$$

Based on the conditional information until t, we know $\mathrm{E}_t[\epsilon^2_{it+1}] = \sigma^2_{it+1}$, $\mathrm{E}_t[\sigma^2_{it+1}] = \sigma^2_{it+1}$, and $\mathrm{E}_t[\chi_{it+1}\epsilon^2_{it+1}] = \frac{1}{2}\sigma^2_{it+1}$, because we made the assumption that ξ^τ_{it} possesses unit variance and is drawn from a symmetric distribution. Furthermore from (7), and the knowledge of all information at time t, we know σ_{it+1}. Using this we obtain

$$\mathrm{E}_t[\sigma^2_{it+2}] = \omega_i + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)\sigma^2_{it+1}\,. \tag{32}$$

Computing the expectation of $\sigma^2_{it+\tau}$, it results in the analogous discrete difference equation as (32), namely

$$\mathrm{E}_t[\sigma^2_{it+\tau}] = \omega_i + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)\mathrm{E}_t[\sigma^2_{it+\tau-1}]. \tag{33}$$

Combining (32) and (33) and iterating the result $(\kappa - 1)$ times, we obtain the prediction equation

$$\begin{aligned} \mathrm{E}_t[\sigma^2_{it+\kappa}] = \; & \omega_i \sum_{j=1}^{\kappa-1} (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)^j \\ & + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)^{\kappa-1} \sigma^2_{it+1}. \end{aligned} \tag{34}$$

When $\gamma_i = 0$, the TARCH(1,1,1) model is reduced to a GARCH(1,1) model for which the prediction equation can be found in [9]. In the case of dynamic conditional correlations, the prediction equations are difficult to derive, because the model is nonlinear: when inspecting (8), we see that $\mathrm{E}_t[\zeta_{t+\kappa}\zeta_{t+\kappa}^T] = \Lambda_{t+\kappa} = Q_{t+\kappa}^{*-1} Q_{t+\kappa} Q_{t+\kappa}^{*-1}$ and thus the model is non-linear. In [4] the authors proposed to use the approximation $\mathrm{E}_t[Q_{t+\kappa}] \approx \mathrm{E}_t[\Lambda_{t+\kappa}]$. This results in the following approximated prediction equation for the DCC(1,1) model

$$\begin{aligned}\mathrm{E}_t[\Lambda_{t+\kappa}] = &\sum_{j=1}^{\kappa-1}(1-\delta-\eta)\overline{Q}(\delta+\eta)^j \\ &+(\delta+\eta)^{\kappa-1}\Lambda_{t+1}\,. \end{aligned} \tag{35}$$

It possesses the same structure as the GARCH(1,1) model and therefore we can derive an equation that is similar to (34), but with $\gamma = 0$. Having computed the prediction of the individual variances and the correlation, we now know the prediction of the conditional covariance matrix.

3.3 Optimal Portfolio Control

The objective of the portfolio optimization is to maximize the mean-variance objective function of the portfolio κ time steps into the future. The objective is given by

$$m_{t+\kappa|t}^{(w)} + \frac{1}{2}\lambda V_{t+\kappa|t}^{(w)}\,, \tag{36}$$

where $\lambda < 0$ is the risk aversion coefficient and κ denotes the time horizon. By inspecting the mean $m_{t+\kappa|t}^{(w)}$ and the variance $V_{t+\kappa|t}^{(w)}$, we note that both are linear-quadratic functions of u_{t+i} for $i = 0, \dots, \kappa-1$. Thus, the optimization is a quadratic program. The optimization problem is given by

$$\begin{aligned} \max \quad & c(t)^T U + \frac{1}{2}U^T D(t) U \\ s.t. \quad & EU \leq e\,, \end{aligned} \tag{37}$$

where $c(t)$ and $D(t)$ can be assembled tediously with (30), (28), and (36), $U = (u_t, u_{t+1}, ..., u_{t+\kappa-1})$, and E and e can be used to impose constraints on the asset allocation. An example for $D(t)$ and $c(t)$ for $\kappa = 2$ is given by

$$D(t) = \begin{bmatrix} (\lambda-1)H_{t+1} & \lambda\psi_{t+1}\rho\nu F^T \\ \lambda F\nu^T\rho^T\psi_{t+1}^T & (\lambda-1)\overline{H}_{t+2}+F\nu\nu^T F^T \end{bmatrix}$$

$$c(t) = \begin{bmatrix} 2\lambda\psi_{t+1}\rho\nu F_B^T + Fx_t + h \\ h + F[Ax_t + b] + 2\lambda F\nu\nu^T F_B \end{bmatrix}.$$

From the example, we see that the decision for the first period is linked with the decision for the second period. In the case of uncorrelated factors and returns ($\rho = 0$) the two decision variables are independent and the optimization is reduced to multiple single period optimizations. Further examples for $c(t)$ and $D(t)$ with $k > 2$ are given in [8], which we omit for brevity. The vector $c(t)$ and the matrix $D(t)$ depend on the last conditional information at time t and have to be computed at every time step.

We solve the optimal control problem by using a strategy taken from model predictive control, consult [10]. At every time step, we compute $c(t)$ and $D(t)$ and solve for U according to (37). From the obtained solution, we apply just the first u_t and disregard all other u_{t+i} for $i = 1, \dots, \kappa-1$. We then repeat the optimization again at time $t+1$ over the same horizon, i.e., we assume that the time horizon is always constant and equals κ. The main advantage of this sliding horizon procedure is, that we thereby introduce a feedback into the portfolio optimization as the decision variable u_t is constantly based on last information regarding expected risks and returns of risky assets. Moreover, by this procedure we also solve the multi-period asset allocation problem.

4 Case Study

In the case study, we use nine US stock market indices and a government bond index as risky assets. The stock market indices are nine direct sub-indices of the S&P 500, Table 1, which are obtained from Thomson DATASTREAM. All time series start on 01/01/1973 and end on 11/28/2003 with a weekly sampling frequency. The bond index is the 10 year constant maturity treasury bond index. For the bank account, we use the 1-month treasury note interest rate. As factors for each stock market index, we use the 10 year treasury bond interest rate, the earnings-price (EP) ratio of the index, past 150 weeks average return of the index, and the past one week return of the index. Thus, every index return is modelled by four factors. The interest rate and the EP ratio are fundamental values, which should influence the expected returns due to their impact on the underlying economics of the industry. The other two factors should rather reflect short-term trends, which are motivated by the behavioral aspects of asset prices. For the 10 year treasury index we use the 10 year, the 1 year, and the 3 month treasury bond interest rates, as well as the average 150 weeks return and the past weeks return of the index. The three interest rates are the fundamental factors and the two return factors are again behaviorally motivated. In total we use 32 factors to model the returns of the 10 risky assets. The reader may consult [11, Chapter 3 and 4] for a good reference on factor models.

4.1 Strategy

We estimate the parameters of the TARCH(1,1,1) model for each of the ten risky assets and the DCC(1,1) model based on last 520 weeks of data at every time step of the simulation. The estimation procedure is a maximum-likelihood estimation as outlined in [5]. Two examples of the dynamic correlations are shown in Figure 1, where the correlation of the Financial Services index and the Bond index and the correlation of Financial Services index and Utilities index are plotted. The factor model and the factor loading matrix are estimated on all past data available at this time in the simulation, using an ordinary least square approach. This is asymptotically correct despite the presence of heteroskedasticity in the innovations, see [12, Chapter 8]. Based on all the parameters and the latest x_t and H_t, we then compute the prediction of the mean and variance κ steps ahead and then assemble the input of the objective function $c(t)$ and $D(t)$. Furthermore, we imposed the constraints that $0 \leq u_{j,t+i} \leq 1$ as well as $0 \leq \sum_{j=1}^{n} u_{j,t+i} \leq 1$. In the case study, we chose $\kappa = 10$ which results in solving a 100 dimensional quadratic program.

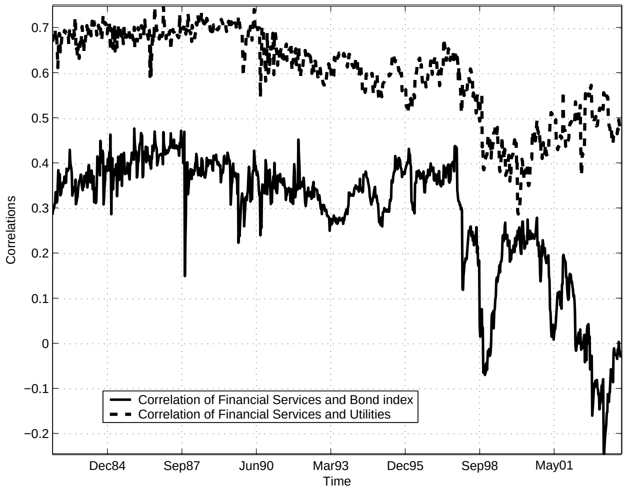

Figure 1. Dynamic correlations

The quadratic program (37) is solved and the optimal asset allocation is applied to the next set of market data. We evaluate three different portfolios, with risk aversion λ of -2, -5, and -8. In this manner, we move through the data set, starting in December 1982, because we use the first 520 weeks of data to estimate the GARCH models until fall 2003. All calculations are computed in MATLAB and the NAG toolbox is used to solve the quadratic program.

4.2 Results

We evaluate our strategy on an out-of-sample test with 1090 weeks. The summary statistics of the risky assets and of the portfolios are given in Table 1, where r denotes the return, σ denotes the volatility, and SR the Sharpe ratio. In Figure 2, the evolution of a 1$ investment in either portfolios with $\lambda = -2$ and $\lambda = -8$ or the S&P 500 index is shown. The portfolio with $\lambda = -5$ has similar evolution as the portfolio with $\lambda = -8$ and is therefore omitted from the Figure. The three portfolios all outperform the S&P 500 index, but the Noncyclical Consumer Goods index possesses the same return as the two best portfolios. In

Table 1. Summary statistics of the 10 indices and the portfolios from Dec. 1982 to Nov. 2003.

Time series	r (%)	σ (%)	SR
Basic Ind.	8.4	20.1	0.13
Cycl. Consumer Goods	7.1	19.6	0.06
Cycl. Services	10.4	19.1	0.24
Finanical Services	12.6	18.7	0.36
Information Technology	10.7	27.4	0.18
Noncyl. Consumer Gds.	13.0	16.4	0.43
Noncyl. Services.	6.8	17.7	0.05
Resources	7.4	18.9	0.08
Utilities	4.2	14.5	-0.12
Treasury Bond	9.4	6.3	0.55
S&P 500	10.1	15.8	0.26
Portfolio $\lambda = -2$	11.5	15.8	0.36
Portfolio $\lambda = -5$	13.1	13.6	0.53
Portfolio $\lambda = -8$	13.2	12.2	0.61

terms of the Sharpe ratio, the portfolios with $\gamma = -5$ and $\gamma = -8$ are slightly better than the government bond index, but markedly better than all of the stock market indices. The portfolio with the lowest risk aversion does not produce the highest return, because in the downturn between 2000 and 2003 it loses more than the two more conservative portfolios.

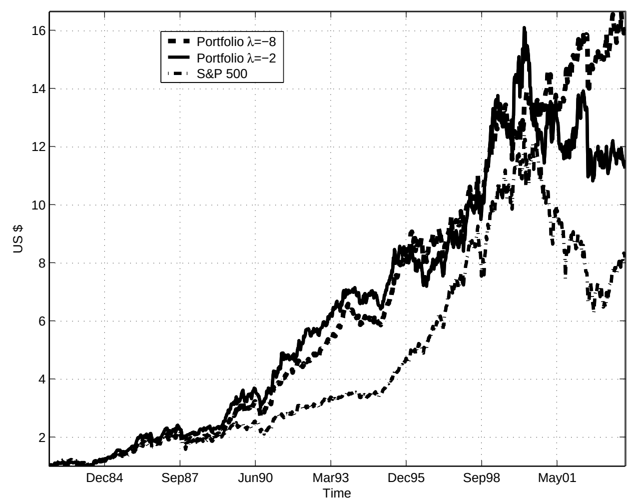

Figure 2. Results of the historical simulation of portfolios with $\lambda = -2$ and $\lambda = -8$.

The portfolios achieve these results, because of the partial predictability of the index returns and due to a superior hedging strategy. The hedging strategy depends on the partial predictability of future volatilities and correlations. The dynamic covariance model allows us to closely track changes in the covariance structure and therefore, improves the risk management aspect of our portfolio construction. In Figure 3, the asset allocation of the portfolio with $\lambda = -8$ into the three main asset classes (stocks, bonds, and cash) is shown. By inspecting Figure 3 one may

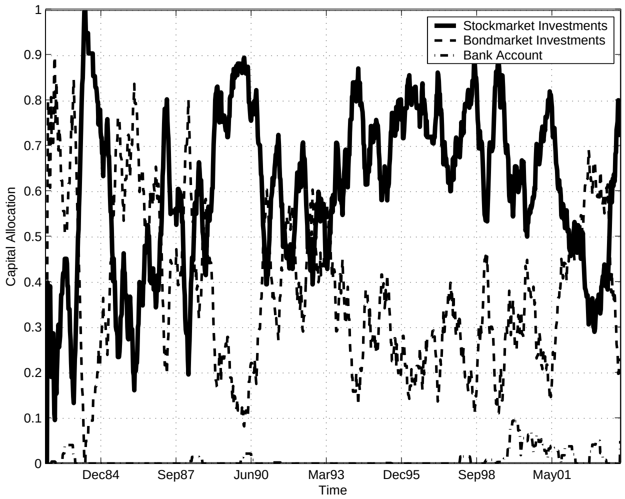

Figure 3. Asset allocation of Portfolio with $\lambda = -8$.

notice the good market-timing capabilities of this portfolio construction. During the stock market downturn from mid 2000 to early 2003, the portfolio shifts its capital from mostly stock market investments into the bond market. Thereby, the portfolio keeps its value and even increases its net worth, because the gains in the bond market are larger than the losses of the stock market investments. In early 2003, the capital is mostly moved back into the stock market and thus the portfolio participates in the stock market upturn. Similar market-timings can be observed in other market cycles, such as during the crises 1998. It is also interesting to observe that not much money is allocated to the bank account, even in very volatile market cycles.

5 Conclusion

In this paper, we present a stochastic modelling framework, where expected returns and risks are modelled realistically with respect to observed market data. Expected returns are modelled using a Gaussian factor model and expected risks are modelled using a multivariate GARCH model. Based on this market model, we state a mean-variance optimization for a long-term investor. In the presence of correlation of the factors and the returns, we show that the long-term asset allocation differs from the single period myopic allocation. We solved the corresponding constrained optimal portfolio control problem, based on the expected return and expected covariance predictions. In the case study we demonstrated the feasibility of this method on a large-scale portfolio decision problem.

References

[1] M. J. Brennan, E. S. Schwartz, & R. Lagnado, Strategic asset allocation, *Journal of Economics Dynamics and Control*, 21, 1997, 1377-1403.

[2] R. Engle, & A. Patton, What is a volatility model?, *Quantitative Finance*, 1, 2001, 237-245.

[3] L. Glosten, R. Jagannathan, & D. Runkle, On the relation between the expected value and the volatility of the nominal excess returns on stocks, *Journal of Finance*, 48, 1993, 1779-1801.

[4] R. Engle, & K. Sheppard, Theoretical and Empirical properties of Dynamic Conditional Correlation Multivariate GARCH, Working Paper, UC San Diego, 2001.

[5] R. Engle, Dynamic Conditional Correlation - A Simple Class of Multivariate GARCH Models, *Journal of Buisness and Economics Statistics*, 20(3), 2002, 339-350.

[6] J. Campbell, & L. Viceira, *Strategic Asset Allocation* (Oxford: Oxford University Press, 2002).

[7] D. G. Luenberger, *Investment Science* (Oxford: Oxford University Press, 1998).

[8] F. Herzog, Strategic Portfolio Optimization in Discrete Time, Working Paper, IMRT ETH Zrich, 2003.

[9] R. Baillie, T. Bollerslev, Prediction in dynamic models with time-dependent conditional variances, *Journal of Econometrics*, 52, 1992, 91-113.

[10] A. Bemporad, M. Morari, V. Duca, & E. Pistikopoulos, The explicit quadratic regulator for constraint systems, *Automatica*, 38, 2002, 3-20.

[11] R. E. Oberuc, *Dynamic Portfolio Theroy and Management* (New York: McGraw-Hill, 2004).

[12] J. Hamilton, *Time Series Analysis* (Princeton: Princeton University Press, 1994).