Ensemble Methods

COMP9417, 22T2

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Arguably the most powerful non *deep-learning* methods, coming close to the performance of neural networks and still winning Kaggle competitions.

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Why?

Quick Recap: Bias and Variance of Estimators

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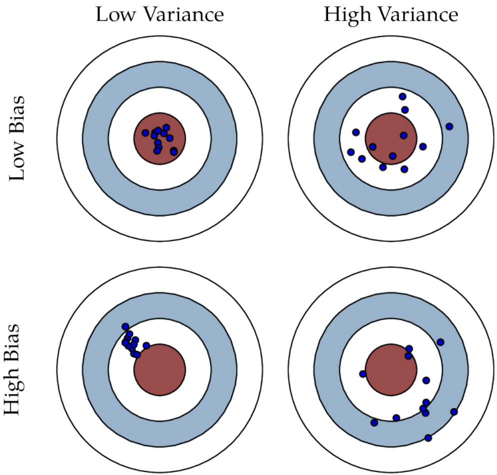
Recall the bias of an estimator $\hat{\theta}$ is defined as:

$$\text{bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta$$

And its variance is defined as:

$$\text{var}(\hat{\theta}) = \mathbb{E}\left[(\theta - \mathbb{E}[\hat{\theta}])^2\right]$$


Low Variance
High Variance
Low Bias
High Bias

Bias-Variance Tradeoff

Bias-Variance Tradeoff

Recall the bias-variance decomposition of the MSE for an estimator $\hat{\theta}$:

$$\mathsf{MSE}(\hat{\theta}) = \mathsf{var}(\hat{\theta}) + \mathsf{bias}(\hat{\theta})^2$$

obviously for the best estimator we need to minimise the variance and minimise the bias.

However, if we try and minimise the bias, we typically also increase variance.

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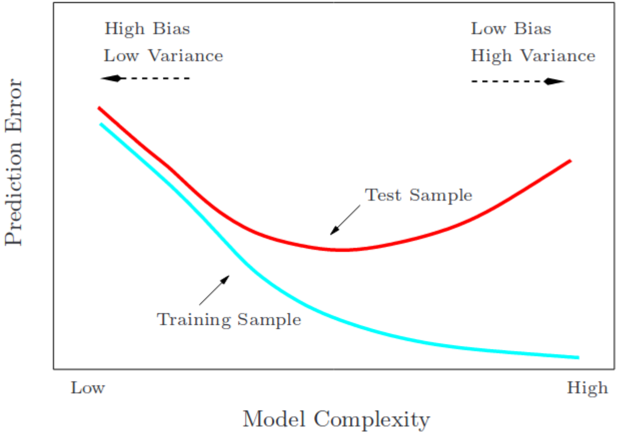

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For example, if we have a dataset $D = (x_i, y_i)$ for $i \in [1, n]$, we might train 4 decision trees on m points (where $m = n/4$) randomly picked from our dataset.

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For example, if we have a dataset $D = (x_i, y_i)$ for $i \in [1, n]$, we might train 4 decision trees on m points (where $m = n/4$) randomly picked from our dataset. We then have a committee of four trees with distinct knowledge on the dataset, which we can then average for our final prediction.

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and the final prediction for a point x is

$$\hat{f}(x) = \frac{1}{B} \sum_{b=1}^{B} \hat{f}^b(x)$$

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Random Forests

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Rationale: if we have strong predictors/features in our dataset, bagged trees will all typically pick the same features, leading to highly correlated predictions within the committee. This methods reduces this correlation and therefore the variance.

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$$C_m(X) = \alpha_1 h_1(X) + \alpha_2 h_2(X) + \ldots + \alpha_m h_m(X)$$

where α_i signifies the influence/weighting we give a model h_i for the final decision.

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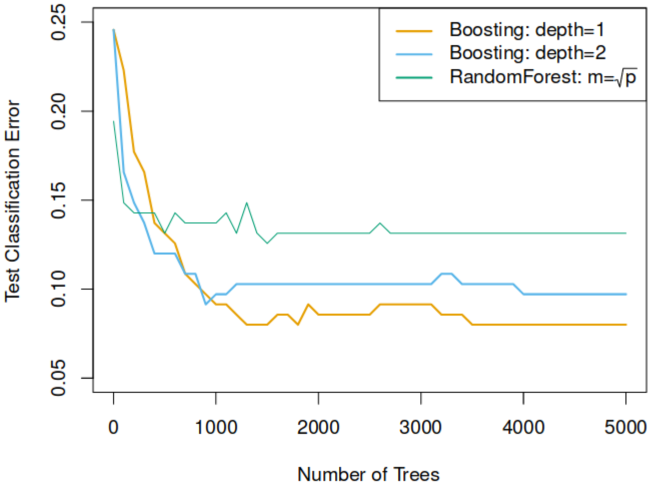
We also define a w_i for each iteration, signifying the weighting of each point. As each subsequent model needs to be an improvement on the last, we use these weights to signify which point the previous model misclassified.


Boosting: depth=1
Boosting: depth=2
RandomForest: $m=\sqrt{p}$
Test Classification Error
0.05
0.10
0.15
0.20
0.25
0
1000
2000
3000
4000
5000
Number of Trees

Adaboost

Let's take a look at the **Ada**ptive **Boost**ing algorithm.

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For a binary classification problem, we'll define the exponential loss as:

$$L(h(x_i), y_i) = e^{-y_i h(x_i)}$$

this loss typically isn't used in practice, but gives us a way of *weighting* how good a model performs on a dataset.

Recall, our boosted model takes the form:
$C_m(X) = \alpha_1 h_1(X) + \alpha_2 h_2(X) + \ldots + \alpha_m h_m(X)$.

So, our total loss will be:

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&= \sum_{i=1}^{n} e^{-y_i C_{m-1}(x_i)} e^{-y_i \alpha_m h_m(x_i)} \\
&= \sum_{i=1}^{n} w_i^m e^{-y_i \alpha_m h_m(x_i)}
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So, our problem is essentially,

$$\arg\min_{\alpha, h} \left(\sum_{y_i = h_m(x_i)} w_i^m e^{-\alpha_m} + \sum_{y_i \neq h_m(x_i)} w_i^m e^{\alpha_m} \right)$$

$$\frac{\partial L}{\partial \alpha} = -e^{-\alpha_m} \sum_{y_i = h_m(x_i)} w_i^m + e^{\alpha_m} \sum_{y_i \neq h_m(x_i)} w_i^m$$

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At the minimum:

$$-e^{-\alpha_m} \sum_{y_i = h_m(x_i)} w_i^m + e^{\alpha_m} \sum_{y_i \neq h_m(x_i)} w_i^m = 0$$

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$$\alpha_m = \frac{1}{2} \log \left(\frac{\sum_{y_i = h_m(x_i)} w_i^m}{\sum_{y_i \neq h_m(x_i)} w_i^m} \right)$$

If we let

$$\epsilon_m = \frac{\sum_{y_i \neq h_m(x_i)} w_i^m}{\sum_{i=1}^{n} w_i^m}$$

We can redefine α_m as:

$$\alpha_m = \frac{1}{2} \log \left(\frac{1 - \epsilon_m}{\epsilon_m} \right)$$

To actually get a form for $w_i^{(m)}$, we can apply the same trick of recursion,

$$\begin{aligned} w_i^{(m)} &= e^{-y_i C_{m-1}(x_i)} \\ &= e^{-y_i (C_{m-2}(x_i) + \alpha_{m-1} h_{m-1}(x_i))} \\ &= w_i^{(m-1)} e^{-y_i \alpha_{m-1} h_{m-1}(x_i)} \end{aligned}$$

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When $y_i \neq h_{m-1}(x_i)$:

$$w_i^{(m)} = w_i^{(m-1)} e^{\alpha_{m-1}}$$

Now we have our definitions, we can define the **Adaboost** algorithm.

If we have a dataset $D = (X, y)$ where $X \in \mathbb{R}^{n \times p}$ and $y \in \mathbb{R}^n$. Where T is our ensemble size and we have a learning algorithm A.

```
$w^{(1)} \leftarrow \frac{1}{n}$
**for** $t = 1, \ldots, T$ **do**
    $M_t \leftarrow A(X, w^{(t)})$
    $\alpha_t \leftarrow \frac{1}{2} \log\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$
    $w_i^{(t+1)} \leftarrow w_j^{(t)} \exp(\alpha_t)$    $j$ where $y_j \neq M_t(x_j)$
    $w_j^{(t+1)} \leftarrow w_j^{(t)} \exp(-\alpha_t)$    $j$ where $y_j = M_t(x_j)$
**end forreturn** $M(X) = \text{sgn}\left(\sum_{t=1}^{T} \alpha_t M_t(X)\right)$
```