# Fused RSOS as Higher Level Non-Unitary Minimal Cosets

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## Introduction: Lattice Models and CFT

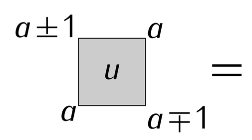
- We are interested in studying the off-critical fused Restricted Solid-on-Solid (RSOS) lattice models.
- The RSOS models are Yang-Baxter integrable and were first introduced by Forrester and Baxter in 1985.
- At criticality, the continuum scaling limit is described by the non-unitary minimal models in conformal field theory (CFT).
- We apply $n \times n$ fusion to the RSOS models and argue that, at criticality, these lattice models relate to higher level non-unitary coset models.

## Restricted Solid-on-Solid Lattice Models

- RSOS models are defined on a square lattice where the heights $a$ at each lattice site take a fixed number of values and heights $a, b$ at adjacent sites differ by one

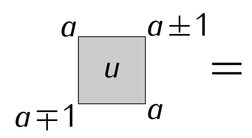
$$a = 1, \dots, m'-1, \qquad m' = 3, 4, 5 \dots, \qquad |a-b| = 1$$

- The Boltzmann weights are



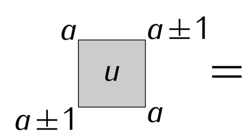
$$W\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\middle| u\right) = s(\lambda - u) \qquad s(u) = \vartheta_1(u,t)/\vartheta_1(\lambda,t)$$

elliptic theta functions



$$W\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\middle| u\right) = -\frac{s((a\pm1)\lambda)}{s(a\lambda)}s(u) \qquad 0<\lambda = \frac{(m'-m)\pi}{m'} < \pi$$

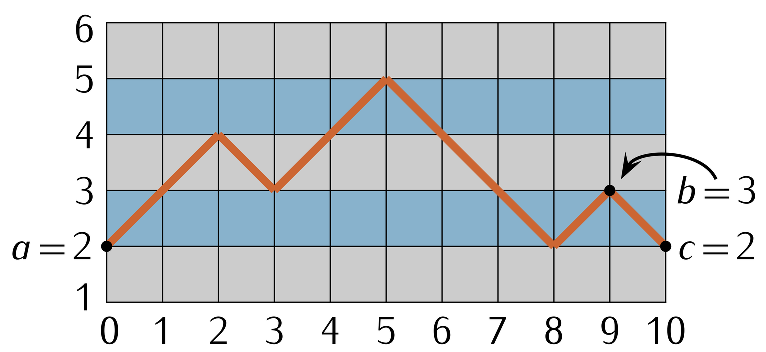
crossing parameter



$$W\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\middle| u\right) = s(a\lambda \pm u) \qquad 2 \le m < m'$$

coprime

- Lattice paths are sequences of $N+2$ heights $\sigma = \{a_0, a_1, \dots, a_N, a_{N+1}\}$ with boundary conditions $(a_0, a_N, a_{N+1}) = (a, b, c)$.




Path with boundary conditions $(a,b,c) = (2,3,2)$ in model with $(m,m') = (3,7)$.

- Local energy functions $H(a_i, a_{i+1}, a_{i+2})$ are calculated by taking the low temperature limit $x = \exp(-2\pi\lambda/\varepsilon) \to 0$, $u/\varepsilon$ fixed, of the Boltzmann weights

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) \sim \frac{g_a g_c}{g_b g_d} e^{-2\pi u H(d,a,b)/\varepsilon}\delta_{a,c}, \qquad g_a = \text{appropriate gauge}, \qquad t = e^{-\varepsilon}$$

- The corner transfer matrix local energy functions are

$$H(a\pm1, a, a\pm1) = \pm(h_{a\pm1} - h_a) \qquad H(a\pm1, a, a\mp1) = 1 - \tfrac{1}{2}(h_{a+1} - h_{a-1})$$

$$h_a = \lfloor a\lambda/\pi \rfloor, \qquad h_{a+1} - h_a = 0, 1 \text{ for } 0 < \lambda < \pi$$

- Local energies take the values $H(a_i, a_{i+1}, a_{i+2}) = 0, \frac{1}{2}, 1$.
- Global energy functions for a path $\sigma$ are defined to be

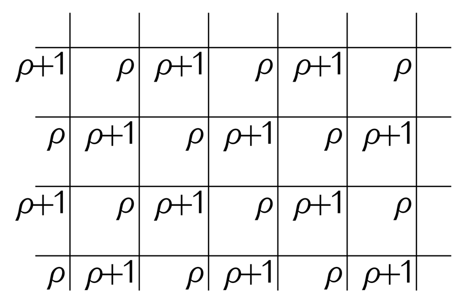
$$E(\sigma) = \sum_{j=1}^{N} j\, H(a_{j-1}, a_j, a_{j+1})$$

## Ground States for RSOS

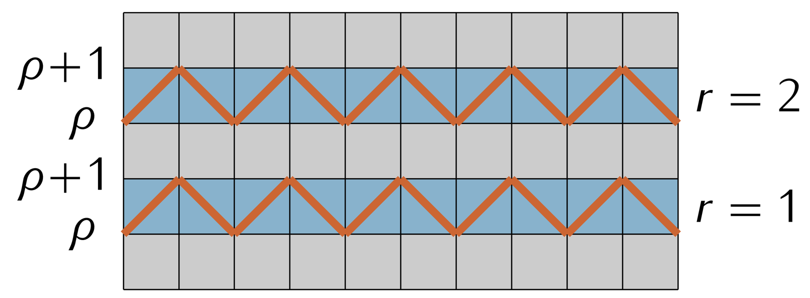
- Paths with zero energy oscillate between $\rho, \rho+1$

$$\rho = \left\lfloor \frac{rm'}{m} \right\rfloor, \qquad r = 1, \dots, m-1 \Leftrightarrow h_{\rho+1} = h_\rho$$

- In the path diagrams, we shade the bands between $\rho$ and $\rho+1$.




Lattice configuration




Path diagram

- For boundary conditions $(a,b,c)$, the lowest energy or ground state is the path which spends most number of steps oscillating between heights $\rho, \rho+1$.
- The 1-D configurational sums encode as a polynomial the energies of all paths $\sigma = \{a_0, \dots, a_N, a_{N+1}\}$ with fixed boundary conditions

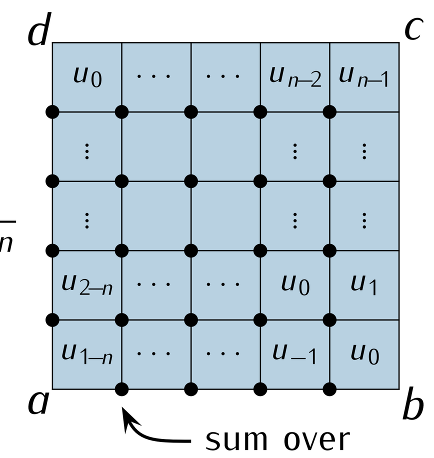
$$X^{(N)}_{a,b,c}(q) = \sum_{\{\sigma\}} q^{E(\sigma)}, \qquad q = e^{-4\pi\lambda/\varepsilon} \qquad a_0 = a,\ a_N = b,\ a_{N+1} = c$$

$$1 \le a_i \le m'-1$$

## $n \times n$ Fused RSOS

- The $n \times n$ fused RSOS models are defined by

$$W^{n,n}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = \frac{1}{\eta^{n,n}}$$

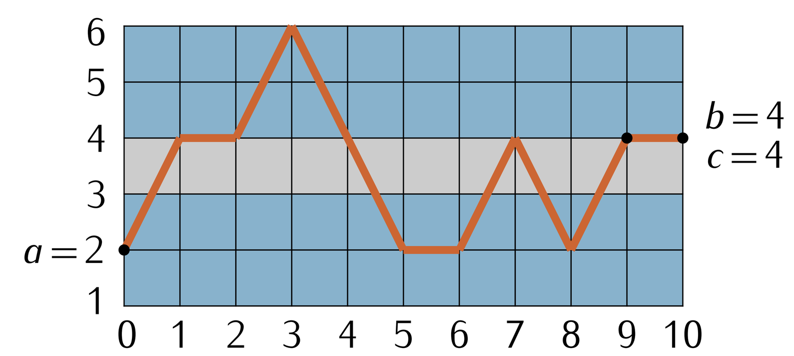



$$u_k = u + k\lambda, \qquad 0 < \lambda < \frac{\pi}{n}$$

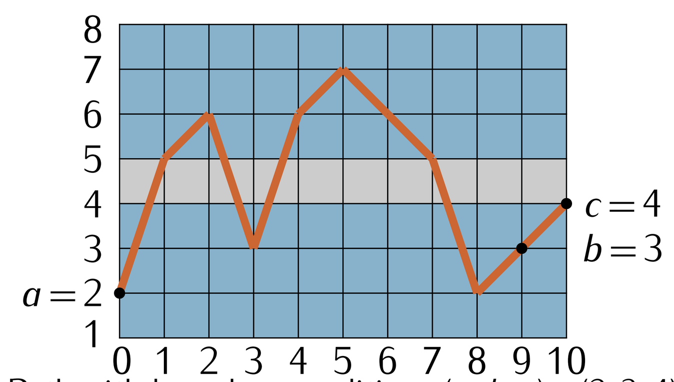
- On the lattice, adjacent heights $a, b = 1, \dots, m'-1$, $m' = 3, 4, \dots$ must satisfy

$$|a-b| = \begin{cases} 0, 2, 4, \dots, n, & n \text{ even} \\ 1, 3, 5, \dots, n, & n \text{ odd} \end{cases} \qquad n+2 \le a+b \le 2m'-n-2$$

- For example, typical paths for $2\times2$ and $3\times3$ fused RSOS are




Path with boundary conditions $(a,b,c) = (2,4,4)$ in $2\times2$ fused model with $(m,m') = (5,7)$.



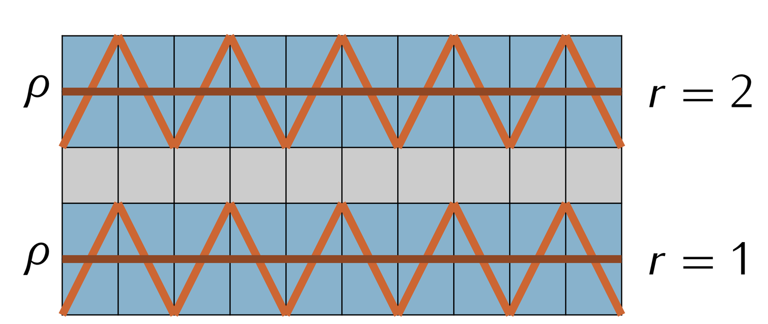

Path with boundary conditions $(a,b,c) = (2,3,4)$ in $3\times3$ fused model with $(m,m') = (7,9)$.

- Local energy functions are calculated the same way as in the $1\times1$ case.
- We have done this for $n = 2, 3$ and written them as differences of $h_a$ functions, which restricts the number of values they take.

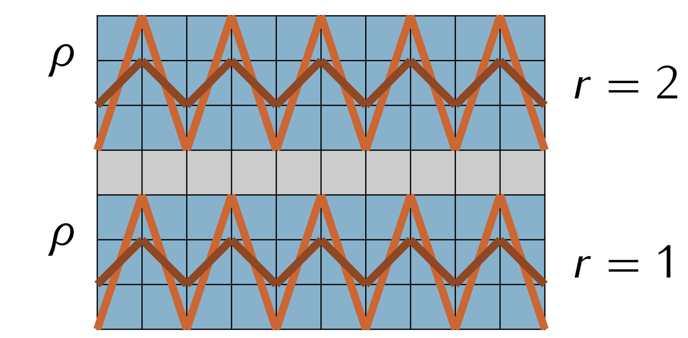
## Ground States for $n \times n$ Fused RSOS

- The lowest energy paths either oscillate within or step flat through the middle of $n$ adjacent shaded bands. These shaded $n$-bands occur when

$$h_a = h_{a+1} = \dots = h_{a+n}$$




Zero energy paths for $2\times2$ fusion of model with $(m,m') = (5,7)$.




Zero energy paths for $3\times3$ fusion of model with $(m,m') = (7,9)$.

- There are always exactly $M-1 = nm-(m-1)m'-1$ shaded $n$-bands.
- Each shaded $n$-band is said to be at height $\rho$, for $r = 1, \dots, M-1$. The values $\rho$ takes for any particular model can be found using the formula from the elementary RSOS case.

## Cosets in Conformal Field Theory

- The non-unitary minimal models $\mathcal{M}(m,m')$ are related to the RSOS models at criticality.
- The boundary conditions in the RSOS models correspond to the Kac labels $r, s$ in the minimal models

$$a = s = 1 \dots, m'-1, \qquad \rho = \lfloor rm'/m \rfloor, \qquad r = 1, \dots, m-1$$

- The 1-D configurational sums are finitised fermionic Virasoro characters.
- The conformal field theory related to the $n\times n$ fused RSOS models is the level-$n$ non-unitary coset model

$$\text{COSET}(k,n): \ \frac{(A_1^{(1)})_k \oplus (A_1^{(1)})_n}{(A_1^{(1)})_{k+n}}, \qquad k = \frac{nM}{M'-M} - 2$$

$$(M, M') = (nm-(n-1)m', m'), \qquad 0 < \lambda = \frac{(m'-m)\pi}{m'} < \frac{\pi}{n}$$

- The formulae for the conformal data are known, and the boundary conditions correspond to the Kac labels, as in the $1\times1$ case. The difference is the extra parameter $\ell$, labelling the possible widths of paths in the ground state bands.
- For $n = 2, 3$, we have checked for various models that the 1-D configurational sums are the finitised fermionic branching functions.
- This confirms that in the continuum scaling limit

$$\boxed{\text{RSOS}(m,m';n) \to \text{COSET}\left(\frac{m'(n+1)-m(n+2)}{m-m'}, n\right)}$$

- The coset construction and the above relationship between $(M,M')$ and $(m,m')$ give the central charges for these models

$$c^{m,m';n} = \frac{3n}{n+2}\left(1 - \frac{2(n+2)(m'-m)^2}{m'(mn-m'(n-1))}\right)$$