Off-critical interfaces in two dimensions. Exact results from field theory

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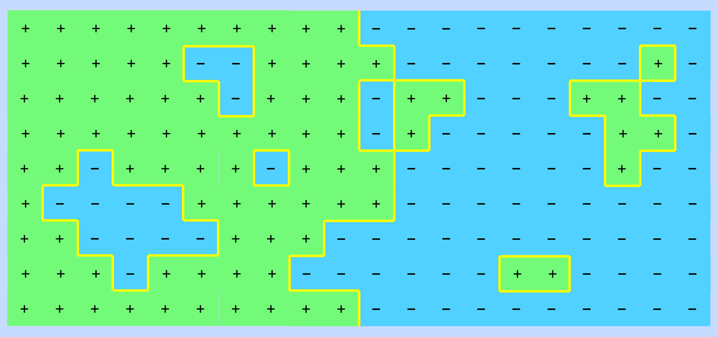
Abstract

Phase separation is the coexistence of two or more thermodynamic phases below criticality. The characterization of the order parameter, the statistical properties of the interfaces and the role of boundaries are a classic topic of statistical mechanics attacked by several approaches: rigorous, numerical and Exact Statistical Mechanics.

Motivations

What is the status of the art from lattice computations? Exact transfer matrix results are available but only for the Ising model in 2d.



...what does field theory tell us?

- $T = T_c$: 2d CFT is successfully applied, interfaces are SLE curves, but very few is known about massive deformations (off-criticality).

the issues

- What about off-criticality? How to avoid lattice calculations and work directly in the continuum?
- and universality classes other than Ising?
- Ising solutions are the only available, is this due to integrability? Does integrability play any role?

the answers

- $T < T_c$: we propose a new and exact approach to phase separation based on local fields [1]. The role of integrability is elucidated and field theory yields exact solutions for a larger variety of models with a simpler language [2], [3], [4].

Theoretical tools

the model(s)

Scaling limit of a ferromagnetic spin model below T_c. How? Analytic continuation of $(1+1)$-relativistic field theory to a $2d$ Euclidean field theory in the plane ($y = -it$). Elementary excitations in $2d$: kinks $|K_{ab}(\theta)\rangle$ interpolating the vacua $|\Omega_a\rangle$ and $|\Omega_b\rangle$. Kinks are particles with energy-momentum

$$(e, p) = m_{ab}(\cosh\theta, \sinh\theta)$$

m_{ab} is the mass, θ is the rapidity.

the dictionary

- vacua set $\{|\Omega_i\rangle\} \longleftrightarrow$ coexisting phases
- kinks excitations $\longleftrightarrow$ domain walls = interfaces
- kink mass $m_{ab} \longleftrightarrow$ interfacial (or surface) tension Σ_{ab}

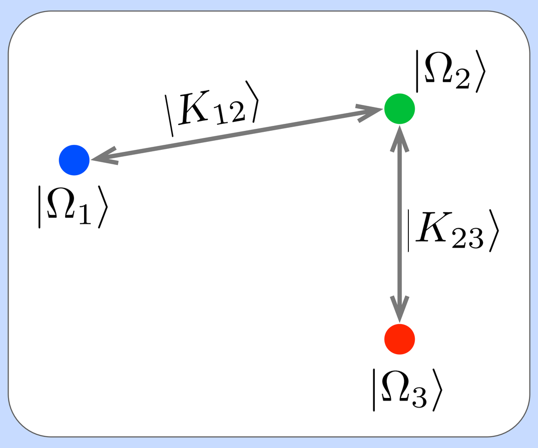


Figure 1: connectivity of a vacuum structure

theoretical framework

Interaction of particles is captured by the analytic S-matrix. The models we are considering (Ising, Potts, ...) are integrable, i.e. the S matrix is known exactly, however we do not need the full knowledge of $S(\theta)$ at $\theta \neq 0$. Our results do not depend on the integrability of a specific model. Essentially, they depend on:

- the vacuum connectivity
- the statistic of excitations for $\theta \to 0$
- presence of eventual boundary bound states

Single interfaces

Adjacent phases (as Ω_1 and Ω_2 in Fig.1). Phase separation is induced by b.dry conditions in the large distance regime $mR \gg 1$

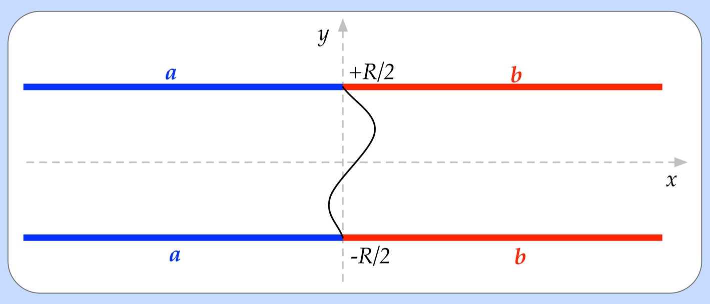


Figure 2: strip with ab boundary cond.

Order parameter profile

$$\langle\sigma(x,y)\rangle_{ab} = \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b}{2} - \frac{\langle\sigma\rangle_a - \langle\sigma\rangle_b}{2}\mathrm{erf}(\chi),$$

$$\chi = \sqrt{\frac{2mR}{R^2 - 4y^2}}\,x.$$

Ising is easily reproduced ($\langle\sigma\rangle_+ = -\langle\sigma\rangle_-$). The interface fluctuates with gaussian-like fluctuations. For the midpoint $(x,0)$:

$$P(x;0) = \sqrt{\frac{2m}{\pi R}}\,\mathrm{e}^{-\frac{2mx^2}{R}}.$$

More interesting is the q-state Potts model; for $q = 3$:

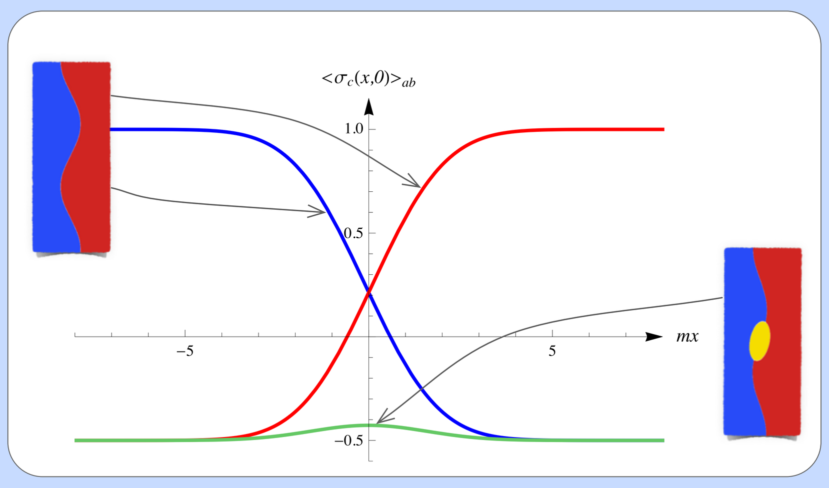


Figure 3: magnetization profiles for the 3-Potts

The interface branches and recombines, producing subleading corrections to the magnetization profile. This phenomenon can be computed exactly [1].

Double interfaces

A natural candidate: tricritical (or dilute) q-state Potts model. q ordered vacua $|\Omega_i\rangle$, 1 disordered vacuum $|\Omega_0\rangle$. The ferromagnetic vacua $|\Omega_i\rangle$ and $|\Omega_j\rangle$ cannot be connected with a single kink, the phase separation pattern will exhibit an intermediate bubble of phase $|\Omega_0\rangle$.

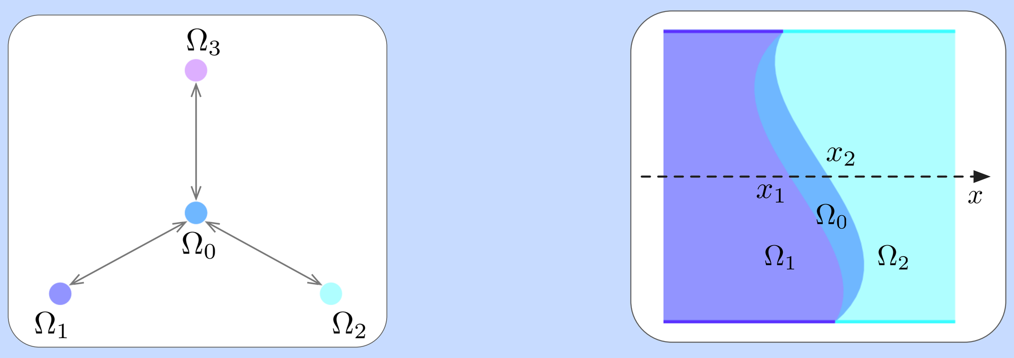


Figure 4: dilute 3-Potts: vacuum connectivity (left); phase separation pattern (right)

The average magnetization can be computed exactly [3]; for the bubble profile:

$$\langle\sigma_3(x,y)\rangle_{12} \propto 1 - \frac{2}{\pi}\mathrm{e}^{-2\chi^2} - \frac{2\chi}{\sqrt{\pi}}\mathrm{erf}(\chi)\,\mathrm{e}^{-\chi^2} + \mathrm{erf}^2(\chi).$$

The two interfaces are mutually avoiding curves with passage probability density

$$P(x_1, x_2; y) = \frac{(\chi_1 - \chi_2)^2}{\pi(\kappa\lambda)^2}\,\mathrm{e}^{-(\chi_1^2+\chi_2^2)},$$

$$\kappa = \sqrt{1 - 4y^2/R^2} \quad , \quad \lambda \equiv \sqrt{R/(2m)}.$$

The contour lines of the magnetization density are ellipses pinned in $(0, \pm R/2)$ with horizontal semi-axis $\sim \sqrt{R}$.

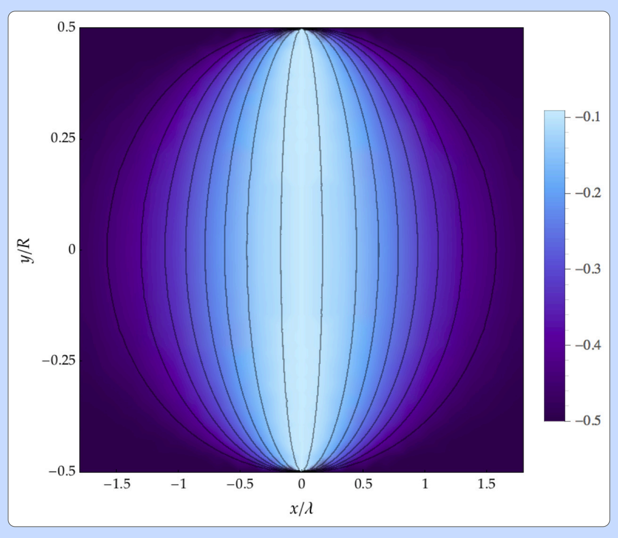


Figure 5: the bubble density $\langle\sigma_3(x,y)\rangle_{12}$.

Interfacial wetting

Another candidate for double interfaces is the Ashkin-Teller model, it is a pair of Ising models coupled through a four-spin interaction of strength J_4. The vacua are (α_1, α_2), $\alpha_j \in \{\pm\}$ being the vacua of the j-th copy.

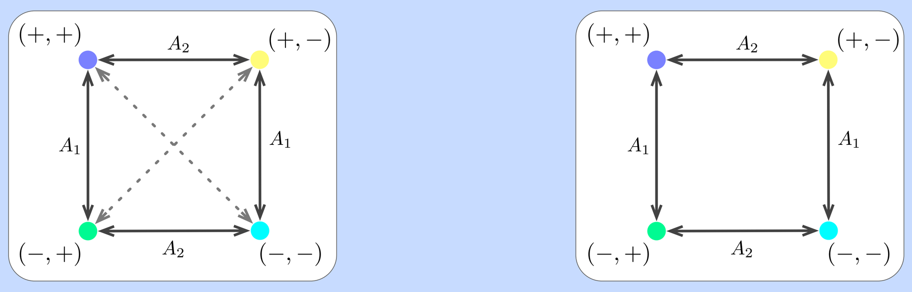
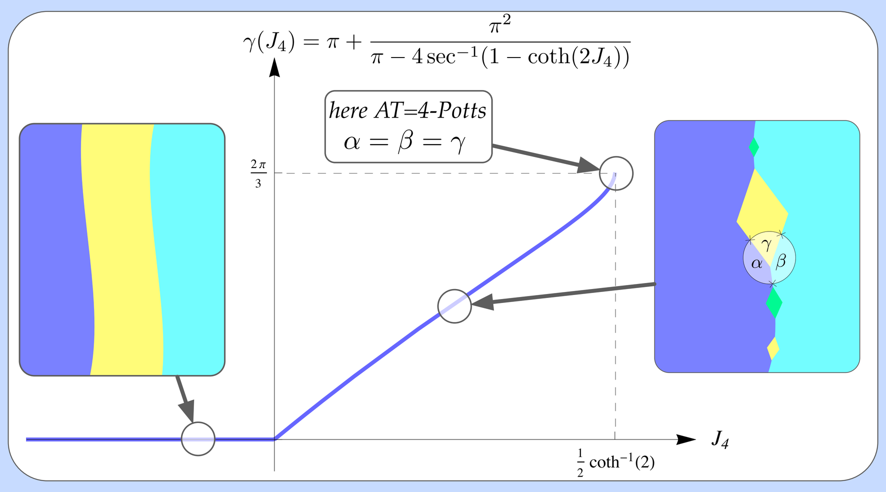


Figure 6: vacuum structure for $J_4 > 0$ (left); $J_4 \leqslant 0$ (right)

The kink A_j flips the vacuum of the j-th Ising. The vacuum structure now depends on J_4. For $J_4 \leqslant 0$ the vacua $(+,+)$ and $(-,-)$ are not adjacent $\Longrightarrow$ double interface



For $J_4 > 0$ bubbles of $(\pm,\mp)$ phase are present at the interface $(+,+)|(-,-)$ with contact angle γ. As $\gamma \to 0$ the phase $(\pm,\mp)$ spreads and wets the $(+,+)|(-,-)$ interface (left).

Wetting & filling

Wetting is the ability of a liquid to maintain a contact angle θ_0 with a surface; as $\theta_0 \to 0$ the drop spreads: this is the wetting transition.

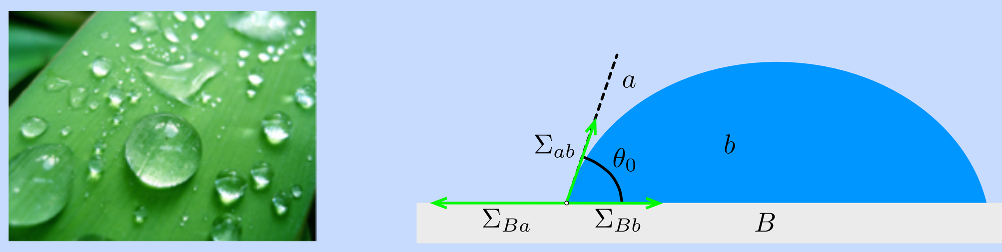


Figure 7: drops on a substrate

The collision of the kink off the boundary may exhibit a bound state pole at the resonance rapidity iu, in this regime the kink adheres to the boundary with a contact angle u [2] that obey the phenomenological Young's law

$$\Sigma_{Ba} = \Sigma_{Bb} + \Sigma_{ab}\cos\theta_0.$$

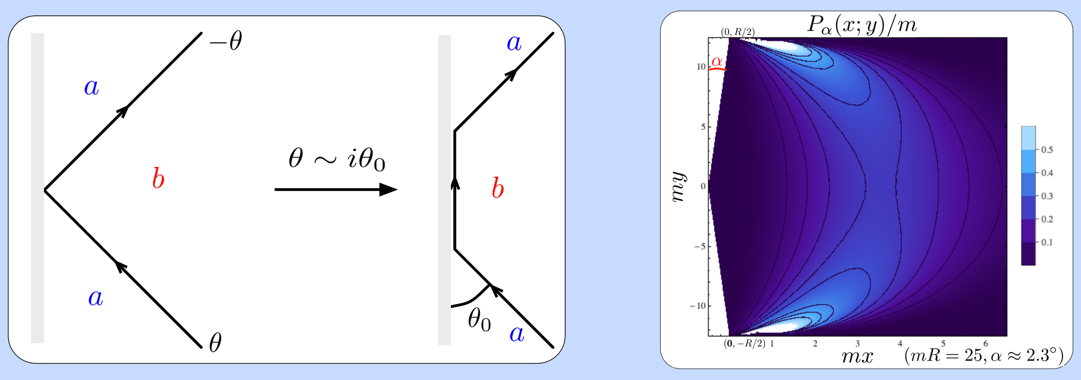


Figure 8: Wetting of a flat substrate (left). Passage probability density for a wedge (right).

The interface fluctuates with passage probability

$$P_\alpha(x;y) = \frac{8\sqrt{2}}{\sqrt{\pi}\,\kappa^3}\left(\frac{m}{R}\right)^{3/2}\frac{\left(x + \frac{R\alpha}{2}\right)^2 - (\alpha y)^2}{1 + mR\alpha^2}\,\mathrm{e}^{-\chi^2},$$

α is the tilt angle of the wedge (opening angle: $\pi - 2\alpha$). The filling transition occurs at $\theta_0(T_\alpha) = \alpha$. Field theory provides the explanation of the *wedge covariance* phenomenon. [4]

Take home message

- ✓ a new method: exact formulation of phase separation and related topics: passage probabilities, interface structure (branching & recombination), interfaces at boundaries, wetting & filling, ...
- ✓ the known solutions (from lattice) for Ising are obtained as a particular case. Different universality classes are examined for the first time directly from field theory
- ✓ the validity of the whole technique does not relies on integrability but rather on the fact that domain walls are particle trajectories
- ✓ although $mR \gg 1$ projects to low energies, relativistic particles are essential for kinematical poles and contact angle

Based on

[1] G. Delfino and J. Viti, *Phase separation and interface structure in two dimensions from field theory*, J. Stat. Mech. P10009 (2012)

[2] G. Delfino and A.S., *Interfaces and wetting transition on the half plane. Exact results from field theory*, J. Stat. Mech. P05010 (2013)

[3] G. Delfino and A.S., *Exact theory of intermediate phases in two dimensions*, Annals of Physics **342**, 171 (2014)

[4] G. Delfino and A.S., *Phase separation in a wedge. Exact results*, Physical Review Letters 113, (2014) 066101