Integrability of six-vertex model and ASEP

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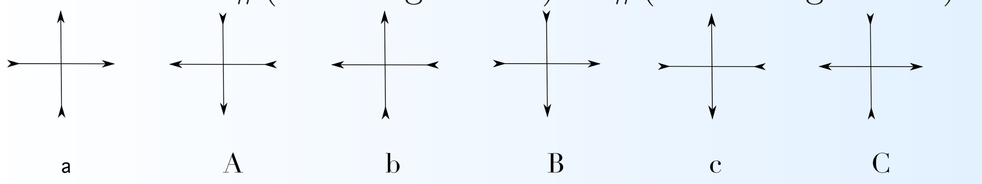
6-vertex model

Periodic $K \times L$ lattice

Vertices with #(incoming arrows) = #(outcoming arrows)

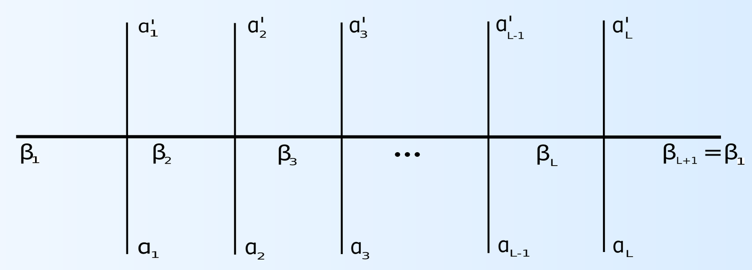


$$Z = \sum e^{-E_i/T} = \sum \prod R_{\beta\alpha}^{\beta'\alpha'}, \quad R : |\beta\rangle \otimes |\alpha\rangle \to R_{\beta\alpha}^{\beta'\alpha'} |\beta'\rangle \otimes |\alpha'\rangle,$$

$$R = \begin{pmatrix} R_{++}^{++} & R_{+-}^{++} & R_{-+}^{++} & R_{--}^{++} \\ R_{++}^{+-} & R_{+-}^{+-} & R_{-+}^{+-} & R_{--}^{+-} \\ R_{++}^{-+} & R_{+-}^{-+} & R_{-+}^{-+} & R_{++}^{-+} \\ R_{++}^{--} & R_{+-}^{--} & R_{-+}^{--} & R_{--}^{--} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & C & B & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

Transfer matrix: $T = tr_{V_0}(R_{01}R_{02}...R_{0L})$

R_{ij} acts as R on $V_i \otimes V_j$ and $\mathbb{I}$ on the others, $Z = tr(T^K)$



Integrable model: $[T, T'] = 0$, where $T' = tr_{V_0}(R'_{01}R'_{02}...R'_{0L})$, $R'(a', b', c')$. It is correct if there exists L such that $\quad RLL = LLR \quad$ or if

$$R''_{ij}R_{ik}R'_{jk} = R'_{jk}R_{ik}R''_{ij} - \qquad \textbf{Yang – Baxter equation}.$$

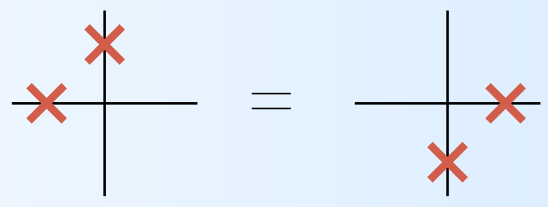
6-vertex model in horizontal field

$\widetilde{R_{\alpha\beta}}(u,h) = g_\beta(h)R_{\alpha\beta}(u)g_\beta(h)$, where $g_\beta = g \otimes \mathbf{1}, g = \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix}$,

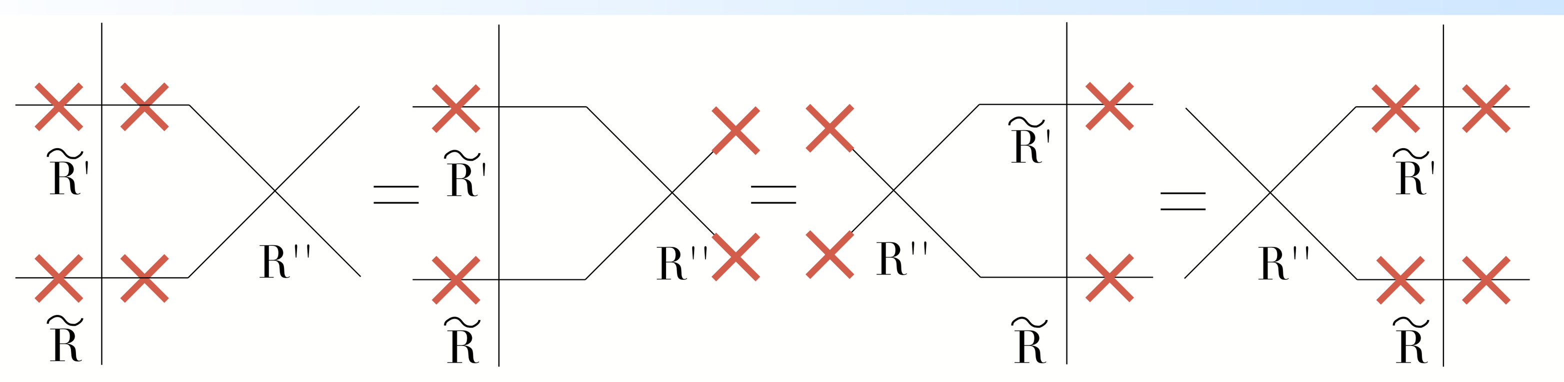
$$\widetilde{R_{\alpha\beta}}(u,h) = \begin{pmatrix} a(u)e^h & 0 & 0 & 0 \\ 0 & b(u)e^h & c & 0 \\ 0 & c & b(u)e^{-h} & 0 \\ 0 & 0 & 0 & a(u)e^{-h} \end{pmatrix};$$

since

$$R(u)(g(h) \otimes g(h)) = (g(h) \otimes g(h))R(u),$$



$$R''_{12}(u'')\widetilde{R}_{13}(u,h)\widetilde{R'}_{23}(u',h) = \widetilde{R'}_{23}(u',h)\widetilde{R}_{13}(u,h)R''_{12}(u'').$$



ASEP as a spin chain

Continuous time, probabilities of jump: qdt to the left, pdt to the right

$$\partial_t \vec{P} = \mathbb{M}\vec{P}, \quad \mathbb{M} = \sum_i^L \mathbb{M}_{i,i+1} = \sum_i^L \mathbf{1}_1 \otimes ... \otimes \mathbf{1}_{i-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p & q & 0 \\ 0 & p & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes ... \otimes \mathbf{1}_L$$

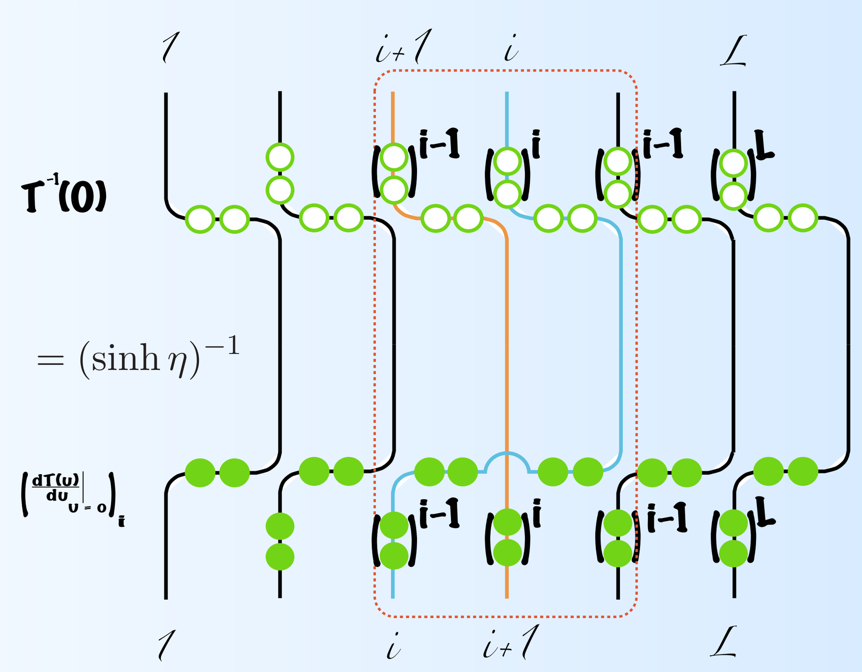
$$\mathbb{M} = \sum_{i=1}^L \left(pS_i^- S_{i+1}^+ + qS_i^+ S_{i+1}^- + \frac{1}{4}S_i^z S_{i+1}^z - \frac{p-q}{4}(S_i^z - S_{i+1}^z) - \frac{1}{4} \right),$$

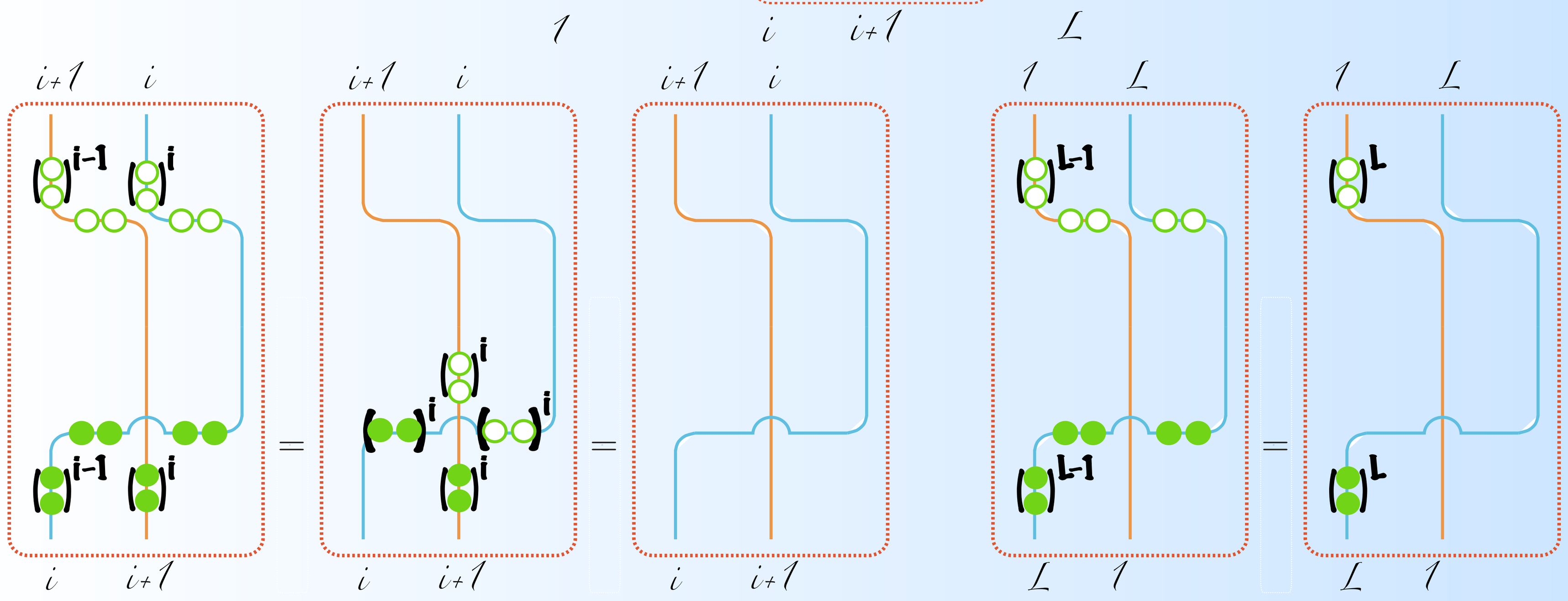
$$H = (\alpha + \alpha^{-1})(\prod_{i=1}^L U_i)(M + \frac{L}{4}\mathbf{1})(\prod_{i=1}^L U_i)^{-1} =$$

$$= \frac{1}{2}\sum_{i=1}^{L-1}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \frac{\alpha + \alpha^{-1}}{2} S_i^z S_{i+1}^z) + \alpha^L S_L^- S_1^+ + \alpha^{-L} S_L^+ S_1^- + \frac{\alpha + \alpha^{-1}}{4} S_L^z S_1^z$$

$$U_i = \mathbf{1_1} \otimes ... \otimes \mathbf{1_{i-1}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{i-1} \end{pmatrix} \otimes \mathbf{1_{i+1}} \otimes ... \otimes \mathbf{1_L}, \quad \alpha = \sqrt{\frac{p}{q}}$$

$$\prod_{i=1}^L (g_i)^{-2(i-1)} T^{-1}(0) \left(\frac{\mathrm{d}}{\mathrm{d}u} T(u) \Big|_{u=0} \right)_i \prod_{i=1}^L (g_i)^{2(i-1)} =$$





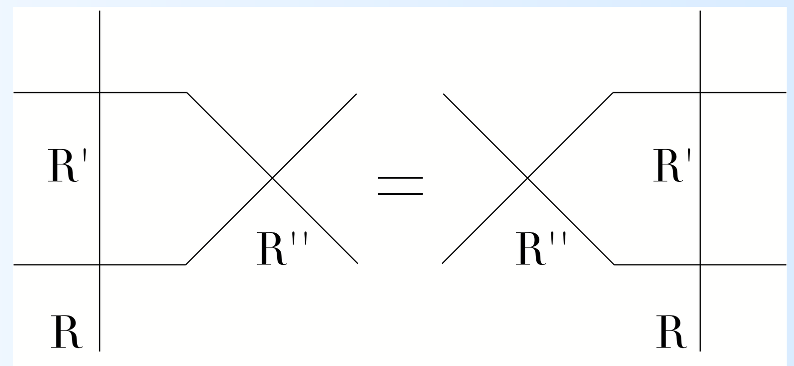
Symmetric 6-vertex model

$R_{\alpha\beta} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$, to solve Y-B we need $\Delta = \frac{a^2+b^2-c^2}{2ab} = \Delta' = \Delta''$,

hyperbolic parametrization with same η satisfies:

$$R_{\alpha\beta}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}, \ \Delta = \sinh\eta,$$

$$R_{12}(u'')R_{13}(u'+u'')R_{23}(u') = R_{23}(u'')R_{13}(u'+u'')R_{12}(u').$$

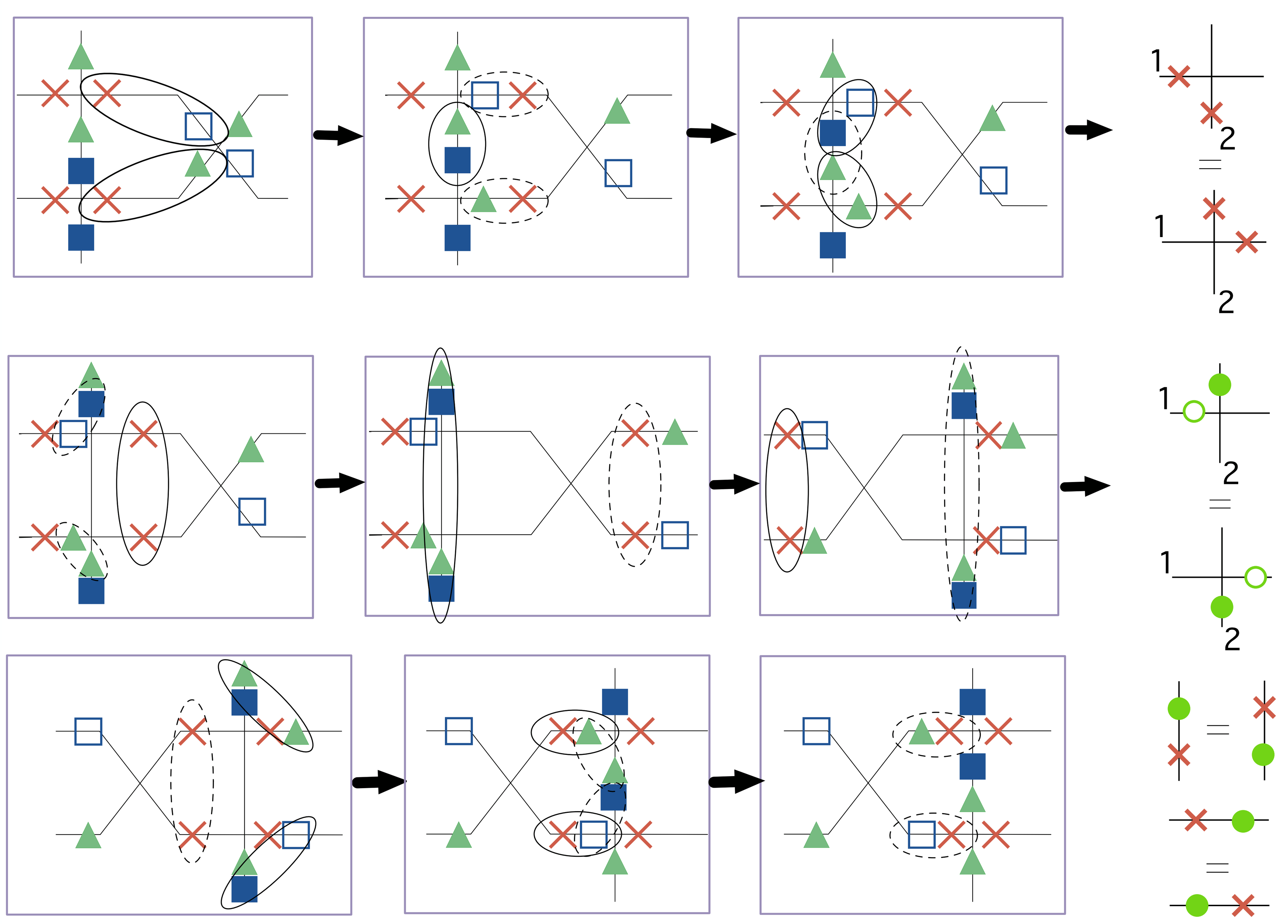


6-vertex in horizontal and vertical fields model

$\widetilde{\widetilde{R}}_{\alpha\beta}(u,h,v) = f_\alpha(v)\widetilde{R}_{\alpha\beta}(u,h)f_\alpha(v)$, where $f_\alpha = \mathbf{1} \otimes f, f = \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}$,

$$\widetilde{\widetilde{R}} = \begin{pmatrix} a(u)e^{h+v} & 0 & 0 & 0 \\ 0 & b(u)e^{h-v} & c & 0 \\ 0 & c & b(u)e^{-h+v} & 0 \\ 0 & 0 & 0 & a(u)e^{-h-v} \end{pmatrix}$$

$$\widetilde{\widetilde{R}}_{12}(-v', v, u-u')\widetilde{\widetilde{R}}_{13}(h,v,u)\widetilde{\widetilde{R}}_{23}(h,v',u') = \widetilde{\widetilde{R}}_{23}(h,v',u')\widetilde{\widetilde{R}}_{13}(h,v,u)\widetilde{\widetilde{R}}_{12}(-v', v, u-u').$$



6-vertex model as a spin chain

Symmetric case:

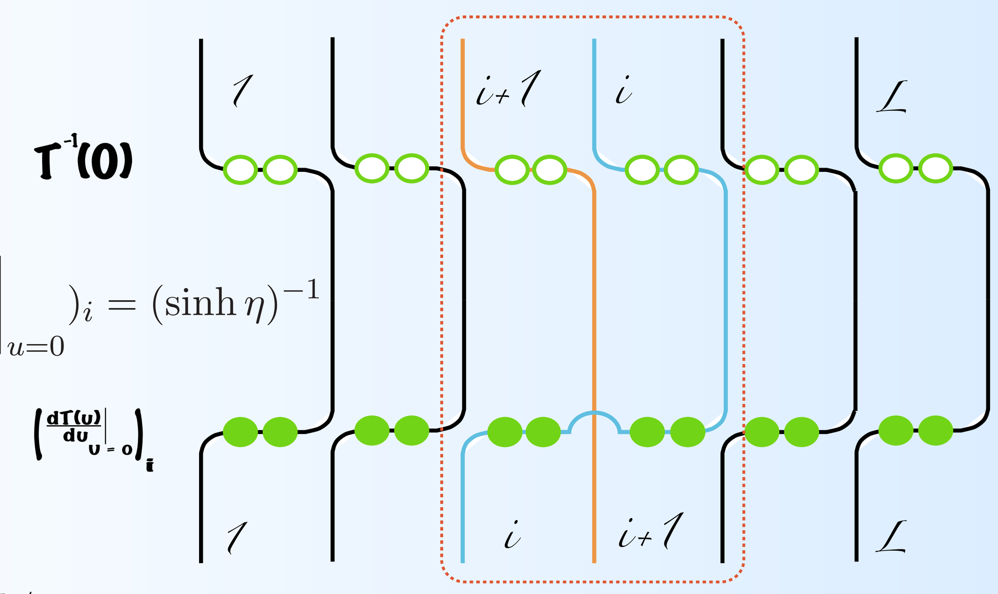
$$H_{XXZ} = -\sinh \frac{\mathrm{d}}{\mathrm{d}u} logT(u)\Big|_{u=0} + N\cosh\eta$$

Asymmetric case:

$$T(u) = exp(v+h)\sum_{j=1}^N \sigma_j^z \widetilde{T(u)},$$

$$H = -V\frac{\mathrm{d}}{\mathrm{d}v}\log[\exp(v+h)\sum_{j=1}^N \sigma_j^z)] - \sinh\frac{\mathrm{d}}{\mathrm{d}u} logT(u)\Big|_{u=0} + N\cosh\eta$$

$$T^{-1}(0)(\frac{\mathrm{d}}{\mathrm{d}u}T(u)\Big|_{u=0})_i = (\sinh\eta)^{-1}$$



$g =$ ● $= \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix} \qquad g^{-1} =$ ○ $=$ ●$^{-1} = \begin{pmatrix} e^{-h/2} & 0 \\ 0 & e^{h/2} \end{pmatrix}$