

Exact results from the replica Bethe ansatz from KPZ growth and random directed polymers

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with : Pasquale Calabrese (Univ. Pise, SISSA)

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most recent: Andrea de Luca (LPTENS,Orsay)

Thimothee Thiery (LPTENS)

- many discrete models in “KPZ class” exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

RBA: method integrable systems (Bethe Ansatz) +disordered systems(replica)

- provides solution directly continuum KPZ eq./DP (at all times)

KPZ eq. is in KPZ class !

- also to discrete models => allowed rigorous replica

Outline:

- KPZ equation, KPZ class, random matrices, Tracy Widom distributions.
- solving KPZ at any time by mapping to directed paths
then using (imaginary time) quantum mechanics
attractive bose gas (integrable) => large time TW distrib. for KPZ height
- droplet initial condition P. Calabrese, PLD, A. Rosso EPL 90 20002 (2010)
P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)
- flat initial condition P. Calabrese, PLD, PRL 106 250603 (2011)
J. Stat. Mech. P06001 (2012)
- KPZ in half space T. Gueudre, PLD, EPL 100 26006 (2012).
- Integrable directed polymer models on square lattice
T. Thiery, PLD, J.Stat. Mech. P10018 (2014) and arXiv1506.05006
- Non-crossing probability of directed polymers
A. De Luca, PLD, arXiv1505.04802.

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- other works/perspectives:
 - V. Dotsenko, H. Spohn, Sasamoto
(math) Amir, Corwin, Quastel, Borodine,..
also G. Schehr, Reymenik, Ferrari, O'Connell..
 - not talk about: stationary initial condition T. Inamura, T. Sasamoto PRL 108, 190603 (2012)
reviews KPZ: I. Corwin, H. Spohn..

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height $h(x,t)$

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x, t)$$

diffusion noise

$$\overline{\eta(x, t)\eta(x', t')} = D\delta(x - x')\delta(t - t')$$

- 1D scaling exponents

$$h \sim t^{1/3} \sim x^{1/2} \quad x \sim t^{2/3}$$

- $P(h=h(x,t))$ non gaussian

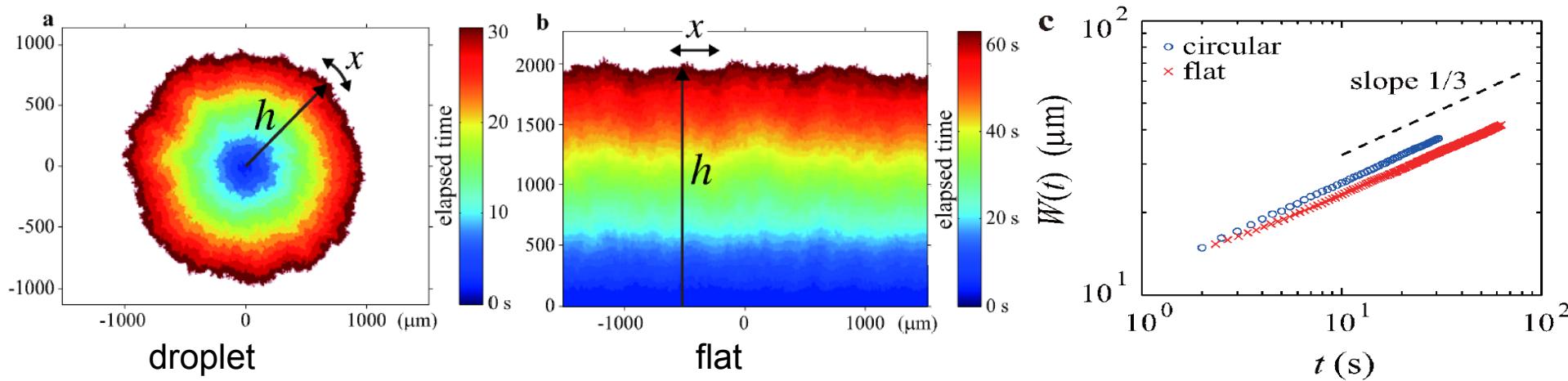
depends on some details of initial condition

flat	$h(x,0) = 0$
wedge	$h(x,0) = -w x $
(droplet)	

$\lambda_0 = 0$ Edwards Wilkinson $P(h)$ gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x, t) \simeq_{t \rightarrow +\infty} v_\infty t + \chi t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

- slow combustion of paper

J. Maunuksela et al. PRL 79 1515 (1997)

- bacterial colony growth

Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)

- fronts of chemical reactions

S. Atis (2012)

- formation of coffee rings via evaporation

Yunker et al. PRL (2012)

Large N by N random matrices H, with Gaussian independent entries

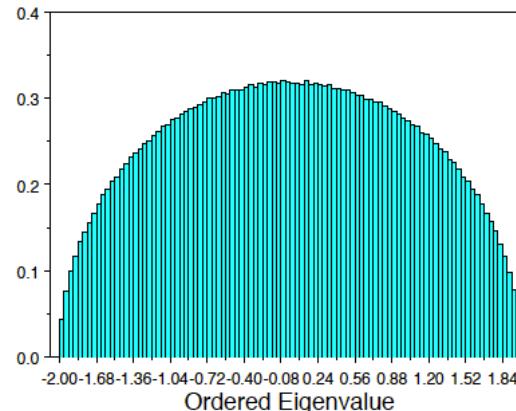
eigenvalues $\lambda_i \quad i = 1,..N$

$$P[\lambda] = c_{N,\beta} \prod_{i < j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:	
1 (GOE)	real symmetric
2 (GUE)	hermitian
4 (GSE)	symplectic

Universality large N :

- DOS: semi-circle law



histogram of
eigenvalues
N=25000

- distribution of the largest eigenvalue

$$H \rightarrow NH$$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

Tracy-Widom distributions (largest eigenvalue of RM)

GOE

$$F_1(s) = \text{Det}[I - K_1]$$

$$K_1(x, y) = \theta(x)Ai(x + y + s)\theta(y)$$

Fredholm
determinants

$$(I - K)\phi(x) = \phi(x) - \int_y K(x, y)\phi(y)$$

GUE

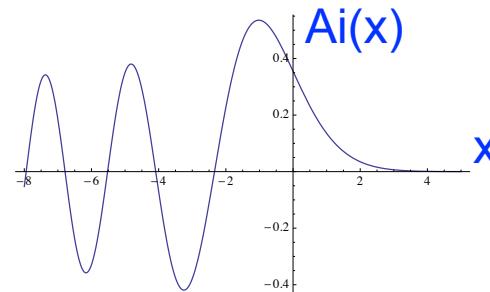
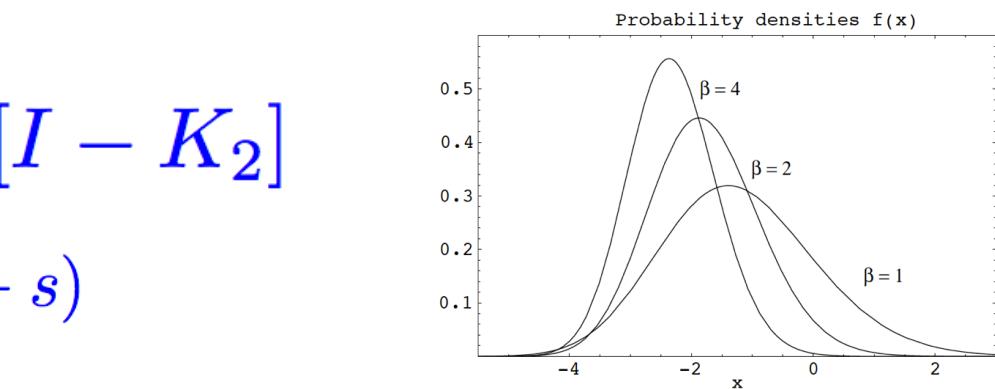
$$F_2(s) = \text{Det}[I - K_2]$$

$$K_2(x, y) = K_{Ai}(x + s, y + s)$$

$$K_{Ai}(x, y) = \int_{v>0} Ai(x + v)Ai(y + v)$$

Ai(x-E)

is eigenfunction E
particle linear potential

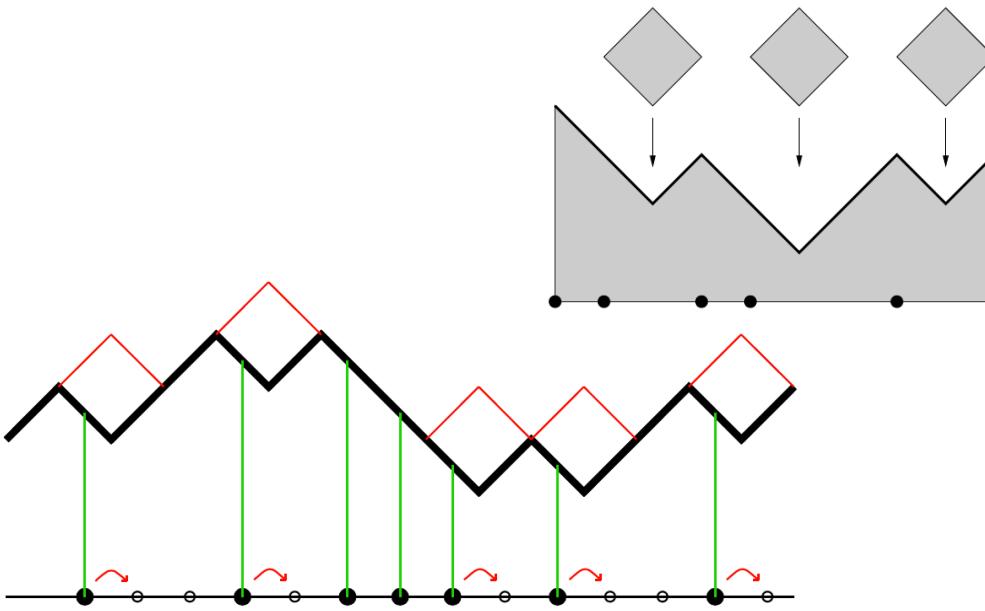


discrete models in KPZ class/exact results

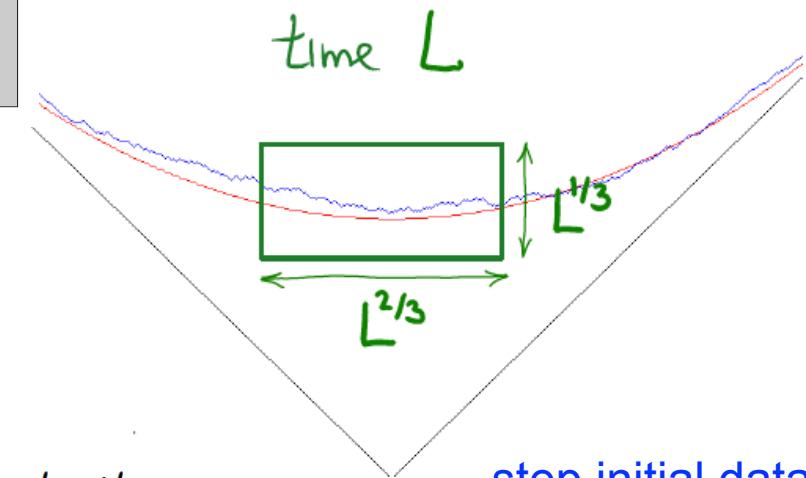
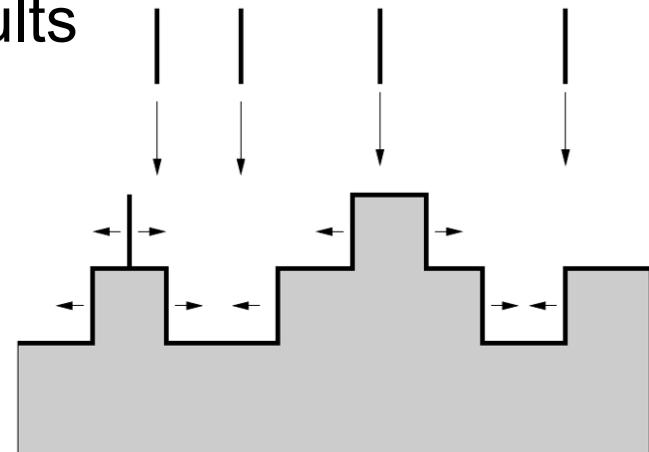
- polynuclear growth model (PNG)

Prahofer, Spohn, Baik, Rains (2000)

- totally asymmetric exclusion process (TASEP)



Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.



step initial data
Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deift, Johansson (1999)

$$h(0, t) \simeq_{t \rightarrow \infty} 2t + t^{1/3}\chi$$

droplet IC

GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations

Airy processes

$A_2(y)$ GUE

$$h(yt^{2/3}, t) \simeq_{t \rightarrow \infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

$A_1(y)$ GOE

- similar results for TASEP

Johansson (1999), ...

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Johansson (1999), ...

question: is KPZ equation in KPZ class ?

KPZ equation

Cole Hopf mapping



Continuum
Directed paths (polymers)
in a random potential



Quantum mechanics
of bosons
(imaginary time)

Kardar 87

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints
Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
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ASEP J. Ortmann, J. Quastel and D. Remenik arXiv1407.8484
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- **Stationary KPZ => Patrik Ferrari's talk**

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x, t)$$

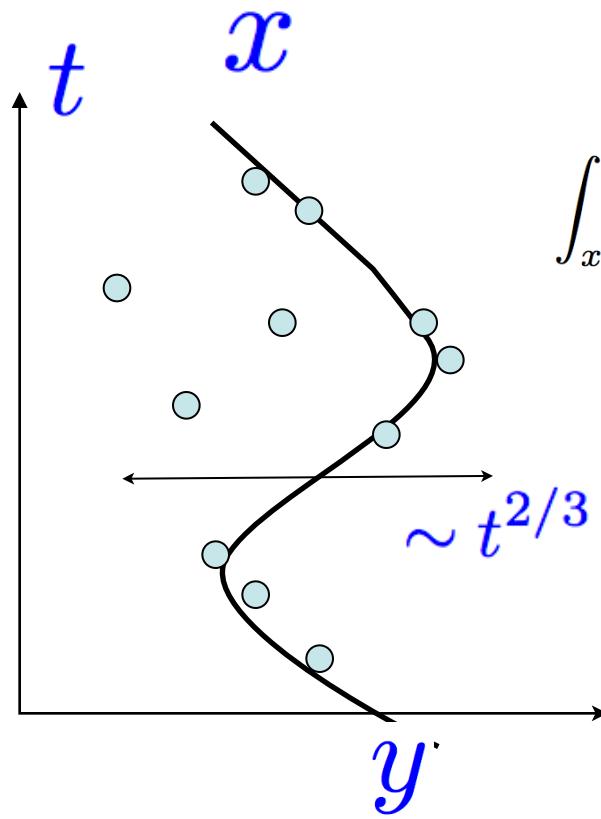
define:

$$Z(x, t) = e^{\frac{\lambda_0}{2\nu} h(x, t)} \quad \lambda_0 h(x, t) = T \ln Z(x, t)$$
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2} \partial_x^2 Z - \frac{V(x, t)}{T} Z \quad \lambda_0 \eta(x, t) = -V(x, t)$$

describes directed paths in random potential $V(x, t)$



$$Z(x, t|y, 0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau) e^{-\frac{1}{T} \int_0^t d\tau \frac{\kappa}{2} (\frac{dx(\tau)}{d\tau})^2 + V(x(\tau), \tau)}$$

$$\overline{V(x, t)V(x', t')} = \bar{c} \delta(t - t') \delta(x - x')$$

Feynman Kac

$$Z(x, y, t=0) = \delta(x - y)$$

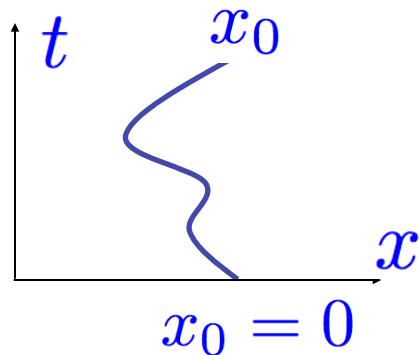
$$\partial_t Z = \frac{T}{2\kappa} \partial_x^2 Z - \frac{V(x, t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

1) DP both fixed endpoints

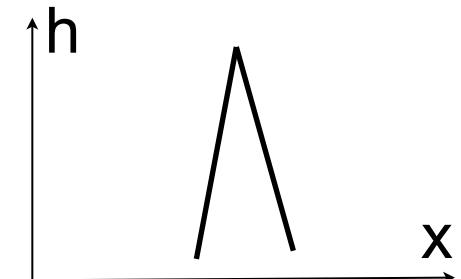
$$Z(x_0, t|x_0, 0)$$



KPZ: narrow wedge \Leftrightarrow droplet initial condition

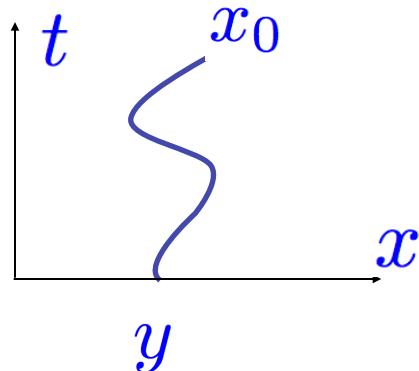
$$h(x, t = 0) = -w|x|$$

$$w \rightarrow \infty$$



2) DP one fixed one free endpoint

$$\int dy Z(x_0, t|y, 0)$$



KPZ: flat initial condition

$$h(x, t = 0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $\overline{Z^n} = \int dZ Z^n P(Z) \quad n \in \mathbb{N}$

“guess” the probability distribution from its integer moments:

$$P(Z) \rightarrow P(\ln Z) \rightarrow P(h)$$

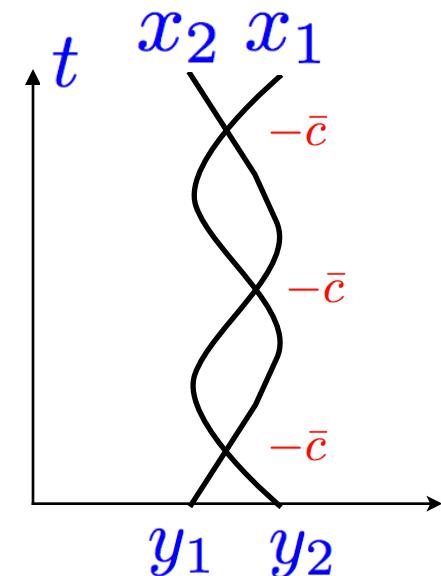
Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, t|y_1, 0) .. Z(x_n, t|y_n, 0)} = \langle x_1, .. x_n | e^{-tH_n} | y_1, .. y_n \rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3 \kappa^{-1} \tilde{x} \quad , \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

drop the tilde..



$$H_n = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

Attractive Lieb-Liniger (LL) model (1963)

what do we need from quantum mechanics ?

- KPZ with droplet initial condition

μ eigenstates

= fixed endpoint DP partition sum

E_μ eigen-energies

$$\overline{Z(x_0 t | x_0 0)^n} = \langle x_0 \dots x_0 | e^{-tH_n} | x_0, \dots x_0 \rangle$$

$e^{-tH} = \sum_{\mu} |\mu\rangle e^{-E_{\mu}t} \langle \mu|$

symmetric states = bosons

$$= \sum_{\mu} \Psi_{\mu}^*(x_0 \dots x_0) \Psi_{\mu}(x_0 \dots x_0) \frac{1}{||\mu||^2} e^{-E_{\mu}t}$$

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- KPZ with droplet initial condition μ eigenstates
- = fixed endpoint DP partition sum E_μ eigen-energies

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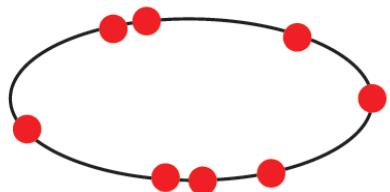
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- flat initial condition

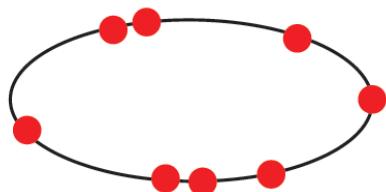
$$\overline{\left(\int_y Z(x_0 t | y 0) \right)^n} = \sum_{\mu} \Psi_{\mu}^*(x_0, \dots x_0) \int_{y_1, \dots y_n} \Psi_{\mu}(y_1, \dots y_n) \frac{1}{||\mu||^2} e^{-E_{\mu} t}$$

LL model: n bosons on a ring with local delta attraction



$$H_n = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

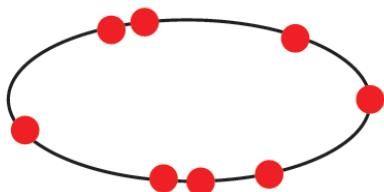
$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^n \lambda_j^2$$

$$A_P = \prod_{n \geq \ell > k \geq 1} \left(1 - \frac{i c \operatorname{sgn}(x_\ell - x_k)}{\lambda_{P_\ell} - \lambda_{P_k}} \right)$$

They are indexed by a set of rapidities $\lambda_1, \dots, \lambda_n$

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which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particles Kardar 87

$$\psi_0(x_1, \dots x_n) \sim \exp\left(-\frac{\bar{c}}{2} \sum_{i < j} |x_i - x_j|\right) \quad E_0(n) = -\frac{\bar{c}^2}{12} n(n^2 - 1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \sim_{t \rightarrow \infty} e^{-t E_0(n)} \sim e^{\frac{\bar{c}^2}{12} n^3 t} \quad \text{exponent } 1/3$$

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can it be continued in n ? NO !

information about the tail
of FE distribution

$$P(f) \sim_{f \rightarrow -\infty} \exp\left(-\frac{2}{3}(-f)^{3/2}\right)$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

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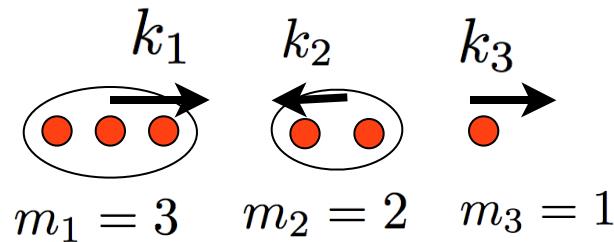
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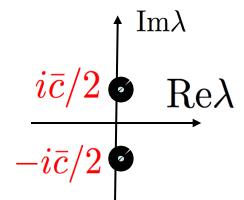
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need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into n_s “strings” each with m_j particles and momentum k_j



$$n = \sum_{j=1}^{n_s} m_j$$



$$E_\mu = \sum_{j=1}^{n_s} \left(m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1) \right)$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_{\mu} \frac{|\Psi_{\mu}(0..0)|^2}{||\mu||^2} e^{-E_{\mu}t}$$

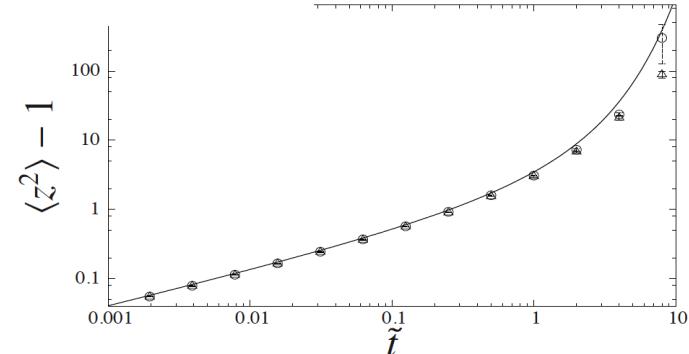
$$\Psi_{\mu}(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^n \frac{n!}{n_s! (2\pi\bar{c})^{n_s}} \sum_{(m_1, \dots m_{n_s})_n} n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k, m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k, m] = \prod_{1 \leq i < j \leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2 / 4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2 / 4}$$



how to get $P(\ln Z)$ i.e. $P(h)$?

$$\ln Z = -\lambda f \quad \lambda = \left(\frac{\bar{c}^2}{4}t\right)^{1/3} \quad f \quad \begin{array}{l} \text{random variable} \\ \text{expected } O(1) \end{array}$$

introduce generating function of moments $g(x)$:

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \rightarrow \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

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what we aim
to calculate=
Laplace transform
of $P(Z)$

what we actually study

so that at large time:

$$\lim_{\lambda \rightarrow \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \dots, m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \leq i < j \leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$



reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$Z(n_s, x) = \sum_{m_1, \dots, m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}} \int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

↑

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \leq i < j \leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

↓ double Cauchy formula

$$\det \left[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)} \right]$$

$$\frac{1}{X} = \int_0^\infty dv e^{-vX}$$

$$= \prod_{i < j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

Results: 1) $g(x)$ is a Fredholm determinant at any time t

$$Z(n_s, x) = \prod_{j=1}^{n_s} \int_{v_j > 0} dv_j \det[K(v_j, v_\ell)] \quad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1, v_2) = - \int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1 - v_2)} \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x) = \text{Det}[I + K] \quad \text{by an equivalent definition of a Fredholm determinant}$$

$$K(v_1, v_2) \equiv \theta(v_1) K(v_1, v_2) \theta(v_2)$$

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2) large time limit $\lambda = +\infty$ $\frac{e^{\lambda y}}{1 + e^{\lambda y}} \rightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

$$g(x) = Prob(f > x = -2^{2/3} s) = \text{Det}(1 - P_s K_{Ai} P_s) = F_2(s)$$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) $g(s=-x)$ is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4m_j k_j^2 \lambda^3 - \lambda m_j s} \\ \times \text{Pf} \left[\begin{pmatrix} \frac{2\pi}{2ik_i} \delta(k_i + k_j) (-1)^{m_i} \delta_{m_i, m_j} + \frac{1}{4} (2\pi)^2 \delta(k_i) \delta(k_j) (-1)^{\min(m_i, m_j)} \text{sgn}(m_i - m_j) & \frac{1}{2} (2\pi) \delta(k_i) \\ -\frac{1}{2} (2\pi) \delta(k_j) & \frac{2ik_i + m_i - 2ik_j - m_j}{2ik_i + m_i + 2ik_j + m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j > 0} \text{Pf}[\mathbf{K}(v_i, v_j)]_{2n_s, 2n_s}$$

$$g_\lambda(s) = \text{Pf}[\mathbf{J} + \mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x + y + s) \check{\theta}(y)$$

Fredholm Pfaffian Kernel at any time t

$$\begin{aligned}
K_{11} &= \int_{y_1, y_2, k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) \left[\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1 + y_2)}) \right. \\
&\quad \left. + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) \right] \\
K_{12} &= \frac{1}{2} \int_y Ai(y + s + v_i) (e^{-2e^{\lambda y}} - 1) \delta(v_j) \\
K_{22} &= 2\delta'(v_i - v_j),
\end{aligned}$$

$$f_k(z) = \frac{-2\pi kz {}_1F_2(1; 2 - 2ik, 2 + 2ik; -z)}{\sinh(2\pi k)\Gamma(2 - 2ik)\Gamma(2 + 2ik)}, \quad (19)$$

$$\begin{aligned}
F(z_i, z_j) &= \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du \\
&\times J_0(2\sqrt{z_1 z_2(1-u)}) [z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)].
\end{aligned}$$

$$g_\lambda(s) = \sqrt{\text{Det}(1 - 2K_{10})} (1 + \langle \tilde{K}|(1 - 2K_{10})^{-1}|\delta\rangle)$$

$$K_{10}(v_1, v_2) = \partial_{v_1} K_{11}(v_1, v_2) \qquad \qquad K_{12}(v_1, v_2) = \tilde{K}(v_1)\delta(v_2)$$

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large time limit

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$$\lim_{\lambda \rightarrow +\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) =$$

$$\theta(y_1 + y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

how to calculate $\int dy_1 \dots dy_n \Psi_\mu(y_1, \dots y_n)$

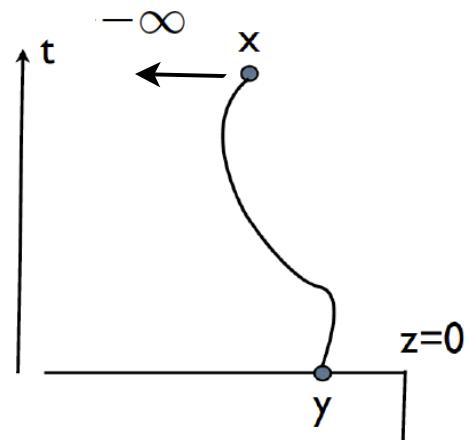
first method: flat as limit of half-flat (wedge)

$$\lim_{x \rightarrow -\infty, w \rightarrow 0} Z_{hs}(x, t) \equiv Z_{flat}(x, t)$$

$$Z_{hs,w}(x, t) = \int_{-\infty}^0 dy e^{wy} Z(x, t|y, 0)$$

$$Z(x, t=0) = \theta(-x) e^{wx}$$

$$\left(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_\alpha e^{wy_\alpha} \right) \Psi_\mu(y_1 \dots y_n) = \sum_P A_P G_{P\lambda}$$



$$\begin{aligned} \Psi_\mu &= \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P\ell} x_\ell} \\ G_\lambda &= \prod_{j=1}^n \frac{1}{jw + i\lambda_1 + \dots + i\lambda_j} \end{aligned}$$

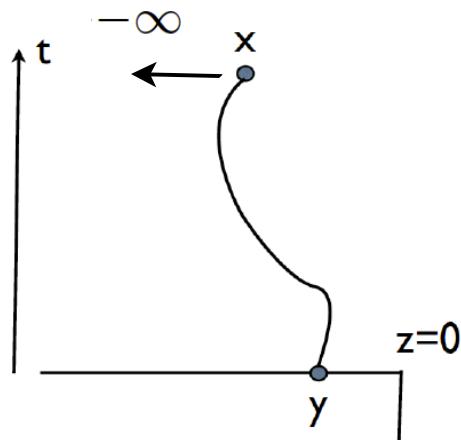
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miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w + i\lambda_\alpha)} \prod_{1 \leq \alpha < \beta \leq n} \frac{2w + i\lambda_\alpha + i\lambda_\beta - 1}{2w + i\lambda_\alpha + i\lambda_\beta}$$

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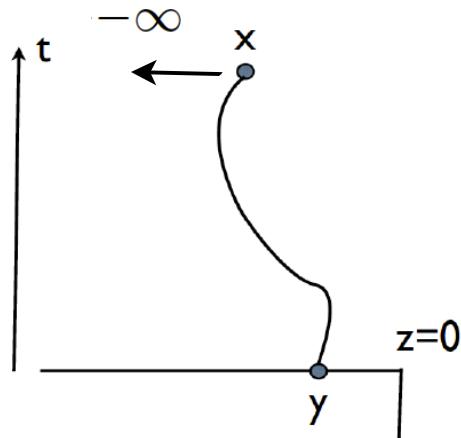
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strings: $\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j+1-2a)$
 $a = 1, \dots, m_j$

$$\int^w \Psi_\mu = n!(-2)^n \prod_{i=1}^{n_s} S_{m_i, k_i}^w \prod_{1 \leq i < j \leq n_s} D_{m_i, k_i, m_j, k_j}^w$$

$$D_{m_1, k_1, m_2, k_2}^w = (-1)^{m_2} \frac{\Gamma(1 - z + \frac{m_1+m_2}{2})\Gamma(z + \frac{m_1-m_2}{2})}{\Gamma(1 - z + \frac{m_1-m_2}{2})\Gamma(z + \frac{m_1+m_2}{2})} \quad z = ik_1 + ik_2 + 2w$$

$$S_{m, k}^w = \frac{(-1)^m \Gamma(z)}{\Gamma(z + m)} \quad z = 2ik + 2w.$$

in double limit

$$\lim_{x \rightarrow -\infty, w \rightarrow 0}$$

$$S_{m_i, k_i}^w \rightarrow \frac{(-1)^{m_i}}{2\Gamma(m_i)} 2\pi \delta(k_i) + s_{m_i, k_i}^0$$

expand the product $\prod_i S_i \prod_{i < j} D_{ij}$
each momentum k_ℓ appears
only in exactly one pole

$$D_{m_i, k_i, m_j, k_j}^w \rightarrow (-1)^{m_i} m_i \delta_{m_i, m_j} 2\pi \delta(k_i + k_j) + d_{m_i, k_i, m_j, k_j}^w$$

pairing of string momenta and pfaffian structure emerges

second method: calculate: $\prod_{\alpha=1}^n \int_0^L dy_\alpha \Psi_\mu(y_1, \dots, y_n) = \langle \Phi_0 | \mu \rangle$

use Bethe equations: $e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$ is the overlap with uniform state

=> integral vanishes for generic state
observe: requires pairs opposite rapidities

$$\Phi_0(x_1, \dots, x_n) = 1$$

Can be seen as interaction quench in Lieb-Liniger model with initial state BEC ($c=0$)

de Nardis et al., arXiv 1308.4310

overlap is non zero only for parity invariant states $\{\lambda_1, -\lambda_1, \dots, \lambda_{n/2}, -\lambda_{n/2}\}$

$$\langle \Phi_0 | \mu \rangle = n! c^{n/2} \prod_{\alpha=1}^{n/2} \frac{1}{\lambda_\alpha^2} \prod_{1 \leq \alpha < \beta \leq n/2} \frac{(\lambda_\alpha - \lambda_\beta)^2 + c^2}{(\lambda_\alpha - \lambda_\beta)^2} \frac{(\lambda_\alpha + \lambda_\beta)^2 + c^2}{(\lambda_\alpha + \lambda_\beta)^2} \times \det G^Q.$$

$$G_{\alpha\beta}^Q = \delta_{\alpha\beta} (L + \sum_{\gamma=1}^{n/2} K^Q(\lambda_\alpha, \lambda_\gamma)) - K^Q(\lambda_\alpha, \lambda_\beta)$$

$$K^Q(x, y) = K(x - y) + K(x + y),$$

Brockmann, arXiv1402.1471.

P. Calabrese, P. Le Doussal, arXiv 1402.1278

$$K(x) = \frac{2c}{x^2 + c^2}.$$

large L limit, overlap for strings
partially recovers the moments Z^n for flat

Summary: we found
for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3} \left(\frac{t}{t^*}\right)^{1/3} \chi$$

χ at large time has the same distribution
as the largest eigenvalue of the GUE

for flat initial conditions
similar (more involved)

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$$t^* = \frac{8(2\nu)^5}{D^2 \lambda_0^4}$$

in addition: $g(x)$ for all times
 $\Rightarrow P(h)$ at all t (inverse LT)

describes full crossover from
Edwards Wilkinson to KPZ

t^* is crossover time scale
large for weak noise, large diffusivity

GSE ?

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GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



$$g(s) = \sqrt{\text{Det}[I + \mathcal{K}]}$$

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$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}\left(\frac{2}{\sqrt{z}}\right) + J_{4ik}\left(\frac{2}{\sqrt{z}}\right) \right) \\ - {}_1F_2(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$$\nabla h(0, t) \text{ fixed}$$

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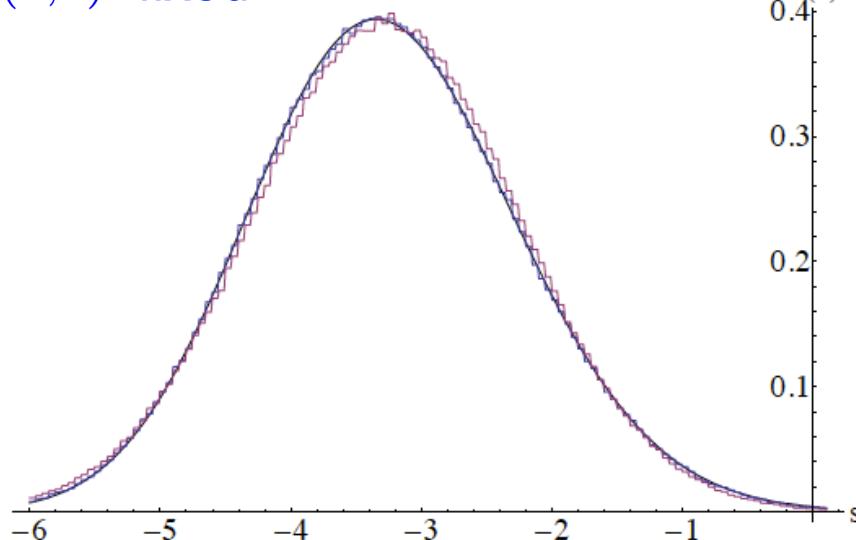
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$$\lim_{\lambda \rightarrow \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$$\nabla h(0, t) \text{ fixed}$$



$$\lambda = \left(\frac{\bar{c}^2 t}{8T^5}\right)^{1/3} = \left(\frac{D\lambda_0^2 t}{8(2\nu)^5}\right)^{1/3}$$

$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

χ_4 distributed as $F_4(s)$

Gaussian Symplectic Ensemble

Integrable directed polymer (DP) on square lattice

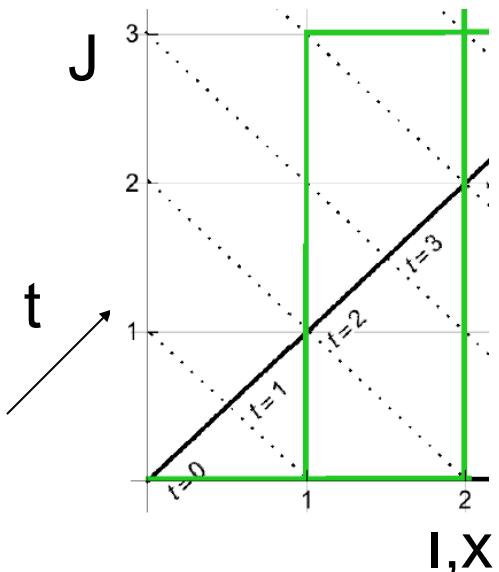
$$Z_t(x) = \sum_{\pi:(0,0) \rightarrow (x,t)} \prod_{(x',t') \in \pi} w_{x',t'}$$

- log-Gamma DP on-site weights $w \in [0, +\infty[$

inverse Gamma distribution $P(w) = \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w}$

Seppalainen (2012) Brunet

COSZ(2011) BCR(2013), Thiery, PLD(2014)



Integrable directed polymer (DP) on square lattice

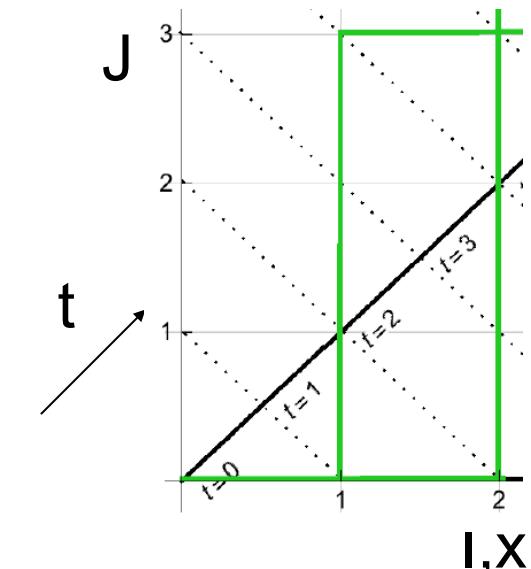
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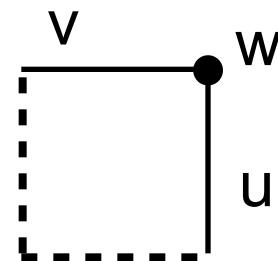
COSZ(2011) BCR(2013), Thiery, PLD(2014)



- Strict-Weak DP $u \in [0, +\infty[$ $v = 1$

Gamma distribution

Corwin, Seppäläinen, Shen(2014) $P(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u}$
O'Connell, Ortmann(2014)



- Beta DP $u, v \in [0, 1]$ $v = 1 - u$

Beta distribution

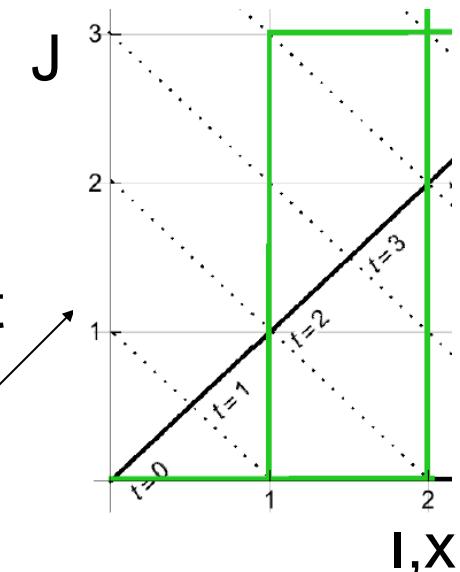
Barraquand, Corwin(2014) $p_{\alpha,\beta}(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1}$ $\alpha > 0$ and $\beta > 0$

Integrable directed polymer (DP) on square lattice

$$Z_t(x) = \sum_{\pi:(0,0) \rightarrow (x,t)} \prod_{(x',t') \in \pi} w_{x',t'}$$

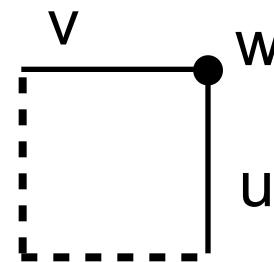
- log-Gamma DP on-site weights $w \in [0, +\infty[$

inverse Gamma distribution $P(w) = \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w}$
 Seppäläinen (2012) Brunet
 COSZ(2011) BCR(2013), Thiery, PLD(2014)



- Strict-Weak DP $u \in [0, +\infty[$ $v = 1$

Gamma distribution $P(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u}$
 Corwin, Seppäläinen, Shen(2014)
 O'Connell, Ortmann(2014)



- Beta DP $u, v \in [0, 1]$ $v = 1 - u$

Beta distribution $p_{\alpha,\beta}(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1}$ $\alpha > 0$ and $\beta > 0$
 Barraquand, Corwin(2014)

- Inverse-Beta DP $u \in [1, +\infty[$ $v \in [0, +\infty[$ $v = u - 1$

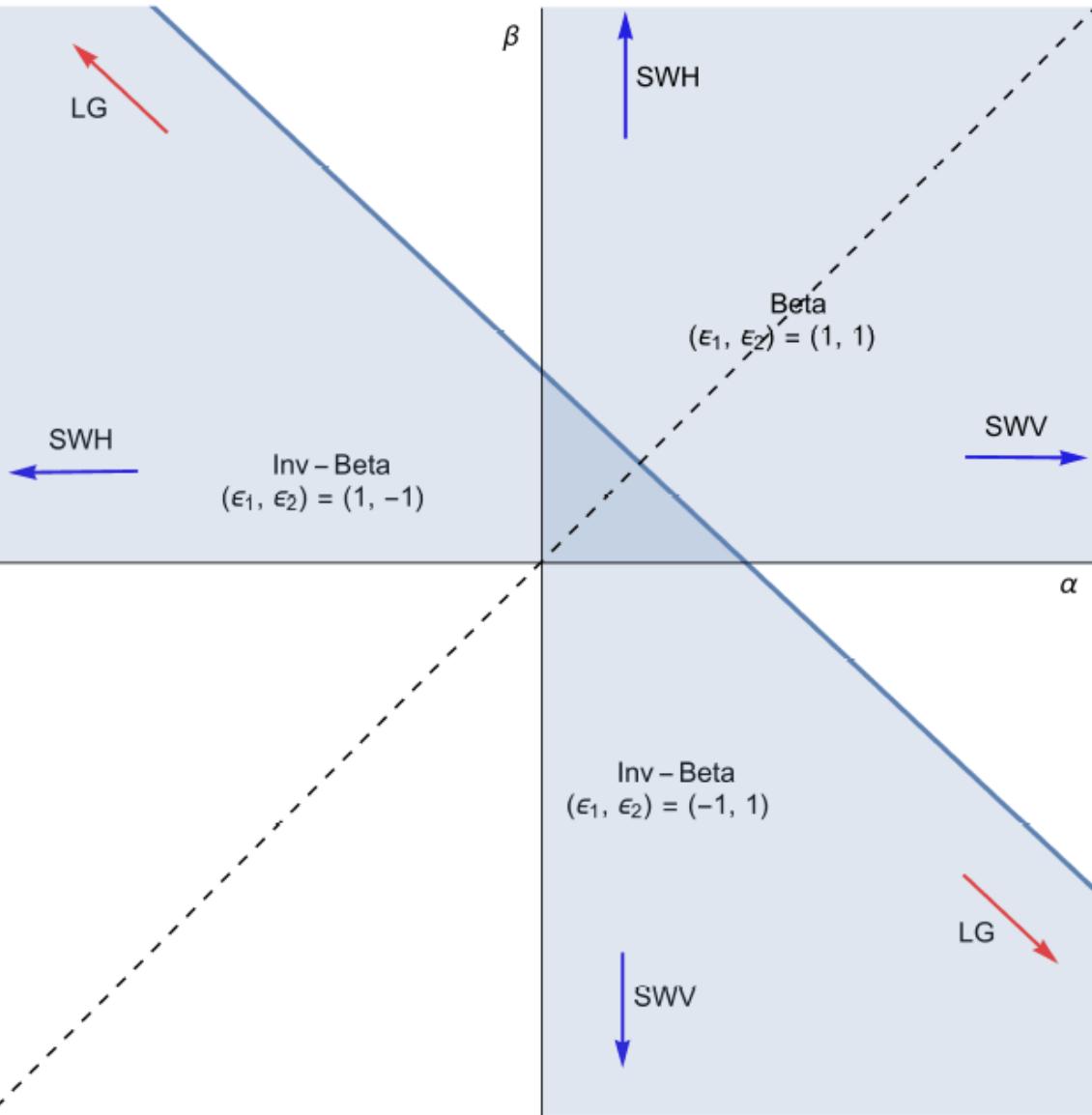
inverse-Beta distribution

T. Thiery, PLD(2015)

$$\tilde{p}_{\gamma,\beta}(u) = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \quad \gamma := 1 - (\alpha + \beta)$$

Integrable directed polymer models on square lattice

T. Thiery, PLD(2015)



What do they have in common ?

$$\overline{u^{n_1}v^{n_2}} = (\epsilon_1)^{n_1}(\epsilon_2)^{n_2} \frac{(\alpha)_{n_1}(\beta)_{n_2}}{(\alpha + \beta)_{n_1+n_2}}$$

$$(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$$

$$(a)_n = a(a+1)\dots(a+n-1)$$

$$\gamma := 1 - (\alpha + \beta)$$

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)}$$

=> Transfer matrix

$$\Lambda_\mu \psi_\mu = T_n \psi_\mu$$

contains all integer
moments $\overline{u^{n_1} v^{n_2}}$

$$n_1 + n_2 = n$$

$$Z_{t+1}(x) = u_{t+1,x} Z_t(x) + v_{t+1,x} Z_t(x-1)$$

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)} \quad \Rightarrow \text{ Transfer matrix} \quad \Lambda_\mu \psi_\mu = T_n \psi_\mu$$

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Bethe Ansatz $\psi_\mu(x_1, \dots, x_n) = \tilde{\psi}_\mu(x_1, \dots, x_n)$ if $x_1 \leq \dots \leq x_n$ $n_1 + n_2 = n$

$$\tilde{\psi}_\mu(x_1, \dots, x_n) = \sum_{\sigma \in S_n} A_\sigma \prod_{i=1}^n z_{\sigma(i)}^{x_i}$$

$$\tilde{\psi}_\mu(x_i, x_i - 1) = a\tilde{\psi}_\mu(x_i, x_i) + b\tilde{\psi}_\mu(x_i - 1, x_i) + c\tilde{\psi}_\mu(x_i - 1, x_i - 1)$$

$$A_\sigma = \epsilon(\sigma) \prod_{1 \leq i < j \leq n} \frac{c + bz_{\sigma(i)} + az_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(j)}}{c + bz_i + az_iz_j - z_j}$$

$$a = \frac{\overline{u^2} - (\overline{u})^2}{(\overline{u})(\overline{v})} \quad b = \frac{2\overline{uv} - (\overline{u})(\overline{v})}{(\overline{u})(\overline{v})} \quad c = \frac{\overline{v^2} - (\overline{v})^2}{(\overline{u})(\overline{v})}$$

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)} \quad \Rightarrow \text{ Transfer matrix } \quad \Lambda_\mu \psi_\mu = T_n \psi_\mu$$

contains all integer moments $\frac{u^{n_1} v^{n_2}}{u^{n_1} v^{n_2}}$

$$Z_{t+1}(x) = u_{t+1,x} Z_t(x) + v_{t+1,x} Z_t(x-1)$$

Bethe Ansatz

$$\psi_\mu(x_1, \dots, x_n) = \tilde{\psi}_\mu(x_1, \dots, x_n) \text{ if } x_1 \leq \dots \leq x_n \quad n_1 + n_2 = n$$

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A. Povolotsky (2013) zero-range process quantum binomial formula

$$BA = aA^2 + bAB + cB^2 \quad (\overline{u}A + \overline{v}B)^m = \sum_{n_1=0}^m \rho^{n_1} \phi_{q,\mu,\nu}(n_1|m) A^{n_1} B^{m-n_1}$$

condition for integrability

$$(\overline{u}A + \overline{v}B)^m = \sum_{n_1=0}^m \overline{u^{n_1} v^{m-n_1}} C_m^{n_1} A^{n_1} B^{m-n_1}$$

$$\boxed{\frac{(\nu; q)_{n_1} (\mu; q)_{n_2}}{(\nu; q)_{n_1+n_2}} \frac{(q; q)_{n_1+n_2}}{(q; q)_{n_1} (q; q)_{n_2}} \frac{1}{C_{n_1+n_2}^{n_1}}}$$

T. Thiery, PLD(2015)

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

Inverse-Beta polymer: main results via Bethe Ansatz

T. Thiery, PLD(2015)

- Laplace-transform representations

$$\overline{e^{-uZ_t(x)}} = \text{Det} (I + K_{tx})$$

$$K_{t,x}^{BA}(z, z') = \int_{2a+\tilde{a}+i\mathbb{R}} dw \frac{1}{4\pi(w-z')} \frac{1}{\sin(\pi(w-z))} u^{w-z} \\ \left(\frac{\Gamma(\gamma+a-w)}{\Gamma(\gamma+a-z)} \right)^{1+x} \left(\frac{\Gamma(z-a)}{\Gamma(w-a)} \right)^{1-x+t} \left(\frac{\Gamma(w-a+\beta)}{\Gamma(z-a+\beta)} \right)^t$$

$$K_{t,x}^{BA} : L^2(a+\tilde{a}+i\mathbb{R}) \rightarrow L^2(a+\tilde{a}+i\mathbb{R}) \quad 0 < a < \min(1, \gamma) \text{ and } 0 < \tilde{a} < \gamma - a.$$

conjecture

$$\overline{e^{-uZ_t(x)}} = \frac{1}{J!} \int_{(iR)^J} \prod_{j=1}^J \frac{dw_j}{2i\pi} \prod_{j \neq k=1}^J \frac{1}{\Gamma(w_j - w_k)} \\ \left(\prod_{j=1}^J u^{w_j-a} \Gamma[a - w_j] \right)^J \left(\frac{\Gamma(\gamma+a-w_j)}{\Gamma(\gamma)} \right)^I \left(\frac{\Gamma(w_j-a+\beta)}{\Gamma(\beta)} \right)^{I+J-2}$$

contains log-Gamma polymer $\beta \rightarrow \infty$

- large time

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\frac{\log Z_t((1/2 + \varphi)t) + tc_\varphi}{\lambda_\varphi} < 2^{\frac{2}{3}} z \right) = F_2(z)$$

optimal angle

$$\varphi^* = -\frac{1}{2} \frac{\psi'(\beta + \gamma/2)}{\psi'(\gamma/2)} < 0$$

$$c_{\varphi^*} = \psi(\gamma/2) - \psi(\beta + \gamma/2)$$

$$\lambda_{\varphi^*} = \left(\frac{t}{8} (\psi''(\beta + \gamma/2) - \psi''(\gamma/2)) \right)^{1/3}$$

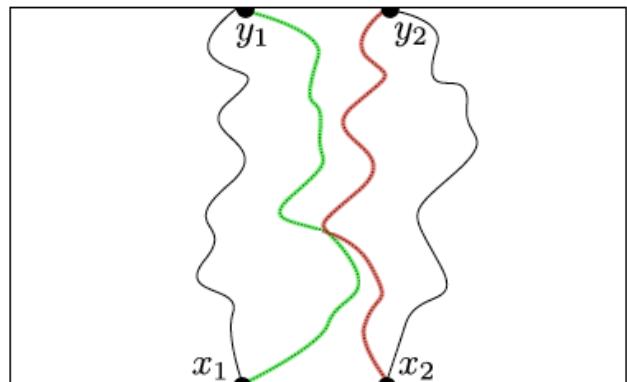
- zero temperature limit

$\gamma = \epsilon\gamma'$ and $\beta = \epsilon\beta'$ with $\epsilon \rightarrow 0$
exponential-Bernoulli bond energies

$\beta'/\gamma' \rightarrow \infty$ exponential site energies
recovers Johansson's 2000 formula

Probability that 2 directed polymers in same disorder do not cross

Andrea de Luca, PLD, arXiv 1505.04802



non-crossing probability

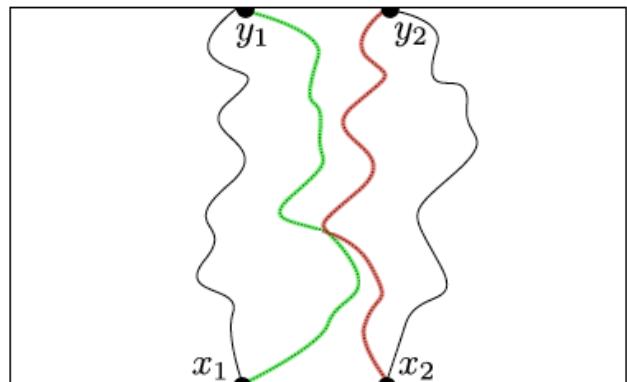
Karlin Mc Gregor

$$p_\eta(x_1, x_2; y_1, y_2 | t) \equiv 1 - \frac{Z_\eta(x_2; y_1 | t) Z_\eta(x_1; y_2 | t)}{Z_\eta(x_1; y_1 | t) Z_\eta(x_2; y_2 | t)}$$

$$p_\eta(t) \equiv \lim_{\epsilon \rightarrow 0} \frac{p_\eta(-\epsilon, \epsilon | -\epsilon, \epsilon; t)}{4\epsilon^2} = \partial_x \partial_y \ln Z_\eta(x; y | t) \Big|_{\substack{x=0 \\ y=0}}$$

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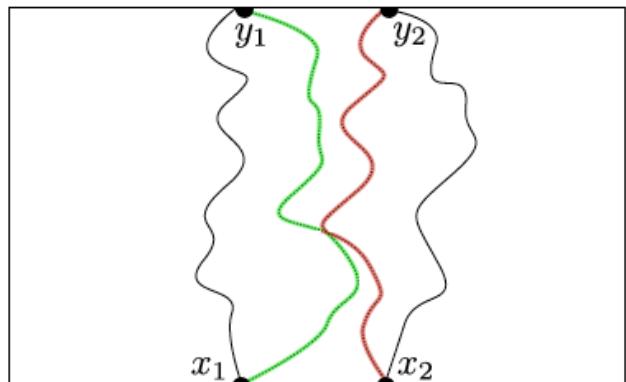
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Moments of the non-crossing probability

$$\overline{p_\eta(t)^m} = \lim_{n \rightarrow 0} \Theta_{n,m}(t)$$
$$\Theta_{n,m}(t) \equiv \lim_{\epsilon \rightarrow 0} \overline{[(2\epsilon)^{-2} Z_\eta^{(2)}(\epsilon)]^m [Z_\eta(0; 0|t)]^{n-2m}}$$
$$Z_\eta^{(2)}(\epsilon) = Z_\eta(\epsilon; \epsilon|t)Z_\eta(-\epsilon; -\epsilon|t) - Z_\eta(-\epsilon; \epsilon|t)Z_\eta(\epsilon; -\epsilon|t)$$

Probability that 2 directed polymers in same disorder do not cross

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non-crossing probability

Karlin Mc Gregor

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Lieb-Liniger with general symmetry (beyond bosons)

$$H_n = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

$$\Theta_{n,m}(t) = \lim_{\epsilon \rightarrow 0} (2\epsilon)^{-2m} \langle \Psi_m(\epsilon) | e^{-tH_n} | \Psi_m(\epsilon) \rangle = \sum_{\mu} \frac{|\mathcal{D}_m \psi_{\mu}(\mathbf{x})|^2}{\|\mu\|^2} e^{-tE_{\mu}}$$

$$|\Psi_m(\epsilon)\rangle = 2^{-m/2} (\otimes_{j=1}^m |\epsilon, -\epsilon\rangle - |-\epsilon, \epsilon\rangle) \otimes |0 \dots 0\rangle$$

$$\mathcal{D}_1 = 2^{-1/2} (\partial_{x_1} - \partial_{x_2}) \Big|_{\mathbf{x}=0}$$

Nested Bethe ansatz

$$\{\mu_1, \dots, \mu_n\} \quad E_\mu = \sum_{j=1}^n \mu_j^2$$

A_Q^P inside irreducible representation of S_n

2-row Young diagram $\xi = (n-m, m)$

$$\psi_\mu(\mathbf{x}) = \sum_{P, Q \in \mathcal{S}_n} \vartheta_Q(\mathbf{x}) A_Q^P \exp \left[i \sum_{j=1}^n x_{Q_j} \mu_{P_j} \right]$$

$$x_{Q_1} \leq x_{Q_2} \dots \leq x_{Q_n}$$

example $n = 8$ and $m = 3$

$$(5,3) \equiv \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 7 & 9 \\ \hline 2 & 4 & 6 & & \\ \hline \end{array} \quad \text{antisymmetry } \mathcal{D}_{m=3} \quad x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4, x_5 \leftrightarrow x_6$$

C-N Yang PRL 19,1312 (1967)

$$\prod_{\substack{b=1 \\ b \neq a}}^m \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^n \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2},$$

$$\prod_{\substack{k=1 \\ k \neq j}}^n \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^m \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

auxiliary spin chain

auxiliary rapidities $\lambda_a \quad a = 1,..m$

=> large L : strings again !
but not all strings contribute !

$$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$$

$$\mu_j^a = k_j + \frac{ic}{2}(m_j + 1 - 2a) + \delta_j^a$$

$$\Theta_{n,1}(t) = \sum_{n_s=1}^n \frac{n! \bar{c}^n}{n_s! (2\pi \bar{c})^{n_s}} \sum_{(m_1, \dots m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j e^{-A_2 t}}{m_j} \Phi(\mathbf{k}, \mathbf{m}) \Lambda_{n,1}(\mathbf{k}, \mathbf{m})$$

LL conserved charges $A_p = \sum_{j=1}^n \mu_j^p$

$$\Lambda_{n,1} = \frac{1}{n(n-1)} \left(nA_2 - A_1^2 + \frac{n^2(n^2-1)}{12} \bar{c}^2 \right).$$

$$\Theta_{n,0}(t) = \mathcal{Z}_n(t) \equiv \mathcal{Z}_n(\mathbf{x} = \mathbf{0}; \mathbf{0}|t) \quad \Lambda_{n,1} \rightarrow \Lambda_{n,0} \equiv 1$$

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$$\text{Introduce GGE partition function } \mathcal{Z}_n^\beta(t) \quad e^{-A_2 t} \Rightarrow e^{-A_2 t + \sum_{p \geq 1} \beta_p A_p}$$

$$A_p \rightarrow \partial_p \equiv \partial_{\beta_p} \quad \Theta_{n,m}(t) = \Lambda_{n,m}(\{\partial_p\})[\mathcal{Z}_n^\beta(t)]$$

$$\lim_{n \rightarrow 0} \partial_{i_1} \dots \partial_{i_k} \frac{\mathcal{Z}_n^\beta(t) - 1}{n} \Big|_{\beta=0} = - \int_0^\infty \frac{du}{u} \partial_{i_1} \dots \partial_{i_k} \text{Det}(1 + \Pi_0 \mathcal{K}_u^\beta \Pi_0)$$

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We calculated the $\Lambda_{n,m}(A_p)$ from the Borodin-Corwin “conjecture”

BC arXiv11114408

$$\Theta_{n,m}(t) = \frac{1}{2^m} \int \frac{dz_1}{2\pi} \dots \int \frac{dz_n}{2\pi} e^{-t \sum_{k=1}^n z_k^2} \quad h(u) = u(u - ic) \\ \times \left(\prod_{1 \leq k < j \leq n} f(z_{kj}) \right) \left(\prod_{q=1}^m h(z_{2q-1, 2q}) \right)$$

$z_{kj} = z_k - z_j$ imaginary part C_j for z_j satisfying $C_{j+1} > C_j + \bar{c}$

Results: - first moment: simple from STS !

$$\overline{p_\eta(t)} = \lim_{n \rightarrow 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$\overline{\ln Z_\eta(x; y|t)} = h(t) - (x - y)^2 / (4t)$$

$$h(t) = \overline{\ln Z_\eta(0; 0|t)}$$

Results: - first moment: simple from STS !

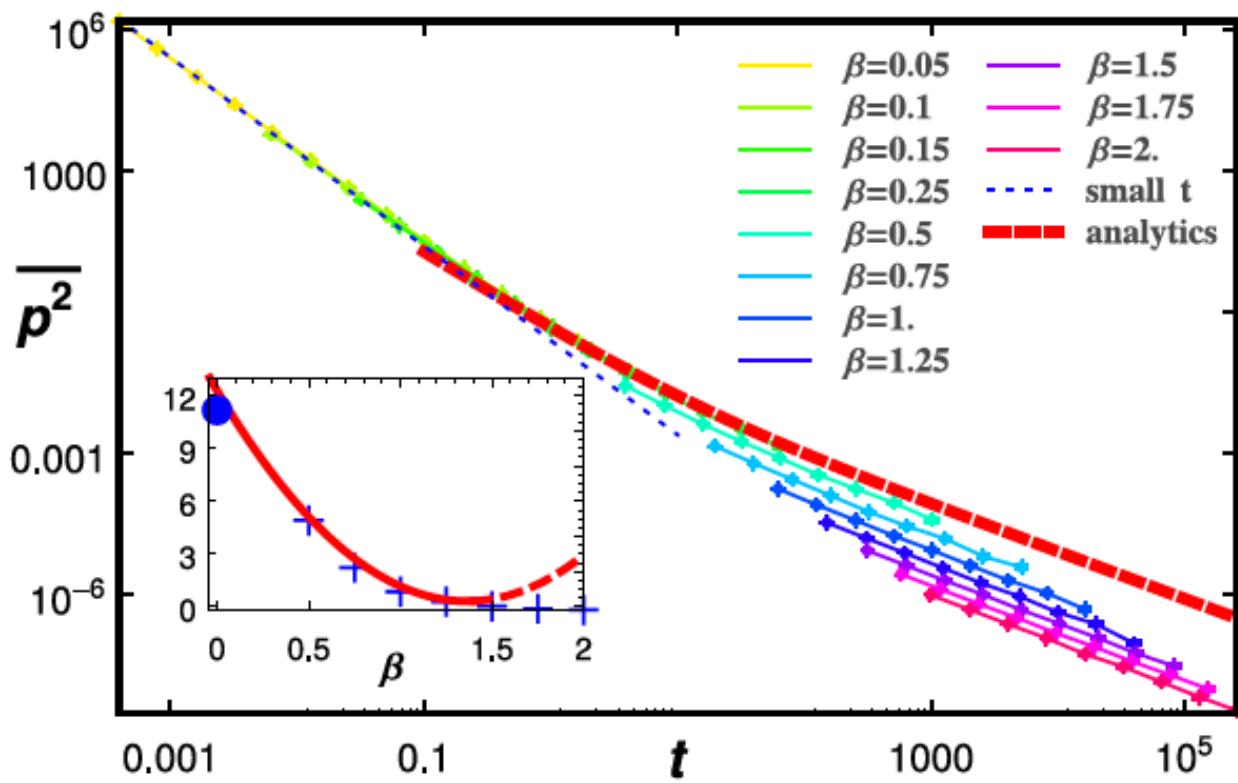
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$$\begin{aligned}\overline{\ln Z_\eta(x; y|t)} &= h(t) - (x-y)^2/(4t) \\ h(t) &= \overline{\ln Z_\eta(0; 0|t)} \\ &\simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2} (\bar{c}^2 t)^{1/3}\end{aligned}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right)h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

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- higher moments

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\bar{\chi}_2\bar{c}^{8/3}}{9t^{5/3}} + \text{complicated}$$

=> conjecture 1

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

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$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

- conjecture 2 $\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3}$ $a = \bar{\chi}_2 - \bar{\chi}'_2 \approx 1.9043$

=> $p(t)$ (sub-)exponentially small for typical environments

and $p_\eta(t) \sim \bar{c}^2$ for a fraction $\sim 1/(\bar{c}^2 t)$ of environments

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$$h(t) = \overline{\ln Z_\eta(0; 0|t)}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right)h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\bar{\chi}_2\bar{c}^{2/3}}{9t^{5/3}}$$

- higher moments

=> conjecture 1

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\bar{\chi}_2\bar{c}^{8/3}}{9t^{5/3}} + \text{complicated}$$

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

- conjecture 2 $\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3}$ $a = \bar{\chi}_2 - \bar{\chi}'_2 \approx 1.9043$

=> $p(t)$ (sub-)exponentially small for typical environments

and $p_\eta(t) \sim \bar{c}^2$ for a fraction $\sim 1/(\bar{c}^2 t)$ of environments

compare with proba $q(t)$ of single DP not crossing a hard-wall at 0 T. Gueudre, PLD 2012

$$\overline{\ln q_\eta(t)} = -(\bar{\chi}_2 - \bar{\chi}_4)(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

$$\overline{\ln p_\eta(t)} < \overline{\ln q_\eta(t)}$$

Perspectives/other works

- replica BA method

stationary KPZ

Sasamoto Inamura

Airy process

$t \rightarrow \infty$ $A_2(y)$

2 space points

$\text{Prob}(h(x_1, t), h(x_2, t))$

Prohlac-Spoohn (2011),
Dotsenko (2013)

2 times

$\text{Prob}(h(0, t), h(0, t'))$

Dotsenko (2013)

endpoint distribution of DP

Dotsenko (2012)

Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP

$q \rightarrow 1$

avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

WASEP

Bose gas

moments as nested contour integrals

- sine-Gordon FT

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- Lattice directed polymers