Stochastic quantum integrable systems in infinite volume

Leonid Petrov

Institute for Information Transmission Problems (Moscow, Russia)
University of Virginia (Charlottesville, VA, USA)

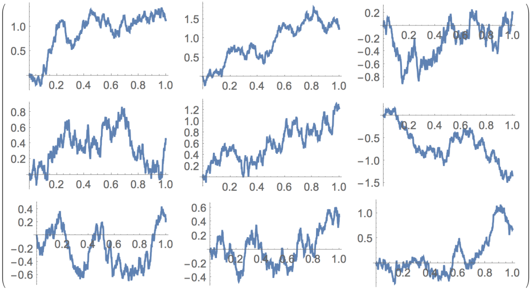
Outline

1. Integrable stochastic particle systems
2. Bethe ansatz eigenfunctions of ASEP
3. Stochastic vertex models

Historic example: Dyson's Brownian motion

Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

$$a_{11}(t) = B_{11}(t), \quad a_{12}(t) = \overline{a_{21}(t)} = \frac{1}{\sqrt{2}} B^r_{12}(t) + \frac{\sqrt{-1}}{\sqrt{2}} B^i_{12}(t), \quad \text{etc.}$$

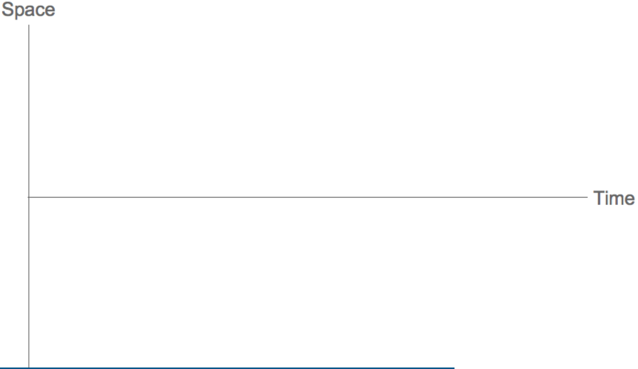


Historic example: Dyson's Brownian motion

Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

Eigenvalues of A are real $\lambda_1(t) \ge \ldots \ge \lambda_N(t)$.

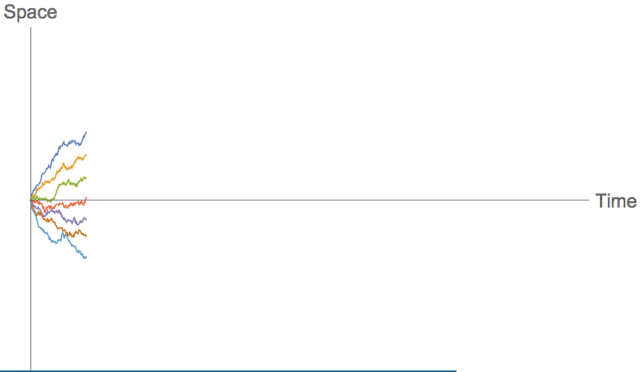
They evolve as a *marginally Markov process* **[Dyson '60s]** — N Brownian motions conditioned to never collide.



Historic example: Dyson's Brownian motion

Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

Eigenvalues of A are real $\lambda_1(t) \ge \ldots \ge \lambda_N(t)$.
They evolve as a *marginally Markov process* **[Dyson '60s]** — N Brownian motions conditioned to never collide.

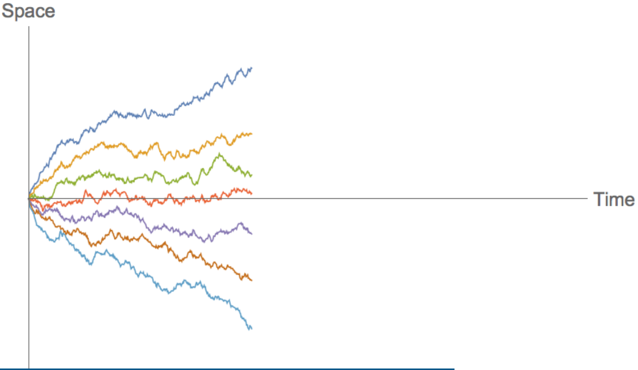


Historic example: Dyson's Brownian motion

Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

Eigenvalues of A are real $\lambda_1(t) \ge \ldots \ge \lambda_N(t)$.

They evolve as a *marginally Markov process* **[Dyson '60s]** — N Brownian motions conditioned to never collide.

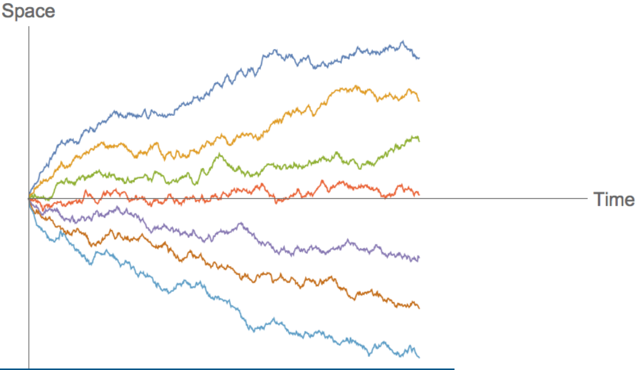


Historic example: Dyson's Brownian motion

Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

Eigenvalues of A are real $\lambda_1(t) \ge \ldots \ge \lambda_N(t)$.

They evolve as a *marginally Markov process* **[Dyson '60s]** — N Brownian motions conditioned to never collide.

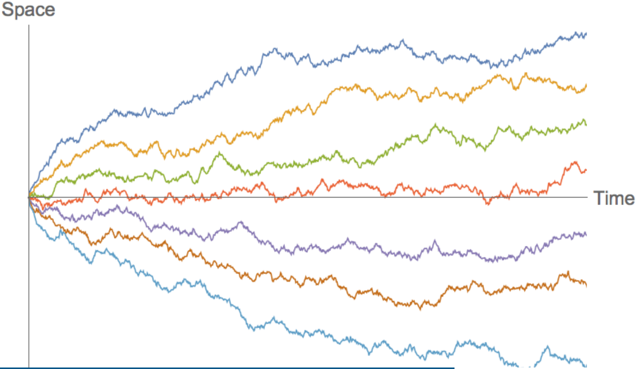


Historic example: Dyson's Brownian motion

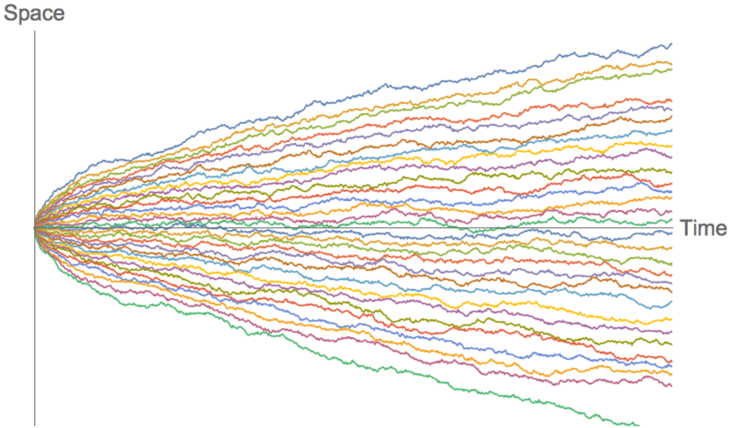
Let A be a random $N \times N$ Hermitian matrix whose entries $a_{ij}(t)$, $i \le j$, evolve as independent *complex* Brownian motions.

Eigenvalues of A are real $\lambda_1(t) \ge \ldots \ge \lambda_N(t)$.

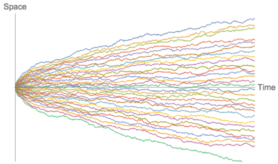
They evolve as a *marginally Markov process* **[Dyson '60s]** — N Brownian motions conditioned to never collide.



Historic example: Dyson's Brownian motion



Historic example: Dyson's Brownian motion



(also looks like a 2d model of statistical mechanics)

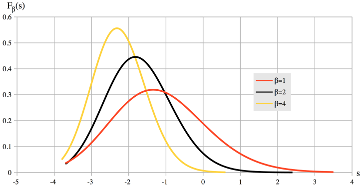
- As $N \to \infty$, there is a *limit shape* for the density of the eigenvalues — a semicircle growing with time **[Wigner's semicircle law, '50s]**. Global *Gaussian Free Field* type behavior and *Tracy–Widom* edge fluctuations are also present **[mostly since '90s, e.g. see book by Anderson–Guionnet–Zeitouni]**.

Historic example: Dyson's Brownian motion

- As $N \to \infty$, there is a *limit shape* for the density of the eigenvalues — a semicircle growing with time **[Wigner's semicircle law, '50s]**. Global *Gaussian Free Field* type behavior and *Tracy–Widom* edge fluctuations are also present **[mostly since '90s, e.g. see book by Anderson–Guionnet–Zeitouni]**.

Theorem: Tracy–Widom fluctuations

$$\mathbb{P}\left[N^{\frac{1}{6}}\left(\frac{1}{\sqrt{t}}\lambda_{\max}(t) - 2\sqrt{N}\right) \le u\right] \to F_{GUE}(u) \quad \text{as } N \to \infty,\ t \text{ fixed.}$$



$\frac{d}{du}F_{GUE}(u)$ in the middle **[Wikipedia]**

Historic example: Dyson's Brownian motion

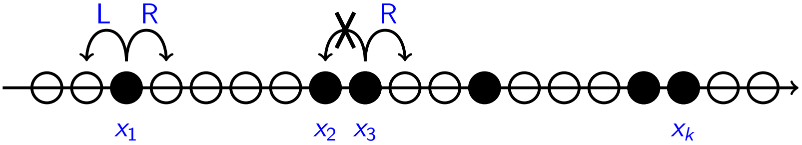
- As $N \to \infty$, there is a *limit shape* for the density of the eigenvalues — a semicircle growing with time **[Wigner's semicircle law, '50s]**. Global *Gaussian Free Field* type behavior and *Tracy–Widom* edge fluctuations are also present **[mostly since '90s, e.g. see book by Anderson–Guionnet–Zeitouni]**.
- Integrability structure: dynamical correlations are *determinantal* ("free fermions") **[Eynard–Mehta '98]**, **[Nagao–Forrester '98]**.
  $+$ connections to *Schur symmetric polynomials*.

Historic example: Dyson's Brownian motion

- As $N \to \infty$, there is a *limit shape* for the density of the eigenvalues — a semicircle growing with time **[Wigner's semicircle law, '50s]**. Global *Gaussian Free Field* type behavior and *Tracy–Widom* edge fluctuations are also present **[mostly since '90s, e.g. see book by Anderson–Guionnet–Zeitouni]**.
- Integrability structure: dynamical correlations are *determinantal* ("free fermions") **[Eynard–Mehta '98]**, **[Nagao–Forrester '98]**.
  $+$ connections to *Schur symmetric polynomials*.
- Dyson's Brownian motion is a *nonlocal particle dynamics*.

Another historic example: ASEP

ASEP (Asymmetric Simple Exclusion Process) — a continuous-time Markov chain on configurations on $\mathbb{Z}$ (at most one particle per site), introduced in **[Spitzer '70]**.

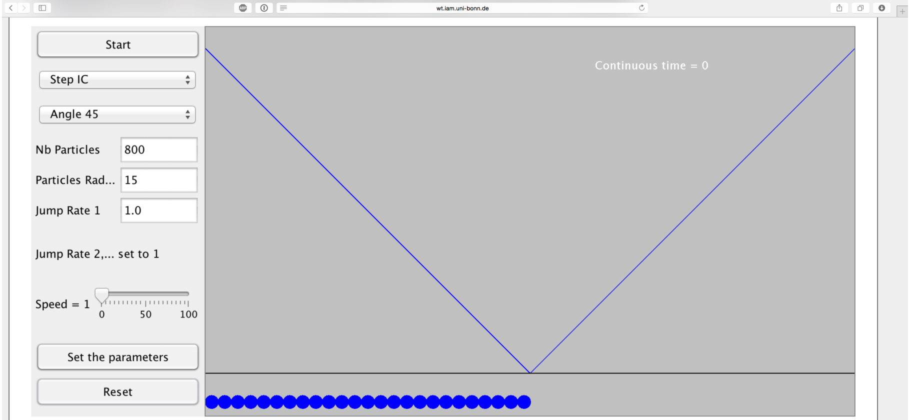


$\mathsf{R} + \mathsf{L} = 1$, $\mathsf{R}/\mathsf{L} = q < 1$.

- *Local* particle dynamics.
- $\mathsf{L} = 0 \Rightarrow$ TASEP, has *determinantal structure* and is connected to *Schur symmetric polynomials*.
- **[Gorin–Shkolnikov '12]** — scaling limit of multilayer TASEP-like processes to Dyson's Brownian motion.
- *No determinantal structure* when $\mathsf{R}, \mathsf{L} > 0$.

Simulation of TASEP: Step IC

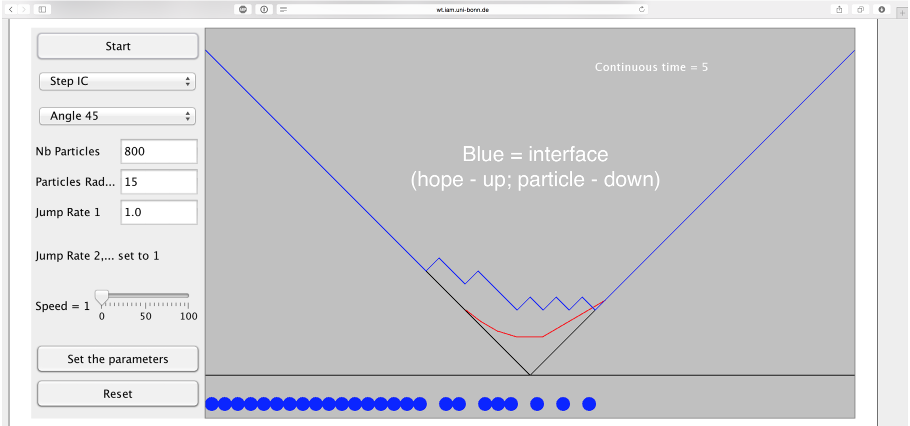
[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial condition*, where particles initially are (for example) at every second site (half => only on negative axis).

Simulation of TASEP: Step IC

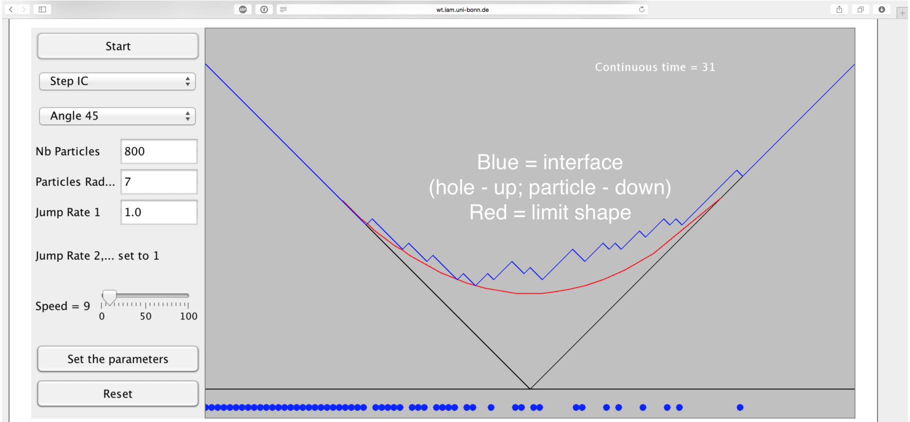
[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial condition*, where particles initially are (for example) at every second site (half => only on negative axis).

Simulation of TASEP: Step IC

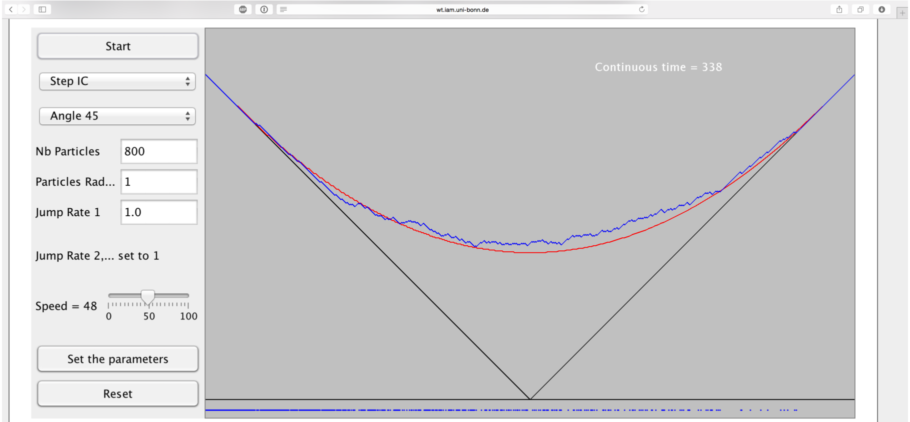
[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial condition*, where particles initially are (for example) at every second site (half => only on negative axis).

Simulation of TASEP: Step IC

[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial condition*, where particles initially are (for example) at every second site (half => only on negative axis).

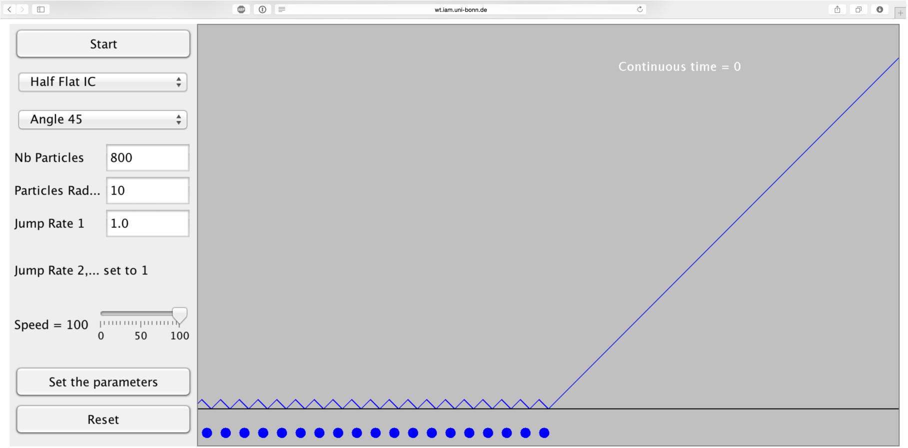
Simulation of TASEP: Step IC

[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/

The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial condition*, where particles initially are (for example) at every second site (half => only on negative axis).

Simulation of TASEP: Half-flat IC

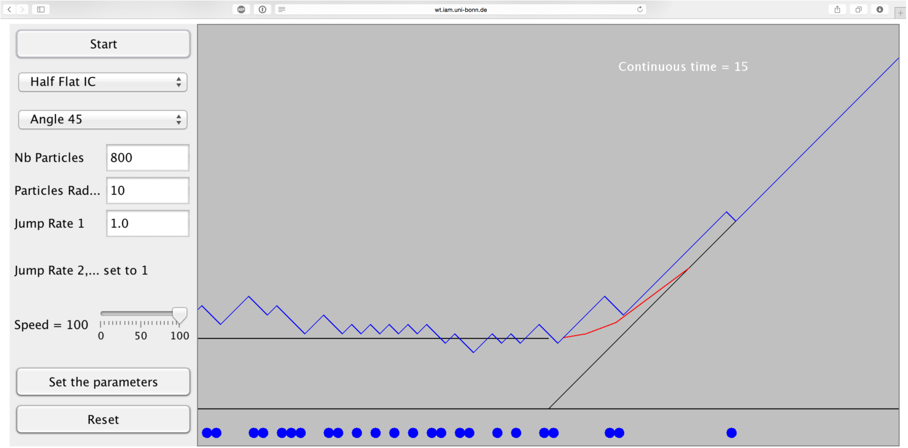
[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.
Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial*

Simulation of TASEP: Half-flat IC

[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/

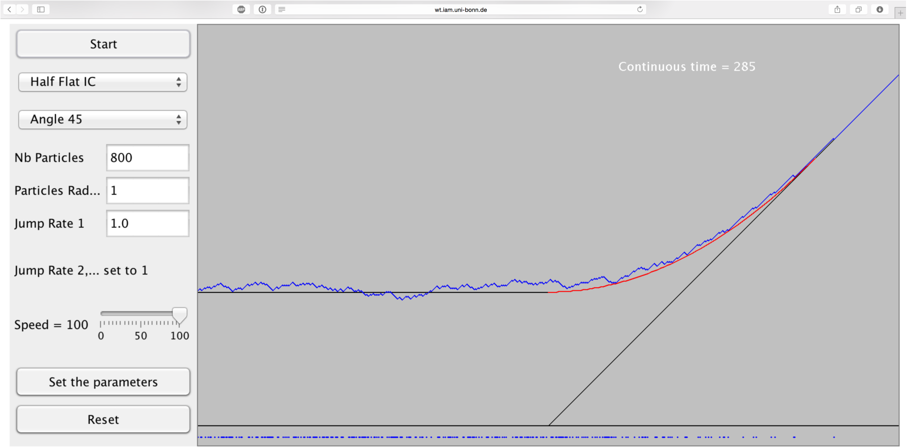


The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.

Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial*

Simulation of TASEP: Half-flat IC

[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/

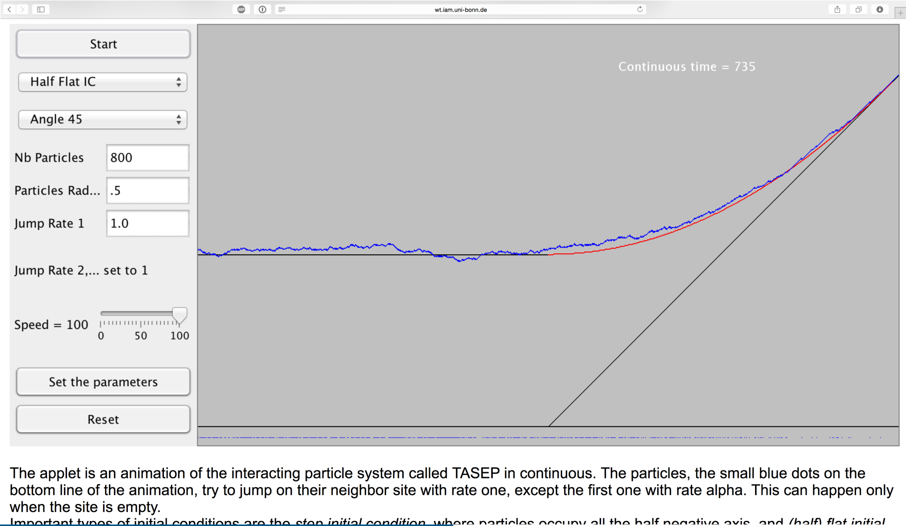


The applet is an animation of the interacting particle system called TASEP in continuous. The particles, the small blue dots on the bottom line of the animation, try to jump on their neighbor site with rate one, except the first one with rate alpha. This can happen only when the site is empty.

Important types of initial conditions are the *step initial condition*, where particles occupy all the half negative axis, and *(half) flat initial*

Simulation of TASEP: Half-flat IC

[Ferrari '08], http://wt.iam.uni-bonn.de/ferrari/research/continoustasep/



ASEP

- As $t \to \infty$, the ASEP interface ($=$ height function) possesses a *limit shape* (evolving in time). *Tracy–Widom fluctuations* around the limiting interface are also present, established for special initial data **[Tracy–Widom '07+]** (TASEP: **[Johansson '99]**).

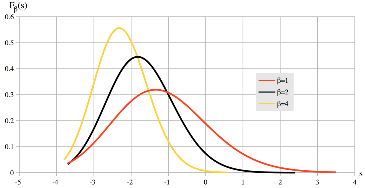
ASEP

- As $t \to \infty$, the ASEP interface ($=$ height function) possesses a *limit shape* (evolving in time). *Tracy–Widom fluctuations* around the limiting interface are also present, established for special initial data **[Tracy–Widom '07+]** (TASEP: **[Johansson '99]**).

Theorem: Tracy–Widom fluctuations

$N_0 := \#$ particles to the left of zero if initially $\mathbb{Z}_+$ is packed, $\mathbb{Z}_-$ empty.

Then $\mathbb{P}\left[\dfrac{N_0(t/(\mathsf{L}-\mathsf{R})) - t/2}{2^{-1/3}t^{1/3}} \geq -u\right] \to F_{GUE}(u)$ as $t \to \infty$



[Wikipedia]

ASEP

- As $t \to \infty$, the ASEP interface ($=$ height function) possesses a *limit shape* (evolving in time). *Tracy–Widom fluctuations* around the limiting interface are also present, established for special initial data **[Tracy–Widom '07+]** (TASEP: **[Johansson '99]**).
- Under a more delicate scaling, the ASEP interface converges to the solution of a (1+1-dimensional) stochastic PDE — the *KPZ equation* **[Sasamoto–Spohn '10]**, **[Amir–Corwin–Quastel '10]**, **[Dotsenko '10+]**, …

$$\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 + \text{space-time white noise.}$$ **[Kardar–Parisi–Zhang '86]**

ASEP

- *Tracy–Widom fluctuations*.
- Convergence to the *KPZ equation*.
- Integrability structure ($\mathsf{R}, \mathsf{L} > 0$): *explicit eigenfunctions* of the Markov generator of the ASEP, obtained by the (coordinate) Bethe ansatz. *No Bethe equations* because lattice is infinite.

  Properties of eigenfunctions allow to compute probability $\mathbb{P}(x_m(t) \le x)$ as a *Fredholm determinant* $\det(1 - K)$, and analyze it asymptotically.

Fredholm determinant is a kind of generating function for minors of K, more precisely,

$$\det(1-K) = 1 - (\text{“sum” of 1-dim diagonal minors}) + (\text{“sum” of 2-dim diagonal minors}) - \ldots$$

Overview: Sources of Integrability in (Stochastic) Interacting Particle Systems

1. **Representation theory / Algebra of symmetric functions**: Schur functions, Schur processes, determinantal structure ("free fermions"), Robinson–Schensted–Knuth correspondence, . . . , Macdonald processes, . . .
   - Dyson's Brownian motion, lozenge tilings, q-TASEP, random polymers, . . .

Overview: Sources of Integrability in (Stochastic) Interacting Particle Systems

1. **Representation theory / Algebra of symmetric functions**: Schur functions, Schur processes, determinantal structure ("free fermions"), Robinson–Schensted–Knuth correspondence, ..., Macdonald processes, ...
   - Dyson's Brownian motion, lozenge tilings, q-TASEP, random polymers, ...
2. **Quantum integrable systems / exactly solvable lattice models in statistical mechanics**: Yang-Baxter relation, Bethe ansatz, Plancherel theory for Bethe ansatz eigenfunctions, Markov duality (incl. quantum group symmetries), ...
   - ASEP / XXZ, six-vertex model, higher spin stochastic vertex models, q-TASEP, random polymers, ...

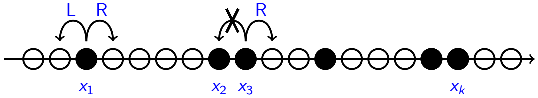
(so Dyson's BM was solved by determinantal point processes, many its discrete relatives can be approached using symmetric polynomials; but ASEP required new ideas based on Bethe ansatz)

Outline

1. Integrable stochastic particle systems
2. Bethe ansatz eigenfunctions of ASEP
3. Stochastic vertex models

Coordinate Bethe ansatz for k-particle ASEP

Let k be the number of particles, $x_1 < x_2 < \ldots < x_k$, and $\mathcal{H}^{(k)}$ be the Markov generator of this k-particle ASEP (i.e., $\mathcal{H}^{(k)}$ is the matrix of jump rates).

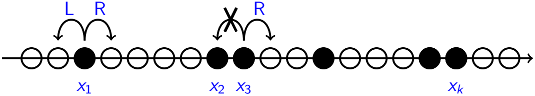


$\boxed{k = 1 :}$

$$\mathcal{H}^{(1)} f(x_1) = \mathsf{R}(f(x_1+1) - f(x_1)) + \mathsf{L}(f(x_1-1) - f(x_1)).$$

Coordinate Bethe ansatz for k-particle ASEP

Let k be the number of particles, $x_1 < x_2 < \ldots < x_k$, and $\mathcal{H}^{(k)}$ be the Markov generator of this k-particle ASEP (i.e., $\mathcal{H}^{(k)}$ is the matrix of jump rates).



$\boxed{k = 1 :}$

$$\mathcal{H}^{(1)} f(x_1) = \mathsf{R}(f(x_1+1) - f(x_1)) + \mathsf{L}(f(x_1-1) - f(x_1)).$$

$\boxed{k = 2,\ x_1 + 1 < x_2:}$

$$\mathcal{H}^{(2)} f(x_1, x_2) = \mathsf{R}(f(x_1+1, x_2) - f(x_1, x_2)) + \mathsf{L}(f(x_1-1, x_2) - f(x_1, x_2)) + \mathsf{R}(f(x_1, x_2+1) - f(x_1, x_2)) + \mathsf{L}(f(x_1, x_2-1) - f(x_1, x_2)) = \left(\mathcal{H}_1^{(1)} + \mathcal{H}_2^{(1)}\right) f(x_1, x_2).$$

Coordinate Bethe ansatz for k-particle ASEP

$\boxed{k = 1:}$

$$\mathcal{H}^{(1)} f(x_1) = \mathsf{R}(f(x_1+1) - f(x_1)) + \mathsf{L}(f(x_1-1) - f(x_1)).$$

$\boxed{k = 2,\ x_1 + 1 < x_2:}$

$$\mathcal{H}^{(2)} f(x_1, x_2) = \mathsf{R}(f(x_1+1, x_2) - f(x_1, x_2)) + \mathsf{L}(f(x_1-1, x_2) - f(x_1, x_2)) + \mathsf{R}(f(x_1, x_2+1) - f(x_1, x_2)) + \mathsf{L}(f(x_1, x_2-1) - f(x_1, x_2)) = \big(\mathcal{H}^{(1)}_1 + \mathcal{H}^{(1)}_2\big) f(x_1, x_2).$$

$\boxed{k = 2,\ x_1 + 1 = x_2:\ x_1 \text{ cannot jump right, } x_2 \text{ cannot jump left}}$

$$\mathcal{H}^{(2)} f(x_1, x_2) = \mathsf{R}(f(x_1, x_2+1) - f(x_1, x_2)) + \mathsf{L}(f(x_1-1, x_2) - f(x_1, x_2)) = \big(\mathcal{H}^{(1)}_1 + \mathcal{H}^{(1)}_2\big) f(x_1, x_2) + \text{discrepancy},$$

$$\boxed{\text{discrepancy} = \mathsf{R} f(x_1+1, x_2) + \mathsf{L} f(x_1, x_2-1) - f(x_1, x_2)}$$

Coordinate Bethe ansatz for k-particle ASEP

When $x_1 + 1 = x_2$,
$\mathrm{discrepancy} = \mathsf{R}f(x_1+1, x_2) + \mathsf{L}f(x_1, x_2-1) - f(x_1, x_2)$ involves values of f outside of the "physical region" $x_1 < x_2$.

Therefore, we can **assign** arbitrary values to f outside this region so that $\mathrm{discrepancy} = 0$. Can do the same for k particles, and the boundary conditions will involve only pairs of neighboring particles (**two-body boundary conditions**).

Coordinate Bethe ansatz for k-particle ASEP

When $x_1 + 1 = x_2$,
$\mathrm{discrepancy} = \mathsf{R}f(x_1+1, x_2) + \mathsf{L}f(x_1, x_2 - 1) - f(x_1, x_2)$ involves values of f outside of the "physical region" $x_1 < x_2$.

Therefore, we can **assign** arbitrary values to f outside this region so that $\mathrm{discrepancy} = 0$. Can do the same for k particles, and the boundary conditions will involve only pairs of neighboring particles (**two-body boundary conditions**).

Proposition: ASEP is integrable in the sense of **[Bethe '31]**

$\mathcal{H}^{(k)} f = \left(\mathcal{H}_1^{(1)} + \ldots + \mathcal{H}_k^{(1)}\right) f$ if f is such that for any i,

$$\mathsf{R}f(\ldots, x_i + 1, x_{i+1}, \ldots) + \mathsf{L}f(\ldots, x_i, x_{i+1} - 1, \ldots) - f(\ldots) = 0$$

whenever $x_i + 1 = x_{i+1}$. **[Schutz et al. since '90s]**, **[Tracy–Widom '07]**.

Coordinate Bethe ansatz for k-particle ASEP

Proposition: ASEP is integrable in the sense of **[Bethe '31]**

$\mathcal{H}^{(k)} f = \big(\mathcal{H}^{(1)}_1 + \ldots + \mathcal{H}^{(1)}_k\big) f$ if f is such that for any i,

$$\mathsf{R} f(\ldots, x_i + 1, x_{i+1}, \ldots) + \mathsf{L} f(\ldots, x_i, x_{i+1} - 1, \ldots) - f(\ldots) = 0$$

whenever $x_i + 1 = x_{i+1}$. **[Schutz et al. since '90s]**, **[Tracy–Widom '07]**.

No surprise: ASEP generator is conjugate to the Hamiltonian of the Heisenberg XXZ quantum spin chain (with $|\Delta| > 1$). The XXX case $\Delta = 1$ (corresponding to $\mathsf{R} = \mathsf{L}$) was studied by Bethe himself.

Eigenfunctions of k-particle ASEP

Diagonalize each $\mathcal{H}_i^{(1)}$ separately, and combine the eigenfunctions to satisfy the two-body boundary conditions.

The sum of one-particle operators has eigenfunctions

$$\sum_{\sigma\in S(k)} A_\sigma(\vec{z}) \prod_{i=1}^{k} \left(\frac{1+z_{\sigma(i)}}{1+z_{\sigma(i)}/q}\right)^{-x_i}, \qquad \vec{z}=(z_1,\ldots,z_k)\in\mathbb{C}^k.$$

These will be eigenfunctions for any choice of $A_\sigma(\vec{z})$.

Then it is possible to choose $A_\sigma(\vec{z})$ to satisfy the two-body boundary conditions, and thus one has

k-particle ASEP eigenfunctions

$$\Psi_{\vec{z}}^{ASEP}(\vec{x}) = \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(B)} - q z_{\sigma(A)}}{z_{\sigma(B)} - z_{\sigma(A)}} \prod_{i=1}^{k} \left(\frac{1+z_{\sigma(i)}}{1+z_{\sigma(i)}/q}\right)^{-x_i}$$

Eigenfunctions of k-particle ASEP

k-particle ASEP eigenfunctions

$$\Psi^{ASEP}_{\vec{z}}(\vec{x}) = \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(B)} - qz_{\sigma(A)}}{z_{\sigma(B)} - z_{\sigma(A)}} \prod_{i=1}^{k} \left(\frac{1+z_{\sigma(i)}}{1+z_{\sigma(i)}/q}\right)^{-x_i}$$

$$\mathcal{H}^{(k)}\Psi^{ASEP}_{\vec{z}} = \underbrace{-\frac{(1-q)^2}{1+q}\sum_{j=1}^{k}\frac{1}{(1+z_j)(1+q/z_j)}}_{ev(\vec{z})}\Psi^{ASEP}_{\vec{z}}$$

Solving Kolmogorov equations for k-particle ASEP

Eigenfunctions $\Psi^{ASEP}_{\vec{z}}(\vec{x})$ help solve the backward and forward Kolmogorov equations with arbitrary initial data — these are systems of first-order linear ODEs with the difference operator $\mathcal{H}^{(k)}$ or its transpose in the right-hand side.

This allows to compute observables $\mathbb{E}_{\vec{x}(0)=\vec{x}}F(\vec{x}(t))$ and transition probabilities $\mathbb{P}_t(\vec{x} \to \vec{y})$.

For instance, $f(t;\vec{y}) := \mathbb{P}_t(\vec{x} \to \vec{y})$ satisfies

Master equation

$$\begin{cases} \dfrac{d}{dt} f(t;\vec{y}) = \displaystyle\sum_{\vec{y}'} f(t;\vec{y}')\mathcal{H}^{(k)}(\vec{y}',\vec{y}), \\ f(0;\vec{y}) = \mathbf{1}_{\vec{y}=\vec{x}}. \end{cases}$$

Solving Kolmogorov equations for k-particle ASEP

$$\frac{d}{dt} f(t;\vec{y}) = \sum_{\vec{y}'} f(t;\vec{y}')\mathcal{H}^{(k)}(\vec{y}',\vec{y}), \qquad f(0;\vec{y}) = \mathbf{1}_{\vec{y}=\vec{x}}$$

Strategy:

1. Come up with direct and inverse *Fourier-like transforms* associated with eigenfunctions $\Psi_{\vec{z}}^{ASEP}(\vec{x})$ (*analogy*: $\Psi_z(x) = e^{zx}$ for the 1d Laplacian on $\mathbb{R}$)
2. Project the initial data $\mathbf{1}_{\vec{y}=\vec{x}}$ onto the eigenfunctions using direct transform
3. Evolve in the $\vec{z}$-space: multiply by $e^{t\cdot ev(\vec{z})}$
4. Reconstruct the solution using inverse transform

Fourier-like transforms for ASEP

Direct transform

$f(\vec{x})$ on $\mathbb{W}^k := \{x_1 < \ldots < x_k\} \subset \mathbb{Z}^k$ is mapped to

$$\left\langle f, \Psi_{\vec{z}} \right\rangle_{\vec{x}} := \sum_{\vec{x}\in\mathbb{W}^k} f(\vec{x})\Psi^{ASEP}_{\vec{z}}(\vec{x})$$

Inverse transform

$G(\vec{z})$ is mapped to

$$\oint .. \oint G(\vec{z}) \prod_{B<A} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^{k} \frac{1-1/q}{(1+z_j)(1+z_j/q)} \left(\frac{1+z_j}{1+z_j/q}\right)^{-x_j} \frac{dz_i}{2\pi\mathbf{i}},$$

integration over small circles around -1.

This can also be regarded as a scalar product of $G(\vec{z})$ with $\Psi^{ASEP}_{\vec{z}}(\vec{x})$, denote it by $\left\langle G, \Psi_{\bullet}(\vec{x}) \right\rangle_{\vec{z}}$

Plancherel theorem

$f(\vec{x}) \mapsto \sum_{\vec{x}\in\mathbb{W}^k} f(\vec{x})\Psi_{\vec{z}}^{ASEP}(\vec{x})$

$G(\vec{z}) \mapsto \oint .. \oint G(\vec{z}) \prod_{B<A} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{1-1/q}{(1+z_j)(1+z_j/q)} \left(\frac{1+z_j}{1+z_j/q}\right)^{-x_j} \frac{dz_i}{2\pi\mathbf{i}}$

Plancherel theorem **[Tracy–Widom '07+], [Borodin–Corwin–P.–Sasamoto '14]**

The direct and inverse transforms are *mutual inverses* on:

- compactly supported functions on $\mathbb{W}^k = \{x_1 < \ldots < x_k\} \subset \mathbb{Z}^k$
- symmetric Laurent polynomials in $(1 + z_i)/(1 + z_i/q)$

(two separate statements)

Plancherel theorem

$f(\vec{x}) \mapsto \sum_{\vec{x}\in\mathbb{W}^k} f(\vec{x})\Psi^{ASEP}_{\vec{z}}(\vec{x})$

$G(\vec{z}) \mapsto \oint .. \oint G(\vec{z}) \prod_{B<A} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^{k} \frac{1-1/q}{(1+z_j)(1+z_j/q)} \left(\frac{1+z_j}{1+z_j/q}\right)^{-x_j} \frac{dz_i}{2\pi\mathbf{i}}$

Plancherel theorem [Tracy–Widom '07+], [Borodin–Corwin–P.–Sasamoto '14]

The direct and inverse transforms are *mutual inverses* on:

- compactly supported functions on $\mathbb{W}^k = \{x_1 < \ldots < x_k\} \subset \mathbb{Z}^k$
- symmetric Laurent polynomials in $(1+z_i)/(1+z_i/q)$

(two separate statements)

- The Bethe ansatz for ASEP is *complete*, i.e., any (nice) initial data is determined by its image in the $\vec{z}$ space.
- The eigenfunctions $\Psi^{ASEP}_{\vec{z}}(\vec{x})$ are *orthogonal*: in the usual sense under $\langle\cdot,\cdot\rangle_{\vec{z}}$, in a generalized sense under $\langle\cdot,\cdot\rangle_{\vec{x}}$.

Next steps towards Tracy–Widom fluctuations

1. For k-particle ASEP, $\mathbb{P}_t(\vec{x} \to \vec{y})$ is given as a k-fold contour integral for *any* initial data $\vec{x}$.

Next steps towards Tracy–Widom fluctuations

1. For k-particle ASEP, $\mathbb{P}_t(\vec{x} \to \vec{y})$ is given as a k-fold contour integral for *any* initial data $\vec{x}$.
2. Use certain *combinatorial summation identities* (following from the Plancherel theory) to compute $\mathbb{P}(x_m(t) \le x)$. Works only for special initial data: $\mathbb{Z}_+$ is packed, $\mathbb{Z}_-$ is empty (number of particles can be taken *infinite*).
   The answer is a sum of k-fold contour integrals over $k \ge m$.

Next steps towards Tracy–Widom fluctuations

1. For k-particle ASEP, $\mathbb{P}_t(\vec{x} \to \vec{y})$ is given as a k-fold contour integral for *any* initial data $\vec{x}$.
2. Use certain *combinatorial summation identities* (following from the Plancherel theory) to compute $\mathbb{P}(x_m(t) \le x)$. Works only for special initial data: $\mathbb{Z}_+$ is packed, $\mathbb{Z}_-$ is empty (number of particles can be taken *infinite*).
   The answer is a sum of k-fold contour integrals over $k \ge m$.
3. Relate this sum over k to a *Fredholm determinant*

$$\det(1 - uK) = \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \oint .. \oint \det(K(z_i, z_j))_{i,j=1}^{k} dz_1 \dots dz_k$$

(integrands in $\mathbb{P}(x_m(t) \le x)$ are determinants by Cauchy determinantal formula)

Next steps towards Tracy–Widom fluctuations

1. For k-particle ASEP, $\mathbb{P}_t(\vec{x} \to \vec{y})$ is given as a k-fold contour integral for *any* initial data $\vec{x}$.
2. Use certain *combinatorial summation identities* (following from the Plancherel theory) to compute $\mathbb{P}(x_m(t) \le x)$. Works only for special initial data: $\mathbb{Z}_+$ is packed, $\mathbb{Z}_-$ is empty (number of particles can be taken *infinite*).
   The answer is a sum of k-fold contour integrals over $k \ge m$.
3. Relate this sum over k to a *Fredholm determinant*
   $$\det(1 - uK) = \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \oint .. \oint \det(K(z_i, z_j))_{i,j=1}^{k} dz_1 \dots dz_k$$
   (integrands in $\mathbb{P}(x_m(t) \le x)$ are determinants by Cauchy determinantal formula)
4. Analyze asymptotics of this Fredholm determinant, and get F_{GUE} in the limit. All boils down to dealing with K which is explicit.
   ($F_{GUE}(u)$ is itself a certain Fredholm determinant)

Last slide about ASEP: Key ingredients for Tracy–Widom fluctuations

1. Nice explicit eigenfunctions (by coordinate Bethe ansatz)
2. Plancherel theory (combinatorics of contour integrals)
3. Fredholm determinantal structure (for *special* initial data)
4. Asymptotics of Fredholm determinants (steepest descent)

What to do with other initial data? — open except for few other cases.

TASEP results and KPZ theory give predictions. In particular, the Tracy–Widom distribution F_{GOE} (corresponding to *real symmetric matrices*) should arise in the limit when the interface is initially "*flat*". **[Corwin's KPZ survey '11]**

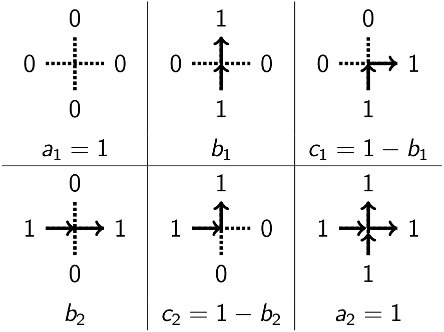
(next — vertex models as particle systems: a similar integrability structure)

⋆ Outline ⋆

1. Integrable stochastic particle systems
2. Bethe ansatz eigenfunctions of ASEP
3. Stochastic vertex models
   - Stochastic six-vertex model
   - Yang-Baxter relation
   - Stochastic higher spin vertex model

Stochastic six-vertex model

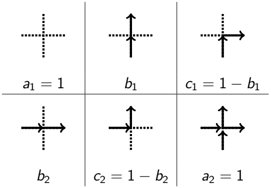
Six vertex model (**"square ice"**) — widely studied integrable lattice model **[book by Baxter]**, **[Reshetikhin '10]**



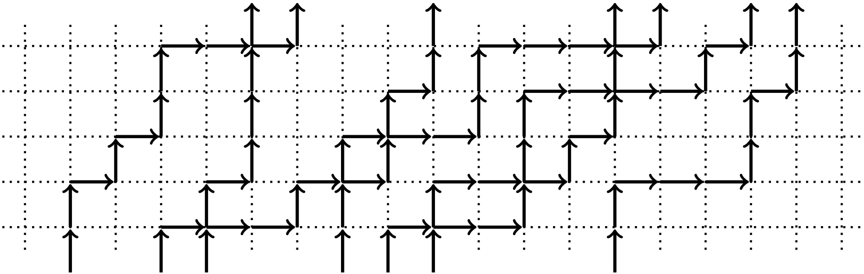
Configurations of **arrows** (**spins**) in a region on the plane. Vertices of 6 types. Weight of a configuration is the product of weights of all vertices.

A special choice of weights makes the behavior of arrows at each vertex *stochastic* **[Gwa–Spohn '92]**, **[Borodin–Corwin–Gorin '14]**

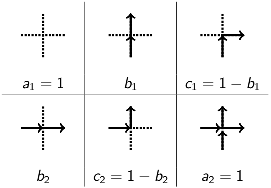
Stochastic six-vertex model



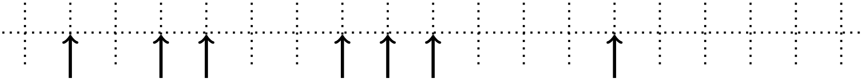
In each horizontal slice, the number of *vertical arrows* is *preserved*.
For finite number k of vertical arrows, the stochastic six-vertex model is well-defined in *infinite horizontal strip* because $a_1 = 1$.



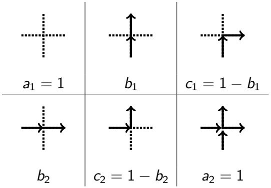
Stochastic six-vertex model: transfer matrix



For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



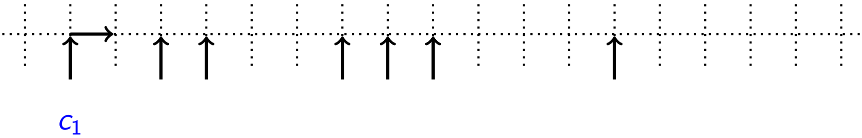
Stochastic six-vertex model: transfer matrix



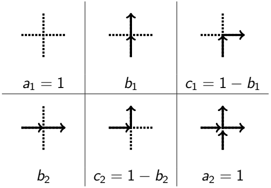
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.

Incoming arrows = input,

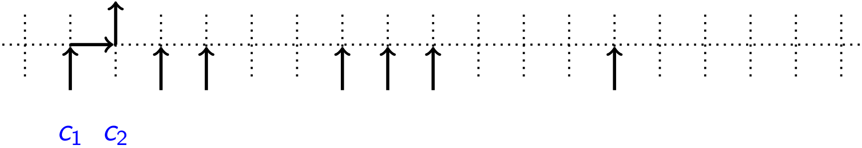
Outgoing arrows = output.



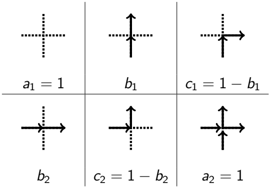
Stochastic six-vertex model: transfer matrix



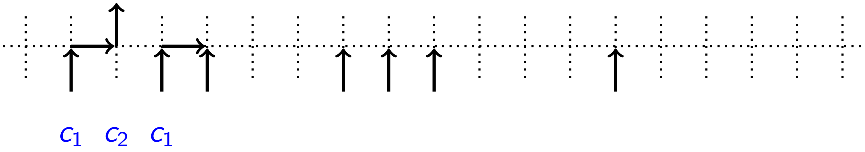
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



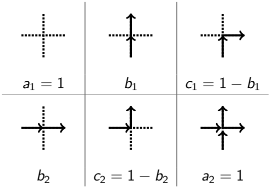
Stochastic six-vertex model: transfer matrix



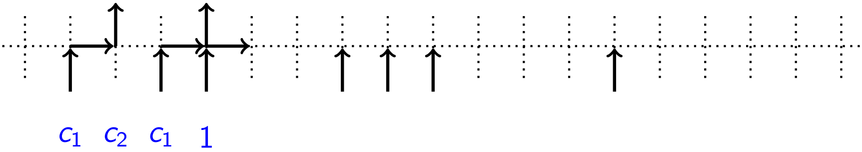
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



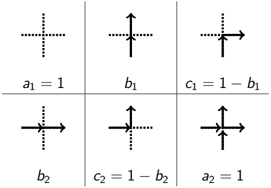
Stochastic six-vertex model: transfer matrix



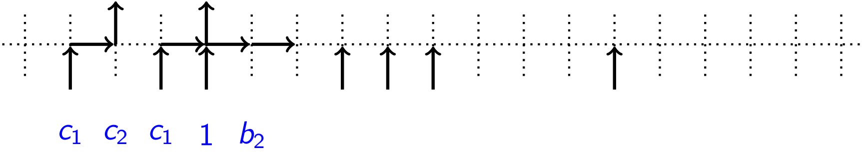
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



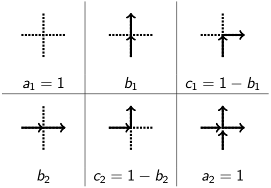
Stochastic six-vertex model: transfer matrix



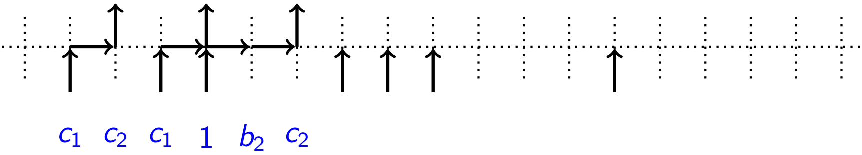
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



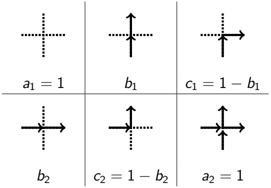
Stochastic six-vertex model: transfer matrix



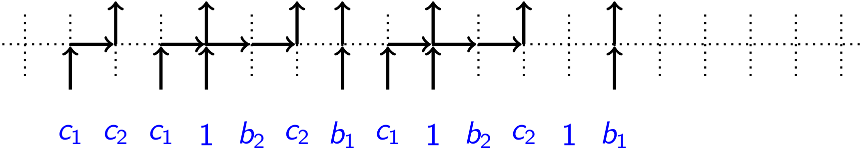
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



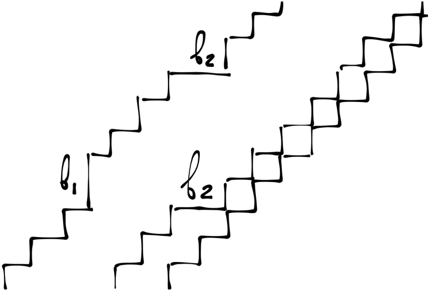
Stochastic six-vertex model: transfer matrix



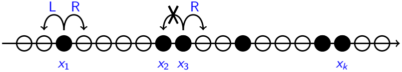
For k vertical spins, the *transfer matrix* $\mathcal{B}$ is a *local stochastic operator*, with left-to-right update.
Incoming arrows = input,
Outgoing arrows = output.



Stochastic six-vertex model: ASEP limit

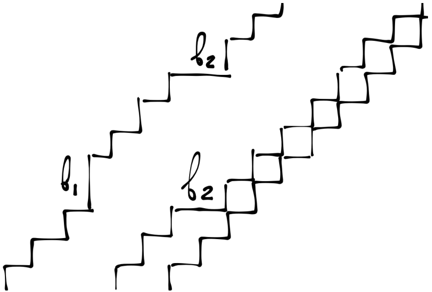


$$q = b_1/b_2$$

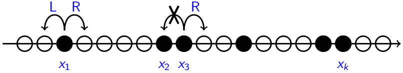


Let $b_1, b_2 \to 0$, subtract diagonal movement, rescale to continuous time $\Rightarrow$ get ASEP (*particles* $=$ *vertical arrows*).

Stochastic six-vertex model: ASEP limit



$q = b_1/b_2$



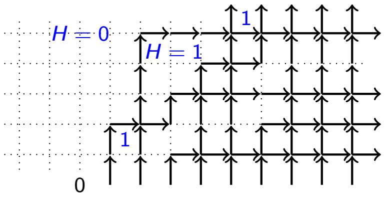
Let $b_1, b_2 \to 0$, subtract diagonal movement, rescale to continuous time $\Rightarrow$ get ASEP (*particles* = *vertical arrows*).

The transfer matrix of the k-particle six-vertex model has the same* eigenfunctions $\Psi^{ASEP}_{\vec{z}}(\vec{x})$, where $x_1 < \ldots < x_k$ are positions of the vertical spins.

* — up to $q \leftrightarrow q^{-1}$

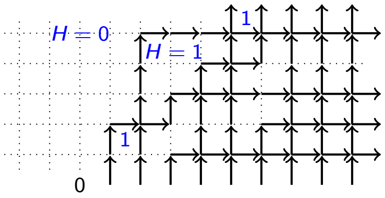
Additional free parameter $\Rightarrow$ *commuting transfer matrices*.

Stochastic six-vertex model with half domain wall



Half domain wall boundary conditions = packed spin configuration to the right of 0. Analogue of step initial data for ASEP.
$H(x, y) := \#$ vertical arrows to the left of (x, y)

Stochastic six-vertex model with half domain wall



Half domain wall boundary conditions = packed spin configuration to the right of 0. Analogue of step initial data for ASEP. $H(x, y) := \#$ vertical arrows to the left of (x, y)

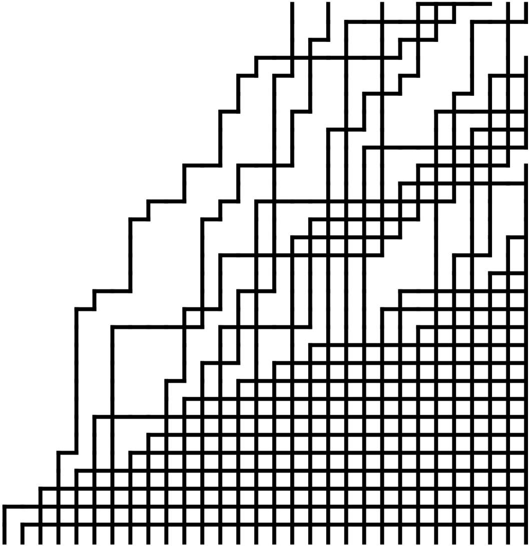
Theorem **[Borodin–Corwin–Gorin '14]**

For $0 < b_2 < b_1 < 1$, $\kappa := (1 - b_1)/(1 - b_2)$, as $L \to \infty$,

- $$H(Lx, Ly)/L \to \mathbf{H}(x, y) := \begin{cases} 0, & x/y < \kappa; \\ \left(\sqrt{y(1 - b_1)} - \sqrt{x(1 - b_2)}\right)^2, & \kappa < x/y < 1/\kappa; \\ x - y, & x/y > 1/\kappa; \end{cases}$$
- $$\mathbb{P}\left[\frac{\mathbf{H}(x, y)L - H(Lx, Ly)}{\sigma_{x,y} L^{1/3}} \le u\right] \to F_{GUE}(u).$$

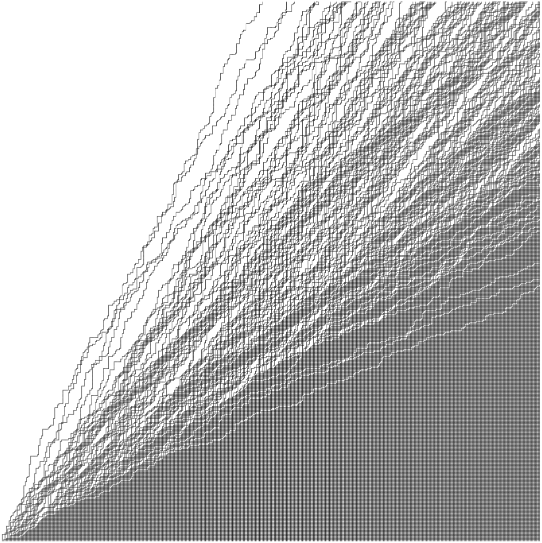
(proof is by methods similar to ASEP: Bethe ansatz, Fredholm determinants...)

Stochastic six-vertex model: Simulations



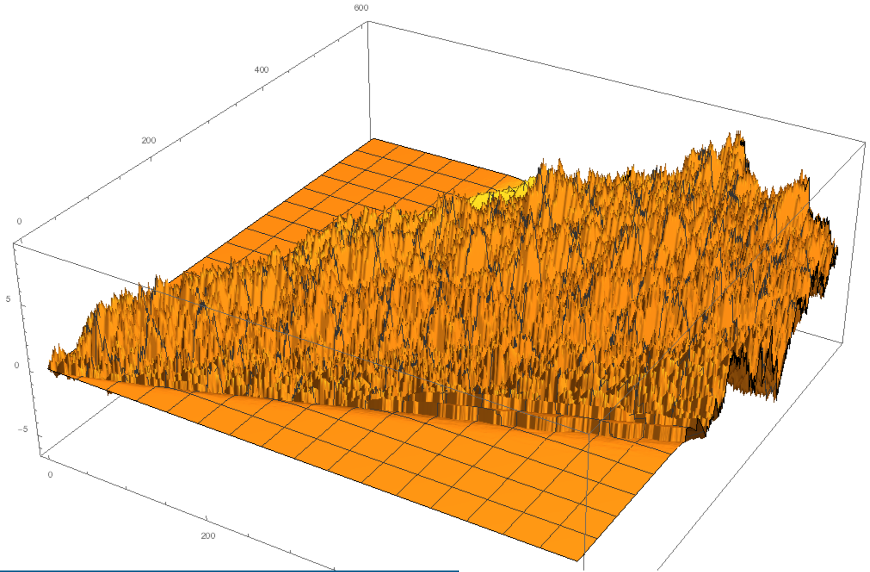
$b_1 = 2/3$ (“up”),
$b_2 = 1/3$ (“right”),
size 30

Stochastic six-vertex model: Simulations

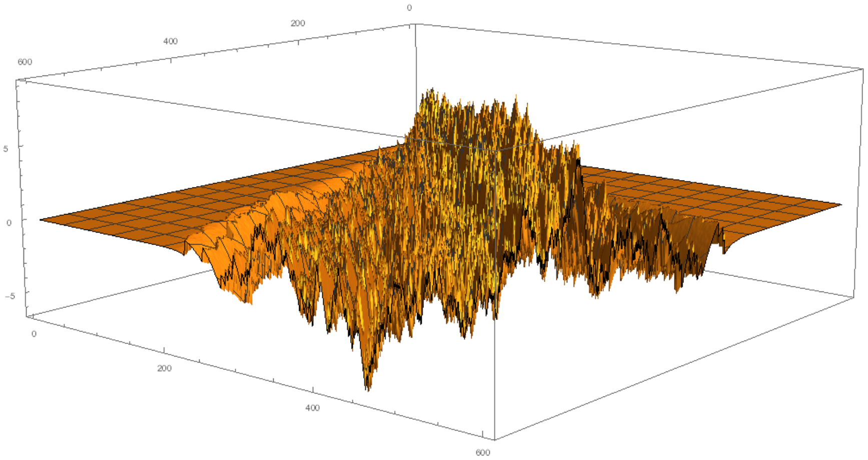


$b_1 = 2/3$ ("up"),
$b_2 = 1/3$ ("right"),
size 400

Stochastic six-vertex model: Fluctuations

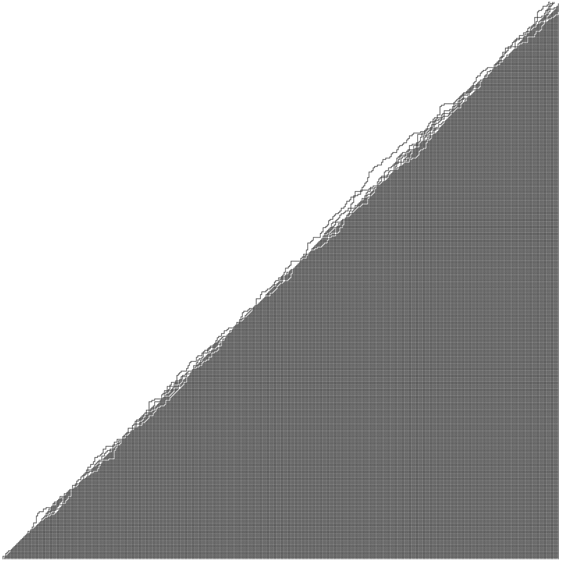


Stochastic six-vertex model: Fluctuations



(Global fluctuations do not seem to be described by a Gaussian Free Field)

Stochastic six-vertex model: Simulations



$b_1 = 1/3$ ("up"),
$b_2 = 1/2$ ("right"),
size 400

⋆ Outline ⋆

1. Integrable stochastic particle systems
2. Bethe ansatz eigenfunctions of ASEP
3. Stochastic vertex models
   - Stochastic six-vertex model
   - Yang-Baxter relation
   - Stochastic higher spin vertex model

Yang-Baxter relation for six-vertex model

Another parametrization of vertex weights, $s := 1/\sqrt{q}$:

$$\boxed{b_1 = \frac{1 - suq}{1 - su}, \qquad b_2 = \frac{-su + q^{-1}}{1 - su}, \qquad \begin{array}{l} 0 < q < 1, \quad u > \sqrt{q} \\ \text{or} \\ q > 1, \quad 0 < u < \sqrt{q} \end{array}}$$

(u — free parameter entering transfer matrix but not eigenfunctions)

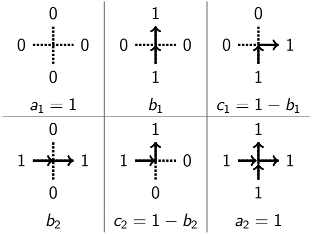
Yang-Baxter relation for six-vertex model

Another parametrization of vertex weights, $s := 1/\sqrt{q}$:

$$b_1 = \frac{1 - suq}{1 - su}, \qquad b_2 = \frac{-su + q^{-1}}{1 - su}, \qquad \begin{array}{l} 0 < q < 1, \quad u > \sqrt{q} \\ \text{or} \\ q > 1, \quad 0 < u < \sqrt{q} \end{array}$$

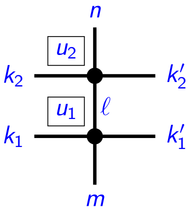
(u — free parameter entering transfer matrix but not eigenfunctions)

$$V_u := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b_2 & 1 - b_2 & 0 \\ 0 & 1 - b_1 & b_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



rows and columns of V_u correspond to $\mathbb{C}^2 \otimes \mathbb{C}^2$, i.e., to incoming / outgoing arrow configurations $00, 01, 10, 11$

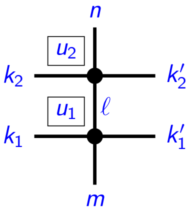
Yang-Baxter relation for six-vertex model



$W^{(m,n)}_{u_1,u_2}$, $m, n \in \{0, 1\}$ — 4×4 matrix corresponding to the weight of this configuration, from (k_1, k_2) to (k_1', k_2')

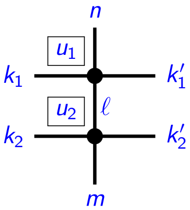
(ℓ is defined uniquely by $m, n, k_{1,2}, k_{1,2}'$)

Yang-Baxter relation for six-vertex model



$W^{(m,n)}_{u_1,u_2}$, $m,n \in \{0,1\}$ — 4×4 matrix corresponding to the weight of this configuration, from (k_1, k_2) to (k_1', k_2')

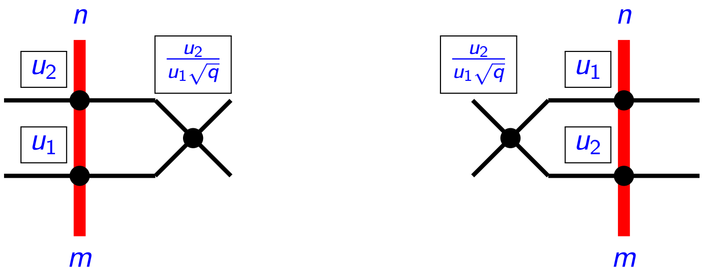
(ℓ is defined uniquely by $m, n, k_{1,2}, k_{1,2}'$)



$\widetilde{W}^{(m,n)}_{u_1,u_2}$, $m,n \in \{0,1\}$

Yang-Baxter relation for six-vertex model

$$W^{(m,n)}_{u_1,u_2}\big(V_{\frac{u_2}{u_1\sqrt{q}}}\big)^{transpose} = \big(V_{\frac{u_2}{u_1\sqrt{q}}}\big)^{transpose}\widetilde{W}^{(m,n)}_{u_1,u_2}, \qquad (u_1,u_2)\to\frac{u_2}{u_1\sqrt{q}}$$



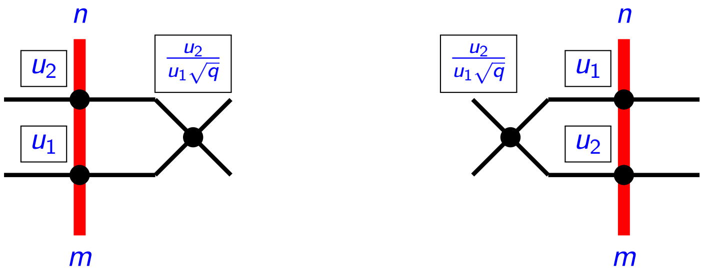
⋆ Outline ⋆

1. Integrable stochastic particle systems
2. Bethe ansatz eigenfunctions of ASEP
3. Stochastic vertex models
   - Stochastic six-vertex model
   - Yang-Baxter relation
   - Stochastic higher spin vertex model

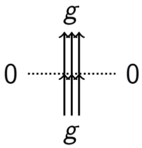
Yang-Baxter relation: solution with higher vertical spins

$$W^{(m,n)}_{u_1,u_2}\big(V_{\frac{u_2}{u_1\sqrt{q}}}\big)^{transpose} = \big(V_{\frac{u_2}{u_1\sqrt{q}}}\big)^{transpose}\widetilde{W}^{(m,n)}_{u_1,u_2}, \qquad m,n\in\mathbb{Z}_{\ge 0}$$

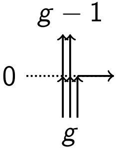
(but each matrix $W^{(m,n)}_{u_1,u_2}$ is still 4×4)



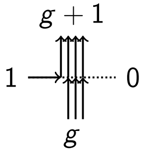
Stochastic higher spin vertex model



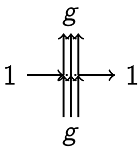
$$\frac{1 - suq^g}{1 - su}$$



$$\frac{-su(1 - q^g)}{1 - su}$$



$$\frac{1 - s^2 q^g}{1 - su}$$

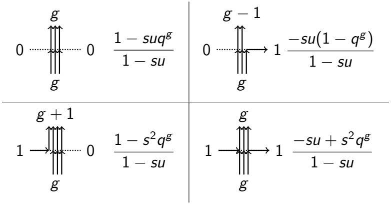


$$\frac{-su + s^2 q^g}{1 - su}$$

- $s = 1/\sqrt{q} \Rightarrow$ stochastic six-vertex model
- $s = 1/(\sqrt{q})^{\mathrm{I}} \Rightarrow$ finitely many vertical spins, $g \in \{0, 1, 2, \ldots, \mathrm{I}\}$
- s generic $\Rightarrow$ infinitely many vertical spins possible

[Mangazeev '14], **[Borodin '14]**, **[Corwin–P. '15]**, **[Borodin–P., in progress '15]**

q-TASEP degeneration

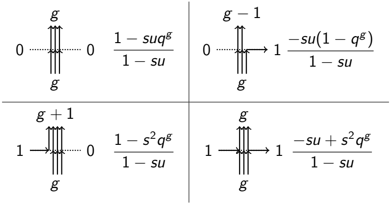


generic s

$s, u \to 0$, $s \ll u$,

continuous time scaling: speed up by $s|u|$

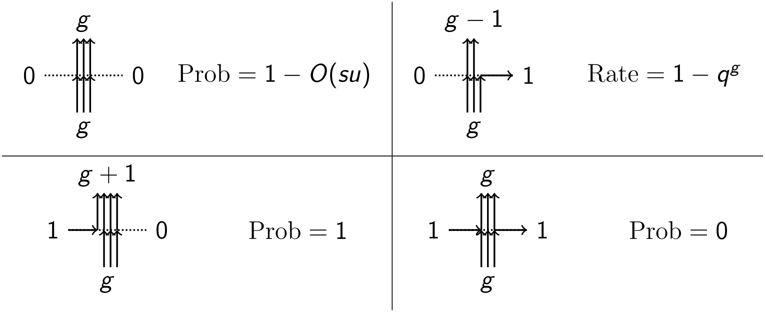
q-TASEP degeneration



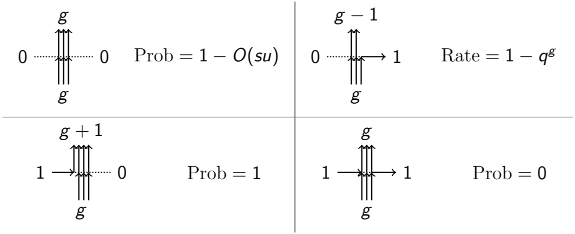
generic s

$s, u \to 0$, $s \ll u$,

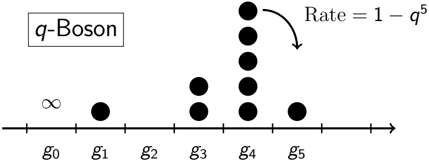
continuous time scaling: speed up by $s|u|$

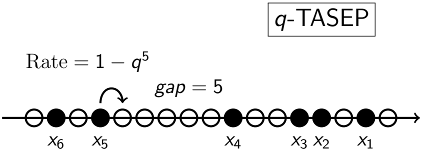


q-TASEP degeneration



Spins are *gaps* in another process (∞ spins at location 0).





[Bogoliubov–Bullough–Timonen '94], **[Bogoliubov–Izergin–Kitanine '98]**, **[Sasamoto–Wadati '98]**, **[Borodin–Corwin '11]**, **[Borodin–Corwin–Sasamoto '12]**, **[Ferrari–Veto '13]**, . . .

Eigenfunctions of the higher spin model

Let $\mathcal{B}^{u,qu}$ be the transfer matrix of the k-particle higher spin model.

Eigenfunctions of $\mathcal{B}^{u,qu}$ — e.g., [Borodin '14]

$$\Psi^{HS}_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \boxed{s^2} z_{\sigma(j)}} \right)^{x_j},$$

$\vec{x} = (x_1 \geq \ldots \geq x_k)$ — positions of k spins.

Eigenfunctions of the higher spin model

Let $\mathcal{B}^{u,qu}$ be the transfer matrix of the k-particle higher spin model.

Eigenfunctions of $\mathcal{B}^{u,qu}$ — e.g., [Borodin '14]

$$\Psi^{HS}_{\vec{z}}(\vec{x}) = \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \boxed{s^2} z_{\sigma(j)}} \right)^{x_j},$$

$\vec{x} = (x_1 \geq \ldots \geq x_k)$ — positions of k spins.

$$\mathcal{B}^{u,qu} \Psi^{HS}_{\vec{z}} = \prod_{i=1}^{k} \frac{1 - qu \cdot sz_i}{1 - u \cdot sz_i} \Psi^{HS}_{\vec{z}}$$

Coordinate Bethe ansatz derivation of $\Psi^{HS}_{\vec{z}}$ — [Povolotsky '13]

(operator $\mathcal{B}^{u,qu}$ is *not* equal to a free operator plus boundary conditions. But it is a *ratio of two such operators* — q-Hahn generators introduced by Povolotsky)

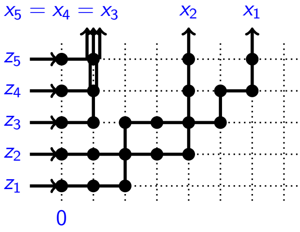
Plancherel theory — [Borodin–Corwin–P.–Sasamoto '14]

Eigenfunctions as partition functions

[Borodin '14]

$\Psi^{HS}_{\vec{z}}(\vec{x})$ is essentially a partition function of configurations of a higher spin vertex model.

(based on turning an algebraic Bethe ansatz expression for eigenfunctions into a coordinate one)

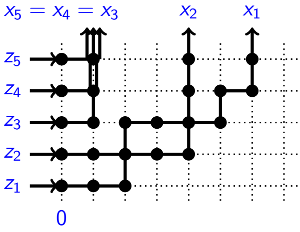


Eigenfunctions as partition functions

[Borodin '14]

$\Psi_{\vec{z}}^{HS}(\vec{x})$ is essentially a partition function of configurations of a higher spin vertex model.

(based on turning an algebraic Bethe ansatz expression for eigenfunctions into a coordinate one)

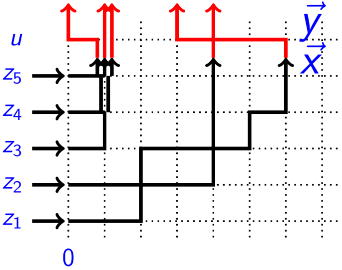


The action of operator $\mathcal{B}^{u,qu} \Leftrightarrow$ adding a top row to this configuration with horizontal arrows *reversed*; but without the incoming left arrow.

Use Yang-Baxter to commute this additional row all the way down. Each commutation spits out a factor $\frac{1-qu\cdot sz_i}{1-u\cdot sz_i}$.

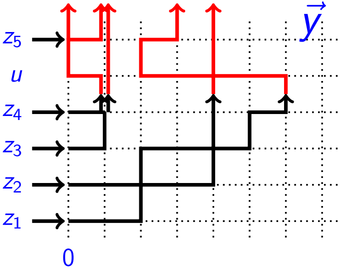
On finite lattice there would be *two terms* of the Yang-Baxter relation, but one of them *dies* in infinite volume limit.

Eigenfunctions as partition functions



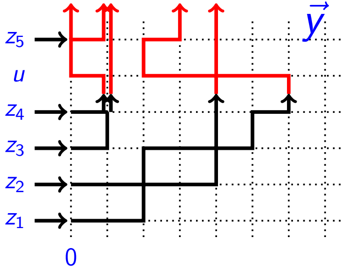
$$\left(\mathcal{B}^{u,qu}\Psi^{HS}_{\vec{z}}\right)(\vec{y}) = \sum_{\vec{x}} \mathcal{B}^{u,qu}(\vec{y},\vec{x})\Psi^{HS}_{\vec{z}}(\vec{x})\Psi^{HS}_{\vec{z}}(\vec{y})$$

Eigenfunctions as partition functions



$$\left(\mathcal{B}^{u,qu}\Psi^{HS}_{\vec{z}}\right)(\vec{y}) = \sum_{\vec{x}} \mathcal{B}^{u,qu}(\vec{y},\vec{x})\Psi^{HS}_{\vec{z}}(\vec{x})\Psi^{HS}_{\vec{z}}(\vec{y})$$

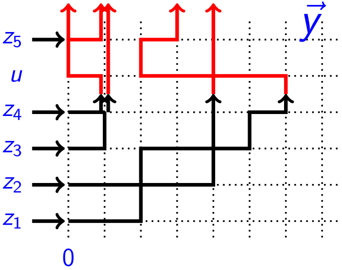
Eigenfunctions as partition functions



etc.

$$\left(\mathcal{B}^{u,qu}\Psi^{HS}_{\vec{z}}\right)(\vec{y}) = \sum_{\vec{x}} \mathcal{B}^{u,qu}(\vec{y},\vec{x})\Psi^{HS}_{\vec{z}}(\vec{x})\Psi^{HS}_{\vec{z}}(\vec{y})$$

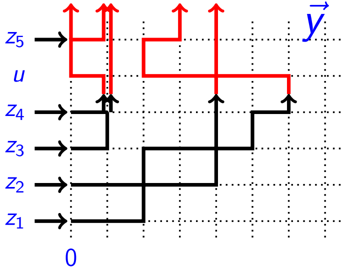
Eigenfunctions as partition functions



etc.

$$\left(\mathcal{B}^{u,qu}\Psi^{HS}_{\vec{z}}\right)(\vec{y}) = \sum_{\vec{x}} \mathcal{B}^{u,qu}(\vec{y},\vec{x})\Psi^{HS}_{\vec{z}}(\vec{x}) = \prod_{i=1}^{k} \frac{1-qu\cdot sz_i}{1-u\cdot sz_i}\Psi^{HS}_{\vec{z}}(\vec{y})$$

Eigenfunctions as partition functions



etc.

$$\left(\mathcal{B}^{u,qu}\Psi^{HS}_{\vec{z}}\right)(\vec{y}) = \sum_{\vec{x}} \mathcal{B}^{u,qu}(\vec{y},\vec{x})\Psi^{HS}_{\vec{z}}(\vec{x}) = \prod_{i=1}^{k} \frac{1-qu\cdot sz_i}{1-u\cdot sz_i}\Psi^{HS}_{\vec{z}}(\vec{y})$$

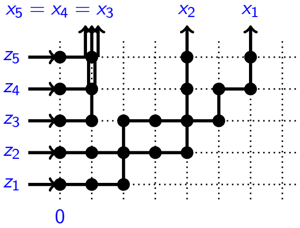
(This is Pieri rule for symmetric rational functions; there are also skew Cauchy identity and Cauchy identity — properties one would expect from symmetric polynomials)

$$\Psi^{HS}_{\vec{z}}(\vec{x}) = \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(A)}-qz_{\sigma(B)}}{z_{\sigma(A)}-z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1-z_{\sigma(j)}}{1-s^2 z_{\sigma(j)}}\right)^{x_j}$$

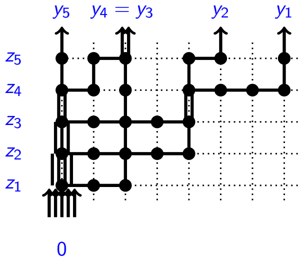
— generalize Hall-Littlewood polynomials

Eigenfunctions as partition functions

Moreover, there is also a Cauchy-type summation identity for the eigenfunctions. For it we need "dual" partition functions G:

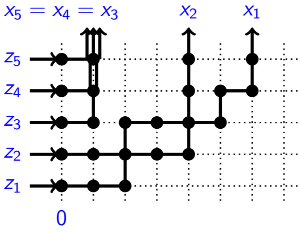


$\Psi^{HS}_{\vec{z}}(\vec{x}) =: F_{\vec{x}}(\vec{z})$

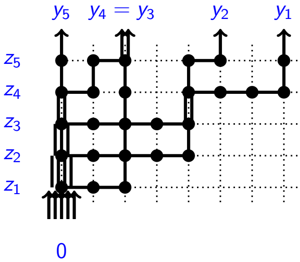


$G_{\vec{y}}(\vec{z})$

Eigenfunctions as partition functions



$\Psi^{HS}_{\vec{z}}(\vec{x}) =: F_{\vec{x}}(\vec{z})$



$G_{\vec{y}}(\vec{z})$

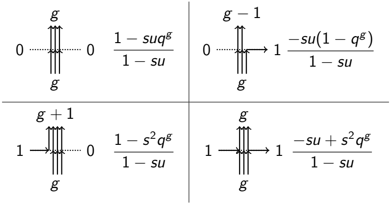
Cauchy identity **[Borodin '14]**

$$\sum_{\vec{x}} c(\vec{x}) F_{\vec{x}}(\vec{z}) G_{\vec{x}}(\vec{w}) = \prod_{i,j} \frac{1 - q z_i w_j}{1 - z_i w_j}$$

($c(\vec{x})$ is product of $(s^2;q)_{g_i}/(q;q)_{g_i}$ over "clusters" of $\vec{x}$)

(can use this identity to define probability distributions on "rainbows" of paths; $\vec{x}$ will be the configuration on the middle horizontal)

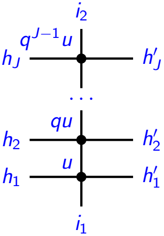
Fusion [Kirillov–Reshetikhin '87], [Corwin–P. '15]



3-parameter family of stochastic models

At most one horizontal arrow per edge

Let $\mathcal{B}^{u,q^J u} := \mathcal{B}^{u,qu}\mathcal{B}^{qu,q^2 u}\ldots\mathcal{B}^{q^{J-1}u,q^J u}$, eigenvalue $\displaystyle\prod_{i=1}^{k}\frac{1-q^J u\cdot sz_i}{1-u\cdot sz_i}$.

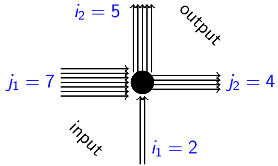


q-exchangeable distribution of $\vec{h}$
$\Rightarrow$ q-exchangeable distribution of $\vec{h}'$

Allows to collapse vertex weights by looking only at $h_1+\ldots+h_J$ and $h'_1+\ldots+h'_J$

Introduces fourth parameter $J\in\mathbb{Z}_{\geq 1}$

General J vertex weights



Weights expressed via general R matrix for $U_q(\widehat{\mathfrak{sl}_2})$; or basic hypergeometric functions; or classical q-Racah orthogonal polynomials **[Mangazeev '14]**, **[Corwin–P. '15]**.

$$V_u(i_1,j_1;i_2,j_2) = \mathbf{1}_{i_1+j_1=i_2+j_2} \frac{(-1)^{i_1} q^{\frac{1}{2}i_1(i_1+2j_1-1)} u^{i_1} s^{j_1+j_2-i_2} (us^{-1};q)_{j_2-i_1}}{(q;q)_{i_2}(su;q)_{i_2+j_2}(q^{\boxed{J}+1-j_1};q)_{j_1-j_2}}$$

$$\times\, {}_4\bar{\phi}_3 \left(\left. \begin{matrix} q^{-i_2}; q^{-i_1}, suq^{\boxed{J}}, qs/u \\ s^2, q^{1+j_2-i_1}, q^{\boxed{J}+1-i_2-j_2} \end{matrix} \right| q, q \right),$$

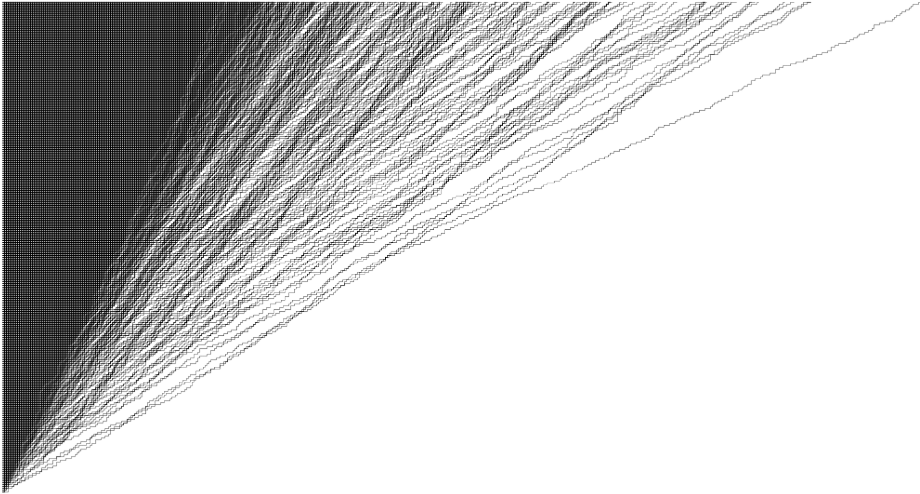
$$\text{where } {}_{r+1}\bar{\phi}_r \left(\left. \begin{matrix} q^{-n}; a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \right| q, z \right) = \sum_{k=0}^{n} z^k \frac{(q^{-n};q)_k}{(q;q)_k} \prod_{i=1}^{r} (a_i;q)_k (b_i q^k; q)_{n-k}$$

(treat q^J as an analytic parameter $\in \mathbb{C}$;

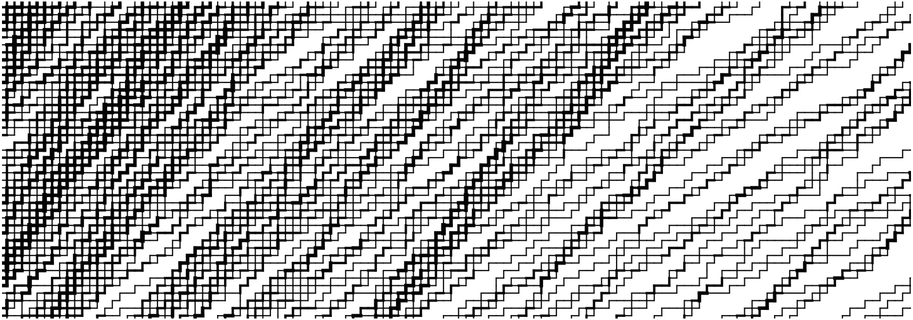
such general J weights lead to other interesting degenerations like the q-Hahn TASEP, ...)

Simulation of higher spin model

(J finite, half domain wall boundary conditions with all arrows incoming from the left)



Simulation of higher spin model

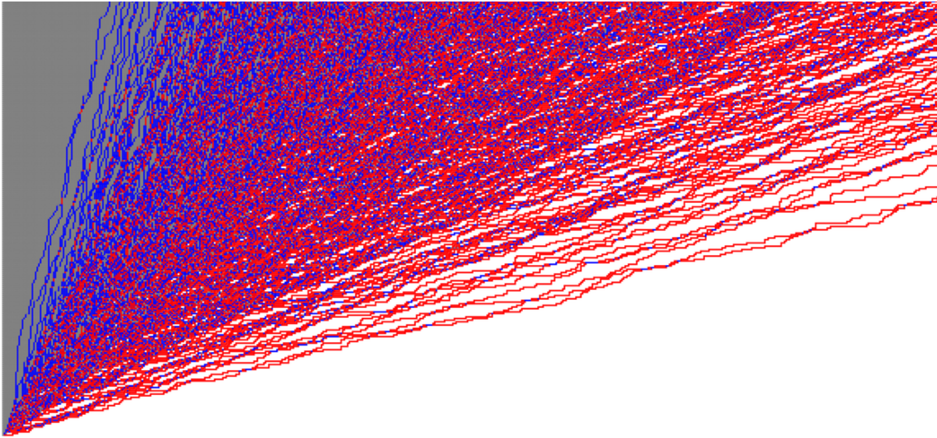


$J = 3$

$s = 1/q^2$

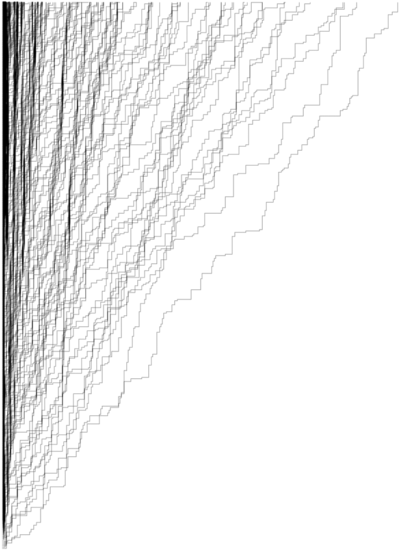
(so at most 4 vertical spins allowed)

Simulation of higher spin model



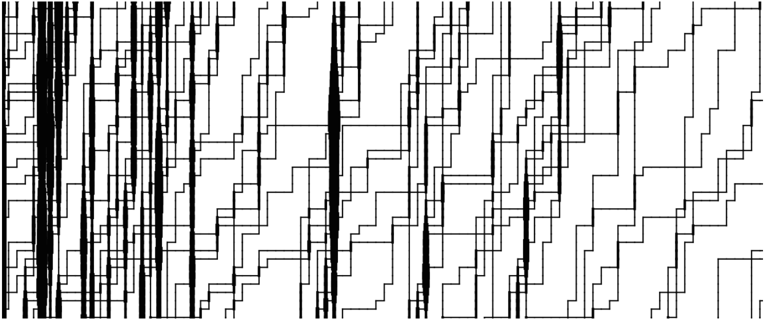
$J = 3$, $s = 1/q^{3/2}$

Simulation of higher spin model

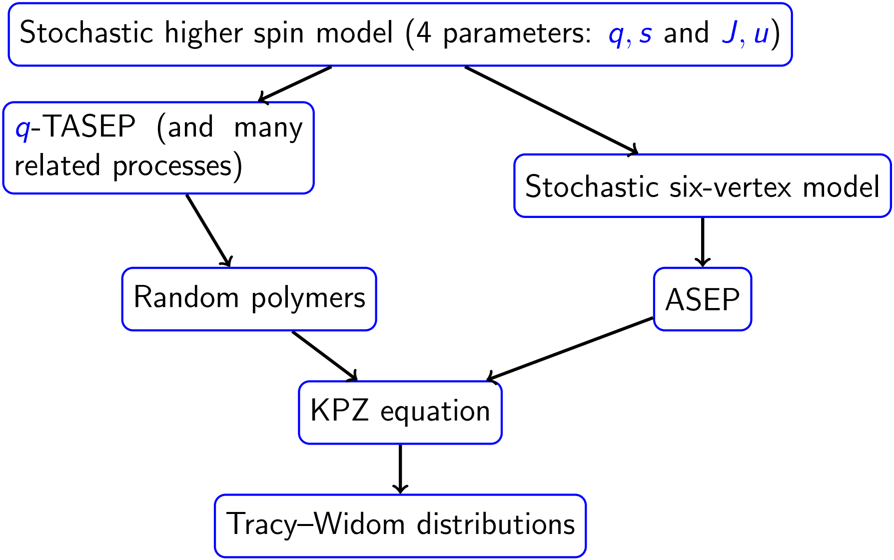


$J = 1$
s generic
(so any number of vertical spins allowed)

Simulation of higher spin model



Various degenerations of higher spin model



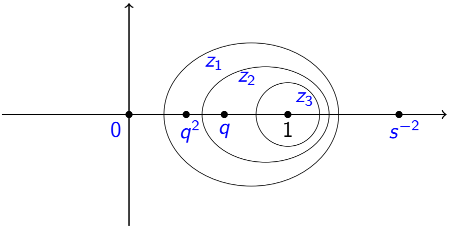
Moment formulas [Corwin–P. '15]

Initially $g_0 = \infty$ and $g_i = 0$ for $i > 0$, s generic. Under $\mathcal{B}^{u,q^J u}$, for any $\ell \geq 1$:

$$\mathbb{E}\left(q^{\ell \sum_{i \geq n} g_i(t)}\right) =$$

$$\frac{(-1)^\ell q^{\frac{\ell(\ell-1)}{2}}}{(2\pi \mathbf{i})^\ell} \oint \ldots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{\ell} \frac{(1 - s^2 z_j)^{\ell-1}}{(1 - z_j)^\ell} \left(\frac{1 - s q^J u z_j}{1 - s u z_j}\right)^t \frac{dz_j}{z_j}$$

The contours encircle 1 and not 0 or $1/s^2$, and z_i contains $q z_j$ for $i < j$



Formula obtained using *Markov self-duality* of the transfer matrix $\mathcal{B}^{u,q^J u}$: it quasi-commutes with the matrix $q^{\sum_{i>j} g_i y_j}$

Leads to Fredholm determinant for the q-Laplace transform of $q^{\sum_{i \geq n} g_i(t)}$

Tracy–Widom asymptotics in case $J = \infty$ in **[Veto '14]**

Moment formulas [Borodin–P., in progress '15]

In model with half domain wall boundary conditions (all arrows income from the left), initially $g_i = 0$ for $i > 0$. For any $\ell \geq 1$:

$$\mathbb{E}\left(q^{\ell \sum_{i \geq n} g_i(t)}\right) =$$

$$\frac{(-1)^\ell q^{\frac{\ell(\ell-1)}{2}}}{(2\pi \mathbf{i})^\ell} \oint .. \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{\ell} \left(\frac{1 - s^2 z_j}{1 - z_j}\right)^{\ell - 1} \left(\frac{1 - s q^J u z_j}{1 - s u z_j}\right)^t \frac{dz_j}{z_j}$$

- Contours are slightly more complicated.
- Formula obtained by studying "algebraic" properties of the Bethe ansatz eigenfunctions $\Psi^{HS}_{\vec{z}}(\vec{x})$ (linking this subject back to theory of symmetric polynomials). Does not involve Markov duality.
- Formula implies the previous one by taking different u_j's at different horizontals, plus a nice limit transition.
- This also leads to Fredholm determinants.

Summary

- There is a *4-parameter family of interacting particle systems* imported from exactly solvable lattice models of statistical mechanics
- Bethe ansatz produces *exact distribution formulas* (moment and Fredholm determinantal formulas) for this system, which lead to *asymptotics* for special initial data
- This particle system leads to symmetric rational functions generalizing the *Hall-Littlewood polynomials*: From Bethe ansatz to symmetric functions (representation theory? — **[Takeyama '14]**)
- The 4-parameter particle system generalizes to most (all?) known integrable interacting particle systems in the KPZ universality class (i.e., which have the *Tracy–Widom fluctuation behavior*)

(many open questions of analytic and algebraic nature)