On the Integrability of B-type KdV Equations

Jianqin Mei
School of Mathematical Sciences
Dalian University of Technology
meijq@dlut.edu.cn

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What is Integrability

How about integrable systems with infinite degrees of freedom?

- Lax integrability : Lax pair
- Painlevé integrability: movable singularities=poles:
- Liouville integrability: infinitely many conserved quantities
- Bilinear integrability : Bilinear form
- Symmetry integrability: infinitely many symmetries
- existence of Backlund transformation, Bi-hamiltonian structure, multi-soliton solutions,...
- C integrability, Darboux integrability,...

Main Question

- How to construct integrable systems?
  - Pseudodifferential operator
  - Lie algebra (Adler,Zakharov-Shabat, Ablowitz-Kaup-Newell-Segur, Wilson, Drinfeld-Soklov and Terng-Uhlenbeck)
- Is a given system of PDE integrable in the sense of soliton theory?

Lie algebra splitting method[Terng, 06',11']

Lie algebra splitting. Let L be a formal Lie group with Lie algebra $\mathcal{L}$. A pair of Lie subalgebras of $\mathcal{L}$, $(\mathcal{L}_+, \mathcal{L}_-)$ is called a splitting of $\mathcal{L}$, if $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$ as the direct sum of linear subspaces.
Vacuum sequence. Let $(\mathcal{L}_+, \mathcal{L}_-)$ be a splitting of $\mathcal{L}$. A vacuum sequence of the splitting is a sequence of elements $J = \{J_i | i \geq 1\}$ in $\mathcal{L}_+$ satisfying
(1) $J_i's$ are linearly independent and generates a maximal abelian subalgebra of $\mathcal{L}_+$.
(2) J_j lies in the polynomial subalgebra generated by J_1.

Lie algebra splitting

For $\mathcal{G} = sl(4, C)$, and

$$\mathcal{L}_+^B = \{\sum_{j\geq 0} A_j \lambda^j | A_j \in sl(4, C)\},$$

$$\mathcal{L}_-^B = \{B((A_1)_+) + \sum_{j<0} A_j \lambda^j | A_j \in sl(4, C)\},$$

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We find that there exist five types of linear operators $B : \mathcal{N}_+ \to \mathcal{N}_-$ which make $(\mathcal{L}_+^B, \mathcal{L}_-^B)$ to be a Lie algebra splitting and give the corresponding affine B-type KdV hierarchies[Mei,12'].

B-type KdV hierarchy

If

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1-k}{k+1} & 0 & 0 & 1 & 0 & \frac{k-1}{k+1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

then $(\mathcal{L}_+^B, \mathcal{L}_-^B)$ becomes a Lie algebra splitting.
Let e_{ij} be the ij-th elementary matrix, $a = e_{41}$, $b = e_{12} + e_{23} + e_{34}$, and $J = az + b$.
From the Lie algebra splitting $(\mathcal{L}_+^B, \mathcal{L}_-^B)$ and vacuum sequence $J = \{J^i | i \geq 1\}$, the B-type KdV hierarchy can be constructed.

B-type KdV Equation

Let $f \in \mathcal{L}_-^B$, and $u_f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 \\ v & 0 & 0 & 0 \\ w & kv & u & 0 \end{pmatrix}$, we have

$$Q(u_f) = MJM^{-1} = a\lambda + \sum_{i\leq 0} Q_i \lambda^i.$$

For example, the flow generated by J^3 is as follows,

$$\begin{cases} u_t = (k+1)v_x \\ v_t = -\dfrac{1}{k+1}(kv_{xx} - v_{xx} - 2w_x) \\ w_t = -\dfrac{k^2+1}{k+1}v_{xxx} + (k+1)(vu_x + 2uv_x) + \dfrac{k-1}{k+1}w_{xx}. \end{cases} \tag{1.1}$$

Lax Integrability

According to the process of constructing KdV hierarchy, we conclude that (1.1) has a $sl(4)$-valued Lax pair,

$$E_x = UE, \qquad U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u & 0 & 1 & 0 \\ v & 0 & 0 & 1 \\ w & kv & u & 0 \end{pmatrix} \tag{2.1.1a}$$

$$E_t = VE, V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ v & 0 & 0 & 1 \\ \lambda - kv_x + w & kv + v & 0 & 0 \\ P & \lambda + v_x + w & 0 \end{pmatrix} \tag{2.1.1b}$$

where $P = (k+1)uv - \frac{1}{k+1}[(k^2+1)v_{xx} + (k-1)w_x]$.

Tau function

Theorem 2.1 [Terng,14']. Let $\mathcal{L}_\pm \in \mathcal{L}$, $(\mathcal{L}_+, \mathcal{L}_-)$ be a splitting, $J = \{J_j | j \geq 1\}$ a vacuum sequence, ω a 2-cocycle on $\mathcal{L}$ compatible with the splitting, and $V(t) = \exp(\Sigma_{j=1}^N t_j J_j)$ the vacuum frame. Let $f \in \mathcal{L}_-$, and

$$V(t)f^{-1} = M^{-1}(t)E(t) \tag{2.1.3}$$

with $M(t) \in \mathcal{L}_-$ and $E(t) \in \mathcal{L}_+$. Then
(1)$(\ln \tau_f)_{t_j} = \langle J_j, M^{-1}\partial_\lambda M \rangle_{-1} = \langle MJ_jM^{-1}, (\partial_\lambda M)M^{-1} \rangle_{-1}$,
(2)$(\ln \tau_f)_{t_1 t_j} = \langle MJ_jM^{-1}, \partial_\lambda J_1 \rangle_{-1}$.

Tau function

By Theorem 2.1 , we have $(\ln \tau_f)_{t_1 t_j} = tr(aQ_j)$. According to the expression of Q_1 , we can give explicit formulas of $(\ln \tau_f)_{t_1 t_1}$ in terms of u_f for the B-type KdV hierarchy and

$$(\ln \tau_f)_{t_1 t_1} = tr(aQ_1) = -\frac{1}{2}u. \tag{2.1.4}$$

Bäcklund Transformation I[Mei,12']

If $\{u_0, v_0, w_0\}$ is a solution of Eqs.(1.1), then

$$u = \frac{2\phi_x^2 - 2\phi_{xx}\phi}{\phi^2} + u_0, v = \frac{2}{k+1}\frac{\phi_x\phi_t - \phi_{xt}\phi}{\phi^2} + v_0,$$
$$w = \frac{(k-1)(-2\phi_x^2\phi_t + 2\phi\phi_x\phi_{xt} + \phi\phi_t\phi_{xx} - \phi_{xxt}\phi^2)}{(k+1)\phi^3} + \frac{\phi_t^2 - \phi\phi_{tt}}{\phi^2} + w_0 \tag{2.2.1}$$

is another new solution of Eqs.(1.1), where ϕ satisfies

$$\begin{cases} -\phi_{xxxxt} - \phi_{ttt} + 4u_0\phi_{xxt} + 2(k+1)v_0\phi_{xxx} + 2u_{0x}\phi_{xt} + 4(k+1)v_{0x}\phi_{xx} = 0, \\ \phi_{xxxx}\phi_t - 2\phi_{xx}\phi_{xxt} + 4\phi_x\phi_{xxxt} + 3\phi_t\phi_{tt} - 8u_0\phi_x\phi_{xt} - 4u_0\phi_{xx}\phi_t \\ -6(k+1)v_0\phi_x\phi_{xx} - 2u_{0x}\phi_x\phi_t - 4(k+1)v_{0x}\phi_x^2 = 0, \\ 2(k+1)v_0\phi_x^3 + 2\phi_x^2(2u_0\phi_t - \phi_{xxt}) + 2\phi_x(\phi_{xx}\phi_{xt} - \phi_{xxx}\phi_t) + \phi_t(\phi_{xx}^2 - \phi_t^2) = 0. \end{cases}$$

Bäcklund Transformation II

Theorem 2.2 Let $(\mathcal{L}_+^B, \mathcal{L}_-^B)$ be a B-type KdV splitting of $\mathcal{L}(sl(n, C))$ with B. Given constants m_{11}, m_{21}, m_{31} and $m_{32} \in \mathbb{C}$, let

$$W(x,t) = (w_0(x,t), w_1(x,t), w_2(x,t), w_3(x,t))$$

be the solution of the system of ODE equations

$$\begin{cases} W^T(x,t)_x = E(x,t)_x E^{-1}(x,t) W^T(x,t) \\ W^T(x,t)_t = E(x,t)_t E^{-1}(x,t) W^T(x,t) \\ W(0,0) = (1, -m_{11}, -m_{21} + m_{11}^2, (m_{32} + m_{21})m_{11} - m_{11}^3 - m_{31}) \end{cases}$$

Bäcklund Transformation II[Mei,15']

$$a_{11}(x,t) = -\frac{w_1}{w_0}, a_{21}(x,t) = \frac{w_1^2}{w_0^2} - \frac{w_2}{w_0}$$
$$a_{31}(x,t) = -a_{32}\frac{w_1}{w_0} + \frac{w_1 w_2}{w_0^2} - \frac{w_3}{w_0}$$
$$a_{32}(x,t) = 0, \qquad a_{43}(x,t) = a_{21}, \quad a_{42}(x,t) = \frac{k-1}{k+1}a_{31} + \frac{k^2+1}{k+1}v,$$

then

$$\tilde{u_1} = 2a_{21} - u_1, \qquad \tilde{u_2} = \frac{k-1}{k+1}u_2 + \frac{2}{k+1}a_{31}. \tag{2.2.2}$$

$$\tilde{u_3} = \frac{2k}{k+1}a_{11}^4 - \frac{4k}{k+1}a_{21}a_{11}^2 + \frac{2a_{11}}{k+1}[(k-1)a_{31} + (k^2+1)u_2]$$
$$+ \frac{2k}{k+1}a_{21}^2 + \frac{k^2+1}{k+1}u_{2,x} - \frac{k-1}{k+1}u_3 - \frac{2kk1}{k+1}.$$

is a new solution of the j-th flow of the B-type KdV hierarchy.

Bilinear Integrability

Theorem 2.3[Mei,15'] Under the transformation

$$u = -2(\ln\phi)_{xx}, v = -\frac{2}{k+1}(\ln\phi)_{xt},$$
$$w = -(\ln\phi)_{tt} - \frac{k-1}{k+1}(\ln\phi)_{xxt},$$

Eqs.(1.1) can be bilinearized into

$$\begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_x D_s)\phi\cdot\phi = 0, \\ (\frac{1}{3}D_x^3 D_t - \frac{1}{2}D_s D_t)\phi\cdot\phi = 0, \end{cases} \tag{2.3}$$

where s is an auxiliary variable.

Soliton Solution

Expand ϕ in the power series of a small parameter ε as follows

$$\phi = 1 + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + \cdots.$$

Substituting it into (2.3), we have

$$\varepsilon : \begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_x D_s)(\phi^{(1)}\cdot 1 + 1\cdot\phi^{(1)}) = 0, \\ (\frac{1}{3}D_x^3 D_t - \frac{1}{2}D_s D_t)(\phi^{(1)}\cdot 1 + 1\cdot\phi^{(1)}) = 0, \end{cases}$$

$$\varepsilon^2 : \begin{cases} (\frac{1}{3}D_x^4 + D_t^2 + D_x D_s)(\phi^{(2)}\cdot 1 + \phi^{(1)}\cdot\phi^{(1)} + 1\cdot\phi^{(2)}) = 0, \\ (\frac{1}{3}D_x^3 D_t - \frac{1}{2}D_s D_t)(\phi^{(2)}\cdot 1 + \phi^{(1)}\cdot\phi^{(1)} + 1\cdot\phi^{(2)}) = 0, \end{cases}$$

$$\varepsilon^3 : \ldots$$
$$\vdots$$

Soliton Solution

We can easily have the N-soliton solution ,

$$\begin{cases} u = -2(\ln\phi)_{xx}, \\ v = -\dfrac{2}{k+1}(\ln\phi)_{xt}, \\ w = -(\ln f)_{tt} - \dfrac{k-1}{k+1}(\ln\phi)_{xxt}, \\ \phi = \displaystyle\sum_{\mu=0,1}\exp(\sum_{i=1}^{N}\mu_i\eta_i + \sum_{1\le i<j}^{N}A_{ij}\mu_i\mu_j), \end{cases} \tag{2.4}$$

where $\eta_i = \alpha_i x + \beta_i t + \gamma_i s + \eta_0^{(i)}, \gamma_i = \frac{2}{3}\alpha_i^3, \quad \alpha_i^4 + \beta_i^2 = 0$, $e^{A_{ij}} = \frac{(\alpha_i-\alpha_j)^2}{\alpha_i^2+\alpha_j^2}, i,j = 1,2,\cdots,N$, the notation $\sum\limits_{\mu=0,1}$ means the sum of all possible combinations of $\mu_1 = 0,1, \ \mu_2 = 0,1, \ \ldots, \ \mu_n = 0,1$.

Riemann theta Function Solution

In fact, we have the following general bilinear form of Eqs.(1.1),

$$\begin{aligned} \mathfrak{L}_1(D_x,D_t,D_s)\phi\cdot\phi &= (\frac{1}{3}D_x^4 + D_t^2 + D_xD_s + c_1)\phi\cdot\phi = 0, \\ \mathfrak{L}_2(D_x,D_t,D_s)\phi\cdot\phi &= (\frac{1}{3}D_x^3D_t - \frac{1}{2}D_sD_t + c_2)\phi\cdot\phi = 0. \end{aligned} \tag{2.5.1}$$

where $c_1 = c_1(x,s), c_2 = c_2(t,s)$ are constants of integration.
To investigate the following Riemann theta function with $N = 1$

$$\phi = \vartheta(\xi,\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi + \pi n^2\tau} \tag{2.5.2}$$

where the phase variable $\xi = \alpha x + \omega t + \gamma s + \delta_0$ and the parameter $\tau < 0$.

Riemann theta Function Solution

Substituting (2.5.2) into (2.5.1) yields an algebraic system, and solving it we can have

$$\omega^2 = \frac{a_1b_2 - a_2b_1}{2ab_1 - 2a_1b}\alpha^4,$$

$$c_1 = \frac{ab_2 - ba_2}{ab_1 - a_1b}\alpha^4, c_2 = -\frac{ab_2 - ba_2}{ab_1 - a_1b}\alpha^3\omega.$$

And the B-type KdV equation has Riemann-theta function 1-periodic solutions.

Riemann theta Function Solution

$$\begin{cases} u = -2(\ln\vartheta(\xi,\tau))_{xx}, \\ v = -\dfrac{2}{k+1}(\ln\vartheta(\xi,\tau))_{xt}, \\ w = -(\ln\vartheta(\xi,\tau))_{tt} - \dfrac{k-1}{k+1}(\ln\vartheta(\xi,\tau))_{xxt}, \end{cases} \tag{2.5.3}$$

where $\xi = \alpha x \pm \sqrt{\frac{a_1b_2-a_2b_1}{2ab_1-2a_1b}}\alpha^2 t + \epsilon_0, \ \epsilon_0 = \gamma s + \delta_0$ and

$$\begin{gathered} a = 8\pi^2\sum_{n=-\infty}^{\infty} n^2\lambda^{2n^2}, b = 2\pi^2\sum_{n=-\infty}^{\infty}(2n-1)^2\lambda^{2n^2-2n+1}, \\ a_1 = \sum_{n=-\infty}^{\infty}\lambda^{2n^2}, b_1 = \sum_{n=-\infty}^{\infty}\lambda^{2n^2-2n+1}, \lambda = e^{\pi\tau} \\ a_2 = \frac{256}{3}\pi^4\sum_{n=-\infty}^{\infty} n^4\lambda^{2n^2}, b_2 = \frac{16}{3}\pi^4\sum_{n=-\infty}^{\infty}(2n-1)^4\lambda^{2n^2-2n+1}. \end{gathered} \tag{2.5.4}$$

Conclusion

Based on Lie algebra splitting method, we have constructed unusual B-type KdV hierarchies and investigate several kinds of integrability,

- Lax integrability
- Backlund transformation
- Bilinear integrability
- N-soliton solutions
- Riemann-theta function solution.

References

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