Off-critical interfaces in two dimensions
Exact results from field theory

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Statistical Mechanics, Integrability and Combinatorics, GGI
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Based on:

- Gesualdo Delfino, AS, *Interfaces and wetting transition on the half plane. Exact results from field theory*, J. Stat. Mech. (2013) P05010
- Gesualdo Delfino, AS, *Exact theory of intermediate phases in two dimensions*, Annals of Physics 342 (2014) 171
- Gesualdo Delfino, AS, *Phase separation in a wedge. Exact results*, Phys. Rev. Lett. 113 (2014) 066101

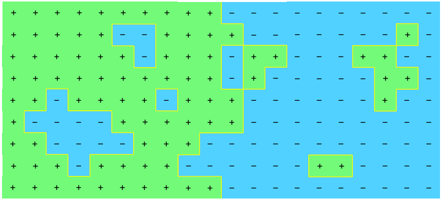
Outline

1. Introduction
2. Simple interfaces
   - average magnetization, passage probability
   - Interface structure; Ising & q-Potts
3. Double interfaces
   - Tricritical q-Potts interfaces
   - Bulk wetting transition & Ashkin-Teller
4. Interfaces at boundaries
   - Wedge geometry
   - Boundary wetting transition & filling transitions
5. Summary & outlook

Interfaces in two dimensions

From lattice

- Exact studies focused so far on $D = 2$ Ising, exploiting lattice solvability



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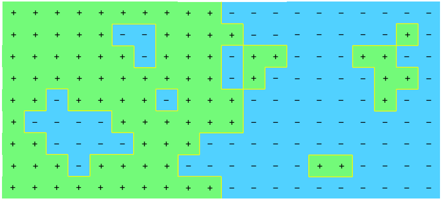

From field theory

- $T = Tc$: Interfaces are conformally invariant random curves described by SLE. Connection with CFT in $D = 2$ applied at criticality but few is known about massive deformations.

Away from criticality? How to avoid lattice calculations and work directly in the continuum for general models? (i.e. scaling q-Potts, Ashkin-Teller,. . .)

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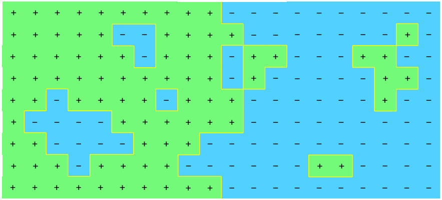

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Away from criticality? How to avoid lattice calculations and work directly in the continuum for general models? (i.e. scaling q-Potts, Ashkin-Teller,. . .)

$\hookrightarrow$ We propose a new approach to phase separation for massive interfaces ($T < Tc$) based on local fields

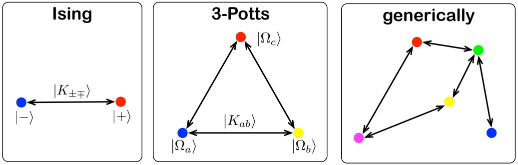
$\hookrightarrow$ Field theory yields general and exact solutions for a wider class of models with a simple language, accounting for interface structure, boundary&bulk wetting, wedge filling

$\hookrightarrow$ application to thermodynamic Casimir forces and its dependence on bc.s (not this talk)

Field-theoretic formulation

Scaling limit of a system of classical statistical mechanics in $2d$ below T_c. $(1+1)$-relativistic field theory analytically continued to a 2-dim Euclidean field theory in the plane $(x, y = -it)$.

- States with minimum energy: degenerate vacua (coexisting phases)



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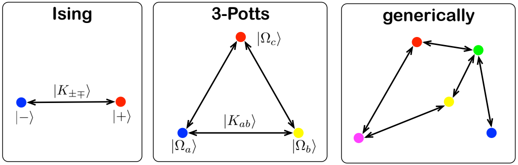


- Elementary excitations: kinks (domain walls or interfaces)

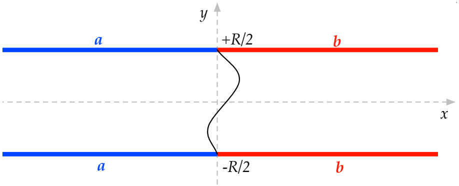
$$|K_{ab}(\theta)\rangle \text{ interpolates between } |\Omega_a\rangle, |\Omega_b\rangle$$

  relativistic particles with $(E, P) = (m_{ab}\cosh\theta, m_{ab}\sinh\theta)$.

- Adjacency structure
  $\Omega_a|\Omega_b$: adjacent $\longrightarrow$ connected by $|K_{ab}\rangle$
  $\Omega_{\bullet}|\Omega_{\bullet}$: not adjacent $\longrightarrow$ connected by $|K_{\bullet\bullet}K_{\bullet\bullet}\rangle$ (the lightest)

Phase separation for adjacent phases

Symmetry breaking boundary conditions: $a \neq b$ with $R/\xi \propto m_{ab}R \gg 1 \longrightarrow$ *single interface*



No phase separation for $a = b$

$$\therefore \langle \sigma_a \rangle = \langle \Omega_a | \sigma(x,y) | \Omega_a \rangle$$

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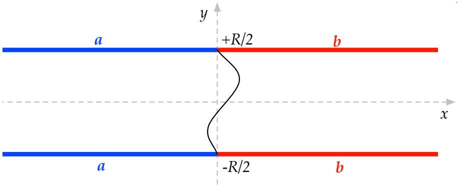

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Boundary states (cf. [Ghoshal-Zamolodchokov] for the translationally invariant case)

$$|\mathcal{B}_{ab}(x,t)\rangle = \mathrm{e}^{-itH+ixP}\left[\int_{\mathbb{R}} \frac{\mathrm{d}\theta}{2\pi} f_{ab}(\theta)|K_{ab}(\theta)\rangle + \sum_{c\neq a,b}\int_{\mathbb{R}^2} \frac{\mathrm{d}\theta \mathrm{d}\theta'}{(2\pi)^2} f^c_{ab}(\theta,\theta')|K_{ac}(\theta)K_{cb}(\theta')\rangle + \dots\right]$$

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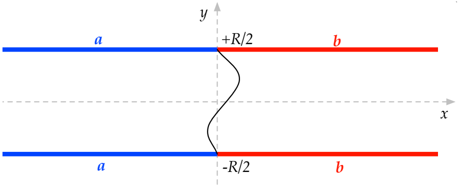

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Partition functions (leading order)

$$\mathcal{Z}_a(R) = \langle \mathcal{B}_a(0, iR/2) | \mathcal{B}_a(0,-iR/2)\rangle \sim \langle \Omega_a | \Omega_a \rangle = 1$$

$$\mathcal{Z}_{ab}(R) = \langle \mathcal{B}_{ab}(0, iR/2) | \mathcal{B}_{ab}(0,-iR/2)\rangle \sim \frac{|f_{ab}(0)|^2}{\sqrt{2\pi m R}} \mathrm{e}^{-mR}$$

Interfacial tension of $\Omega_a|\Omega_b$

$$\Sigma_{ab} = -\lim_{R\to\infty} \frac{\mathcal{Z}_{ab}(R)}{\mathcal{Z}_a(R)} = m$$

$\hookrightarrow mR \to \infty \Longrightarrow$ projection to low-energy physics: $\theta \ll 1$

Single interfaces: order parameter profile

One-point function of the spin operator along the horizontal axis $(x, y = 0)$

$$\begin{aligned}
\langle\sigma(x,0)\rangle_{ab} &= \frac{1}{\mathcal{Z}_{ab}}\langle\mathcal{B}_{ab}(0,iR/2)|\sigma(x,0)|\mathcal{B}_{ab}(0,-iR/2)\rangle \\
&\simeq \frac{|f_{ab}(0)|^2}{\mathcal{Z}_{ab}}\int_{\mathbb{R}^2}\frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{(2\pi)^2}\,\mathcal{M}^{\sigma}_{ab}(\theta_1|\theta_2)\mathrm{e}^{-mR\left(1+\frac{\theta_1^2+\theta_2^2}{4}\right)-imx\theta_{12}}
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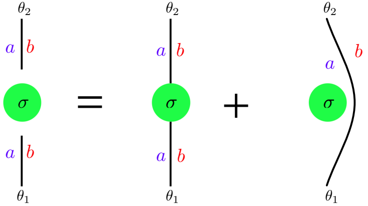
- Matrix element: 2-kink Form Factor + disconnected



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\mathcal{M}^{\sigma}_{ab}(\theta_1|\theta_2) &= \langle K_{ab}(\theta_1) | \sigma(0,0) | K_{ba}(\theta_2) \rangle \\
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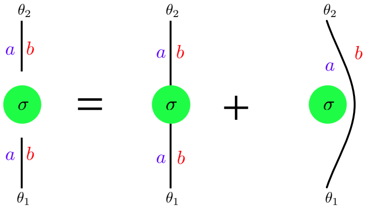

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- Crossing symmetry
  Two kinks can annihilate $\longrightarrow$ kinematic pole of the FF: *does not require integrability*
  [Berg-Karowski-Weisz '78; Smirnov 80's; Delfino-Cardy '98]

$$K_{ab}(\theta_1) + K_{ba}(\theta_2) \to \emptyset \qquad \text{as} \qquad \theta_1 - \theta_2 \to i\pi$$

$$\rightsquigarrow -i\,\mathrm{Res}_{\theta=i\pi} F^{\sigma}(\theta) = \langle\sigma\rangle_a - \langle\sigma\rangle_b$$

Single interfaces (cont'd)

low-energy expansion

$$F^{\sigma}_{aba}(\theta + i\pi) = \boxed{\frac{i\Delta\langle\sigma\rangle}{\theta}} + \sum_{k=0}^{\infty} c^{(k)}_{ab}\theta^k$$

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$$\sigma(x,0)\rangle_{ab} = \langle\sigma\rangle_a + \frac{i\Delta\langle\sigma\rangle}{2}\int_{\mathbb{R}} \frac{\mathrm{d}\theta}{\theta}\mathsf{e}^{-\frac{\theta^2}{2}+i\eta\theta} + \dots \qquad \left(\eta \equiv \frac{x}{\lambda}, \qquad \lambda \equiv \sqrt{\frac{R}{2m}}\right)$$

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The simple pole is essential but it needs to be regularized ($\lim_{\epsilon\to 0}\frac{1}{\theta \pm i\epsilon} = \mp\pi i\delta(\theta) + \mathcal{P}\frac{1}{\theta}$).

Single interfaces (cont'd)

Final result: [Delfino-Viti 12]

$$\langle\sigma(x,0)\rangle = \boxed{\frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b}{2} - \frac{\langle\sigma\rangle_a - \langle\sigma\rangle_b}{2}\,\mathrm{erf}\,(\eta)} + c_{ab}^{(0)}\sqrt{\frac{2}{\pi m R}}\,\mathrm{e}^{-\eta^2} + \dots$$

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- the non-local term is generated by the pole. It reflects non-locality of kinks w.r.t. spin field
- subleading local corrections $\propto c_{ab}^{(k)}$: *interface structure*
- extend the derivation to $y \neq 0$: replacement $\eta \to \chi \equiv \eta/\kappa$, $(\kappa \equiv \sqrt{1 - 4y^2/R^2})$.

  The profile depends only on $\chi \implies$ Contour lines are arcs of ellipses. [Delfino-AS, 14]

$$\frac{x^2}{\frac{R}{2m}(\mathrm{const.})} + \frac{y^2}{\left(\frac{R}{2}\right)^2} = 1$$

- Midpoint fluctuation $\sim \sqrt{R}$

Examples: broken $\mathbb{Z}_2$ & broken S_q

- Ising model: $\langle\sigma\rangle_+ = -\langle\sigma\rangle_-$

$$\boxed{\langle\sigma(x,y)\rangle_\mp = \langle\sigma\rangle_\pm \mathsf{erf}(\chi)}$$

Perfect match with scaling of lattice solution, cf [Abraham, 81].
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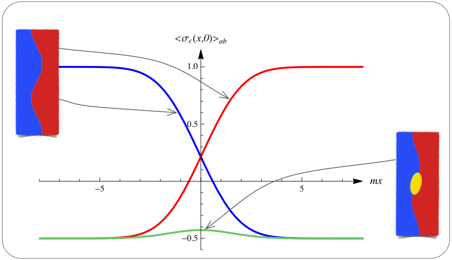
- q-state Potts model: The scattering theory is integrable [Chim-Zamolodchikov] and Form Factors are known [Delfino-Cardy]

$$\begin{aligned}
\sigma_c(x) &= \delta_{s(x),c} - \frac{1}{q} \\
\langle\sigma_c\rangle_a &= \frac{q\delta_{ac}-1}{q-1}M \\
c_{ab,c}^{(0)} &= \left[2 - q(\delta_{ac}+\delta_{bc})\right]MB(q)
\end{aligned}$$

with $B(3) = \dfrac{1}{4\sqrt{3}}, B(4) = \dfrac{1}{3\sqrt{3}}$.



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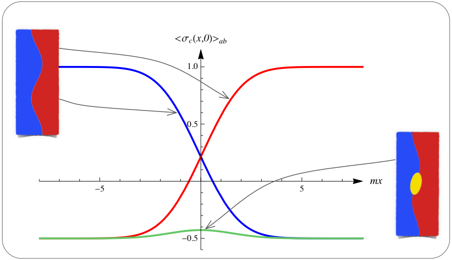

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$\hookrightarrow$ Branching is a general phenomenon not due to integrability
$\hookrightarrow$ For integrable theories we can compute the amplitude of the island (i.e. $B(q)$)

Passage probability and interface structure

The interface will cross the horizontal axis $(y = 0)$ in $x \in (u, u + \mathrm{d}u)$, with passage probability $p(u; 0)du$, how is the magnetization affected in x?

$$\langle \sigma(x, 0) \rangle_{ab} = \int_{\mathbb{R}} \mathrm{d}u \, \sigma_{ab}(x|u) p(u; 0)$$

$$\sigma_{ab}(x|u) = \boxed{\theta(u - x)\langle\sigma\rangle_a + \theta(x - u)\langle\sigma\rangle_b} + A_{ab}^{(0)} \delta(x - u) + A_{ab}^{(1)} \delta'(x - u) + \dots$$

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Matching with field theory yields

$$p(x;y) = \frac{1}{\sqrt{\pi}\kappa\lambda}\mathrm{e}^{-\chi^2} \Longrightarrow \text{Gaussian Bridge } (*)$$

$$A_{ab}^{(0)} = \frac{c_{ab}^{(0)}}{m} \Longrightarrow \text{Bifurcation amplitude}$$

($*$) rigorously known for Ising and Potts [Greenberg, Joffe, '05; Campanino, Joffe, Velenik, '08]

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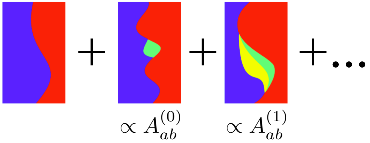
"RG" perspective: large R/ξ expansion

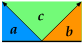
- $R/\xi = \infty$: *sharp interface picture*
- $R/\xi \gg 1$: *proliferation of inclusions: bubbles of different phases*



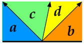
Double interfaces (I)

If the vacua $|\Omega_a\rangle$ and $|\Omega_b\rangle$ cannot be connected by a single kink

$|\mathcal{B}_{ab}\rangle = \sum_{c\neq a,b} \Big[$



$+ \sum_{d\neq c,b}$



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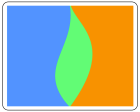

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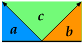

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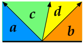

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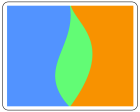
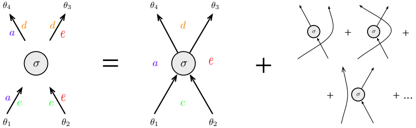


4-kink matrix element

$$\langle K_{bd}(\theta_3)K_{da}(\theta_4)|\sigma|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle =$$



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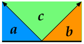

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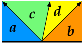
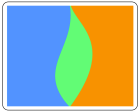
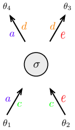


$$+ \dots \Big]$$

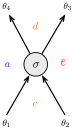


4-kink matrix element

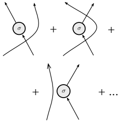
$$\langle K_{bd}(\theta_3)K_{da}(\theta_4)|\sigma|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle =$$



$$=$$



$$+$$



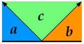
Connected part: low-energy limit

$$\mathcal{M}^{\sigma,\text{conn}}_{ab,cd}(\theta_1,\theta_2|\theta_3,\theta_4) = \big[2\langle\sigma\rangle_c - \langle\sigma\rangle_a - \langle\sigma\rangle_b\big]\frac{\theta_{12}\theta_{34}}{\theta_{13}\theta_{14}\theta_{23}\theta_{24}}$$

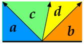
Double interfaces (I)

If the vacua $|\Omega_a\rangle$ and $|\Omega_b\rangle$ cannot be connected by a single kink

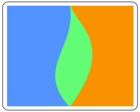
$$|\mathcal{B}_{ab}\rangle = \boxed{\sum_{c\neq a,b}\Big[}$$



$$+\sum_{d\neq c,b}$$



$$+\dots\Big]$$



4-kink matrix element

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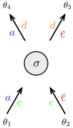
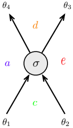
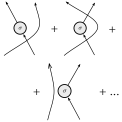


$$=$$



$$+$$



Connected part: low-energy limit

$$\mathcal{M}^{\sigma,\text{conn}}_{ab,cd}(\theta_1,\theta_2|\theta_3,\theta_4) = \Big[2\langle\sigma\rangle_c - \langle\sigma\rangle_a - \langle\sigma\rangle_b\Big]\frac{\theta_{12}\theta_{34}}{\theta_{13}\theta_{14}\theta_{23}\theta_{24}}$$

this structure is inherited from the kinematic poles

- Average spin field

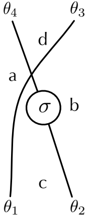
$$\langle\sigma(x,y)\rangle^{\text{conn}} \sim \int_{\mathbb{R}^4} \mathrm{d}\theta_1\dots\mathrm{d}\theta_4\,\mathcal{M}_{ab,cd}(\theta_1,\theta_2|\theta_3,\theta_4)\,Y^-(\theta_1)Y^-(\theta_2)Y^+(\theta_3)Y^+(\theta_4)$$

$$Y^{\pm}(\theta) = \exp\Big[-\frac{1\pm\epsilon}{2}\theta^2 \pm i\eta\theta\Big]$$

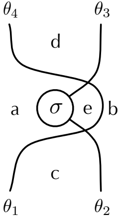
then: regularization and integration over all the rapidities

Double interfaces (II)

- Disconnected parts: each annihilation (leg contraction) produces a Dirac delta



$$= \quad 2\pi\delta(\theta_{13})\,\frac{i(\langle\sigma_c\rangle - \langle\sigma_b\rangle)}{\theta_{24}}$$



$$= \quad 2\pi\delta(\theta_{14}) \sum_{e\neq a,b} S_{ab}^{ce}(0) S_{ab}^{ed}(0)\,\frac{i(\langle\sigma_a\rangle - \langle\sigma_e\rangle)}{\theta_{23}}$$

then: sum up all the contributions

Double interfaces (III)

- For arbitrary models [Delfino-AS, 14]

$$\langle\sigma(x,y)\rangle_{ab} = \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b - 2\langle\sigma\rangle_c}{4}\mathcal{G}(\chi) - \frac{\langle\sigma\rangle_a - \langle\sigma\rangle_b}{2}\mathcal{L}(\chi) + \frac{\langle\sigma\rangle_a + \langle\sigma\rangle_b + 2\langle\sigma\rangle_c}{4}$$

$$\begin{aligned} \mathcal{G}(\chi) &= -\frac{2}{\pi}\mathrm{e}^{-2\chi^2} - \frac{2\chi}{\sqrt{\pi}}\mathrm{e}^{-\chi^2} + \mathrm{erf}^2(\chi) \\ \mathcal{L}(\chi) &= -\frac{\chi}{\sqrt{\pi}}\mathrm{e}^{-\chi^2} + \mathrm{erf}(\chi) \end{aligned}$$

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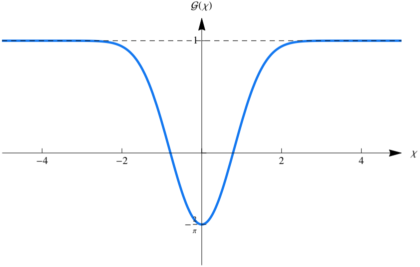
$$\mathcal{L}(\chi) = -\frac{\chi}{\sqrt{\pi}}\mathrm{e}^{-\chi^2} + \mathrm{erf}(\chi)$$

Universal scaling form. Specific features of the models enters through the vev.s $\langle\sigma_\alpha\rangle_\beta$

- A "forced" example: Ising bubble (we have only two vacua!)

$$\langle\sigma(x,y)\rangle_{\pm\pm} = \langle\sigma\rangle_\pm \mathcal{G}(\chi)$$

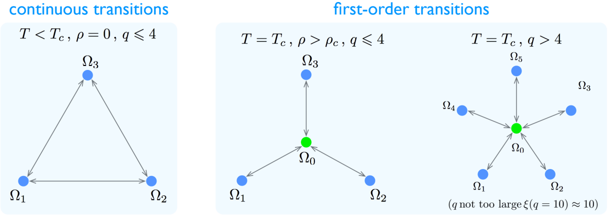


perfect match with lattice Ising [Abraham-Upton, 93]

Tricritical q-state Potts

Annealed vacancies are allowed (if no vacancies: pure q-state Potts).

- vacua connectivity



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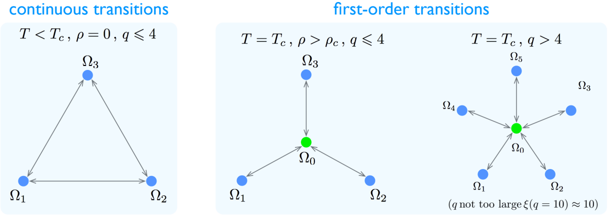

Dilute regime: Star-graph-like vacua structures. The continuum limit is described by an integrable scattering theory whose spectrum is known. Elementary excitations: K_{i0}, K_{0j}. The process

$$|K_{i0}\rangle + |K_{0j}\rangle \longrightarrow |\tilde{K}_{ij}\rangle$$

cannot take place (absence of a pole of S^{00}_{ij} in the physical strip [Delfino, '99]) $\longrightarrow$ the vacua connectivity for the dilute case is a star graph.

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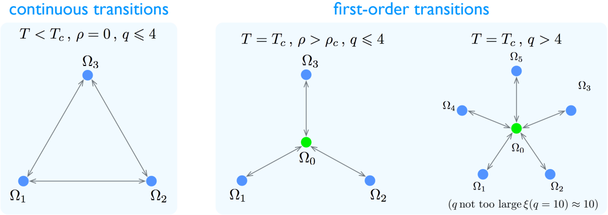

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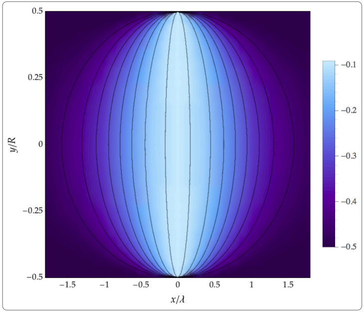
- Order parameter profiles

$$\langle \sigma_1(x,y)\rangle_{12} = \frac{\langle\sigma_1\rangle_1}{2}\left[\frac{q-2}{2(q-1)}(1+\mathcal{G}(\chi)) + \frac{q}{q-1}\mathcal{L}(\chi)\right] \qquad \text{(smooth-step-like)}$$

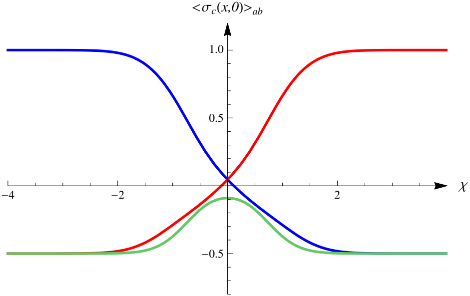
$$\langle \sigma_3(x,y)\rangle_{12} = -\frac{\langle\sigma_1\rangle_1}{2(q-1)}\left[1+\mathcal{G}(\chi)\right] \qquad \text{(bubble-like)}$$

Tricritical q-state Potts

Dilute 3-Potts: plot of $\langle \sigma_0(x,y) \rangle_{12}$



o.p. profiles $\langle \sigma_k(x,y) \rangle_{12}$



Dilute case: the bubble is not suppressed for $mR \gg 1$ (cf. pure 3-Potts)

Tricritical q-state Potts

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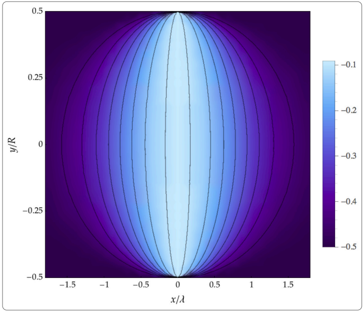

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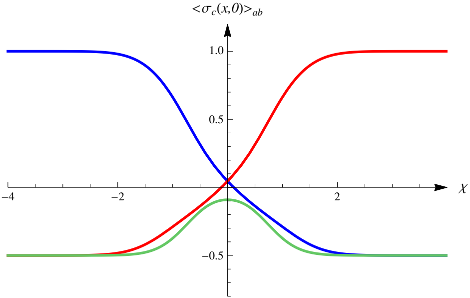

Dilute case: the bubble is not suppressed for $mR \gg 1$ (cf. pure 3-Potts)

- Passage probability matches field theory with

$$P(x_1, x_2; y=0) = \frac{2m}{\pi R}\boxed{(\eta_1-\eta_2)^2}\,\mathrm{e}^{-(\eta_1^2+\eta_2^2)}$$

the interfaces $\Omega_1|\Omega_0$, $\Omega_0|\Omega_2$ are mutually avoiding curves anchored in $(0, \pm R/2)$.

Bulk wetting transition: Ashkin-Teller (I)

Ising spins σ, τ on a lattice

$$\mathcal{H}_{AT} = -\sum_{\langle x_1, x_2 \rangle} \left[J\sigma(x_1)\sigma(x_2) + J\tau(x_1)\tau(x_2) + J_4\sigma(x_1)\sigma(x_2)\tau(x_1)\tau(x_2) \right]$$

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scaling AT(J_4) renormalizes into Sine-Gordon(β) $\Longrightarrow$ $J_4 \leftrightarrow \beta$ & kinks $\leftrightarrow$ solitons

$$\frac{4\pi}{\beta^2} = 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\tanh 2J_4}{\tanh 2J_4 - 1}\right)$$ on square lattice [Kadanoff]

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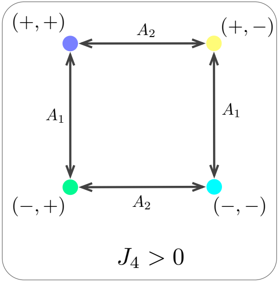
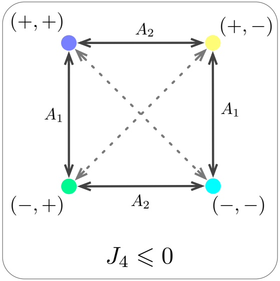
- Vacua connectivity





We can tune J_4 to change the vacua connectivity and the phase separation pattern $\rightarrow$ Transition!

Bulk wetting transition: Ashkin-Teller (II)

- Bulk wetting transition

$J_4 > 0$: drops of $\pm\mp$ phase are adsorbed along $(++)|(--)$ with contact angle γ
$J_4 \to 0^+$, $\gamma \to 0^+$: wetting
$J_4 \leqslant 0$: drops spreading, $(++)|(--)$ is wetted by $\pm\mp$ ($\gamma = 0$)

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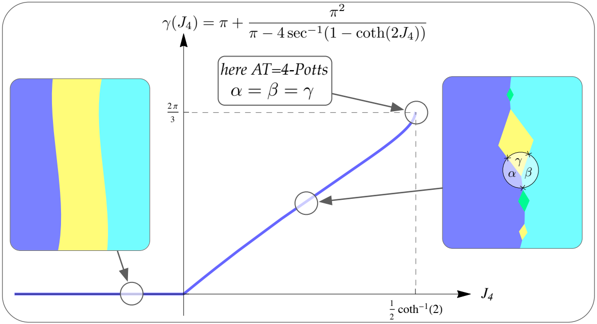

- Decoupling point $J_4 = 0$ Ising results are recovered
- Equilibrium condition for the triple line $\Longrightarrow$ contact angle

$$\gamma = 2\pi \frac{4\pi - \beta^2}{8\pi - \beta^2}$$

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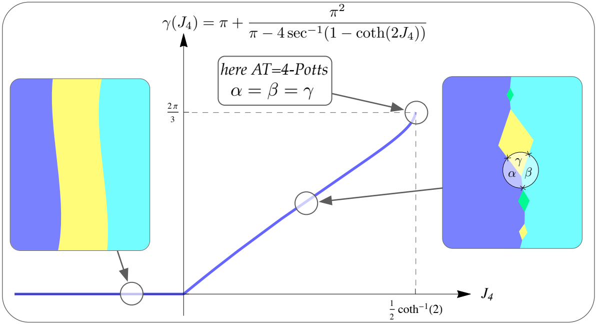

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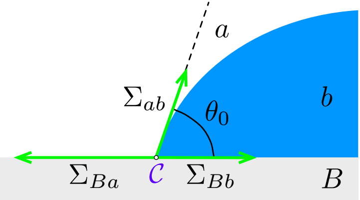
- Observables are sensitive only of the interaction sign: from $J_4 < 0$ to $J_4 > 0$

$$\begin{aligned}
\langle \sigma_i(x,y)\rangle_{(++,--)} \propto \mathcal{L}(\chi) &\quad\longrightarrow\quad \propto \mathrm{erf}(\chi)\\
\langle \sigma\tau(x,y)\rangle_{(++,--)} \propto \mathcal{G}(\chi) &\quad\longrightarrow\quad \propto \mathrm{erf}^2(\chi)\\
P(x;y) = (\chi_1-\chi_2)^2 p(\chi_1)p(\chi_2) &\quad\longrightarrow\quad = p(\chi_1)p(\chi_2)
\end{aligned}$$

Interfaces at boundaries

Phenomenological description in terms of contact angle and surface tensions



equilibrium condition for the contact line $\mathcal{C}$:

$$\boxed{\Sigma_{Ba} = \Sigma_{Bb} + \Sigma_{ab} \cos\theta_0} \qquad \text{(Young's law, 1802)}$$

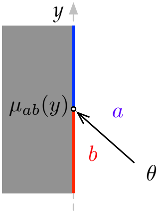
$\hookrightarrow \theta_0 \to 0$: wetting transition (spreading of the drop)

Interfaces at boundaries

Boundary field theory

[Delfino-AS, J Stat Mech '13]

- Vertical b.dry. Pinned interface selected with a b.dry changing field $\mu_{ab}(y)$: switches from B_a to B_b



$$ {}_0\langle\Omega_a|\mu_{ab}(y)|K_{ba}(\theta)\rangle_0 = \mathrm{e}^{-my\cosh\theta}\mathcal{F}_0^{\mu}(\theta) $$

linear behavior for small rapidities:

$$ \mathcal{F}_0^{\mu}(\theta) = c\theta + o(\theta) $$

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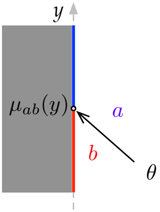

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$$

linear behavior for small rapidities:

$$
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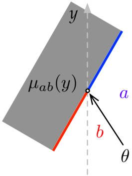
$$

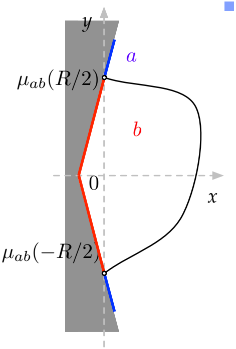
- Tilted b.dry: take an imaginary Lorentz boost ($\mathcal{B}_\Lambda : \theta \to \theta + \Lambda$)



$$
\mathcal{B}_{-i\alpha} : \mathcal{F}_0^{\mu}(\theta) \longrightarrow \mathcal{F}_\alpha^{\mu}(\theta) = \mathcal{F}_0^{\mu}(\theta + i\alpha)
$$

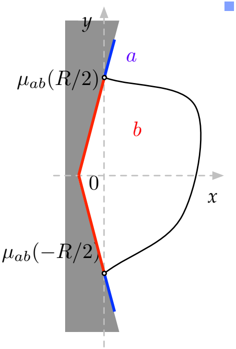
at small rapidities: $\quad \mathcal{F}_\alpha^{\mu}(\theta) \simeq c(\theta + i\alpha)$

Interfaces in a shallow wedge



- Order parameter in the wedge [Delfino-AS, PRL '13]

Interfaces in a shallow wedge



- Order parameter in the wedge [Delfino-AS, PRL '13]

$$\langle\sigma(x,y)\rangle_{W_{aba}} = \frac{{}_\alpha\langle\Omega_a|\mu_{ab}(0,R/2)\sigma(x,y)\mu_{ba}(0,-R/2)|\Omega_a\rangle_\alpha}{{}_\alpha\langle\Omega_a|\mu_{ab}(0,R/2)\mu_{ba}(0,-R/2)|\Omega_a\rangle_\alpha}$$

$$(\alpha \ll 1) \qquad = \langle\sigma\rangle_b + (\langle\sigma\rangle_a - \langle\sigma\rangle_b)\left[\mathrm{erf}(\chi) - \frac{2}{\sqrt{\pi}}\frac{\chi + \sqrt{2mR}\frac{\alpha}{\kappa}}{1+mR\alpha^2}\mathrm{e}^{-\chi^2}\right]$$

$\hookrightarrow$ recover results for lattice Ising with $\alpha = 0$ [Abraham, '80]

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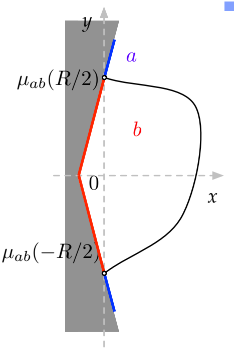

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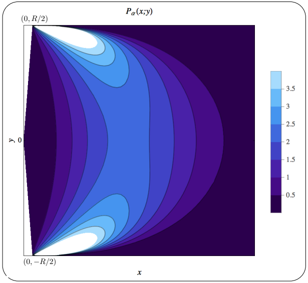
- Passage probability density

$$P(x;y) = \frac{8\sqrt{2}}{\sqrt{\pi}\kappa^3}\left(\frac{m}{R}\right)^{\frac{3}{2}}\frac{(x+\alpha R/2)^2 - (\alpha y)^2}{1+mR\alpha^2}\mathrm{e}^{-\chi^2}$$

  - Vanishes along the boundary.
  - Midpoint fluctuations $\sim \sqrt{R}$.



Boundary wetting & filling transitions

- Half plane

  The boundary amplitude may exhibit a simple pole at $\theta = i\theta_0$

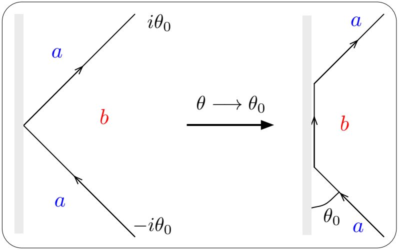
  kink + boundary $\rightarrow$ bound state $|\Omega_a\rangle'$

  with binding energy: $E_0' - E_0 = m\cos\theta_0$

  kink unbinding $\rightarrow$ wetting transition

  $$\theta_0(T_0) = 0 \quad , \quad T_0 < T_c$$

  resonant angle $\longleftrightarrow$ contact angle



Boundary wetting & filling transitions

- Half plane

The boundary amplitude may exhibit a simple pole at $\theta = i\theta_0$

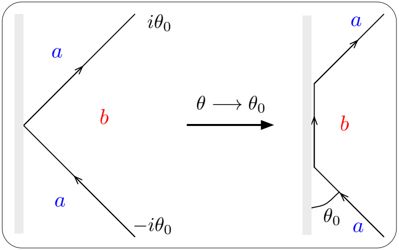
kink + boundary → bound state $|\Omega_a\rangle'$

with binding energy: $E_0' - E_0 = m\cos\theta_0$

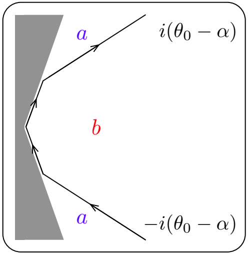
kink unbinding → wetting transition

$$\theta_0(T_0) = 0 \quad , \quad T_0 < T_c$$

resonant angle ⟷ contact angle



- Wedge



Lorentz invariance

$$\theta_0 \to \theta_0 - \alpha \qquad \text{(wedge covariance)}$$

condition encountered in effective hamiltonian theories

Kink unbinding ⟶ filling condition

$$E_\alpha' - E_\alpha = m\cos(\theta_0 - \alpha) \longrightarrow \theta_0(T_\alpha) = \alpha$$

condition known from macroscopic thermodynamic arguments [Hauge '92]

Summary & outlook

- A new method: exact and general field-theoretic formulation of phase separation and related issues (passage probabilities, interface structure (branching), interfaces at boundaries, wetting & filling)
- Phase separation is investigated for general models for the first time directly in the continuum, the known solutions from lattice for Ising are recovered as a particular case.
- Extended observables (interfaces) captured by local fields
- The validity of the technique *does not* rely on integrability but rather on the fact that domain walls are particle trajectories
- Although $mR \gg 1$ projects to low energies, relativistic particles are essential for kinematical poles and contact angles

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Perspectives

- Extensions to higher dimensions are possibile (e.g. 3D XY vortex profile [Delfino, 14]); what about more vortices?
- Connection with critical point &SLE?
- Different geometries
- ...

Thank you for your attention!