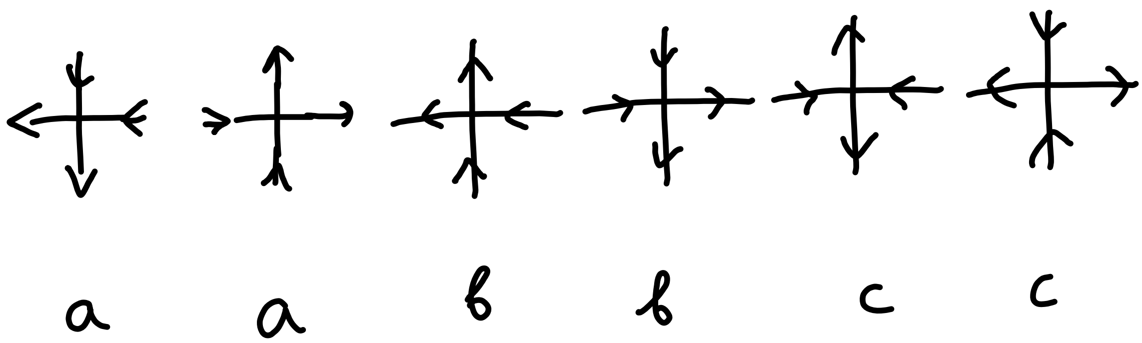
On integrability of limit shapes

Integrable models in statistical mechanics.

- Solvable
- Positive real forms of quantizations of classical integr. systems

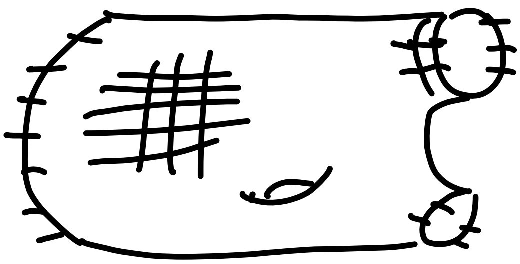
Main example: the 6-vertex model

States and weights



The partition function:

- Σ - genus g, $\varphi: \mathbb{Z}^2 \to \Sigma$



boundary edges

$\Rightarrow \Gamma \subset \Sigma$

- B: arrows on boundary edges $\longrightarrow \mathbb{R}_{\geq 0}$

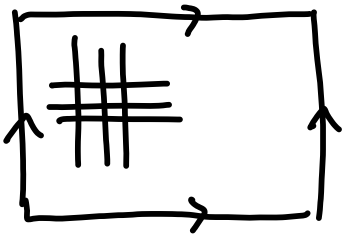
- $$Z_B(\Gamma \subset \Sigma) = \sum_{\substack{\text{arrows on edges} \\ \{\sigma\}}} B(\sigma_\partial) \prod_v w_v(\sigma_v)$$

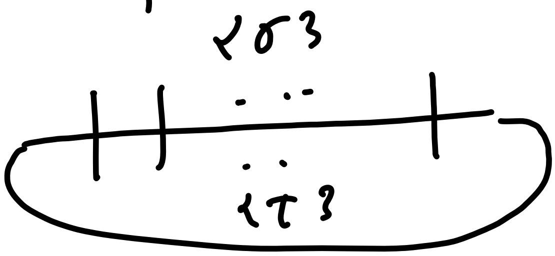
Ex: $B(\{\sigma\}) = \delta_{\{\sigma\}\{\tau\}}$ Dirichlet

- Local correlation functions:

$$\langle \sigma_{e_1} \dots \sigma_{e_k} \rangle = \frac{1}{Z_B(\Gamma \subset \Sigma)} \sum_{\{\sigma\}} B(\sigma_\partial) \prod_v w_v(\sigma_v) \, \sigma_{e_1}, \dots, \sigma_{e_k}$$

• Torus, periodic boundary conditions

M  N , Transfer-matrix:

$$Z^{\{\sigma\}}_{\{\tau\}} =$$ 

$$e_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}, \quad e_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

$$t = \{ Z^{\{\sigma\}}_{\{\tau\}} \} = \mathrm{tr}_0 (R_{01} \cdots R_{0N})$$

on $(\mathbb{C}^2)^{\otimes N}$

$$R : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$$

Yang-Baxter equation:

$$R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u)$$

Here weights a, b, c are parametrized as

$$a(u) = \mathrm{sh}(u+\eta), \quad b(u) = \mathrm{sh}\, u, \quad c(u) = \mathrm{sh}\,\eta$$

Commutativity of transfer-matrices

$$t(u) = \mathrm{tr}_0 \left(R_{01}(u) \ldots R_{0N}(u) \right)$$

$$[t(u), t(v)] = 0$$

By analogy with integrability in classical Hamiltonian mechanics we can say "many quantum integrals of motion" = "integrable"

<u>More precizely</u>:

Higher spin 6-vertex model:

$$R^{(1,\ell)}(u): \quad \mathbb{C}^2 \otimes \mathbb{C}^{\ell+1}$$

$$R^{(1,\ell)}(u) = \begin{pmatrix} zK - z^{-1}K^{-1} , & F \\ E , & zK^{-1} - z^{-1}K \end{pmatrix}$$

$$K e_n = q^{\frac{\ell}{2}-n} e_n , \quad E e_n = (q^n - q^{-n}) e_{n-1} ,$$

$$F e_n = (q^{\ell-n} - q^{-\ell+n}) e_{n-1} , \qquad q = e^{\eta}$$

$$U_q(sl_2) : \quad KE = qEK , \quad KF = q^{-1} FK ,$$

$$EF - FE = (q - q^{-1})(K^2 - K^{-2}) ,$$

As $q \to 1$ $(\eta \to 0)$, $U_q(sl_2) \to C(SL_2^*)$

Poisson algebra

$$\{K, E\} = EK , \quad \{K, F\} = FK$$

$$\{E, F\} = K^2 - K^{-2}$$

sympl. leaf of SL_2^*

$$\mathrm{End}(\mathbb{C}^{\ell+1}) \xrightarrow[\eta \to 0]{R = \ell\eta - \text{fixed}} C(S_R)$$

$$t^{(\ell_1,\dots,\ell_N)}(u) \text{ “}\longrightarrow\text{” } t_{cl}^{(R_1\cdots R_N)}(u) = \mathrm{tr}\left(\mathcal{L}_1^{(R_1)}(u)\cdots\mathcal{L}_N^{(R_N)}(u)\right)$$

“$\longrightarrow$” = · the semiclassical limit
· the number of degrees of freedom $\to \infty$.

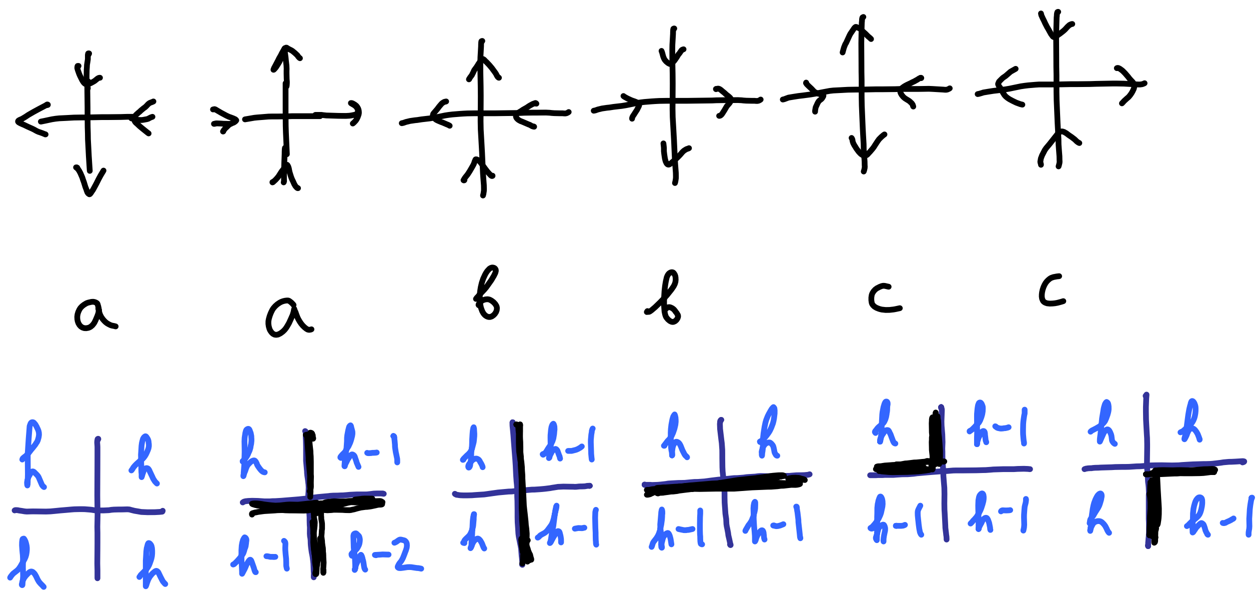
6-vertex $\hookrightarrow$ higher spin 6-vertex $\xrightarrow{\eta\to 0}$ classical integrable spin chain

We can also acheave “∞-many degrees of freedom” by passing to “$N \to \infty$” limit in the 6-vertex model.

<u>Will we have integrability there?</u>

Limit shape phenomenon

a) The height function



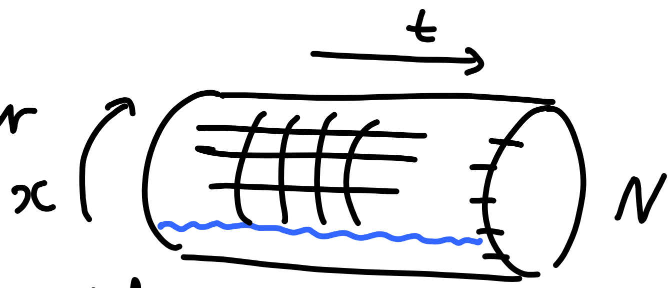
(i) 6-vertex model on $D_\varepsilon = D \cap \varphi_\varepsilon(\mathbb{Z}^2)$

φ_ε - step ε square grid

D - connected simply connected

h : faces of $D_\varepsilon \longrightarrow \mathbb{Z}$

(ii) On a cyllinder

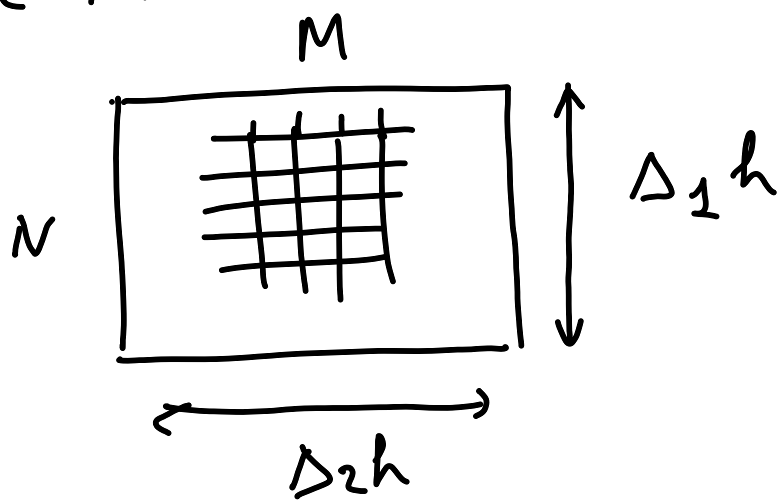


$h_{0,t} - h_{N,t} = \Delta h$

does not depend on t

$h : C \setminus \text{branch cut} \to \mathbb{Z}$

(iii) On a torus



Partition functions

(i) $$Z_{D_\varepsilon}(h_B) = \sum_{h_{inner}} \prod_v w_v(h_v)$$

(ii) $Z_{C_\varepsilon}(h_\ell^+, h_\ell^-) =$

$$= \sum_{h_{inner}} \prod_v w_v(h_v)$$

Constraints on $h^\ell_\pm$: $\Delta h^\ell_+ = \Delta h^\ell_-$

(iii) $Z_{T_\varepsilon}(\Delta_1 h, \Delta_2 h) =$

$$= \sum_{\substack{\{h\} \\ \Delta_1 h,\ \Delta_2 h \text{ - fixed}}} \prod_v w_v(h_v)$$

(b) $\varepsilon \longrightarrow 0$

(iii) $T = \varepsilon M$, $L = \varepsilon N$ fixed

$$s_1 = \varepsilon \Delta_1 h, \quad s_2 = \varepsilon \Delta_2 h \quad - \text{fixed}$$

$\varepsilon \to 0$, s_1, s_2 – fixed

magnetizations ↑ vertical horizontal

$$Z_{T_\varepsilon}(s_1, s_2) \simeq \exp\left(\frac{TL}{\varepsilon^2} \sigma(s_1, s_2) + o\left(\frac{1}{\varepsilon^2}\right)\right)$$

Cohn-Kenyon-Propp type arguments.

From Bethe ansatz and standard accopaning hypothesis:

$$\sigma(s_1, s_2) = \sigma(s_1, s_2; u, \eta)$$

given by solutions to linear integral equations

If (H, V) is a region D_1 or D_2 the corresponding translationally invariant Gibbs measure has the slope (h, v) given by (3). In this phase the system is disordered, which means that local correlation functions decay as a power of the distance $d(e_i, e_j)$ between e_i and e_j when $d(e_i, e_j) \to \infty$.

In the regions D_1 and D_2 the free energy is given by [SY]:

$$
\begin{aligned}
f(H,V) &= \min\Big\{\min_{\alpha}\Big\{E_1 - H - (1-2\alpha)V - \frac{1}{2\pi i}\int_C \ln\big(\frac{b}{a} - \frac{c^2}{ab - a^2 z}\big)\rho(z)dz\Big\}, \\
&\qquad \min_{\alpha}\Big\{E_2 + H - (1-2\alpha)V - \frac{1}{2\pi i}\int_C \ln\big(\frac{a^2 - c^2}{ab} + \frac{c^2}{ab - a^2 z}\big)\rho(z)dz\Big\}\Big\},
\end{aligned} \tag{7}
$$

where $\rho(z)$ can be found from the integral equation

$$
\rho(z) = \frac{1}{z} + \frac{1}{2\pi i}\int_C \frac{\rho(w)}{z - z_2(w)}dw - \frac{1}{2\pi i}\int_C \frac{\rho(w)}{z - z_1(w)}dw, \tag{8}
$$

in which

$$
z_1(w) = \frac{1}{2\Delta - w}, \qquad z_2(w) = -\frac{1}{w} + 2\Delta.
$$

$\rho(z)$ satisfies the following normalization condition:

$$
\alpha = \frac{1}{2\pi i}\int_C \rho(z)dz.
$$

The contour of integration C (in the complex z-plane) is symmetric with respect to the conjugation $z \to \bar{z}$, is dependent on H (see Appendix B) and is defined by the condition that the form $\rho(z)dz$ has purely imaginary values on the vectors tangent to C:

$$
\mathrm{Re}(\rho(z)dz)\Big|_{z \in C} = 0.
$$

The formula (7) for the free energy follows from the Bethe Ansatz diagonalization of the row-to-row transfer-matrix. Its derivation is outlined in Appendix B. It relies on a number of conjectures that are supported by numerical and analytical evidence and in physics are taken for granted. However, there is no rigorous proof.

$\sigma(s_1, s_2)$ = Legendre transform of $f(H,V)$

$$\Delta = \frac{a^2+b^2-c^2}{2ab},$$

$\Delta < -1$ $\qquad$ $|\Delta| < 1$ $\qquad$ $\Delta > 1$

To avoid analytical complications, assume that we are here

Important fact (follows from integral equations)

$$\mathrm{Hess}(\sigma) = \partial_1\partial_1\sigma\, \partial_2\partial_2\sigma - (\partial_1\partial_2\sigma)^2$$

does not depend on u.
Here ∂_1, ∂_2 are derivatives in the 1st and 2nd arguments

(Noh, Kim)

(ii) Arguments similar to the variational principle in domino tilings by

Cohn, Kenyon, Propp

Assume: $L = N\varepsilon$, $T = M\varepsilon$ fixed

$\varepsilon h_b^+ \longmapsto \varphi^+$, $\varepsilon h_b^- \longmapsto \varphi^-$, $|\partial_x \varphi^\pm| \leq 1$

$$Z_{C_\varepsilon}(h_b^+, h_b^-) \underset{\varepsilon \to 0}{\simeq} \exp\left(\frac{1}{\varepsilon^2} \int_D \sigma(\partial_x h^c, \partial_t h^c)\, dx\, dt + o\left(\frac{1}{\varepsilon^2}\right)\right)$$

Here h^c is the minimizer of

$$S[h] = \int_D \sigma(\partial_x h, \partial_t h)\, dx\, dt$$

Subject to constraints $|\partial_x h|, |\partial_t h| \leq 1$

$$h\Big|_{t=0} = \varphi^-(x), \quad h\Big|_{t=T} = \varphi^+(x)$$

on each $U \subset D$ where the minimizer is smooth, it satisfies

$$\partial_x \Big(\partial_1 \sigma (\partial_x h, \partial_t h) \Big) + \partial_t \Big(\partial_2 \sigma (\partial_x h, \partial_t h) \Big) = 0$$

(i) Similarly, for a connected simply-connected $D \subset \mathbb{R}^2$, assuming $\varepsilon h_\partial \xrightarrow[\varepsilon \to 0]{} \varphi$ we have:

$$Z_{D_\varepsilon}(h_\partial) \simeq \exp\Big(\frac{1}{\varepsilon^2} \iint_D \sigma(\partial_x h^c, \partial_y h^c)\, dx\, dy + o\big(\tfrac{1}{\varepsilon^2}\big) \Big)$$

where h^c is the minimizer of

$$S[h] = \iint_D \sigma(\partial_x h, \partial_y h)\, dx\, dy$$

subject to constraints $|\partial_x h|, |\partial_y h| < 1$

$$h\big|_{\partial D} = \varphi \, ,$$

The integrability of limit shapes

The limit shape phenomenon can be regarded as a version of semiclassical limit: Deterministic PDE emerged from a random variable. The ranom variable is integrable in a sence of commuting transfer-matrices etc.

Is the limit shape PDE is integrable?

(a) Hamiltonian formulation:

$$S[h] = \int_0^T \int_0^L \sigma(\partial_x h, \partial_t h)\, dx\, dt$$

Legendre transform gives the Hamiltonian

$$H(\pi, \partial_x h) = \max\left(\pi \xi - \sigma(\xi, \partial_x h)\right)$$

$$= \sigma^*(\pi, \partial_x h),$$

The action functional in the Hamiltonian framework:

$$S(\pi, h) = \int_0^T \int_0^L \pi \, \partial_t h \, dx \, dt -$$

$$- \int_0^T \int_0^L H(\pi, \partial_x h) \, dx \, dt$$

Euler-Lagrange equations:

$$\begin{cases} \partial_t h(x,t) - \dfrac{\partial H}{\partial \pi}\big(\pi(x,t), \partial_x h(x,t)\big) = 0 \\[2ex] \partial_t \pi(x,t) - \dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial q}\big(\pi(x,t), \partial_x h(x,t)\big)\right) = 0 \end{cases}$$

These are equations for the flow lines of the Hamiltonian vector field generated by

$$H_u[\pi, h] = \int_0^L \sigma_u^*\big(\pi(x), \partial_x h(x)\big)\, dx$$

with $\{\pi(x), h(y)\} = \delta(x-y)$

Thm (A. Sridhar, N.R.) Hamiltonians H_u form Poisson commutative family:

$$\{H_u, H_v\} = 0$$

$$\{H_u, H_v\} = \int \left(\frac{\delta H_u}{\delta \pi(x)} \frac{\delta H_v}{\delta h(x)} - \frac{\delta H_u}{\delta \pi(x)} \frac{\delta H_v}{\delta h(x)} \right) dx$$

$$= \int \Big(-\frac{d}{dx}\big(\partial_2 \sigma_v^*(\pi, h_x)\big)\partial_1 \sigma_u^*(\pi, h_x) +$$

$$+ \frac{d}{dx}\big(\partial_2 \sigma_u^*(\pi, h_x)\big)\partial_1 \sigma_v^*(\pi, h_x)\Big)\, dx =$$

$$= \cdots = \int \Big(\Big(\partial_1\partial_2 \sigma_u^*(\pi, h_x)\partial_1 \sigma_v^*(\pi, h_x)\, \pi_x +$$

$$+ \partial_2\partial_2 \sigma_u^*(\pi, h_x)\partial_1 \sigma_v^*(\pi, h_x)\, h_{xx}\Big) - \big(\sigma_u \leftrightarrow \sigma_v\big)\Big)\, dx$$

Here $\partial_i \sigma_u(s_1, s_2) = \frac{\partial}{\partial s_i}\sigma_u(s_1, s_2)$

Direct computation:

$$\delta_\pi \{H, \tilde{H}\} = \int \left(\partial_1^2 \sigma_u^* \partial_2^2 \tilde{\sigma}_v^* - \partial_2^2 \sigma_u^* \partial_1^2 \sigma_v^*\right) h_{xx} \, \delta\pi \, dx$$

$$\delta_h \{H, \tilde{H}\} = \int \left(\partial_1^2 \sigma_u^* \partial_2^2 \sigma_v^* - \partial_2^2 \sigma_u^* \partial_1^2 \sigma_v^*\right) h_{xx} \, \delta h_x \, dx$$

Lemma. $\quad \dfrac{\sigma_{22}^*}{\sigma_{12}^*} = -\operatorname{Hess}(\sigma)$

(simple fact about Legendre transform)

This lemma implies $\delta_\pi \{H, \tilde{H}\} =$
$= \delta_h \{H, \tilde{H}\} = 0$

Together with

$$\{H_u, H_v\}\Big|_{\pi = h_x = 0} = 0$$

we proved that $\{H_u, H_v\} = 0$ ∎

Thus, we have ∞-many integrals.

Some natural questions:

(a) Integrable or not? Conjecture: Yes

(b) Lax pair?

(c) Soliton solutions? Special solutions?

(d) Can this family of integrals be written is a better (simpler) way?