Random matrices and spin chains

Miguel Tierz
Departamento de Análisis Matemático
Mathematics Faculty
Universidad Complutense de Madrid
13/05/2015

GGI-Firenze

tierz@mat.ucm.es

Outline

- Spin chain

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators
  - Double scaling-limit and third order phase transition

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators
  - Double scaling-limit and third order phase transition
- Generalized spin chain

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators
  - Double scaling-limit and third order phase transition
- Generalized spin chain
  - Adding interactions: random matrix representation

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators
  - Double scaling-limit and third order phase transition
- Generalized spin chain
  - Adding interactions: random matrix representation
  - Gauge theory interpretation

Outline

- Spin chain
  - XX spin chain and definition of thermal correlation functions
  - Random matrix expression for the correlators
  - Double scaling-limit and third order phase transition
- Generalized spin chain
  - Adding interactions: random matrix representation
  - Gauge theory interpretation
- Conclusions and Outlook

- Consider a 1d infinite Heisenberg XX spin chain

$$\hat{H} = -\frac{1}{2}\sum_i \left(\sigma_i^- \otimes \sigma_{i+1}^+ + \sigma_i^- \otimes \sigma_{i-1}^+\right) + \frac{h}{2}\sum_i (\sigma_i^z - 1),$$

  summation is over all lattice sites. Recall $\sigma_i^{\pm} = \left(\sigma_i^x \pm i\sigma_i^y\right)/2$ and $[\sigma_i^+, \sigma_k^-] = \sigma_i^z \delta_{ik}$, $[\sigma_i^z, \sigma_k^{\pm}] = \pm 2\sigma_i^{\pm}\delta_{ik}$. Operators nilpotent $(\sigma_i^{\pm})^2 = 0$.
- Define its thermal correlation functions

$$F_{\alpha_1,\ldots,\alpha_K;\gamma_1,\ldots,\gamma_K}(\beta) = \langle \Uparrow | \sigma_{\alpha_1}^+ \cdots \sigma_{\alpha_N}^+ \mathrm{e}^{-\beta \hat{H}} \sigma_{\gamma_1}^- \cdots \sigma_{\gamma_N}^- | \Uparrow \rangle.$$

- $|\Uparrow\rangle$ denotes the ferromagnetic state: all the spins up $|\Uparrow\rangle = \otimes_i |\uparrow\rangle_i$. Satisfies $\sigma_k^+ |\Uparrow\rangle = 0$ for all k. The state is normalized $\langle \Uparrow | \Uparrow \rangle = 1$

- Work by Bogoliubov, Malyshev, Pronko, ... show that the thermal correlation functions can be written as a unitary matrix model

$$F_{\alpha;\gamma}(\beta) = \int_{-\pi}^{\pi} d\varphi_1 \cdots \int_{-\pi}^{\pi} d\varphi_N \prod_{1\leq j<k\leq N} \left| \mathrm{e}^{i\varphi_k} - \mathrm{e}^{i\varphi_j} \right|^2$$
$$\times \prod_{j=1}^{N} \mathrm{e}^{-\beta\cos\varphi_i} \overline{\hat{s}_{\widetilde{\alpha}}\left(e^{i\varphi_1}, \ldots, e^{i\varphi_N}\right)} \hat{s}_{\widetilde{\gamma}}\left(e^{i\varphi_1}, \ldots, e^{i\varphi_N}\right),$$

- The partitions on the Schurs and the pattern of flipped is spins are related by $\widetilde{\alpha}_i = \alpha_i - N + i$ and $\widetilde{\gamma}_i = \gamma_i - N + i$. The case without Schur polynomials follows by choosing $\{\alpha_i\}_{i=1}^{N} = N - i$ and $\{\delta_i\}_{i=1}^{N} = N - i$. Corresponds to a partition function in gauge theory.
- This is also the integral representation of the determinant of a Toeplitz minor.

XX Spin chain

Random matrix description, particular cases and gauge theory (David Pérez-García and MT, Phys. Rev. X 4, 021050)

- This model appears in gauge theory in two different ways: 1) in a lattice study of $U(N)$ 2d Yang-Mills theory (with Wilson lattice action) (Gross-Witten 1980) 2) Leutwyler-Gasser (87) and Leutwyler-Smilga (92) found essentially the same matrix integral in a effective field theory description of low-energy QCD at finite volume.
- The work (1) implies a 3rd order phase transition for the correlator (next slide), whereas the connection with QCD is:

$$\langle \ldots, \uparrow, \underbrace{\downarrow, \ldots, \downarrow}_{N_f}, \underbrace{\uparrow, \ldots, \uparrow}_{\nu} | \mathrm{e}^{-\beta \hat{H}_{XX}} | \underbrace{\downarrow, \downarrow, \ldots, \downarrow}_{N_f}, \uparrow, \ldots \rangle = Z^{\mathrm{eff}}_{\nu, N_f}(m)$$

where $Z^{\mathrm{eff}}_{\nu, N_f}(m)$ denotes the partition function and $\beta = mV\Sigma$.

- One of the central aspects of matrix models is the study of double-scaling limits: $N \to \infty$ whereas some parameter of the potential $\to 0$.
- In our context, we have that $N \to \infty$ and $\beta^{-1} \to 0$ such that $N/\beta = \lambda = \text{cte}$. It is a most unusual scaling limit from the point of view of the spin chain: Order of the correlator goes to ∞ while $T \to 0$.
- In this limit, the GW model is known to have a weak-coupling ($\lambda < 1$) and a strong-coupling ($\lambda > 1$) phase. At $\lambda = 1$ there is a third-order phase transition (third derivative of free energy is discontinuous at $\lambda = 1$).

- The free energy of the model in the two phases

$$\lim_{N\to\infty} F(\lambda) = \begin{Bmatrix} \frac{1}{4\lambda^2} & \lambda \geq 1 \\ \frac{1}{\lambda} + \frac{1}{2}\log\lambda - \frac{3}{4} & \lambda < 1 \end{Bmatrix},$$

  where $F(\lambda) = \frac{1}{N^2}\ln Z_N$ and $\lambda = N/\beta$.
- Remnants of the phase transition for finite N have also been explored. There is of course no phase transition but there is a cross-over between the two regimes. Appreciable even for $U(2)$ (two spins flipped). For example, in the strong-coupling phase $|F_N(\lambda) - F(\lambda)| \leq Ce^{-cN}$.

- We consider the XX Hamiltonian extended to admit generic interactions, denoted by a_j, to infinitely many neighbors

$$\hat{H}_{\mathrm{Gen}} = -\sum_i \sum_{j\in\mathbb{Z}}' a_j \left(\sigma_i^- \otimes \sigma_{i+j}^+\right) + \frac{h}{2}\sum_i (\sigma_i^z - 1).$$

- We have the following commutation relations

$$[\sigma_j^+, \hat{H}] = -\sigma_j^z \sum_{i\in\mathbb{Z}} a_i \sigma_{j+i}^+ - h\sigma_j^+$$

which give for the correlator $G_{jl}(\beta) = \langle \Uparrow | \sigma_j^+ \mathrm{e}^{-\beta \hat{H}_{\mathrm{Gen}}} \sigma_l^- | \Uparrow \rangle$ $(N = 1)$ the following differential-difference equation:

$$\frac{d}{d\beta} G_{jl}(\beta) = \sum_{i\in\mathbb{Z}} a_i G_{j+i,l}(\beta) + hG_{jl}(\beta).$$

- The final general result is

$$
\begin{aligned}
&\langle \Uparrow | \sigma_{j_1}^{+} \cdots \sigma_{j_N}^{+} \mathrm{e}^{-\beta \hat{H}_{Gen}} \sigma_{l_1}^{-} \cdots \sigma_{l_N}^{-} | \Uparrow \rangle = \\
&\frac{\mathrm{e}^{\beta h N}}{(2\pi)^N n!} \int_{-\pi}^{\pi} d\varphi_1 \cdots \int_{-\pi}^{\pi} d\varphi_N \prod_{1 \le j < k \le N} \left| e^{i\varphi_k} - e^{i\varphi_j} \right|^2 \\
&\times \left(\prod_{j=1}^{N} g_\beta(\varphi_j) \right) \overline{\hat{s}_\mu \left(e^{i\varphi_1}, \ldots, e^{i\varphi_N} \right)} \hat{s}_\lambda \left(e^{i\varphi_1}, \ldots, e^{i\varphi_N} \right),
\end{aligned}
$$

with

$$
g_\beta(\lambda) = g_0(\lambda) \exp\left(\beta \sum_{i \in \mathbb{Z}} a_i \lambda^i \right).
$$

- We consider Chern-Simons theory on a three-manifold M and for a gauge group G, with action

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

  where A is a connection on M and $k \in \mathbb{Z}$.

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)},$$

  defines a topological invariant.

Unitary matrix model

M. Romo and MT, Phys. Rev. D 86, 045027 (2012), R. Szabo and MT, Nucl. Phys. B 876 (2013) 234-308

- U(N) Chern-Simons theory on S^3 has a random matrix description. For the partition function:

$$Z_{\mathrm{CS}}^{U(N)}\left(S^3\right) = \int_0^{2\pi} \prod_{j=1}^{N} \frac{\mathrm{d}\theta_j}{2\pi}\, \Theta(\mathrm{e}^{\mathrm{i}\theta_j}|q) \prod_{k<l} \left|\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_l}\right|^2,$$

  where the weight function is Jacobi's third theta function

$$\omega\left(\theta\right) = \Theta(\mathrm{e}^{\mathrm{i}\theta}|q) = \sum_{n=-\infty}^{\infty} \mathrm{q}^{n^2/2} \mathrm{e}^{in\theta},$$

- For Wilson loops we have precisely insertions of Schur polynomials on the integrand. Exactly as in the spin chain description.

- Hence, to reproduce Chern-Simons theory we have to consider the generalized spin chain in the specific setting where the resulting generating function is the theta function

$$g(\varphi) = \Theta_3(\mathrm{e}^{\mathrm{i}\varphi}|q) = \sum_{n=-\infty}^{\infty} \mathrm{q}^{n^2/2}\mathrm{e}^{in\varphi},$$

- The Fourier coefficients of $\ln \Theta_3(\mathrm{e}^{\mathrm{i}\varphi}|q)$ can be easily obtained by using the Jacobi triple product identity, giving

$$a_k = 1/(2k \sinh(kg_s/2)), \tag{1}$$

with k denoting positive and negative integers and with q-parameter $q = \exp(-g_s)$. We have to choose $\beta = 1$ in the thermal average because plugging (1) in the generating function gives $\Theta_3^{\beta}(\mathrm{e}^{\mathrm{i}\varphi}|q)$.

- Particularizing to obtain the CS matrix model, we have that the topological S-matrix (Hopf link in CS theory)

$$S_{\lambda\mu} = \langle \Uparrow | \sigma_{j_1}^{+} \cdots \sigma_{j_N}^{+} \mathrm{e}^{-\hat{H}_{CS}} \sigma_{l_1}^{-} \cdots \sigma_{l_N}^{-} | \Uparrow \rangle,$$

where

$$\hat{H}_{CS} = -\sum_{i,k\in\mathbb{Z}_+} \frac{1}{2k\sinh\left(\frac{kg_s}{2}\right)} \left(\sigma_i^{-} \otimes \sigma_{i+k}^{+} + \sigma_i^{-} \otimes \sigma_{i-k}^{+}\right) + \frac{h}{2}\sum_i (\sigma_i^z - 1).$$

Summary

- Thermal correlation functions of 1d spin chain models admit random matrix representations

Summary

- Thermal correlation functions of 1d spin chain models admit random matrix representations
- The matrix models also appear in problems of gauge theory (low-energy, finite volume QCD; Chern-Simons theory). The double-scaling limit of the matrix model is interesting: 3rd order phase transitions. This implies a phase transition in the correlator of the spin chain (exotic: the order of the correlator to ∞ and $T \to 0$).

Summary

- Thermal correlation functions of 1d spin chain models admit random matrix representations
- The matrix models also appear in problems of gauge theory (low-energy, finite volume QCD; Chern-Simons theory). The double-scaling limit of the matrix model is interesting: 3rd order phase transitions. This implies a phase transition in the correlator of the spin chain (exotic: the order of the correlator to ∞ and $T \to 0$).
- Realistic chains are finite, which implies a discretization of the matrix model. The relative error with the infinite chain is exponentially small (Baik-Liu) and hence the correlators may be measured experimentally. Relationship between finite and infinite chains well-known in the double scaling limit (Baik-Liu).

Open problems

- Study the XXZ model instead because then the model has an stochastic interpretation in terms of an exclusion process. Add additional interactions (simplest case: interaction to next to nearest-neighbours) and use the corresponding matrix model formulation of the correlator to obtain analytical results for the (generalized) exclusion processes.

Open problems

- Study the XXZ model instead because then the model has an stochastic interpretation in terms of an exclusion process. Add additional interactions (simplest case: interaction to next to nearest-neighbours) and use the corresponding matrix model formulation of the correlator to obtain analytical results for the (generalized) exclusion processes.
- Unitary matrix models with polynomial potentials $V(U) = \sum_{k=-m}^{m} t_k \mathrm{Tr} U^k$ have been studied in the large N limit, with analytical results in terms of Painlevé equations.

Open problems

- Study the XXZ model instead because then the model has an stochastic interpretation in terms of an exclusion process. Add additional interactions (simplest case: interaction to next to nearest-neighbours) and use the corresponding matrix model formulation of the correlator to obtain analytical results for the (generalized) exclusion processes.
- Unitary matrix models with polynomial potentials $V(U) = \sum_{k=-m}^{m} t_k \mathrm{Tr} U^k$ have been studied in the large N limit, with analytical results in terms of Painlevé equations.
- Phase transitions are very sensitive to the form of the potential of the unitary random matrix ensemble. Study if the associated spin chains have a phase transition in the thermal correlator.

Open problems

- Study the XXZ model instead because then the model has an stochastic interpretation in terms of an exclusion process. Add additional interactions (simplest case: interaction to next to nearest-neighbours) and use the corresponding matrix model formulation of the correlator to obtain analytical results for the (generalized) exclusion processes.
- Unitary matrix models with polynomial potentials $V(U) = \sum_{k=-m}^{m} t_k \mathrm{Tr} U^k$ have been studied in the large N limit, with analytical results in terms of Painlevé equations.
- Phase transitions are very sensitive to the form of the potential of the unitary random matrix ensemble. Study if the associated spin chains have a phase transition in the thermal correlator.
- Adler-van Moerbecke (2004) have very similar integral rep. for non-intersecting walkers, using Virasoro action on 2d Fourier series in Schur polynomials. It would be interesting to compare.