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Homework 2 – Digital Image Processing

Question 1:

The action of the camera can be expressed as $y(x) = (f * p)(x)$. Where $f(x)$ is the continuous scene, and p is the PSF of the camera. Using the same logic we can express l[n] the low-res image, using PSF $p_L(x)$ as:

$$l[n] = (f * p_L)(n) = \int f(z) p_L(n-z) dz$$

and h[n]- the high-res image can be expressed using $p_H(x)$ PSF, as:

$$h[n] = (f * p_H)(n) = \int f(z) p_H(n-z) dz$$

Since $h(x)$ is being sampled on $\frac{1}{\alpha}Z^2$, where $\alpha \in N$ so in order to perform up-sampling:

$$h[n] = h\left(\frac{1}{\alpha}n\right) = \int f(z) p_H\left(\frac{1}{\alpha}n - z\right) dz$$

we can notice that in order to get this high res $p_H(x) = \alpha p_L(\alpha x)$ must be true.

Question 2:

$l[n]$ can be expressed using $h[n]$ and the discrete kernel k[n] ass follows:

$$l[n] = \downarrow_\alpha (h * k)[n] = \downarrow_\alpha \sum_m h[m]k[n-m] = \sum_m h[m]k[\alpha n - m]$$

Question 3:

Untill now we got:

1. $l[n] = (f * p_L)(n) = \int f(z) p_L(n-z) dz.$
2. $h[n] = (f * p_H)(n) = \int f(z) p_H(n-z) dz.$
3. $l[n] = \sum_m h[m]k[\alpha n - m]$

k[n] is a discrete low pass filter in $\frac{1}{\alpha}Z^2$, in the continues domain: k[n] = k($\frac{1}{\alpha}n$), substituting this into 3 we get:

$$l[n] = \downarrow_\alpha (h * k)[n] = \downarrow_\alpha \sum_m h(\frac{1}{\alpha}m) k\left(n - \frac{1}{\alpha}m\right)$$

By substituting 1, 2 into 3 we get:

$$\int f(z) p_L(n-z) dz = \sum_m \left\{ \int f(z) p_H\left(\frac{1}{\alpha}m - z\right) dz \right\} k\left(n - \frac{1}{\alpha}m\right)$$

$$(f * p_L)(n) = \sum_{m \in \frac{1}{\alpha}\mathbb{Z}^2} (f * p_H)(m)k(n-m)$$

$$(f * p_L)[n] = \sum_{m \in \frac{1}{\alpha}\mathbb{Z}^2} (f * p_H)[m]k[n-m]$$

$$(f * p_L)[n] = (f * k * p_H)[n]$$

$$(f * p_L)[n] - (f * k * p_H)[n] = \big(f * (\, p_L - k * p_H)\big)[n] = 0$$

Assuming the equation holds for every possible f then we get:

$$p_L(x) = (k * p_H)(x)$$

Question 4:

We already noticed in section 1 that: $p_H(x) = \alpha p_L(\alpha x)$, applying Fourier transform on both sides and using the linearity property and stretching property we get:

$$P_H(w) = \alpha \frac{1}{\alpha} P_L(\alpha^{-1} w) = P_L\left(\frac{w}{\alpha}\right)$$

Now let us apply forier transfrom on both sides on the equation above:

$$\mathcal{F}\{p_L(x)\}(w) = \mathcal{F}\{(k * p_H)(x)\}(w)$$

$$P_L(w) = \mathcal{F}\{k\}(w) \cdot \mathcal{F}\{p_H\}(w) = K(w) \cdot P_H(w)$$

Since $P_H(w) \neq 0 \; for \; w \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ we can write:

$$K(w) = \frac{P_L(w)}{P_H(w)} = \frac{P_L(w)}{P_L\left(\frac{w}{\alpha}\right)}$$

Question 5:

We will first check if the assumption holds for $p_L = sinc$, the Fourier transform of for sinc is $Box(w)$, so we can write:

$$K(w) = \frac{P_L(w)}{P_L\left(\frac{w}{\alpha}\right)} = \frac{Box(w)}{Box\left(\frac{w}{\alpha}\right)} = Box(w)$$

This is justified because:

$$Box\left(\frac{w}{\alpha}\right) = \begin{cases} 0 \; if \; w > \dfrac{\alpha}{2} \\ 0.5 \; if \; w = \dfrac{\alpha}{2} \\ 1 \; if \; w < \dfrac{\alpha}{2} \end{cases}$$

And since $w \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ the division is justified (none zero values).

Let us apply $\mathcal{F}^{-1}\{K(w)\} = \mathcal{F}^{-1}\{Box(w)\}$=sinc, as we can see if $p_L = sinc$ the assumption holds.

For p_L is an isotropic Gaussian, i.e. $p_L = \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right) = \frac{1}{(2\pi)^{\frac{d}{2}}|\sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2\sigma^2}\|x\|^2\right)$, the Fourier transform of this Gaussian would be:

$$\begin{aligned}
\mathcal{F}\{p_L(x)\}(w) = P_L(w) &= \int_{-\infty}^{\infty} \exp(-2\pi i\omega x)\, p_L(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{d}{2}}|\sigma|^{\frac{1}{2}}}\exp(-2\pi i\omega x)\exp\left(-\frac{1}{2\sigma^2}\|x\|^2\right)dx \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}|\sigma|^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp(-2\pi i\omega x)\exp\left(-\frac{1}{2\sigma^2}\|x\|^2\right)dx \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}|\sigma|}\sqrt{2\pi}|\sigma|\cdot e^{-2\sigma^2\pi^2\|w\|^2} = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\cdot e^{-2\sigma^2\pi^2\|w\|^2}
\end{aligned}$$

Let us calculate $K(w)$:

$$\begin{aligned}
K(w) = \frac{P_L(w)}{P_L\left(\frac{w}{\alpha}\right)} &= \frac{\left(\frac{1}{(2\pi)^{\frac{d-1}{2}}}\cdot e^{-2\sigma^2\pi^2\|w\|^2}\right)}{\frac{1}{(2\pi)^{\frac{d-1}{2}}}\cdot e^{-2\sigma^2\pi^2\left\|\frac{w}{\alpha}\right\|^2}} = \exp\left(-2\sigma^2\pi^2\left(\|w\|^2 - \frac{\|w\|^2}{\alpha}\right)\right) \\
&= \exp\left(-2\sigma^2\pi^2\|w\|^2\left(1-\frac{1}{\alpha}\right)\right)
\end{aligned}$$

As we can conclude from calculations above, $K(w)$ is not entirlty equal to $P_L(w)$, therefore they are not equal in the time domain also. But still the isotopic gaussian represents a better $p_L(x)$, since it has a smoother decay towards high frequencies, and is not flat on the entire support like the sinc, therefore is a more likley $p_L(x)$.

Question 6:

We need to derive an objective function whose minimum argument corresponds to the ML estimation of k given l . Let us write:

$$D(Q_k||Q_{\hat{k}}) = E_{k\sim Q_k}\log\frac{Q_k}{Q_{\hat{k}}} = E_{k\sim Q_k}\log Q_k - E_{k\sim Q_k}\log Q_{\hat{k}}$$

So the MLE estimator of k given l is:

$$\begin{aligned}
\hat{k}_{MLE} &= \underset{\hat{k}}{argmin}\, D(Q_k||Q_{\hat{k}}) = \underset{\hat{k}}{argmin}\, E_{k\sim Q_k}\log Q_k - E_{k\sim Q_k}\log Q_{\hat{k}} = \underset{\hat{k}}{argmin}\, -E_{l\sim Q_k}\log Q_{\hat{k}} = \\
&= \underset{\hat{k}}{argmin} -E_{l\sim P_{l|k}(l|k)}\log P_{l|k}(l|\hat{k}) =
\end{aligned}$$

As we have seen in the lectures, this can be approximated to:

$$\begin{aligned}
&= \underset{\hat{k}}{argmin}\; L(q_1, q_2, \dots q_M|\hat{k}) = \underset{\hat{k}}{argmin}\; -\frac{\log P_{l|k}(q_1, q_2, \dots q_M|\hat{k})}{M} \\
&= \underset{\hat{k}}{argmax}\, P_{l|k}(q_1, q_2, \dots q_M|\hat{k}) = \underset{\hat{k}}{argmax}\prod_{i=1}^{M} P(q_i|\,\hat{k}) =
\end{aligned}$$

Using the Total Probability Equation, we can write:

$$= \underset{\hat{k}}{argmax} \prod_{i=1}^{M} \sum_{j=1}^{N} P(q_i|\hat{k}, p_j)P(p_j)$$

We are given that $n_i \sim \mathcal{N}(0, \sigma_N)$, so by substituting $q_i = \downarrow_\alpha (p_{j_i} * \hat{k}) + n_i$, we get:

$$= \underset{\hat{k}}{argmax} \prod_{i=1}^{M} \sum_{j=1}^{N} P_{n_i}\left(q_i - \downarrow_\alpha (p_j * \hat{k})\right) P(p_j)$$

We assumed that j_i is uniformly distributed over {1, 2, …, N}, therefore $P(p_j) = \frac{1}{N}$, therefore:

$$\hat{k}_{MLE} = \prod_{i=1}^{M} \sum_{j=1}^{N} P_{n_i}\left(q_i - \downarrow_\alpha (p_j * \hat{k})\right) \frac{1}{N} = \frac{1}{N} \prod_{i=1}^{M} \sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - \downarrow_\alpha (p_j * \hat{k})\right\|^2}{2\sigma_N^2}\right\}$$

Question 7:

As we know, $\hat{k}_{MAP} = \underset{\hat{k}}{argmax}\, P(l|k) = \underset{\hat{k}}{argmax}\, P(k|l)P(k) = \underset{\hat{k}}{argmax} \log (P(k|l)P(k)) = \underset{\hat{k}}{argmax} \log\ P(k|l) + \log\ P(k))$, therefore using the calculations above we can write:

$$\hat{k}_{MAP} = \underset{\hat{k}}{argmax} \log \prod_{i=1}^{M} \sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - \downarrow_\alpha (p_j * \hat{k})\right\|^2}{2\sigma_N^2}\right\} + \log P(k)$$

$$= \underset{\hat{k}}{argmax} \sum_{i=1}^{M} \log \left(\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - \downarrow_\alpha (p_j * \hat{k})\right\|^2}{2\sigma_N^2}\right\}`\right) + \log P(k)$$

We know that $Dk \sim N(0, \sigma_D)$ where D is some operator. Therefore we can say:

$$k \sim D^{-1}N(0, \sigma_D)$$

$$k \sim N(0, D^{-1}\sigma_D {D^{-1}}^T)$$

Let us define $C^{-1} = D^{-1}\sigma_D {D^{-1}}^T, i.e. C = D^T\sigma_D D$, therefore we can now say that:

$$P(k) = const\ exp(-\frac{\|Ck\|^2}{2})$$

By substituting this into $\hat{k}_{MAP}$ we get:

$$\hat{k}_{MAP} = \underset{\hat{k}}{argmax} \sum_{i=1}^{M} \log \left(\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - \downarrow_\alpha (p_j * \hat{k})\right\|^2}{2\sigma_N^2}\right\}\right) + \log const\ exp(-\frac{\|Ck\|^2}{2})$$

$$= \underset{\hat{k}}{argmax} \sum_{i=1}^{M} \log \left(\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - \downarrow_\alpha (p_j * \hat{k})\right\|^2}{2\sigma_N^2}\right\}\right) - \frac{\|Ck\|^2}{2}$$

Let us define $R_j k = \downarrow_\alpha (p_j * \hat{k})$, i.e. R_j is a circulant matrix that does the convolution with p_j and the down sampling, now we get:

$$\hat{k}_{MAP} = \underset{\hat{k}}{argmax} \sum_{i=1}^{M} \log \left(\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}\right) - \frac{\|Ck\|^2}{2}$$

Question 8:

To calculate derivative of the function above, we will define some functions and construct $\hat{k}_{MAP}$ out of these functions:

- $f(k) = \frac{\|Ck\|^2}{2}, \frac{\partial f(k)}{\partial k} = C^T Ck$
- $g(k,j) = \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}, \frac{\partial g(k,j)}{\partial k} = \frac{R_j^T}{\sigma_N^2}(q_i - R_j k) g(k,j)$
- $h(k,j) = \log \sum_{j=1}^{N} g(k,j), \frac{\partial h(k,j)}{\partial k} = \frac{1}{\sum_{j=1}^{N} g(k,j)} \sum_{j=1}^{N} \frac{\partial g(k,j)}{\partial k}$
- $w(k,j) = \sum_{i=1}^{M} h(k,j) - f(k)$

$$\hat{k}_{MAP} = \underset{\hat{k}}{argmax} \sum_{i=1}^{M} h(k,j) - f(k)$$

Now we can write:

$$\begin{aligned}\frac{\partial w(k,j)}{\partial k} &= \sum_{i=1}^{M} \frac{\partial h(k,j)}{\partial k} - \frac{\partial f(k)}{\partial k} \\ &= \sum_{i=1}^{M} \frac{1}{\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}} \sum_{j=1}^{N} \frac{R_j^T}{\sigma_N^2}(q_i - R_j k) \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\} - C^T Ck \\ &= \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{R_j^T(q_i - R_j k)}{\sigma_N^2 \sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\} - C^T Ck \\ &= \sum_{i,j} \frac{R_j^T(q_i - R_j k)}{\sigma_N^2 \sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\} - C^T Ck =\end{aligned}$$

By defining $\frac{\exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}}{\sum_{j=1}^{N} \exp\left\{-\frac{\left\|q_i - R_j k\right\|^2}{2\sigma_N^2}\right\}} = w_{i,j}$ we get:

$$\frac{\partial w(k,j)}{\partial k} = \sum_{i,j} \frac{R_j^T(q_i - R_j k)}{\sigma_N^2} w_{i,j} - C^T Ck$$

To calculate $\hat{k}_{MAP}$ we need to set $\frac{\partial w(k,j)}{\partial k} = 0$, and find k:

$$\sum_{i,j} \frac{R_j^T(q_i - R_j k)}{\sigma_N^2} w_{i,j} - C^T Ck = 0$$

$$\sum_{i,j} \frac{R_j^T(q_i - R_j k)}{\sigma_N^2} w_{i,j} - C^T C k = \left(-\sum_{i,j} \frac{R_j^T R_j}{\sigma_N^2} w_{i,j}\right) k + \sum_{i,j} R_j^T q_i - C^T C k = 0$$

$$\left(\frac{1}{\sigma_N^2} \sum_{i,j} R_j^T R_j w_{i,j} + C^T C\right) k = \sum_{i,j} R_j^T q_i$$

$$k = \left(\frac{1}{\sigma_N^2} \sum_{i,j} R_j^T R_j w_{i,j} + C^T C\right)^{-1} \sum_{i,j} R_j^T q_i$$

Now we can write an iterative algorithm to recover k:

- $initialize\ k\ with\ a\ delta\ function$
- $for\ T = 1, \ldots T\ do$:
  1. Calculate $r_j^\alpha = R_j k$
  2. Calculate $\frac{\exp\left\{-\frac{\left\|q_i - r_j^\alpha\right\|^2}{2\sigma_N^2}\right\}}{\sum_{j=1}^N \exp\left\{-\frac{\left\|q_i - r_j^\alpha\right\|^2}{2\sigma_N^2}\right\}} = w_{i,j}$
  3. Update $k = \left(\frac{1}{\sigma_N^2} \sum_{i,j} R_j^T R_j w_{i,j} + C^T C\right)^{-1} \sum_{i,j} R_j^T q_i$

Where $D = laplacian\ operator\ as\ circulant\ matrix.$

And we chose a Laplacian operator, since incase of a gaussian distribution of, using a Laplacian operator on a gaussian will also give a smoother gaussian.

Question 9:

Assuming $f\left(\frac{x}{\alpha}\right)$ is the zoomed in version of our scene, and it is being captured using $p_L(x)$ on the lattice Z^2, the resulting image can be expressed as an integral over $f\left(\frac{x}{\alpha}\right)$ and $p_H(x)$ as follows:

$$z[n] = \int f\left(n - \frac{x}{\alpha}\right) p_L(x) dx = \int \frac{1}{\alpha} f\left(n - \frac{x}{\alpha}\right)\ p_H\left(\frac{x}{\alpha}\right) dx = \int \frac{1}{\alpha} f(n - x)\ p_H(x) dx = (f * p_H)[n]$$

This can be justified because of the relation between p_H and p_L that is:

$$p_H(x) = \alpha p(\alpha x)$$
$$p_L(x) = \frac{1}{\alpha} p_H\left(\frac{x}{\alpha}\right)$$

Question 10:

As we calculated in previous sections:

$$h[n] = (f * p_H)(n) = \int f(z) p_H(n - z) dz$$

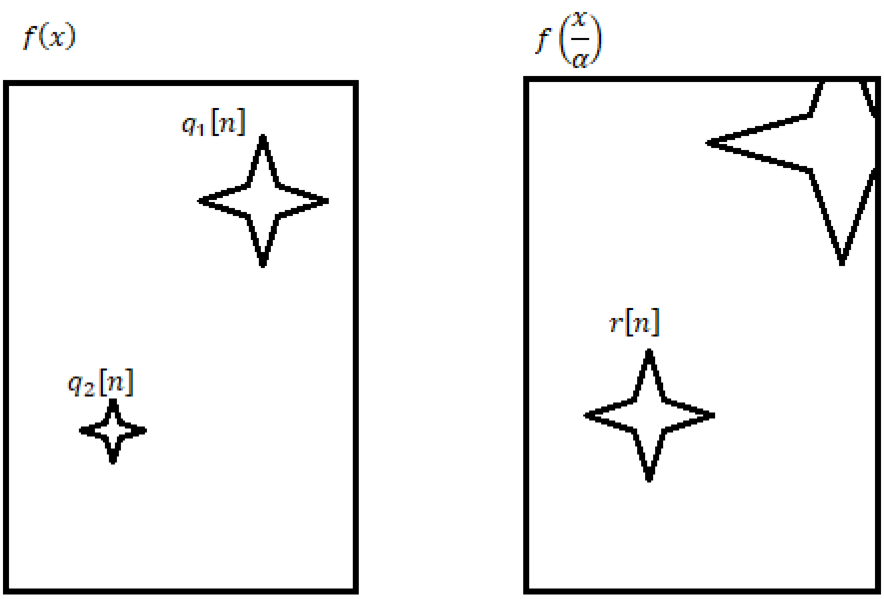
$$l[n] = \downarrow_\alpha (h * k)[n]$$

We can continue with this as follows:

$$l[n] = \downarrow_{\alpha} (h * k)[n] = \downarrow_{\alpha} (f * p_H * k)[n] = \downarrow_{\alpha} (z * k)[n]$$

Question 11:

By defining $r[n], q_1[n], q_2[n]$ as similar patches from l the low-res image on both scenes $f\left(\frac{x}{\alpha}\right)$ *and* $f(x)$, assuming $r[n], q_1[n]$ are larger than $q_2[n]$ by factor α (see image for demonstration):



From the image above we notice that certain patterns in $f\left(\frac{x}{\alpha}\right)$ such as r are approximately equal to other patterns in $f(x)$ such as q_1, this is true because of the reoccurring patterns in the original scene. We can mathematically express:

$$q_2[n] = \int f(x) p_L(n - x) dx = (f * p_L)[n]$$

$$r[n] = \int f\left(\frac{x}{\alpha}\right) p_L(n - x) dx = \int \alpha f(x) p_L(n - \alpha x) dx = \int f(x) p_H\left(\frac{n}{\alpha} - x\right) dx$$

$$= \uparrow_{\alpha} \int f(x) p_H(n - x) dx = \uparrow_{\alpha} (f * p_H)[n]$$

As we conclude from the calculations above we can relate r to the high-res image of the continuous scene, and q can be related with the low-res one. By mathematical manipulations on the above we can get:

$$q[n] = \sum r[m] k[\alpha n - m] = \downarrow_{\alpha} (r * k)[n]$$

Therefore to obtain an approximation of the set $\{p_1, p_2, \dots, p_N\}$ we can use the values in $f(\frac{x}{\alpha})$ with low-res psf, which according to the above are approximate to $f(x)$ with high-res psf, and restore the kernel as we suggested in the algorithm on section 8, and use that to recover h.

Question 12:

We can recover h given l, k, by mathematically calculating:

$$\underset{h}{\mathrm{a}rgmin} \left\| \downarrow_{\alpha} \left(h * \hat{k}\right) - l \right\|_2^2 + \lambda P(h)$$

For some priority P and constant λ. This would be a good recover, since it minimizes the mean square error between the low-res image and the low-res image we get from applying the kernel we calculated in the algorithm we suggested.