# CMPE 362
# Digital Image Processing

## Frequency Domain Techniques – Part I

# Image processing techniques

1. Spatial domain operations
   - operate on the pixels of an image
2. Frequency domain operations
   - operate on the Fourier transform of an image
     1) Compute the Fourier transform of an image
     2) Operate
     3) Take the inverse transform

# Fourier Transform

- Named after Joseph Fourier (1768-1830)
- Any periodic function can be represented as a weighted sum of sines and/or cosines of different frequencies.
- Fourier's ideas were initially met with skepticism.
  - At the time it first appeared, the concept that complicated functions can be represented as a sum of simple sines and cosines was not at all intuitive.

# A sum of sines

Our building block: $A\sin(\omega x+\phi)$

Add enough of them to get any periodic function f(x)

# Our building block: $A\sin(\omega x + \phi)$

https://www.desmos.com/calculator

sin(x) → T = 2$\pi$, f = 1/2$\pi$, A = 1

sin(2$\pi$x) → T = 1, f = 1, A = 1

sin(2$\pi$2x) → T = 0.5, f = 2, A = 1

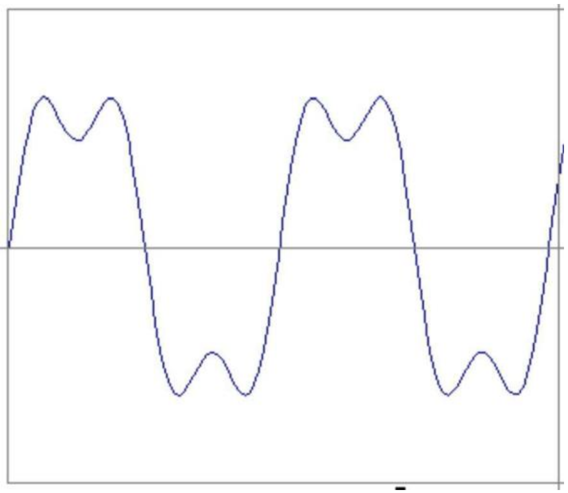
2 sin(2$\pi$2x) → T = 0.5, f = 2, A = 2

sin(2$\pi$x + 1) → T = 1, f = 1, A = 1, $\phi$ = 1

sin(2$\pi$x + 2) → T = 1, f = 1, A = 1, $\phi$ = 2

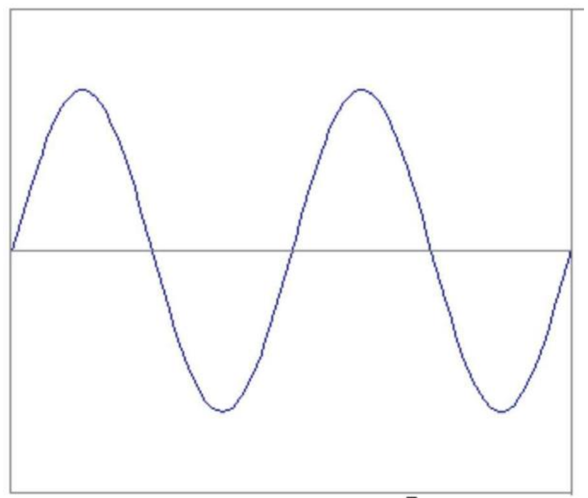
# Frequency Spectra

Example: $g(x) = \sin(2\pi f x) + (1/3)\sin(2\pi(3f)x)$



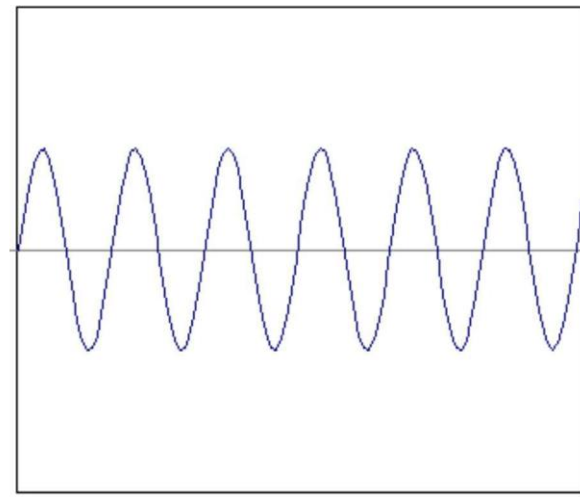
$g(x)$

=

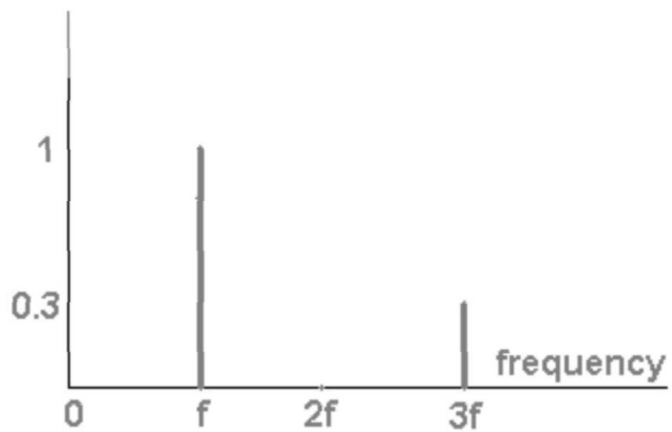


$\sin(2\pi f x)$

+

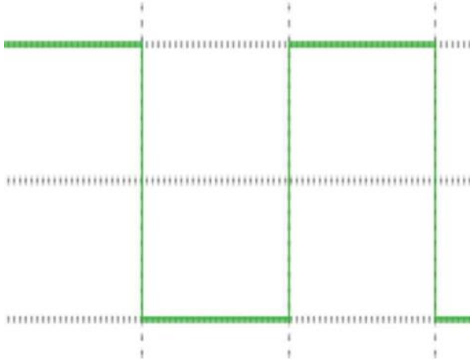


$(1/3)\sin(2\pi(3f)x)$



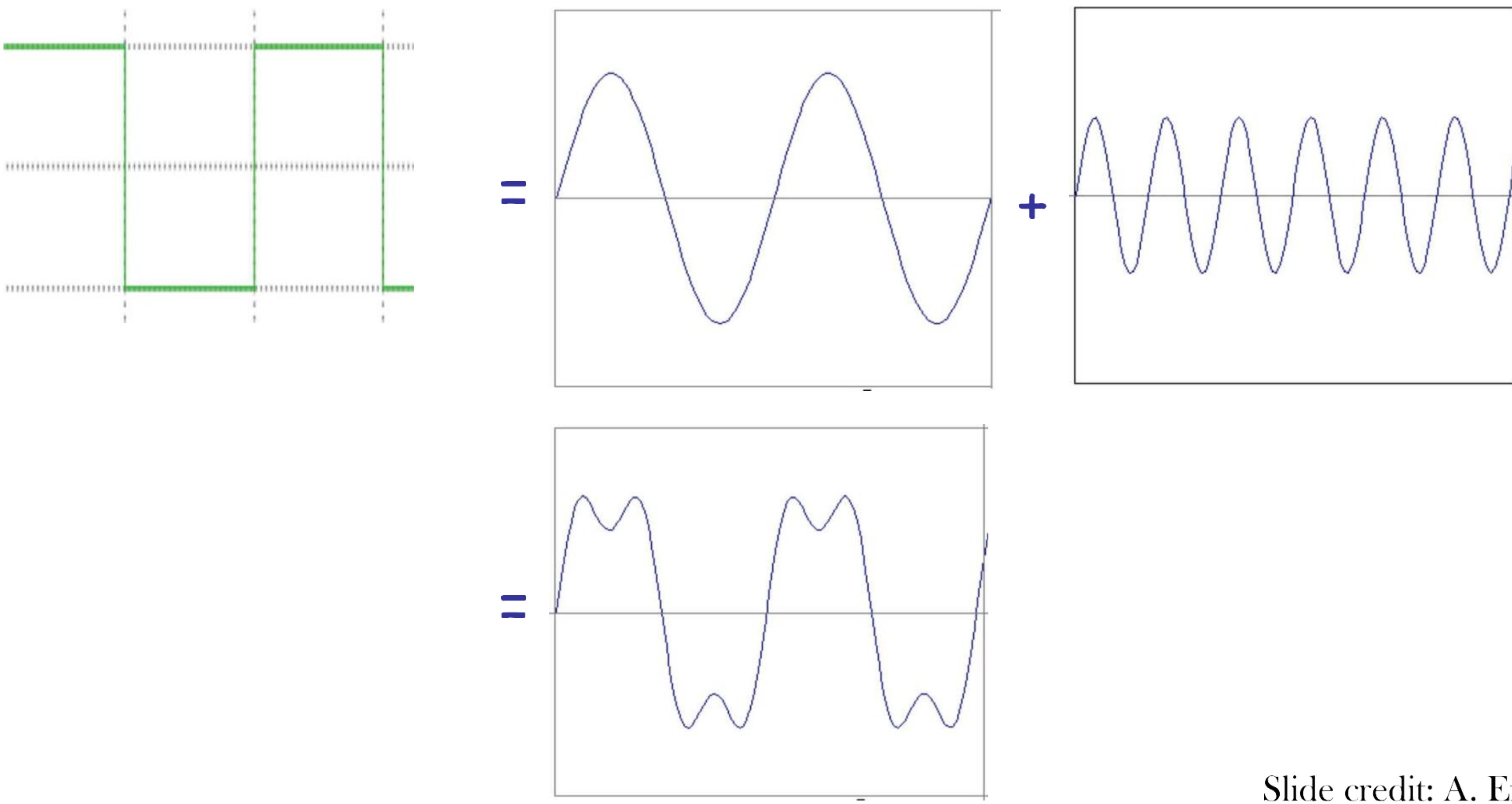

Slide credit: A. Efros

# Frequency Spectra



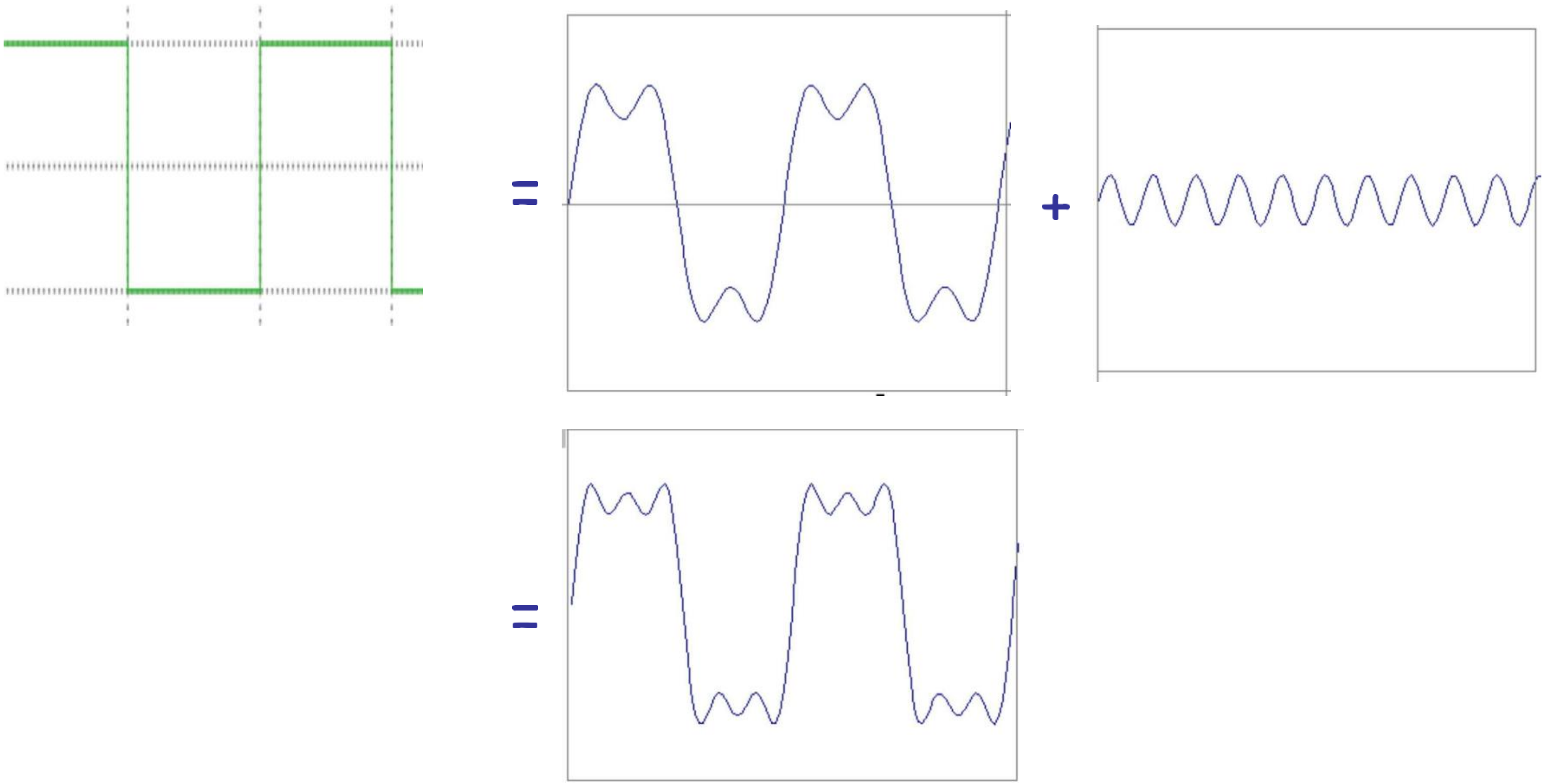
Slide credit: A. Efros

# Frequency Spectra



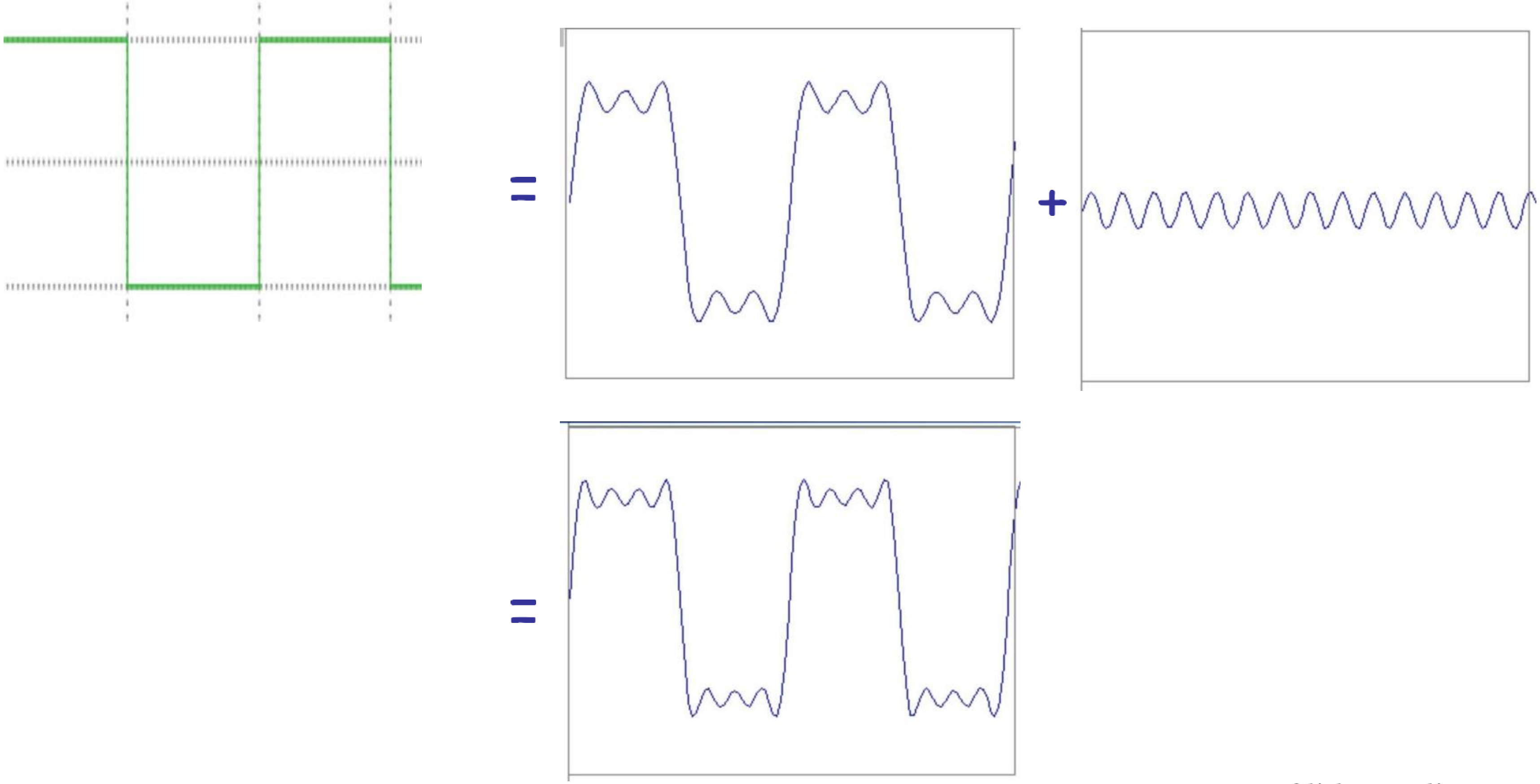
Slide credit: A. Efros

# Frequency Spectra



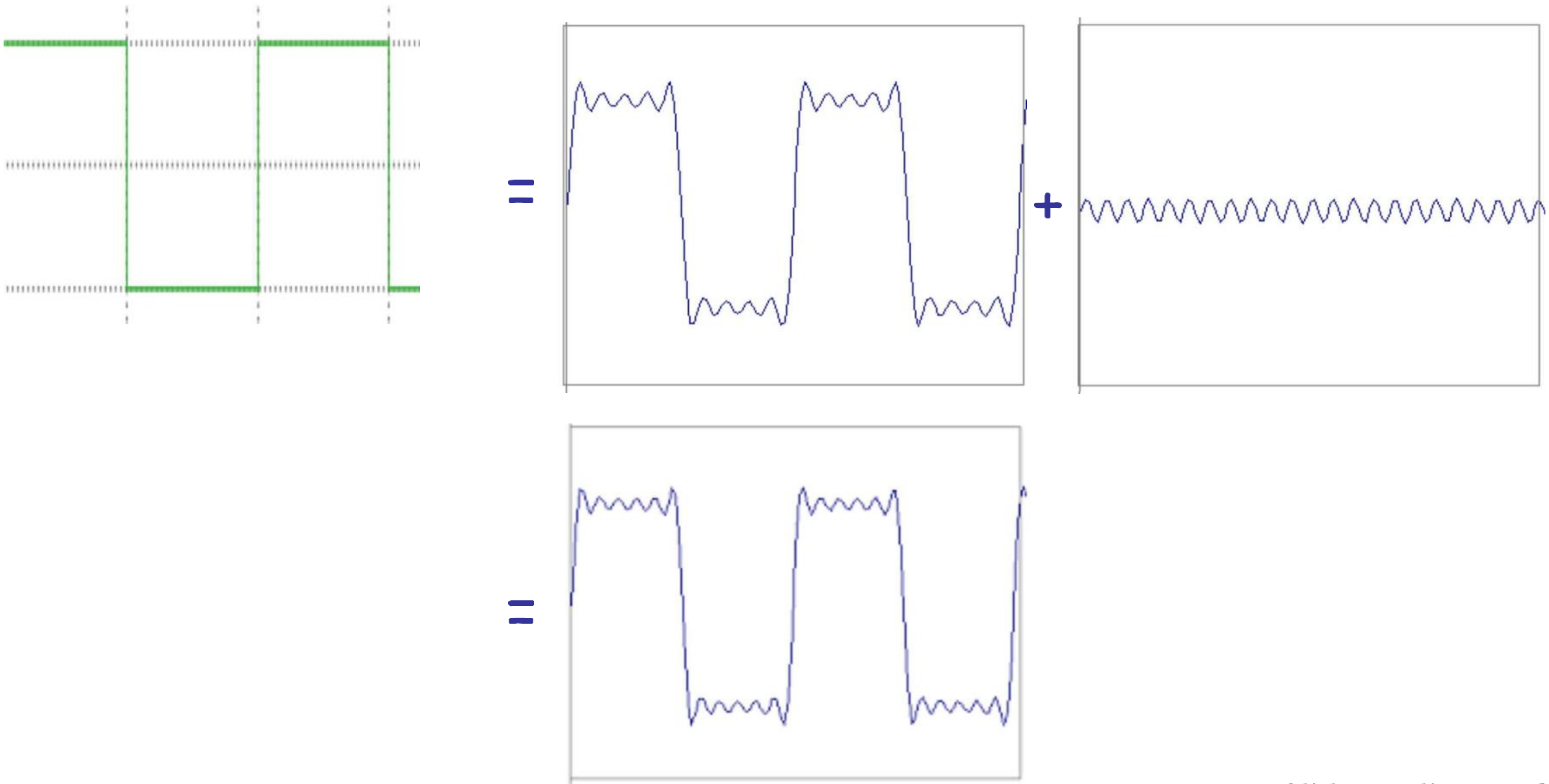
Slide credit: A. Efros

# Frequency Spectra



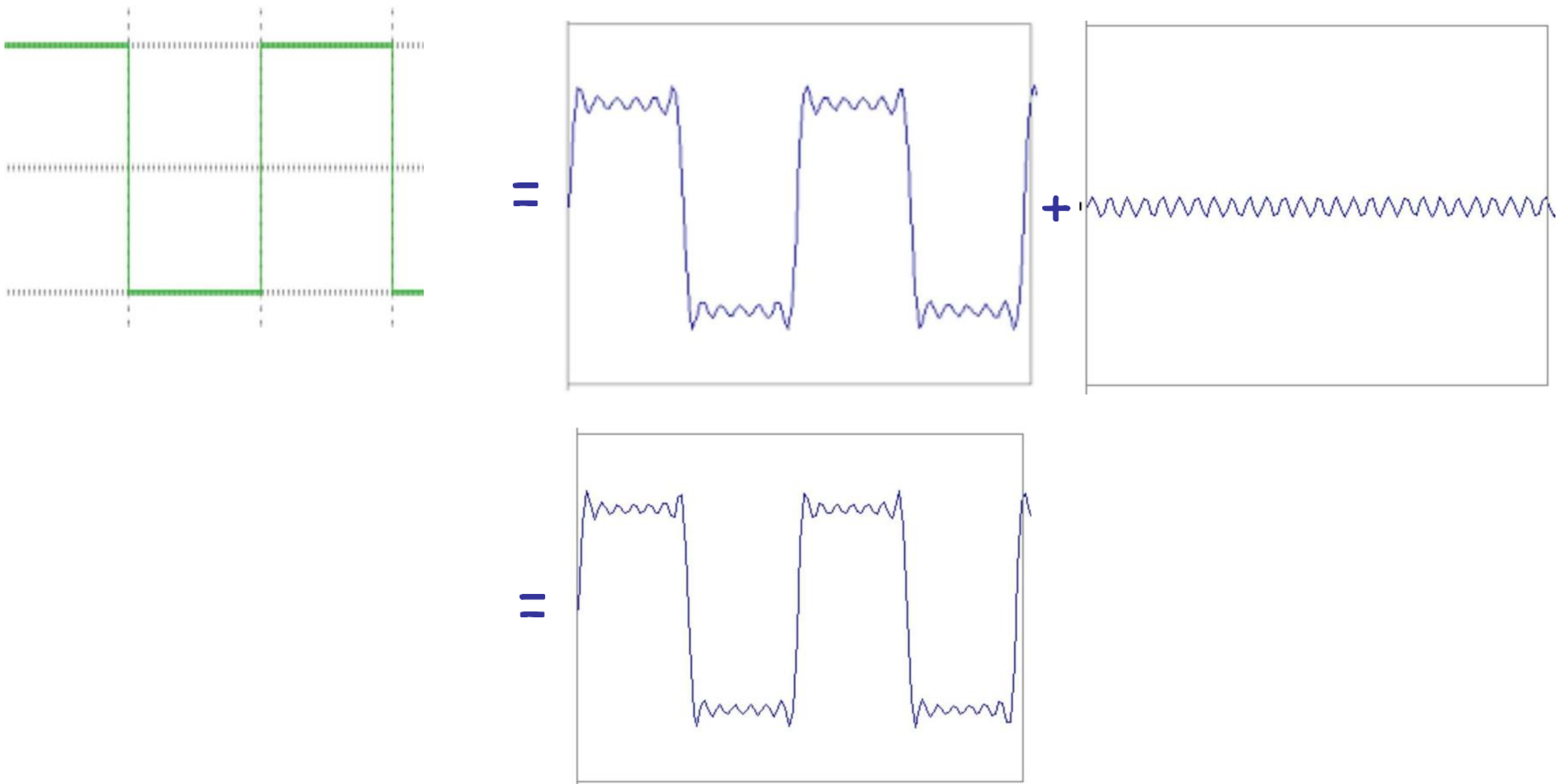
Slide credit: A. Efros

# Frequency Spectra



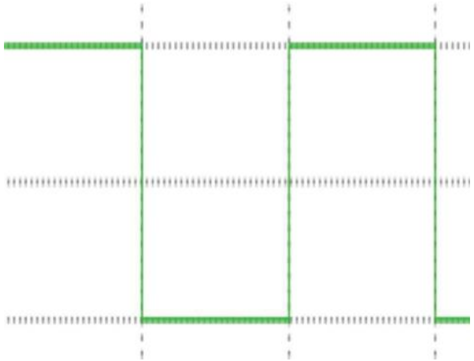

Slide credit: A. Efros

# Frequency Spectra



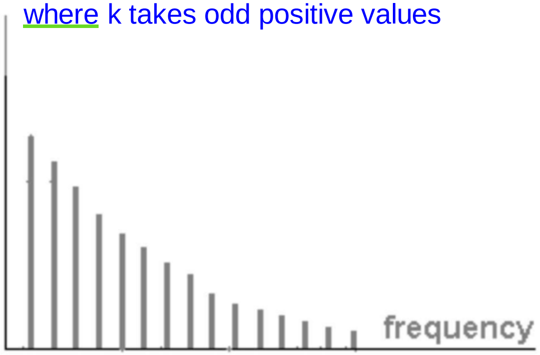

Slide credit: A. Efros

# Frequency Spectra



$$= \boxed{A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k t)}$$

where k takes odd positive values



# Fourier Transform

We want to understand the frequency *w* of our signal. So, let's reparametrize the signal by *w* instead of *x*:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(w)$$

For every *w* from 0 to inf, ***F(w)*** holds the amplitude *A* and phase *f* of the corresponding sine $A\sin(\omega x + \phi)$

- How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

$$F(w) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(x)$$

Slide credit: A. Efros

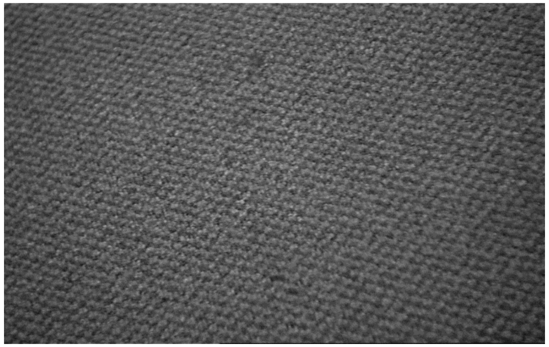
# Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$

Phase: $\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$

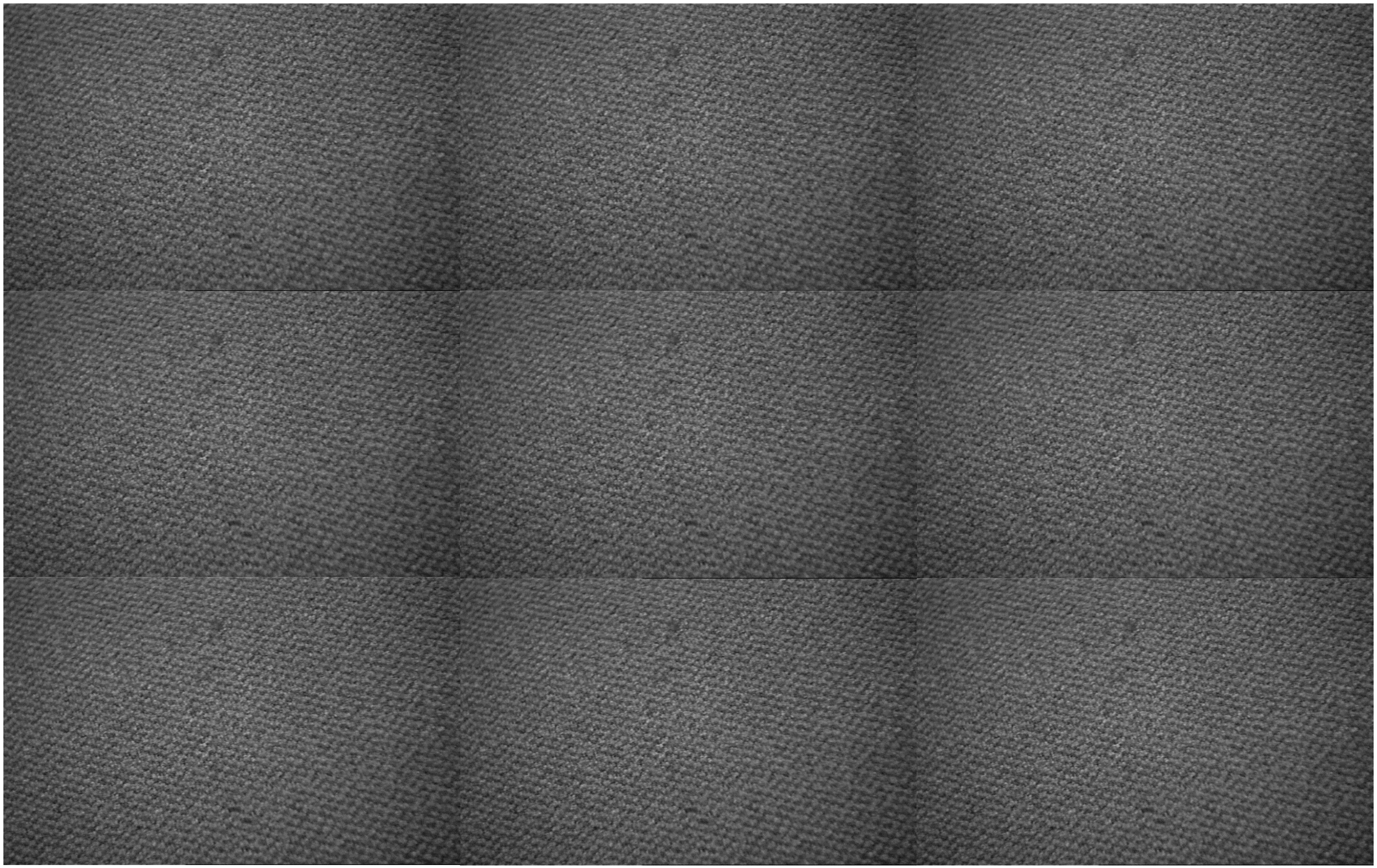
We treat images as infinite-size, continuous periodic functions.



# We treat images as infinite-size, continuous periodic functions.

...

...



...

...

# Discrete Fourier transform

- Forward transform

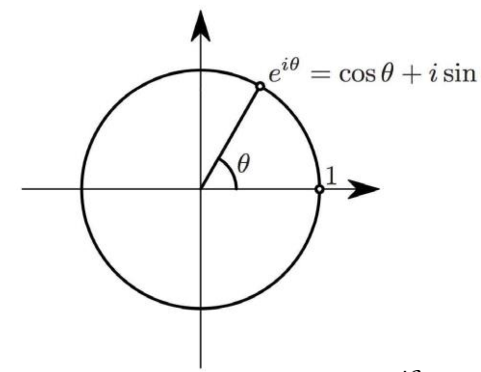
$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

for $u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$

- Inverse transform

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

for $x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$



Euler's definition of $e^{i\theta}$

*u*, *v* : the transform or frequency variables
*x*, *y* : the spatial or image variables

Slide credit: B. Freeman and A. Torralba

What is F(0, 0)?

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

for $u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$

F(0, 0) refers to the mean intensity of the image.

# The Fourier Transform

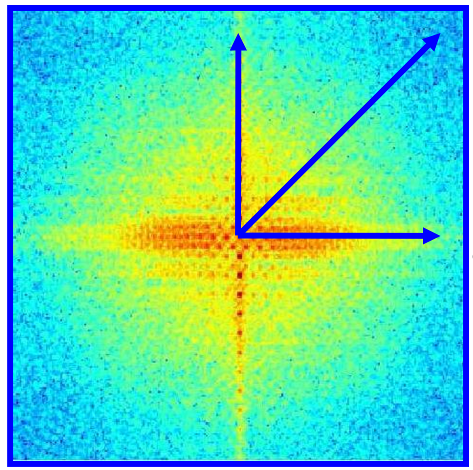
- Represent function on a new basis
  - Think of functions as vectors, with many components
  - We now apply a linear transformation to transform the basis
    - dot product with each basis element
- In the expression, u and v select the basis element, so a function of x and y becomes a function of u and v
- basis elements have the form $e^{-i2\pi(ux+vy)}$

Slide credit: S. Thrun

# How to interpret 2D Fourier Spectrum

Vertical orientation

0



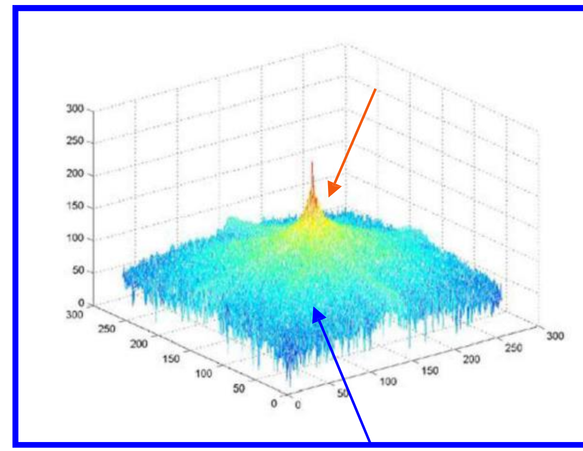
45 deg.

Horizontal orientation

0 $f_{max}$

fx in cycles/image

Low spatial frequencies



High spatial frequencies

Log power spectrum

Slide credit: B. Freeman and A. Torralba

Fourier basis element

$$e^{-i2\pi(ux+vy)}$$

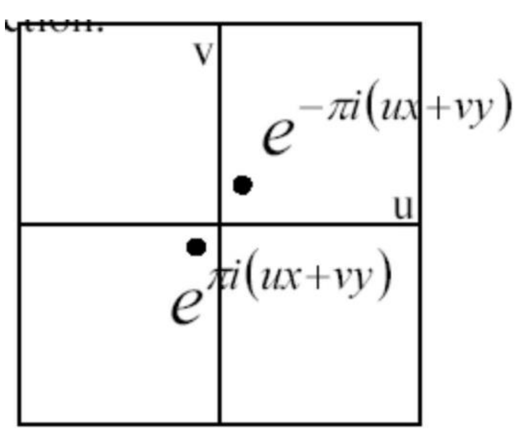
example, real part

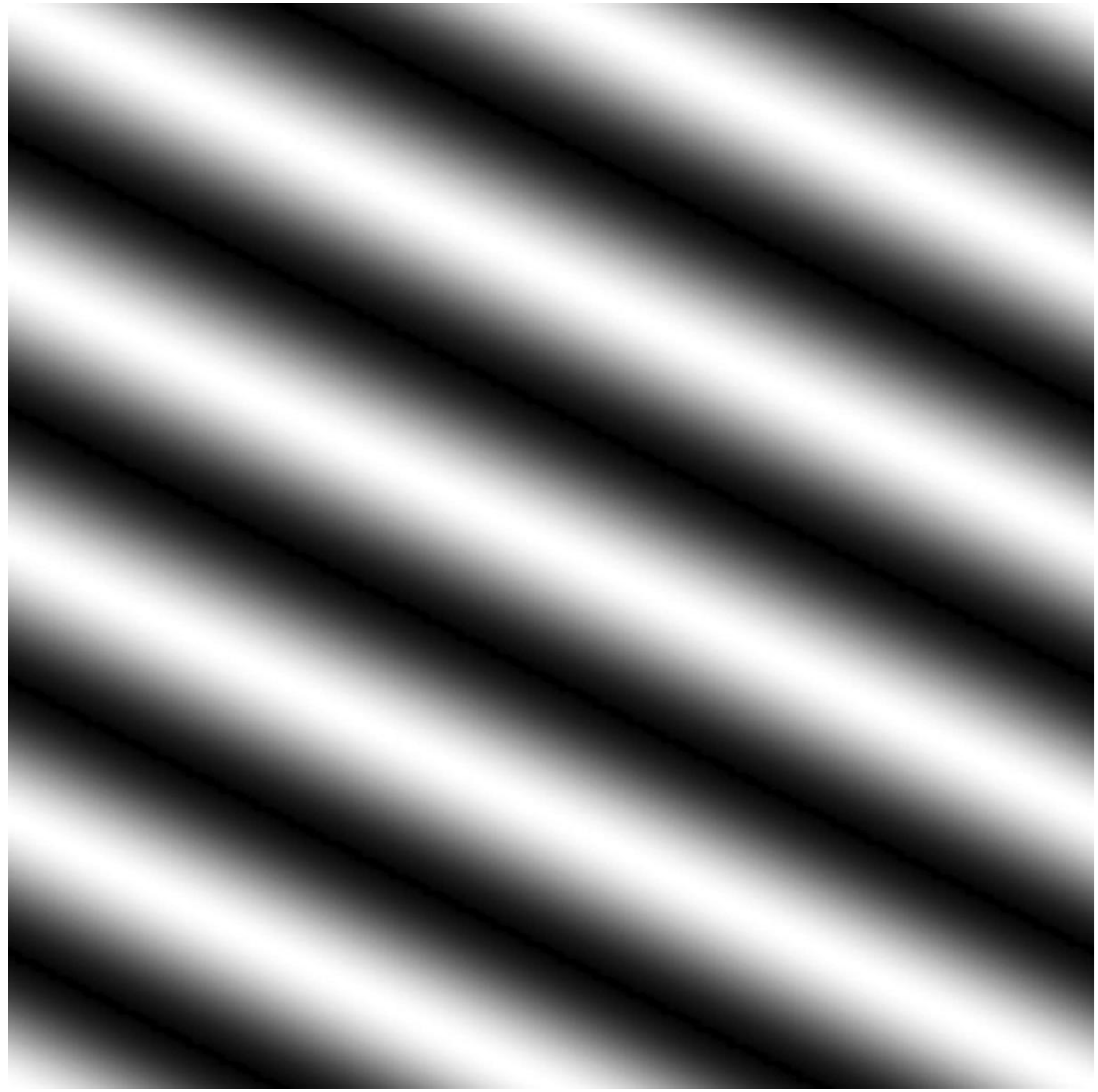
$F^{u,v}(x,y)$

$F^{u,v}(x,y)$=const. for $(ux+vy)$=const.

Vector (u,v)

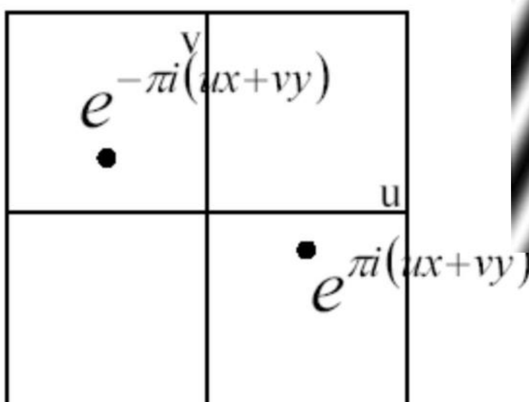
- Magnitude gives frequency
- Direction gives orientation.



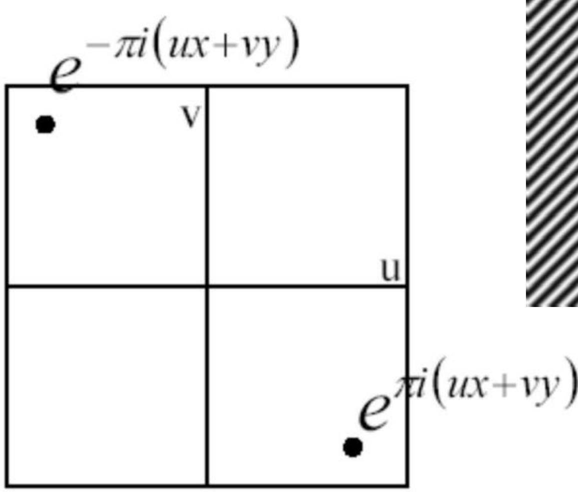
Slide credit: S. Thrun

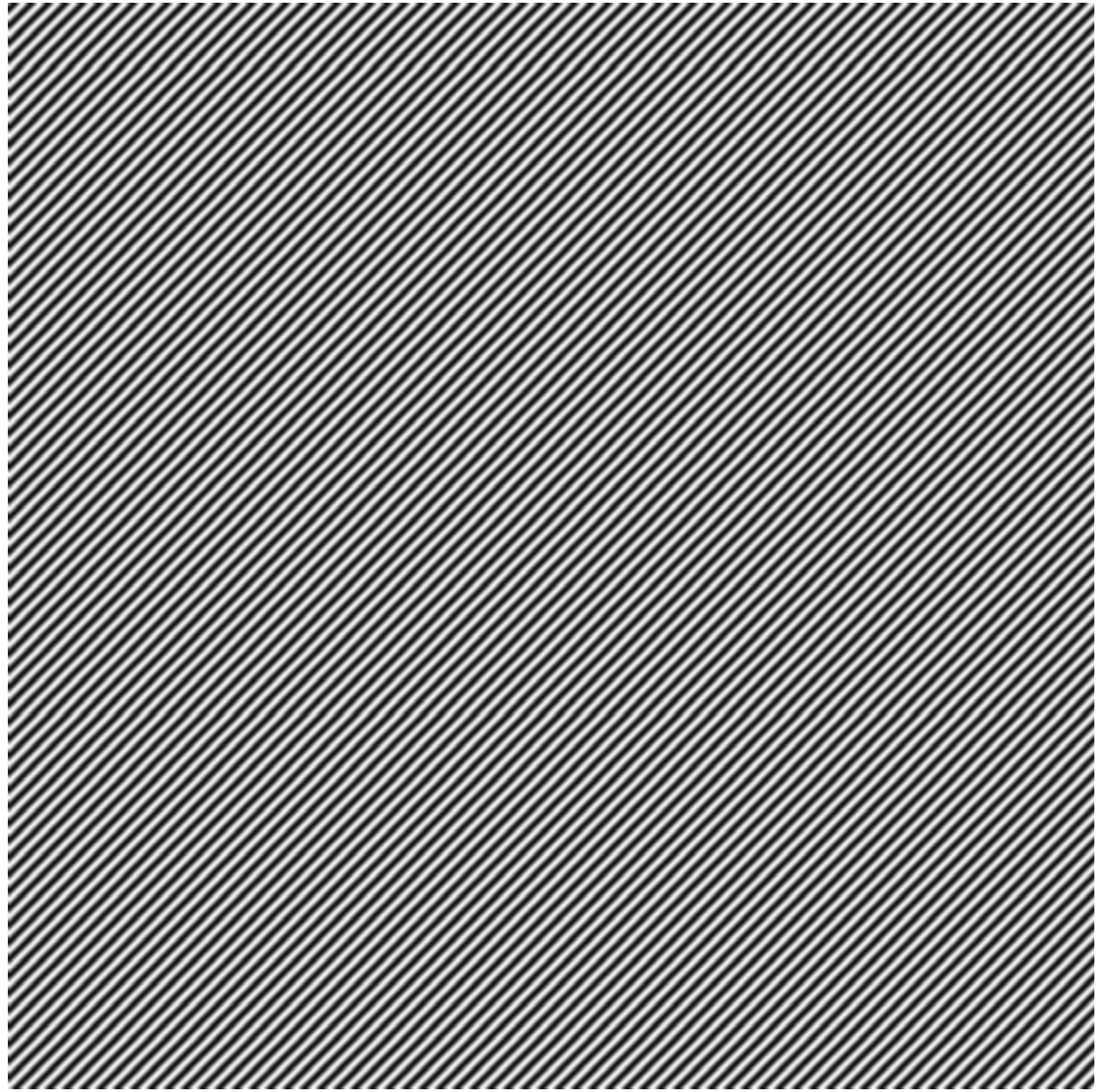
Here u and v are larger than in the previous slide.




Slide credit: S. Thrun

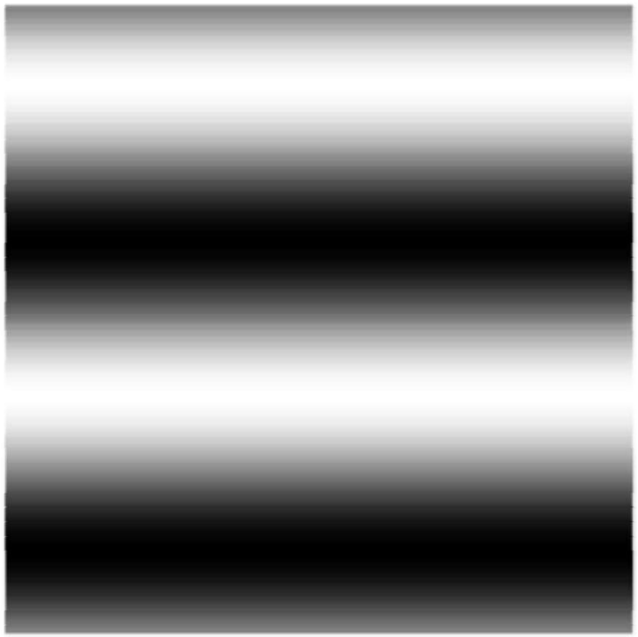
And larger still...

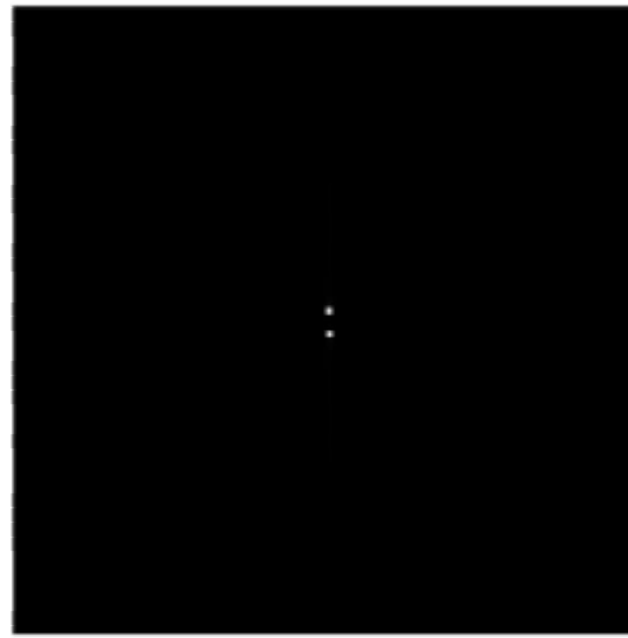


Slide credit: S. Thrun

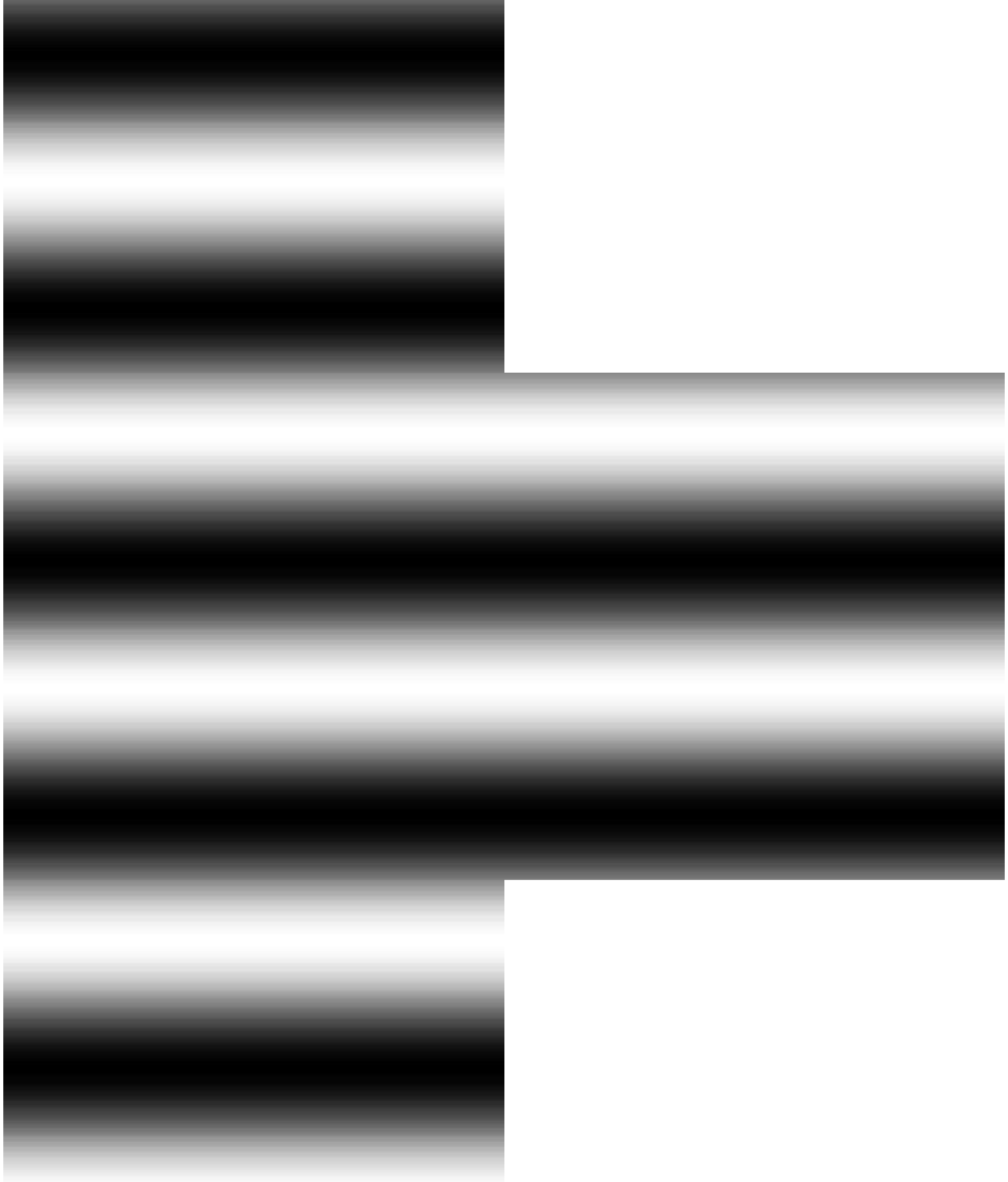
# 2D FFT





Sinusoid with frequency = 1 and its FFT

Slide credit: M. H. Yang

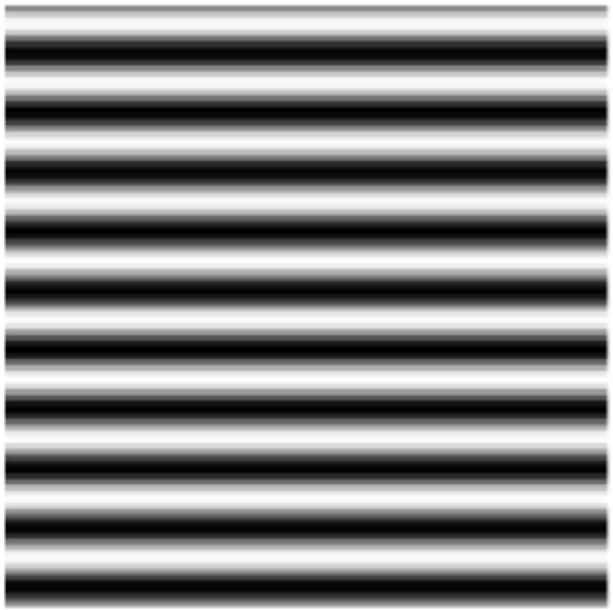


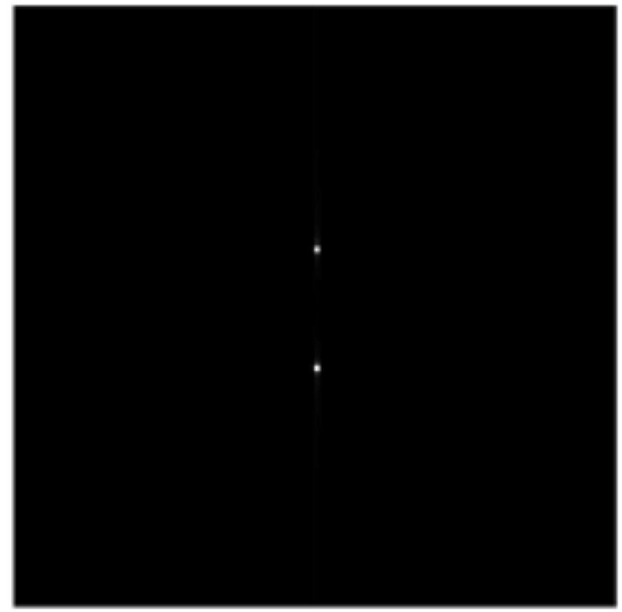
# 2D FFT

Sinusoid with frequency = 3 and its FFT

Slide credit: M. H. Yang

# 2D FFT

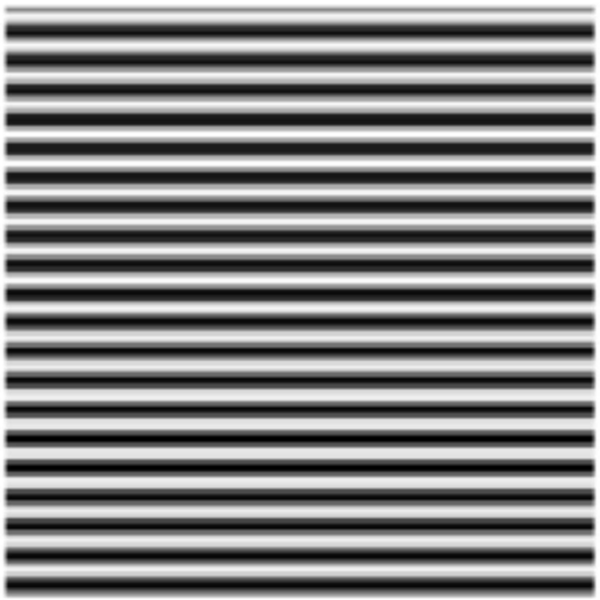


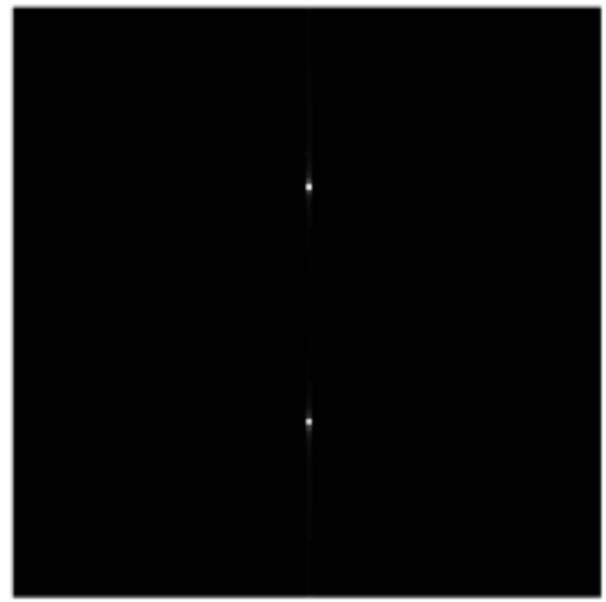


Sinusoid with frequency = 5 and its FFT

Slide credit: M. H. Yang

# 2D FFT

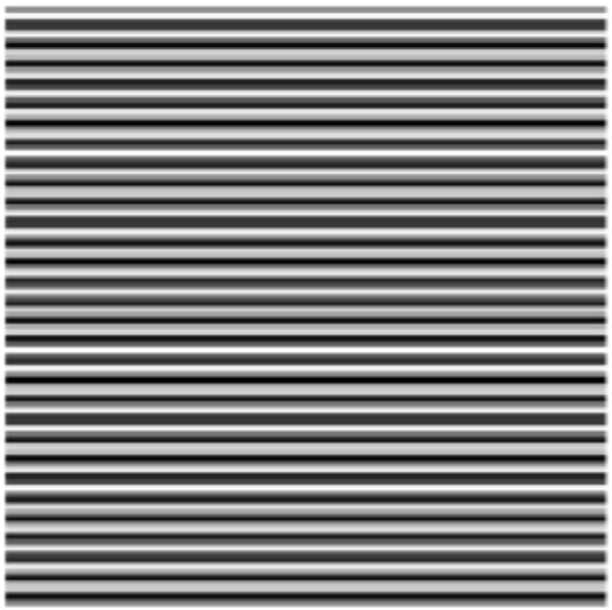
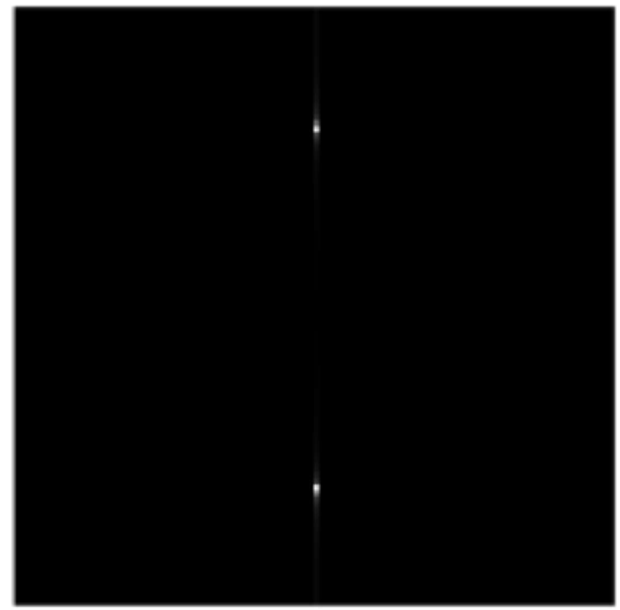




Sinusoid with frequency = 10 and its FFT

Slide credit: M. H. Yang

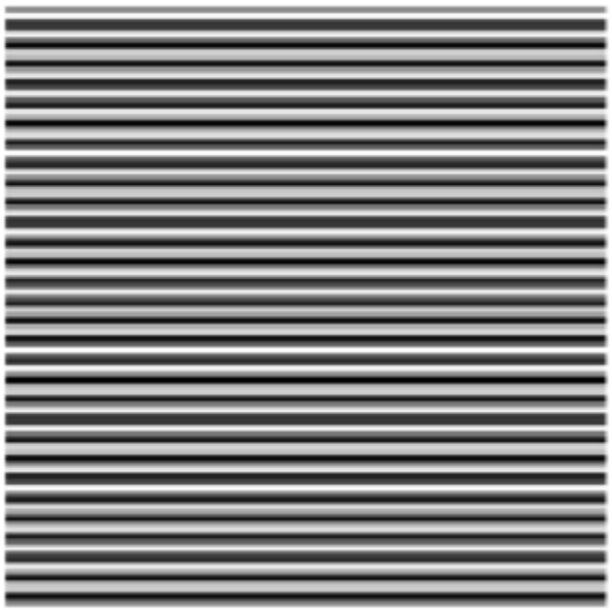
# 2D FFT

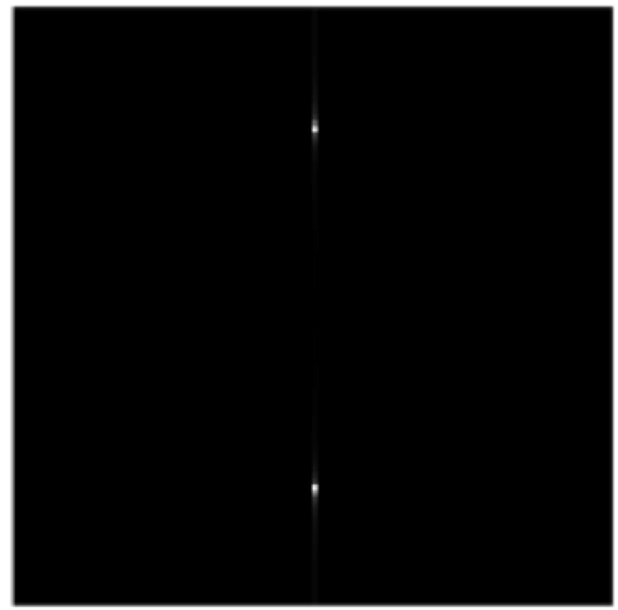
 

Sinusoid with frequency = 15 and its FFT

Slide credit: M. H. Yang

# 2D FFT

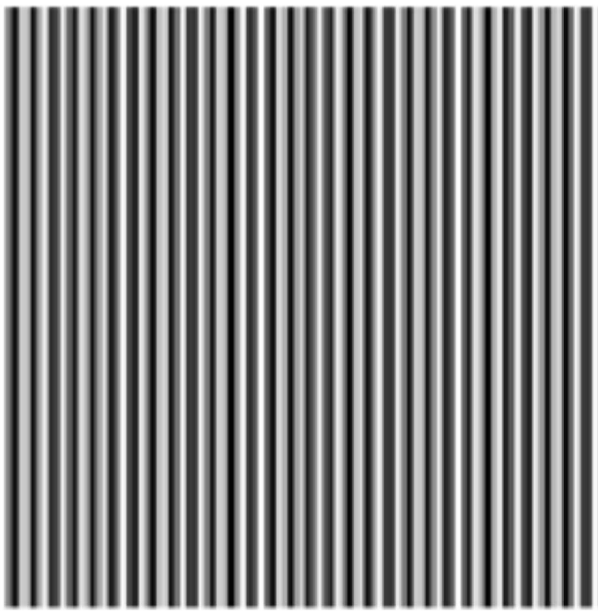


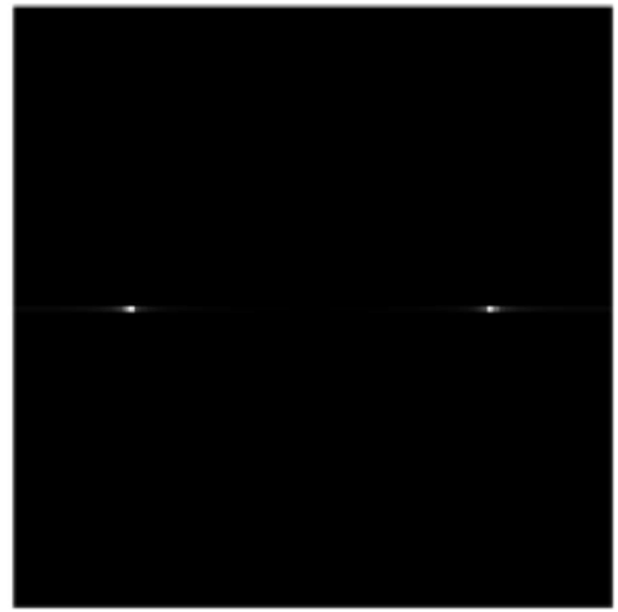


Sinusoid with frequency = 15 and its FFT

Slide credit: M. H. Yang

# 2D FFT





Sinusoid with frequency = 15 and its FFT

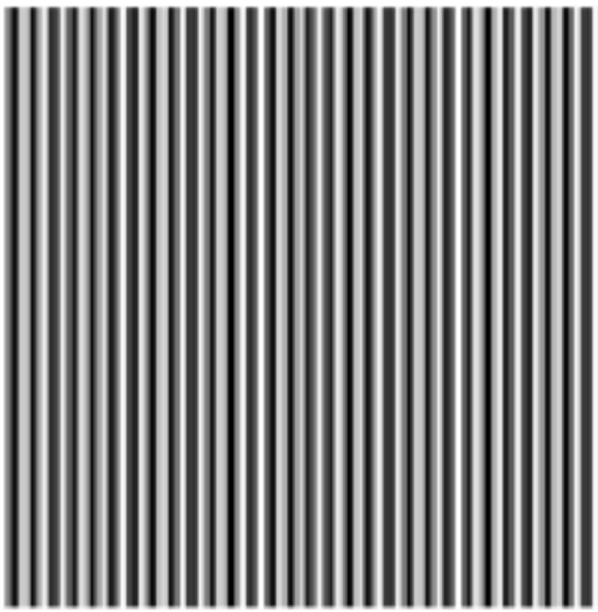
Slide credit: M. H. Yang

# Rotation

Sinusoid rotated at 30 degrees and its FFT

Slide credit: M. H. Yang

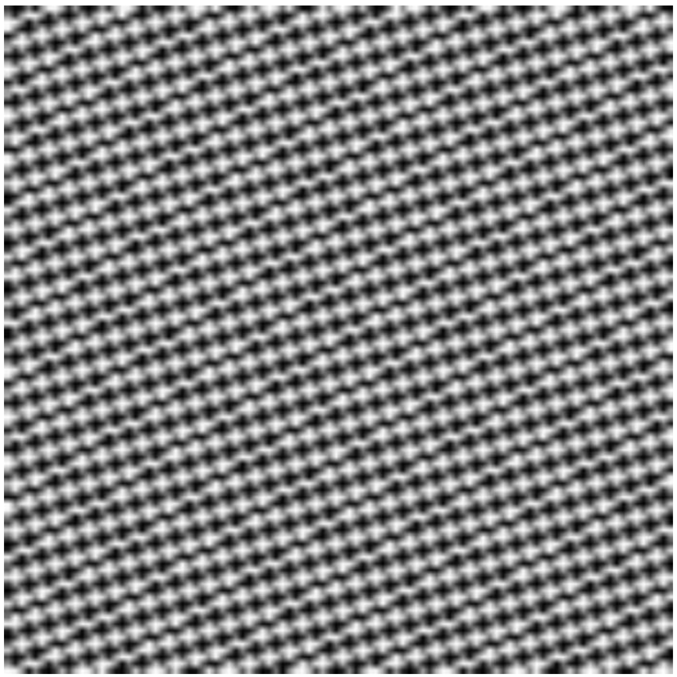
# 2D FFT

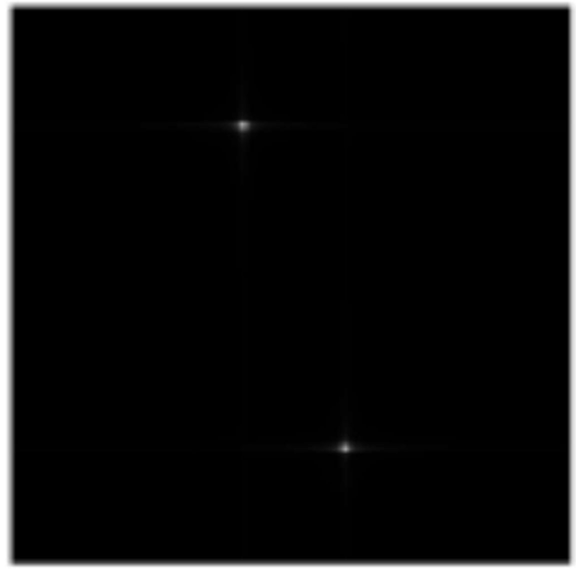


Sinusoid with frequency = 15 and its FFT

Slide credit: M. H. Yang

# 2D FFT



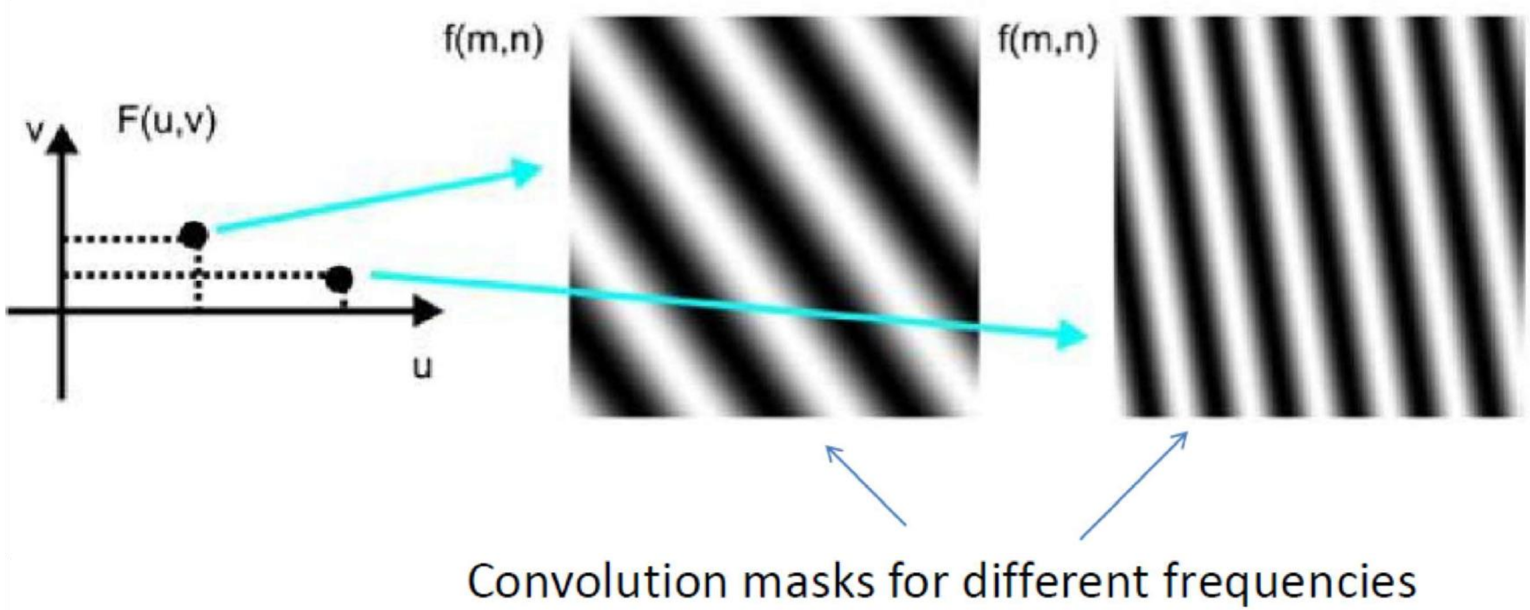


Sinusoid rotated at 60 degrees and its FFT

Slide credit: M. H. Yang

# 2D FFT

$$F(u,v) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi(xu/M + yv/N)}$$

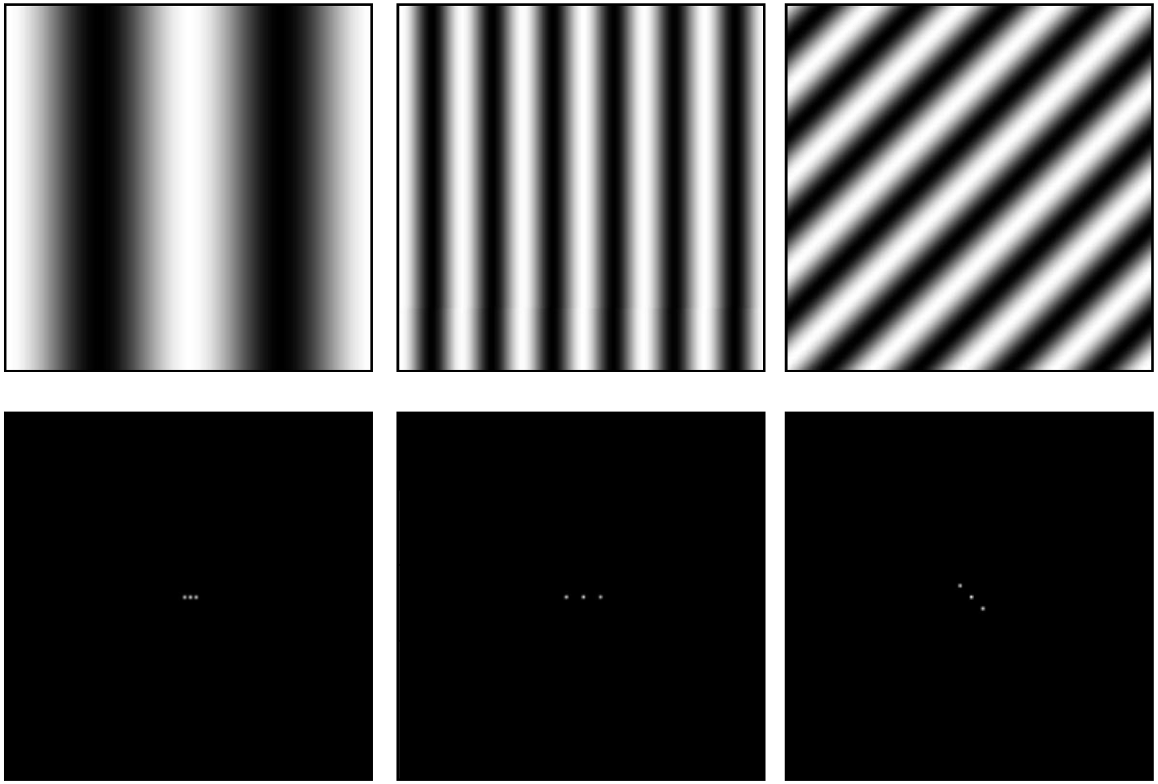



Slide credit: M. H. Yang

# Fourier analysis in images

Intensity Image

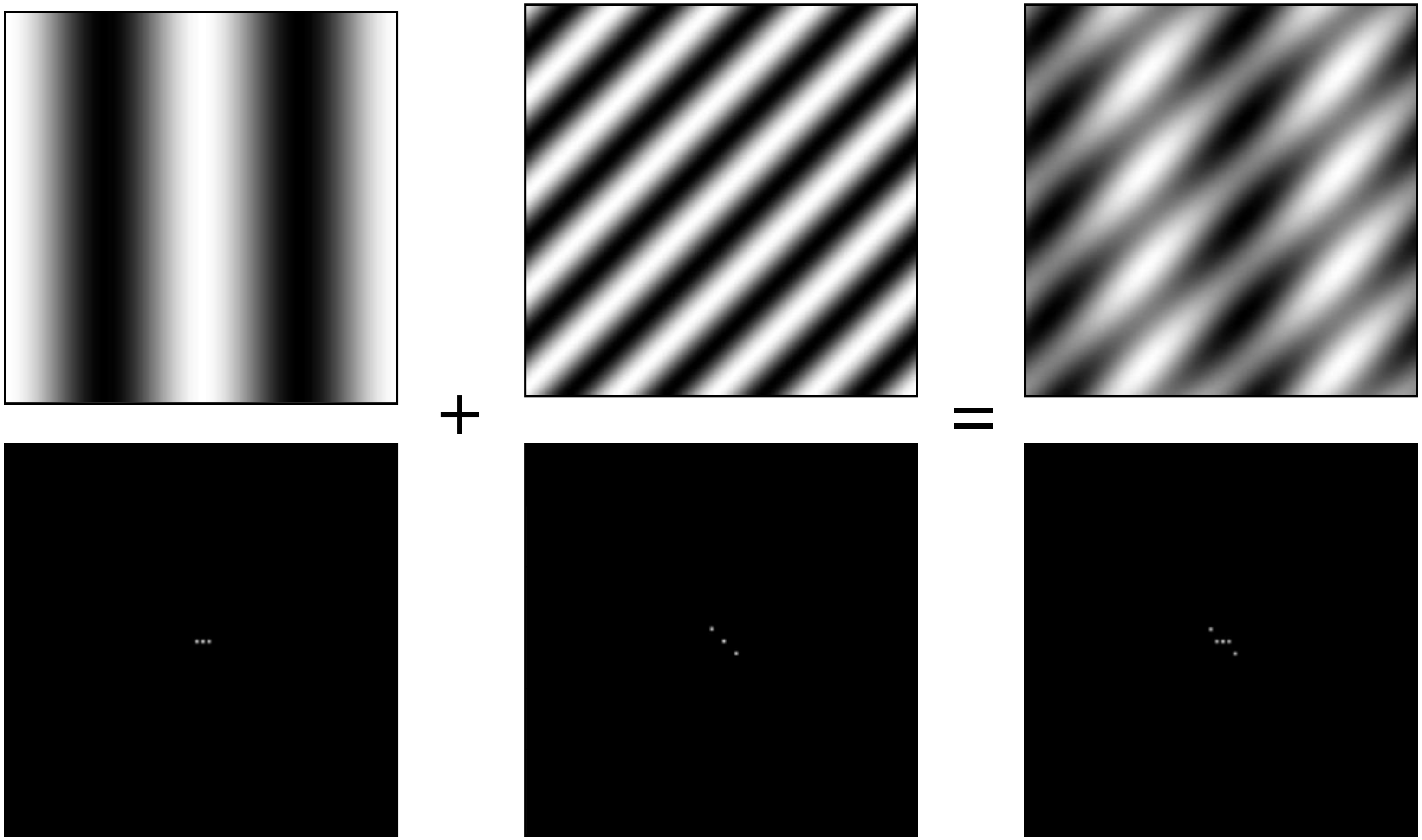
Fourier Image



http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html

Slide credit: A. Efros

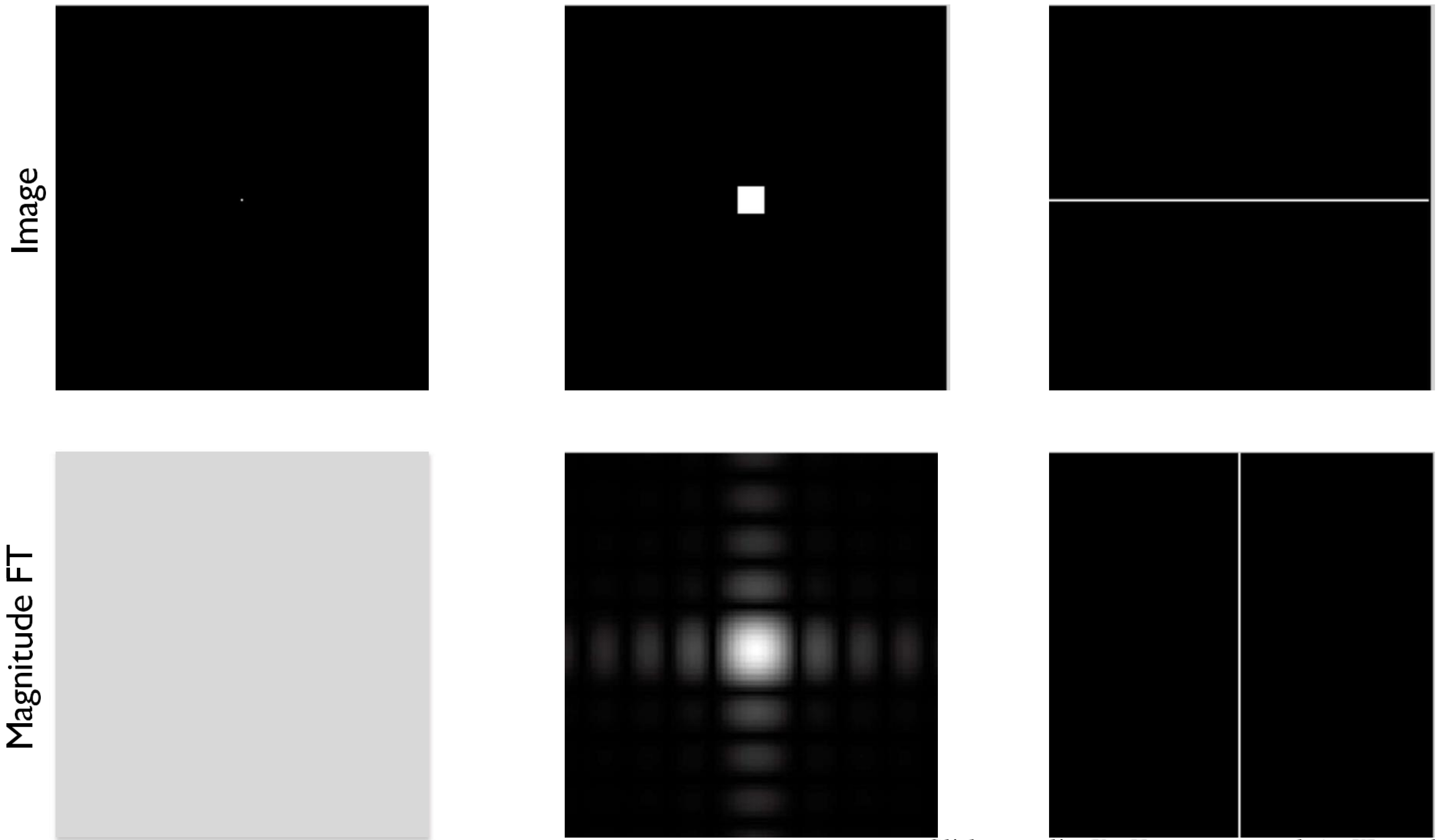
# Signals can be composed




http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html

Slide credit: A. Efros

# Some important Fourier Transforms

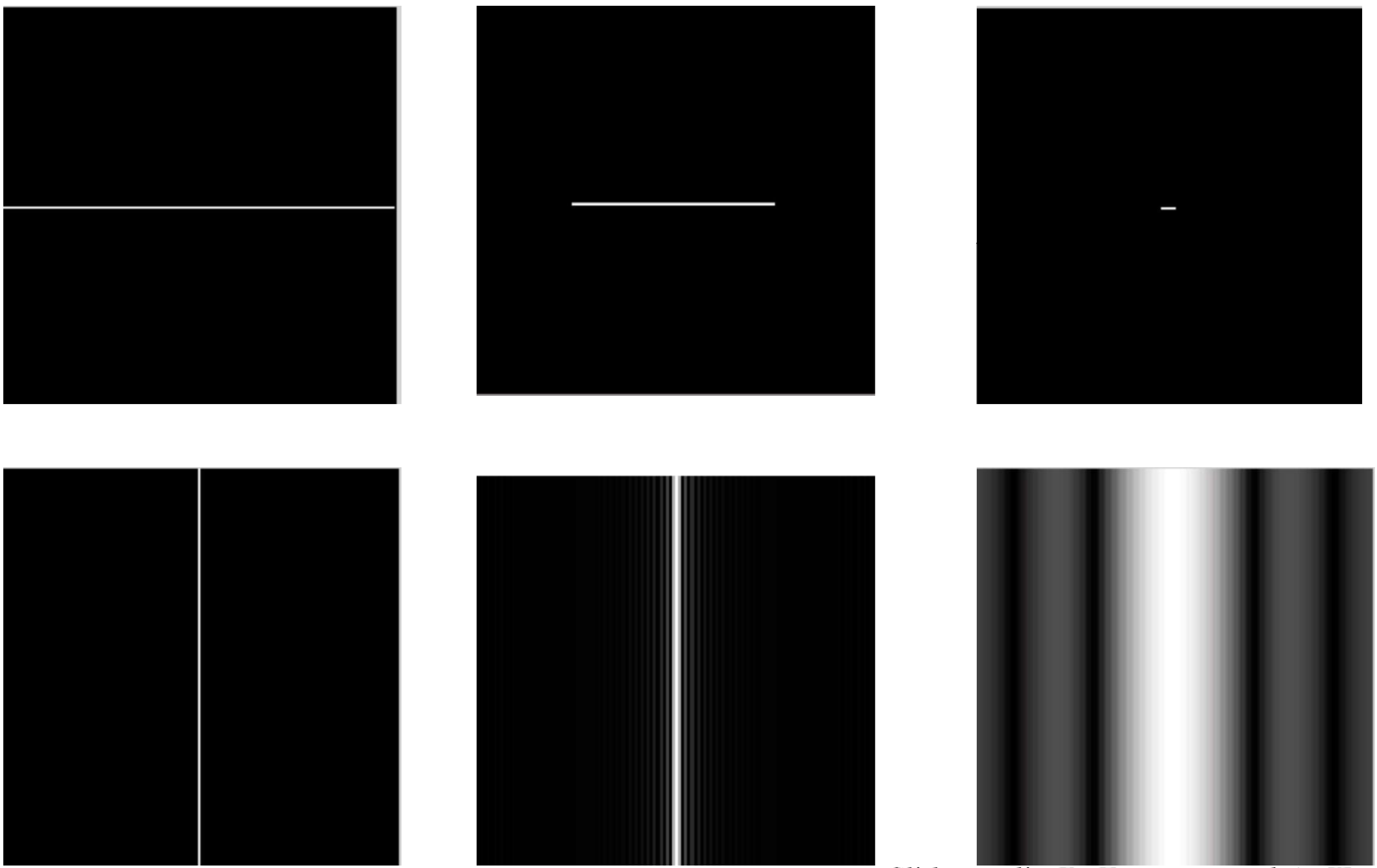



Slide credit: B. Freeman and A. Torralba

# Some important Fourier Transforms

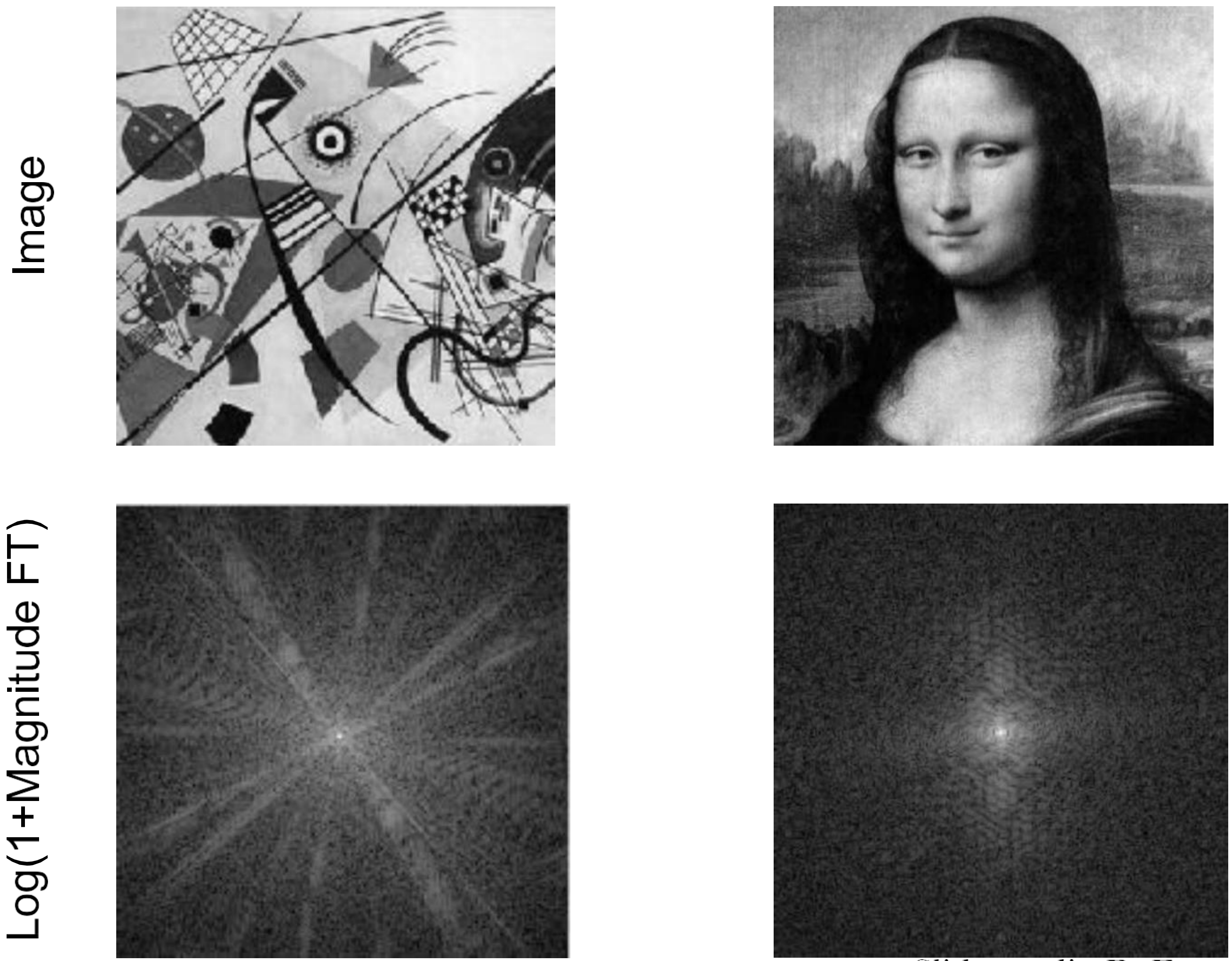
Image

Magnitude FT



Slide credit: B. Freeman and A. Torralba

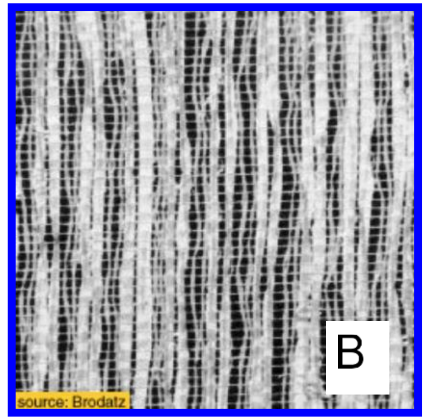
# The Fourier Transform of some well-known images



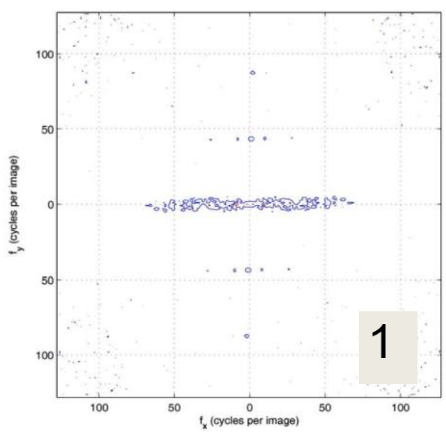

Slide credit: B. Freeman and A. Torralba

# Fourier Amplitude Spectrum

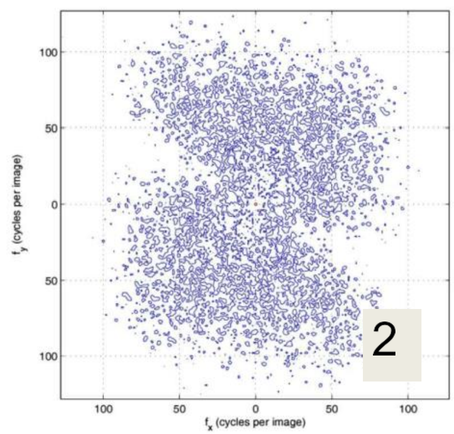
Poll

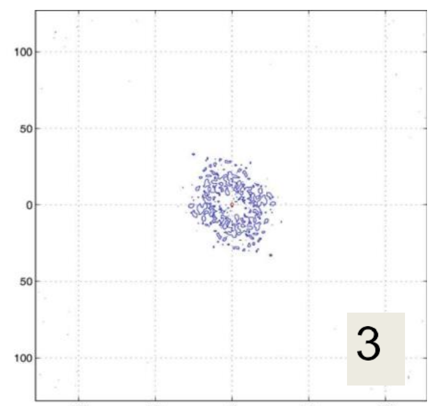





fx(cycles/image pixel size)



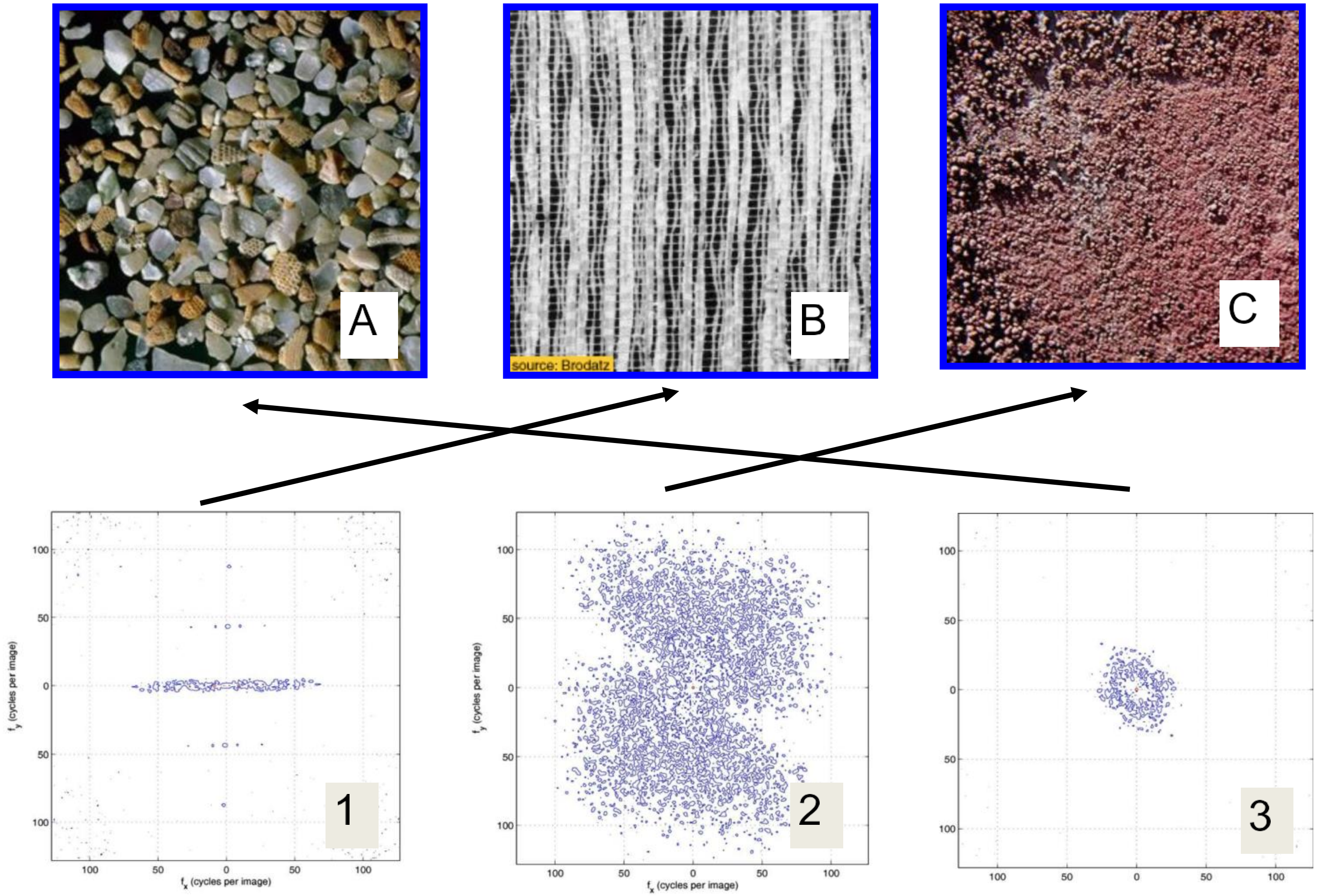

fx(cycles/image pixel size)




fx(cycles/image pixel size)

Slide credit: B. Freeman and A. Torralba

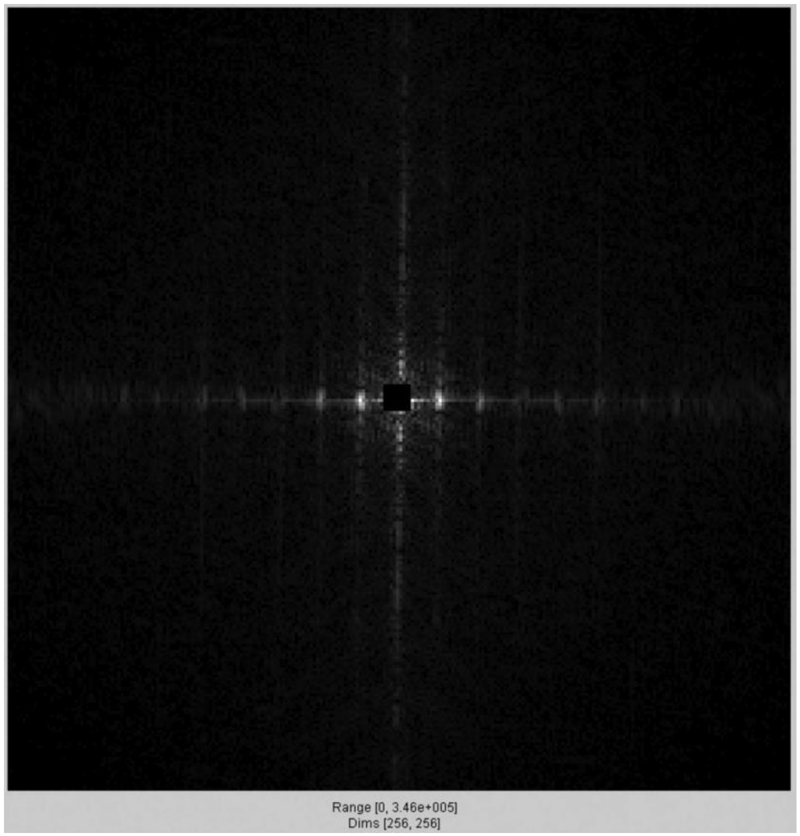
# Fourier Amplitude Spectrum




fx(cycles/image pixel size) fx(cycles/image pixel size) fx(cycles/image pixel size)

Slide credit: B. Freeman and A. Torralba

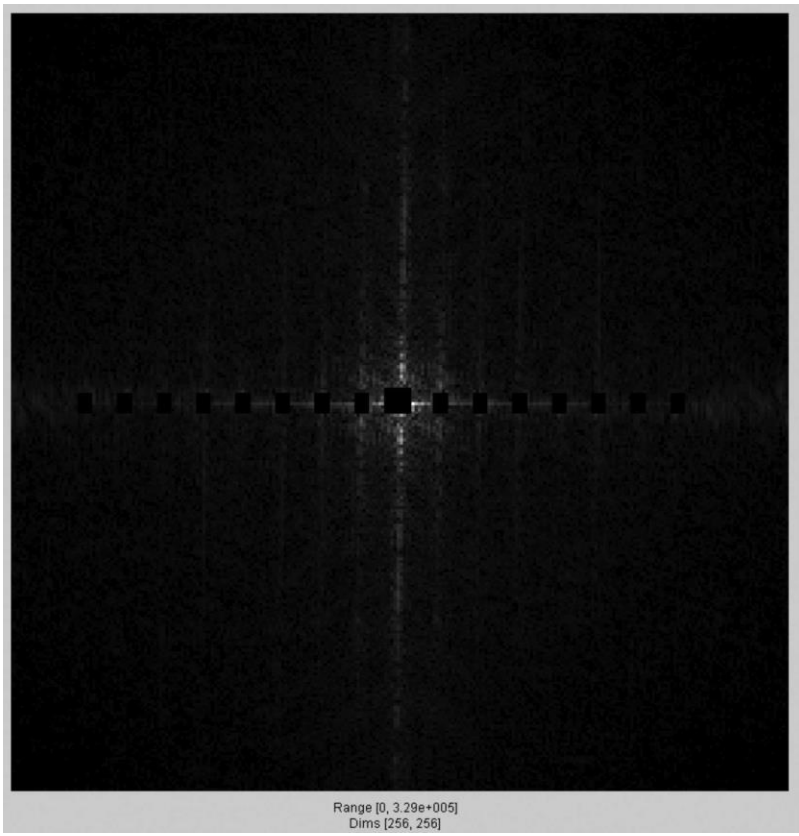
# Fourier transform magnitude

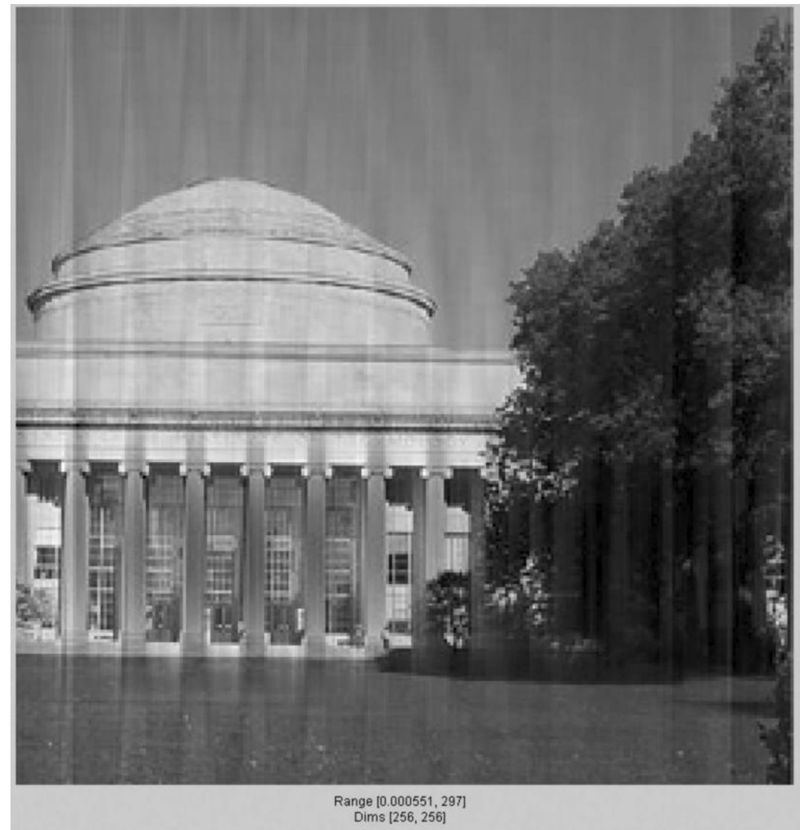



What in the image causes the dots?

Slide credit: B. Freeman and A. Torralba

# Masking out the fundamental and harmonics from periodic pillars




Slide credit: B. Freeman and A. Torralba

# The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$F[g * h] = F[g].F[h]$$

→ Convolution in spatial domain is equivalent to multiplication in frequency domain.

- The Fourier transform of the product of two functions is the convolution of their Fourier transforms

$$F[g.h] = F[g] * F[h]$$

→ Multiplication in spatial domain is equivalent to convolution in frequency domain.

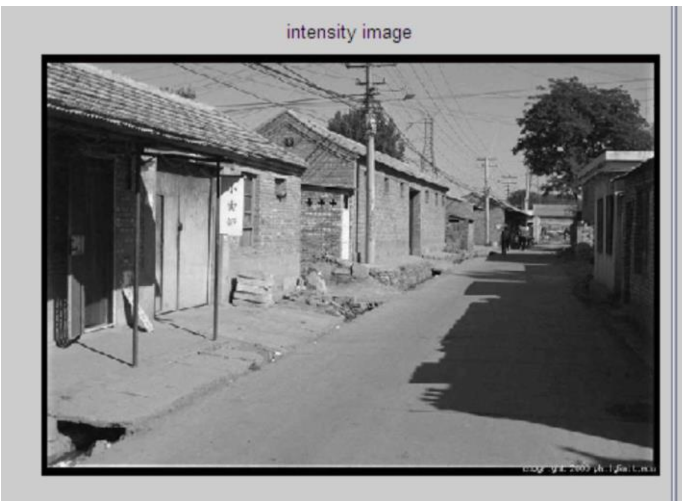
# Properties of Fourier Transform

- Linearity:
  $$F[a\,g(x) + b\,h(x)] = a\,F[g(x)] + b\,F[h(x)]$$

- Fourier transform of a real signal is symmetric about origin.

# Filtering in spatial domain

| | | |
|---|---|---|
| 1 | 0 | -1 |
| 2 | 0 | -2 |
| 1 | 0 | -1 |




*

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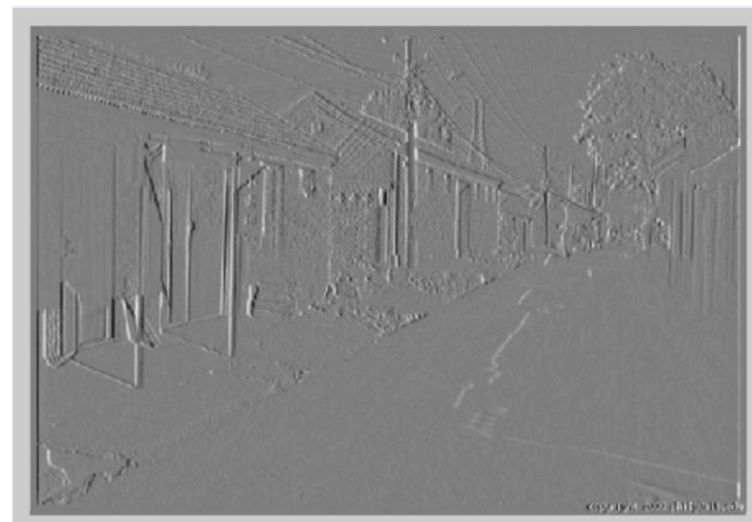
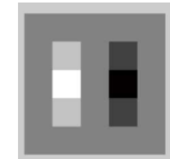
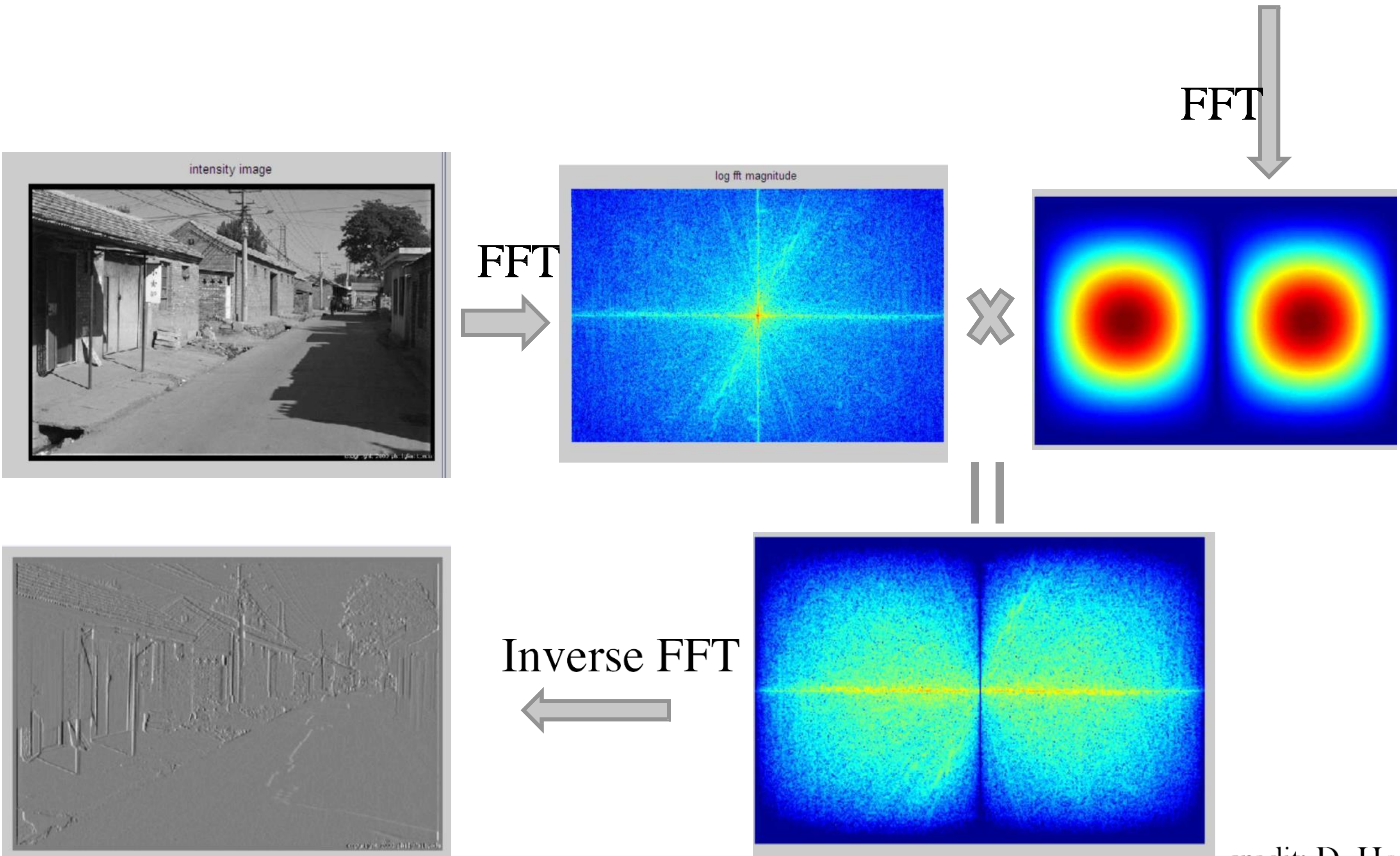


Slide credit: D. Hoiem

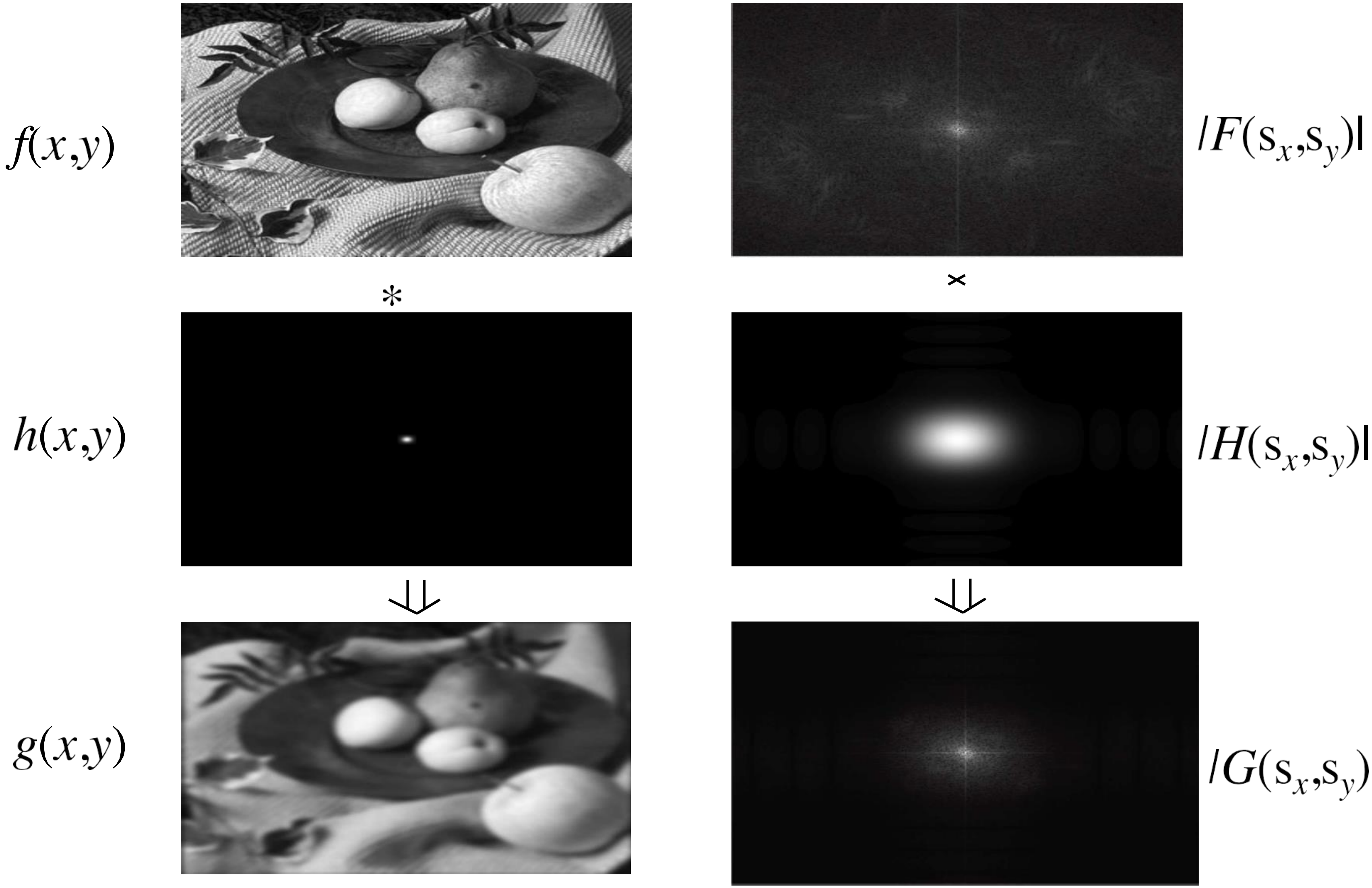
# Filtering in frequency domain






Slide credit: D. Hoiem

# 2D convolution theorem example

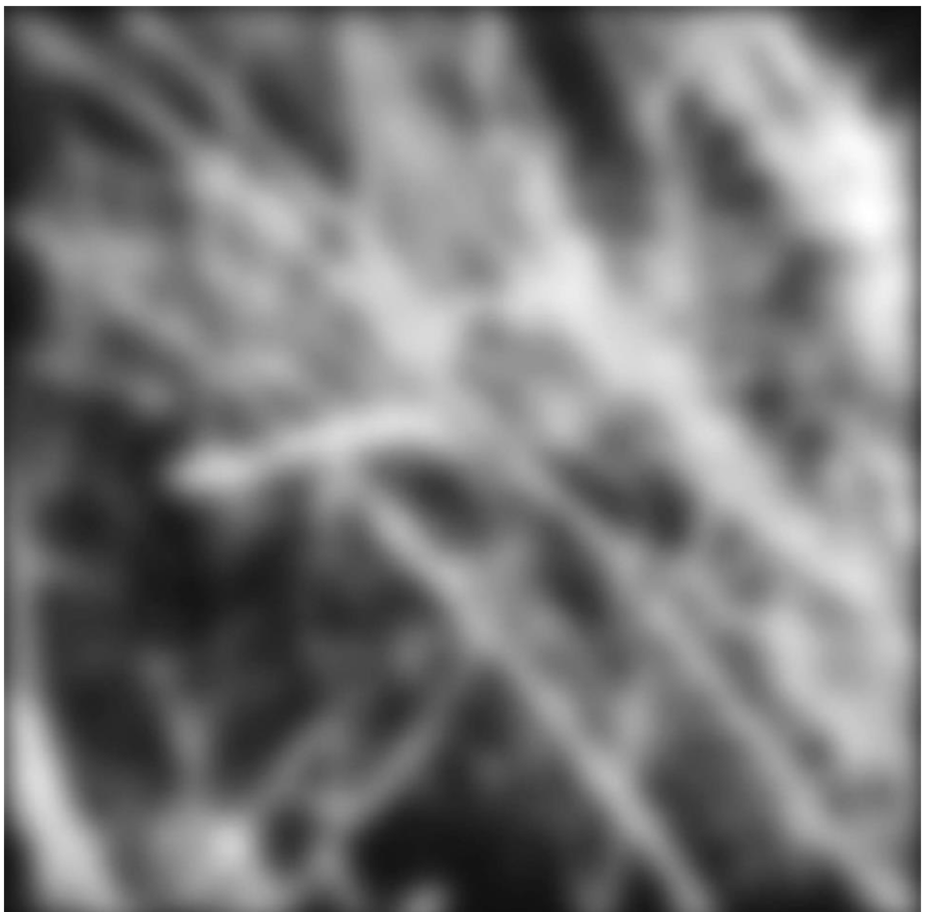



Slide credit: A. Efros

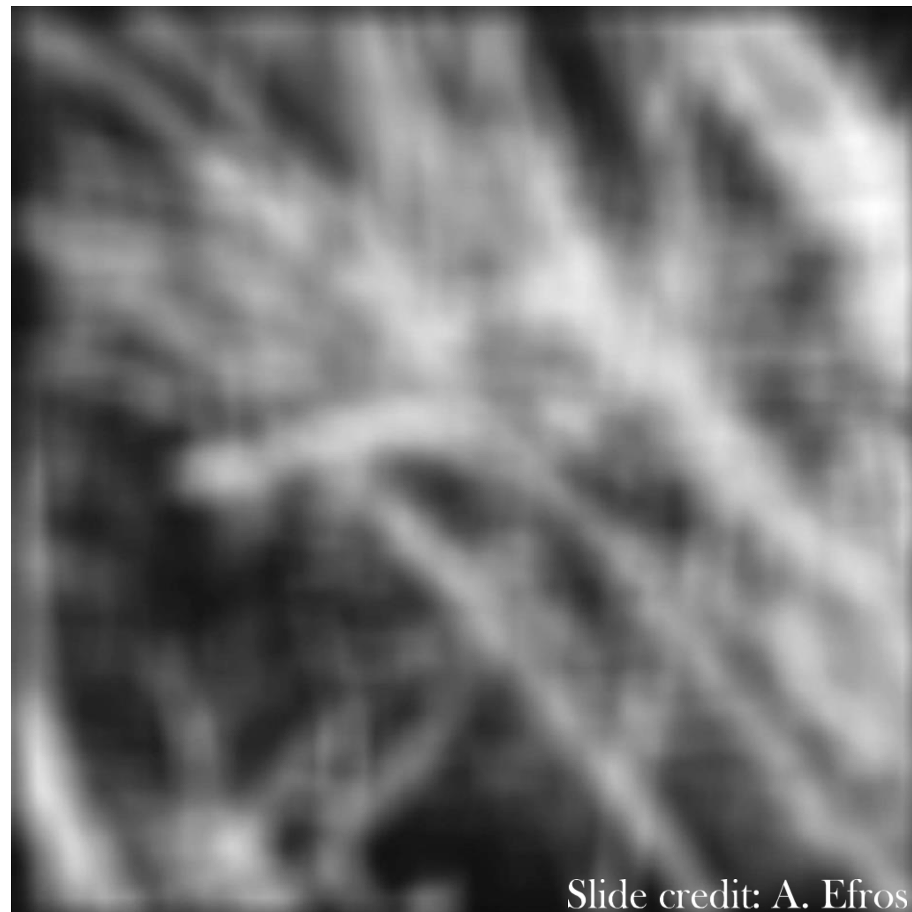
# Filtering

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

Gaussian 



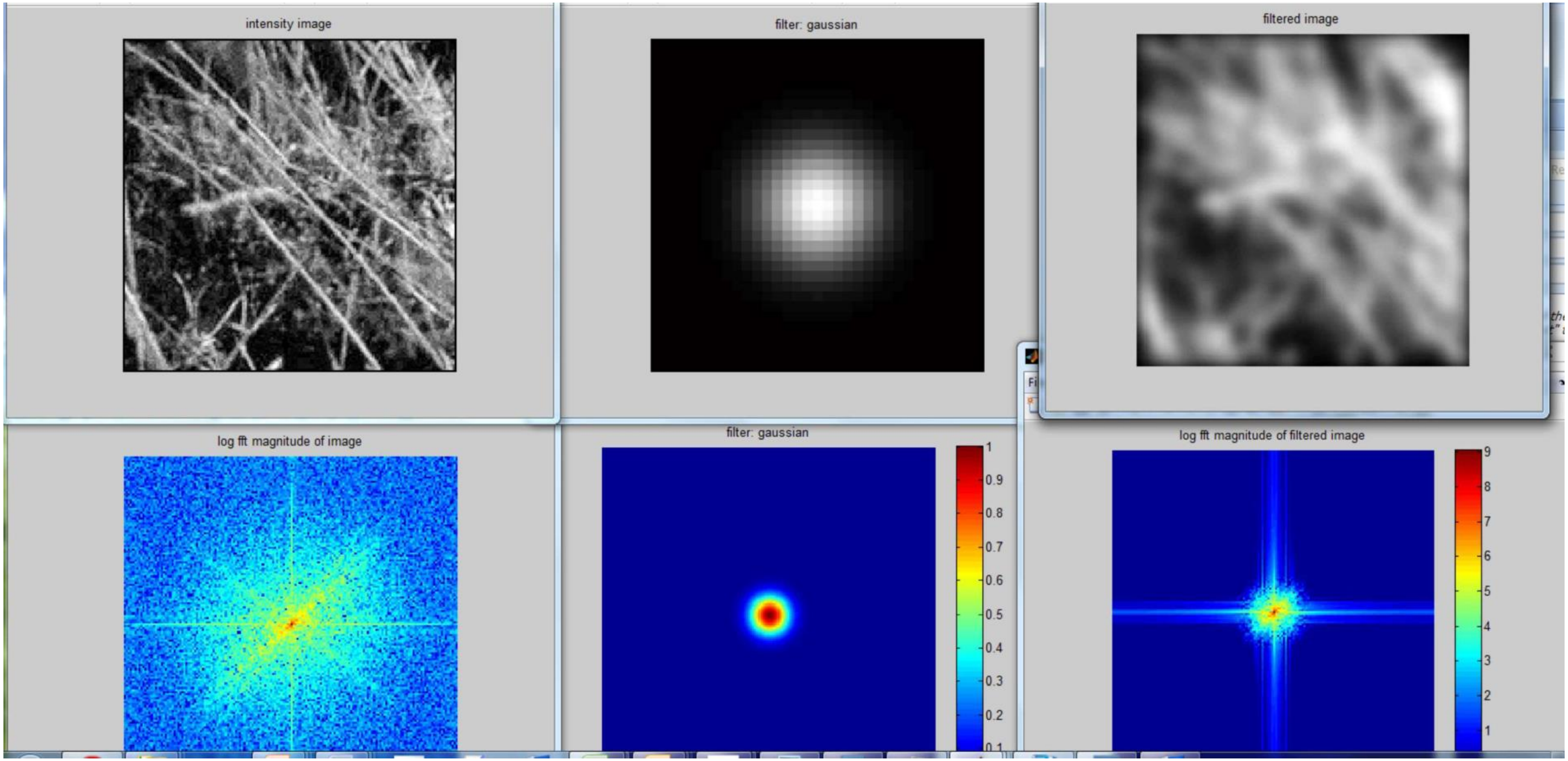
Box filter 



Slide credit: A. Efros

# Filtering

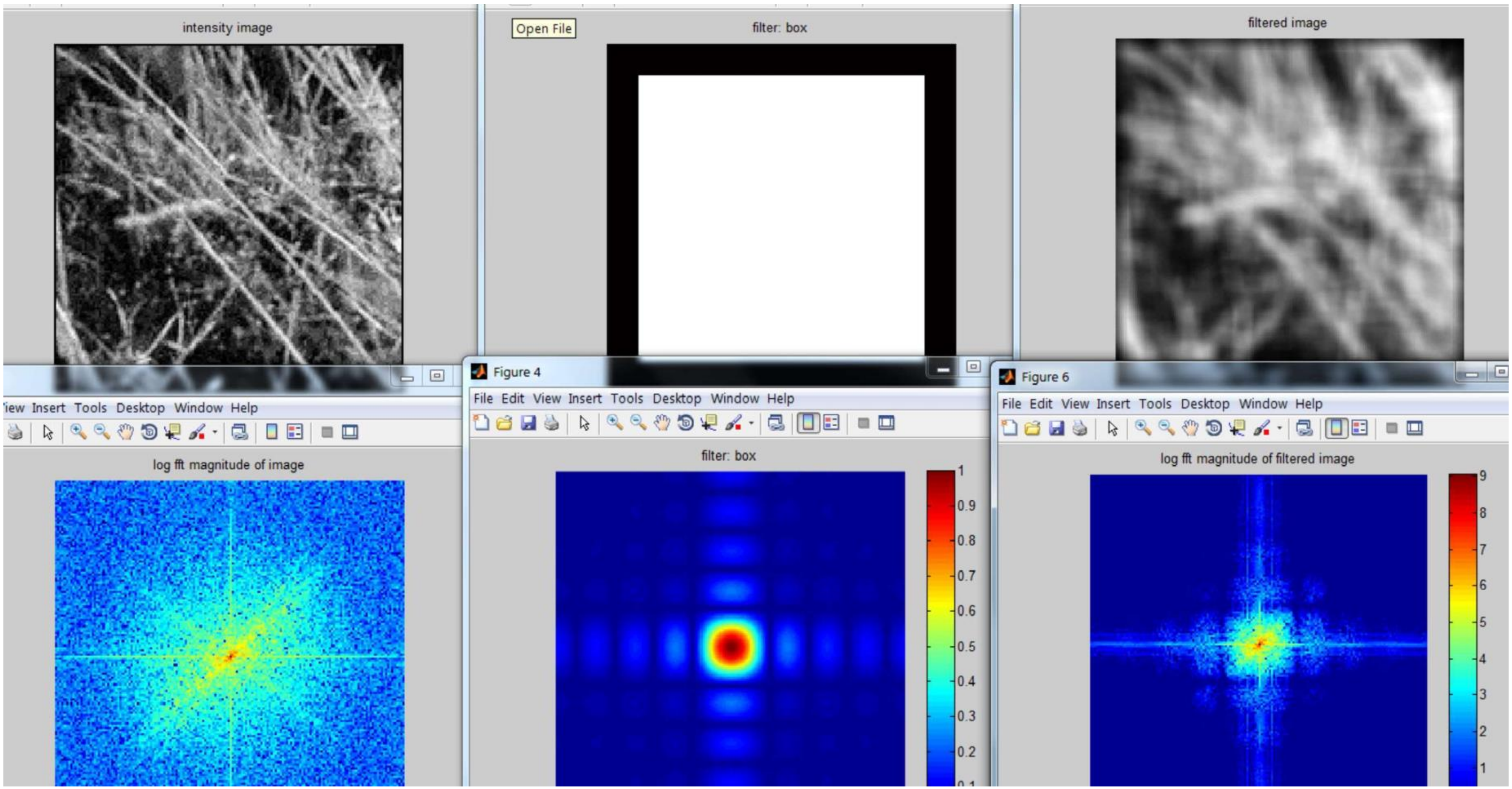
Gaussian




Slide credit: A. Efros

# Filtering

## Box Filter



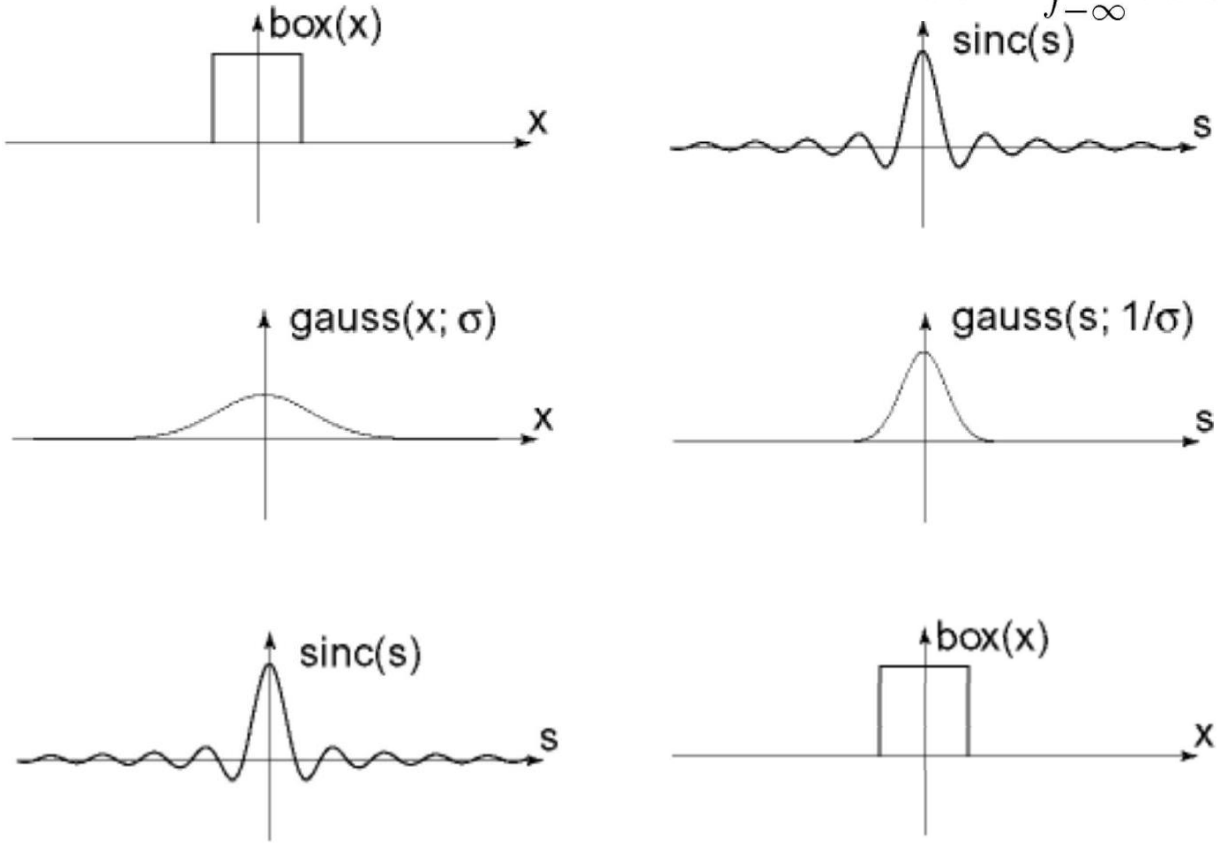

Slide credit: A. Efros

# Fourier Transform pairs

Spatial domain

Frequency domain

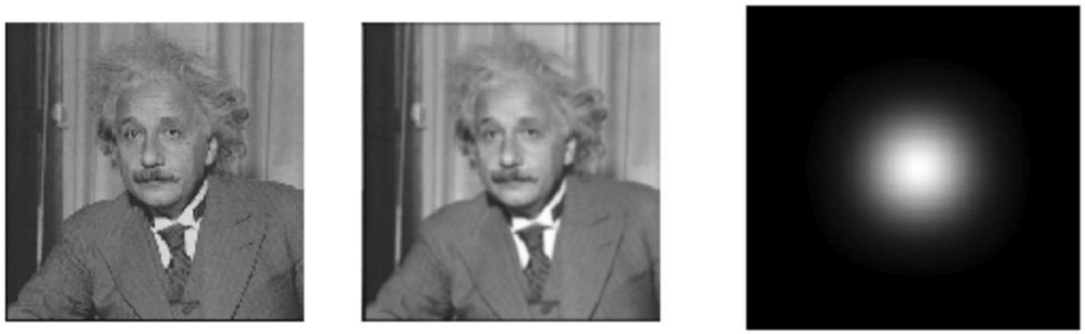
$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$




Slide credit: A. Efros

# Low-pass, Band-pass, High-pass filters

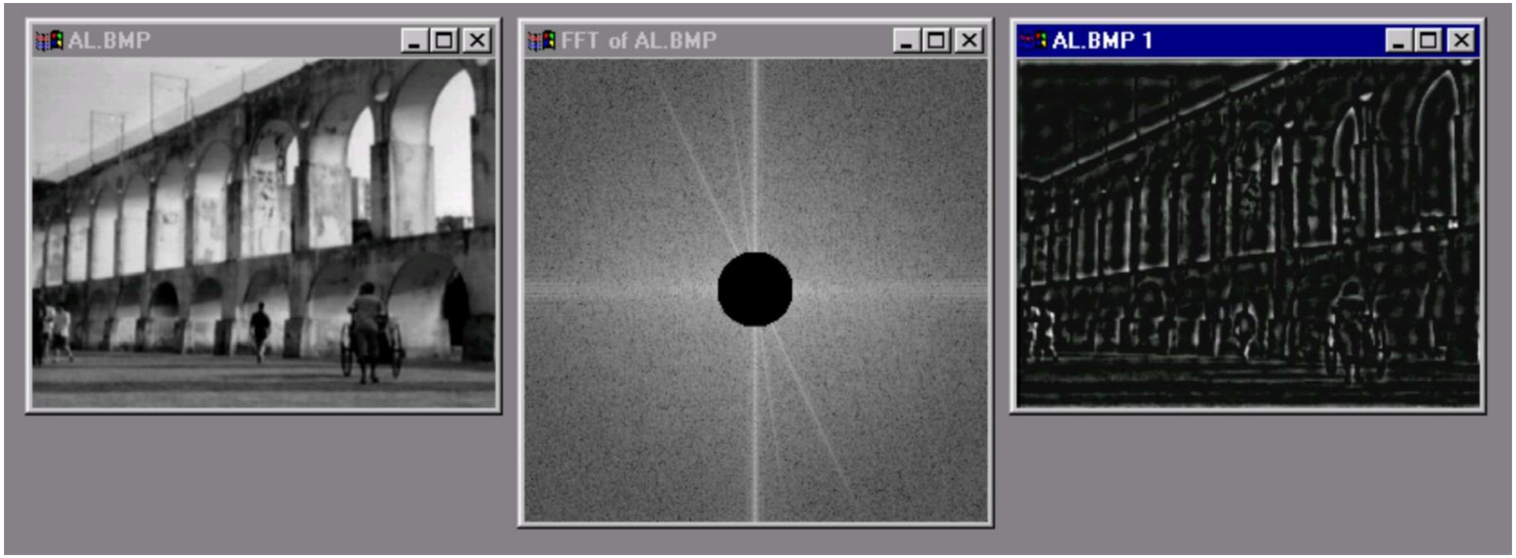
low-pass:



High-pass / band-pass:

Slide credit: A. Efros

# Edges in images




Slide credit: A. Efros

# Phase and Magnitude

- Curious fact
  - all natural images have about the same magnitude transform
  - hence, phase seems to matter, but magnitude largely doesn't
- Demonstration
  - Take two pictures, swap the phase transforms, compute the inverse - what does the result look like?

Image with cheetah phase (and zebra magnitude)

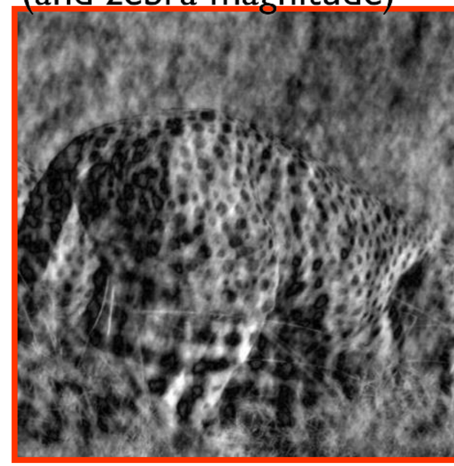
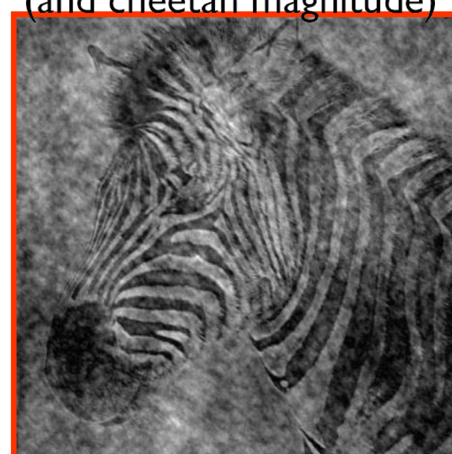


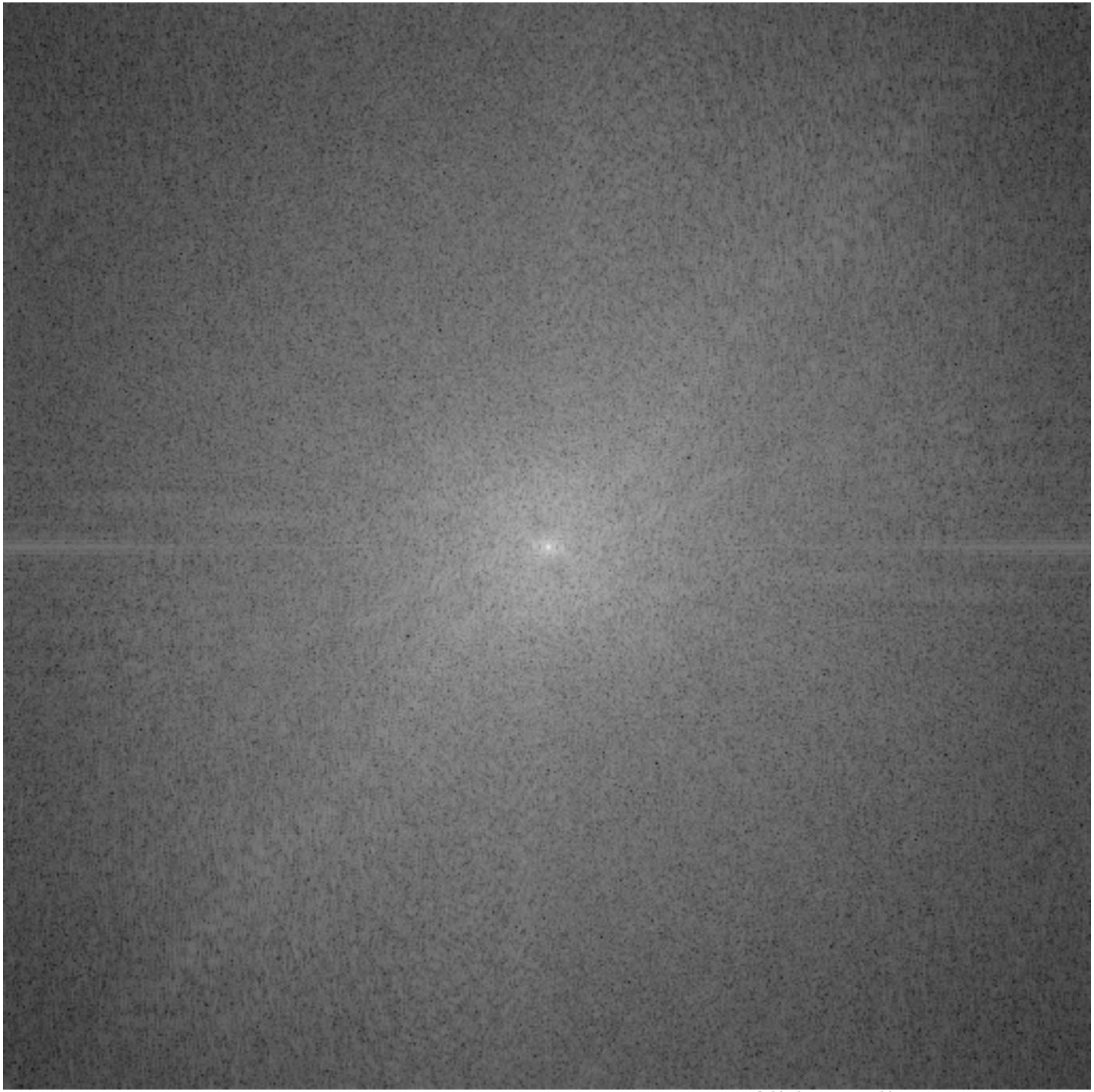
Image with zebra phase (and cheetah magnitude)



Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

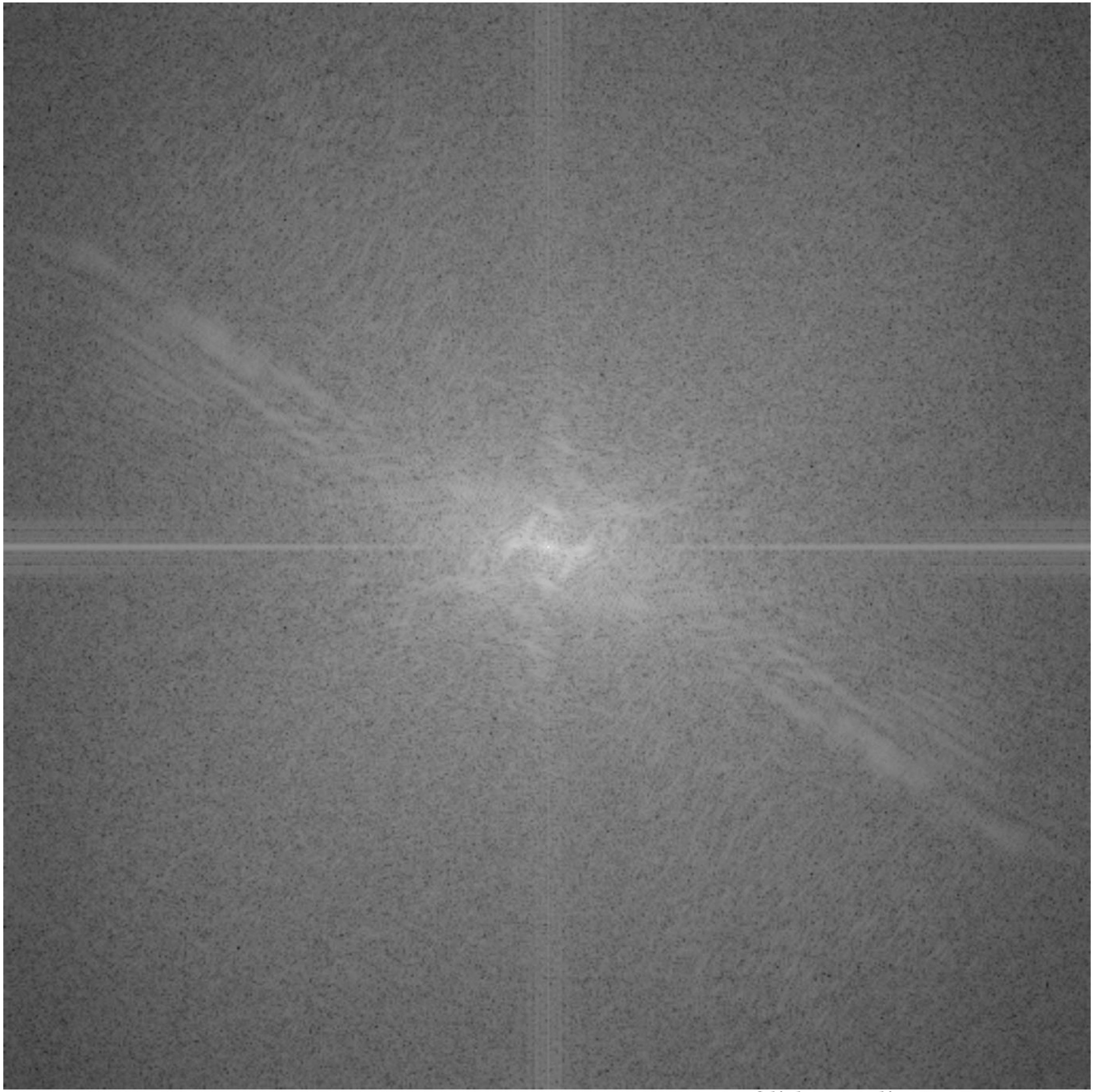
This is the magnitude transform of the cheetah picture



Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

This is the magnitude transform of the zebra picture

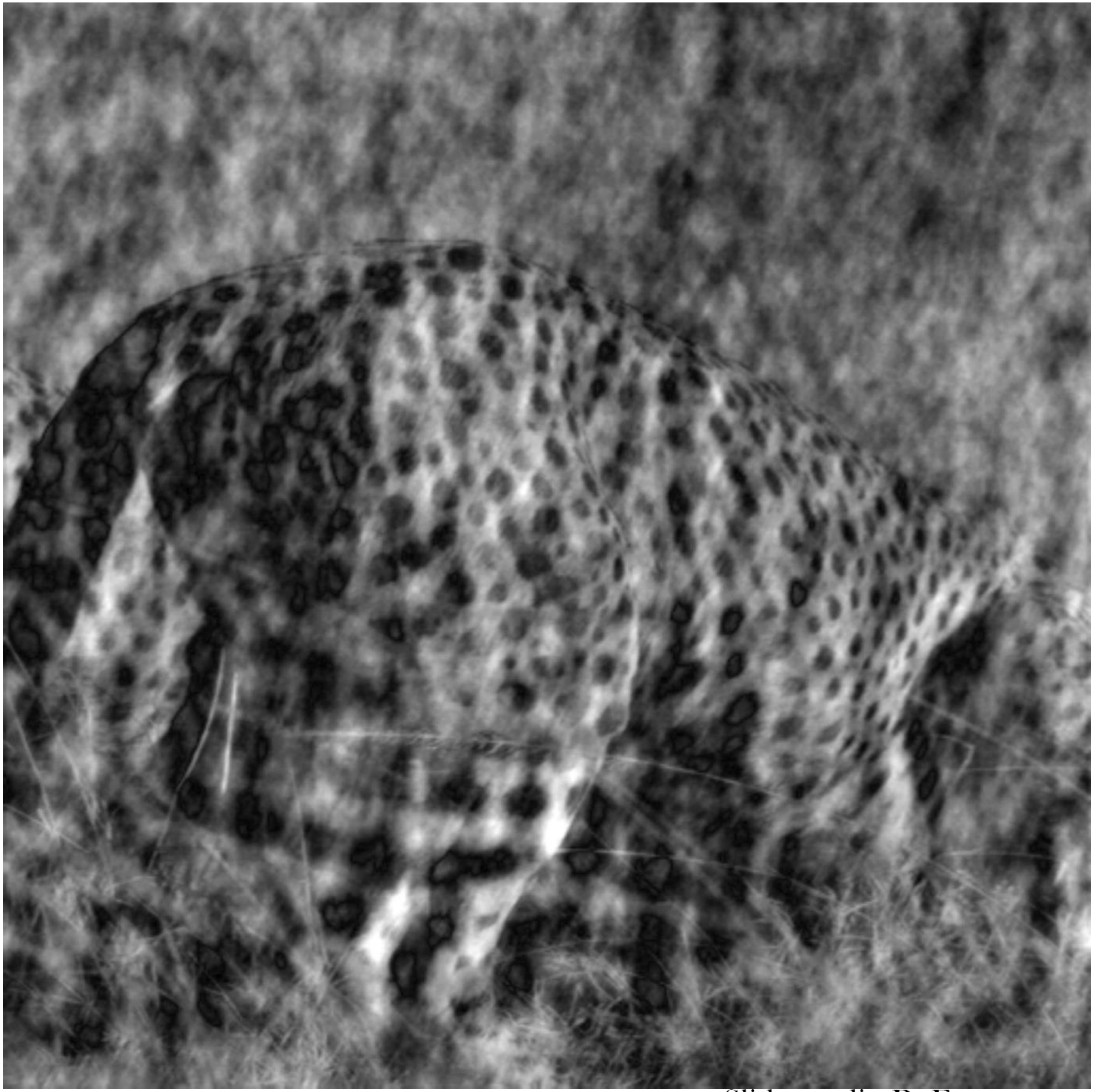


Slide credit: B. Freeman and A. Torralba

Reconstruction with zebra phase, cheetah magnitude

Slide credit: B. Freeman and A. Torralba

Reconstruction with cheetah phase, zebra magnitude



Slide credit: B. Freeman and A. Torralba