Measure Theory

1 probability spaces

Basic definitions. Monotonicity. TEst 2

Theorem 1. $(\Omega, \mathcal{F}, \mu)$

1. *If* $A \subset \bigcup_{i=1}^{\infty} A_i$*, then* $\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$
2. *If* $A_1 \subset A_2 \subset \cdots$ *and* $A = \bigcup_{i=1}^{\infty} A_i$ *then* $\mu(A_i) \uparrow \mu(A)$
3. *If* $A_1 \supset A_2 \supset \cdots$ *and* $A = \bigcap_{i=1}^{\infty} A_i$ *then* $\mu(A_i) \downarrow \mu(A)$

Proof.

1. Take $A_i' = A \cap A_i$, $B_1 = A_1'$, $B_n = A_n' - \bigcup_{i=1}^{n-1} A_i$. Then B_i are disjoint, $\bigcup_{i=1}^{\infty} B_i = A$ and $B_i \subset A_i$ which gives

$$\mu(A) = \sum_{i=1}^{\infty} \mu(B_i) \le \sum_{i=1}^{\infty} \mu(A_i)$$

2. Write $B_1 = A_1$, $B_i = A_i - A_{i-1}$. Then the B_is are disjoint, and $\bigcup_{i=1}^{\infty} B_i = A$, so

$$\mu(A) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_n)$$

3. $A_1 - A_n \uparrow A_1 - A$ so $\mu(A_1 - A_n) \uparrow \mu(A_1 - A)$. But $\mu(A_1 - A_n) = \mu(A_1) - \mu(A_n)$ and similar for the RHS, so $\mu(A_n) \downarrow \mu(A)$.

□

Discrete probability spaces. σ-field generated by a collection of subsets.

Definition 1. F is called a Stieltjes measure function if

1. F is nondecreasing
2. F is continuous from above: $\lim_{y\downarrow x} F(y) = F(x)$

Theorem 2. *Given a Stieltjes function* F*, there is a unique measure* μ *on* $(\mathbb{R}, \mathcal{B}$ *such that*

$$\mu((a,b]) = F(b) - F(a)$$

Definition 2. $\mathcal{S}$ is a semialgebra if

1. it is closed under intersections
2. If $S \in \mathcal{S}$, then S^c is a finite union of elements in $\mathcal{S}$

Definition 3. $\mathcal{A}$ is an algebra if it is closed under finite unions and complements