Non Parametric Methods

COMP9417, 23T2

1. Non Parametric Methods
2. Decision Trees
3. k-NN
4. Linear Smoothing

Section 1

Non Parametric Methods

Non Parametric Methods

Parametric modelling

We make assumptions on the type of function which our data takes.

- Linear regression
- Perceptron
- Logistic regression

Non parametric modelling

We make no assumptions on the underlying function and purely use our datapoints as guides for inference.

- k-Nearest neighbours
- Local regression
- Decision Trees

Section 2

Decision Trees

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A tree-like model used for both **regression** and **classification**.

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- Interpretable
- Useful when used in ensemble learning (we'll come back to this notion)

Disadvantages:

- Tend to overfit data
- Often innacurate in their most basic form

Entropy

Entropy essentially measures the *uncertainty* or *surprise* of a random variable.

We define the entropy for a set S,

$$H(S) = \sum_{x \in X} -p(x) \log p(x)$$

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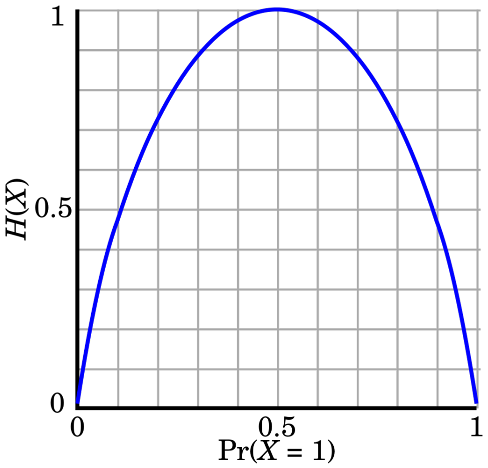
Say we have a random variable $X \sim \text{Bernoulli}(p)$. We can define the entropy of X:

$$H(x) = -(1-p)\log(1-p) - p \log p$$


1
0.5
0
0
0.5
1
$H(X)$
$\Pr(X = 1)$

Gain

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If we have a dataset S with a feature A,

$$\text{Gain}(S, A) = H(S) - \sum_{v \in V_A} \frac{|S_v|}{|S|} H(S_v)$$

Basic Example

Say we have a dataset as follows: $[29+, 35-]$:

- A1 $\sim$ T: $[21+, 5-]$ F: $[8+, 30-]$
- A2 $\sim$ T: $[18+, 33-]$ F: $[11+, 2-]$

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$$
\begin{aligned}
H(S) &= \sum_{x \in X} -p(x) \log p(x) \\
&= -\frac{29}{29+35} \log\left(\frac{29}{29+35}\right) - \frac{35}{29+35} \log\left(\frac{35}{29+35}\right) \\
&= 0.9936
\end{aligned}
$$

Dataset: $[29+, 35-]$:

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$$\begin{aligned} H(S_{A_{1,T}}) &= -\frac{21}{26}\log\left(\frac{21}{26}\right) - \frac{5}{26}\log\left(\frac{5}{26}\right) \\ &= 0.7063 \end{aligned}$$

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$$\begin{aligned} H(S_{A_{1,F}}) &= -\frac{8}{38}\log\left(\frac{8}{38}\right) - \frac{30}{38}\log\left(\frac{30}{38}\right) \\ &= 0.7425 \end{aligned}$$

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$$\begin{aligned} H(S_{A_{2,T}}) &= -\frac{18}{51}\log\left(\frac{18}{51}\right) - \frac{33}{51}\log\left(\frac{33}{51}\right) \\ &= 0.9366 \end{aligned}$$

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$$\begin{aligned} H(S_{A_{2,F}}) &= -\frac{11}{13}\log\left(\frac{11}{13}\right) - \frac{2}{13}\log\left(\frac{2}{13}\right) \\ &= 0.4674 \end{aligned}$$

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$$

$$
\begin{aligned}
&= H(S) - \frac{26}{64} H(A_{1,T}) - \frac{38}{64} H(A_{1,F}) \\
&= 0.2658
\end{aligned}
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$$\begin{aligned}\text{Gain}(S, A_2) &= H(S) - \frac{51}{64} H(A_{2,T}) - \frac{13}{64} H(A_{2,F}) \\ &= 0.1643\end{aligned}$$

ID3 Algorithm

Basically what we just did:

- Calculate the entropy for each attribute $a \in A$.
- Split on the attribute with the maximum Gain. This means creating a decision tree node using that attribute.
- Recurse on this new subset of the data.

Regression Trees

Regression trees split the dataset up into regions and fit separate models to each region.

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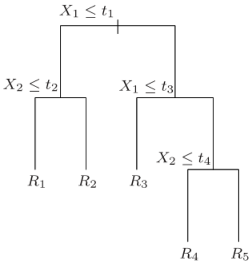

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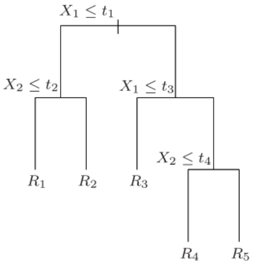

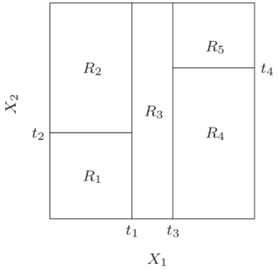

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$$\min_{j,s}\left[\min_{c_1}\sum_{y_i \in R_1(j,s)}(y_i - c_1)^2 + \sum_{y_i \in R_2(j,s)}\min_{c_2}(y_i - c_2)^2\right]$$

Where $\hat{c_1} = \text{ave}(y_i | x_i \in R_1), \quad \hat{c_2} = \text{ave}(y_i | x_i \in R_2)$.

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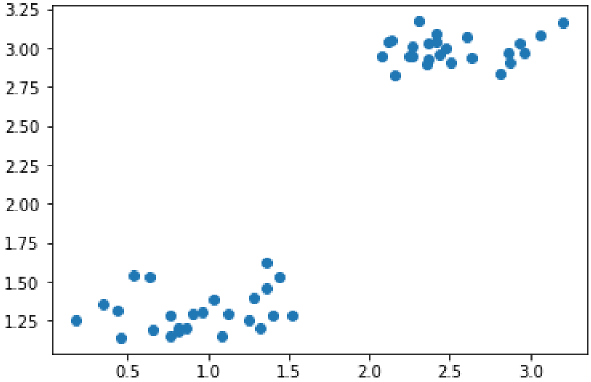
Where $\hat{c}_1 = \text{ave}(y_i | x_i \in R_1), \quad \hat{c}_2 = \text{ave}(y_i | x_i \in R_2)$.

This essentially finds regions (R_1 and R_2) with the minimum variance.



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Section 3

k-NN

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Regression If we define the set K as the k-nearest neighbours of a point X_i, then our k-NN estimate is:

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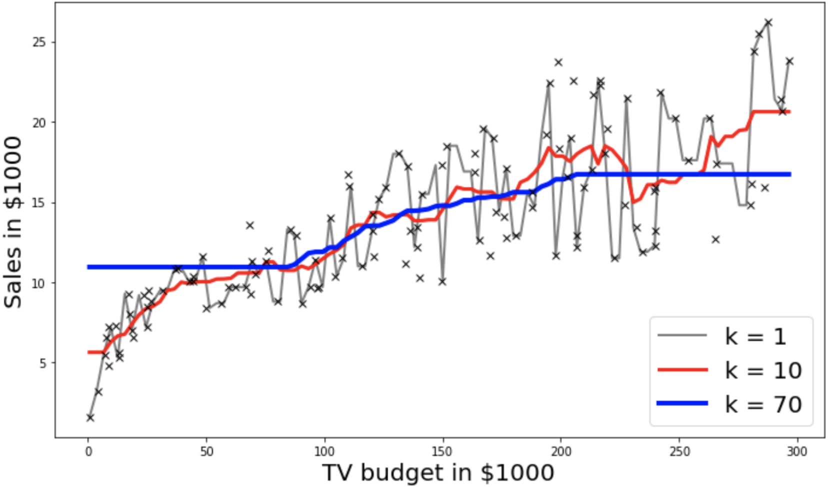
Classification we assign X_i the majority class in K.


Sales in $1000
TV budget in $1000
k = 1
k = 10
k = 70
0
50
100
150
200
250
300
5
10
15
20
25

An obvious limitation:

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A not-so obvious limitation:

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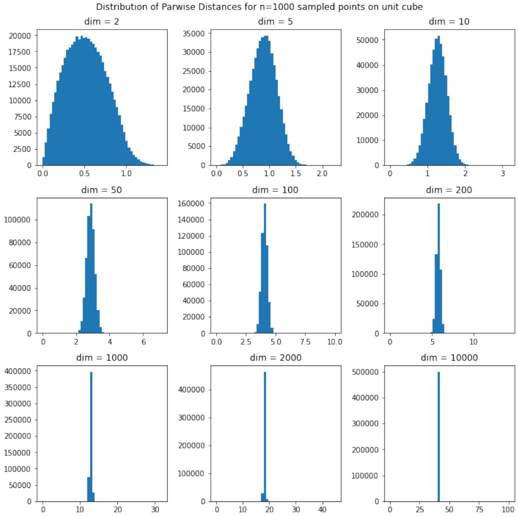
The most common are:

- Distances between points breaking down
- The need for even *more* data


Distribution of Parwise Distances for n=1000 sampled points on unit cube
dim = 2
dim = 5
dim = 10
dim = 50
dim = 100
dim = 200
dim = 1000
dim = 2000
dim = 10000

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- *1 binary variable* - 2 unique combinations - 20 samples

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- *2 binary variables* - 4 unique combinations - 40 samples

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- *k binary variables* - 2^k unique combinations - 10×2^k samples

Say we have a classification task and we want 10 samples per unique combination of variables for a comprehensive data set.

- *1 binary variable* - 2 unique combinations - 20 samples
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- *k binary variables* - 2^k unique combinations - 10×2^k samples

So for 20 features, we need $10 \times 2^{20} = 10485760$ data points!

Section 4

Linear Smoothing

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If we define h as the smoothing parameter and K as the kernel, the Linear Smoothing estimate is:

$$\hat{y}_i = \frac{\sum_{j=1}^{n} K\left(\frac{\|x_i - x_j\|}{h}\right) y_i}{\sum_{j=1}^{n} K\left(\frac{\|x_i - x_j\|}{h}\right)}$$

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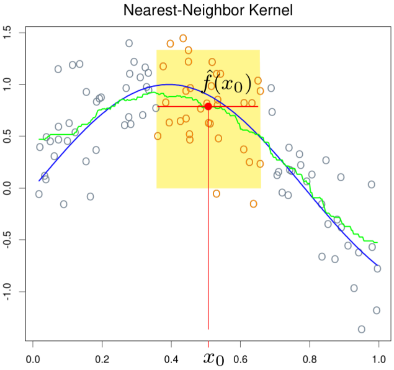
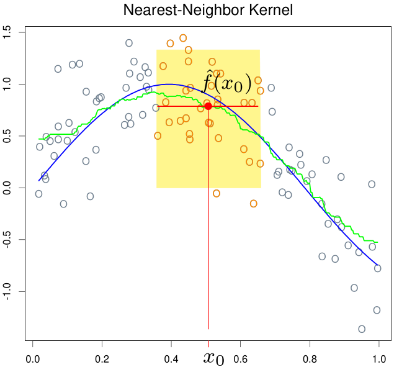
As $h \to 0$ our distances have a higher variance. If $h \to \infty$ have a lower variance, and our model is in turn smoother.


Nearest-Neighbor Kernel
$\hat{f}(x_0)$
x_0
0.0
0.2
0.4
0.6
0.8
1.0
-1.0
-0.5
0.0
0.5
1.0
1.5


Nearest-Neighbor Kernel
1.5
1.0
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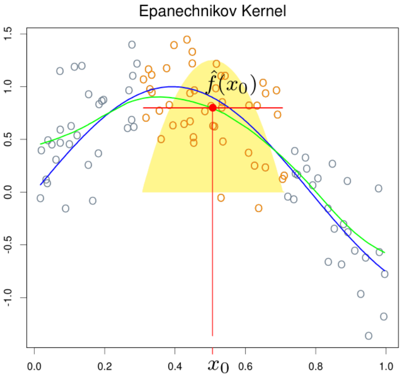
$\hat{f}(x_0)$


Epanechnikov Kernel
1.5
1.0
0.5
0.0
-0.5
-1.0
0.0
0.2
0.4
x_0
0.6
0.8
1.0
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