Kernel Methods

COMP9417, 23T1

Section 1

Kernel Methods

Section 2

Primal vs. Dual Algorithms

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The *dual* view of a problem is simply just another way to view a problem mathematically.

Instead of pure parameter based learning (i.e minimising a loss function etc.), dual algorithms introduce **instance-based** learning.

This is where we 'remember' mistakes in our data and adjust the corresponding weights accordingly.

We then use a *similarity function* or **kernel** in our predictions to weight the influence of the training data on the prediction.

In the primal problem, we typically learn parameters:

$$\mathbf{w} \in \mathbb{R}^p$$

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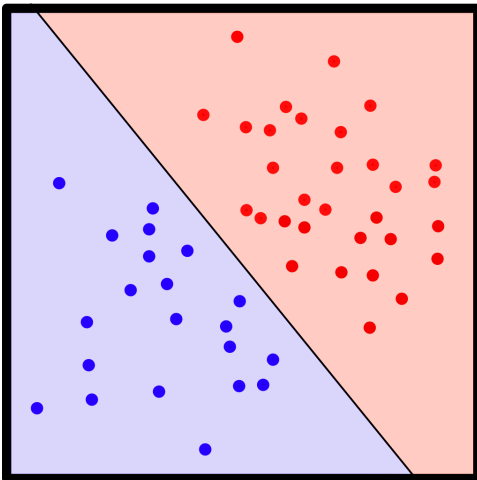
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$$\alpha_i \qquad \text{for } i \in [1, n]$$

meaning we learn parameters for each of the n **data-points**.

α_i represents the *importance* of a data point (x_i, y_i).

What do we mean by importance?

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The Dual/Kernel Perceptron

Provide an explanation of how the dual version of the perceptron relates to the original.

Kernel Methods   Primal vs. Dual Algorithms   Transformations   The Kernel Trick   Support Vector Machines   Question 7   Extension: The RBF Kernel

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Recall the *primal* perceptron:

$converged \leftarrow 0$
while not $converged$ **do**
     $converged \leftarrow 1$
     for $x_i \in X, y_i \in y$ **do**
         if $y_i w \cdot x_i \leq 0$ **then**
             $w \leftarrow w + \eta y_i x_i$
             $converged \leftarrow 0$
         end if
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end while

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If we define the number of iterations the perceptron makes as $K \in \mathbb{N}^+$ and assume $\eta = 1$. We can derive an expression for the final weight vector $w^{(K)}$:

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If we define the number of iterations the perceptron makes as $K \in \mathbb{N}^+$ and assume $\eta = 1$. We can derive an expression for the final weight vector $w^{(K)}$:

$$w^{(K)} = \sum_{i=1}^{N} \sum_{j=1}^{K} \mathbf{1}\{y_i w^{(j)} x_i \leq 0\} y_i x_i$$

We can simply our expression and take out the indicator variable:

$$w^{(K)} = \sum_{i=1}^{N} \sum_{j=1}^{K} \mathbf{1}\{y_i w^{(j)} x_i \leq 0\} y_i x_i$$

$$= \sum_{i=1}^{N} \alpha_i y_i x_i$$

where α_i is the number of times the perceptron makes a mistake on a data point (x_i, y_i).

If we sub in $w^{(K)} = \sum_{i=1}^{N} \alpha_i y_i x_i$. We get the algorithm for the **dual** perceptron.

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Gram Matrix

The Gram matrix represents the *inner product* of two vectors.

For a dataset X we define $G = X^T X$. That is:

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$$G = \begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \cdots & \langle x_1, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{bmatrix}$$

$$G_{i,j} = \langle x_i, x_j \rangle$$

Section 3

Transformations

Transformations

How do we go about solving **non-linearly separable** datasets with linear classifiers?

Transformations

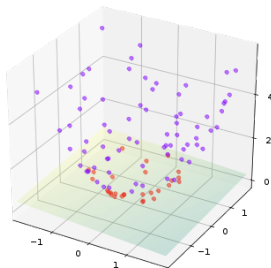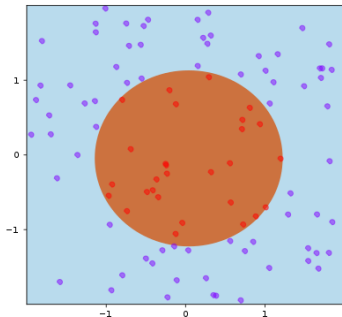How do we go about solving **non-linearly separable** datasets with linear classifiers?

Project them to higher dimensional spaces through a transformation $\phi : \mathbb{R}^p \to \mathbb{R}^k$.
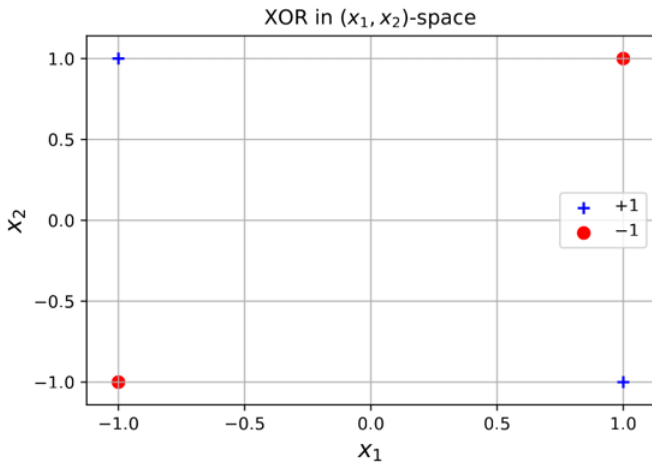
Transformations

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Project them to higher dimensional spaces through a transformation $\phi : \mathbb{R}^p \to \mathbb{R}^k$.

Let's revisit the XOR.

XOR in (x_1, x_2)-space

Extend the dual perceptron to learn the XOR function.

A solution:

Kernel Methods   Primal vs. Dual Algorithms   **Transformations**   The Kernel Trick   Support Vector Machines   Question 7   Extension: The RBF Kernel

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A solution:

For our input vectors in the form $\mathbf{x} = [x_1, x_2]^T$, use a transformation:

$$\phi(\mathbf{x}) = \begin{bmatrix} 1 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \end{bmatrix}$$

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$$
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For the negative class:

$$\phi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_{2,6} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right)_{2,6} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

For the positive class:

$$\phi\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)_{2,6} = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

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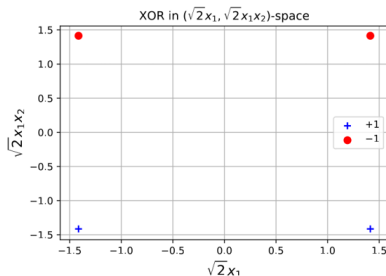
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XOR in $(\sqrt{2}x_1, \sqrt{2}x_1x_2)$-space

To use the **dual perceptron** on our transformed data, we simply need to redefine it.

$converged \leftarrow 0$

while not $converged$ **do**

    $converged \leftarrow 1$

    for $x_i \in X, y_i \in y$ **do**

        if $y_i \sum_{j=1}^{N} \alpha_j y_j x_j \cdot x_i \leq 0$ **then**

            $\alpha_i \leftarrow \alpha_i + 1$

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        if $y_i \sum_{j=1}^{N} \alpha_j y_j \phi(x_j) \cdot \phi(x_i) \leq 0$ **then**

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Recall the transformation $\phi : \mathbb{R}^p \to \mathbb{R}^k$.

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$$G = \begin{bmatrix} \langle \phi(x_1), \phi(x_1) \rangle & \langle \phi(x_1), \phi(x_2) \rangle & \cdots & \langle \phi(x_1), x_n \rangle \\ \langle \phi(x_2), \phi(x_1) \rangle & \langle \phi(x_2), \phi(x_2) \rangle & \cdots & \langle \phi(x_2), \phi(x_n) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi(x_n), \phi(x_1) \rangle & \langle \phi(x_n), \phi(x_2) \rangle & \cdots & \langle \phi(x_n), \phi(x_n) \rangle \end{bmatrix}$$

the Gram matrix becomes costly to compute.

Section 4

The Kernel Trick

The Kernel Trick

Show how computational issues in the previous section can be mitigated by using the Kernel trick.

Recall the transformation to the XOR data:

$$\phi(\mathbf{x}) = \begin{bmatrix} 1 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \end{bmatrix}$$

The Kernel Trick

Show how computational issues in the previous section can be mitigated by using the Kernel trick.

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$$\phi(\mathbf{x}) \cdot \phi(\mathbf{y}) = \begin{bmatrix} 1 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2}y_1 \\ \sqrt{2}y_2 \\ y_1^2 \\ y_2^2 \\ \sqrt{2}y_1y_2 \end{bmatrix}$$

Kernel Methods
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Primal vs. Dual Algorithms
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Transformations
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The Kernel Trick
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Support Vector Machines
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Question 7
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Extension: The RBF Kernel
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$$\phi(\mathbf{x}) \cdot \phi(\mathbf{y}) = 1 + 2x_1y_1 + 2x_2y_2 + x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2$$

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$$\phi(\mathbf{x}) \cdot \phi(\mathbf{y}) = 1 + 2(x_1y_1 + x_2y_2) + (x_1y_1 + x_2y_2)^2$$

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So our Gram matrix is:

$$G = \begin{bmatrix} k(x_1, x_1) & k(x_1, x_2) & \cdots & k(x_1, x_n) \\ k(x_2, x_1) & k(x_2, x_2) & \cdots & k(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_n, x_1) & k(x_n, x_2) & \cdots & k(x_n, x_n) \end{bmatrix}$$

$$\phi(\mathbf{x}) \cdot \phi(\mathbf{y}) = 1 + 2x_1y_1 + 2x_2y_2 + x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2$$

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Why is this useful?

$$\phi(\mathbf{x}) \cdot \phi(\mathbf{y}) = 1 + 2x_1y_1 + 2x_2y_2 + x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2$$
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Why is this useful? We've essentially gotten a 6-dimensional transformation with the cost of a 2-dimensional dot-product.

The kernel perceptron is now defined as:

$converged \leftarrow 0$

while not $converged$ **do**

    $converged \leftarrow 1$

    for $x_i \in X, y_i \in y$ **do**

        if $y_i \sum_{j=1}^{N} \alpha_j y_j k(x_j, x_i) \leq 0$ **then**

            $\alpha_i \leftarrow \alpha_i + 1$

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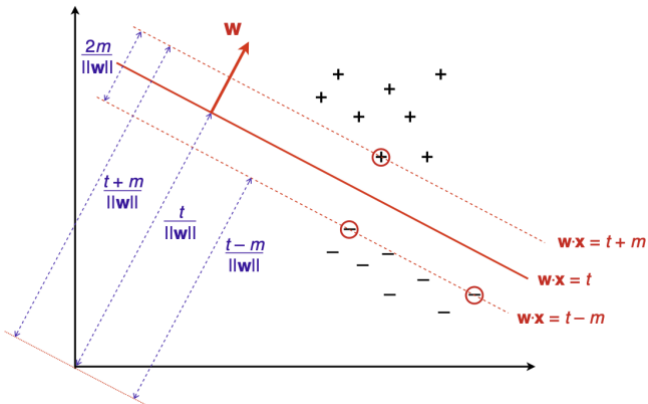
        end if

    end for

end while

Section 5

Support Vector Machines

Support Vector Machines

The basic SVM is a linear classifier defined by:

$$\arg\min_{w,t} \frac{1}{2}\|w\|^2 \qquad\qquad \text{subject to } y_i(\langle x_i, w\rangle - t) \geq m$$

where t is the line's intercept, and we a consider a margin m. Typically, we'll see $m = 1$ for a standardised dataset.

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where t is the line's intercept, and we a consider a margin m. Typically, we'll see $m = 1$ for a standardised dataset.

This formulation means that we find the **maximal margin** classifier for the dataset.

Aside: Lagrangian Dual Problem

Say we have a problem as follows:

$$\max_{x,y} xy \qquad\qquad \text{subject to } x + y = 4$$

we can also consider the constraint as $x + y - 4 = 0$.

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We can set up the Lagrangian dual and *move* the constraint into the function itself:

$$\Lambda(x, y, \lambda) = xy + \lambda(x + y - 4)$$

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$$\Lambda(x, y, \lambda) = xy + \lambda(x + y - 4)$$

To solve this, we can calculate $\frac{\partial L}{\partial x}$, $\frac{\partial L}{\partial y}$ and $\frac{\partial L}{\partial \lambda}$ and solve the remaining system of equations.

The General Form of a Dual Problem

If we have a problem:

$$\underset{x}{\arg\min}\, f(x)$$

$$\text{subject to } g_i(x) \leq 0, \qquad\qquad i \in \{1, \ldots, n\}$$

The General Form of a Dual Problem

If we have a problem:

$$\arg\min_x f(x)$$
$$\text{subject to } g_i(x) \leq 0, \qquad\qquad i \in \{1, \dots, n\}$$

The general *dual* problem is:

$$\Lambda(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{i=1}^{n} \lambda_i g_i(x_i)$$

The Dual Problem for SVM

If we take the general SVM problem ($m = 1$):

$$\arg\min_{w,t} \frac{1}{2}\|w\|^2 \qquad\qquad \text{subject to } y_i(\langle x_i, w\rangle - t) \geq 1$$

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If we take the general SVM problem ($m = 1$):

$$\underset{w,t}{\arg\min} \frac{1}{2}\|w\|^2 \qquad \text{subject to } y_i(\langle x_i, w \rangle - t) - 1 \geq 0$$

From the general form, we can take the vector α to form the dual problem:

$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 + \left(-\sum_{i=1}^{n} \alpha_i y_i(\langle x_i, w \rangle - t) - 1)\right)$$

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$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 - \sum_{i=1}^{n} \alpha_i y_i(w \cdot x_i) + t\sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \alpha_i$$

$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 + \left(-\sum_{i=1}^{n}\alpha_i y_i(\langle x_i, w\rangle - t) - 1\right)$$

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$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 - w \cdot \sum_{i=1}^{n}\alpha_i y_i x_i + t\sum_{i=1}^{n}\alpha_i y_i + \sum_{i=1}^{n}\alpha_i$$

Let's try and optimise the Lagrangian Λ w.r.t w,

$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 - w \cdot \sum_{i=1}^{n} \alpha_i y_i x_i + t \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \alpha_i$$

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We can see that at $\dfrac{\partial \Lambda}{\partial w} = 0$

$$w = \sum_{i=1}^{n} \alpha_i y_i x_i$$

Repeating a similar process for t,

$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 - w \cdot \sum_{i=1}^{n} \alpha_i y_i x_i + t \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \alpha_i$$

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Repeating a similar process for t,

$$\Lambda(w, t, \alpha) = \frac{1}{2}\|w\|^2 - w \cdot \sum_{i=1}^{n} \alpha_i y_i x_i + t \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \alpha_i$$

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The Dual Problem for SVM

We've derived that for an optimal solution, $\sum_{i=1}^{n} \alpha_i y_i = 0$ and $w = \sum_{i=1}^{n} \alpha_i y_i x_i$

$$\Lambda(w, t, \alpha) = \frac{1}{2} \|w\|^2 - w \cdot \sum_{i=1}^{n} \alpha_i y_i x_i + t \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \alpha_i$$

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$$\Lambda(w, \alpha) = \frac{1}{2} w^T w - w^T w + \sum_{i=1}^{n} \alpha_i$$

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$$\Lambda(w, \alpha) = \frac{1}{2} w^T w - w^T w + \sum_{i=1}^{n} \alpha_i$$

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$$\Lambda(w, \alpha) = -\frac{1}{2}w^T w + \sum_{i=1}^{n} \alpha_i$$

$$\Lambda(\alpha) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j (x_i \cdot x_j) + \sum_{i=1}^{n} \alpha_i$$

Our final problem now has relaxed constraints:

$$\Lambda(\alpha) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j (x_i \cdot x_j) + \sum_{i=1}^{n} \alpha_i$$

$$\text{subject to } \sum_{i=1}^{n} \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \text{ for } i = 1, \ldots, n$$

Section 6

Question 7

Question 7

Given data $\mathbf{X}$ and targets $\mathbf{y}$, with transformed data $\mathbf{X}'$.

$$\mathbf{X} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

solve the SVM problem by hand.

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The steps given are:

1. Set up the Gram matrix for labelled data
2. Set up the expression to be minimised
3. Take partial derivatives
4. Set to zero and solve for each multiplier
5. Solve for w
6. Solve for t
7. Solve for m

1. Set up the Gram matrix for labelled data

The Gram matrix is just the product $\mathbf{X}'(\mathbf{X}')^T$.

$$\mathbf{X}'(\mathbf{X}')^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

1. Set up the Gram matrix for labelled data

The Gram matrix is just the product $\mathbf{X}'(\mathbf{X}')^T$.

$$\mathbf{X}'(\mathbf{X}')^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 & -3 \\ 5 & 5 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

❷ Set up the expression to be minimised

Recall the dual problem for the SVM:

$$\operatorname*{arg\,min}_{\alpha_1,\ldots,\alpha_n} -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j (x_i \cdot x_j) + \sum_{i=1}^{n}\alpha_i$$

$$\text{subject to } \sum_{i=1}^{n}\alpha_i y_i = 0$$

$$\alpha_i \geq 0 \text{ for } i = 1,\ldots,n$$

❷ Set up the expression to be minimised

Recall the dual problem for the SVM:

$$\arg\min_{\alpha_1,\alpha_2,\alpha_3} -\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\alpha_i\alpha_j\mathbf{G}[i,j] + \sum_{i=1}^{3}\alpha_i$$

$$\text{subject to } \sum_{i=1}^{3}\alpha_i y_i = 0$$

$$\alpha_i \geq 0 \text{ for } i = 1,\ldots,3$$

Recall the gram matrix:

$$\mathbf{G} = \begin{bmatrix} 10 & 5 & -3 \\ 5 & 5 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

$$\operatorname*{arg\,min}_{\alpha_1, \alpha_2, \alpha_3} -\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j \mathbf{G}[i,j] + \sum_{i=1}^{3} \alpha_i$$

Recall the gram matrix:

$$\mathbf{G} = \begin{bmatrix} 10 & 5 & -3 \\ 5 & 5 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

$$\underset{\alpha_1,\alpha_2,\alpha_3}{\arg\min} -\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j \mathbf{G}[i,j] + \sum_{i=1}^{3} \alpha_i$$

$$\underset{\alpha_1,\alpha_2,\alpha_3}{\arg\min} -\frac{1}{2} \left(10\alpha_1^2 + 10\alpha_1\alpha_2 - 6\alpha_1\alpha_3 + 5\alpha_2^2 - 2\alpha_2\alpha_3 + \alpha_3^2 \right) + \alpha_1 + \alpha_2 + \alpha_3$$

If we look at the constraints $\left(\sum_i \alpha_i y_i = 0\right)$,

$$\sum_{i=1}^{3} \alpha_i y_i = 0$$

Therefore if we substitute in $\alpha_3 = \alpha_1 + \alpha_2$, out final maximisation problem becomes:

If we look at the constraints $\left(\sum_i \alpha_i y_i = 0 \right)$,

$$\sum_{i=1}^{3} \alpha_i y_i = 0$$

$$\alpha_1 + \alpha_2 - \alpha_3 = 0$$

$$\alpha_3 = \alpha_1 + \alpha_2$$

Therefore if we substitute in $\alpha_3 = \alpha_1 + \alpha_2$, out final maximisation problem becomes:

$$\underset{\alpha_1, \alpha_2}{\arg \min} -\frac{1}{2} \left(10\alpha_1^2 + 10\alpha_1\alpha_2 - 6\alpha_1(\alpha_1 + \alpha_2) + 5\alpha_2^2 - 2\alpha_2(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2)^2 \right) +$$

$$\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)$$

If we look at the constraints $\left(\sum_i \alpha_i y_i = 0 \right)$,

$$\sum_{i=1}^{3} \alpha_i y_i = 0$$

$$\alpha_1 + \alpha_2 - \alpha_3 = 0$$

$$\alpha_3 = \alpha_1 + \alpha_2$$

Therefore if we substitute in $\alpha_3 = \alpha_1 + \alpha_2$, out final maximisation problem becomes:

$$\underset{\alpha_1, \alpha_2}{\arg\min} -\frac{1}{2} \left(5\alpha_1^2 + 4\alpha_1\alpha_2 + 3\alpha_2^2 \right) + 2\alpha_1 + 2\alpha_2$$

③ Take partial derivatives

$$\underset{\alpha_1, \alpha_2}{\arg\min} -\frac{1}{2}\left(5\alpha_1^2 + 4\alpha_1\alpha_2 + 3\alpha_2^2\right) + 2\alpha_1 + 2\alpha_2$$

③ Take partial derivatives

$$\underset{\alpha_1, \alpha_2}{\arg\min} -\frac{1}{2}\left(5\alpha_1^2 + 4\alpha_1\alpha_2 + 3\alpha_2^2\right) + 2\alpha_1 + 2\alpha_2$$

$$\frac{\partial}{\partial \alpha_1} = -5\alpha_1 - 2\alpha_2 + 2$$

$$\frac{\partial}{\partial \alpha_2} = -2\alpha_1 - 4\alpha_2 + 2$$

④ Set to zero and solve for each multiplier

For α_1,

$$-5\alpha_1 - 2\alpha_2 + 2 = 0$$

$$\alpha_1 = -\frac{(2\alpha_2 - 2)}{5}$$

④ Set to zero and solve for each multiplier

For α_1,

$$-5\alpha_1 - 2\alpha_2 + 2 = 0$$
$$\alpha_1 = -\frac{(2\alpha_2 - 2)}{5}$$

For α_2,

$$-2\alpha_1 - 4\alpha_2 + 2 = 0$$
$$\frac{2}{5}(2\alpha_2 - 2) - 4\alpha_2 + 2 = 0$$
$$2\alpha_2 - 2 - 10\alpha_2 + 5 = 0$$
$$\alpha_2 = \frac{3}{8} \qquad \alpha_1 = \frac{1}{4} \qquad \alpha_3 = \frac{5}{8}$$

⑤ Solve for w

What did we define w as for the dual problem?

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5 Solve for w

What did we define w as for the dual problem?

$$w = \sum_{i=1}^{n} \alpha_i y_i \mathbf{x_i}$$

5. Solve for w

What did we define w as for the dual problem?

$$w = \sum_{i=1}^{n} \alpha_i y_i \mathbf{x_i}$$

So, in this case:

$$
\begin{aligned}
w &= \frac{1}{4}x_1 + \frac{3}{8}x_2 + \frac{5}{8}x_3 \\
&= \frac{1}{4} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{8} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{5}{8} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

6 Solve for t

The constraint $y_i(\langle w, x_i \rangle - t) = 1$ for all support vectors. We can use the 3rd data point:

$$y_3(\langle w, x_3 \rangle - t) = 1$$
$$-\left(\frac{1}{2} - t\right) = 1$$
$$t = \frac{3}{2}$$

7 Solve for m

$$m = \frac{1}{\|w\|} = \frac{2}{\sqrt{5}}$$

Section 7

Extension: The RBF Kernel

Extension: The RBF Kernel

A popular Kernel is the Radial Basis Function kernel, defined below:

$$K(x, y) = \exp\left(-\frac{\|x - y\|^2}{2\sigma^2}\right)$$

for scalar values:

$$K(x, y) = \exp\left(-\frac{(x - y)^2}{2\sigma^2}\right)$$

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$$K(x, y) = \exp\left(\frac{(x - y)^2}{2\sigma^2}\right)$$

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$$= \exp\left(\frac{-x^2 + 2xy - y^2}{2\sigma^2}\right)$$

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$$= \exp\left(\frac{-x^2}{2\sigma^2}\right) \exp\left(\frac{-y^2}{2\sigma^2}\right) \exp\left(\frac{xy}{\sigma^2}\right)$$

$$K(x, y) = \exp\left(\frac{(x - y)^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{-x^2 + 2xy - y^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{-x^2}{2\sigma^2}\right)\exp\left(\frac{-y^2}{2\sigma^2}\right)\exp\left(\frac{xy}{\sigma^2}\right)$$

$$= \exp\left(\frac{-x^2}{2\sigma^2}\right)\exp\left(\frac{-y^2}{2\sigma^2}\right)\sum_{i=1}^{\infty}\frac{(xy)^k}{\sigma^{2k}k!}$$

By definition

$$\langle \phi(x), \phi(y) \rangle = \exp\left(\frac{-x^2}{2\sigma^2}\right) \exp\left(\frac{-y^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{(xy)^k}{\sigma^{2k} k!}$$

So, our basis transformation is:

$$\phi(x) = \exp\left(\frac{-x^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{x^k}{\sigma^k \sqrt{k!}}$$

What does this represent?

By definition

$$\langle \phi(x), \phi(y) \rangle = \exp\left(\frac{-x^2}{2\sigma^2}\right) \exp\left(\frac{-y^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{(xy)^k}{\sigma^{2k} k!}$$

So, our basis transformation is:

$$\phi(x) = \exp\left(\frac{-x^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{x^k}{\sigma^k \sqrt{k!}}$$

What does this represent? A projection to infinite dimensions!