Spontaneous Breaking of U(N) Symmetry in Invariant Matrix Models

Fabio Franchini

INFN, Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy

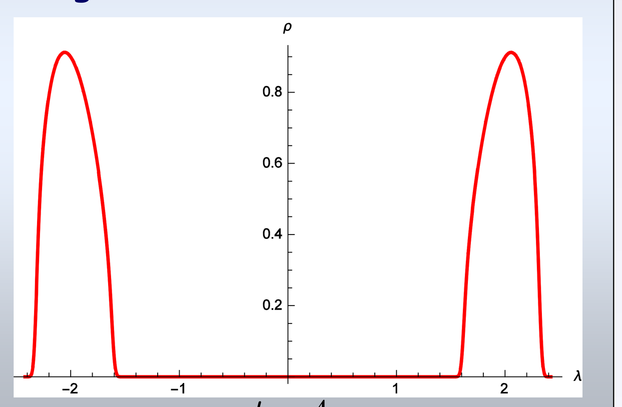
Abstract

We introduce the study of the eigenvectors of a random matrix, to better understand the relation between localization and eigenvalue statistics. Traditionally, the requirement of base invariance has lead to the conclusion that invariant models describe extended systems. We show that deviations of the eigenvalue statistics from the Wigner-Dyson universality reflects itself on the eigenvector distribution. In particular, gaps in the eigenvalue density spontaneously break the U(N) symmetry to a smaller one. Models with log-normal weight, such as those emerging in Chern-Simons and ABJM theories, break the U(N) in a critical way, resulting into a multi-fractal eigenvector statistics. These results pave the way to the exploration of localization problems using random matrices via the study of new classes of observables and potentially to novel, interdisciplinary, applications of matrix models.

Double Well Matrix Models

$$V_{2\mathrm{W}}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

- Disjoint (two-cuts) support of eigenvalue distribution for $t > 2$
- Half of eigenvalues around each minima $\pm t$ (assume N even)
- U(N) symmetry broken into U(N/2) x U(N/2)



$t = 4$

Symmetry Breaking Term: Double Well Case

- Introduce an explicit symmetry breaking term
- Want to favor alignment of the eigenvectors of $\mathbf{M}$ along those of a given Hermitian matrix $\mathbf{S}$
- Most natural choice: $V_{\mathrm{br}} = \mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$ (but too complicated to handle)
- Introduce

$$W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}\left[V_{2\mathrm{W}}(\mathbf{M}) + J|\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}|\right]}$$

- Double well case: $\mathbf{S}$ with two sets of N/2 degenerate eigevanlues $\pm\tau$

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathbf{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Use generating function to calculate (dis-)order parameter:

$$\left.\frac{dW(J)}{dJ}\right|_{J=0} = \langle \mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S} \rangle$$

$$\boxed{\begin{array}{c}\lim\limits_{N\to\infty}\lim\limits_{J\to 0}\frac{dW}{dJ} \neq 0 \\ \lim\limits_{J\to 0}\lim\limits_{N\to\infty}\frac{dW}{dJ} = 0\end{array}}$$

- Remark: order parameter vanishes for symmetry broken

  $\Rightarrow$ U(N) symmetry broken into U(N/2) x U(N/2)

- Corrections to SSB as $e^{-N\,J\left|\lambda_j^{(1)} - \lambda_l^{(2)}\right|}$: contributions from instantons exchanging 2 eigenvalues between wells

  $\rightarrow$ instantons progressively restore the broken symmetries, but are suppressed for large N (and large distances)

Introduction

- Base invariant matrix models:

$$d\mu(\mathbf{M}) = e^{-N\mathrm{Tr}V(\mathbf{M})}\,d\mathbf{M}$$
$$= d\mathbf{U}\prod_{j<l}(\lambda_j - \lambda_l)^2 e^{-N\sum_j V(\lambda_j)}\prod_j d\lambda_j$$

- Eigenvector distribution independent from weight $V(x)$

  $\Rightarrow$ Uniform eigenvector distribution

  $\Rightarrow$ Delocalized phases, Porter-Thomas Distribution

$$\mathcal{P}\left(|U_{ij}|^2\right) = N\exp\left[-N|U_{ij}|^2\right]$$

- Localization by non-invariant ensembles (Banded Matrices)

$$d\mu(\mathbf{M}) \propto e^{-\sum_{j,l} A_{jl}\,|M_{jl}|^2} \Rightarrow \langle M_{nm}^2\rangle = A_{nm}^{-1}$$

  - $A_{nm} = e^{|n-m|/B}$ Localized (Fyodorov & Mirlin, PRL '91)
  - $A_{nm} = 1 + \frac{(n-m)^2}{B^2}$ Critical (multi-fractal) (Evers & Mirlin, PRL '00)

- But limited tractability (numerics or perturbative regimes)

> Can a non-trivial (non Wigner-Dyson) eigenvalue distribution trigger a spontaneous breaking of U(N) symmetry and lead to (partially) localized eigenstates?

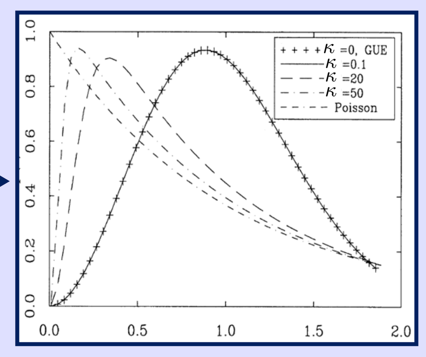
- Like for a ferromagnet, base invariance means that no direction over the N-dimensional unit sphere of the Hilbert space is preferred, but a gap in the eigenvalue distribution freezes the motion of eigenvectors in certain directions

  $\Rightarrow$ U(N) SSB breaks ergodicity

Weakly Confined Matrix Models

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}V(\mathbf{M})}\ ,\ V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa}\ln^2|\lambda|$$

- Arise in localization limit of Chern-Simons/ABJM
- Soft confinement sets them apart from usual polynomial potentials
- Intermediate level spacing
- Same eigenvalue correlations as Power-law Banded Matrices



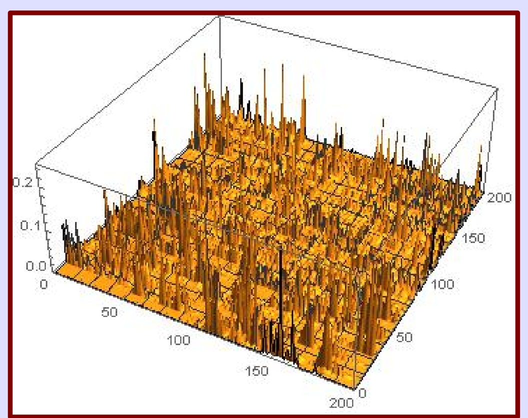
- Complex energy landscape with many metastable saddles:

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U}\int_{\lambda>0} d^N\lambda\,\Delta(\{\lambda\})\,e^{-\frac{1}{2\kappa}\sum_j \ln^2\lambda_j} \qquad \boxed{\lambda_j = e^{\kappa x_j}}$$

$$\propto \int d^N x_j \prod_{n<m}\left(e^{\kappa x_n} - e^{\kappa x_m}\right)^2 e^{-\frac{\kappa}{2}\sum_{l=1}^N [x_l^2 - 2x_l]} \qquad \boxed{q \equiv e^{-\kappa}}$$

$$\propto e^{\frac{\kappa}{6}N(4N^2-1)}\,(2\pi\kappa)^{N/2}\,N!\prod_{n=1}^{N-1}(1-q^n)^{N-n}$$

- Each term of the expansion of the product
  - Different equilibrium conf.
  - Same leading energy
  - q^j: instanton fugacity
  - Instanton $\leftrightarrow$ symmetries
  - Different U(N) breaking



- Typical saddles correspond to multi-fractal spontaneous breaking of rotational symmetry!
- SSB as full-Replica Symmetry Breaking

General Considerations

- The de Haar measure not flat in eigenvalue-eigenvector coordinates

$$\begin{aligned} ds^2 &= \mathrm{Tr}\,(dM)^2 \\ &= \sum_{j=1}^N (d\lambda_j)^2 + 2\sum_{j>l}^N (\lambda_j - \lambda_l)^2\left|\left(\mathbf{U}^\dagger d\mathbf{U}\right)_{jl}\right|^2 \end{aligned}$$

- If two eigenvalues are distant, even a small angular change can produce a large ds
- Dyson Brownian motion representation

Delta-Correlated, independent, stochastic sources

$$\begin{aligned} d\lambda_j &= -\frac{dV(\lambda_j)}{d\lambda_j}\,dt + \frac{\beta}{2N}\sum_{l\neq j}\frac{dt}{\lambda_j - \lambda_l} + \frac{1}{\sqrt{N}}\,dB_j \\ d\vec{\psi}_j &= -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j - \lambda_l)^2}\,\vec{\psi}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}}{\lambda_j - \lambda_l}\,\vec{\psi}_l \end{aligned}$$

- If two sets of eigenvalues are separated by a gap of the order of unity, the evolution of the eigenvectors toward the subspace spanned by eigenvectors belonging to the distant eigenvalues is suppressed

  $\Rightarrow$ eigenvectors cannot spread ergodically over the whole Hilbert space

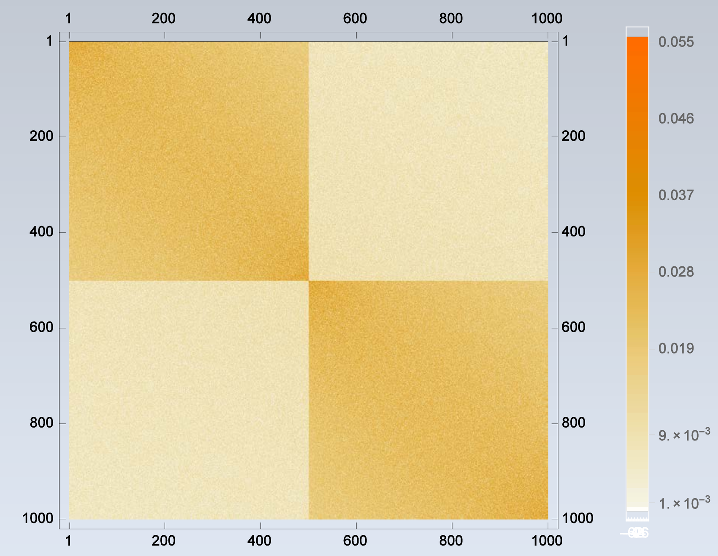
Effect of a Small Perturbation: The Double Well Case

- How do eigenvectors respond to a perturbation?

$$\left.\begin{array}{r}\mathbf{M} = \mathbf{U}^\dagger\mathbf{\Lambda}\mathbf{U} \\ \mathbf{M} + \Delta\mathbf{M} = \mathbf{U}'^\dagger\mathbf{\Lambda}'\mathbf{U}'\end{array}\right\}\ \tilde{\mathbf{U}} = \mathbf{U}'\mathbf{U}^\dagger$$

Order N non-zero elements independently sampled from a Gaussian distribution (mean 0, width $\sqrt{N}$)

- We study the perturbed eigenvectors in the basis where $\mathbf{M}$ is diagonal
- Block structure $\Rightarrow$ SSB

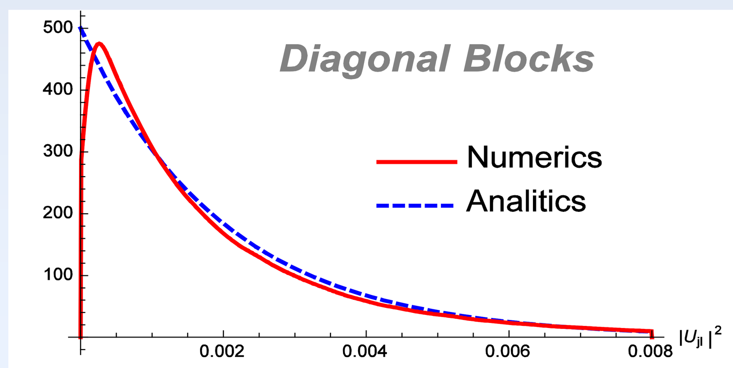


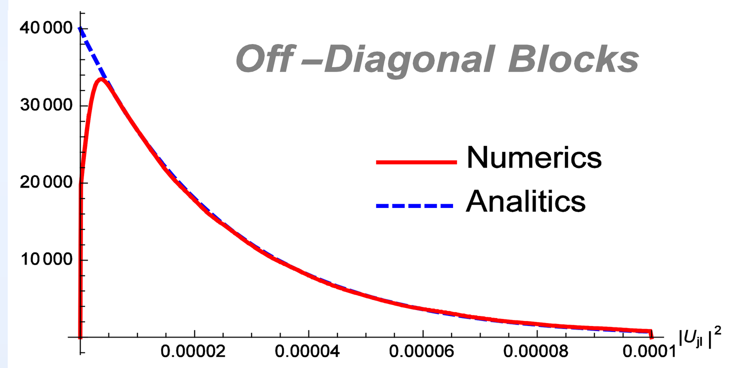
- Diagonal and Off-diagonal elements follow two different distributions

$$\chi_{\mathrm{D}} = \frac{N}{2}$$

$$\mathcal{P}\left(|U_{ij}|^2\right) = \chi e^{-\chi|U_{ij}|^2}$$

$$\chi_{\mathrm{OD}} = \frac{2tN^2}{n}$$

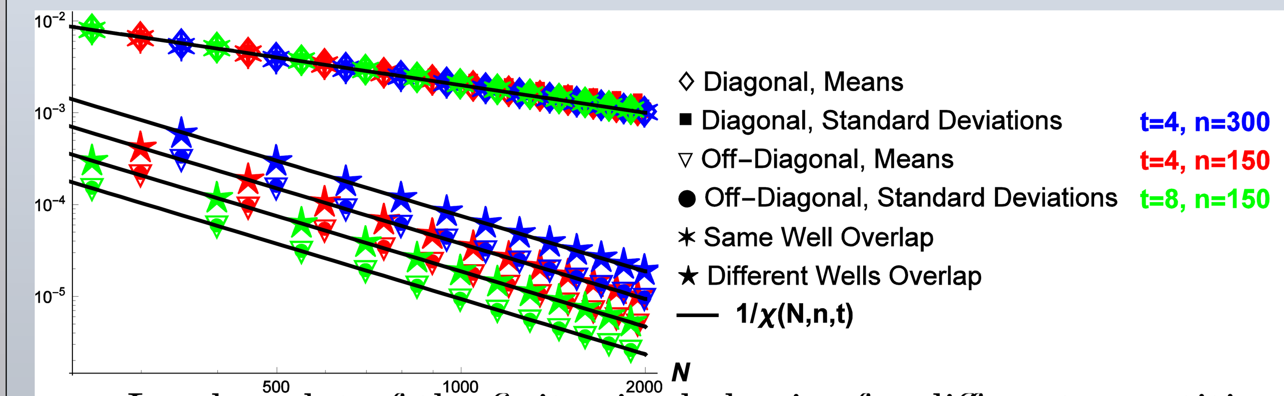




Distribution of diagonal and off-diagonal elements of a typical unitary matrix ($\Delta\mathbf{M}$ has $N\times n$ non-zero elements, with $N = 1000$, $n = 150$; $t = 4$)

- Overlap between eigenstates: $O_{jl} = \sum_m^N \left|\tilde{U}_{mj}\right|^2\left|\tilde{U}_{ml}\right|^2$

$$\begin{aligned} \langle|O_{jl}|\rangle_{\mathrm{D}} &= \langle|\tilde{U}_{jl}|\rangle_{\mathrm{D}} = \langle|\Delta\tilde{U}_{jl}|\rangle_{\mathrm{D}} = \frac{1}{\chi_{\mathrm{D}}} \\ \langle|O_{jl}|\rangle_{\mathrm{OD}} &= 2\langle|\tilde{U}_{jl}|\rangle_{\mathrm{OD}} = 2\langle|\Delta\tilde{U}_{jl}|\rangle_{\mathrm{OD}} = \frac{2}{\chi_{\mathrm{OD}}} \end{aligned}$$



Log-log plot of the finite size behavior for different quantities, averaged over several realizations of the applied perturbation: notice the remarkable aggreement with the analytical expectations

- Off-diagonal blocks suppressed by a power of N: in the thermodynamic limit the eigenvectors are localized over a N/2-dimensional sphere

Conclusions & Outlook

- A gap in the eigenvalue distribution induces a spontaneous breaking of U(N) symmetry
- 3 arguments provided: ✓ Plausibility by geometric reasoning,
  ✓ Explicit analytical construction with symmetry breaking term,
  ✓ Numerical experiment to study finite size behavior.

- F.F.: arXiv:1412.6523 **On the Spontaneous Breaking of U(N) symmetry in invariant Matrix Models**
- F.F.: arXiv:1503.03341 **Toward an invariant matrix model for the Anderson Transition**

- Eigenvectors corresponding to distant eigenvalues cannot mix: breaking of ergodicity in invariant matrix models
- At finite N: suppression of off-diagonal block of unitary matrices/suppression of spillage of eigenvectors out of localization basin
- Applications:
  - ❑ Characterization of critical behavior at the birth of a cut as a phase transition to lower symmetry
  - ❑ Invariant Matrix models to describe Anderson Metal/Insulator transition (Weakly Confined Matrix Models)
  - ❑ Overlaps and IPR alone cannot detect localization: new approach based on response to perturbation
  - ❑ Matrix models from localization limit of string theories (ABJM): new SSB mechanism for fundamental physics and holographic applications (AdS/CFT, AdS/CMT, QGP...)
  - ❑ Opens matrix models techniques to the study of a whole new set of problems related to eigenvectors

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